ADDIS ABABA UNIVERSITY

SCHOOL OF GRADUATE STUDIES

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GRADUTE SEMINAR REPORT

ON

FEEDBACK IN CONTROL THEORY AND OPTIMAL CONTROL

(Submitted in partial fulfillment of M.Sc.Degree in Mathematics)

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Table of Contents

Chapter 1

Basics of Control Theory

1.1 Control Systems

A system is a collection of matter, parts or components which are included inside a specified boundary.

A control system is a device or set of devices to manage, command, direct or regulate the behavior of other devices or systems in order to get the desired system response or output. The system controls the variable output to the desired value by applying proper input or controlling signal to the system input terminals. The input-output relationship of the system represents the cause and effect relationship of the system and mathematically it represents the process in the system by which the input signal through system parameters controls the output signal to have the desired output.

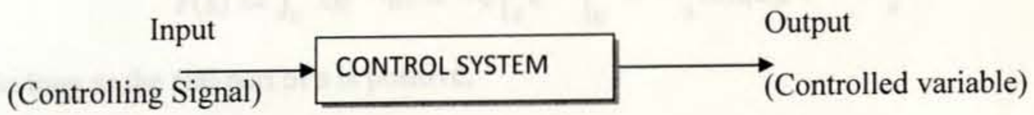


Fig1.1 Basic Control System

Plant: is the part of a system which is to be controlled.

Open Loop Control System: is a system in which the control action is independent of the system output. It uses a controlling device to control the system process so as to obtain the desired output.

Closed Loop Control System (Feedback control system): is a system in which the control action is somehow dependent upon the system output. A closed loop control system measures the actual system output, compares it with the input and determines the error which is then used for controlling the system output to have the desired value.

1.2. Review of Laplace Transforms

Ordinary and partial differential equations describe the way certain quantities vary with time, such as the current in an electric circuit, the oscillation of a vibrating membrane, or the flow of heat through an insulated conductor. These equations are generally coupled with initial conditions that describe the state of the system at time t = 0. A very powerful technique for solving these problems is that of the Laplace Transform, which literally transforms the original differential equation in to an elementary algebraic expression. This latter can then simply be transformed once again, in to the solution of the original problem. This technique is known as the "Laplace Transform method."

We define the Laplace Transform of a function$f\colon [0,\infty) \to \mathbb{C}$ as

$$\mathcal{L}(f(t)) := \int_0^{\infty} e^{-st} f(t)dt \quad \text{where } s \in \mathbb{C}. \tag{1.1}$$

The symbol $\mathcal{L}$ is the Laplace transformation, which acts on functions f and generates a new function, $F(s) := \mathcal{L}(f(t))$

Example 1.1: Consider the step function $f(t)$

$$f(t) = \begin{cases} 0 & \text{for } t < 0 \\ C(\text{constant}) & \text{for } t \geq 0 \end{cases}.$$

Then the Laplace Transform of f is

$$F(s) := \int_0^{\infty} ce^{-st}dt = -c\left[\frac{1}{s}e^{-st}\right]_0^{\infty} = -\frac{c}{s}\lim_{t\to\infty} e^{-st} + \frac{c}{s}.$$

So long as the real part of s is positive,

$$\lim_{t\to\infty} e^{-st} = 0.$$

So we have

$$F(s) = \frac{c}{s}.$$

1.3 Transfer Function and Block Diagrams

With appropriate approximations and using idealized components, we can often describe the dynamic behavior of a system by a linear n[th] order differential equation.

In general form, the nth order system having a single input u and a single output y has an associated differential equation

$$a_0D^n y + a_1D^{n-1}y + \ldots + a_n y = b_0D^m u + b_1D^{m-1}u + \ldots + b_m u \quad (n \geq m) \tag{1.2}$$

Given the input u and the initial conditions $y^{(i)}(0) = \alpha_i$, $i = \{0,1,2,\ldots,n-1\}$, the output response is found by solving the equation (1.2).

Assuming all initial values α_i equal to zero, and taking the Laplace transform equation (1.2) becomes

$$(a_0s^n + a_1s^{n-1} + \ldots + a_n)Y(s) = (b_0s^m + b_1s^{m-1} + \ldots + b_m)U(s), \tag{1.3}$$

where $Y(s) := \mathcal{L}(y(t))$ and $U(s) := \mathcal{L}(u(t))$.

Definition:

The transfer function G of a linear autonomous system is the quotient of Laplace transform of the output and the Laplace transform of the input, with the restriction that all initial conditions are zero, i.e.

$$G(s) := \frac{Y(s)}{U(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \ldots + b_m}{a_0 s^n + a_1 s^{n-1} + \ldots + a_n}. \tag{1.4}$$

Example 1.2:

The transfer function G(s) of the system described by the equation

$$\ddot{y} + 3\dot{y} + 2y = u,$$

where u and y are the input and the output respectively is given by $G(s) = \frac{Y(s)}{U(s)}$.

Taking the Laplace transform on both sides and initial values are zero,

$$\mathcal{L}[\ddot{y} + 3\dot{y} + 2y] = \mathcal{L}[u],$$

or

$$(s^2 + 3s + 2)Y(s) = U(s).$$

So we have

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^2+3s+2} = \frac{1}{(s+2)(s+1)}.$$

If the input to the system is $u(t) := \delta(t)$,the unit impulse function, then $U(s) = 1$

and $Y(s) = G(s)$.

Taking the inverse Laplace Transforms, we get $y(t) = g(t)$, where $g(t) := \mathcal{L}^{-1}[G(s)]$ is called the impulse response or the weighting function for the system.

Remark: At least theoretically we can obtain the transfer function, G(s), of the system by taking the Laplace transform of its response to a unit impulse assuming all initial conditions to be zero.

Block Diagram

A Block diagram is a pictorial representation of the functions performed by each component of a system. It is especially useful when transfer functions are used. Basically, Block Diagrams indicate the transfer function of the component or the system in a block. Arrows are used to show the connection between several blocks.

Example 1.3:

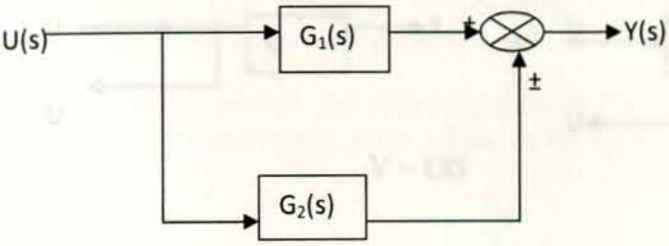


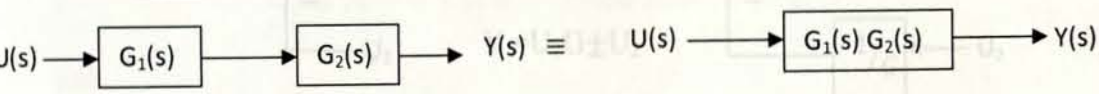
Fig 1.2 Block diagram

Reduction of Block Diagrams

A system consisting of several components can be reduced to a system consisting only of the "system transfer function" or a given block diagram can be transformed in to a different equivalent form.

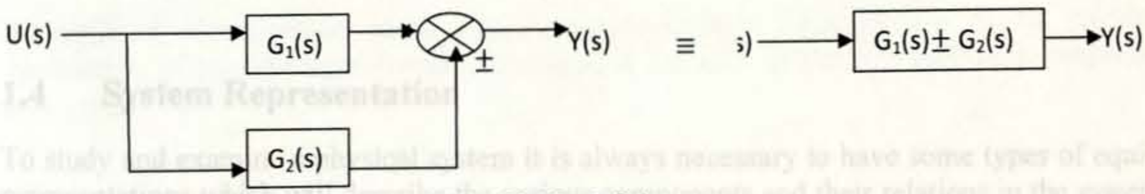
Block diagram reduction techniques

Rule (1): Combining blocks in cascade



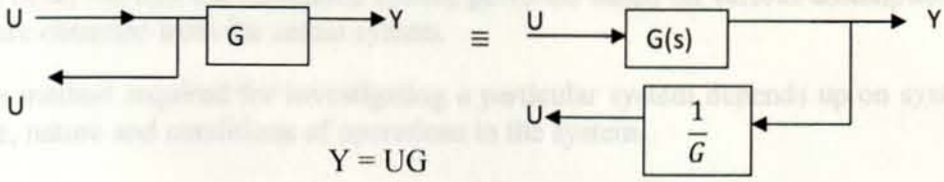
$$Y = UG_1\,G_2$$

Rule (2): Combining blocks in parallel

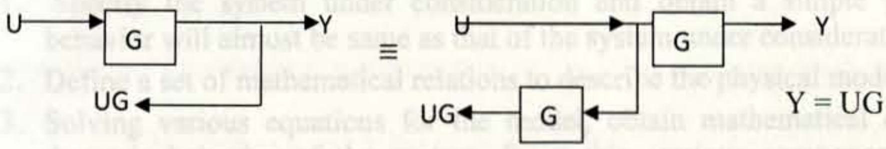


$$Y = U\,(G_1 \pm G_2)$$

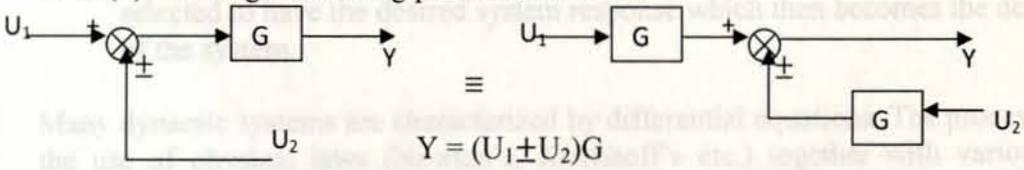
Rule (3): Moving a pick-off point after a block



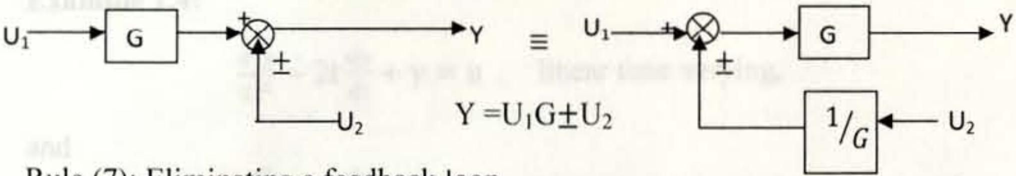
Rule (4): Moving a pick-off point ahead of a block



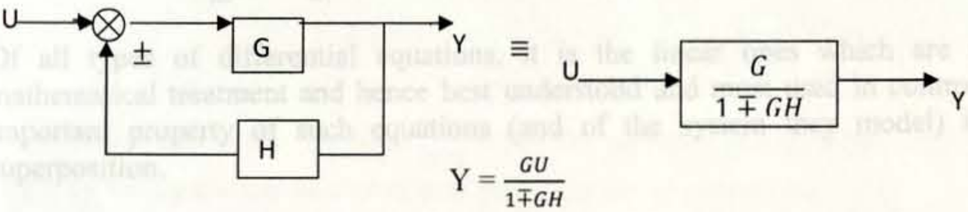
Rule (5): Moving a summing point after a block



Rule (6): Moving a summing point ahead of a block



Rule (7): Eliminating a feedback loop



1.4 System Representation

To study and examine a physical system it is always necessary to have some types of equivalent representations which will describe the various components and their relations in the system. We call this representation "Model of the System".

Models could be graphical like block diagrams or mathematical relations like integro-differential equations, transfer functions or matrices.

The performance of the system is investigated from these models by performing various mathematical operations to have certain variables in terms of system parameters.

It's observed that the calculated system performs based on various assumptions are almost same as are obtained from the actual system.

The method required for investigating a particular system depends up on system properties like size, nature and conditions of operations in the system.

Procedure:

1. Specify the system under consideration and obtain a simple physical model whose behavior will almost be same as that of the system under consideration.
2. Define a set of mathematical relations to describe the physical model.
3. Solving various equations for the model, obtain mathematical equations to study the dynamic behavior of the system. From this, various components and parameters are selected to have the desired system response which then becomes the design specifications of the system.

Many dynamic systems are characterized by differential equations. The process involved, that is, the use of physical laws (Newton's, Kirchhoff's etc.) together with various assumptions of linearity is known as Mathematical Modeling.

Example 1.4:

$$\frac{d^2y}{dt^2} - 2t\frac{dy}{dt} + y = u \quad \text{linear time varying,}$$

and

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 3y = u \quad \text{linear time invariant (autonomous).}$$

Of all types of differential equations, it is the linear ones which are most amenable to mathematical treatment and hence best understood and most used in control theory. The most important property of such equations (and of the system they model) is the principle of superposition.

Principle of superposition states that the response of a linear system to the simultaneous application of two different forcing functions, is the sum of the two individual responses and conversely.

$$\frac{d^2y}{dt^2} - 2t\frac{dy}{dt} + y = u. \tag{1.5}$$

In the equation (1.2) let y_1 and y_2 be the responses to Au_1 and Bu_2 respectively where A and B are constants.

Then we have

$$\frac{d^2y_1}{dt^2} - 2t\frac{dy_1}{dt} + y_1 = Au_1,$$

and
$$\frac{d^2y_2}{dt^2} - 2t\frac{dy_2}{dt} + y_2 = Bu_2.$$

Adding the two equations, we obtain

$$\frac{d^2(y_1+y_2)}{dt^2} - 2t\frac{d(y_1+y_2)}{dt} + y_1 + y_2 = Au_1 + Bu_2.$$

This implies $y_1 + y_2$ is the system response of the input function $Au_1 + Bu_2$.

Example 1.5:

The translational mechanical system shown in Fig 1.3 is described by the equations:

$$M_1 \frac{d^2x_1}{dt^2} + f_1\frac{dx_1}{dt} + k_1(x_1 - x_2) = f(t)$$

and

$$M_2 \frac{d^2x_2}{dt^2} + f_2\frac{dx_2}{dt} + (k_1 + k_2)x_2 - k_1x_1 = 0,$$

where M_1, f_1, k_1, x_1 and M_2, f_2, k_2, x_2 are mass, friction, stiffness (spring constant) and displacement respectively and f(t) is the applied force in the system.

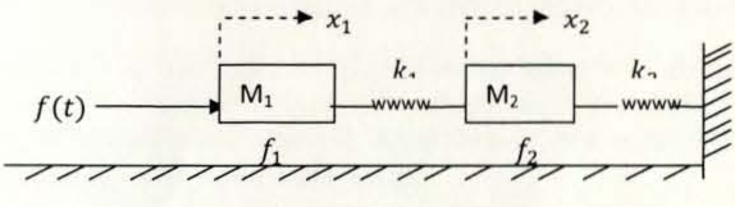


Fig 1.3

To determine the transfer function $\frac{X_1(s)}{F(s)}$ when the system is at rest, before applying the force f(t), we get by taking the Laplace transform with zero initial conditions,

$$M_1s^2X_1(s) + f_1sX_1(s) + k_1X_1(s) - k_2X_2(s) = F(s)$$

and

$$M_2s^2X_2(s) + f_2sX_2(s) + (k_1{+}k_2)X_2(s) - k_1X_1(s) = 0.$$

This is the same as

$$X_1(s)[M_1s^2 + f_1s + k_1] - k_2X_2(s) = F(s) \qquad (1.6)$$

and
$$k_1X_1(s) - [M_2s^2 + f_2s + (k_1{+}k_2)]X_2(s) = 0\,. \qquad (1.7)$$

Eliminating $X_2(s)$ from equations (1.6) and (1.7) the value of $X_1(s)$ is given by

$$X_1(s) = \frac{F(s)[M_2 s^2 + f_2(s) + (k_1 + k_2)]}{[M_2 s^2 + f_2(s) + (k_1 + k_2)][M_1 s^2 + f_1(s) + k_1] - k_1 k_2} .$$

So we have

$$\frac{X_1(s)}{F(s)} = \frac{M_2 s^2 + f_2(s) + (k_1 + k_2)}{[M_2 s^2 + f_2(s) + (k_1 + k_2)][M_1 s^2 + f_1(s) + k_1] - k_1 k_2} .$$

Chapter 2

State Space Representation

2.1 Introduction

So far, the analysis and performance of control systems was based on the transfer function and the basis of transfer function is

1. The initial conditions in system equation are considered as zero.
2. Applicable only to single-input-single-output (SISO) systems.
3. Applicable only to linear time invariant system equations.
4. The analysis of system output for a specified input is obtained. No information regarding internal state of the system is available.

Because of the above limitations, a more general mathematical representation of a control system is required which covers these limitations of the transfer function approach.

The modern approach known as state space analysis is preferred to transfer function approach, which considers analysis in the time domain and overcomes the limitations of transfer function approach.

The state space mathematical model of a control system takes in to account the following points.

1. The initial conditions of the system are taken in to consideration since the behavior of a system in the time domain is dependent on its past history.
2. The mathematical model is in the form of first order differential equations of linear time invariant or time variant systems.
3. The analysis is carried out in time domain.
4. The mathematical model covers both SISO and MIMO systems.
5. The concept of state space model forms the basis for analysis of advanced control systems.
6. The state space model gives complete description of the system.

It should be made clear that the state space approach need not replace the classical transfer function approach. The transfer function approach gives a physical insight in to the system for preliminary design of control systems through a simple mathematical model.

2.2 State Space Forms

Now we consider a more general case - a system characterized by an n^{th} order differential equation. We shall also assume that the system we are dealing with are autonomous, which implies that the free system (when the input is zero) does not depend explicitly on time.

The system equation has the form

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_{n-1}\dot{y} + a_n y = u. \tag{2.1}$$

It is assumed that $y(0), \dot{y}(0), \ldots, y^{(n-1)}(0)$ are known.

If we define

$$x_1 := y, \quad x_2 := \dot{y}, \quad \ldots, \quad x_n := y^{(n-1)},$$

then we can write equation (2.1) as a system of n simultaneous differential equations, each of order 1, namely

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = x_3,$$

$$\vdots$$

$$\dot{x}_{n-1} = x_n,$$

and from equation (2.1) we get

$$\dot{x}_n = -a_n x_1 - a_{n-1} x_2 - \ldots - a_1 x_n + u.$$

This can be written as a vector-matrix differential equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u. \tag{2.2}$$

This is the same as

$$\dot{x} = Ax + Bu, \tag{2.3}$$

where A, x and B are the corresponding matrices in equation (2.2).

The output of the system is $y = x_1$ and is given in matrix form as

$$y = [1\ 0\ 0\ \ldots\ 0][x_1\ x_2\ x_3\ \ldots\ x_n]^T,$$

that is

$$y = Cx, \tag{2.4}$$

where

$$C = [1\ 0\ 0\ \cdots\ 0].$$

The combination of equations (2.3) and (2.4) in the form

$$\dot{x} = Ax + Bu$$

and $$y = Cx\,, \tag{2.5}$$

is known as the state equation of the system considered and the matrix A in equation (2.3) is said to be in companion form.

The components of x are the state variables $x_1, x_2, \ldots, x_n$. They can be considered as the coordinate axes of an n-dimensional space called the state space. Any state of the system is represented by a point in the state space. The choice of the state variables is not unique. Depending on the choice of the state variables the associated matrix A will be changed. In many applications, it is useful to have the state variables decoupled, that is, $\dot{x}_i$ is a function of x_i and the input only. This implies that the matrix *A* in the state equations is in a diagonal form.

The Transformation from the Companion to the Diagonal State form

Since the choice of the state variables is not unique, for some choice of state variable say

$$\dot{x} = Ax + Bu$$

and $$y = Cx\,, \tag{2.6}$$

where A is a matrix of order $n \times n$ and B and C are matrices of appropriate order.

Now consider any non-singular matrix T of order $n \times n$ such that

$$x = Tz. \tag{2.7}$$

Then z is also a state vector and the state equation in (2.6) can be written as

$$T\dot{z} = ATz + Bu$$

and $$y = CTz\,,$$

or as

$$\dot{z} = A_1 z + B_1 u$$

and $$y = C_1 Z, \tag{2.8}$$

where $A_1 := T^{-1}AT,\ B_1 := T^{-1}B$ and $C_1 := CT$.

The transformation defined by equation (2.7) is called a state-transformation and the matrices A and A_1 are similar. It is especially interesting to transform from the Companion form A to the diagonal form A_1.

For this purpose, the matrix T is constructed as a matrix of Eigen-vectors of A. Here we are assuming that the Eigen-values of A are distinct and so the corresponding Eigen-vectors are linearly Independent which again implies that the matrix T is non-singular.

Controllability Observability and Stability

Controllability: State controllability condition implies that it is possible by admissible inputs to steer the states from any initial value to any final value within some finite time period. A continuous time-invariant linear state-space model is controllable if and only if

$$\text{rank}\,[B \;\; AB \;\; A^2B \;\; \dots \;\; A^{n-1}B] = n\,,$$

where A, B and n are as in equation (2.5).

Observability: It is a measure for how well internal states of a system can be inferred by knowledge of its external outputs. The observability and controllability of a system are mathematical duals (i.e., as controllability provides that an input is available that brings any initial state to any desired final state, observability provides that knowing an output trajectory provides enough information to predict the initial state of the system).

A continuous time-invariant linear state-space model is observable if and only if

$$\text{rank}[C \;\; CA \;\; \dots \;\; CA^{n-1}]^T = n\;,$$

where A, B and n are as in equation (2.5).

Stability: A linear time invariant (LTI) system is stable (asymptotically stable) if all the eigenvalues of the state matrix A in the state equation have negative real parts.

Example 2.1:

Let a system be described by the differential equation

$$\ddot{y} + 2\dot{y} - 3y = u\,.$$

First we want to give the companion form of the state space representation.

We let

$$x_1 := y \quad \text{and} \quad x_2 := \dot{y}\,.$$

Then

$$\dot{x}_1 = x_2$$

and

$$\dot{x}_2 = -2x_2 + 3x_1 + u\,.$$

So we have

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ 3 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u,$$

and

$$y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}.$$

The matrix $A := \begin{bmatrix}0 & 1\\ 3 & -2\end{bmatrix}$ is in companion form.

To transform A in to a diagonal form, we solve the characteristic equation of A i.e.

$$|A - \lambda I| = 0, \text{ where } A = \begin{bmatrix}0 & 1\\ 3 & -2\end{bmatrix},$$

then and we get $\lambda_1 = 1$ and $\lambda_2 = 3$ are the Eigenvalues of A and the corresponding Eigen-vectors are

$$v_1 = \begin{pmatrix}1\\ 1\end{pmatrix} \text{ and } v_2 = \begin{pmatrix}1\\ -3\end{pmatrix}.$$

Thus,

$$T = \begin{bmatrix}1 & 1\\ 1 & -3\end{bmatrix} \text{ and } T^{-1} = \begin{bmatrix}\frac{3}{4} & \frac{1}{4}\\ \frac{1}{4} & \frac{-1}{4}\end{bmatrix}.$$

Now transforming the state vector x in to z using the relation $x = Tz$,

$$A_1 = T^{-1}AT,$$

or

$$A_1 = \begin{bmatrix}\frac{3}{4} & \frac{1}{4}\\ \frac{1}{4} & \frac{-1}{4}\end{bmatrix}\begin{bmatrix}0 & 1\\ 3 & -2\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -3\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & -3\end{bmatrix},$$

which is in diagonal form.

$$B_1 = T^{-1}B \text{, that is, } B_1 = \begin{bmatrix}\frac{3}{4} & \frac{1}{4}\\ \frac{1}{4} & \frac{-1}{4}\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix} = \begin{bmatrix}\frac{1}{4}\\ \frac{-1}{4}\end{bmatrix},$$

and

$$C_1 = CT \text{, so that } C_1 = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -3\end{bmatrix} = \begin{bmatrix}1 & 1\end{bmatrix}.$$

Therefore the state equation in diagonal form is given by

$$\dot{z} = A_1 z + B_1 u$$

and $$y = C_1 z,$$

where A_1, B_1 and C_1 are as indicated.

To check the controllability of the system we where $A = \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $C = [1 \quad 0]$.

We have

$$\text{rank } [B \quad AB] = \text{rank} \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} = 2 \text{ (full rank)}.$$

Therefore the system is controllable.

To check the observability of the system we have

$$\text{rank} \begin{bmatrix} C \\ CA \end{bmatrix} = \text{rank} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 2 \text{ (full rank)}.$$

Hence the system is observable.

Since both eigenvalues of the system matrix A are positive and real, the system is unstable.

Example 2.2:

The Newton's laws of motion for an object moving horizontally on a plane and attached to a wall with a spring is

$$m\ddot{y}(t) = u(t) - k_1\dot{y}(t) - k_2 y(t),$$

where y, u, k_1, k_2 and m are the position, viscous friction coefficient, spring constant and mass of the object respectively.

Now let

$$x_1 := y \quad \text{and} \quad x_2 := \dot{y}.$$

This implies

$$\dot{x}_1 = x_2$$

and $$\dot{x}_2 = \ddot{y} = \frac{1}{m}(u(t) - k_1 x_2(t) - k_2 x_1(t)).$$

The state equation would become

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-k_2}{m} & \frac{-k_1}{m} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u(t)$$

and

$$y(t) = [1 \quad 0] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

The system is controllable because

$$[B \quad AB] = \left[\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ \frac{-k_2}{m} & \frac{-k_1}{m} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} \right] = \begin{bmatrix} 0 & \frac{1}{m} \\ \frac{1}{m} & \frac{k_1}{m^2} \end{bmatrix},$$

which has full rank for all k_1 and m.

It is also observable because

$$CA = [0 \quad 1],$$

So, $\begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, this also has a full rank.

Next we consider state equations where the input function involves derivatives. We consider the system equation

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}\dot{y} + a_n y = b_0 u^{(m)} + b_1 u^{(m-1)} + \cdots + b_{n-1}\dot{u} + b_n u. \qquad (2.9)$$

Then we define

$$x_1 = y - k_0 u$$
$$x_2 = \dot{x}_1 - k_1 u$$
$$x_3 = \dot{x}_2 - k_2 u$$
$$\vdots$$
$$x_n = \dot{x}_{n-1} - k_{n-1} u,$$

where

$$k_0 = b_0$$
$$k_1 = b_1 - a_1 b_0$$

(2.10)

$$k_2 = (b_2 - a_2 b_0) - a_1 k_1$$

$$k_3 = (b_3 - a_3 b_0) - a_2 k_1 - a_1 k_2$$

$$\vdots$$

$$k_n = (b_n - a_n b_0) - a_{n-1} k_1 - a_{n-2} k_2 - a_1 k_{n-1}.$$

The state equation of the system in equation (2.10) becomes

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ \vdots \\ k_n \end{bmatrix} u \qquad (2.11)$$

$$y = [1\ 0\ 0\ \ \dots\ 0][x_1\ \ x_2\ \ x_3\ \dots\ x_n]^T + k_0 u,$$

i.e.

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du,$$

where A, B, C and D are the corresponding matrices in equation (2.11).

Example 2.3:

Let the system be given by the equation

$$\ddot{y} + 2\dot{y} - y = 2\ddot{u} - u + 3u\,. \qquad (2.12)$$

Then, we have

$$a_1 = 2, \text{ and } a_2 = -1,$$

and

$$b_0 = 2\,, b_1 = -1 \text{ and } b_2 = 3.$$

From equation (2.10),

$$k_0 = 2, \ k_1 = -5 \ \text{ and } \ k_2 = 15.$$

Hence equation (2.11) becomes

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -5 \\ 15 \end{bmatrix} u$$

and $$y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 2u.$$

2.3 Using the Transfer Function to Define State Variables

It is sometimes possible to define suitable state variables by considering the partial fraction expansion of the transfer function.

Example 2.4:

Given the system differential equation

$$\ddot{y} + 3\dot{y} + 2y = \dot{u} + 3u\,,$$

the corresponding transfer function is

$$G(s) = \frac{Y(s)}{U(s)} = \frac{s+3}{(s+1)(s+2)}\,,$$

this can be written as

$$G(s) = \frac{2}{s+1} - \frac{1}{s+2}\,.$$

Hence $Y(s) = X_1(s) + X_2(s)$ (the choice of state variables is fixed here)

where $X_1(s) = \frac{2U(s)}{s+1}$ and $X_2(s) = \frac{-U(s)}{s+2}$.

On taking the inverse Laplace transforms, we obtain

$$\dot{x}_1 = -x_1 + 2u \text{ and } \dot{x}_2 = -2x_2 - u.$$

That is

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-1 & 0\\ 0 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}2\\ -1\end{bmatrix}u\,,$$

and $$y = [1 \quad 1]\begin{bmatrix}x_1\\ x_2\end{bmatrix}.$$

2.4 Solution of State Equations

2.4.1 Direct solution

Given a state equation

$$\dot{x} = Ax + Bu\,,$$

we need to find **x** at any time t, given u(t) (for all t) and the value of x at some specified time t_0, say $x(t_0) = x_0$.

At this point it is necessary to define a matrix function, that is, exponential of a matrix, and the derivative and the integral of a matrix.

Definition:

For every matrix $A \in \mathbb{R}^{n\times n}$,

$$e^A := \sum_0^\infty \frac{1}{n!}A^n = A^0 + A + \frac{1}{2!}A^2 + \cdots$$
$$= I_n + A + \frac{1}{2!}A^2 + \cdots \ .$$

Also if A has distinct eigenvalues, say $\lambda_1, \lambda_2, \dots, \lambda_n$, then there exists a non- singular matrix P (a matrix of the corresponding eigenvectors of A) such that

$$\Lambda := P^{-1}AP = \mathrm{diag}\{\lambda_1, \lambda_2, \dots, \lambda_n\},$$

then we have

$$A = P\Lambda P^{-1}.$$

This implies

$$A^2 = (P\Lambda P^{-1})(P\Lambda P^{-1}) = P\Lambda^2 P^{-1}.$$

In general

$$A^n = P\Lambda^n P^{-1} \qquad (n = 1,2, \dots).$$

For any matrix A, a matrix polynomial f(A) is defined as

$$f(A) := a_0A^n + a_1A^{n-1} + \cdots + a_{n-1}A + a_nI_n$$
$$= a_0P\Lambda^nP^{-1} + a_1P\Lambda^{n-1}P^{-1} + \cdots + a_{n-1}P\Lambda P^{-1} + a_nI_n$$
$$= P[a_0\Lambda^n + a_1\Lambda^{n-1} + \cdots + a_{n-1}\Lambda + a_nI_n]P^{-1}$$
$$= P\, f(\Lambda)P^{-1} = P\,\mathrm{diag}\{f(\lambda_1), f(\lambda_2), \dots, f(\lambda_n)\}P^{-1} .$$

In particular for $f(A) = e^{At}$, we obtain

$$e^{At} = P\,\mathrm{diag}\{f(\lambda_1), f(\lambda_2), \dots, f(\lambda_n)\}P^{-1}$$
$$= P\,\mathrm{diag}\{e^{\lambda_1 t}, e^{\lambda_2 t}, \dots, e^{\lambda_n t}\}P^{-1} .$$

Definition:

Let $A(t) := [a_{ij}(t)],\ t \in \mathbb{R}$. Then

(1) $\frac{d}{dt}A(t) := \left[\frac{d}{dt}(a_{ij}(t))\right]$,

(2) $\int A(t)dt := [\int a_{ij}(t)dt]$.

We now turn to our original problem, to solve equation (2.11). We write the equation in the form

$$\dot{x} - Ax = Bu.$$

Multiplying both sides by the integrating factor e^{-At}, we obtain

$$e^{-At}(\dot{x} - Ax) = e^{-At}Bu,$$

or $$\frac{d}{dt}[e^{-At}x] = e^{-At}Bu.$$

Integrating the result between 0 and t gives

$$e^{-At}x(t) - x(0) = \int_0^t e^{-A\tau}Bu(\tau)d\tau,$$

so that $$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau. \tag{2.13}$$

Example 2.5:

Consider a system characterized by the state equation

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 2\\ -1 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u.$$

If the input function $u(t) = 1$, for $t \geq 0$ and $x(0) = \begin{bmatrix}1\\ -1\end{bmatrix}$, to find the state x of the system at time t, first we need to evaluate e^{At}.

The eigenvalues of $A = \begin{bmatrix}0 & 2\\ -1 & -3\end{bmatrix}$ are $\lambda_1 = -1$ and $\lambda_2 = -2$.

It can be verified that $P = \begin{bmatrix}2 & 1\\ -1 & -1\end{bmatrix}$ and $P^{-1} = \begin{bmatrix}1 & 1\\ -1 & -2\end{bmatrix}$

Thus $e^{At} = P\,\text{diag}\{e^{\lambda_1 t}, e^{\lambda_2 t}\}\,P^{-1} = \begin{bmatrix}2 & 1\\ -1 & -1\end{bmatrix}\begin{bmatrix}e^{-t} & 0\\ 0 & e^{-2t}\end{bmatrix}\begin{bmatrix}1 & 1\\ -1 & -2\end{bmatrix}$

$$= \begin{bmatrix}2e^{-t} - e^{-2t} & 2e^{-t} - 2e^{-2t}\\ e^{-2t} - e^{-t} & 2e^{-2t} - e^{-t}\end{bmatrix}.$$

So we have

$$e^{At}x(0) = \begin{bmatrix}2e^{-t} - e^{-2t} & 2e^{-t} - 2e^{-2t}\\ e^{-2t} - e^{-t} & 2e^{-2t} - e^{-t}\end{bmatrix}\begin{bmatrix}1\\ -1\end{bmatrix} = e^{-2t}\begin{bmatrix}1\\ -1\end{bmatrix}.$$

The second term of equation (2.13) is

$$\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau = \int_0^t e^{A(t-\tau)}\begin{bmatrix}0\\ 1\end{bmatrix}d\tau$$

$$= \int_0^t \begin{bmatrix} 2e^{-(t-\tau)} - 2e^{-2(t-\tau)} \\ 2e^{-2(t-\tau)} - e^{-(t-\tau)} \end{bmatrix} d\tau$$

$$= \begin{bmatrix} 1 - 2e^{-t} + e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}.$$

Therefore, the state of the system at time t is

$$x(t) = e^{-2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 - 2e^{-t} + e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

$$= \begin{bmatrix} 1 - 2e^{-t} + 2e^{-2t} \\ e^{-t} - 2e^{-2t} \end{bmatrix}$$

Remark: The matrix e^{At} in the solution equation (2.13) is commonly known as state-transition matrix and is denoted by $\Phi(t)$.

2.4.2 Solution of the State Equation by Laplace Transforms

Definition:

For $x(t) := \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$, we define $\mathcal{L}[\mathbf{x}(t)] := \begin{bmatrix} \mathcal{L}[x_1(t)] \\ \mathcal{L}[x_2(t)] \\ \vdots \\ \mathcal{L}[x_n(t)] \end{bmatrix} = \begin{bmatrix} X_1(s) \\ X_2(s) \\ \vdots \\ X_n(s) \end{bmatrix} = X(s).$

From this definition one can easily see that

$$\mathcal{L}[\dot{x}(t)] = \begin{bmatrix} \mathcal{L}[\dot{x}_1(t)] \\ \mathcal{L}[\dot{x}_2(t)] \\ \vdots \\ \mathcal{L}[\dot{x}_n(t)] \end{bmatrix} = \begin{bmatrix} sX_1(s) - x_1(0) \\ sX_2(s) - x_2(0) \\ \vdots \\ sX_n(s) - x_n(0) \end{bmatrix} = sX(s) - x(0).$$

Now we can solve the equation (2.12) by taking the Laplace transform of

$$\dot{x} = Ax + Bu .$$

Then we obtain

$$sX(s) - x(0) = AX(s) + BU(s) ,$$

where

$$U(s) = \mathcal{L}[u(t)] \quad \text{and} \quad X(s) = \mathcal{L}[x(t)].$$

This implies

$$[sI - A]X(s) = x(0) + BU(s).$$

Provided that s is not an eigenvalue of A, $[sI - A]$ is non-singular, so that the above equation can be solved as follows:

$$X(s) = [sI - A]^{-1}x(0) + [sI - A]^{-1}BU(s) \tag{2.14}$$

One can find $x(t)$ by taking the inverse Laplace transform of equation 2.14.

The matrix $[sI - A]^{-1}$ is known as resolvent matrix of the system.

Example 2.6:

Given the system

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-1 & 0\\ 2 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u \text{ with } x(0) = \begin{bmatrix}-1\\ 2\end{bmatrix} \text{ and } u(t) = 1 \text{ for } t \geq 0.$$

For this system $sI - A = \begin{bmatrix}s+1 & 0\\ -2 & s+2\end{bmatrix}$,

so that

$$[sI - A]^{-1} = \frac{1}{(s+1)(s+2)}\begin{bmatrix}s+2 & 0\\ 2 & s+1\end{bmatrix} = \begin{bmatrix}\frac{1}{s+1} & 0\\ \frac{2}{(s+1)(s+2)} & \frac{1}{s+2}\end{bmatrix}.$$

To evaluate the inverse transform, one should express each element of the matrix as a partial fraction, that is, as

$$[sI - A]^{-1} = \begin{bmatrix}\frac{1}{s+1} & 0\\ \frac{2}{s+1} - \frac{2}{s+2} & \frac{1}{s+2}\end{bmatrix}.$$

Then from equation 2.14 we get

$$\begin{aligned} X(s) &= [sI - A]^{-1}x(0) + [sI - A]^{-1}BU(s) \\ &= \begin{bmatrix}\frac{1}{s+1} & 0\\ \frac{2}{s+1} - \frac{2}{s+2} & \frac{1}{s+2}\end{bmatrix}\begin{bmatrix}-1\\ 2\end{bmatrix} + \frac{1}{s}\begin{bmatrix}\frac{1}{s+1} & 0\\ \frac{2}{s+1} - \frac{2}{s+2} & \frac{1}{s+2}\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix} \\ &= \begin{bmatrix}\frac{-1}{s+1}\\ \frac{-2}{s+1} + \frac{4}{s+2} + \frac{1}{s(s+2)}\end{bmatrix} = \begin{bmatrix}\frac{-1}{s+1}\\ \frac{-2}{s+1} + \frac{4}{s+2} + \frac{1}{2s} - \frac{1}{2(s+2)}\end{bmatrix}.\end{aligned}$$

Hence

$$x(t) := \mathcal{L}^{-1}\{X(s)\} = \begin{bmatrix}-e^{-t}\\ \frac{1}{2} - 2e^{-t} + \frac{7}{2}e^{-2t}\end{bmatrix}.$$

2.5 Transfer Matrix

The matrix relating the Laplace transform of the output Y(s) to that of the input U(s) of the state space representation of a control system is known as transfer matrix.

Given the state equations of a system as

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du , \qquad (2.15)$$

where x is a state vector, u is input vector, and y is output vector.

Taking the Laplace transform on both sides of equations (2.15), the following equations are obtained

$$sX(s) - x(0) = AX(s) + BU(s)$$
$$Y(s) = CX(s) + DU(s) \qquad (2.16)$$

Assuming the initial conditions are zero as is the case in transfer function, i.e. $x(0) = 0$, we get

$$sX(s) = AX(s) + BU(s)$$

or

$$X(s) = [sI - A]^{-1}BU(s) ,$$

provided that sI − A is non singular. (2.17)

This implies that $\frac{Y(s)}{U(s)} = \frac{CX(s)+DU(s)}{U(s)} = \frac{C[sI-A]^{-1}BU(s)+DU(s)}{U(s)}$.

Thus the transfer matrix G(s) is given by

$$G(s) := \frac{Y(s)}{U(s)} = C[sI - A]^{-1}B + D. \qquad (2.18)$$

If D is a null matrix then equation (2.18) becomes

$$G(s) = \frac{Y(s)}{U(s)} = C[sI - A]^{-1}B. \qquad (2.19)$$

Example 2.7:

A system is described as follows:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

and $y = \begin{bmatrix} 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Here $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 5 \end{bmatrix}$ and $D = 0$.

Therefore, the transfer matrix is $G(s)$ is given by

$$G(s) = \frac{Y(s)}{U(s)} = C[sI - A]^{-1}B = [2 \quad 5]\begin{bmatrix} s-1 & 1 \\ 1 & s-2 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{2s+1}{(s-1)(s-2)-1},$$

or

$$\frac{Y(s)}{U(s)} = \frac{2s+1}{s^2+3s+1}.$$

Note that the system is unstable because the poles are both positive i.e., $\frac{3+\sqrt{5}}{2}$ and $\frac{3-\sqrt{5}}{2}$. We shall see in the following section that introducing feedback in to the system makes the system stable.

2.6 Feedback State Space Representation

A common method for feedback is to multiply the output $y(t)$ by a matrix K and setting this as the input to the system i.e., $u(t) = Ky(t)$. Since the values of K are unrestricted the values can easily be negated for negative feedback.

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{2.20}$$

replacing $u(t)$ by $Ky(t)$ equation 2.19 becomes

$$\begin{aligned} \dot{x}(t) &= Ax(t) + BKy(t) \\ \text{and} \quad y(t) &= Cx(t) + DKy(t). \end{aligned} \tag{2.21}$$

Solving the output equation for $y(t)$ and substituting in the state equation results in

$$\begin{aligned} \dot{x}(t) &= (A + BK(I - DK)^{-1}C)x(t) \\ \text{and} \quad y(t) &= (I - DK)^{-1}Cx(t). \end{aligned} \tag{2.22}$$

The advantage of this is that the eigenvalues of $\widetilde{A} := A + BK(I - DK)^{-1}C$ can be controlled by setting K appropriately through eigendecomposition of $A + BK(I - DK)^{-1}C$. This assumes that the open-loop system is controllable or that the unstable eigenvalues of A can be made stable through appropriate choice of K. One fairly common simplification to this system is removing *D* and setting *C* to identity, which reduces the equations to

$$\dot{x}(t) = (A + BK)x(t) \tag{2.23}$$

and $\quad y(t) = x(t).$

This reduces the necessary eigendecomposition only to $A + BK$.

Example 2.8:

In example 2.7 if we introduce a feedback i.e., $u(t) = Ky(t)$, then the state equation becomes

$$\dot{x}(t) = (A + BK(I - DK)^{-1}C)x(t)$$

and $\quad y(t) = (I - DK)^{-1}Cx(t).$

This implies

$$\dot{x}(t) = \left(\begin{pmatrix}0 & 1\\ 1 & 2\end{pmatrix} + K\begin{pmatrix}1\\0\end{pmatrix}(2 \quad 5)\right)x(t) = \begin{pmatrix}2K & 5K+1\\ 1 & 2\end{pmatrix}x(t)$$

and $\quad y(t) = (2 \quad 5)x(t).$

Then $\tilde{A} := \begin{pmatrix}2K & 5K+1\\ 1 & 2\end{pmatrix}$, the characteristic equation, $|sI - \tilde{A}| = 0$,becomes

$$s^2 - 2(K+1)s - (K+1) = 0, \tag{2.24}$$

solving for s , we get

$$s = (K+1) \pm \sqrt{(K+1)(K+2)}. \tag{2.25}$$

Since the eigenvalues of $\tilde{A}$ can be controlled by setting K appropriately, if we want to modify one of the poles to $s = -2$ (say the one with minus sigh) then K becomes $-7/3$ and the other eigenvalue (pole) become $s = -2/3$ (solving equation (2.24)). Hence the feedback is stable unlike the open loop system. One advantage of a feedback is stabilizing the unstable system. But here we need to make sure that the open loop system is controllable, i.e., rank [B, AB] = n (full rank).

Example 2.9:

Consider the system

$$\dot{x} = \begin{pmatrix}1 & 1\\ 0 & 2\end{pmatrix}x + \begin{pmatrix}1\\0\end{pmatrix}u \tag{2.26}$$

and $\quad y = (1 \quad 0)x.$

We can easily see that one cannot modify the pole at s = 2.

We have $A = \begin{pmatrix}1 & 1\\ 0 & 2\end{pmatrix}$, $B = \begin{pmatrix}1\\0\end{pmatrix}$, $C = (1 \quad 0)$ and $D = 0$.

Introducing the feedback $u = Ky$, the equations (2.25) becomes

$$\dot{x} = \begin{pmatrix}1 & 1\\ 0 & 2\end{pmatrix}x + \begin{pmatrix}1\\0\end{pmatrix}Ky \tag{2.27}$$

and $\quad y = (1 \quad 0)x.$

This implies

$$\dot{x} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} K(1 \quad 0)x = \begin{pmatrix} 1+K & 1 \\ 0 & 2 \end{pmatrix} x .$$

Hence the state matrix of the feedback system becomes $\tilde{A} := \begin{pmatrix} 1+K & 1 \\ 0 & 2 \end{pmatrix}$.

The characteristics equation is

$$\det(sI - \tilde{A}) = 0,$$

or

$$(s - (1+k))(s-2) = 0 . \tag{2.28}$$

Thus the pole at $s = 2$ cannot be modified while the pole at $s = 1$ can be modified by appropriate choice of K. Since $2 > 0$, the feedback loop remains unstable. The reason for this is that the open loop system is uncontrollable since $\mathrm{rank}[B, \; AB] = 1$(not full).

Feedback with set point (reference) input

In addition to feedback, an input $\mathbf{r}(t)$ can be added such that $\mathbf{u(t) = Ky(t) + r(t)}$.

$$\dot{x}(t) = Ax(t) + Bu(t)$$

and

$$y(t) = Cx(t) + Du(t) . \tag{2.29}$$

Replacing $u(t)$ by $Ky(t) + r(t)$ we get

$$\begin{aligned} \dot{x}(t) &= Ax(t) + BKy(t) + Br(t) \\ y(t) &= Cx(t) + DKy(t) + Dr(t) . \end{aligned} \tag{2.30}$$

Solving the output equation for $y(t)$ and substituting in the state equation results in

$$\dot{x}(t) = (A + BK(I + DK)^{-1}C)x(t) + B(I + K(I + DK)^{-1}D)r(t) \tag{2.31}$$

$$y(t) = (I - DK)^{-1}Cx(t) + (I - DK)^{-1}Dr(t) .$$

One fairly common simplification to this system is removing *D*, which reduces the equations to

$$\dot{x}(t) = (A + BKC)x(t) + Br(t) \tag{2.32}$$

$$y(t) = Cx(t) .$$

Chapter 3

Feedback Characteristics of Control Systems

3.1 Introduction

Open and closed-loop controls

The basis for analysis of a control system is the foundation provided by linear system theory, which assumes a cause-effect relationship for the components of a system. A component or process to be controlled can be represented by a block. Each block possesses an input (cause) and output (effect). The input-output relation represents the cause-and-effect relationship of the process, which in turn represents a processing of the input signal to provide an output signal variable, often with power amplification. An open-loop control system utilizes a controller or control actuator in order to obtain the desired response.

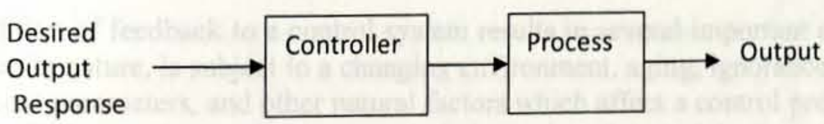


Fig 3.1 Open-loop control system

In contrast to an open-loop control system, a closed-loop control system utilizes an additional measure of the actual output in order to compare the actual output with the desired output response. A standard definition of a feedback control system is a control system which tends to maintain a prescribed relationship of one system variable to another by comparing functions of these variables and using the difference as a means of control. In the case of the driver steering an automobile, the driver uses his or her sight to visually measure and compare the actual location of the car with the desired location. The driver then serves as the controller, turning the steering wheel. The process represents the dynamics of the steering mechanism and the automobile response.

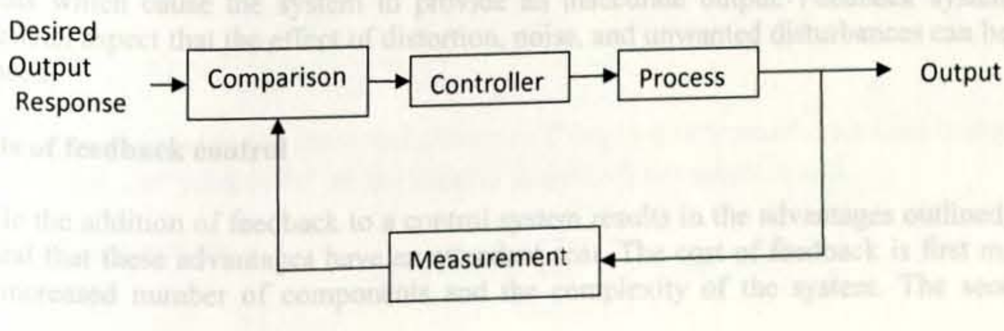


Fig. 3.2 Feedback control system

A feedback control system often uses a function of a prescribed relationship between the output and reference input to control the process. Often, the difference between the output of the process under control and the reference input is amplified and used to control the process so that the difference is continually reduced. The feedback concept has been the foundation for control system analysis and design.

Applications of feedback systems

Familiar control systems have the basic closed-loop configuration. For example, a refrigerator has a temperature setting for desired temperature, a thermostat to measure the actual temperature and the error, and a compressor motor for power amplification. Other examples in the home are the oven, furnace, and water heater. In industry, there are controls for speed, process temperature and pressure, position, thickness, composition, and quality, among many others. Feedback control concepts have also been applied to mass transportation, electric power systems, automatic warehousing and inventory control, automatic control of agricultural systems, biomedical experimentation and biological control systems, and social, economic, and political systems.

Advantages of feedback control

The addition of feedback to a control system results in several important advantages. A process, whatever its nature, is subject to a changing environment, aging, ignorance of the exact values of the process parameters, and other natural factors which affect a control process. In the open-loop system, all these errors and changes result in a changing and inaccurate output. However, a closed-loop system senses the change in the output due to the process changes and attempts to correct the output. The sensitivity of a control system to parameter variations is of prime importance. A primary advantage of a closed-loop feedback control system is its ability to reduce the system's sensitivity.

One of the most important characteristics of control systems is their transient response, which often must be adjusted until it is satisfactory. If an open-loop control system does not provide a satisfactory response, then the process must be replaced or modified. By contrast, a closed-loop system can often be adjusted to yield the desired response by adjusting the feedback loop parameters.

A second important effect of feedback in a control system is the control and partial elimination of the effect of disturbance signals. Many control systems are subject to extraneous disturbance signals which cause the system to provide an inaccurate output. Feedback systems have the beneficial aspect that the effect of distortion, noise, and unwanted disturbances can be effectively reduced.

Costs of feedback control

While the addition of feedback to a control system results in the advantages outlined above, it is natural that these advantages have an attendant cost. The cost of feedback is first manifested in the increased number of components and the complexity of the system. The second cost of feedback is the loss of gain. Usually, there is open-loop gain to spare, and one is more than willing to trade it for increased control of the system response. Finally, a cost of feedback is the introduction of the possibility of instability. While the open-loop system is stable, the closed-loop system may not be always stable.

Stability of closed-loop systems

The transient response of a feedback control system is of primary interest and must be investigated. A very important characteristic of the transient performance of a system is the stability of the system. A stable system is defined as a system with a bounded system response. That is, if the system is subjected to a bounded input or disturbance then the response of the system is bounded in magnitude, this system is said to be stable.

The concept of stability can be illustrated by considering a right circular cone placed on a plane horizontal surface. If the cone is resting on its base and is tipped slightly, it returns to its original equilibrium position. This position and response is said to be stable. If the cone rests on its side and is displaced slightly, it rolls with no tendency to leave the position on its side. This position is designated as neutral stability. On the other hand, if the cone is placed on its tip and released, it falls onto its side. This position is said to be unstable.

The stability of a dynamic system is defined in a similar manner. The response to a displacement, or initial condition, will result in a decreasing, neutral, or increasing response.

3.2 Sensitivity of Control Systems to Parameter Variation

One of the important properties of negative feedback systems is the reduction in the sensitivity to variations in the parameters of the forward path. In the design of control systems, it is important that the transfer function of the closed loop system be relatively insensitive to small changes in the values of the parameters of the components in the forward path of the system.

Let μ be a parameter of $G(s)$. Then the sensitivity of $G(s)$ with respect to the parameter μ is defined as

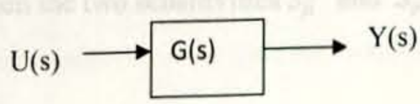
$$S_{\mu}^{G} := \lim_{\Delta\mu \to 0} \frac{\left(\frac{\Delta G}{G}\right)}{\left(\frac{\Delta \mu}{\mu}\right)},$$

which can also be written as

$$S_{\mu}^{G} = \frac{d \ln G}{d \ln \mu} = \frac{dG/G}{d\mu/\mu} = \frac{\mu}{G}\frac{dG}{d\mu}, \qquad (3.4)$$

which is considered as the fractional change in G due to a very small fractional change in μ. This is defined as the "sensitivity" of the transfer function G to variations inμ.

For open loop system

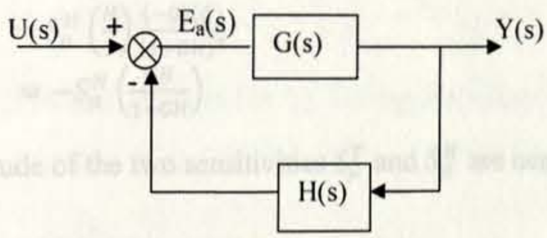


$$Y(s) = G(s)U(s)$$

Therefore

$S_{\mu}^{G} = \frac{\mu}{G}\frac{dG}{d\mu}$, since G is the system transfer function.

For a closed loop system



$$Y(s) = G(s)E_a(s),$$

where E_a is the actuating error and

$$E_a(s) = U(s) - H(s)Y(s).$$

This implies

$$Y(s) = \frac{G(s)}{1+G(s)H(s)}U(s), \tag{3.2}$$

and the actuating error is given by

$$E_a(s) = \frac{1}{1+G(s)H(s)}U(s). \tag{3.3}$$

In order to reduce the error, the loop gain $G(s)H(s)$ should be made large over the range of frequencies of interest, i.e., $|G(s)H(s)| \gg 1$.

The transfer function of a closed loop system is given by

$$T(s) = \frac{Y(s)}{U(s)} = \frac{G(s)}{1+G(s)H(s)}. \tag{3.5}$$

Hence the sensitivity of the closed loop system to variations in μ is given by

$$S_{\mu}^{T} = \frac{\mu}{T}\frac{dT}{d\mu} = \left(\frac{\mu(1+GH)}{G}\right)\left(\frac{dT}{dG}\right)\left(\frac{dG}{d\mu}\right)$$

$$= \left(\frac{\mu(1+GH)}{G}\right)\left(\frac{1}{1+GH}\right)^2\left(\frac{dG}{d\mu}\right)$$

$$= \left(\frac{\mu}{G}\right)\left(\frac{dG}{d\mu}\right)\left(\frac{1}{1+GH}\right).$$

Therefore, the relationship between the two sensitivities S_{μ}^{G} and S_{μ}^{T} is given by

$$S_{\mu}^{T} = \frac{S_{\mu}^{G}}{1+G(s)H(s)}. \tag{3.6}$$

Thus, feedback has reduced the sensitivity to variation in μ by the factor $1/[1 + G(s)H(s)]$, which is small over the range of frequencies of interest.

But it should be clear that feedback does not reduce the sensitivity to variations in the parameter of the feedback path. To show this, let μ be a parameter of $H(s)$. Then we have

$$S_\mu^T = \frac{\mu}{T}\frac{dT}{d\mu} = \left(\frac{\mu}{H}\right)\left(\frac{dH}{d\mu}\right)\left(\frac{H}{T}\right)\left(\frac{dT}{dH}\right)$$

$$= S_\mu^H\left(\frac{H}{T}\right)\frac{(-G)(G)}{(1+GH)^2}$$

$$= -S_\mu^H\left(\frac{GH}{1+GH}\right) \tag{3.7}$$

It follows that the magnitude of the two sensitivities S_μ^T and S_μ^H are nearly equal when the loop gain $G(s)H(s) \gg 1$.

Due to the improvement in sensitivity of feedback there is a loss of system gain. The open loop system has a gain $G(s)$ while the gain of the closed loop system is $G(s)/[1 + G(s)H(s)]$. So by using feedback, the system gain is reduced by the same factor as by which the sensitivity of the system to parameter variations is reduced. Sufficient amount of open loop gain can, however, be easily built in to a system so that we can afford to lose some gain to achieve improvement in sensitivity.

Example 3.1:

In the closed loop system shown below $G_1(s) = \frac{1}{s+1}$, $G_2(s) = \frac{1}{s+2}$ and $H(s) = \frac{1}{s}$, find the sensitivities $S_{G_1}^T$ and $S_{G_2}^T$ and the sensitivity of the open loop system.

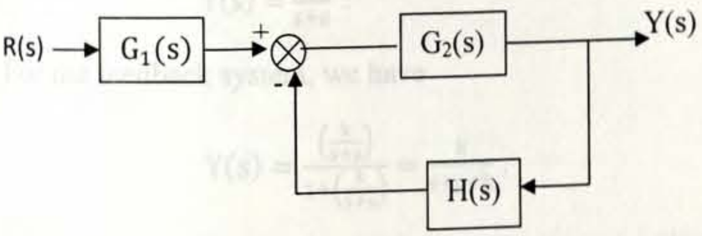


Fig 3.3 Closed loop system

The closed loop transfer function is given by

$$T(s) = \frac{G_1(S)G_2(S)}{1+G_2(S)H(s)}.$$

The sensitivity of the system to variations on G_1 is calculated as

$$S_{G_1}^T = \frac{G_1}{T}\frac{\partial T}{\partial G_1} = \left[\frac{1+G_2H}{G_2}\right]\left[\frac{G_2(1+G_2H)}{(1+G_2H)^2}\right] = 1.$$

This is clearly because there is no feedback on the transfer function G_1.

The sensitivity of the system to variations on G_2 is calculated as

$$S_{G_2}^{T} = \frac{G_2}{T}\frac{\partial T}{\partial G_2} = \frac{(1+G_2H)}{G_1}\left[\frac{G_1}{(1+G_2H)^2}\right]$$

$$= \frac{1}{1+G_2H} = \frac{s(s+2)}{(s+1)^2} = 1 - \frac{1}{(s+1)^2}.$$

The open loop system transfer function is given by $T(s) = G_1(S)G_2(s)$.

The sensitivity of $S_{G_2}^{T}$ is then calculated as $S_{G_2}^{T} = \frac{G_2}{T}\frac{\partial T}{\partial G_2} = \frac{G_2}{T}G_1 = 1$.

3.3 Controls over System Dynamics by Using Feedback

Consider a simple feedback system shown below:

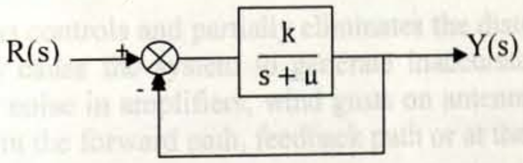


Fig 3.4 Feedback control system

The open loop transfer function of the system is

$$G(s) = \frac{k}{s+\mu}.$$

The output of the non feedback system to an impulse input i.e., $R(s) = 1$, is given by

$$Y(s) = \frac{k}{s+\mu}. \tag{3.8}$$

For the feedback system, we have

$$Y(s) = \frac{\left(\frac{k}{s+\mu}\right)}{1+\left(\frac{k}{s+\mu}\right)} = \frac{k}{s+\mu+k}. \tag{3.9}$$

Note that in the Fig. 3.4 the feedback path has a transfer function $H(s) = 1$.

In order to find the impulse response of the system we take the inverse Laplace Transform of the equations (3.8) and (3.9), from which we have

$$y(t) = ke^{-\mu t} \tag{3.10}$$

and $$y(t) = ke^{-(\mu+k)t}, \tag{3.11}$$

respectively, corresponding to the non-feedback system and the feedback system.

Also from the transfer functions of the non-feedback system and feedback system we can see that $-\mu$ and $-(\mu + k)$ are the respective zeros (poles) of their characteristics equations. And the nature of their response is an exponential decay with a time constant $\tau = \frac{1}{\mu}$ and $\tau = \frac{1}{(k+\mu)}$ for the non-feedback and feedback systems respectively. For positive values of k, the time constant of the feedback system is reduced and hence the control in the dynamics of the system. As k increases, the system dynamics becomes faster and so the transient response decays more rapidly.

It can be seen that feedback controls the dynamics of the system by adjusting the location of the poles. As a result feedback increases the possibility of instability of systems.

3.4 Control of Disturbance Signal in feedback Control Systems

Feedback in systems controls and partially eliminates the disturbance signal in the system. These disturbance signals cause the system to generate inaccurate output. Examples of disturbance signals are thermal noise in amplifiers, wind gusts on antenna of radar systems etc. Disturbance may be introduced in the forward path, feedback path or at the output of the system.

3.4.1 Disturbance in the forward path of the system

These disturbances may be either due to the properties of the elements present in the forward path or can be due to the surrounding condition changes of forward path elements.

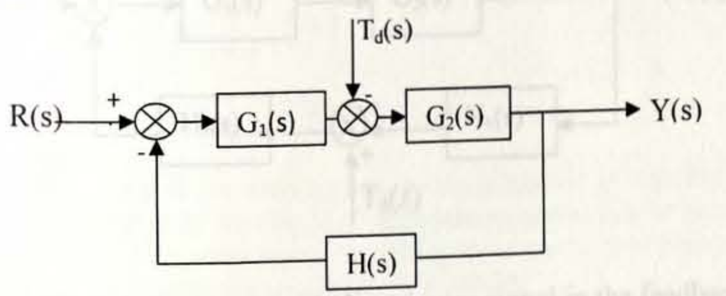


Fig.3.5 A system with disturbance signal in the forward path

Now we have, that the ratio of $Y(s)$ to disturbance $T_d(s)$ is

$$\frac{Y_d(s)}{T_d(s)} = \frac{-G_2(s)}{1+G_1(s)G_2(s)H(s)}. \quad (3.12)$$

Note that here we assume the input $R(s)$ is zero.

If $G_1(s)G_2(s)H(s) \gg 1$, then we have from equation (3.12)

$$M_d(s) := \frac{Y_d(s)}{T_d(s)} \approx \frac{-G_2(s)}{G_1(s)G_2(s)H(s)} = \frac{-1}{G_1(s)H(s)}.$$

Hence we have

$$Y_d(s) = \frac{-T_d(s)}{G_1(s)H(s)} . \qquad (3.13)$$

From equation (3.13), it is clear that if $G_1(s)$ is made very large the effect of disturbance $T_d(s)$ on the output will be very small. Therefore, it is always desired to have large $G_1(s)$, so that the output may not change due to disturbances in the forward path of the system. The sensitivity of the system for changes in the forward path transfer function $G_2(s)$, for disturbance input in the system, we have

$$S_{G_2}^{M_d} = \left(\frac{\partial M_d}{\partial G_2}\right)\left(\frac{G_2}{M_d}\right) = \frac{1}{1+G_1G_2H} \approx 0,$$

if $G_1(s)$ is made very large.

3.4.2. Disturbance in the feedback path

These disturbances are produced due to changes in the feedback path elements like sensing devices. In Fig 2.8, the transfer function relating the disturbance $T_d(s)$ with the output $Y_d(s)$ is given by

$$M_d(s) = \frac{Y_d(s)}{T_d(s)} = \frac{-G_1(s)G_2(s)H_2(s)}{1+G_1(s)G_2(s)H_1(s)H_2(s)} \qquad (3.14)$$

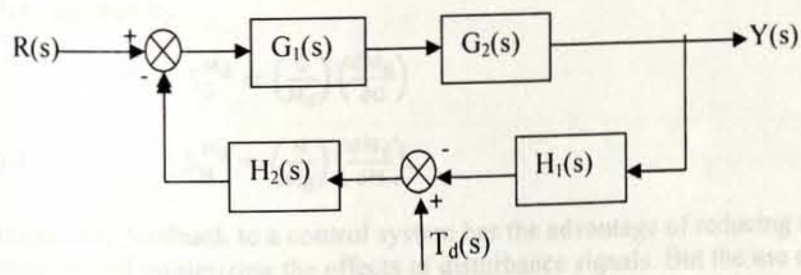


Fig.3.6 A system with disturbance signal in the feedback path

If $G_1(s)G_2(s)H_2(s)H_1(s) \gg 1$ equation 3.14 reduces to

$$\frac{Y_d(s)}{T_d(s)} = \frac{-1}{H_1(s)}, \qquad (3.15)$$

or

$$Y_d(s) = \frac{-T_d(s)}{H_1(s)} . \qquad (3.16)$$

Thus by making proper value of $H_1(s)$ the effect of disturbance in feedback path $T_d(s)$, is reduced at the output of the system. The sensitivity of the system for disturbances due to feedback path is given by

$$S_{H_1}^{M_d} = \left(\frac{\partial M_d}{\partial H_1}\right)\left(\frac{H_1}{M_d}\right).$$

3.4.3. Disturbance at the output of the system

Disturbance at the output of the system are introduced at the output terminals of the system as in Fig 3.7.

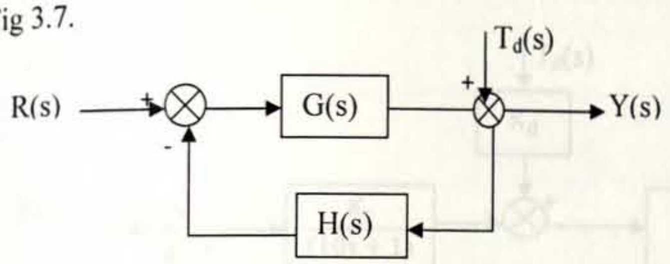


Fig 3.7 A system with disturbance at the output

We have

$$M_d(s) = \frac{Y_d(s)}{T_d(s)} = \frac{1}{1+G(s)H(s)}. \tag{3.17}$$

For $G(s)H(s) \gg 1$, the equation (45) reduces to

$$Y_d(s) = \frac{1}{G(s)H(s)} T_d(s).$$

Therefore the effect of disturbance on the output can be controlled by changing the values of $G(s)$ and $H(s)$. The sensitivity of the system to the disturbance at the output terminal for variations in $G(s)$ and $H(s)$ is given by

$$S_G^{M_d} = \left(\frac{G}{M_d}\right)\left(\frac{\partial M_d}{\partial G}\right) \tag{3.18}$$

and

$$S_H^{M_d} = \left(\frac{H}{M_d}\right)\left(\frac{\partial M_d}{\partial H}\right). \tag{3.19}$$

Introducing feedback to a control system has the advantage of reducing sensitivity, improving transient response and minimizing the effects of disturbance signals. But the use of feedback increases the number of components of the system, hence increasing the complexity of the system. Also it reduces the gain of the system and introduces the possibility of instability.

Example 3.2 :

An automatic speed control of an automobile is described in the block diagram of Fig 3.8.

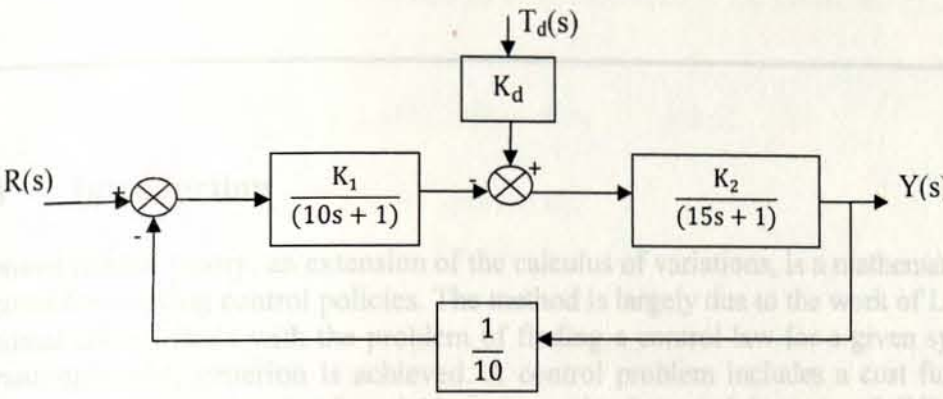


Fig 3.8

To determine the effect of disturbance $T_d(s) = \frac{10}{s}$ on the output, we let $G_1 = \frac{K_2}{(15s+1)}$, $G_2 = \frac{1}{10}$ and $G_3 = \frac{K_1}{(10s+1)}$.

From the block diagram in Fig 3.8 we get

$$\frac{Y_d(s)}{K_d T_d(s)} = \frac{G_1(s)}{1-G_1(s)G_2(s)G_3(s)} = \frac{\frac{K_2}{(15s+1)}}{1-\left(\frac{K_2}{15s+1}\right)\left(\frac{1}{10}\right)\left(\frac{K_1}{10s+1}\right)}.$$

This implies

$$Y_d(s) = \frac{K_d T_d(s) G_1(s)}{1-G_1(s)G_2(s)G_3(s)} = \frac{\frac{10K_d}{s}\frac{K_2}{(15s+1)}}{1-\left(\frac{K_2}{15s+1}\right)\left(\frac{1}{10}\right)\left(\frac{K_1}{10s+1}\right)}.$$

Taking the inverse Laplace Transform, we get

$$y_d(t) = \frac{1}{-10+K_1K_2}(10K_2(P_1P_2P_3P_4)K_d),$$

where $$P_1 = -10 + \frac{1}{\sqrt{5+12K_1K_2}},$$

$$P_2 = e^{\frac{-1}{300}(25+\sqrt{25+60K_1K_2})t},$$

$$P_3 = -5\sqrt{5} - 2\sqrt{5}K_1K_2 + 5\sqrt{5+12K_1K_2} + e^{\frac{1}{30}\sqrt{1+\frac{12K_1K_2}{5}}t},$$

$$P_4 = 2\sqrt{5}K_1K_2 + 5(\sqrt{5} + \sqrt{5+12K_1K_2}).$$

Chapter 4

Feedback in Optimal Control

4.1 Introduction

Optimal control theory, an extension of the calculus of variations, is a mathematical optimization method for deriving control policies. The method is largely due to the work of Lev Pontryagin. Optimal control deals with the problem of finding a control law for a given system such that a certain optimality criterion is achieved. A control problem includes a cost functional that is a function of state and control variables. An optimal control is a set of differential equations describing the paths of the control variables that minimize the cost functional. The optimal control can be derived using Pontryagin's maximum principle (a necessary condition).

We begin with a simple example. Consider a car traveling on a straight line through a hilly road. Then we question how the driver presses the accelerator pedal in order to minimize the total traveling time? Clearly in this example, the term control law refers specifically to the way in which the driver presses the accelerator and shifts the gears. The "system" consists of both the car and the road, and the optimality criterion is the minimization of the total traveling time. Control problems usually include additional constraints. For example the amount of available fuel might be limited; the accelerator pedal cannot be pushed through the floor of the car, speed limits, etc.

A proper cost functional is a mathematical expression giving the traveling time as a function of the speed, geometrical considerations, and initial conditions of the system. It is often the case that the constraints are interchangeable with the cost functional.

4.2. Necessary conditions of Optimality

Theorem 4.1:

Let $a, b, \alpha, \beta \in \mathbb{R}$, $a < b$, and L be continuously partially differentiable with regard to the first two variables. In addition, let

$$f(y) = \int_a^b L(y, \dot{y}, t)dt, \quad \text{and} \quad S = \{y \in C^{(1)}[a, b] | y(a) = \alpha, \ y(b) = \beta\}.$$

If y_0 is a minimum point of f on S, then y_0 satisfies

$$\frac{d}{dt} L_{\dot{y}}(y_0, \dot{y}_0, t) = L_y(y_0, \dot{y}_0, t) \,. \tag{4.1}$$

The equation (4.1) is called Euler-Lagrange Differential Equation (ELDE).

Theorem 4.2:

Let $a, b, \alpha \in \mathbb{R}$, $a < b$ and L be continuous and continuously partially differentiable with regard to the first two components and y_0 be a solution of (P), a variational problem with one free end point.

$$\text{(P)} \qquad f(y) := \int_a^b L(y(t), \dot{y}(t), t)dt \to \min, \qquad y \in R,$$

$$R := \{y \in C^{(1)}[a, b]^n | y(a) = \alpha\}.$$

Then

1. $\frac{d}{dt} L_{\dot{y}}(y_0(t), \dot{y}_0(t), t) = L_y(y_0(t), \dot{y}_0(t), t),$ (ELDE)
2. $L_{\dot{y}}(y_0(b), \dot{y}_0(b), b) = 0.$

The second necessary condition is called transversality condition (TR).

Theorem 4.3:

Let $a, b \in \mathbb{R}$, consider the optimal control problem

$$f(x, u) := \int_a^b k(x(t), u(t), t)dt \to \min, \qquad (x, u) \in R$$

$$R := \{(x, u) \in K \times Q | \dot{x}(t) = \varphi(x(t), u(t), t), \ t \in [a, b]\}$$
$$K \subseteq RCS^{(1)}[a, b]^n,$$
$$Q := \{u \in RS[a, b]^m | u(t) \in U(t), \ t \in [a, b]\}.$$

Let $\lambda \in RS[a, b]^n$, $x \in K$ be fixed and u^* be a minimum point of the function $f_\lambda(x, u)$ on Q,

where

$$f_\lambda(x, u) := f(x, u) + \int_a^b < \lambda(x, \dot{x}, u, t), \dot{x}(t) - \varphi(x(t), u(t), t) > dt$$
$$= \int_a^b k(x(t), u(t), t)dt + \int_a^b < \lambda(x, \dot{x}, u, t), \dot{x}(t) - \varphi(x(t), u(t), t) > dt$$
$$= \int_a^b [< \lambda(x, \dot{x}, u, t), \dot{x}(t) - \varphi(x(t), u(t), t) > + k(x(t), u(t), t)]dt,$$

then

$$L(x, \dot{x}, u^*, \lambda, t) = \min_{u \in Q} L(x, \dot{x}, u, \lambda, t) \qquad \text{for all } t \in [a, b]$$

This condition is called minimum condition (condition MIN).

Theorem 4.4: (Sufficient condition)

Le X be a vector space, $U \subseteq X$ be a convex set and $f: U \to \mathbb{R}$ be a convex function. Then for any algebraically local minimum x_0 of f on U, it follows that x_0 is a minimize (global minimum) of f on U.

General Nonlinear Optimal Control

A more abstract framework of general nonlinear optimal control problem goes as follows. Minimize the cost functional

$$f(x,u) := \Phi(x(a),a,x(b),b) + \int_a^b L(x(t),u(t),t)dt\,, \quad (x,u) \in R \tag{4.2}$$

$$R := \{(x,u) \in K \times Q \mid \dot{x}(t) = \Psi(x(t),u(t),t),\ t \in [a,b]\},$$

$$K := \{x \in RCS^{(1)}[a,b]^n \mid \phi(x(a),a,x(b),b) \le 0\},$$

$$Q := \{u \in RS[a,b]^m \mid u(t) \in U(t) \subseteq \mathbb{R}^m,\ t \in [a,b]\},$$

where $x(t)$ is the state, $u(t)$ is the control, t is the independent variable and a is the initial time, and b is the terminal time. The terms Φ and L are called the endpoint cost and Lagrangian, respectively.

4.3 Quadratic Optimal Control

A special case of the general nonlinear optimal control problem is the quadratic optimal control problem (also called the linear quadratic optimal control). This problem is stated as follows.

Minimize the quadratic cost functional

$$f(x,u) := \frac{1}{2}x^T(b)Sx(b) + s^Tx(b) + \int_a^b \left(\frac{1}{2}x^TCx + c^Tx + \frac{1}{2}u^TDu + d^Tu\right)dt, \quad (x,u) \in R$$

$$\begin{aligned} R &:= \{(x,u) \in K \times Q \mid \dot{x} = Ax + Bu\}, \\ K &:= \{x \in RCS^{(1)}[a,b]^n \mid x(a) = \alpha\}, \\ Q &:= RS[a,b]^m. \end{aligned} \tag{4.3}$$

In equation (48), we have that

$S \in \mathbb{R}^{n\times n}$ be a positive semi definite, symmetrical matrix,
$A \in RS[a,b]^{n\times n}$,
$B \in RS[a,b]^{n\times m}$,
$C \in RS[a,b]^{n\times n}$ be a positive semi definite and symmetrical matrix for all $t \in [a,b]$,
$D \in RS[a,b]^{m\times m}$ be a positive definite and symmetrical matrix for all $t \in [a,b]$,
$s \in \mathbb{R}^n$, $c \in RS[a,b]^n$, $d \in RS[a,b]^m$.

Then we have the Lagrange function

$$L(x,\dot{x},u,\lambda)=\tfrac{1}{2}x^{T}Cx+c^{T}x+\tfrac{1}{2}u^{T}Du+d^{T}u+\lambda^{T}[\dot{x}-Ax-Bu] \tag{4.4}$$

$$=\frac{1}{2}x^{T}Cx+c^{T}x+\lambda^{T}\dot{x}-\lambda^{T}Ax+\frac{1}{2}u^{T}Du+d^{T}u-\lambda^{T}Bu.$$

Let

$$G(x,\dot{x},\lambda):=\tfrac{1}{2}x^{T}Cx+c^{T}x+\lambda^{T}\dot{x}-\lambda^{T}Ax,$$
$$W(u,\lambda):=\tfrac{1}{2}u^{T}Du+d^{T}u-\lambda^{T}Bu.$$

Clearly G is convex with regard to x and $\dot{x}$ and W is convex with regard to u. Hence, as sufficient condition, for (x^{*},u^{*}) to be a solution of the optimal control problem:

i) From the condition MIN, we get

$$\min_{u\in Q}W(u,\lambda)=\min_{u\in Q}\{\tfrac{1}{2}u^{T}Du+d^{T}u-\lambda^{T}Bu\}=\tfrac{1}{2}u^{*T}Du^{*}+d^{T}u^{*}-\lambda^{T}Bu^{*}.$$

But by the convexity and differentiability of W we get (by taking the derivative w.r.t u^{*})

$$Du^{*}+d-B^{T}\lambda=0\,,$$

and since $\det(D)\neq 0$ (D is positive definite) we have

$$u^{*}=D^{-1}B^{T}\lambda-D^{-1}d\,. \tag{4.5}$$

ii) If we substitute in the differential equation $\dot{x}=Ax+Bu$, then

$$\dot{(x^{*})}=Ax^{*}+BD^{-1}B^{T}\lambda-BD^{-1}d\;,\; x^{*}(a)=\alpha\;. \tag{4.6}$$

iii) By ELDE

i.e. $\frac{d}{dt}(L_{\dot{x}})=L_{x},$

we get

$$\dot{\lambda}=-A^{T}\lambda+Cx^{*}+c. \tag{4.7}$$

iv) By the Transversality condition TR we get

$$\lambda(b)=-Sx^{*}(b)-s. \tag{4.8}$$

So we have from equations (4.6) and (4.7) the system of differential equations

$$\begin{pmatrix}\dot{x}\\ \dot{\lambda}\end{pmatrix} = \begin{pmatrix}A & BD^{-1}B^T\\ C & -A^T\end{pmatrix}\begin{pmatrix}x\\ \lambda\end{pmatrix} + \begin{pmatrix}-BD^{-1}d\\ c\end{pmatrix},\ x(a) = \alpha,\ \lambda(b) = -Sx^*(b) - s.$$

Particular Case: $s = 0,\ c = 0$ and $d = 0$, in equation (4.3) the cost functional becomes

$$f(x,u) = \frac{1}{2}x^T(b)Sx(b) + \frac{1}{2}\int_a^b (x^TCx + u^TDu)\,dt \to \min, \qquad (x,u) \in R,$$

then by the transformation

$$\lambda(t) = -P(t)x(t), \qquad P \in C^{(1)}[a,b]^{n\times n}, \tag{4.9}$$

we get by substituting in differential equation (4.7):

$$\dot{\lambda} = (A^TP + C)x. \tag{4.10}$$

By the product rule we get from equation (4.9)

$$\dot{\lambda} = -\dot{P}x - P\dot{x}$$

or

$$\dot{\lambda} + \dot{P}x + P\dot{x} = 0. \tag{4.11}$$

In equation (4.11), we substitute equations (4.6) and (4.10). Then we get

$$\left(A^TP + C + \dot{P} + PA - PBD^{-1}B^TP\right)x = 0. \tag{4.12}$$

Clearly, $x = 0$ is a trivial solution. Also equation (.12) is satisfied if

$$\dot{P} = P(BD^{-1}B^T)P - A^TP - PA - C\,. \tag{4.13}$$

Equation (4.13) is known as a Riccati differential equation.

Moreover, if we choose

$$P(b) = S, \tag{4.14}$$

then the transversality condition in equation (4.8) is satisfied and the system of differential equations (4.13) and (4.14) has a uniquely determined solution P^*.

If we have P^*, then we can calculate x^* as a solution of the system of differential equations (4.6) and (4.9).

$$\dot{x} = [A - (BD^{-1}B^T)P^*]x, \quad x(a) = \alpha.$$

Then we can calculate λ^* by

$$\lambda^* = -P^*x^*, \quad \lambda^*(b) = -Sx^*(b).$$

Finally, we can calculate u^* by equation (4.5):

$$u^* = -D^{-1}B^TP^*x^*. \tag{4.15}$$

So, we have that (x^*, u^*) is a solution of our optimal control problem.

It is especially interesting to note that in equation (4.15) the optimal control input is described in the form of a feedback by a linear function of the state variable x^*.

The matrix $K = -D^{-1}B^TP^*$ is called feedback matrix.

Example 4.1:

In order to solve the quadratic optimal control be given by

$$f(x,u) = x^2(1) + \int_0^1 \left(x^2(t)+\frac{1}{2}u^T(t)u\right)dt \to \min, \quad (x,u) \in R$$

$$R := \{(x,u) \in K \times Q \mid \dot{x}(t) = \frac{-3}{2}x(t) + u(t), \quad t \in [0,1]\},$$
$$K := \{x \in RCS^{(1)}[0,1] | x(0) = 3\}$$
$$Q := RS[0,1]$$

We have $S = 2, s = 0, \ c = 0, \ d = 0, C = 2, D = -1, A = \frac{-3}{2}, \ B = 1, n = 1, m = 1.$

By the transformation

$$\lambda(t) = -P(t)x(t), \quad P \in C^{(1)}[0,1],$$

we have the Ricatti differential equation,

$$\dot{P} = P(BD^{-1}B^T)P - A^TP - PA - C,$$

That is

$$\dot{P} = P(1.(-1)^{-1}1^T)P - \left(\frac{-3}{2}\right)^T P - \left(\frac{-3}{2}\right)P - 2\,,$$

or $$\dot{P} = -P^2 + 3P - 2 = -P^2 + 3P - 2.$$

When we solve the Riccati differential equation we get

$$p(t) = \frac{2e^{t}-c_1}{e^{t}-c_1},$$

and if $\quad P(b) = S$ i.e. $P(1) = 2$, then $c_1 = 0$. That is $p^*(t) = 2$.

To find $x(t)$, we have $\dot{x} = [A - (BD^{-1}B^T)P^*]x$ and $x(a) = \alpha$.

This is the same as

$$\dot{x} = [\tfrac{-3}{2} - (1(-1)^{-1}1^T)2]x \quad \text{and} \quad x(0) = 1,$$

or

$$\dot{x} = \tfrac{1}{2}x \quad \text{and} \quad x(0) = 1.$$

Then we get

$$x^*(t) := x(t) = e^{\frac{1}{2}t-1}.$$

To find $\lambda(t)$ we use the equation $\lambda(t) = -P^*(t)x^*(t)$.

Hence, $\lambda(t) = -2e^{\frac{1}{2}t-1}$.

Finally to find $u(t)$, we use the equation

$$u^*(t) = -D^{-1}B^TP^*(t)x^*(t).$$

Thus

$$u^* = -(-1)^{-1}1^T(2)e^{\frac{1}{2}t-1} = 2e^{\frac{1}{2}t-1}.$$

Therefore, the solution of the quadratic optimal control problem is (x^*, u^*).

Remove["Global`*"];

<u>Mathematica Program for the paper</u>

```
(*Input*)

a0 = 1; b0 = 1;
coleft = {1, 3, 2}; (*coefficients of left side of differential equation*)
coright = {1, 3}; (*coefficients of right side of differential eqution*)
(*u[t_]:=t^2;*)    (*it is better to let u in a general form and define u later in the program*)

(*------------------------------------------------------------*)
(*initialisations*)
n = Length[coleft];
m = Length[coright];
Table[a_i = coleft[[i]], {i, 1, n}];
Table[b_i = coright[[i]], {i, 1, m}];

f[y_] := Sum[a_i * D[y[t], {t, n - i}], {i, 1, n}];

g[u_] := Sum[b_i * D[u[t], {t, m - i}], {i, 1, m}];

Print["The differential equation is given by:"];
Print["          ", f[y], " = ", g[u]];

(*------------------------------------------------------------*)
(* initial conditions *)
iniy = Table[(D[y[t], {t, i}] /. {t -> 0}) -> 0, {i, 0, n - 1}];

iniu = Table[(D[u[t], {t, i}] /. {t -> 0}) -> 0, {i, 0, m - 1}];

Iniy = Join[{LaplaceTransform[y[t], t, s] -> Y[s]}, iniy];
Iniu = Join[{LaplaceTransform[u[t], t, s] -> U[s]}, iniu];
(*------------------------------------------------------------*)
```

```
Remove["Global`*"];

(*Input*)

a0 = 1; b0 = 1;
coleft = {1, 2, 2};  (*coeffincentes of left side of differential equation*)
coright = {1, 3}; (*coefficicients of right side of differential eqution*)
(*u[t_]=t^3;*)    (*it is better to let u in a general form and define u later in the prgram*)

(*-----------------------------------------------------*)
(*initialization*)
n = Length[coleft];
m = Length[coright];
Table[ai = coleft[[i]], {i, 1, n}];
Table[bi = coright[[i]], {i, 1, m}];
f[y_] = Sum[ai *D[y[t], {t, n-i}], {i, 0, n}];
g[u_] = Sum[bi *D[u[t], {t, m-i}], {i, 0, m}];
Print["The differential equation is given by:"];
Print["       ", f[y], " = ", g[u]];

(*-----------------------------------------------------*)
(* initial conditions *)
iniy = Table[(D[y[t], {t, i}] /. {t→0}) → 0, {i, 0, n-1}];

iniu = Table[(D[u[t], {t, i}] /. {t→0}) → 0, {i, 0, m-1}];

inily = Join[{LaplaceTransform[y[t], t, s] → yY[s]}, iniy];
inilu = Join[{LaplaceTransform[u[t], t, s] → uU[s]}, iniu];
(*-----------------------------------------------------*)
```

```
(*calculation of Laplace Transform of the differential equation*)
lefts = LaplaceTransform[Sum[a_i * D[y[t], {t, n - i}], {i, 0, n}], t, s] /. inily // Simplify;

rights = LaplaceTransform[Sum[b_i * D[u[t], {t, m - i}], {i, 0, m}], t, s] /. inilu // Simplify;
Print["The Lagrange Transform of the differential equation is given by:"];
Print[""];
Print["        ", lefts, " = ", rights];
Print[""];

(*-----------------------------------------------------*)
(*calculation of transfer function G (s) *)
denomi = Coefficient[lefts, yY[s]];
numer = Coefficient[rights, uU[s]];
gG[s] = numer / denomi;
Print["The transfer function is given by:"]
Print[""];
Print["     G(s) = ", gG[s]];
Print["The characteristic polynomial is given by:"]
Print["        ", denomi];
(*-----------------------------------------------------*)
(* roots of the denominator *)
ro = Solve[denomi == 0, s];
ro = s /. ro;
Print["The roots of the characteristic polynomial are;"];
Table[ Print["p(", i, ") = ", ro[[i]]], {i, 1, Length[ro]}];
re = Table[Re[ro[[i]]], {i, 1, Length[ro]}];
If[And[Table[re[[i]] < 0, {i, 1, Length[re]}]] == {True},
  Print["The System is asymptotical stable."]];
If[And[Table[re[[i]] > 0, {i, 1, Length[re]}]] == {True}, Print["The System is unstable."]];

(*-----------------------------------*)
(*Solution of differential Equation*)
(*Definition and u *)
u[t_] = e^-t;
Print["The function u is given by:"];
Print["          u(t) = ", u[t]];
```

```
(*---------------------------------------*)
ini = Table[(D[y[t], {t, i}] /. {t → 0}) == 0, {i, 0, Length[coleft] - 1}];
ini1 = Join[{f[y] == g[u]}, ini];
Print["The solution of the initial value problem:"];
Print["         ", f[y], " = ", g[u]];
Print["          ", Table[ini[[j]], {j, 1, Length[ini]}]];
DSolve[ini1, y[t], t];
y[t_] = y[t] /. %[[1]];
Print["is given by:"];
Print["    y(t) = ", y[t]];
Plot[y[t], {t, 0, 10}];
(*----------------------------------------------------------*)
<< LinearAlgebra`MatrixManipulation`;
id = IdentityMatrix[n - 1];
o = Table[{0}, {i, 1, n - 1}];
α = -Reverse[coleft];
β = AppendRows[o, id];
aA = AppendColumns[β, {α}];
bB = Append[o, {1}];
xx = Table[{x_i}, {i, 1, n}];
xxx = Table[{ẋ_i}, {i, 1, n}];
y = Flatten[Prepend[o, 1]].xx;

Print["If g(u) = u, then the state space Form is given by:"];
Print[xxx // MatrixForm, " = ", aA // MatrixForm, xx // MatrixForm, " + ", bB // MatrixForm, "u"];
Print["y = ", Flatten[Prepend[o, 1]], xx // MatrixForm];
(*-----------------------------------------------------*)
(*calculation of diagonal form*)
ev = Eigenvalues[aA];
Print["The eigenvalues are given by:"];
Print["      ", ev];
ei = Transpose[Eigenvectors[aA]];
ei // MatrixForm;
iei = Inverse[ei].aA.ei // FullSimplify;
% // MatrixForm;
zz = Table[{z_i}, {i, 1, n}];
zzz = Table[{ż_i}, {i, 1, n}];
y = Flatten[Prepend[o, 1]].xx;
Print["The diagonal form is given by:"];
Print[zzz // MatrixForm, " = ", iei // MatrixForm, zz // MatrixForm, " + ",
  iei.bB // MatrixForm, "u"];
Print["y = ", Flatten[Prepend[o, 1]].iei, zz // MatrixForm];
```

```
(*SENSITIVITY OF A NEGATIVE FEED BACK SYSTEM*)

Remove["Global`*"];
gG[S_]=s^3;(*TRANSFER FUNCTION ON THE FORWARD PATH*)
hH[s_]=1/s^2;(*TRANSFER FUNCTION ON THE FEEDBACK PATH*)
tTc[s_]=gG[s]/(1+gG[s]*hH[s]);(*Transfer function of a closed
loop system*)
tTo[s_]=gG[s];(*Transfer function of an open loop system*)

(*______________________________________________________________*)
(*The loop gain of the closed system*)
 loopG[s_]=gG[s]*hH[s];

 (*_____________________________________________________________*)
 (*Sensitivity of the closed loop system to variations on G(s)*)
 sSensicG=(gG[s]/tTc[s])*((D[tTc[s],s])/(D[gG[s],s]));
(*__________________________________________________________*)
(*Sensitivity of the closed loop system to variations on H(s)*)
sSensicH=(hH[s]/tTc[s])*((D[tTc[s],s])/(D[hH[s],s]));
(*_____________________________________________________________*)
(*Sensitivity of Open loop *)
sSensiO=(gG[s]/tTo[s])*((D[tTo[s],s])/(D[gG[s],s]));
(*__________________________________________________________*)

Print["THe loop gain of the closed system is
" , loopG[s]];
Print["The sensitivity of the system to changes in G(s) is
" , Apart[sSensicG]];
Print["The sensitivity of the system to changes in H(s) is
" , Apart[sSensicH]];
Print["The sensitivity of the open loop system is
", Apart[sSensiO]];
```

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