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On The Fundamental Solution of Fractional Diffusion-Wave Equation

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BY

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Abstract

In this thesis, explicit closed form for the fundamental solution of multi-dimensional space-time fractional diffusion-wave equation

$${}^{c}D_{t}^{\beta}u + (-\triangle)^{\frac{\alpha}{2}}u = 0$$

with Cauchy data is considered. The Mellin transform is employed along with the Mellin-Barnes transform for specific values of α and β to achieve the representation of the fundamental solution in closed form.

Contents

Chapter 1

Introduction

Fractional Calculus is the generalization of calculus to non integer order. It is the study of an extension of integer order derivative and integrals to non integer order. Fractional calculus can be applied on both ordinary and partial differential equations. In this generalization integer order calculus is the special case. A differential equation is fractional if it involves an operator that can considered to be between $(n-1)^{st}$ and n^{th} order differential operator for some positive integer n and it is said to be of fractional order if this operator is the highest order operator in the equation. Nowadays fractional partial differential equation(FPDE) has attracted the attention of several researchers and has found numerous applications in real world phenomenon. Some of its well known applications are in anomalous diffusion , wave propagation , modeling different materials. Moreover the fractional diffusion-wave equation is widely used in different areas of engineering and science. The main focus of this thesis is to find explicit closed form formula for the fundamental solution of the multi-dimensional space- and time- fractional diffusion-wave equation.Representation of the fundamental solution of the multi-dimensional space- and time- fractional diffusion-wave equation in terms of known special functions is obtained from Mellin-Barnes integral by employing the method of Mellin transform. The necessary and/or sufficient conditions for the positivity of the fundamental solution of the multi-dimensional diffusion-wave equation is still an open problem. The fundamental solution of multi-dimensional space- and time- fractional diffusion-wave equation and its new properties was discussed briefly in [14] and also some of its closed forms was considered in [14]. Some additional closed forms of the fundamental solution to the multi-dimensional space- and time- fractional diffusion-wave equation was discussed in [13]. Moreover, the list of diffusion equation , space-fractional diffusion equation, time-fractional diffusion-wave equation and one dimensional α -fractional wave equation was mentioned and discussed in [11]. The fundamental solution to the time-fractional diffusion-wave equation is solved for the case of $\alpha \in (0, 2]$ in terms of the special function of Fox H- function in [9].

The rest of this thesis is organized as follows, in the next chapter we will discuss some basic mathematical functions that have a significant role in fractional calculus this includes radial function, Gamma function, the Error function and the Mittage-Leffler function. Then we will see some basic properties of fractional differentiation and integration with the most commonly used definitions of fractional differentiation and integration. Finally we focus on the application of fractional calculus on both ordinary and partial differential equation and we deal with the main results of the closed form formula for the fundamental solution.

Chapter 2

Preliminaries

In what follows we visit some special functions that may arise in connection with fractional calculus and hence with fractional differential equations and fractional integral equations.

2.1 Basic Definitions and Special Functions

2.1.1 Gamma Function

The Gamma function which essentially plays the role of generalized factorial is defined as

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, \qquad x \in \mathbb{R}_+$$

By using its recursion relations we obtain

$$\Gamma(x+1) = x\Gamma(x) \qquad x \in \mathbb{R}_+$$

$$\Gamma(x) = (x-1)! \qquad x \in \mathbb{N}$$

One can alternatively define the Gamma function by

$$\Gamma(x) = \lim_{n\to\infty} \frac{n! n^x}{x(x+1)\cdots(x+n)}$$

Indeed this is obtained by applying integration by parts (n-times) and recalling that

$$e^{-t} = \lim_{n\to\infty} \left(1 - \frac{t}{n}\right)^n$$

2.1.2 Beta Function

The Beta function is basically useful for computing fractional derivatives of a power function and is given by the definite integral

$$B(u,w) = \int_0^1 (1-v)^{w-1} v^{u-1} dv \qquad \text{for } u > 0 \text{ and } w > 0$$

The Beta function can also be written in terms of Gamma function as

$$B(u,w) = \frac{\Gamma(u)\Gamma(w)}{\Gamma(u+w)} \qquad \text{for } u > 0 \text{ and } w > 0$$

This relation between the beta and the gamma functions which is established by Laplace transform of convolutions, evidently shows that the beta function is symmetric,

$$B(u,w) = B(w,u)$$

2.1.3 The Error Function

The Error function is given by (see [15])

$$Erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt, \qquad x \in \mathbb{R} \tag{2.1}$$

The complementary Error function (Erfc) is closely related to the error function and hence can be written in terms of the Error function as

$$Erfc(x) = 1 - Erf(x)$$

As a result of (2.1) we have

$$Erf(0) = 0 \qquad \text{and} \qquad Erf(\infty) = 1$$

2.1.4 Mittag-Leffler Functions

The Mittag-Leffler functions are one of the basic functions that evolve with the solution of fractional differential equations.

The one parameter Mittage-Leffler function can be defined as (see [10])

$$E_\beta(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\beta n + 1)}, \qquad \beta > 0$$

while the two parameter Mittage-Leffler function is given by

$$E_{\beta,\gamma}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\beta n + \gamma)}, \qquad \beta, \gamma > 0$$

The two parameter Mittage-Leffler functions that oftentimes arise in the solution of fractional differential equations have some interesting properties worth mentioning.

1. The series associated with its definition is uniformly convergent with the radius of convergent $r = \infty$

2. If both parameters are unit, i.e $\alpha = 1 = \beta$ then

$$E_{1,1}(x) = e^x$$

3. If $\beta = 1$ then

$$E_{\alpha,1} = E_\alpha(x)$$

4. From the definition follows the following relation

$$E_{\alpha,\beta}(x) = \frac{1}{\Gamma(\beta)} + xE_{\alpha,\alpha+\beta}(x)$$

$$E_{\alpha,\beta}(x) = \beta E_{\alpha,\beta+1}(x) + \alpha x \frac{d}{dx} E_{\alpha,\beta+1}(x) \text{and}$$

$$\left(\frac{d}{dx}\right)^m \left[x^{\beta-1} E_{\alpha,\beta}(x^\alpha)\right] = x^{\beta-m-1} E_{\alpha,\beta-m}(x^\alpha), \beta - m > 0, m = 0, 1, 2, \cdots$$

Since the power series in the definition of $E_{\beta,\gamma}$ is uniformly convergent, term by term differentiation yields

$$E^{[n]}_{\beta,\gamma}(x) = \sum_{k=0}^{\infty} \frac{(k+n)!}{k!} \frac{t^k}{\Gamma(\beta k + \beta n + \gamma)}$$

Furthermore, specific values of β and γ leads to well known functions. We shall mention a few of them

$$E_{2,1}(x^2) = Cosh(x)$$

$$E_{2,2}(x^2) = \frac{Sinh(x)}{x}$$

$$E_{\frac{1}{2},1}(x) = e^{x^2} erfc(-x)$$

2.1.5 Radial Function

A radial function is a function

$$\eta : \mathbb{R}_+ \to \mathbb{R}$$

satisfying

$$\eta(x; x_o) := \eta(|x - x_o|)$$

for some points x_o in some subset $E \subset \mathbb{R}^n$ and $|\cdot|$ denotes standard Euclidean norm in $\mathbb{R}^n$. This concept is of paramount importance when dealing with problems that involve circular symmetry, mainly due to the fact that the dependence is only on the radial distance.

2.2 Fractional Derivative and Integration

In this subsection we will see the Riemann-Liouville and Caputo's definition of fractional integrals and fractional derivatives.

Definition 1 (Fractional Integral (Riemann-Liouville)). *This definition is one of the familiar definitions of fractional integration and is given by*

$$\mathcal{J}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x - \tau)^{\alpha - 1} f(\tau) d\tau$$

Here, $\mathcal{J}^{\alpha}$ denotes the fractional integral of order $\alpha > 0$

Definition 2 (Fractional Derivative (Riemann-Liouville)). *The Riemann-Liouville fractional derivative is defined as*

$$D^{\gamma}g(x) = \frac{1}{\Gamma(m-\gamma)}\frac{d^m}{dx^m}\int_0^x \frac{1}{(x-t)^{1+\gamma-m}}g(t)dt \qquad m-1<\gamma<m$$

D^{γ} denotes the fractional derivative of order $\gamma > 0$

Definition 3 (Fractional Derivative (Caputo)). [1] *For $m-1<\beta<m$ the Caputo fractional derivative is given by*

$$^{c}D^{\beta}g(x) = \frac{1}{\Gamma(m-\beta)}\int_0^x \frac{1}{(x-t)^{1+\beta-m}}g^{m}(t)dt \qquad m-1<\beta<m$$

In particular when $0<\beta<1$ the Caputo fractional derivative becomes

$$^{c}D^{\beta}f(x) = \frac{1}{\Gamma(1-\beta)}\int_0^x \frac{1}{(x-t)^{\beta}}f^{'}(t)dt$$

Definition 4 (Euler's Generalization). *Euler generalized the formula*

$$\frac{d^n x^m}{dx^n} = m(m-1)\cdots(m-n+1)x^{m-n}$$

by using the property of gamma function

$$m(m-1)\cdots(m-n+1) = \frac{\Gamma(m+1)}{\Gamma(m-n+1)}$$

and then arrived at

$$\frac{d^n x^m}{dx^n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)}x^{m-n}$$

2.3 Properties of fractional derivative and integration

In these section we are going to see some of the basic properties of Riemann-Lioville integral and derivative. For a function $h \in \mathcal{L}^1(\mathbb{R})$ and $\alpha, \beta \geq 0$ the

[1] Relation between Caputo and Riemann-Liouville derivatives

$$({}^{c}D^{\alpha}f)(x) = \left(D^{\alpha}\left(f(t) - \sum_{k=0}^{n-1}\frac{f^{[k]}(0)}{k!}t^k\right)\right)(x), \quad n-1<\alpha<n,\ x>0$$

following properties hold true

Iteration property

$$\mathcal{J}^{\alpha}\mathcal{J}^{\beta}h(x) = \mathcal{J}^{\alpha+\beta}h(x)$$

Commutative property

$$\mathcal{J}^{\alpha}\mathcal{J}^{\beta}h(x) = \mathcal{J}^{\beta}\mathcal{J}^{\alpha}h(x)$$

For any two functions $h(x)$ and $g(x)$ on the closed interval $[c,d]$, K $\in \mathbb{R}$ and $0 < \alpha < \beta$ the following holds

$$D^{\alpha}(h+g)(x) = D^{\alpha}h(x) + D^{\alpha}g(x)$$

$$D^{\alpha}(Kh)(x) = KD^{\alpha}h(x)$$

$$D^{0}h(x) = h(x)$$

Illustration

1, Unit Function[2]
For $f(x) = 1,\ \alpha > 0$ we have

$$\frac{d^{\alpha}f(x)}{dx^{\alpha}} = \frac{d^{\alpha}1}{dx^{\alpha}} = \frac{x^{-\alpha}}{\Gamma(1-\alpha)}$$

2, Identity Function
For $f(x) = x, \alpha > 0$ we have

$$\frac{d^{\alpha}f(x)}{dx^{\alpha}} = \frac{d^{\alpha}x}{dx^{\alpha}} = \frac{x^{1-\alpha}}{\Gamma(2-\alpha)}$$

3, Exponential Function

[2] If k is a non-zero constant then

$$D^{\alpha}k = \frac{kt^{-\alpha}}{\Gamma(1-\alpha)} \neq 0 = \ {}^{c}D^{\alpha}k$$

Fractional deriviative of the function $f(x) = e^x$, $\alpha > 0$ is

$$\frac{d^\alpha e^x}{dx^\alpha} = \sum_{k=0}^{\infty} \frac{x^{k-\alpha}}{\Gamma(k-\alpha+1)}$$

4, For the monomial x^ν and $\beta > 0$ [3]

$$D^\beta x^\nu = \frac{\Gamma(\nu+1)}{\Gamma(\nu-\beta+1)} x^{\nu-\beta} \qquad \nu \geq 0 \qquad 0 < \beta < 1$$

For $\beta = \frac{1}{2}$

$$\begin{aligned}
D^{\frac{1}{2}} x^0 &= D^{1-\frac{1}{2}} x^0 = D^1 \left[D^{\frac{-1}{2}} x^0 \right] = D \left[2\sqrt{\frac{x}{\pi}} \right] = \frac{1}{\sqrt{x\pi}} \\
D^{\frac{1}{2}} x^1 &= D^{1-\frac{1}{2}} x^1 = D^1 \left[D^{\frac{-1}{2}} x^1 \right] = D \left[\frac{4}{3}\sqrt{\frac{x^3}{\pi}} \right] = 2\sqrt{\frac{x}{\pi}}
\end{aligned}$$

5, For $\beta > 0$, the fractional integral of the monomial x^ν is given by

$$\mathcal{J}^\beta x^\nu = \frac{\Gamma(\nu+1)}{\Gamma(\nu+\beta+1)} x^{\nu+\beta}$$

For $\beta = \frac{1}{2}$

$$\begin{aligned}
\mathcal{J}^{\frac{1}{2}} x^0 &= \frac{\Gamma(1)}{\Gamma(\frac{3}{2})} x^{0+\frac{1}{2}} = 2\sqrt{\frac{x}{\pi}} \\
\mathcal{J}^{\frac{1}{2}} x^1 &= \frac{\Gamma(2)}{\Gamma(\frac{5}{2})} x^{1+\frac{1}{2}} = \frac{4}{3}\sqrt{\frac{x^3}{\pi}} \\
\mathcal{J}^{\frac{1}{2}} x^2 &= \frac{\Gamma(3)}{\Gamma(\frac{7}{2})} x^{2+\frac{1}{2}} = \frac{16}{15}\sqrt{\frac{x^5}{\pi}}
\end{aligned}$$

[3]

$$D_t^\alpha [f(at)] = a^\alpha D_x^\alpha [f(x)] \;;\; x = at$$

Chapter 3

Transforms and Fundamental Solution

Integral transform such as Mellin, Fourier and Mellin-Barnes are useful for the construction of solution of differential equations that involve fractional derivatives of Riemann-Liouville and Caputo type(Kilbas).

3.1 Integral Transforms

3.1.1 Fourier Transform

The fact that Fourier transform is used for solving partial differential equations is well known. In what follows we shall recall some of the properties of the classical Fourier transform (FT) so as to pave the way for introducing the fractional Fourier Transform (FrFT).

The Fourier transform in one dimension

The Fourier transform of a function $f(t)$ of real variable $t \in \mathbb{R}$ is defined by

$$(\mathcal{F}f)(x) = \mathcal{F}[f(t)](x) = \hat{f}(t) = \int_{-\infty}^{\infty} e^{ixt} f(t)dt, \qquad x \in \mathbb{R} \tag{3.1}$$

The inverse Fourier transform is given by

$$(\mathcal{F}^{-1}g)(x) = \mathcal{F}^{-1}[g(t)](x) = \frac{1}{2\pi}\hat{g}(-x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixt} g(t)dt \tag{3.2}$$

The integral in (3.1) and (3.2) converges absolutely for functions $f, g \in \mathcal{L}^1(\mathbb{R})$ and in the norm of the space $\mathcal{L}^2(\mathbb{R})$ for $f, g \in \mathcal{L}^2(\mathbb{R})$. Furthermore it has the following properties.

1. $\mathcal{F}^{-1}\mathcal{F}f = f$
2. $\mathcal{F}\mathcal{F}^{-1}g = g$

If we use $\mathcal{F}^n = \mathcal{F}F^{n-1}(n = 1, 2, \cdots)$ for the composition with $\mathcal{F}^0 = I_d$ we have the following properties.

1. $(\mathcal{F}^2 f)(x) = f(-x)$
2. $(\mathcal{F}^3 f)(x) = \check{f}(-x)$
3. $(\mathcal{F}^4 f)(x) = f(x)$

From this we observe that the composition of Fourier Transform, $\mathcal{F}^n$ $(n \in \mathcal{N})$ can be identified with a rotation through an angel $\theta = n\frac{\pi}{2}$ in a plane.

The following properties are related to transform of derivatives and convolutions.

1. $\mathcal{F}[D^n f(t)](x) = (-ix)^n (\mathcal{F}f)(x) \qquad n \in \mathbb{N}$
2. $D^n(\mathcal{F}f)(x) = \mathcal{F}[(it)^n f(t)](x) \qquad n \in \mathbb{N}$
3. $\mathcal{F}(f * g)(x) = (\mathcal{F}f)(x)(\mathcal{F}g)(x)$

The Fourier transform in n-dimensions

The n-dimensional Fourier transform of a function $f(t)$ of $t \in \mathbb{R}^n$ is given by

$$(\mathcal{F}f)(x) = \mathcal{F}[f(t)](x) = \hat{f}(x) = \int_{\mathbb{R}^n} e^{ix\cdot t} f(t)dt \qquad x \in \mathbb{R}^n \tag{3.3}$$

The pertinent inverse transform is given by

$$(\mathcal{F}^{-1}g)(x) = \mathcal{F}^{-1}[g(t)](x) = \frac{1}{2\pi}\hat{g}(-x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-ixt} g(t)dt \tag{3.4}$$

It has the same properties as the one dimensional case.

We remark that, if $\triangle$ is the n-dimensional Laplace operator

$$\triangle = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

then

$$(\mathcal{F}\triangle f(t))(x) = |x|^2(\mathcal{F}f)(x) \qquad x \in \mathbb{R}^n$$

The Fractional Fourier Transform

The fractional Fourier transform of order α $(0 < \alpha < 1)$ of a function $f \in \mathcal{L}^1(\mathbb{R})$ is given by

$$(\mathcal{F}_\alpha f)(x) = \int_{-\infty}^{\infty} e_\alpha(x,t)f(t)dt, x \in \mathbb{R}$$

Where

$$e_\alpha(x,t) = \begin{cases} e^{-i|x|^{\frac{1}{\alpha}}t}, & x < 0 \\ e^{i|x|^{\frac{1}{\alpha}}t}, & x \geq 0 \end{cases}$$

Here, the kernel e_α coincides with kernel of the usual Fourier transform, whenever $\alpha = 1$. Recalling that

$$x = \begin{cases} |x|, & x \geq 0 \\ -|x|, & x < 0. \end{cases}$$

We have

$$\begin{aligned} e_1(x,t) &= \begin{cases} e^{-i|x|t}, & x < 0 \\ e^{i|x|t}, & x \geq 0 \end{cases} \\ &= e^{ixt} \end{aligned}$$

The fractional Fourier transform of order 1 is the usual Fourier transform. In other words the usual Fourier transform can be seen as a special case of Fractional Fourier transform.

3.1.2 Mellin Transform

Let f be a function of real variable $x \in (0,\infty)$. If $x^{s-1}f(x) \in \mathcal{L}^1_{loc}(\mathcal{R}_+)$ then the Mellin transform of f is defined as (see [16])

$$(Mf)(s) = \int_0^{\infty} x^{s-1}f(x)dx := \check{f}(s), \ \ s \in \mathcal{C}$$

and the inverse Mellin transform is given by

$$(M^{-1}g)(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}g(s)ds, \ \ \text{c = Re(s)}$$

Properties (for $n \in \mathbb{N}$)

1. Scaling property
$$(Mf(nt))(s) = n^{-s}\check{f}(s)$$

2. Multiplication by t^n
$$(Mt^n f(t))(s) = \check{f}(s+n)$$

3. Transform of Derivative
$$(MD^n f)(s) = (-1)^n \frac{\Gamma(s)}{\Gamma(s-n)}\check{f}(s-n)$$

4. Derivative of Transform
$$D^n(Mf)(s) = M((\log x)^n f)(s-1)$$

5. Inverse
$$M^{-1}Mf = f \ \text{ and } \ MM^{-1}g = g$$

Mellin and Fourier Transform

By suitable change of variables, the Mellin integral transform can be converted into Fourier transform. To this end, if we set

$$x = e^t$$

We have

$$\begin{aligned}
\int_0^\infty x^{s-1}f(x)dx &= \int_{-\infty}^{\infty} e^{st}f(e^t)dt \\
&= \int_{-\infty}^{\infty} e^{i\gamma t}\underbrace{(e^{\varsigma t}f(e^t))}_{g(t)}dt \quad s = \varsigma + i\gamma \\
&= \int_{-\infty}^{\infty} e^{i\gamma t}g(t)dt
\end{aligned}$$

And hence [1]

$$(Mf)(s) = (\mathcal{F}g)(\gamma)$$

Mellin transform of fractional order

For $\quad 0 \leq n-1 < \alpha < n, \quad n \in \mathbb{N}$

$$M(D^\alpha f(t))(s) = \frac{\Gamma(1-s+\alpha)}{\Gamma(1-s)} \check{f}(s-\alpha)$$

where D^α is the Riemamm-Liovuille fractional derivative. Furthermore, for any $\alpha > 0$

$$M(t^\alpha f(t))(s) = \check{f}(s+\alpha)$$

And

$$M(f(t^\alpha))(s) = \frac{1}{\alpha} \check{f}\left(\frac{s}{\alpha}\right)$$

3.2 Fundamental Solution of Diffusion Equation

The n-dimensional Diffusion equation is given by

$$u_t(x,t) = \triangle u(x,t), \qquad x \in \mathbb{R}^n, \qquad t > 0 \tag{3.5}$$

Equation (3.5) can be written as

$$u_t - \triangle u = 0$$

For steady state

$$\triangle u = 0$$

i.e

$$u_{x_1 x_1} + u_{x_2 x_2} + \cdots + u_{x_n x_n} = 0$$

If we replace each $u_{x_n x_n}$ by $(x_n)^2$ then we have

$$(x_1)^2 + (x_2)^2 + \cdots + (x_n)^2 = |x|^2$$

[1]Mellin Transform is related to Laplace Transform

$$t = e^{-x} \Rightarrow (Mf(t))(s) = (\mathcal{L}f(e^{-x}))(s)$$

Hence, for the equation

$$u_t - \triangle u = 0$$

we have

$$t - |x|^2 = 0$$

This implies that

$$\frac{|x|}{\sqrt{t}} = 1$$

And hence we may take

$$u(x,t) = \frac{1}{t^{\frac{n}{2}}} v\left(\frac{x}{\sqrt{t}}\right) \tag{3.6}$$

for a given function

$$v : \mathbb{R}^n \to \mathbb{R}$$

If we further assume that v is radial

$$v = w(|y|)$$

and is a particular solution of equation (3.5) and then plugging equation (3.6) in to equation (3.5) leads to

$$\begin{aligned}
\left(\frac{1}{t^{\frac{n}{2}}} v\left(\frac{x}{\sqrt{t}}\right)\right)_t &= \triangle\left(\frac{1}{t^{\frac{n}{2}}} v\left(\frac{x}{\sqrt{t}}\right)\right) \\
\frac{-n}{2} t^{\frac{-n}{2}-1} v(y) + t^{\frac{-n}{2}} \left(v\left(\frac{x}{\sqrt{t}}\right)\right)_t &= t^{\frac{-n}{2}} \left(\frac{1}{\sqrt{t}}\right)\left(\frac{1}{\sqrt{t}}\right) \triangle_y v(y)
\end{aligned}$$

and

$$\begin{aligned}
\left(v\left(\frac{x}{\sqrt{t}}\right)\right)_t &= \left(v\left(\frac{x_1}{\sqrt{t}}, \frac{x_2}{\sqrt{t}}, \cdot, \frac{x_n}{\sqrt{t}}\right)\right) \\
&= v_{y1}\left(\frac{-x_1}{2t\sqrt{t}}\right) + \cdots + v_{yn}\left(\frac{-x_n}{2t\sqrt{t}}\right) \\
&= \frac{-1}{2t}\left(v_{y1}\left(\frac{x_1}{\sqrt{t}}\right) + \cdots + v_{yn}\left(\frac{x_n}{\sqrt{t}}\right)\right)
\end{aligned}$$

Therefore

$$\left(v\left(\frac{x}{\sqrt{t}}\right)\right)_t = \frac{-1}{2t} D_y v \cdot y$$

But then we have

$$\begin{aligned}
\frac{-n}{2} t^{\frac{-n}{2}-1} v + t^{\frac{-n}{2}} \frac{-1}{2t} D_y v \cdot y &= t^{\frac{-n}{2}-1} \triangle_y v \\
\frac{-n}{2} t^{\frac{-n}{2}-1} v - \frac{1}{2} t^{\frac{-n}{2}-1} D_y v \cdot y &= t^{\frac{-n}{2}-1} \triangle_y v \\
\frac{-n}{2} v - \frac{1}{2} D_y v \cdot y &= \triangle_y v \\
\frac{n}{2} v + \frac{1}{2} D v \cdot y + \triangle v &= 0
\end{aligned}$$

Since v is radial

$$v(y) = w(|y|) = w(r)$$

for some $w : \mathbb{R}^n \to \mathbb{R}$ so we have

$$Dv = w^{'}(r) \frac{y}{|y|}$$

$$\triangle v = w^{''} + \left(\frac{n-1}{r} \right) w^{'}$$

It then follows that

$$\begin{aligned}
\frac{n}{2} w + \frac{1}{2} w^{'} \frac{y}{|y|} \cdot y + w^{''} + \left(\frac{n-1}{r} \right) w^{'} &= 0 \\
\frac{n}{2} w + \frac{1}{2} w^{'} r + w^{''} + \left(\frac{n-1}{r} \right) w^{'} &= 0
\end{aligned}$$

$$\frac{1}{2} \left(w^{'} r + wn \right) + w^{''} \cdot 1 + \frac{n-1}{r} w^{'} = 0 \tag{3.7}$$

In order to apply product rule, let us multiply equation (3.7) by the integrating factor r^{n-1}, as a result equation (3.7) becomes

$$\frac{1}{2} \left(w^{'} r^{n} + wnr^{n-1} \right) + w^{''} r^{n-1} + w^{'} (n-1) r^{n-2} = 0$$

$$\begin{aligned}
\frac{1}{2} (r^{n} w)^{'} + \left(w^{'} r^{n-1} \right)^{'} &= 0 \\
\left(\frac{1}{2} r^{n} w + w^{'} r^{n-1} \right)^{'} &= 0 \\
\frac{1}{2} w r^{n} + w^{'} r^{n-1} &= c
\end{aligned}$$

Since we are looking for a particular solution let us take $c = 0$, and then

$$\begin{aligned}
\frac{1}{2}wr^n + w^{'}r^{n-1} &= 0 \\
w^{'}r^{n-1} &= \frac{-1}{2}wr^n \\
\frac{w^{'}}{w} &= \frac{-1}{2}r \\
\ln|w| &= \frac{-r^2}{4} + c \\
w(r) &= ce^{\frac{-r^2}{4}}
\end{aligned}$$

Consequently

$$\begin{aligned}
v(y) &= w(|y|) \\
&= ce^{\frac{-|y|^2}{4}}
\end{aligned}$$

Finally,

$$\begin{aligned}
u(x,t) &= \frac{1}{t^{\frac{n}{2}}}v\left(\frac{x}{\sqrt{t}}\right) \\
&= \frac{1}{t^{\frac{n}{2}}}Ce^{\frac{\left|-\frac{x}{\sqrt{t}}\right|^2}{4}}
\end{aligned}$$

Thus, as anticipated

$$u(x,t) = \frac{C}{t^{\frac{n}{2}}}e^{\frac{-|x|^2}{4t}}$$

is the solution for the n-dimesional Diffusion equation

Chapter 4

Space- and Time- Fractional Diffusion-Wave Equation

The behaviour of solution of fractional Cauchy problem

$$\begin{cases} D_t^{\beta} u(x,t) + (-\triangle)^{\frac{\alpha}{2}} u(x,t) = 0, & (x,t) \in \mathbb{R}^n \times \mathbb{R}_+, 0 < \beta \leq 2 : 1 < \alpha \leq 2 \\ \text{u(x,0) = f(x)} \end{cases}$$

is dictated by the fundamental solution $\Phi_{\alpha,\beta,n}(x,t)$. Representation of the fundamental solution in terms of known special functions is obtained from Mellin-Barnes integral by employing the method of Mellin transform. The necessary and/or sufficient conditions for the positivity of the fundamental solution of the multi-dimensional diffusion-wave equation is still an open problem (to the best of our knowledge).

4.1 The Wave and Diffusion Equation

Here, we see space- and time- fractional diffusion-wave equation in n-dimensions

$$D_t^{\beta} u(x,t) = -(-\triangle)^{\frac{\alpha}{2}} u(x,t), \qquad (x,t) \in \mathbb{R}^n \times \mathbb{R}_+, 0 < \alpha, \beta \leq 2. \tag{4.1}$$

where D_t^{β} is the Caputo time-fractional derivative of order $\beta > 0$ where by

$$D_t^{\beta} u(x,t) = \left(\mathcal{J}_t^{n-\beta} \frac{\partial^n u}{\partial t^n} \right)(t), \qquad n-1 < \beta \leq n, \;\; n \in \mathbb{N}$$

and $(-\triangle)^{\frac{\alpha}{2}}$ is the fractional Laplacian.

If we take specific values of α and β we will recover the usual wave equation and diffusion equation as can be seen hereunder. In this regard, on the one hand, if $\alpha = 2$ and $\beta = 1$

$$\begin{aligned} D_t^1 u(x,t) &= -(-\triangle)^{\frac{2}{2}} u(x,t) \\ u_t(x,t) &= -(-\triangle) u(x,t) \\ \Rightarrow u_t(x,t) &= \triangle u(x,t) \end{aligned}$$

$$u_t(x,t) = \triangle u(x,t), \qquad x \in \mathbb{R}^n, \qquad t > 0 \tag{4.2}$$

Equation (4.2) is n-dimensional Diffusion equation and $\triangle$ is the usual n-dimensional Laplacian. On the other hand, if $\alpha = 2 = \beta$

$$\begin{aligned} D_t^2 u(x,t) &= -(-\triangle)^{\frac{2}{2}} u(x,t) \\ u_{tt}(x,t) &= -(-\triangle) u(x,t) \\ \Rightarrow u_{tt}(x,t) &= \triangle u(x,t) \end{aligned}$$

$$u_{tt}(x,t) = \triangle u(x,t), \qquad x \in \mathbb{R}^n, \qquad t > 0 \tag{4.3}$$

Equation (4.3) is n-dimensional wave equation with $\triangle$ again the n-dimensional Laplacian. Thus, equation (4.2) and equation (4.3) are special cases of the multi-dimensional space- and time- fractional diffusion-wave equation.

The time - fractional diffusion - wave equation

$$D_t^{\beta} u(x,t) = \triangle u(x,t), \;\; x \in \mathbb{R}^n, \; 0 < \beta \leq 2$$

is also a special case of (4.1) in which $\alpha = 2$. Now let us see some relevant Lemma's which have significant role for the derivation of the fundamental solution

Lemma 1. *For $\mu > -1$ and $\alpha > 0$*

$${}^{c}D_t^{\alpha} t^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} t^{\mu-\alpha}$$

where the derivative is in the sense of Caputo.

Proof. Let $f(t) = t^{\mu}$. From the definition of Caputo fractional derivative

$$^{c}D_{t}^{\alpha}f(t) =_{c} D_{t}^{\alpha}t^{\mu} = \frac{1}{\Gamma(n-\alpha)}\int_{0}^{t}(t-\tau)^{n-\alpha-1}(\tau^{\mu})^{(n)}d\tau \tag{4.4}$$

From the Euler's formula of power function, we have

$$\frac{d^{n}}{d\tau^{n}}(\tau^{\mu}) = \frac{\Gamma(\mu+1)}{\Gamma(\mu-n+1)}\tau^{\mu-n} \tag{4.5}$$

Then substituting equation (4.5) into equation (4.4) renders

$$\begin{aligned}
^{c}D_{t}^{\alpha}t^{\mu} &= \frac{1}{\Gamma(n-\alpha)}\int_{0}^{t}(t-\tau)^{n-\alpha-1}\frac{\Gamma(\mu+1)}{\Gamma(\mu-n+1)}\tau^{\mu-n}d\tau \\
&= \frac{\Gamma(\mu+1)}{\Gamma(\mu-n+1)}\frac{1}{\Gamma(n-\alpha)}\int_{0}^{t}(t-\tau)^{n-\alpha-1}\tau^{\mu-n}d\tau
\end{aligned}$$

Now let us substitute $\tau = \lambda t$ and $d\tau = td\lambda$ for $0 \leq \lambda \leq 1$ then we have

$$\begin{aligned}
^{c}D_{t}^{\alpha}t^{\mu} &= \frac{\Gamma(\mu+1)}{\Gamma(\mu-n+1)}\frac{1}{\Gamma(n-\alpha)}\int_{0}^{1}(t-\lambda t)^{n-\alpha-1}(\lambda t)^{\mu-n}td\lambda \\
&= \frac{\Gamma(\mu+1)}{\Gamma(\mu-n+1)}\frac{1}{\Gamma(n-\alpha)}\int_{0}^{1}(t(1-\lambda))^{n-\alpha-1}(\lambda)^{\mu-n}(t)^{\mu-n}td\lambda \\
&= \frac{\Gamma(\mu+1)}{\Gamma(\mu-n+1)}\frac{1}{\Gamma(n-\alpha)}\int_{0}^{1}(1-\lambda)^{n-\alpha-1}\lambda^{\mu-n}t^{n-\alpha-1}t^{\mu-n}td\lambda \\
&= \frac{\Gamma(\mu+1)}{\Gamma(\mu-n+1)}\frac{t^{\mu-\alpha}}{\Gamma(n-\alpha)}\int_{0}^{1}(1-\lambda)^{n-\alpha-1}\lambda^{\mu-n}d\lambda
\end{aligned}$$

Consequently

$$^{c}D_{t}^{\alpha}t^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-n+1)}\frac{t^{\mu-\alpha}}{\Gamma(n-\alpha)}B(\mu-n+1, n-\alpha) \tag{4.6}$$

where

$$B(\mu-n+1, n-\alpha) = \int_{0}^{1}(1-\lambda)^{n-\alpha-1}\lambda^{\mu-n}d\lambda$$

which is the beta function and

$$B(\mu-n+1, n-\alpha) = \frac{\Gamma(\mu-n+1)\Gamma(n-\alpha)}{\Gamma(\mu-\alpha+1)} \tag{4.7}$$

Finally substituting equation (4.7) into equation (4.6) leads to

$$\begin{aligned} {}^{c}D_t^{\alpha} t^{\mu} &= \frac{\Gamma(\mu+1)}{\Gamma(\mu-n+1)} \frac{t^{\mu-\alpha}}{\Gamma(n-\alpha)} \frac{\Gamma(\mu-n+1)\Gamma(n-\alpha)}{\Gamma(\mu-\alpha+1)} \\ &= \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} t^{\mu-\alpha} \end{aligned}$$

Therefore,

$${}^{c}D_t^{\alpha} t^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} t^{\mu-\alpha}$$

□

Lemma 2. *Consider the Riemann-Liouville fractional integral of order $\alpha > 0$,*

$${}_{o}\mathcal{J}_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau$$

and $D = \frac{d}{dt}$ the usual differential operator of order one. Then

$$D[{}_{o}\mathcal{J}_t^{\alpha} f(t)] -_{o} \mathcal{J}_t^{\alpha}[Df(t)] = f(0) \frac{t^{\alpha-1}}{\Gamma(\alpha)}$$

Proof. Since

$$D[{}_{o}\mathcal{J}_t^{\alpha} f(t)] -_{o} \mathcal{J}_t^{\alpha}[Df(t)] = \frac{d}{dt} \left[\frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau \right] - \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f'(\tau) d\tau$$

We have

$$D[{}_{o}\mathcal{J}_t^{\alpha} f(t)] -_{o} \mathcal{J}_t^{\alpha}[Df(t)] = \frac{1}{\Gamma(\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau - \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f'(\tau) d\tau$$

Applying integration by parts, along with the change of variable

$$u = (t-\tau)^{\alpha-1}, du = -(\alpha-1)(t-\tau)^{\alpha-2} d\tau$$

and

$$dv = f'(\tau) d\tau, v = f(\tau)$$

since, $(t-\tau)^{\alpha-1}$ and $\frac{d}{dt}(t-\tau)^{\alpha-1}$ are continuous in the variable t. Then we can differentiate under the integral, we have

$$D[{}_o\mathcal{J}_t^\alpha f(t)] -{}_o \mathcal{J}_t^\alpha[Df(t)] = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{d}{dt}(t-\tau)^{\alpha-1}f(\tau)d\tau -$$
$$\frac{1}{\Gamma(\alpha)}\left[(t-\tau)^{\alpha-1}f(\tau)|_0^t - \int_0^t -(\alpha-1)(t-\tau)^{\alpha-2}f(\tau)d\tau\right]$$

But then

$$D[{}_o\mathcal{J}_t^\alpha f(t)] -{}_o \mathcal{J}_t^\alpha[Df(t)] = \frac{1}{\Gamma(\alpha)}\int_0^t (\alpha-1)(t-\tau)^{\alpha-2}f(\tau)d\tau -$$
$$\frac{1}{\Gamma(\alpha)}\left[-t^{\alpha-1}f(0) + \int_0^t (\alpha-1)(t-\tau)^{\alpha-2}f(\tau)d\tau\right]$$

And hence

$$D[{}_o\mathcal{J}_t^\alpha f(t)] -{}_o \mathcal{J}_t^\alpha[Df(t)] = f(0)\frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)}\int_0^t (\alpha-1)(t-\tau)^{\alpha-2}f(\tau)d\tau -$$
$$\frac{1}{\Gamma(\alpha)}\int_0^t (\alpha-1)(t-\tau)^{\alpha-2}f(\tau)d\tau$$

Therefore,

$$D[{}_o\mathcal{J}_t^\alpha f(t)] -{}_o \mathcal{J}_t^\alpha[Df(t)] = f(0)\frac{t^{\alpha-1}}{\Gamma(\alpha)}$$

Identifying special cases of

$${}^cD_t^\beta u(x,t) = -(-\triangle)^{\frac{\alpha}{2}}u(x,t),\ \ x\in\mathbb{R}^n, t>0$$

and characterizing those cases for which the fundamental solution can be expressed in terms of known special functions is the topic of current research. In some of the cases the fundamental solution can be expressed in terms of the Mittage-Leffler functions and this can be attained by using the Mellin transform. □

Lemma 3. *Let $\alpha > 0$ and the initialization be at $a = 0$. Then the Mellin transform of the Riemann-Liouville fractional integral is given by*

$$M[{}_o\mathcal{J}_t^\alpha f(t)] = \frac{\Gamma(1-s-\alpha)}{\Gamma(1-s)}\check{f}(s+\alpha)$$

where $\check{f}(s)$ is the Mellin transform of $f(t)$

Proof. Using the definition of Riemann-Liouville fractional integral

$$_o\mathcal{J}_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau$$

And the change of variable $\tau = xt$ and $d\tau = tdx$ for $0 \leq x \leq 1$ leads to

$$\begin{aligned}
_o\mathcal{J}_t^\alpha f(t) &= \frac{1}{\Gamma(\alpha)}\int_0^1 (t-xt)^{\alpha-1} f(xt)tdx \\
&= \frac{1}{\Gamma(\alpha)}\int_0^1 (t(1-x))^{\alpha-1} f(xt)tdx \\
&= \frac{1}{\Gamma(\alpha)}\int_0^1 t * t^{\alpha-1}(1-x)^{\alpha-1} f(xt)dx
\end{aligned}$$

But then

$$_o\mathcal{J}_t^\alpha f(t) = \frac{t^\alpha}{\Gamma(\alpha)}\int_0^1 f(xt)h(x)dx \tag{4.8}$$

where

$$h(x) = \begin{cases} (1-x)^{\alpha-1} & \text{if } 0 \leq x < 1 \\ 0 & \text{if } x \geq 1. \end{cases}$$

Now let us compute the Mellin transorm

$$\begin{aligned}
M[_o\mathcal{J}_t^\alpha f(t)] &= M\left[\frac{t^\alpha}{\Gamma(\alpha)}\int_0^1 f(xt)h(x)dx\right] \\
&= \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha+s-1}dt\int_0^\infty f(tx)h(x)dx
\end{aligned}$$

Since $h(x)$, $t^{\alpha+s-1}$ and $f(xt)$ are differentiable functions we can change the order of integration we have

$$M[_o\mathcal{J}_t^\alpha f(t)] = \frac{1}{\Gamma(\alpha)}\int_0^\infty h(x)dx\int_0^\infty f(tx)t^{\alpha+s-1}dt$$

If we set

$$y = tx$$

Then

$$dy = xdt$$

$$\Rightarrow \frac{dy}{x} = dt$$

As a result of this we have

$$\begin{aligned}
M[{}_o\mathcal{J}_t^\alpha f(t)] &= \frac{1}{\Gamma(\alpha)} \int_0^\infty h(x)dx \int_0^\infty f(y) \left(\frac{y}{x}\right)^{\alpha+s-1} \frac{dy}{x} \\
&= \frac{1}{\Gamma(\alpha)} \int_0^\infty h(x)dx \int_0^\infty y^{\alpha+s-1} f(y) x^{-s-\alpha} dy \\
&= \frac{1}{\Gamma(\alpha)} \int_0^\infty x^{-s-\alpha} h(x)dx \int_0^\infty y^{\alpha+s-1} f(y)dy \\
&= \frac{1}{\Gamma(\alpha)} \int_0^1 x^{-s-\alpha}(1-x)^{\alpha-1}dx \int_0^\infty y^{\alpha+s-1} f(y)dy \\
&= \frac{1}{\Gamma(\alpha)} B(1-s-\alpha, \alpha)\check{f}(s+\alpha) \\
&= \frac{1}{\Gamma(\alpha)} \frac{\Gamma(1-s-\alpha)\Gamma(\alpha)}{\Gamma(1-s)} \check{f}(s+\alpha) \\
&= \frac{\Gamma(1-s-\alpha)}{\Gamma(1-s)} \check{f}(s+\alpha)
\end{aligned}$$

Therefore,

$$M[{}_o\mathcal{J}_t^\alpha f(t)] = \frac{\Gamma(1-s-\alpha)}{\Gamma(1-s)} \check{f}(s+\alpha)$$

□

Theorem 1. *For $\alpha > 0, \lambda \in \mathbb{R}$ and $t \geq 0$,*

$$\varphi(t) = E_\alpha(\lambda t^\alpha)$$

is a solution of the fractional eigenvalue problem.

$${}^cD^\alpha \varphi(t) = \lambda \varphi(t)$$

where the derivative is in Caputo sense.

Proof.

$$E_\alpha(\lambda t^\alpha) = \sum_{k=0}^{\infty} \frac{(\lambda t^\alpha)^k}{\Gamma(\alpha k + 1)}$$

The Caputo fractional derivative of $f(t)$ with

$$n - 1 < \alpha < n, n \in \mathbb{N}$$

is given by

$$^{c}D^{\alpha}f(t) = \, ^{c}\mathcal{J}^{n-\alpha c}D^{n}f(t)$$

where $^{c}\mathcal{J}$ stand for Caputo fractional integration. For $f(t) = x(t)$ we have

$$^{c}D^{\alpha}x(t) = \, ^{c}D^{\alpha}E_{\alpha}(\lambda t^{\alpha}) =^{c} D^{\alpha}\left[\sum_{k=0}^{\infty}\frac{\lambda^{k}t^{\alpha k}}{\Gamma(\alpha k+1)}\right]$$

Thus

$$\begin{aligned}
^{c}D^{\alpha}x(t) &= \, ^{c}\mathcal{J}^{n-\alpha}D^{n}\left[\sum_{k=0}^{\infty}\frac{\lambda^{k}t^{\alpha k}}{\Gamma(\alpha k+1)}\right] \\
&= \, ^{c}\mathcal{J}^{n-\alpha}\left[\sum_{k=0}^{\infty}\frac{\lambda^{k}D^{n}t^{\alpha k}}{\Gamma(\alpha k+1)}\right] \qquad \text{by lemma 2} \\
&= \, ^{c}\mathcal{J}^{n-\alpha}\left[\sum_{k=1}^{\infty}\frac{\lambda^{k}t^{\alpha k-n}}{\Gamma(\alpha k+1-n)}\right] \\
&= \sum_{k=1}^{\infty}\frac{\lambda^{k}\, ^{c}\mathcal{J}^{n-\alpha}t^{\alpha k-n}}{\Gamma(\alpha k+1-n)} \\
&= \sum_{k=1}^{\infty}\frac{\lambda^{k}t^{\alpha k-n+n-\alpha}}{\Gamma(\alpha k+1-n+n-\alpha)} \\
&= \sum_{k=1}^{\infty}\frac{\lambda^{k}t^{\alpha k-\alpha}}{\Gamma(\alpha k-\alpha+1)}
\end{aligned}$$

Hence, for $k \to k+1$

$$\begin{aligned}
^{c}D^{\alpha}\varphi(t) &= \sum_{k=0}^{\infty}\frac{\lambda^{k+1}t^{\alpha k}}{\Gamma(\alpha k+1)} \\
&= \lambda\sum_{k=0}^{\infty}\frac{\lambda^{k}t^{\alpha k}}{\Gamma(\alpha k+1)} \\
&= \lambda\sum_{k=0}^{\infty}\frac{(\lambda t^{\alpha})^{k}}{\Gamma(\alpha k+1)}
\end{aligned}$$

Therefore,

$$^{c}D^{\alpha}\varphi(t) = \lambda\varphi(t)$$

This completes the proof of the claim that $\varphi(t)$ is a solution of the differential equation $^{c}D^{\alpha}\varphi(t) = \lambda\varphi(t)$ □

Theorem 2. *If $1 < \alpha \leq 2$ and $\lambda \in \mathbb{R}$ then the fundamental system of solution of the Caputo fractional differential equation*

$$^{c}D_t^{\alpha} y(x) - \lambda y(x) = 0, \qquad x > 0$$

is given by $\{y_1, y_2\}$ where $y_1(x) = E_\alpha(\lambda x^\alpha)$ and $y_2(x) = xE_{\alpha,2}(\lambda x^\alpha)$

Proof. For $1 < \alpha \leq 2$ and $\lambda \in \mathbb{R}$ we can write $^{c}D_t^{\alpha} y(x)$ in the following equivalent equation

$^{c}D_t^{\alpha} y_1(x) = ^{c}\mathcal{J}_t^{2-\alpha c}D_t^2 y_1(x)$, $^{c}\mathcal{J}_t^{1-\alpha}$ stands for Caputo fractional integral

$$\begin{aligned}
^{c}D_t^{\alpha} y(x) &= {}^{c}\mathcal{J}_t^{2-\alpha c}D_t^2 E_\alpha(\lambda x^\alpha) \\
&= {}^{c}\mathcal{J}_t^{2-\alpha c}D_t^2 \left[\sum_{k=0}^{\infty} \frac{(\lambda x^\alpha)^k}{\Gamma(\alpha k+1)}\right] \\
&= {}^{c}\mathcal{J}_t^{2-\alpha c}D_t^2 \left[\sum_{k=0}^{\infty} \frac{\lambda^k x^{\alpha k}}{\Gamma(\alpha k+1)}\right] \\
&= {}^{c}\mathcal{J}_t^{2-\alpha} \left[\sum_{k=0}^{\infty} \frac{\lambda^k \, {}^{c}D_t^2 x^{\alpha k}}{\Gamma(\alpha k+1)}\right] \qquad \text{by Lemma 2} \\
&= {}^{c}\mathcal{J}_t^{2-\alpha} \left[\sum_{k=1}^{\infty} \frac{\lambda^k}{\Gamma(\alpha k+1)} \frac{\Gamma(\alpha k+1)}{\Gamma(\alpha k-1)} x^{\alpha k-2}\right] \\
&= {}^{c}\mathcal{J}_t^{2-\alpha} \left[\sum_{k=1}^{\infty} \frac{\lambda^k}{\Gamma(\alpha k-1)} x^{\alpha k-2}\right]
\end{aligned}$$

$$\begin{aligned}
^{c}D_t^{\alpha} y(x) &= \sum_{k=1}^{\infty} \frac{\lambda^k}{\Gamma(\alpha k-1)} {}^{c}\mathcal{J}_t^{2-\alpha} x^{\alpha k-1} \qquad \text{by the Caputo fractional integral} \\
&= \sum_{k=1}^{\infty} \frac{\lambda^k}{\Gamma(\alpha k-1)} \frac{\Gamma(\alpha k-1)}{\Gamma(\alpha k-\alpha+1)} x^{\alpha k-\alpha} \qquad \text{k}\rightarrow\text{k+1} \\
&= \sum_{k=0}^{\infty} \frac{\lambda^{k+1}}{\Gamma(\alpha(k+1)-\alpha+1)} x^{\alpha(k+1)-\alpha} \\
&= \sum_{k=0}^{\infty} \frac{\lambda^{k+1} x^{\alpha k}}{\Gamma(\alpha k+1)} \\
&= \lambda \sum_{k=0}^{\infty} \frac{\lambda^k x^{\alpha k}}{\Gamma(\alpha k+1)}
\end{aligned}$$

Then,

$$^{c}D_t^{\alpha}y_1(x) - \lambda y_1(x) = 0$$

And hence, we have shown that $y_1(x)$ is a solution for the given fractional differential equation.

Similarly for

$$\begin{aligned}
y_2(x) &= xE_{\alpha,2}(\lambda x^{\alpha}) \\
&= x\sum_{k=0}^{\infty}\frac{(\lambda x^{\alpha})^k}{\Gamma(\alpha k+2)} \\
&= \sum_{k=0}^{\infty}\frac{\lambda^k x^{\alpha k+1}}{\Gamma(\alpha k+2)}
\end{aligned}$$

if $1 < \alpha \leq 2$ and $\lambda \in \mathbb{R}$ we can write $^{c}D_t^{\alpha}y_2(x)$ in the following equivalent equation

$^{c}D_t^{\alpha}y_2(x) = ^{c}\mathcal{J}_t^{2-\alpha c}D_t^2 y_2(x)$, $^{c}\mathcal{J}_t^{1-\alpha}$ stands for Caputo fractional integral

$$\begin{aligned}
^{c}D_t^{\alpha}y_2(x) &= {}^{c}\mathcal{J}_t^{2-\alpha c}D_t^2 xE_{\alpha,2}(\lambda x^{\alpha}) \\
&= {}^{c}\mathcal{J}_t^{2-\alpha c}D_t^2\left[\sum_{k=0}^{\infty}\frac{x(\lambda x^{\alpha})^k}{\Gamma(\alpha k+2)}\right] \\
&= {}^{c}\mathcal{J}_t^{2-\alpha c}D_t^2\left[\sum_{k=0}^{\infty}\frac{\lambda^k x^{\alpha k+1}}{\Gamma(\alpha k+2)}\right] \\
&= {}^{c}\mathcal{J}_t^{2-\alpha}\left[\sum_{k=0}^{\infty}\frac{\lambda^{k c}D_t^2 x^{\alpha k+1}}{\Gamma(\alpha k+2)}\right] \qquad \text{by Lemma 2} \\
&= {}^{c}\mathcal{J}_t^{2-\alpha}\left[\sum_{k=1}^{\infty}\frac{\lambda^k}{\Gamma(\alpha k+2)}\frac{\Gamma(\alpha k+2)}{\Gamma(\alpha k)}x^{\alpha k-1}\right] \\
&= {}^{c}\mathcal{J}_t^{2-\alpha}\left[\sum_{k=1}^{\infty}\frac{\lambda^k}{\Gamma(\alpha k)}x^{\alpha k-1}\right]
\end{aligned}$$

$$\begin{aligned}
{}^{c}D_t^{\alpha} y_2(x) &= \sum_{k=1}^{\infty} \frac{\lambda^k}{\Gamma(\alpha k)} {}^{c}\mathcal{J}_t^{2-\alpha} x^{\alpha k-1} && \text{by the Caputo fractional integral} \\
&= \sum_{k=1}^{\infty} \frac{\lambda^k}{\Gamma(\alpha k)} \frac{\Gamma(\alpha k)}{\Gamma(\alpha k-\alpha+2)} x^{\alpha k-\alpha+1} && \text{k}\rightarrow\text{k+1} \\
&= \sum_{k=0}^{\infty} \frac{\lambda^{k+1}}{\Gamma(\alpha(k+1)-\alpha+2)} x^{\alpha(k+1)-\alpha+1} \\
&= \sum_{k=0}^{\infty} \frac{\lambda^{k+1} x^{\alpha k+1}}{\Gamma(\alpha k+2)} \\
&= \lambda x \sum_{k=0}^{\infty} \frac{\lambda^k x^{\alpha k}}{\Gamma(\alpha k+2)}
\end{aligned}$$

Then,

$${}^{c}D_t^{\alpha} y_2(x) - \lambda y_2(x) = 0$$

Evidently, we have shown that $y_2(x)$ is a solution of the given fractional differential equation. The fact that y_1 and y_2 are linearly independet can be establised by appealing to thier Wronskian. Therefore, the fundamental system of solutions of the fractional differential equation

$${}^{c}D_t^{\alpha} y_2(x) - \lambda y_2(x) = 0$$

is given by $\{y_1(x), y_2(x)\}$ where $y_1(x) = E_{\alpha}(\lambda x^{\alpha})$ and $y_2(x) = xE_{\alpha,2}(\lambda x^{\alpha})$

$\square$

4.2 Fundamental Solution of Space-Time Fractional Diffusion-Wave Equation

For the equation (4.1) with prescribed Cauchy data

$$u(x,0) = g(x), x \in \mathbb{R}^n$$

and β such that $0 < \beta \leq 1$, when we apply the multi-dimensional Fourier transform, the partial differential equation reduces to the following fractional differential equation in the Fourier domain,

$$\begin{cases} D_t^{\beta}\widehat{\Phi}_{\alpha,\beta,n}(k,t) + |k|^{\alpha}\widehat{\Phi}_{\alpha,\beta,n}(k,t) = 0, \\ \widehat{\Phi}_{\alpha,\beta,n}(k,0) = 1 \end{cases}. \tag{4.9}$$

And for $1 < \beta \leq 2$, the Fourier transform of the same problem along with the initial values

$$u(x,0) = g(x)$$

$$u_t(x,0) = 0$$

leads to

$$\begin{cases} D_t^{\beta}\widehat{\Phi}_{\alpha,\beta,n}(k,t) + |k|^{\alpha}\widehat{\Phi}_{\alpha,\beta,n}(k,t) = 0, \\ \widehat{\Phi}_{\alpha,\beta,n}(k,0) = 1, \\ \frac{\partial}{\partial t}\widehat{\Phi}_{\alpha,\beta,n}(k,0) = 0 \end{cases}.$$

The solution of equation (4.1) has the form

$$\widehat{\Phi}_{\alpha,\beta,n}(k,t) = E_{\beta}(-|k|^{\alpha}t^{\beta}) \tag{4.10}$$

Since $\widehat{\Phi}_{\alpha,\beta,n}$ is integrable, the inverse Fourier transform of equation (4.10) has the following representation

$$\Phi_{\alpha,\beta,n}(k,t) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ik.x}E_{\beta}(-|k|^{\alpha}t^{\beta})dx \tag{4.11}$$

Since the Mittage-Leffler is radial function, we know that

$$\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ik.x}g(|k|)dk = \frac{|x|^{1-\frac{n}{2}}}{(2\pi)^{\frac{n}{2}}}\int_0^{\infty} g(\tau)\tau^{\frac{n}{2}}J_{\frac{n}{2}-1}(\tau|x|)d\tau \tag{4.12}$$

Then equation (4.11) becomes

$$\Phi_{\alpha,\beta,n}(x,t) = \frac{|x|^{1-\frac{n}{2}}}{(2\pi)^{\frac{n}{2}}}\int_0^{\infty} E_{\beta}(-\tau^{\alpha}t^{\beta})\tau^{\frac{n}{2}}J_{\frac{n}{2}-1}(\tau|x|)d\tau \tag{4.13}$$

Now equation (4.13) can be transformed to a Mellin-Barnes integral and the resulting Mellin-Barnes integral representation of the fundamental solution is given by

$$\Phi_{\alpha,\beta,n}(x,t) = \frac{1}{\alpha}\frac{|x|^{-n}}{\pi^{\frac{n}{2}}}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma\left(\frac{n}{2}-\frac{s}{2}\right)\Gamma\left(\frac{s}{\alpha}\right)\Gamma\left(1-\frac{s}{\alpha}\right)}{\Gamma\left(1-\frac{\beta}{\alpha}s\right)\Gamma\left(\frac{s}{2}\right)}\left(\frac{2t^{\frac{\beta}{\alpha}}}{|x|}\right)^{-s}ds \tag{4.14}$$

The above equation (4.14) can be written in another equivalent representation by substituting $s \to -s$ and then $s \to s - n$. As a result, it has the following form,

$$\Phi_{\alpha,\beta,n}(x,t) = \frac{1}{\alpha}\frac{|x|^{-n}}{\pi^{\frac{n}{2}}}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma\left(\frac{n}{2}+\frac{s}{2}\right)\Gamma\left(-\frac{s}{\alpha}\right)\Gamma\left(1+\frac{s}{\alpha}\right)}{\Gamma\left(1+\frac{\beta}{\alpha}s\right)\Gamma\left(-\frac{s}{2}\right)}\left(\frac{|x|}{2t^{\frac{\beta}{\alpha}}}\right)^{-s}ds \tag{4.15}$$

provided that $\frac{n-1}{2} < -c < min\{\alpha, n\}$ and

$$\Phi_{\alpha,n}(x,t) = \frac{1}{\alpha}\frac{t^{-\frac{\beta n}{\alpha}}}{4\pi^{\frac{n}{2}}}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{n}{\alpha}-\frac{s}{\alpha}\right)\Gamma\left(1-\frac{n}{\alpha}+\frac{s}{\alpha}\right)}{\Gamma\left(1-\frac{\beta}{\alpha}n+\frac{\beta}{\alpha}s\right)\Gamma\left(\frac{n}{2}-\frac{s}{2}\right)}\left(\frac{|x|}{2t^{\frac{\beta}{\alpha}}}\right)^{-s}ds \tag{4.16}$$

under the condition, $max\{n-\alpha, 0\} < c < n$.

From the property of Mellin-Barnes integral transform, we recall that explicit representation of the fundamental solution, $\Phi_{\alpha,\beta,n}$ has distinct closed form as based on the relation between the parameter pairs, α and β. In what follows, we shall take a closer look at this scenario.

Closed Form Formula For $\beta = \frac{1}{2}\alpha$

Theorem 3. *The fundamental solution of the multi-dimensional space- and time-fractional diffusion-wave equation for the case* $\beta = \frac{1}{2}\alpha$ *and* $n = 2$ *with* $0 < \alpha \leq 2$ *is given by*

$$\Phi_{\alpha,\frac{1}{2}\alpha,2}(x,t) = \frac{1}{4\pi t}\left[\frac{|x|}{2t^{1/2}}\right]^{\alpha-2}\sum_{k=0}^{\infty}\frac{1}{\Gamma\left(\frac{\alpha}{2}k+\frac{\alpha}{2}\right)}\left[-\left(\frac{|x|}{2t^{1/2}}\right)^{\alpha}\right]^{k} \tag{4.17}$$

Proof. In the Mellin-Barnes representation of the fundamental solution of the multi-dimensional time- and space- fractional diffusion-wave equation

(4.16) if we plug

$$\beta = \frac{1}{2}\alpha, \text{ and } n = 2$$

then it becomes

$$\begin{aligned}
\Phi_{\alpha,\frac{1}{2}\alpha,2}(x,t) &= \frac{1}{\alpha}\frac{t^{-1}}{4\pi}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{2}{\alpha}-\frac{s}{\alpha}\right)\Gamma\left(1-\frac{2}{\alpha}+\frac{s}{\alpha}\right)}{\Gamma\left(1-1+\frac{s}{2}\right)\Gamma\left(1-\frac{s}{2}\right)}\left(\frac{|x|}{2t^{1/2}}\right)^{-s}ds \\
&= \frac{1}{\alpha}\frac{t^{-1}}{4\pi}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{2}{\alpha}-\frac{s}{\alpha}\right)\Gamma\left(1-\frac{2}{\alpha}+\frac{s}{\alpha}\right)}{\Gamma\left(\frac{s}{2}\right)\Gamma\left(1-\frac{s}{2}\right)}\left(\frac{|x|}{2t^{1/2}}\right)^{-s}ds \\
&= \frac{1}{\alpha}\frac{t^{-1}}{4\pi}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma\left(\frac{2}{\alpha}-\frac{s}{\alpha}\right)\Gamma\left(1-\frac{2}{\alpha}+\frac{s}{\alpha}\right)}{\Gamma\left(1-\frac{s}{2}\right)}\left(\frac{|x|}{2t^{1/2}}\right)^{-s}ds
\end{aligned}$$

In order to make it simpler let us substitute $s \to \alpha s$ and $ds \to \alpha ds$ then the above becomes

$$\begin{aligned}
\Phi_{\alpha,\frac{1}{2}\alpha,2}(x,t) &= \frac{1}{\alpha}\frac{t^{-1}}{4\pi}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma\left(\frac{2}{\alpha}-s\right)\Gamma\left(1-\frac{2}{\alpha}+s\right)}{\Gamma\left(1-\frac{\alpha}{2}s\right)}\left(\frac{|x|}{2t^{1/2}}\right)^{-\alpha s}\alpha ds \\
&= \frac{t^{-1}}{4\pi}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma\left(\frac{2}{\alpha}-s\right)\Gamma\left(1-\frac{2}{\alpha}+s\right)}{\Gamma\left(1-\frac{\alpha}{2}s\right)}\left[\left(\frac{|x|}{2t^{1/2}}\right)^{\alpha}\right]^{-s}ds
\end{aligned}$$

To obtain the series representation of the above expression, we should apply the residual of gamma function and the cauchy residue theorems. So the contour of integration on the righthand side of the above expression can be transformed into $L_{-\infty}$ starting and ending at $-\infty$, encircling all poles

$$s_k = \frac{2}{\alpha} - k - 1$$

of the function

$$\Gamma\left(1 - \frac{2}{\alpha} + s\right)$$

The series representation will be

$$
\begin{aligned}
\Phi_{\alpha,\frac{1}{2}\alpha,2} &= \frac{1}{4\pi t}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{\Gamma(k+1)}{\Gamma\left(\frac{\alpha}{2}k+\frac{\alpha}{2}\right)}\left(\frac{|x|}{2t^{1/2}}\right)^{\alpha+\alpha k-2} \\
&= \frac{1}{4\pi t}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{k!}{\Gamma\left(\frac{\alpha}{2}k+\frac{\alpha}{2}\right)}\left(\frac{|x|}{2t^{1/2}}\right)^{\alpha+\alpha k-2} \\
&= \frac{1}{4\pi t}\sum_{k=0}^{\infty}(-1)^k\frac{1}{\Gamma\left(\frac{\alpha}{2}k+\frac{\alpha}{2}\right)}\left[\left(\frac{|x|}{2t^{1/2}}\right)^{\alpha}\right]^k\left(\frac{|x|}{2t^{1/2}}\right)^{\alpha-2} \\
&= \frac{1}{4\pi t}\left(\frac{|x|}{2t^{1/2}}\right)^{\alpha-2}\sum_{k=0}^{\infty}\frac{\left[-\left(\frac{|x|}{2t^{1/2}}\right)^{\alpha}\right]^k}{\Gamma\left(\frac{\alpha}{2}k+\frac{\alpha}{2}\right)}
\end{aligned}
$$

□

From the above theorem we can see that for $\alpha = 2$ the solution of multi-dimensional space-time diffusion-wave equation will be

$$
\begin{aligned}
\Phi_{2,1,2}(x,t) &= \frac{1}{4\pi t}\left(\frac{|x|}{2t^{1/2}}\right)^{2-2}\sum_{k=0}^{\infty}\frac{\left[-\left(\frac{|x|}{2t^{1/2}}\right)^{2}\right]^k}{\Gamma\left(\frac{2}{2}k+\frac{2}{2}\right)} \\
&= \frac{1}{4\pi t}\sum_{k=0}^{\infty}\frac{\left[-\left(\frac{|x|^2}{4t}\right)\right]^k}{\Gamma(k+1)} \\
&= \frac{1}{4\pi t}\sum_{k=0}^{\infty}\frac{\left[-\left(\frac{|x|^2}{4t}\right)\right]^k}{k!}
\end{aligned}
$$

The above equation can be written as

$$\Phi_{2,1,2}(x,t) = \frac{1}{4\pi t}e^{-\frac{|x|^2}{4t}}$$

This is solution for the standard diffusion equation.

4.3 Closed Form Formula For $\beta = \frac{5}{2}\alpha$

Theorem 4. *The fundamental solution of the multi-dimensional space- and time-fractional diffusion-wave equation for the case $\beta = \frac{5}{2}\alpha$ under the* *condition* $0 < \alpha \leq \frac{4}{5}$ *is given by:*

$$\Phi_{\alpha,\frac{5}{2}\alpha,2}(x,t) = \frac{4\pi}{\sqrt{5}|x|^2}\sum_{k=0}^{\infty}\frac{\left[-\left(\frac{2}{|x|}\left(\frac{t}{5}\right)^{\frac{5}{2}}\right)^{\alpha}\right]^k}{\Gamma\left(\frac{1}{5}+\frac{\alpha}{2}k\right)\Gamma\left(\frac{2}{5}+\frac{\alpha}{2}k\right)\Gamma\left(\frac{3}{5}+\frac{\alpha}{2}k\right)\Gamma\left(\frac{4}{5}+\frac{\alpha}{2}k\right)\Gamma\left(-\frac{\alpha}{2}k\right)}$$

Proof. In the Mellin-Barnes representation (4.16) when we substitute

$$\beta = \frac{5}{2}\alpha, \text{ and } \ n = 2$$

we have

$$\Phi_{\alpha,\frac{5}{2}\alpha,2}(x,t) = \frac{1}{\alpha}\frac{t^{-\frac{5}{2}2}}{4\pi}\frac{1}{2\pi}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{2}{\alpha}-\frac{s}{\alpha}\right)\Gamma\left(1-\frac{2}{\alpha}+\frac{s}{\alpha}\right)}{\Gamma\left(1-\frac{5}{2}2+\frac{5}{2}s\right)\Gamma\left(1-\frac{s}{2}\right)}\left(\frac{|x|}{2t^{\frac{5}{2}}}\right)^{-s}ds$$

$$= \frac{1}{\alpha}\frac{t^{-5}}{4\pi}\frac{1}{2\pi}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{2}{\alpha}-\frac{s}{\alpha}\right)\Gamma\left(1-\frac{2}{\alpha}+\frac{s}{\alpha}\right)}{\Gamma\left(-4+\frac{5}{2}s\right)\Gamma\left(1-\frac{s}{2}\right)}\left(\frac{|x|}{2t^{\frac{5}{2}}}\right)^{-s}ds \qquad (4.18)$$

Employing the multiplication formula of the gamma function which is given by

$$\Gamma(ns) = n^{ns-\frac{1}{2}}(2\pi)^{\frac{1-n}{2}}\prod_{k=0}^{n-1}\Gamma\left(s+\frac{k}{n}\right), \qquad n = 2,3,4,5,...$$

We have

$$\begin{aligned}\Gamma\left(-4+\frac{5}{2}s\right) &= \Gamma\left[5\left(\frac{-4}{5}+\frac{1}{2}s\right)\right]\\ &= 5^{5\left(\frac{-4}{5}+\frac{1}{2}s\right)-\frac{1}{2}}(2\pi)^{-2}\prod_{k=0}^{4}\Gamma\left(\frac{-4}{5}+\frac{1}{2}s+\frac{k}{5}\right)\end{aligned}$$

Consequently

$$\Gamma\left(-4+\frac{5}{2}s\right) = 5^{-4+\frac{5}{2}s-\frac{1}{2}}(2\pi)^{-2}\Gamma\left(\frac{-4}{5}+\frac{1}{2}s\right)\Gamma\left(\frac{-3}{5}+\frac{1}{2}s\right)\Gamma\left(\frac{-2}{5}+\frac{1}{2}s\right)$$
$$\Gamma\left(\frac{-1}{5}+\frac{1}{2}s\right)\Gamma\left(\frac{1}{2}s\right) \qquad (4.19)$$

and then substituting equation (4.19) into equation (4.18) we obtain the following expression

$$\begin{aligned}
\Phi_{\alpha,\frac{5}{2}\alpha,2}(x,t) &= \frac{t^{-5}}{\alpha 4\pi}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{5^{\frac{9}{2}-\frac{5}{2}s}(2\pi)^2\Gamma\left(\frac{2}{\alpha}-\frac{s}{\alpha}\right)\Gamma\left(1-\frac{2}{\alpha}+\frac{s}{\alpha}\right)\left(\frac{|x|}{2t^{\frac{5}{2}}}\right)^{-s}}{\lambda}ds \\
&= \frac{t^{-5}}{\alpha 4\pi}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{5^{\frac{9}{2}}5^{\frac{-5}{2}s}(2\pi)^2\Gamma\left(\frac{2}{\alpha}-\frac{s}{\alpha}\right)\Gamma\left(1-\frac{2}{\alpha}+\frac{s}{\alpha}\right)\left(\frac{|x|}{2t^{\frac{5}{2}}}\right)^{-s}}{\lambda}ds \\
&= \frac{5^{\frac{9}{2}}4\pi^2}{\alpha t^5 4\pi}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma\left(\frac{2}{\alpha}-\frac{s}{\alpha}\right)\Gamma\left(1-\frac{2}{\alpha}+\frac{s}{\alpha}\right)\left(\frac{|x|}{2}\left(\frac{5}{t}\right)^{\frac{5}{2}}\right)^{-s}}{\lambda}ds \\
&= \frac{1}{\alpha}\frac{\pi 5^{\frac{9}{2}}}{t^5}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma\left(\frac{2}{\alpha}-\frac{s}{\alpha}\right)\Gamma\left(1-\frac{2}{\alpha}+\frac{s}{\alpha}\right)\left(\frac{|x|}{2}\left(\frac{5}{t}\right)^{\frac{5}{2}}\right)^{-s}}{\lambda}ds
\end{aligned}$$

where $\lambda = \Gamma\left(\frac{-4}{5}+\frac{1}{2}s\right)\Gamma\left(\frac{-3}{5}+\frac{1}{2}s\right)\Gamma\left(\frac{-2}{5}+\frac{1}{2}s\right)\Gamma\left(\frac{-1}{5}+\frac{1}{2}s\right)\Gamma\left(1-\frac{1}{2}s\right)$.

Now let us substitute $s \to \alpha s$ and $ds \to \alpha ds$ and set

$$\omega = \Gamma\left(\frac{-4}{5}+\frac{\alpha}{2}s\right)\Gamma\left(\frac{-3}{5}+\frac{\alpha}{2}s\right)\Gamma\left(\frac{-2}{5}+\frac{\alpha}{2}s\right)\Gamma\left(\frac{-1}{5}+\frac{\alpha}{2}s\right)\Gamma\left(1-\frac{\alpha}{2}s\right)$$

But then we have

$$\begin{aligned}
\Phi_{\alpha,\frac{5}{2}\alpha,2}(x,t) &= \frac{1}{\alpha}\frac{\pi 5^{\frac{9}{2}}}{t^5}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma\left(\frac{2}{\alpha}-\frac{\alpha s}{\alpha}\right)\Gamma\left(1-\frac{2}{\alpha}+\frac{\alpha s}{\alpha}\right)\left(\frac{|x|}{2}\left(\frac{5}{t}\right)^{\frac{5}{2}}\right)^{-\alpha s}}{\omega}\alpha ds \\
&= \frac{\pi 5^{\frac{9}{2}}}{t^5}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma\left(\frac{2}{\alpha}-s\right)\Gamma\left(1-\frac{2}{\alpha}+s\right)\left[\left(\frac{|x|}{2}\left(\frac{5}{t}\right)^{\frac{5}{2}}\right)^{\alpha}\right]^{-s}}{\omega}ds
\end{aligned}$$

In order to obtain the series representation of the above integral we apply the residual of the gamma function and the Cauchy residue theorem. So the contour of integration on the right hand side of the above formula can be transformed to $L_{+\infty}$ starting and ending at ∞ encircling all poles

$$S_k = \frac{2}{\alpha} + k$$

of the gamma function

$$\Gamma\left(\frac{2}{\alpha} - s\right)$$

and setting

$$\tau = \Gamma\left(\frac{1}{5}+\frac{\alpha}{2}k\right)\Gamma\left(\frac{2}{5}+\frac{\alpha}{2}k\right)\Gamma\left(\frac{3}{5}+\frac{\alpha}{2}k\right)\Gamma\left(\frac{4}{5}+\frac{\alpha}{2}k\right)\Gamma\left(-\frac{\alpha}{2}k\right)$$

we have

$$\begin{aligned}
\Phi_{\alpha,\frac{5}{2}\alpha,2}(x,t) &= \frac{\pi 5^{\frac{9}{2}}}{t^5}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{\Gamma(k+1)\left[\left(\frac{|x|}{2}\left(\frac{5}{t}\right)^{\frac{5}{2}}\right)^{\alpha}\right]^{\frac{-2}{\alpha}-k}}{\tau} \\
&= \frac{\pi 5^{\frac{9}{2}}}{t^5}\left[\frac{|x|}{2}\left(\frac{5}{t}\right)^{\frac{5}{2}}\right]^{-2}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{k!\left[\left(\frac{2}{|x|}\left(\frac{t}{5}\right)^{\frac{5}{2}}\right)^{\alpha}\right]^{k}}{\tau} \\
&= \frac{\pi 5^{\frac{9}{2}}}{t^5}\left[\frac{2}{|x|}\left(\frac{t}{5}\right)^{\frac{5}{2}}\right]^{2}\sum_{k=0}^{\infty}\frac{(-1)^k\left[\left(\frac{2}{|x|}\left(\frac{t}{5}\right)^{\frac{5}{2}}\right)^{\alpha}\right]^{k}}{\tau} \\
&= \frac{\pi 5^{\frac{9}{2}}}{t^5}\left[\frac{4}{|x|^2}\frac{t^5}{5^5}\right]\sum_{k=0}^{\infty}\frac{\left[-\left(\frac{2}{|x|}\left(\frac{t}{5}\right)^{\frac{5}{2}}\right)^{\alpha}\right]^{k}}{\tau} \\
&= \frac{4\pi}{\sqrt{5}|x|^2}\sum_{k=0}^{\infty}\frac{\left[-\left(\frac{2}{|x|}\left(\frac{t}{5}\right)^{\frac{5}{2}}\right)^{\alpha}\right]^{k}}{\tau}
\end{aligned}$$

These is a series representation of $\Phi_{\alpha,\frac{5}{2}\alpha,2}(x,t)$ $\square$

Chapter 5

Conclusion

In this thesis, the main focus is to establish explicit closed form for the fundamental solution of multi-dimensional space-time fractional diffusion-wave equation with particular case when $\beta = \frac{5}{2}\alpha$ and $n = 2$. Thus, for $\beta = \frac{5}{2}\alpha$ the fundamental solution is expressed as

$$\Phi_{\alpha,\frac{5}{2}\alpha,2}(x,t) = \frac{4\pi}{\sqrt{5}|x|^2} \sum_{k=0}^{\infty} \frac{\left[-\left(\frac{2}{|x|}\left(\frac{t}{5}\right)^{\frac{5}{2}}\right)^{\alpha}\right]^k}{\Gamma\left(\frac{1}{5}+\frac{\alpha}{2}k\right)\Gamma\left(\frac{2}{5}+\frac{\alpha}{2}k\right)\Gamma\left(\frac{3}{5}+\frac{\alpha}{2}k\right)\Gamma\left(\frac{4}{5}+\frac{\alpha}{2}k\right)\Gamma\left(-\frac{\alpha}{2}k\right)}$$

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Bibliography

[1] Al-Bassam, M. A., *On fractional analysis and its applications,* in: Manocha,H. L.(Ed.),Modern Analysis and its Applications, Prentice-Hall,New Delhi, (1983),269-307.

[2] Al-Bassam, M. A., *On fractional calculus and its applications to the theroy of ordinary differential equatins of generalized order*, in: Nonlinear Analysis and Applications. Vol. 80 of Lect. Notes Pure Math., Dekker, New York,(1981), 305-331.

[3] Alonson, J., *On differential equations of fractional order*, University Cincinnati,Diss. Abstrs., 26(4),(1964) 947-1017.

[4] Al-Saqabi, B. N., *Solution of a class of differintegral equations by means of Riemann-Liouville operator*, J. Fract. Calc., 8, (1995) 95-102.

[5] Bafket, R. L. and Torvik, P. J. *A theorical basis for the application of fractional calculus to viscoelasticity* ,J. Rheol., 27(3),(1983) 201-210.

[6] Boyadjiev, L. and Luchko, Y. *Mellin integral transform approach to analyze the multi-dimensional diffusion-wave equations*.Chaos Solitons Fractals,102,(2017), 127-134.

[7] Boyadjiev, L. and Luchko, Y. *Multi-dimensional α-fractional diffusion-wave equation and some properties of its fundamental solution*.Comput. Math, Appl. 73,(2017), 2561-2572.

[8] Chevillard, S. *The functions erf and erfc computed with arbitrary precision and explicit error bounds,* information and computation 216,(2012) 72-95.

[9] Diethelm, K. *Analysis of fractional differential equations*, Journal of fractional calculus, 7, (1995), 89-100.

[10] Emilia, B. and Ivan, B. *Subordination Approach to Space-Time Fractional Diffusion*,Mathematics, 7,(2019),4-15.

[11] Erdelyi, A., *Fractional integral of generalied functions*, J. Ausrtal. Math. Soc., 14(1),(1972) 30-37.

[12] Ferreira, M., Vieira, N. *Fundamental solutions of the time fractional diffusion-wave and parabolic Dirac operators*, J.Math.Anal.Appl. 447, (2016), 329-353.

[13] Gorenflo, R., Loutchko, J., Luchko, Y. *Computation of the Mittag Leffler function and its derivatives*, Fract Calc Appl Anal, (2002).

[14] Graf, U. *Introduction to Hyperfunctions and Their Integral Transforms*, (2010).

[15] Kilbas, A. A., Srivastava, H. M and Trujillo, J. J. *Theory and Application of Fractional Differential Equations*. 204, (2006) Elsevier Science B. V., Amesterdam, The Netherlands.

[16] Lakshmianthan, V. and Vasta, A. S. *Basic theory of fractional differential equations*, Nonlinear Analysis: Theory Methods and Applications, 69(8),(2008),262-267.

[17] Luchko, Y. *Multi-dimesional fractional wave equation and some propeties of its fundamental solution*,Common Appl. Ind. Math. 6, (2014) e-485.

[18] Luchko, Y. *A New Fractional Calculus Model for the Two-dimensional Anomalous Diffusion and its Analysis*,Math. Model. Nat. Phenom. 11(3), (2016), 117.

[19] Luchko, Y. *On some New properties of the fundamental solution to the Multidimensional space and time fractional Diffusion Wave equation*,Mathematics 5, (2017), 76.

[20] Luchko, Y. *Subordination Principles for the Multi-Dimensional Space-Time-Fractional Diffusion-Wave Equation*,Math.AP.(2018).

[21] Lyubomir, B., Luchko, Y. *Multi-dimensional α-fractional diffusion-wave equation and some properties of its fundamental solution*, computers and mathematics with application. 73, (2017), 2561-2572.

[22] Mainardi, F., Luchko, Y., Pagnini, G. *The fundamental solution of the space-time fractional diffusion equation*, Fract. Calc. Appl. Anal. 4, (2001), 153-192.

[23] Mainardi, F. *Fractional relaxation-oscillation and fractional diffusion-wave phenomena.* Chaos Solitons Fractals. 7, (1996), 1461-1477.

[24] Mariam, A., Mohammed, M. *Fractional Differential Equations and Applications*. 7, (2020).

[25] Miller, K. S., Ross, B. *An Introduction to the Fractional Calculus and Fractional Differential equations*, Wiley, New York, (1993).

[26] Oldham, K. B., Spanier, J. *The Fractional Calculus*, Acadamic Press, New York, (1974).

[27] Pagnini, G. *The M-Wright function as a generalization of the Gaussian density fro fractional diffusion processes*, Fract.Calc.Appl.Anal.16,(2013),436-453.

[28] Pagnini, G. *Subordination formulae for space-time fractional diffusion processes via Mellin convolution*,Recent Advances in Mathematics, Statistics and Economics,ISBN: 978-1-61804-225-5.

[29] Podlubny, I. *Fractional Differential Equations*, Academic Press,San Diego, (1999).

[30] Samko, S.G., Kilbas, A. A., Marichev, O. I. *Fractional Integrals and Derivatives, Theory and Applications*, Gordon and Breach, Amesterdam, The Netherlands, (1993).