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ON BESSEL FUNCTIONS AND THE MODIFIED BESSEL FUNCTIONS AND ITS PROPERTIES

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Abstract

In this paper we study Bessel differential equation of the form

$$x^2y'' + xy' + (x^2 - v^2)y = 0,$$

and the modified Bessel equation of the form

$$x^2y'' + xy' - (x^2 + v^2)y = 0$$

Along with the corresponding Bessel functions

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+n}}{2^{2m+n} m!\,(n+m)!}$$

and the modified Bessel function

$$I_v(x) = \sum_{m=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2m+v}}{m!\,\Gamma(m+v+1)}$$

and the pertinent properties.

Introduction

A differential equation of the form

$$y'' + \frac{1}{x}y' + \left(1 - \frac{v^2}{x^2}\right)y = 0$$

where$v$ is arbitrary real or complex number is called a ***Bessel equation*** and its solution is known as Besselfunction.

Bessel equation arises in problems involving vibrations, or heat conduction in regions possessing circular symmetry. Therefore Bessel functions have many applications in physics and engineering in connection with the propagation of wave, elasticity, fluid motion and especially in many problems of potential theory and diffusion involving cylindrical symmetry.

This paper consists of three chapters, each of one divided into several section. In the first chapter we present an introduction to special function of second order (Bessel) homogenous equation, basic definition and concepts, variable coefficients second order linear ordinary differential equations.

In the second chapter we give a solution of Bessel equations including Bessel function of first kind, Bessel function the second kind and modified Bessel function of the first and second kind.

In the third chapter we includes some properties of Bessel functions and their proofs, Such as generating function, recurrence formulas for Bessel polynomials, orthogonality and integral representation of Bessel function.

CHAPTER-ONE

1. DEFINITIONS AND PRELIMINARY CONCEPTS

1.1 Power series

The ***power series method*** is the standard method for solving linear ODE_s with ***variable*** coefficients. It gives solutions in the form of power series. These series can be used for computing values, graphing curves , proving formulas, and exploring properties of solutions, as we shall see. In this section we begin by explaining the idea of the power series method.

A power series is a series of functions in powers of $(x - x_0)$ is an infinite series of the form $\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots$ (1.1)

where x is variable, $a_0, a_1, a_2, \ldots$ are constants, called the ***coefficients*** of the series. x_0 is a constant, called the ***center*** of the series. In particular, if $x_0 = 0$, we obtain a power series in powers of x.
$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$ (1.2)

We assume that all variables and constants are real. We note that the term "power series" usually refers to a series of the form (1.1) (or1.2) but ***does not include*** series of negative or fractional powers of x.

1.1.1.Intervals of convergence

One of the most useful tests for absolute converges of a power Series is the ratio test. If $a_n \neq 0$, and if for a fixed value of x, $\lim_{n\to\infty} \left|\frac{a_{n+1}(x-x_0)^{n+1}}{a_n(x-x_0)^n}\right| = |x - x_0| \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$
$= |x - x_0|L$, where $L=\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$.Therefore ,we see that the power series converges absolutely for $|x - x_0| < \frac{1}{L} = R$ and diverges for $|x - x_0| > \frac{1}{L} = R$. The range of x for which the series converges, $x_o - R < x < x_0 + R$ is called the ***interval of convergence*** of the series and R is called its ***radius of convergence***.

Theorem*:* For a given power series $\sum_{n=0}^{\infty} c_n (x - x_0)^n$ there are only three possibilities.

(i) The series Converge only for the single value $x = x_0$
(ii) The series Converge if $|x - x_0| < R$ and diverges if $|x - x_0| > R$.
(iii) The series Converge absolutely for all values of x, that is for $-\infty < x < \infty$.

We give below examples of each three types of series. For convenience we have taken $x_0 = 0$.

Example1.1 Determine the radius of convergence and interval of convergence for the following power series:

a)$\sum_{n=0}^{\infty} n!\,(2x+1)^n$

Solution: We'll start this example with the ratio test

L$=\lim_{n\to\infty}\left|\frac{(n+1)!(2x+1)^{n+1}}{n!(2x+1)^n}\right|$

$$=|(2x+1)|\lim_{n\to\infty}(n+1)$$

The limit is infinite, we have L=$\infty > 1.$ provided x$\neq -\frac{1}{2}$.

So, this power series will only converge if x=$-\frac{1}{2}$.

b)$\sum_{n=1}^{\infty}\frac{(-1)^n\,n}{4^n}(x+3)^n$

Solution: We know that this power series converge for $x=-3$,

L=$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)(x+3)^{n+1}}{4^{n+1}}\frac{4^n}{(-1)^n(n)(x+3)^n}\right|$

$$=\lim_{n\to\infty}\left|\frac{-(n+1)(x+3)}{4n}\right|$$

$=|x+3|\lim_{n\to\infty}\frac{n+1}{4n}$,

$\frac{1}{4}|x+3|<1 \Rightarrow |x+3|<4\,, series\ converges$

$\frac{1}{4}|x+3|>1 \Rightarrow |x+3|>4\,, series\ diverges$

The radius of converges for this power series is R=4.Thus, the interval of convergence is -7$< x < 1$

c)$\sum_{n=0}^{\infty}\frac{(x-6)^n}{n^n}$

Solution: In this example the root test seems more appropriate. So,

$$\text{L}=\lim_{n\to\infty}\left|\frac{(x-6)^n}{n^n}\right|^{\frac{1}{n}} =\lim_{n\to\infty}\left|\frac{(x-6)}{n}\right| =|x-6|\lim_{n\to\infty}\frac{1}{n} =0.$$

So, since L=0$<$ 1 regardless of the value of x this power series will converge for every x. In this case we say that the radius of convergence isR=and interval of convergence is -$\infty < x < \infty$.

1.2.SECOND- ORDER EQUATION

A second-order linear differential equation

$$b_2(x)y'' + b_1(x)y' + b_0(x)y = g(x) \quad (1.3)$$

Has a variable coefficients when $b_2(x), b_1(x), and b_0(x)$ are *not* all constants or constant multiples of one another.

If $b_2(x)$ is not zero in a given interval, then we can divide by it and rewrite equation(1.3) as

$$y'' + p(x)y' + Q(x)y = \emptyset(x) \quad (1.4)$$

Where$p(x) = \frac{b_1(x)}{b_2(x)}$, $Q(x) = \frac{b_0(x)}{b_2(x)}$,and$\emptyset(x) = \frac{g(x)}{b_2(x)}$

1,2.1. ANALYTIC FUNCTIONS AND ORDINARY POINTS

A function f(x) is ***analytic*** at $x_o$if its Taylor series about x_o,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)(x - x_0)^n}{n!}$$

Converges to $f(x)$ in some neighborhood of x_0.

Polynomials, sin$x$, cos$x$, ande^x are analytic everywhere; so too are sums, difference, and product of these functions, quotients and any two of these functions are analytic at all points where the denominator is not zero.

The point $x_o$is an***ordinary point***of the differential equation (1.4) if both P(x) and Q(x) are analytic at x_o.If either of these functions is not analytic at x_o, then $x_o$is a singular point of (1.4).

1.3.SOLUTIONSAROUND THE ORIGIN OF HOMOGENEOUS EQUATIONS.

Equations (1.3) is homogenous when g(x)=0,in which case equation (1.4) specializes to

$$y'' + p(x)y' + Q(x)y = 0 \quad (1.5)$$

Theorem 1.1 If x=0 is an ordinary point of equation (1.5), then the general solution in an interval containing this point has the form

$$y=\sum_{n=0}^{\infty} a_n x^n = a_0 y_1(x) + a_1 y_2(x) \quad (1.6)$$

Where a_0 and a_1 are arbitrary constants and$y_1(x)and\ y_2(x)$ are linearly independent functions analytic at x=0

To evaluate the coefficients a_n in the solution furnished by Theorem (1.1) , use the following five-stepProcedure known as the power series method.

Step -1 Substitute into the left side of the homogenous differential equation the power series

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + a_{n+1}x^{n+1} + a_{n+2}x^{n+2} + \cdots \quad (1.7)$$

Together with the power series for

$$y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots + na_nx^{n-1} + (n+1)a_{n+1}x^n + (n+2)a_{n+2}x^{n+1} + \cdots \quad (1.8)$$

and

$$y'' = 2a_2 + 6a_3x + 12a_4x^2 + \cdots n(n-1)a_nx^{n-2} + (n+1)(n)a_{n+1}x^{n-1} + (n+2)(n+1)a_{n+2}x^n + \cdots \quad (1.9)$$

Step-2 collect powers of x and set the coefficients of each powers of x equal to zero.

Step-3 The equation obtained by setting the coefficients of x^n to zero in step 2 well contain a_j terms for a finite number of j values. Solve this equation for the $a_j$terms having the largest subscript. The resulting equation is known as the ***recurrence formula*** for the given differential equation.

Step- 4 Use the recurrence formula to sequentially determine a_j (j=2,3,4…) interims of $a_{0\ and}\ a_1$.

Step-5 Substitute the coefficients determined in step- 4 in to equation (1.7) and rewrite the solution in the form of equation(1.6).

The power series method is only applicable when x=0is an ordinary point. Although a differential equation must be in the form of equation (1.4)to determine whether x=0 is an ordinary point, once this condition is verified , the power series method can be used on either form (1.3)or(1.4). If p(x) and Q(x) in (1.4) are quotients of polynomials, it is often simpler first to multiply through by the lowest common denominator, there by clearing fractions, and then to apply the power series method to the resulting equation in the form of Equation (1.3).

1.4.REGULAR SINGULARPOINTS

Definition1.1If$x = x_o$is a singularity of (1.5) and if the multiplication of p(x) by $(x - x_o)$ and Q(x) by $(x - x_0)^2$ result in functions,each of which is analytic at $x = x_o$,then the point $x = x_o$ is called a ***regular singularity*** of (1.5).

Definition*1.2** If$x = x_o$ is a singularity of (1.5) and if the multiplication of P(x) by (x$-x_o$) and Q(x) by $(x - x_0)^2$ result in functions one or both of which are not analytic at $x = x_o$, then the point $x = x_o$ is called an ***irregular singular of (1.5).

Definition1.3: if x_0 is a regular singular point of (1.5),the indicial equation for this point is

$$r(r-1) + p_0 r + q_0 = 0$$

where$p_0 = \lim_{x \to x_0}(x - x_0)p(x)$, $q_0 = \lim_{x \to x_0}(x - x_0)^2 q(x)$

The roots of the indicial equation are called the exponents (indices) of the singularity x_0.

Example1.4:Determine whether x=o, x=1, x=2 are ordinary point or regular singular point or irregular singular point of the differential equation.

$$y'' + \frac{1}{x(x-1)^2}y' + \frac{x+1}{x(x-1)^3}y = 0.$$

Solution; Here P(x)= $\frac{1}{x(x-1)^2}$ $and\ Q(x) = \frac{x+1}{x(x-1)^3}$

P(x) and Q(x) are analytic everywhere except where x=0, 1. Hence the only singular points are 0 and 1. Therefore, 2is an ordinary point.

a) Consider the singular point x=0,the function x P(x)= $\frac{1}{(x-1)^2}$ and $x^2 Q(x) = \frac{x(x+1)}{(X-1)^3}$ are analytic at x=0 $\Rightarrow$ x=0 is a regular singular point of the equation.
b) Consider singular point x=1,the function (x-1) P(x) = (x-1)$\frac{1}{x(x-1)^2} = \frac{1}{x(x-1)}$ is not analytic at x=1 so, that x=1 is an irregular singular point of the equation.

1.4.1. Method of Frobeninus

Theorem 1.2. If $x_0 = 0$ is a regular singular point of (1.5), then the equation has at least one

solution of the form

$$y = x^{\lambda} \sum_{n=0}^{\infty} a_n x^n. \qquad (1.10)$$

Where λ and a_n $(n = 0, 1, 2, \ldots)$ are constants. This solution is valid in an interval $0 < x < R$ for some real number R.

To evaluate the coefficients a_n and λ in Theorem (1.2**),** one proceeds asin the power series method describedabove. The infinite series

$$y = x^{\lambda} \sum_{n=0}^{\infty} a_0 x^n = \sum_{n=0}^{\infty} a_n x^{n+\lambda}.$$

$$= a_0 x^{\lambda} + a_1 x^{\lambda+1} + a_2 x^{\lambda+2} + \cdots + a_{n-1} x^{\lambda+n-1} + a_n x^{\lambda+n} + a_{n+1} x^{\lambda+n+1} + \cdots$$

With its derivatives

$$y' = \lambda a_0 x^{\lambda-1} + (\lambda+1)\, a_1 x^{\lambda} + (\lambda+2)\, a_2 x^{\lambda+1} + \cdots + (\lambda+n-1)\, a_{n-1} x^{\lambda+n-2} + (\lambda+n)\, a_n x^{\lambda+n-1} + (\lambda+n+1)\, a_{n+1} x^{\lambda+n} + \cdots$$

and

$$y'' = \lambda(\lambda-1) a_0 x^{\lambda-2} + (\lambda+1)(\lambda)\, a_1 x^{\lambda-1} + (\lambda+2)(\lambda+1) a_2 x^{\lambda} + \cdots + (\lambda+n-1)(\lambda+n-2)\, a_{n-1} x^{\lambda+n-3} + (\lambda+n)(\lambda+n-1) a_n x^{\lambda+n-2} + (\lambda+n+1)(\lambda+n) a_{n+1} x^{\lambda+n-1} + \cdots$$

are substituted in to equation(1.5).Terms with like powers of x are collected together and equal to zero. When this is done for x^n the resulting equation is recurrence formula. A quadratic equation in λ, called the indicial equation, arises when the coefficient of x^0 is set to zero and $a_0$is left arbitrary.

1.5.General Solution with Frobenius Method

The method for obtaining this second solution depends on the relationship between the two roots of the indicial equation.

Theorem1.3:Letx_0 be a regular singular point for $y'' + py' + qy = 0$ and let r_1 and $r_2$be the roots of the associated indicial equations , where $r_1 > r_2$

i. If $r_1 - r_2$ is not an integer**,** then thereexist two linearly independent solutions of the form

   a) $y_1(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+r_1}, \qquad a_0 \neq 0.$

   b) $y_2(x) = \sum_{n=0}^{\infty} b_n (x - x_0)^{n+r_2}, \qquad b_0 \neq 0.$

ii. If $r_1 = r_2$, there exist two linearly independent solution of the form (1.11)

   a) $y_1(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+r_1}, \qquad a_0 \neq 0$

   b) $y_2(x) = y_1(x) \ln(x - x_0) + \sum_{n=0}^{\infty} b_n (x - x_0)^{n+r_2}, \qquad b_0 \neq 0.$

iii. If $r_1 - r_2$is a positive integer, then there exist two linearly independent solutions of the form

   a) $y_1(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+r_1}, \qquad a_0 \neq 0.$

   b) $y_2(x) = C_1 y_1(x) \ln(x - x_0) + \sum_{n=0}^{\infty} b_n (x - x_0)^{n+r_2}, \qquad b_0 \neq 0.$

1.6. The Gamma Function

The gamma function which is used in series expansion of the Bessel function, Before we use the method of Frobenious to construct the solutions of Bessel's equations, it will be useful for us to make a couple of definitions.

Definition1.3. For each $x > 0$, the Gamma function is defined by:

$$\Gamma(x) = \int_0^\infty t^{x-1}\, e^{-t}\, dt \tag{1.13}$$

Theorem1.4. The Gamma function has the following proof:

$$\Gamma(1) = 1$$

$\Gamma(x+1) = x\,\Gamma\,(x)$ (1.14)

$\Gamma(n+1) = n!$, for $\forall\, n \in \mathbb{Z}^+$

To prove $\Gamma(x+1) = x\,\Gamma\,(x)$ we use integration by parts as follows:

$$\Gamma(x+1) = \int_0^\infty t^{x+1-1}\, e^{-t}\, dt = \lim_{r\to\infty} \int_0^r t^x\, e^{-t}\, dt$$

$$= \lim_{r\to\infty}[-t^x e^{-t}|_0^r + \ x\int_0^r t^{x-1}\, e^{-t}\, dt]$$

$$= \lim_{r\to\infty}(-r^x e^{-r} + 0) + x\int_0^\infty t^{x-1}\, e^{-t}\, dt$$

$$= 0 + x\int_0^\infty t^{x-1}\, e^{-t}\, dt$$

$$= x\,\Gamma\,(x)$$

The result $\lim_{r\to\infty}\ r^x\, e^{-r} = 0$ is easily obtained by first writing $r^x e^{-r}$as $\frac{r^x}{e^r}$ and then using L'HOSPITA'S rule.

$\Gamma(1) = 1$ (1.15)

To prove$\Gamma(1) = 1$ as follows:-

$$\Gamma(1) = \int_0^\infty t^{1-1}\, e^{-t}\, dt = \lim_{r\to\infty} \int_0^r e^{-t}\, dt = \lim_{r\to\infty} \quad -e^{-t}|_0^r$$

$$= \lim_{r\to\infty}(-e^{-r} + 1) = \lim_{r\to\infty} \quad \left(\frac{-1}{e^r} + 1\right) = 0 + 1 = 1$$

Now we compute some values of the gamma function.

$$\Gamma(1) = 1$$

By using the fundamental property of Γ, we get easily its values at the positive integers.

$$\Gamma(2) = \Gamma(1+1) = 1\Gamma(1) = 1 \times 1 = 1 = 1!$$

$$\Gamma(3) = \Gamma(2+1) = 2\Gamma(2) = 2 \times 1 = 2!$$

$\Gamma(4) = \Gamma(3+1) = 3\Gamma(3) = 3 \times 2 \times 1 = 3!$, …etc

$\Gamma(n+1) = n!$, for $\forall\, n \in \mathbb{Z}^+$.(1.16) (1.16)

Others values of the gamma function can be found with various degrees of difficulty. From the value

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \qquad (1.17)$$

To prove this, first we consider the equation (1.13)

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty q^{-\frac{1}{2}} e^{-q}\, dq$$

If we introduce the new variable

$Q = \sqrt{q}$ so that $dQ = \frac{1}{2} q^{-\frac{1}{2}} dq$, this integral becomes

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-Q^2}\, dQ$$

We can also write this integral in terms of another new variable, $Q = \tilde{Q}$, to obtain

$$\left\{\Gamma\left(\frac{1}{2}\right)\right\}^2 = \left(2\int_0^\infty e^{-Q^2}\, dQ\right)\left(2\int_0^\infty e^{-\tilde{Q}^2}\, d\tilde{Q}\right)$$

since the limits are independent, we can combine the integrals as

$$\left\{\Gamma\left(\frac{1}{2}\right)\right\}^2 = 4\int_0^\infty \int_0^\infty e^{-(Q^2+\tilde{Q}^2)}\, dQ d\tilde{Q}$$

If we now change to standard polar coordinates we have$dQd\tilde{Q} = rdrd\theta, where$
$Q = r\cos\theta \; and \; \tilde{Q} = \mathrm{r}\sin\theta$, and hence

$$\left\{\Gamma\left(\frac{1}{2}\right)\right\}^2 = 4\int_{Q=0}^{\frac{\pi}{2}} \int_{r=0}^\infty e^{-r^2}\, rdrd\theta$$

The limits of integration give us the positive quadrant of the $(Q, \tilde{Q})$ −plane, as required. Performing the integration over Q we have

$\left\{\Gamma\left(\frac{1}{2}\right)\right\}^2 = 2\pi\int_0^\infty re^{-r^2}\, dr,$

and integrating with respect to r gives

$$\left\{\Gamma\left(\frac{1}{2}\right)\right\}^2 = 2\pi\left[\frac{-1}{2}e^{-r^2}\right]_0^\infty = 2\pi\left(\frac{1}{2}\right) = \pi$$

Finally, we have $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ and the basic property we find.

$$\Gamma\left(\frac{3}{2}\right) = \left(\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\times\sqrt{\pi} = \frac{\sqrt{\pi}}{2}$$

$$\Gamma\left(\frac{5}{2}\right) = \left(\frac{3}{2}+1\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}\times\frac{\sqrt{\pi}}{2} = \frac{3}{4}\sqrt{\pi}.$$

Although we have defined the gamma function for $x > 0$, it is possible to extend its definition to all real numbers other than $0, -1, -2, -3, \ldots$ in such a way that the basic property continues to hold. To do so, we write the basic property as

$$\Gamma(x) = \frac{1}{x}\Gamma(x+1)(1.18)$$

and then define the value of the gamma function at x from its value at x +1.

For example, we have

$$\Gamma\left(\frac{-1}{2}\right) = -2\Gamma\left(\frac{1}{2}\right) = -2\sqrt{\pi} \; and \; \Gamma\left(\frac{-3}{2}\right) = \frac{-2}{3}\Gamma\left(\frac{-1}{2}\right) = \frac{4}{3}\sqrt{\pi}$$

In general using the gamma function, we shall simplify the form of the solution of Bessel equation.

CHAPTER TWO

2. BESSEL FUNCTIONS AND MODIFIED BESSEL FUNCTIONS

Bessel functions is named after the German Mathematician and astronomer Friedrch Bessel, who first used them to analyze planetary orbits, Bessel functions occur in many other physical problems, usually in a cylindrical coordinates.

The equation

$$x^2y'' + xy' + (x^2 - v^2)y = 0 \qquad (2.1)$$

Where v is a none negative constant, is called the Bessel equation of order v.

2.1. Solution of Bessel's equation

The general solution of equation (2.1) in case where v is any real number is given by

$$y = C_1J_V(x) + C_2Y_V(x) \qquad (2.2)$$

where $J_v(x)$ is the Bessel polynomial of order v (also known as the Bessel function of the first kind) and $Y_v(x)$ is the Bessel function of second kind.

The point$x_o = 0$is a regular singular pot. We shall use the method of Frobenius to solve this equation.

Thus, we seek solutions of the form

$$y(x) = x^r \sum_{m=0}^{\infty} a_m x^m = \sum_{m=0}^{\infty} a_m x^{m+r} \text{ , } x > 0 \qquad (2.3)$$

With $a_m \neq 0$.

Differentiation of (2.3) term by term yields

$$y'(x) = \sum_{m=0}^{\infty}(m + r)a_m x^{m+r-1}$$

Similarly, we obtain

$$y''(x) = \sum_{m=0}^{\infty}(m + r)(m + r - 1)a_m x^{m+r-2}$$

Now substituting the power series expansion for $y(x),\ y'(x) and\ y''\ (x)$in the Bessel's differential equation (2.1) we get

$$x^2 \sum_{m=0}^{\infty}(m + r)(m + r - 1)a_m x^{m+r-2} + x \sum_{m=0}^{\infty}(m + r)a_m x^{m+r-1} + (x^2 - v^2) \sum_{m=0}^{\infty} a_m x^{m+r} = 0$$

$\Rightarrow \sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r} + \sum_{m=0}^{\infty}(m+r)a_m x^{m+r} + \sum_{m=0}^{\infty} a_m x^{m+r+2} -$

$\sum_{m=0}^{\infty} v^2 a_m x^{m+r} = 0$ (2.4)

$\Rightarrow \sum_{m=0}^{\infty}[(m+r)^2 - v^2]a_m x^{m+r} + \sum_{m=0}^{\infty} a_m x^{m+r+2} = 0$

$\Rightarrow \sum_{m=0}^{\infty}[(m+r)^2 - v^2]a_m x^{m+r} + \sum_{m=2}^{\infty} a_{m-2} x^{m+r} = 0$ (2.5)

Let $m = 0 \; and \; m = 1$

$$(r^2 - v^2)a_0 x^r + [(1+r)^2 - v^2]a_1 x^{1+r} + \sum_{m=2}^{\infty}[(m+r)^2 - v^2)a_m + a_{m-2}]x^{m+r} = 0$$

a) $(r^2 - v^2)a_0 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. (m = 0)$

b) $[(1+r)^2 - v^2]a_1 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (m = 1)$ (2.6)

c) $[(m+r)^2 - v^2]a_m + a_{m-2} = 0 \ldots\ldots\ldots\ldots\ldots (m \geq 2)$

From (2.6a) we obtain the ***indicial equation***by

$a_0 \neq 0, (r^2 - v^2) = 0 \Rightarrow (r+v)(r-v) = 0$ (2.7)

The roots are $r_1 = v \; (v \geq 0) \; and \; r_2 = -v$

Let as consider the two cases:

Case 1: $r = v, \; v \geq 0$

For $r = v$ equation (2.6b) reduce to

$(2v + 1)a_1 = 0$

$2v + 1 \neq 0, since \; v \geq 0$

$\Rightarrow a_1 = 0$

Substituting $r = v$ in equation (2.6c) we obtain;

$[(m+v)^2 - v^2)]a_m + a_{m-2} = 0$ (2.8)

Since $a_1 = 0$and$v \geq 0$, it follows from (2.8) that $a_3 = a_5 = a_7 \ldots = a_{2n+1} = 0$ Hence we have to deal only with ***even-number***coefficients :

solving for a_{2m} gives the ***recursion formula***

$$a_{2m} = \frac{-a_{2m-2}}{2^2 m(m+v)} \; , m = 1, 2, \ldots \quad (2.9)$$

From (2.9) we can now determine $a_2, a_4, a_6, \dots$ successively. This gives

$$m = 1, \qquad a_2 = \frac{-a_0}{2^2(v+1)}$$

$$m = 2, \qquad a_4 = \frac{-a_2}{2^2 2(v+2)} = \frac{-1}{2^2 2(v+2)} \times \frac{-a_0}{2^2(v+1)} = \frac{a_0}{2^4 2(v+1)(v+2)}$$

$$m = 3, \qquad a_6 = \frac{-a_4}{2^2 3(v+3)} = \frac{-1}{2^2 3(v+3)} \times \frac{a_0}{2^4.2!(v+1)(v+2)} = \frac{-a_0}{2^6 3!\,(v+1)(v+2)(v+3)}$$

and so on.

In general

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m} m!(v+1)(v+2)\dots(v+m)}, \; m = 1, 2, \dots \qquad (2.10)$$

Bessel Functions $J_n(x)$ for Integral $v = n$.

Integral values for v are denoted by n. This is standard. For $v = n$ the relation (2.10) becomes

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m} m!(n+1)(n+2)\dots(n+m)}, \; m = 1, 2, \dots \qquad (2.11)$$

a_0 is still arbitrary, so that the series (2.3) with these coefficients would contain this arbitrary factor a_0. This would be a highly impractical situation for developing formulas or computing values of this new function. Accordingly we have to make a choice. $a_0 = 1$ would be possible, but more practical turns out to be

$$a_0 = \frac{1}{2^n n!} \qquad (2.12)$$

because then $n!\,(n+1)\dots(n+m) = (n+m)!$ *in* (2.11), so (2.11) simply becomes

$$a_{2m} = \frac{(-1)^m}{2^{2m+n}(m!)(n+m)!} \qquad m = 1, 2, \dots \; (2.13)$$

This simplicity of the denominator of (2.13) partially motivates the choice (2.12). With these coefficients and $r_1 = v = n$ we get from (2.3) a particular solution of (2.1), denoted by $J_n(x)$ and given by

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+n}}{2^{2m+n} m!(n+m)!} \qquad (2.14)$$

$J_n(x)$ is called the ***Bessel function of the first kind of order n***.

The series (2.14) converges for all x, as the ratio test shows.

Example 1. Bessel function $J_0(x)$ and $J_1(x)$.

For $n = 0$, we obtain from (2.14) the ***Bessel function of order 0***:

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2} = 1 - \frac{x^2}{2^2(1!)^2} + \frac{x^4}{2^4(2!)^2} - \frac{x^6}{2^6(3!)^2} + - \cdots$$

which looks similar to a cosine function.

For $n = 1$,we obtain the Bessel function of order 1:

$$J_1(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{2^{2m+1} m!(m+1)!} = \frac{x}{2} - \frac{x^3}{2^3 1!2!} + \frac{x^5}{2^5 2!3!} - \frac{x^7}{2^7 3!4!} + - \cdots$$

which looks similar to a sine function.

Bessel functions $J_v(x)$ for any $v \geq 0$

We now extend our discussion from integer $v = n$ to any $v \geq 0$. All we need is an extension of the factorials in (2.12) and (2.14) to any v. This is done by the gamma function.

Now, in (2.12) we had $a_0 = \frac{1}{2^n n!}$. This is $\frac{1}{2^n \Gamma(n+1)}$ by (1.16). It suggests choosing for any v,

$$a_0 = \frac{1}{2^v \Gamma(v+1)} \qquad (2.15)$$

Then (2.10) becomes

$$a_{2m} = \frac{(-1)^m}{2^m m!(v+1)(v+2)\dots(v+m)2^v \Gamma(v+1)}$$

But (1.13) gives in the denominator
$(v+1)\Gamma(v+1) = \Gamma(v+2), (v+2)\Gamma(v+2) = \Gamma(v+3)$and so on, so that
$(v+1)(v+2)\dots(v+m)\Gamma(v+1) = \Gamma(v+m+1)$ hence because of our choice (2.15) of a_0 the coefficients (2.10) simply are

$$a_{2m} = \frac{(-1)^m}{2^{2m+v} m!\, \Gamma(v+m+1)} \qquad (2.16)$$

with these coefficients and $r = r_1 = v$ we get from (2.3) a particular solution of (2.1), denoted by $J_v(x)$ and given by

$$J_v(x) = x^v \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+v} m!\, \Gamma(v+m+1)} \qquad (2.17)$$

$J_v(x)$ is called the ***Bessel function of the first kind of order*** v. The series (2.17) converges for all x.

General Solution for Non integer v. Solution J_{-v}.

For a general solution, in addition to J_v we need a second linearly independent solution. For v is not an integer this is easy. Replacing v by $-v$ in (2.17),we have

$$J_{-v}(x) = x^{-v} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m-v} m!\Gamma(m-v+1)} \qquad (2.18)$$

J_v *and* J_{-v} are linearly independent only when v is not an integer. In fact when v is a positive integer, we observe that $m - v + 1 \leq 0$ for $m = 0,1,\dots,v-1$, and the coefficients (2.18) are not even defined for $m = 0,1,\dots,v-1$, because the gamma function is not defined at 0 and negative integers.

Theorem 2.1.(General solution of Bessel's Equation)

If v is not an integer, a general solution of Bessel's equation for all $x \neq 0$ is

$$y(x) = C_1 J_v(x) + C_2 J_{-v}(x) \tag{2.19}$$

But v is an integer then (2.19) is not a general solution because of linear dependence:

Theorem 2.2. **Linear Dependency of Bessel Functions** J_n and J_{-n}

For integer $v = n$ the Bessel functions $J_n(x)$ *and* $J_{-n}(x)$ are linearly dependent, because

$$J_{-n}(x) = (-1)^n J_n \qquad (n = 1, 2, \dots)$$

Proof*:* $J_{-n}(x) = \sum_{m=n}^{\infty} \frac{(-1)^m x^{2m-n}}{2^{2m-n} m!(m-n)!}$

$$= \sum_{s=0}^{\infty} \frac{(-1)^{n+s} x^{2s+n}}{2^{2s+n}(n+s)!s!} \, (m = n + s)$$

$$= (-1)^n \sum_{s=0}^{\infty} \frac{(-1)^s x^{2s+n}}{2^{2s+n}(n+s)!s!}$$

$$= (-1)^n J_n(x).$$

Note that $J_{-n}(x) = (-1)^n J_n(x)$ shows to us that n is an integer then $J_n(x)$ *and* $J_{-n}(x)$ are linearly dependent and then $C_1 J_n(x) + C_2 J_{-n}(x)$ cannot be a general solution of Bessel differential equation.

When index is half integer:

We start by using the expression of Bessel function of the first kind as

$$J_v(x) \quad = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+v+1)} \left(\frac{x}{2}\right)^{2m+v}$$

And calculate for half integer value as $v = \frac{1}{2}$, we get

$$J_{\frac{1}{2}}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\frac{3}{2})} \left(\frac{x}{2}\right)^{2m+\frac{1}{2}}$$

on opening the summation we get

$$J_{\frac{1}{2}}(x) = \left(\frac{x}{2}\right)^{\frac{1}{2}} \left[\frac{1}{\Gamma\left(\frac{3}{2}\right)} - \frac{1}{\Gamma\left(1+\frac{3}{2}\right)} \left(\frac{x}{2}\right)^2 + \frac{1}{2!\Gamma\left(2+\frac{3}{2}\right)} \left(\frac{x}{2}\right)^4 - \cdots\right]$$

$$= \frac{\left(\frac{x}{2}\right)^{\frac{1}{2}}}{\Gamma\left(\frac{3}{2}\right)}\left[1 - \frac{1}{\left(\frac{3}{2}\right)}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)}\left(\frac{x}{2}\right)^4 - \cdots\right]$$

$$= \frac{\left(\frac{x}{2}\right)^{\frac{1}{2}}}{\Gamma\left(\frac{3}{2}\right)}\left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right]$$

Since $\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$, we use this as well as multiply and divide by $\sqrt{x}$ to get

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}x\left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right]$$

$$= \sqrt{\frac{2}{\pi x}}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right]$$

$$= \sqrt{\frac{2}{\pi x}}\ \sin(x) \qquad (2.20)$$

Similarly, we calculate for negative $v\ and\ evaluate J_{-v}(x)$.

$$J_{-v}(x) = \sum_{m=0}^{\infty}\frac{(-1)^m}{m!\ \Gamma(m - v + 1)}\left(\frac{x}{2}\right)^{2m-v}$$

Substituting half integer value of $v = \frac{-1}{2}$ in equation (2.17), we get

$$J_{\frac{-1}{2}}(x) = \sum\nolimits_{m=0}^{\infty}\frac{(-1)^m}{m!\ \Gamma(m + \frac{1}{2})}\left(\frac{x}{2}\right)^{2m-\frac{1}{2}}$$

on opening the summation we get

$$J_{\frac{-1}{2}}(x) = \left(\frac{x}{2}\right)^{\frac{-1}{2}}\left[\frac{1}{\Gamma\left(\frac{1}{2}\right)} - \frac{1}{\Gamma\left(1+\frac{1}{2}\right)}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\Gamma\left(2+\frac{1}{2}\right)}\left(\frac{x}{2}\right)^4 - \cdots\right]$$

$$J_{\frac{-1}{2}}(x) = \frac{\left(\frac{x}{2}\right)^{\frac{-1}{2}}}{\Gamma\left(\frac{1}{2}\right)}\left[1 - \frac{1}{\left(\frac{1}{2}\right)}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)}\left(\frac{x}{2}\right)^4 - \cdots\right]$$

$$= \frac{\left(\frac{x}{2}\right)^{\frac{-1}{2}}}{\Gamma\left(\frac{1}{2}\right)}\left[1 - \frac{2}{4}x^2 + \frac{4}{16\cdot 3\cdot 2}x^4 - \cdots\right]$$

Since $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, we use this to get

$$J_{\frac{-1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left[1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots\right]$$

$$= \sqrt{\frac{2}{\pi x}}\ \cos(x) \qquad (2.21)$$

From the expressions of $J_{\frac{1}{2}}(x)$ in equation (2.20) and $J_{\frac{-1}{2}}(x)$ in equation (2.21) we see that the two functions $J_{\frac{1}{2}}(x)\ and\ J_{\frac{-1}{2}}(x)$ are linearly independent.

2.2. Bessel Function of the second kind $Y_v(x)$

From the last section we know that $J_v(x)$ and J_{-v} form a basis of solutions of Bessel equation, provided v is not an integer. But when $v$is an integer, these two solutions are linearly dependent on any interval. Hence to have a general solution also when $v = n$ is an integer, we need a second linearly independent solution besides J_n. This solution is called a ***Bessel function of the secondkind*** and is denoted by Y_n . We shall now derive such a solution, beginning with the case $n = 0$.

Bessel Equations of order zero.

The Bessel Equation of order zero is

$$x^2y'' + xy' + x^2y = 0 \tag{2.22}$$

We assume solutions have the form

$$y(x) = \Phi(r,\ x) = \sum_{n=0}^{\infty} a_n\, x^{r+n}, for\ a_0 \neq 0, x > 0 \tag{2.23}$$

Taking the derivatives,

$$y(x) = \sum_{n=0}^{\infty} a_n\, x^{r+n},\ y'(x) = \sum_{n=0}^{\infty}(r+n)\, a_n\, x^{r+n-1},$$

$$y''(x) = \sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\, x^{r+n-2}$$

Substituting these in to the differential equation, we obtain

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\, x^{r+n} + \sum_{n=0}^{\infty}(r+n)\, a_n x^{r+n} + \sum_{n=0}^{\infty} a_n\, x^{r+n+2} = 0.$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n[(r+n)(r+n-1) + (n+r)]\, x^{r+n} + \sum_{n=0}^{\infty} a_n\, x^{r+n+2} = 0$$

$$\Rightarrow a_0[r(r-1) + r]x^r + a_1[r(r+1) + (r+1)]x^{r+1} +$$

$$\sum_{n=2}^{\infty}\{a_n[(n+r)(n+r-1) + (n+r)] + a_{n-2}\}\, x^{r+n} = 0. \tag{2.24}$$

The roots of the indicial equation $r(r-1) + r = 0\ are\ r_1 = 0$and $r_2 = 0$; hence we have the case of equal roots.

The recurrence relation is

$$a_n(r) = \frac{-a_{n-2}(r)}{(n+r)(n+r-1)+(n+r)} = \frac{-a_{n-2}(r)}{(n+r)^2},\ n \geq 2 \tag{2.25}$$

To determine $y_1(x)$ we set r equal to 0. Then from Equation (2.25) it follows that for the coefficients of x^{r+1} to be zero we must choose $a_1 = 0$. Hence from Equation (2.25) $a_3 = a_5 = a_7 = \cdots = a_{2n+1} = \cdots = 0$.

Further

$$a_n(0) = \frac{-a_{n-2}(0)}{n^2}, n = 2,\ 4,\ 6, \ldots$$

or letting $n = 2m$,

$$a_{2m}(0) = \frac{-a_{2m-2}(0)}{(2m)^2},\ m = 1,\ 2,\ 3, \ldots$$

Thus $a_2(0) = \frac{-a_0}{2^2}$,

$$a_4(0) = \frac{-a_2(0)}{(2\cdot2)^2} = \frac{a_0}{(2\cdot2)^2 2^2} = \frac{a_0}{(2)^4(2\cdot1)^2}$$

$$a_6(0) = \frac{-a_4(0)}{(2\cdot3)^2} = \frac{-a_0}{(2\cdot3)^2 2^4 (2\cdot1)^2} = \frac{-a_0}{2^6(3\cdot2\cdot1)^2}, \ldots \text{etc.}$$

$a_{2m}(0) = \frac{(-1)^m a_0}{2^{2m}(m!)^2}, m = 1, 2,\ 3, \ldots$ (2.26)Hence, $y_1(x) = a_0\left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2}\right],\ x > 0$ (2.27)The function in brackets is known as the Bessel function of the first kind of order zero, and denoted by, $J_0(x)$. In this example we will determine$y_2(x)$by computing$a'_n(0)$. From the coefficient of x^{r+1} in equation (2.24) that $(r+1)^2 a_1(r) = 0$ Thus $a_1(r) = 0$for all r near$r = 0$.so not only does$a_1(0) = 0$, *but also* $a'_1(0) = 0$. From the recurrence relation (2.25) it followsthat $a'_3(0) = a'_5(0) = \cdots = a'_{2n+1}(0) = \cdots = 0$; hence we need only compute $a'_{2m}(0),\ m = 1,2, \ldots$

from equation (2.24) we have,

$$a_{2m}(r) = \frac{-a_{2m-2}(r)}{(2m+r)^2}, m = 1, 2, \quad 3, \ldots$$

Hence $a_2(r) = \frac{-a_0}{(2+r)^2}$

$$a_4(r) = \frac{-a_2(r)}{(4+r)^2} = \frac{a_0}{(4+r)^2(2+r)^2}$$

$$a_6(r) = \frac{-a_4}{(6+r)^2} = \frac{-a_0}{(6+r)^2(4+r)^2(2+r)^2}, \ldots \text{etc.}$$

$$a_{2m}(r) = \frac{(-1)^m a_0}{(2m+r)^2(2m-2+r)^2 \ldots (4+r)^2(2+r)^2},\ m = 1, 2, 3, \ldots \qquad (2.28)$$

To computation of a'_{2m} can be carried out most conveniently by noting that if

$$f(x) = (x-\alpha_1)^{\beta_1}(x-\alpha_2)^{\beta_2}(x-\alpha_3)^{\beta_3} \ldots (x-\alpha_n)^{\beta_n},$$

and if $x$is not equal to $\alpha_1, \alpha_2, \ldots, \alpha_n,$ then

$$f'(x) = \beta_1(x-\alpha_1)^{\beta_1 - 1}\left[(x-\alpha_2)^{\beta_2} \ldots (x-\alpha_n)^{\beta_n}\right] +$$

$$\beta_2(x-\alpha_2)^{\beta_2-1}\left[(x-\alpha_1)^{\beta_1} \ldots (x-\alpha_n)^{\beta_n} + \cdots\right]$$

$$\frac{f'(x)}{f(x)} = \frac{\beta_1}{x-\alpha_1} + \frac{\beta_2}{x-\alpha_2} + \cdots + \frac{\beta_n}{x-\alpha_n}$$

Applying this result to $a_{2m}(r)$ from Equation (2.28),

$$\frac{a'_{2m}(r)}{a_{2m}(r)} = -2\left(\frac{1}{2m+r} + \frac{1}{2m-2+r} + \cdots + \frac{1}{2+r}\right),$$

and setting r equal to 0, we obtain

$$a'_{2m}(0) = -2\left[\frac{1}{2m} + \frac{1}{2(m-1)} + \frac{1}{2(m-2)} + \cdots + \frac{1}{2}\right] a_{2m}(0)$$

Substituting for $a_{2m}(0)$ from equation (2.26), and letting

$$H_m = \frac{1}{m} + \frac{1}{m-1} + \frac{1}{m-2} + \cdots + \frac{1}{2} + 1 \qquad (2.29)$$

We obtain, finally,

$$a'_{2m}(0) = -H_m \frac{(-1)^m a_0}{2^{2m}(m!)^2}, \; m = 1, 2, 3, \ldots \qquad (2.30)$$

The second solution of Bessel equation of order zero is obtained by setting $a_0 = 1$, and substituting for $y_1(x) and\ a'_{2m}(0) = b_{2m}(0)$ in equation (1.9b).we obtain

$$y_2(x) = J_0(x) \ln x + \sum_{m=1}^{\infty} \frac{(-1)^{m+1} H_m}{2^{2m}(m!)^2} x^{2m}, \; x > 0 \qquad (2.31)$$

If we replace y_2 by an independent particular solution of the form$a(y_2 + bJ_0)$, where $a \neq 0$and b are constants. It is customary to choose $a = \frac{2}{\pi}$ *and* $b = \gamma - \ln 2$, where the number $\gamma = 0.57721566490 \ldots$ is so called Euler constant.

In place of y_2, the second solution is usually taken to be a certain linear combination of $J_0 and\ y_2$. It is known as the ***Bessel function of the second kind of order zero***, and is denoted by $Y_0(x)$.

$$Y_0(x) = \frac{2}{\pi}[y_2(x) + (\gamma - \ln 2)J_0(x)] \qquad (2.32)$$

Substituting for $y_2(x)$ in equation (2.32), we obtain

$$Y_0(x) = \frac{2}{\pi}\left[(\gamma - \ln\frac{x}{2})J_0(x) + \sum_{m=1}^{\infty}\frac{(-1)^{m+1}H_m}{2^{2m}(m!)^2}x^{2m}\right],\ \ x > 0. \qquad (2.33)$$

The general solution of the Bessel equation of order zero for x>0 is given by

$y = C_1J_0(x) + C_2Y_0(x)$.

For many purposes, it is convenient to take the linear combination.

$$Y_v(x) = \frac{\cos(v\pi)J_v(x) - J_{-v}(x)}{\sin(v\pi)}$$

As the second independent solution instead of$J_{-v}(x)$. This is known as the ***Bessel function ofsecond kind of order v***. Thus, the complete solution of Bessel's equation in alternative form is

$$y(x) = C_1J_v(x) + C_2Y_v(x).$$

Bessel Equation of order one half.

This example illustrates in which the roots of the indicial equation differ by a positive integer, but there is no logarithmic term in the 2nd solution.

The Bessel Equation of order one-half is

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0 \qquad (2.34)$$

We assume solutions have the for

$$y(x) = \sum_{n=0}^{\infty} a_n\, x^{r+n}, for\ a_0 \neq 0, x > 0$$

Substituting these in to the differential equation, we obtain

$$\sum_{n=0}^{\infty}\left[(r+n)(r+n-1) + (r+n) - \frac{1}{4}\right]a_n\, x^{r+n} + \sum_{n=0}^{\infty} a_n\, x^{r+n+2} = 0.$$

$$\Rightarrow\left(r^2-\frac{1}{4}\right)a_0x^r+\left[(r+1)^2-\frac{1}{4}\right]a_1x^{r+1}+\sum_{n=0}^{\infty}\left\{\left[(r+n)^2-\frac{1}{4}\right]a_n+a_{n-2}\right\}x^{r+n}=\ 0 \quad (2.35)$$

The roots of the indicial equation are $r_1=\frac{1}{2},\ r_2=-\frac{1}{2}$; hence the roots differ by an integer.

The recurrence relation is

$$\left[(r+n)^2-\frac{1}{4}\right]a_n=-a_{n-2},\ n\geq 2 \qquad (2.36)$$

Corresponding to the larger root $r_1=\frac{1}{2}$ we find from the coefficient of x^{r+1} in equation (2.35) that

$a_1=0$. Hence, from equation (2.36),

$$a_3=a_5=\cdots=a_{2n+1}=\cdots=0.$$

Further, for $r_1=\frac{1}{2}$

$$a_n=\frac{-a_{n-2}}{n(n+1)}\ ,\ n=2,\ 4,\ 6,\ldots.$$

Or letting $n=2m$

$$a_{2m}=\frac{-a_{2m-2}}{2m(2m+1)}\ ,\ m=1,\ 2,\ 3,\ldots.$$

Thus $a_2=\frac{-a_0}{3\cdot 2}\quad=\frac{-a_0}{3!}$

$$a_4=\frac{-a_0}{5\cdot 4}\quad=\frac{a_0}{5\cdot 4\cdot 3!}=\frac{a_0}{5!},a_6=\frac{-a_0}{6!}\ldots$$

$$a_{2m}=\frac{(-1)^m a_0}{(2m+1)!}\ ,\ m=1,\ 2,\ 3,\ldots. \qquad (2.37)$$

Hence, taking $a_1=1$, we obtain

$$y_1(x)=a_0x^{1/2}\left[\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)!}x^{2m}\right],x>0$$

$$=a_0x^{-1/2}\left[\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+1}}{(2m+1)!}\right],x$$

$$=\sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{x}}\left[\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+1}}{(2m+1)!}\right]$$

$$=\sqrt{\frac{2}{\pi x}}\,sinx\,,x>0.$$

The Bessel function of the first kind of order one-half, $J_{1/2}$, is defined as

$$J_{\frac{1}{2}}(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} sinx\ , x > 0.$$

Now consider the case $r_2 = \frac{-1}{2}$. We know that

$$\left(r^2 - \frac{1}{4}\right) a_0 x^r + \left[(r+1)^2 - \frac{1}{4}\right] a_1 x^{r+1} + \sum_{n=2}^{\infty} \left\{\left[(r+n)^2 - \frac{1}{4}\right] a_n + a_{n-2}\right\} |x^{r+n} = 0$$

Since$r_2 = \frac{-1}{2}$, $a_1 =$ arbitrary.

For the even coefficients,

$$a_{2m} = \frac{-a_{2m-2}}{(\frac{-1}{2}+2m)^2 - \frac{1}{4}} = \frac{-a_{2m-2}}{2m(2m-1)}\ , m = 1, 2,\ \ldots$$

It follows that

$$a_2 = \frac{-a_0}{2!}\ , a_4 = \frac{-a_2}{4 \cdot 3} \quad = \frac{a_0}{4!}, \ldots$$

and

$$a_{2m} = \frac{(-1)^m a_0}{(2m)!} \quad , m = 1, 2, \ldots$$

For odd coefficients,

$$a_{2m+1} = \frac{a_{2m-1}}{\left(\frac{-1}{2}+2m+1\right)^2 - \frac{1}{4}} = \frac{-a_{2m-1}}{2m(2m+1)}\ , m = 1, 2, \ldots$$

It follows that

$$a_3 = \frac{-a_1}{3!}\ , a_5 = \frac{-a_3}{5 \cdot 4} \quad = \frac{a_1}{5!}, \ldots$$

and

$$a_{2m+1} = \frac{(-1)^m a_1}{(2m+1)!} \quad , m = 1, 2, \ldots$$

There fore

$$y_2(x) = x^{-1/2}\left[a_0 \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} + a_1 \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!}\right], x > 0$$

$$= x^{-1/2}\,[a_0 cosx + a_1 sinx], x > 0$$

The second solution is usually taken to be the function

$$J_{\frac{-1}{2}}(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} cosx\,, x > 0.$$

Where$a_0 = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}$ $and\ a_1 = 0.$

The general solution of Bessel's equation of order one-half is

$$y(x) = C_1 J_{\frac{1}{2}}(x) + C_2 J_{-\frac{1}{2}}(x).$$

2.3 Modified Bessel functions of the first and the second kind

Bessel differential equation of order$n$ is given by :

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right)y = 0 \quad (2.38)$$

Replacing the independent variable x in Bessel'sequation by it changes the differential equation as follows:

Now putting x=it or t=-ix, we have

$$\frac{dx}{dt} = i, \qquad \frac{dt}{dx} = \frac{1}{i}$$

Thus, $\frac{dy}{dx} = \frac{dt}{dx}\frac{dy}{dt} = \frac{1}{i}\frac{dy}{dt}$

$$\frac{d^2y}{dx^2} = -\frac{d^2y}{dt^2}$$

Thus, (2.38) becomes

$$-\frac{d^2y}{dt^2} - \frac{1}{t}\frac{dy}{dt} + \left(1 + \frac{n^2}{t^2}\right)y = 0 \text{or} \frac{d^2y}{dt^2} + \frac{1}{t}\frac{dy}{dt} - \left(1 + \frac{n^2}{t^2}\right)y = 0$$

Whichis called the***modified Bessel's equation***.

Definition3.1: A differential equation of the form

$$\frac{d^2y}{dt^2}+\frac{1}{t}\frac{dy}{dt}-\left(1+\frac{n^2}{t^2}\right)y=0 \quad (2.39)$$

is called the modified Bessel equation.

Since $y=c_1J_n(x)+c_2Y_n(x), (where\ c_1\ and\ c_2\ are\ arbitrary)$ is the general solution of the Bessel equation, therefore the solution of the modified Bessel equation is obtained by putting x= it that is

$$y=c_1J_n(it)+c_2Y_n(it) (2.40)$$

We know that $J_v(x)=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(m+v+1)}(\frac{x}{2})^{2m+v}$

Equation (2.40) provides a solution of modified Bessel equation in complex form. But it is desirable to have a real form of solution. Now putting x=it:

Consider $J_v(x)=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(m+v+1)}\left(\frac{x}{2}\right)^{2m+v}$

$$J_v(it)=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(m+1+v)}\left(\frac{it}{2}\right)^{2m+v}$$

$$=\sum_{m=0}^{\infty}\frac{(-1)^m(i)^{2m+v}}{m!\Gamma(m+v+1)}\left(\frac{t}{2}\right)^{2m+v}$$

$$=\sum_{m=0}^{\infty}\frac{(-1)^m(i^2)^m\cdot i^v}{m!\Gamma(m+v+1)}\left(\frac{t}{2}\right)^{2m+v}$$

$$=i^v\sum_{m=0}^{\infty}\frac{(-1)^{2m}}{m!\Gamma(m+v+1)}\left(\frac{t}{2}\right)^{2m+v}$$

$$=i^v\sum_{m=0}^{\infty}\frac{1}{m!\Gamma(m+v+1)}\left(\frac{t}{2}\right)^{2m+v}=i^v\sum_{m=0}^{\infty}\frac{\left(\frac{t}{2}\right)^{2m+v}}{m!\Gamma(m+v+1)}$$

Then define a function which is called the ***modified Bessel function of the first kind of order*** $\boldsymbol{v}$.

Thus replacing t by x,we get

$$I_v(x)=i^{-v}J_v(ix)=\sum_{m=0}^{\infty}\frac{\left(\frac{x}{2}\right)^{2m+v}}{m!\Gamma(m+v+1)}$$

Which is a real function and which is a solution of the modified Bessel equation(2.38). Notation I for this function reflects the method of its definition, and it means "the function of imaginary argument." For negative values of parameter $-v$, define second solution of the modified Bessel equation as:

$$I_{-v}(x) = i^v J_{-v}(ix) = \sum_{m=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2m-v}}{m!\Gamma(-v+m+1)} \tag{2.42}$$

Functions $I_v(x) and\ I_{-v}(x)$ are linearly independent and form a fundamental set, then the general solution of the modified Bessel equation of non- integer order is given by

$$y(x) = C_1 I_v(x) + C_2 I_{-v}(ix) \qquad v \neq integer \tag{2.43}$$

In the case of integer orders, function $I_{-n}(x)$ is the same as function $I_n(x)$. Indeed, when v is changed for n in equation (2.42),

$$I_{-n}(x) = i^n J_{-n}(ix) = i^n(-1)^n J_n(ix) = i^n(-1)^n i^n[i^{-n} J_n(ix)]$$
$$= (i^2)^n(-1)^n I_n(x) = (-1)^n(-1)^n I_n(x) = (-1)^{2n} I_n(x) = I_n(x).$$

For integer order $v = n,\ \ n = 0, 1, 2, 3, \ldots$, the second solution of the modified Bessel equation is defined with the help of the ***modified Bessel function of the second kind of order*** *v*:

$$k_v(x) = \frac{\pi}{2}\frac{I_v(x) - I_{-v}(x)}{\sin(vx)} \tag{2.44}$$

as the limit of

$$k_n(x) = \lim_{v \to n} k_v(x) \tag{2.45}$$

CHAPTER THREE

3. PROPERTIES OF THE BESSEL FUNCTIONS

In this chapter we will prove some properties of Bessel functions.

3.1. Generating function

Many facts about Bessel functions can be proved by using its generating function. Here we want to determine the generating function.

Theorem 2.1. we have

$$e^{\frac{x}{2}(z-z^{-1})} = \sum_{n=-\infty}^{\infty} J_n(x) z^n \qquad (3.1)$$

i.e, $e^{\frac{x}{2}(z-z^{-1})}$ is the generating function of $J_n(x)$.

$$e^{\frac{x}{2}(z-z^{-1})} = e^{\frac{x}{2}z} \cdot e^{\frac{-x}{2z}}$$

$$= \sum_{m=0}^{\infty} \frac{\left(\frac{x}{2}z\right)^m}{m!} \cdot \sum_{k=0}^{\infty} \frac{\left(\frac{-x}{2z}\right)^k}{k!}$$

$$= \sum_{m=0}^{\infty} \frac{\left(\frac{x}{2}\right)^m z^m}{m!} \cdot \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^k z^{-k}}{k!}$$

$$= \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{k+m} z^{m-k}}{m! \;\; k!}$$

We now make the change of index $n = m - k$. Because of the range of values on m and k, it follows that $-\infty < n < \infty and$ thus

$$= \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k+n} z^n}{k!(k+n)!}$$

$$= \sum_{n=-\infty}^{\infty} \left[\sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k+n}}{k!(k+n)!}\right] z^n$$

Hence, $e^{\frac{x}{2}(z-z^{-1})} = \sum_{n=-\infty}^{\infty} J_n(x) z^n$

Lemma 3.1 we have

$$\cos(x) = J_0(x) + 2\sum_{n=1}^{\infty}(-1)^n J_{2n}(x)$$

$$\sin(x) = 2\sum_{n=0}^{\infty}(-1)^n J_{2n+1}(x)$$

$$1 = J_0(x) + 2\sum_{n=1}^{\infty} J_{2n}(x)$$

Proof: Directly from (3.1) with

$$z = e^{i\emptyset}, \qquad z^n = e^{in\emptyset}$$

By using generating function

$$,e^{x\left(\frac{z-z^{-1}}{2}\right)} = \sum_{n=-\infty}^{\infty} J_n(x) z^n$$

$$e^{x\,\frac{(z-z^{-1})}{2}} = e^{\frac{x}{2}}\left(e^{i\emptyset} - e^{-i\emptyset}\right) = e^{\frac{x}{2}}(\cos\emptyset + isin\emptyset) - (cos\emptyset - isin\emptyset) = e^{\frac{x}{2}}(2isin\emptyset) =$$

$$e^{ixsin\emptyset} \Rightarrow e^{ixsin\emptyset} = \sum_{n=-\infty}^{\infty} J_n(x) e^{in\emptyset}$$

$$\Rightarrow \cos(xsin\emptyset) + i\sin(xsin\emptyset) = \sum_{n=-\infty}^{\infty} J_n(x)\,(\cos(n\emptyset) + isin(n\emptyset))$$

Equating the real and imaginary parts:

$\cos(xsin\emptyset) = \sum_{n=-\infty}^{\infty} J_n(x)\,(\cos(n\emptyset)$and $\sin(x\ \sin\emptyset) = \sum_{n=-\infty}^{\infty} J_n(x)\,sin(n\emptyset)$

$$\sum_{n=-\infty}^{\infty} J_n(x)\,(\cos(n\emptyset) = J_0(x) + 2J_2(x)\cos(2\emptyset) + 2J_4(x)cos\emptyset + \cdots$$

$$= J_0(x) + 2[J_2(x)\cos(2\emptyset) + J_4(x)\cos(4\emptyset) + \cdots]$$

$$Therefore, \cos(x sin\emptyset) = J_0(x\ ) + 2\sum_{n=1}^{\infty} J_{2n}(x)\cos(2n\emptyset)$$

For $\emptyset = \frac{\pi}{2}$,

$$\cos(x sin\emptyset)\ \frac{\pi}{2}) = J_0(x) + 2\sum_{n=1}^{\infty} \mathrm{J}_{2\mathrm{n}}(x)\cos(2n\frac{\pi}{2}) = J_0(x) + 2\sum_{n=1}^{\infty} J_{2n}(x)\cos(\pi n)$$

$$\cos(x)\ =\ J_0(x) + 2\sum_{n=1}^{\infty} \mathrm{J}_{2\mathrm{n}}(x)\,(-1)^n$$

$$\cos(x) = J_0(x) + 2\sum_{n=1}^{\infty} (-1)^n \mathrm{J}_{2\mathrm{n}}(x)$$

$\sum_{n=-\infty}^{\infty} J_n(x)\ sin(n\emptyset)$=$2J_1(x)\sin(\emptyset) + 2J_3(x)\sin(3\emptyset) + \cdots$

$$= 2[J_1(x)\sin(\emptyset) + J_3(x)\sin(3\emptyset) + \cdots]$$

$$\sin(x sin\emptyset)\ =\ \ 2\sum_{n=0}^{\infty} J_{2n+1}(x)\sin((2n+1)\emptyset)$$

$$\sin\left(x sin\frac{\pi}{2}\right) = 2\sum_{n=0}^{\infty} J_{2n+1}(x)\sin\left((2n+1)\frac{\pi}{2}\right)$$

$\sin(\mathrm{x})\ = 2\sum_{n=0}^{\infty} J_{2n+1}(x)\ sin(n\pi + \frac{\pi}{2})$

$\sin(\ x)\ = 2\ \sum_{n=0}^{\infty} (-1)^n J_{2n+1}(x)$

From(3.2) we have, $\cos(x sin\emptyset) = J_0(x) + 2\sum_{n=1}^{\infty} J_n(x)\ \ \cos(2n\emptyset)$

For $\emptyset = 0$, $\cos(\ 0) = J_0(x) + 2\sum_{n=1}^{\infty} J_{2n}(x)\ \cos(0)$

$1 = J_0(x) + 2\sum_{n=1}^{\infty} J_{2n}(x)$.

This gives the results with $\theta = {}^{\pi}/_{2}$ and $\theta = 0$ respectively.

Lemma3.2 we have

$$J_n(-x) = J_{-n}(x) = (-1)^n J_n(x) \; for\; all\; n \in \mathcal{Z}.$$

Proof

$$J_n(x) = \textstyle\sum_{k=0}^{\infty} \frac{(-1)^k (\frac{x}{2})^{2k+n}}{k!(k+n)!}$$

$$J_n(-x) = \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{-x}{2})^{2k+n}}{k!\,(k+n)!} = \sum_{k=0}^{\infty} \frac{(-1)^k (-1)^{2k+n} (\frac{x}{2})^{2k+n}}{k!\,(k+n)!}$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k (-1)^{2k} (-1)^n (\frac{x}{2})^{2k+n}}{k!\,(k+n)!}$$

$$= (-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{x}{2})^{2k+n}}{k!\,(k+n)!}$$

$$= (-1)^n J_n(x) = \begin{cases} J_n(x)\; if\; n\; is\; even \\ -J_n(x)\; if\; n\; is\; odd \end{cases}$$

Similarly

$$J_{-n}(x) = \textstyle\sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k-n}}{k!\,(k-n)!}$$

Changing the index $m = k - n \Rightarrow k = m + n$

$$\mathrm{J_{-n}(x)} = \textstyle\sum_{m=0}^{\infty} \frac{(-1)^{m+n} \left(\frac{x}{2}\right)^{2m+n}}{m!(m+n)!} = \textstyle\sum_{m=0}^{\infty} \frac{(-1)^m (-1)^n \left(\frac{x}{2}\right)^{2m+n}}{m!(m+n)!}$$

$$= (-1)^n \textstyle\sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{2m+n}}{m!(m+n)!}$$

$$= (-1)^n J_n(x)$$

Hence $J_n(-x) = J_{-n}(x) = (-1)^n J_n(x)$

Lemma 3.3 we have for any $n \in \mathcal{Z}$.

$$2J'_n(x) = J_{n-1}(x) - J_{n+1}(x)$$

$$\frac{2n}{x} J_n(x) = J_{n+1}(x) + J_{n-1}(x)$$

$$\frac{d}{dx}\left(x^n J_n(x)\right) = x^n J_{n-1}$$

Proof:- We have $e^{\frac{x}{2}(z-z^{-1})} = \sum_{n=-\infty}^{\infty} J_n(x) z^n$

differentiating with respect to x.

$$\frac{d}{dx} \textstyle\sum_{n=-\infty}^{\infty} J_n(x)\, z^n = \frac{d}{dx}\left(e^{\frac{x}{2}(z-z^{-1})}\right)$$

$$= \frac{1}{2} z e^{\frac{x}{2}(z-z^{-1})} - \frac{1}{2} \cdot \frac{1}{z} e^{\frac{x}{2}(z-z^{-1})}$$

$$= \frac{1}{2} z \sum_{n=-\infty}^{\infty} J_n(x)\, z^n = -\frac{1}{2} \cdot \frac{1}{z} \sum_{n=-\infty}^{\infty} J_n(x)\, x^n$$

$$= \frac{1}{2} \sum_{n=-\infty}^{\infty} J_{n-1}(x)\, z^n = -\frac{1}{2} \sum_{n=-\infty}^{\infty} J_{n+1}(x)\, x^n$$

$$\Rightarrow \frac{d}{dx} \sum_{n=-\infty}^{\infty} J_n(x)\, z^n = \frac{1}{2} \sum_{n=-\infty}^{\infty} J_{n-1}(x)\, z^n = -\frac{1}{2} \sum_{n=-\infty}^{\infty} J_{n+1}(x)\, x^n$$

$$\Rightarrow J'_n(x) z^n = \frac{1}{2} J_{n-1}(x) z^n - \frac{1}{2} J_{n+1}(x) z^n$$

Therefore: $2J'_n(x) = J_{n-1}(x) - J_{n+1}(x)$ … (1)

Again differentiating with respect to z.

$$\frac{d}{dt} \sum_{n=-\infty}^{\infty} J_n(x)\, z^n = \frac{d}{dx}\left(e^{\frac{x}{2}(z-z^{-1})}\right)$$

$$= \frac{x}{2}\left(1 + \frac{1}{z^2}\right) e^{\frac{x}{2}(z-z^{-1})}$$

$$= \frac{x}{2} e^{\frac{x}{2}(z-z^{-1})} + \frac{x}{2z^2} e^{\frac{x}{2}(z-z^{-1})}$$

$$= \frac{x}{2} \sum_{n=-\infty}^{\infty} J_n(x)\, z^n + \frac{x}{2} z^{-2} \sum_{n=-\infty}^{\infty} J_n(x)\, z^n$$

$$= \frac{x}{2} \sum_{n=-\infty}^{\infty} J_n(x)\, z^n + \frac{x}{2} \sum_{n=-\infty}^{\infty} J_n(x)\, z^{n-2}$$

$$= \frac{x}{2} \sum_{n=-\infty}^{\infty} J_n(x)\, z^n + \frac{x}{2} \sum_{n=-\infty}^{\infty} J_{n+2}(x)\, z^n$$

Therefore, $\frac{d}{dz} \sum_{n=-\infty}^{\infty} J_n(x)\, z^n = \frac{x}{2} \sum_{n=-\infty}^{\infty} J_n(x)\, z^n + \frac{x}{2} \sum_{n=-\infty}^{\infty} J_{n+2}(x)\, z^n$

$$\sum_{n=-\infty}^{\infty} nJ_n(x)\, z^{n-1} = \frac{x}{2} \sum_{n=-\infty}^{\infty} J_n(x)\, z^n + \frac{x}{2} \sum_{n=-\infty}^{\infty} J_{n+2}(x)\, z^n$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} nJ_{n+1}(x)\, z^n = \frac{x}{2} \sum_{n=-\infty}^{\infty} J_n(x)\, z^n + \frac{x}{2} \sum_{n=-\infty}^{\infty} J_{n+2}(x)\, z^n$$

$$\Rightarrow nJ_{n+1}(x) = \frac{x}{2} J_n(x) + \frac{x}{2} J_{n+2}(x)$$

Multiplying both sides by $\frac{2}{x}$ and change the index $n - 1\ by$.

$\Rightarrow \frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x)$ … (2)

Adding (1) and (2) and multiplied by $\frac{x^n}{2}$

$$\begin{cases} 2J'_n(x) = J_{n-1}(x) - J_{n+1}(x) \\ \frac{2n}{x} J_n(x) = J_{n+1}(x) + J_{n-1}(x) \end{cases}$$

$$\Rightarrow 2J'_n(x) + \frac{2n}{x} J_n(x) = 2J_{n-1}(x)$$

$$\Rightarrow \frac{x^n}{2}\left[2J'_n(x) + \frac{2n}{x} J_n(x) = 2J_{n-1}(x)\right]$$

$$\Rightarrow x^n J'_n(x) + nx^{n-1} J_n(x) = x^n J_{n-1}(x)$$

$$\Rightarrow \frac{d}{dx}\left(x^n J_n(x)\right) = x^n J_{n-1}(x)$$

3.2 Recurrence Formula for Bessel's Polynomial

The Bessel polynomial follows same recurrence relation among themselves. We study some of them important ones;

1. $\boldsymbol{xJ'_k(x) = kJ_k(x) - xJ_{k+1}(x)}$(3.2)

From the definition of Bessel polynomial we write

$$J_k(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(k+m+1)} \left(\frac{x}{2}\right)^{2m+k}$$

Differentiating with respect to x

$$J'_k(x) = \sum_{m=0}^{\infty} (2m+k)\frac{1}{2}\frac{(-1)^m}{m!\Gamma(k+m+1)} \left(\frac{x}{2}\right)^{2m+k-1}$$

multiplying by x throughout we get

$$xJ'_k(x) = \sum_{m=0}^{\infty} \frac{x}{2}(2m+k)\ \frac{(-1)^m}{m!\Gamma(k+m+1)} \left(\frac{x}{2}\right)^{2m+k-1}$$

$$= \sum_{m=0}^{\infty} (2m+k)\frac{(-1)^m}{m!\Gamma(k+m+1)} \left(\frac{x}{2}\right)^{2m+k}$$

separating the terms on RHS we get

$$xJ'_k(x) = k\sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(k+m+1)} \left(\frac{x}{2}\right)^{2m+k} + \sum_{m=0}^{\infty} 2m\ \frac{(-1)^m}{m!\Gamma(k+m+1)} \left(\frac{x}{2}\right)^{2m+k}$$

$$xJ'_k(x) = kJ_k(x) + \sum_{m=0}^{\infty} \frac{(-1)^m\,2m}{m!\Gamma(k+m+1)} \left(\frac{x}{2}\right)^{2m+k}$$

The second term of RHS has m which is a positive integer, so $m! = \Gamma(m+1) = m\Gamma(m)$. We write the second term as

$$2^{\text{nd}}\text{ term} = \sum_{m=0}^{\infty} \frac{(-1)^m\,2m}{m\Gamma(m)\Gamma(k+m+1)} \left(\frac{x}{2}\right)^{2m+k}$$

$$= \sum_{m=0}^{\infty} \frac{2(-1)^m}{\Gamma(m)\Gamma(k+m+1)} \left(\frac{x}{2}\right)^{2m+k}$$

For$m = 0;\ we\ have\ \ \Gamma(0) = -\infty$ and so that its contribution to the 2nd terms is zero. In fact the contribution in the series actually starts from $m = 1$. Thus we should write the second term as

$$2^{\text{nd}}\text{ term} = \sum_{m=1}^{\infty} \frac{2(-1)^m}{\Gamma(m)\Gamma(k+m+1)} \left(\frac{x}{2}\right)^{2m+k}$$

Let as put $m = \ell + 1$ to compare this term with$J_k(x)$; to get

$$2^{\text{nd}} \text{ term} = \sum_{\ell=0}^{\infty} \frac{2(-1)^{\ell+1}}{\Gamma(\ell+1)\Gamma(k+\ell+1+1)} \left(\frac{x}{2}\right)^{2(\ell+1)+k}$$

$$= \sum_{\ell=0}^{\infty} \left(\frac{x}{2}\right) \frac{2(-1)(-1)^{\ell}}{\Gamma(\ell+1)\Gamma(k+\ell+1+1)} \left(\frac{x}{2}\right)^{k+2\ell+1}$$

So we see that

$$(-1)^{\ell+1} = (-1)^1(-1)^{\ell} = -(-1)^{\ell}$$

$$\left(\frac{x}{2}\right)^{k+2+2\ell} = \frac{x}{2}\left(\frac{x}{2}\right)^{k+1+2\ell}$$

$$2^{\text{nd}} \text{ term} = \sum_{\ell=0}^{\infty} \frac{x}{2} \frac{-2(-1)^{\ell}}{\Gamma(\ell+1)\Gamma(k+1+\ell+1)} \left(\frac{x}{2}\right)^{k+1+2\ell}$$

$$= \sum_{\ell=0}^{\infty} -x \frac{(-1)^{\ell}}{\Gamma(\ell+1)\Gamma(k+1+\ell+1)} \left(\frac{x}{2}\right)^{k+1+2\ell}$$

$$= -x \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}}{\ell!\,\Gamma(k+1+\ell+1)} \left(\frac{x}{2}\right)^{k+1+2\ell} = -xJ_{k+1}(x)$$

Therefore $xJ'_k(x) = kJ_k(x) - xJ_{k+1}(x)$

The second recurrence relation is:

2. $\boldsymbol{xJ'_k(x) = -kJ_k(x) + xJ_{k-1}}(x)$ (3.3)

Again using the definition of Bessel polynomial we write,

$$J_k(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(k+m+1)} \left(\frac{x}{2}\right)^{2m+k}$$

Differentiating with respect to x

$$J'_k(x) = \sum_{m=0}^{\infty} \frac{k+2m}{2} \cdot \frac{(-1)^m}{m!\,\Gamma(k+m+1)} \left(\frac{x}{2}\right)^{2m+k-1}$$

Multiplying by x throughout we get

$$xJ'_k(x) = \sum_{m=0}^{\infty} \frac{x(k+2m)}{2} \cdot \frac{(-1)^m}{m!\,\Gamma(k+m+1)} \left(\frac{x}{2}\right)^{2m+k-1}$$

separating the terms on RHS by writing

$(k + 2m) = (-k + 2k + 2m)$ we get

$$xJ'_k(x) = \sum_{m=0}^{\infty} \frac{x}{2}(-k + 2k + 2m) \cdot \frac{(-1)^m}{m!\,\Gamma(k + m + 1)} \left(\frac{x}{2}\right)^{2m+k-1}$$

$$= \sum_{m=0}^{\infty} (-k + 2k + 2m) \cdot \frac{(-1)^m}{m!\Gamma(k+m+1)} \left(\frac{x}{2}\right)^{2m+k}$$

$$= -k \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(k+m+1)} \left(\frac{x}{2}\right)^{2m+k} + \sum_{m=0}^{\infty} \frac{(-1)^m 2(k+m)}{m!\,(k+m)\Gamma(k+m)} \left(\frac{x}{2}\right)^{2m+k}$$

$$xJ'_k(x) = -kJ_k(x) + \ 2^{nd} term$$

The second term on the RHS we multiply and divide by x

$$2^{\text{nd}} \text{ term} = \sum_{m=0}^{\infty} \frac{x}{x} \frac{(-1)^m\, 2(k+m)}{m!(k+m)\Gamma(k+m)} \left(\frac{x}{2}\right)^{2m+k} = \sum_{m=0}^{\infty} \frac{2}{x} \cdot x \ \frac{(-1)^m}{m!\Gamma(k+m)} \left(\frac{x}{2}\right)^{2m+k}$$

$$= \sum_{m=0}^{\infty} x \ \left(\frac{x}{2}\right)^{-1} \cdot \frac{(-1)^m}{m!\Gamma(k+m)} \left(\frac{x}{2}\right)^{2m+k}$$

$$= x \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(k+m)} \left(\frac{x}{2}\right)^{2m+k-1}$$

$$= xJ_{k-1}(x)$$

Thus the 2^{nd} recurrence relation is obtained by substituting the above 2^{nd} term.

$$xJ'_k(x) = -kJ_k(x) + xJ_{k-1}(x)$$

The third recurrence relation is:

3. $2J'_k(x) = J_{k-1}(x) - J_{k+1}(x)$ (3.3)

We use the first recurrence relation

$$xJ'_k(x) = \ kJ_k(x) - xJ_{k+1}(x)$$

And rewrite the second recurrence relation as

$$xJ'_k(x) = \ -kJ_k(x) + xJ_{k-1}(x)$$

Add the two relation to obtain the third recurrence relation.

$$2xJ'_k(x) = \ xJ_{k-1}(x) - xJ_{k+1}(x)$$

$$2J'_k(x) = \ J_{k-1}(x) - J_{k+1}(x)$$

The fourth recurrence relation is

4. $J_{k+1}(x) + J_{k-1}(x) = \frac{2k}{x} J_k(x)$ (3.4)

Subtract the two recurrence relations (3.2) and (3.3**) to** get the fourth recurrence relation.

$$-\begin{cases} xJ'_k(x) = \ kJ_k(x) - xJ_{k+1}(x) \\ xJ'_k(x) = \ -kJ_k(x) + xJ_{k-1}(x) \end{cases}$$

$$\Rightarrow 0 = 2J_k(x) - xJ_{k+1}(x) - xJ_{k-1}(x)$$
$$\Rightarrow 2J_k(x) = x\left(J_{k+1}(x) + J_{k-1}(x)\right)$$
$$\Rightarrow J_{k+1}(x) + J_{k-1}(x) = \frac{2}{x}J_k(x)$$

Anther compact form (read k=v) of recurrence relation is:

5. $\frac{d}{dx}[x^{-v}J_v(x)] = -x^{-v}J_{v+1}(x)$ Proof:-

$$J_v(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+v}}{2^{2m+v} m!\Gamma(v+m+1)}$$

Multiply both sides by x^{-v}, we have

.
$$x^{-v}J_v(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+v} m!\Gamma(m+v+1)}$$

Differentiate with respect to x

$$\frac{d}{dx}[x^{-v}J_v(x)] = \sum_{m=0}^{\infty} \frac{(-1)^m 2m\, x^{2m-1}}{2^{2m+v} m!\Gamma(m+v+1)}$$

$$= x^{-v}\sum_{m=0}^{\infty} \frac{(-1)^m 2m\, x^{2m+v-1}}{2^{2m+v} m(m-1)!\Gamma(m+v+1)}$$

$$= x^{-v}\sum_{m=0}^{\infty} \frac{(-1)^m\, x^{2m+v-1}}{2^{2m+v-1}(m-1)!\Gamma(m+v+1)}$$

replace m by m+1

Therefore $\frac{d}{dx}[x^{-v}J_v(x)] = x^{-v}\sum_{m=0}^{\infty} \frac{(-1)^{m+1}\, x^{2m+v+1}}{2^{2m+v+1} m!\Gamma(m+(v+1)+1)}$

$$= -x^{-v}\sum_{m=0}^{\infty} \frac{(-1)^m\, x^{2m+v+1}}{2^{2m+v+1} m!\Gamma(m+(v+1)+1)}$$
$$= -x^{-v}J_{v+1}(x)$$

***Example 1*:**The forth recurrence relation is used to evaluate$J_{\frac{3}{2}}(x)$,$J_{\frac{-3}{2}}(x)$,…etc.

$$J_{v-1}(x) + J_{v+1}(x) = \frac{2v}{x}J_v(x)$$

We know the above values of $J_{\frac{1}{2}}(x)\ and\ J_{\frac{-1}{2}}(x)$which are

$$J_{\frac{-1}{2}}(x) = \sqrt{\frac{2}{\pi x}}cosx \quad , \qquad J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}sinx$$

In the recurrence relation we use $v = \frac{1}{2}$ to determine the values of $J_{\frac{3}{2}}(x)$

$$J_{\frac{1}{2}-1}(x) + J_{\frac{1}{2}+1}(x) = \frac{1}{x}J_{\frac{1}{2}}(x)$$

$$\Rightarrow J_{\frac{-1}{2}}(x) + J_{\frac{3}{2}}(x) = \frac{1}{x}J_{\frac{1}{2}}(x)$$

$$\Rightarrow \sqrt{\frac{2}{\pi x}}cosx + J_{\frac{3}{2}}(x) = \frac{1}{x}\sqrt{\frac{2}{\pi x}}sinx$$

$$\Rightarrow J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x^3}}sinx - \sqrt{\frac{2}{\pi x}}cosx$$

In the recurrence relation we use $v = \frac{-1}{2}$ to determine the values of $J -_{\frac{3}{2}(x)}$

.

$$J_{\frac{-1}{2}-1}(x) + J_{\frac{-1}{2}+1}(x) = \frac{-1}{x}J_{\frac{-1}{2}}(x)$$

$$\Rightarrow J_{\frac{-3}{2}}(x) + J_{\frac{1}{2}}(x) = \frac{1}{x}J_{\frac{-1}{2}}(x)$$

$$J_{\frac{-3}{2}}(x) + \sqrt{\frac{2}{\pi x}}sinx = \frac{1}{x}\sqrt{\frac{2}{\pi x}}cosx$$

$$J_{\frac{-3}{2}}(x) = \sqrt{\frac{2}{\pi x^3}}cosx - \sqrt{\frac{2}{\pi x}}sinx$$

3.3. Integral Representation of Bessel function

Statement: Integral Representation of Bessel function has the following form.

$$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(n\theta - \text{x}\sin\theta)\, d\theta = 0 \qquad \forall\, n = 0, 1, 2, \dots \quad (3.5)$$

We start from the generating function.

$$e^{\frac{x}{2}(t-t^{-1})} = \sum_{n=-\infty}^{\infty} \mathrm{J_n(x)}\, t^n$$

Let $t = e^{i\theta}$, so that the LHS is

$$e^{\frac{x}{2}(t-t^{-1})} = e^{\frac{x}{2}(e^{i\theta}-e^{-i\theta})} = e^{ixsin\theta} = \cos(xsin\theta) + isin(xsin\theta) \qquad (3.6)$$

The RHS can be expressed as

$$\sum_{n=-\infty}^{\infty} \mathrm{J_n(x)}\, t^n = \sum_{n=-\infty}^{\infty} \mathrm{J_n(x)}\left(e^{i\theta}\right)^n = \sum_{n=-\infty}^{\infty} \mathrm{J_n(x)}\,(cosn\theta + isinn\theta)$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} \mathrm{J_n(x)}\, t^n = J_0(x) + [J_{-1}(x) + J_1(x)]\, cos\theta + [J_{-2}(x) + J_2(x)]\, cos2\theta + \cdots$$
$$+ i\{[\, J_1(x) - J_{-1}(x)]\, sin\theta + [J_2(x) - J_{-2}(x)]\, sin2\theta + \cdots\}$$

Now since $J_{-1}(x) = -J_1(x), J_{-2}(x) = (-1)^2 J_2(x), i.e J_{-n}(x) = (-1)^n J_n(x)$using we get

$$\Rightarrow \sum_{n=-\infty}^{\infty} \mathrm{J_n(x)} = [J_0(x) + 2J_0(x)cos2\theta + 2J_4(x)cos4\theta + \cdots]$$
$$+ i[2J_1(x)sin\theta + 2J_3(x)sin3\theta + \cdots]$$

Or in compact form we can write

$$\Rightarrow \sum_{n=-\infty}^{\infty} \mathrm{J_n(x)}\, t^n = J_0(x) + 2\sum_{k=1}^{\infty} \mathrm{J_{2k}(x)}\, cos2k\theta + 2i\sum_{k=1}^{\infty} \mathrm{J_{2k-1}(x)} \sin{(2k-1)\theta} \quad (3.7)$$

Equation (1) and (2) we get

$$\cos(xsin\theta) + isin(xsin\theta) = J_0(x) + 2\sum_{k=1}^{\infty} \mathrm{J_{2k}(x)}\, cos2k\theta + 2i\sum_{k=1}^{\infty} \mathrm{J_{2k-1}(x)} \sin{(2k-1)\theta}$$

Equating the real and imaginary parts we get

$$\cos(xsin\theta) = J_0(x) + 2\sum_{k=1}^{\infty} \mathrm{J_{2k}(x)}\, cos2k\theta \qquad (3.8)$$

and

$$sin(xsin\theta) = 2\sum_{k=1}^{\infty} \mathrm{J_{2k-1}(x)} \sin{(2k-1)\theta} \qquad (3.9)$$

The series on the right in equation (3.8) and equation (3.9) are just the Fourier expansion of the functions on the left. Now we multiply both sides of equation(3.8) by$cosn\theta$ and both sides of equation (3.9) by $sinn\theta$ and integrate each identity with respect to $\theta$from 0 to π. Since we know that

$$\int_0^\pi \cos n\theta\;\, cosn\theta\, d\theta = \begin{cases} 0, if\ n \neq 2k,\ \forall\, n \\ \frac{\pi}{2}, if\ n = 2k,\ \forall\, n \end{cases} \qquad (3.10)$$

$$\int_0^\pi \sin(2k-1)\theta \sin n\theta\, d\theta = \begin{cases} \frac{\pi}{2}, if\ n = 2k-1,\ \forall\, n \\ 0, if\ n \neq 2k-1,\ \forall\, n \end{cases}$$

from equation (3.8)

$\int_0^{\pi}\cos n\theta\ \cos(x\sin\theta)\,d\theta = J_0(x)\int_0^{\pi}\cos n\theta\,d\theta + 2\sum_{k=1}^{\infty} J_{2n}(x)\int_0^{\pi}\cos 2k\theta\cos n\theta\,d\theta$ (3.11)The first integer disappears for all values of integral n

$$\int_0^{\pi}\cos n\theta\,d\theta = 0$$

The second integral disappears only if $n \neq 2k\ so\ if\ n = 2k$

$$\int_0^{\pi}\cos n\theta\ \cos n\theta\,d\theta = \frac{\pi}{2} \quad \forall\, n, n{:}\, even = 2k$$

$$\int_0^{\pi}\cos n\theta\ \cos(x\sin\theta)\,d\theta = \begin{cases}\pi J_n(x) & \forall\, n{:}\ even\\ 0 & \forall\, n{:}\ odd\end{cases} \tag{3.12}$$

From equation (3.9)

$$\int_0^{\pi}\sin n\theta\ \sin(x\sin\theta)\,d\theta = 2\sum_{k=1}^{\infty} J_{2k-1}(x)\int_0^{\pi}\sin(2k-1)\theta\sin n\theta\,d\theta$$

$$\int_0^{\pi}\sin n\theta\ \sin(x\sin\theta)\,d\theta = \begin{cases}\pi J_n(x) & \forall\, n{:}\ odd\\ 0 & \forall\, n{:}\ even\end{cases} \tag{3.13}$$

Adding the two expansions equation (3.12) and equation (3.13) and dividing by πwe have for all integral values of n

$$\pi J_n(x) = \int_0^{\pi}[\cos n\theta\ \cos(x\sin\theta) + \sin n\theta\sin(x\sin\theta)]\,d\theta \tag{3.14}$$

Since for every values of n, one of the integrals vanishes, and the other (remaining one) contributes to $J_n(x)$. Finally using the cosine formula for difference of two quantities we get

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi}\cos(n\theta - x\sin\theta)\,d\theta, \quad \forall\, n = 0, 1, 2, \ldots$$

Example 1. Show that the functions $J_n(x)$ are oscillating with infinite zeros.

Step1: the Bessel functions have oscillating behavior. They have infinite number of zeros which are unequally spaced. The zeros are not periodic like the sine and cosine. Let us annualize what a combination of Bessel functions yield by using equation (3.8) and equation (3.9).

$$\text{Cos}\,(x\sin\theta) = J_0(x) + 2\sum_{k=1}^{\infty} J_{2k}(x)\cos 2k\theta$$

$$\sin(x\sin\theta) = 2i\sum_{k=1}^{\infty} J_{2k-1}(x)\sin(2k-1)\theta$$

If we put $\theta = \frac{\pi}{2}$ in both the equations we get

$$\text{Cos}\,x = J_0(x) - 2J_2(x) + 2J_4(x) - 2J_6(x) + \ldots$$

$$\sin x = 2[J_1(x) - J_3(x) + J_5(x) - \cdots]$$

If we put $\theta = 0$ in the equation (3.8) we get

$$1 = J_0(x) + 2J_2(x) + 2J_4(x) + \ldots$$

So we see that some combination of Bessel functions gives 1, another combination gives $cosx$ and yet another combination yields $sinx$. Both $\sin x$ and $cosx$ are periodic functions with infinite zeros. Thus Bessel functions are oscillating with infinite zeros.

3.4 Orthogonality properties of Bessel's polynomials

Statement: if λand μ are roots of the equation $J_n(\alpha) = 0$ then condition of orthogonality of Bessel's function over the interval (0, 1) with weight function x is

$$\int_0^1 x\, J_n(\lambda x) J_n(\mu x) dx = 0 \qquad \text{for } \lambda \neq \mu$$

With the condition of normalization

$$\int_0^1 x\, [J_n(\lambda x)]^2\, dx = \frac{1}{2} J^2{}_{n+1}(x) dx = 0 \qquad \text{for } \lambda = \mu$$

Both the above conditions can be combined to write the condition of orthogonality as

$$\int_0^1 x\, [J_n(\boldsymbol{\lambda x})]^2\, dx = \frac{1}{2} J^2{}_{\boldsymbol{n+1}}(\boldsymbol{x}) \boldsymbol{\delta_{\lambda\mu}}$$

Proof: The second order Bessel's differential equation is

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$$

Let as change the independent variable x toλxand x to μx,where x is a constant, the resulting equations are

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (\lambda^2 x^2 - n^2)y = 0$$

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (\mu^2 x^2 - n^2)y = 0$$

We see that $y_1 = J_n(\lambda x)$ and $y_2 = J_n(\mu x)$are the solutions of the equations

$$x^2 \frac{d^2y_1}{dx^2} + x\frac{dy_1}{dx} + (\lambda^2 x^2 - n^2)y_1 = 0 \qquad \text{(R-1)}$$

and

$$x^2 \frac{d^2y_2}{dx^2} + x\frac{dy_2}{dx} + (\lambda^2 x^2 - n^2)y_2 = 0 \qquad \text{(R-2)}$$

Multiply equation (R-1) with y_2 and equation (R-2) by y_1 and subtract to get

$$(\mu^2 - \lambda^2)xy_1y_2 = x\frac{d}{dx}\left[y_2\frac{dy_1}{dx} - y_1\frac{dy_2}{dx}\right] + \left[y_2\frac{dy_1}{dx} - y_1\frac{dy_2}{dx}\right]$$

$$= \frac{d}{dx}\left[x\left(y_2\frac{dy_1}{dx} - y_1\frac{dy_2}{dx}\right)\right]$$

Integrating as we omit the constant of integration we get

$$(\mu^2-\lambda^2)\int_0^1 xy_1y_2dx = \int_0^1 \frac{d}{dx}\left[x\left(y_2\frac{dy_1}{dx}-y_1\frac{dy_2}{dx}\right)\right]dx$$

$$= \left[x\left(y_2\frac{dy_1}{dx}-y_1\frac{dy_2}{dx}\right)\right]\Big|_0^1$$

Using $y_1 = J_n(\lambda x) and\ y_2 = J_n(\mu x)$ and dividing by $(\mu^2-\lambda^2)$we get

$$(\mu^2-\lambda^2)\int_0^1 x\,J_n(\lambda x)J_n(\mu x)dx = \left[x\left(J_n(\mu x)\frac{dJ_n(\lambda x)}{dx}-J_n(\lambda x)\frac{dJ_n(\mu x)}{dx}\right)\right]\Big|_0^1$$

$$\int_0^1 x\,J_n(\lambda x)J_n(\mu x)dx = \frac{\lambda J_n(\mu)J'_n(\lambda)-\mu J_n(\lambda)J'_n(\mu)}{(\mu^2-\lambda^2)} \text{for} \lambda\neq\mu \quad \text{(R-3)}$$

Thus the orthogonality condition implies that if $\mu\ and\ \lambda$ are two zeros of $J_n(x)$that is $J_n(\lambda)=0$ and$J_n(\mu)=0$, then for λ≠ μ

$$\int_0^1 x\,J_n(\lambda x)J_n(\mu x)dx = 0 \qquad \text{for } \lambda\neq\mu$$

To reach the normalization condition let us reconsider equation (R-3). If λ= μ then if we take limits then we get $\frac{0}{0}$ form in the RHS. So we first apply L'Hospital rule (differentiate only with respect to μ keeping λ as constant) and find

$$\int_0^1 x\,J^2{}_n(\lambda x)dx = \lim_{\mu\to\lambda}\frac{\lambda J'_n(\mu)J'_n(\lambda)-J_n(\lambda)J'_n(\mu)-\mu J_n(\lambda)J''_n(\mu)}{2\mu}$$

$$= \frac{\lambda J'_n(\mu)J'_n(\lambda)-J_n(\lambda)J'_n(\mu)-\mu J_n(\lambda)J''_n(\mu)}{2\lambda}$$

Now since$J_n(\boldsymbol{\lambda})=0$ we have

$$\int_0^1 x\,J^2{}_n(\lambda x)dx = \frac{\lambda J'_n(\lambda)J'_n(\lambda)}{2\lambda} \quad \text{(R-4)}$$

We use the first recurrence relation

$$xJ'_n(x) = nJ_n(x) - xJ_{n+1}(x)$$

change the independent variable x to λx

$$\lambda xJ'_n(\lambda x) = nJ_n(x\lambda) - x\lambda J_{n+1}(x\lambda)$$

For$x=1$, we have

$$\lambda J'_n(\lambda) = nJ_n(\lambda) - \lambda J_{n+1}(\lambda) = -\lambda J_{n+1}(\lambda)\text{since}J_n(\lambda)=0$$

using $J'_n(x) = -J_{n+1}(\lambda)$ in equation (R-4)

$$\int_0^1 x\,J^2{}_n(\lambda x)dx = \frac{1}{2}J^2{}_{n+1}(\lambda)$$

Now since $J_{n+1}(\lambda)\neq 0$ because λ is the zero of $J_n(\lambda)$. Thus the above is the normalization condition.

Summary

Bessel's equation was introduced and series solution were obtained by Frobenius method for Bessel function $J_v(x)$ of the first kind of order v. When n is an integer $J_n(x)$ and $J_{-n}(x)$are linearly dependent. The second linearly independent solution $Y_v(x)$ can be constructed that for all v is linearly independent of $J_v(x)$, so the general solution of Bessel's equation can always be written $Y(x) = AJ_v(x) + BY_v(x)$, where A and B are arbitrary constants. The function $Y_v(x)$ is called a Bessel function of the second kind of order v.The general solution of Modified Bessel function was also expressed in terms of $I_v(x)\ and K_v(x)$.

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