QUANTUM HALL EFFECT IN GRAPHENE

By

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Table of Contents

List of Figures

Abstract

We studied Quantum Hall Effect with and without magnetic field in graphene. The quantum Hall effect is a quantum-mechanical version of the Hall effect, observed in graphene, in which the Hall conductance (σ_H) takes on the quantized values (νe^2/h) with $\nu = \pm 4(n+\frac{1}{2})$ an integer (integer quantized Hall Effect in graphene). Our study confirmed that there is a zero energy Landau level in graphene with magnetic fields. The fractional quantum Hall effect in graphene is observed as the result of coulomb interaction and correlation of electrons.

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Moreda Eticha
October, 2015

Chapter 1

Introduction

Two-dimensional electron system is a structures in which the electronic behavior is essentially two-dimensional (2D). This is to say that the carriers are confined in 2D. So that their motion in one direction is restricted. Because of this restriction we have only two-dimensional momentum in k-space. Two-dimensional electron system (2DES) is a semiconductor interface, particularly that GaAs/AlGaAs heterointerface, has been the stage of various experiment in condensed matter physics [1]. 2D electrons at GaAs/AlGaAs interface have the energy dispersion $\epsilon(k) = \hbar^2 k^2/2m$ and behave as free electrons with the effective mass $m = 0.067m_e$, where m_e is the bare electron mass.

A common examples of 2D systemes are Metal-oxide Semiconductor Field Effect Transistores (MOSFET), Superlattice [2] and electrons on the surface of liquid helium.
In MOSFET, inversion layers are formed at the interface between a semiconductor and an insulator or between two semiconductors, with one of them acting as an insulator [3]. The system in which the Quantum Hall effect (QHE) was discovered has Si for the semiconductor, SiO_2 for the insulator. An electric field transverse to the interface attracts electrons from the semiconductor material. These electrons go dawn to the quantum well created by the electric field and the interface, according to the inversion layer principle. The motion transverse to the interface is quantized and thus has a fundamental rigidity that restricts motional degree of freedom in this direction.

As the wave length of the these electrons are long, an effective mass approximation with parabolic bands is used. The periodic lattice potential gives rise to energy bands, and the slowly varying electric potential then is regarded as bending these bands. The basic working principle of MOSFET is that the electric field is applied to the gate electrode and as the result there is a change in the properties of the carriers in the silicon layer immediately below the gate (the conductance in the source-drain channel) [4]. The current that flows between source and drain can thus be controlled by the voltage applied to the metal electrode [5]. The oxide prevents a current flowing between the gate and the silicon, which would reduce the performance of the switch.

Superlattice structure is another type of two-dimensional electron system is formed in the heterostructures of two semiconductors. Superlattice structures are can be grown in semiconductor using molecular beam epitaxy (MBE) technique or by metal-organic chemical vapor deposition (MOCVD). For example, in $GaAs/Ga1-xAlxAs$ super-lattice, a certain controlled number of layers of $GaAs$ is followed by an almost perfectly matched sequence of layers of $GaAlAs$. The $GaAlAs$ is deliberately doped n-type, which puts mobile electrons into its conduction band. These electrons will migrate to fill the few holes on the top of the $GaAs$ valence band but most of them will end up in states near the bottom of the $GaAs$ conduction band. However, there is a positive charge left on the donor impurities which attracts these electrons to the interface and bends the bands in the process. This is the source of the electric field in this system. The transfer of electrons from $GaAlAs$ to $GaAs$ will continue until the dipole layer formed from the positive donors and the negative inversion layer is sufficiently strong,(charge density$\sim 10^{11}cm^{-2}$) [6].

The liquid Helium surface is also example of two-dimensional system. There exists a potential barrier of about 1eV in the surface of liquid Helium which prevents electrons from transmitting into the liquid. On the other hand, the potential attracts the electrons in the surface, resulting a 2D electron system (charge density $\sim 10^{9}cm^{-2}$). Quantum Hall effect has been observed in the first two types which are MOSFET and supper lattice [7].

A two-dimensional electron gas (2DEG) is a gas of electrons free to move in two dimensions, say (x, y), but tightly confined in the third dimension (z). The effective mass approximation, the dispersion relation for a 2DEG is parabolic and the eigen energies of the electrons with the momentum k are [8]

$$E = \epsilon_n + \frac{\hbar^2}{2m}(k_x^2 + k_y^2) \tag{1.0.1}$$

where the index n numbers successive subbands, n is the lowest energy of electrons in the n-th subband, and m is the effective mass of an electron. At low concentrations and low temperatures, electrons populate only the lowest subband. In this case the electrons do not participate in the dynamics of orthogonal to the plane and the dynamics is that of a real 2DEG (two-dimensional electron gas in the quantum limit). In such a system, electron states can be described as freely propagating plane waves and their in-plane normalized wavefunctions are

$$\psi(x,y) = \frac{1}{\sqrt{s}}\exp(i(k_x x + k_y y)) \tag{1.0.2}$$

, where s is normalized

The motion of an electron in a magnetic field $\vec{B}$ experiences a force called Lorentz force.

$$\mathbf{F} = m\dot{\vec{v}}(t) = e[\vec{E}(t) + \vec{v}\times\vec{B}]. \tag{1.0.3}$$

The external magnetic field applied perpendicularly to a current carrying conductor deflects the charge carries to one side of the conductor (Fig.1.1(b)). Due to this Lorentz force opposite charges accumulate on the opposite side of the conductor. The result is an imbalance distribution of charge carriers on the conductor's surface. This separation of charges establishes an electric field that opposes further separation of charge. This gives rise to electric potential, called Hall voltage,V_H between the sides of the conductor. As shown in Fig.1.1(c) the Hall field is transverse to both the current and the external magnetic field. As long as charges flow, a steady electric potential exists called the Hall voltage and the resistivity of the conductor depends linearly on the magnetic field strength. This is known as the Classical Hall Effect [2].

$$V_H \alpha I, \mathbf{B}. \tag{1.0.4}$$

This observation is now easily explained by the Lorentz force interaction which states that a charged particle moving in a magnetic field experiences a torque given by is characterized with the cyclotron motion with the cyclotron frequency $\omega_c = eB/m$. An electron occasionally experiences a scattering at the scattering rate τ^{-1} and diffuses away from the initial position. A magnetic field sets a length scale as well, namely the spatial extension of wave functions in the magnetic field [1]. It is given by the magnetic length $l_B^2 = \hbar/eB$, which is the unit span of the quantized cyclotron motion. Also of importance is the cyclotron radius r_c, i.e. the radius of the circle the electrons follow in a magnetic field: $r_c = k_F l_B^2$

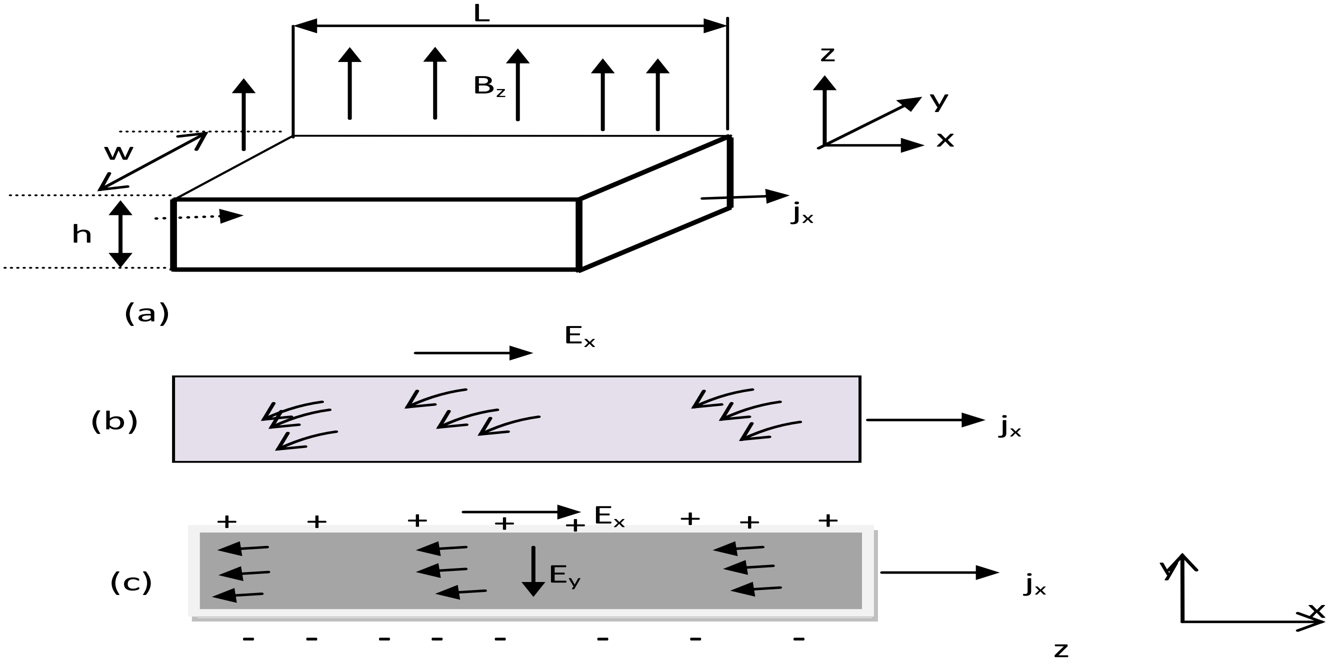

Figure 1.1: Geometry of the Hall effect.

The history of the quantum Hall effect has continually been marked by unexpected new experimental discoveries [9]. The quantum Hall effect (QHE) was first discovered in 1980, when Klaus von Klitzing demonstrated quantization in the Hall resistance of a 2D electron system, at high magnetic fields [10]. Observation of this entirely new quantum phenomenon quickly gave birth to a rich new field of physics [11]. Moreover, von Klitzings discovery of exact quantization in the Hall resistance in units h/e^2 , depending only on fundamental constants, gave us a new standard of resistance [12]. In Chapter 2 we present the Quantum Hall Effect; both the Integral Quantum Hall Effect (IQHE) and the Fractional Quantum Hall Effect (FQHE). In Chapter 3 we discuss the properties of Graphene and in Chapter 4 we present the QHE in graphene. Finally, Chapter 5 summarizes our findings.

Chapter 2

Quantum Hall Effect

2.1 Landau level

In a large magnetic field ($\vec{B}$), electrons e tend to have orbits which oscillate in time with an angular frequency $\omega_c = eB/m$. If the relaxation time for the scattering is τ, then the behavior of electrons depends up on whether $\omega_c\tau$ greater than or smaller than one [9]. If $\omega_c\tau \gg 1$ the electrons form periodic orbits. As we will see later their energy levels are quantized in units of $\hbar\omega_c$ which are called Landau level. These orbits are simple to describe in two-dimensions. The canonical momentum is, as a consequence:

$$\begin{aligned} \vec{p} = &\quad m\vec{v} - e\vec{A} \\ = &\quad \hat{p} - e\vec{A}, \end{aligned} \tag{2.1.1}$$

where, A is vector potential, $\hat{p} = -i\hbar\nabla$ is the electron momentum operator.

Due to the fact that the electron has a negative charge the Hamiltonian of a particle having a spin is

$$\hat{H} = \frac{1}{2m}(\hat{p} - e\vec{A})^2 - \frac{e}{m}\vec{S}.\vec{B} + U(x, y, z). \tag{2.1.2}$$

If we consider the motion of a free electron in a constant magnetic field

$$\hat{H} = \frac{1}{2m}(\hat{p} - e\vec{A})^2 - \frac{e}{m}\vec{S}.\vec{B}, \tag{2.1.3}$$

where, $\vec{S}$ is electron spin.

The stationary-state Schroedinger equation for free particle in a magnetic field is given by [13].

$$\frac{1}{2m}(\hat{p} - e\vec{A})^2\Psi(x, y, z) - \frac{e}{m}\vec{S}.\vec{B}\Psi(x, y, z) = E\Psi(x, y, z) \tag{2.1.4}$$

where $\Psi(x, y, z)$ is the electron wave function, E is the energy eigenvalue. The magnetic field introduced through a vector potential is

$$\vec{B} = \nabla \times \vec{A}. \tag{2.1.5}$$

When the magnetic field is a constant there are many different ways to choose a vector potential to satisfy this relation. If the magnetic field is along the z-direction, then the following choices all give the correct magnetic field [8].

$$\begin{aligned} \vec{A} = &\quad B(-y, 0, 0), \\ = &\quad B(0, x, 0), \\ = &\quad \frac{B}{2}(-y, x, 0). \end{aligned} \tag{2.1.6}$$

Each of the different vector potential satisfy the guage condition. Since the eigenvalue do not depend on this choices of $\vec{A}$, they are called guage invariant. The second guage in Eqn. (2.1.6) is chosen, for no particular reason and we ignore the magnetic interaction part with spin, so that the Schrdinger equation is

$$\frac{1}{2m}[\hat{p}_x^2 + (\hat{p}_y - eBx)^2 + \hat{p}_z^2]\Psi(x, y, z) = E\Psi(x, y, z). \tag{2.1.7}$$

Since the operator in Eqn. (2.1.7) does not contain y and z

$$[\hat{p}_z, B] = [\hat{p}_y, B] = 0. \tag{2.1.8}$$

This leads us to choose the wave function which has a plane wave dependence on y and z coordinates, i.e.

$$\Psi(x, y, z) = \Psi(x)\exp(i(k_y y + k_z z)). \tag{2.1.9}$$

Then seek a solution of Schrdinger equation in Eqn. (2.1.9) form. By Substituting Eqn. (2.1.9) into Eqn. (2.1.7) we get;

$$\frac{1}{2m}[-\hbar^2\frac{\partial^2}{\partial x^2} + (-i\hbar\frac{\partial}{\partial y} - eBx)^2 + \hat{p}_z^2]\Psi(x)\exp(i(k_y y + k_z z)) = E\Psi(x)\exp(i(k_y y + k_z z)). \tag{2.1.10}$$

When the exponential term is canceled throughout, we obtain

$$\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x) + \frac{1}{2m}(\hbar^2 k_y^2 - 2\hbar k_y(eBx) + (eBx)^2)\Psi(x) = (E - \frac{\hat{p}_z^2}{2m})\Psi(x)$$

$$\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x) + \frac{1}{2}\omega_c^2 m[(\frac{\hbar k_y}{eB})^2 - (\frac{\hbar k_y}{eB})x + x^2]\Psi(x) = \varepsilon\Psi(x),$$

where $\varepsilon = E - \frac{\hat{p}_z^2}{2m}$,

$$\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x) + \frac{1}{2}\omega_c^2 m[x_o^2 - 2x_o x + x^2]\Psi(x) = \varepsilon\Psi(x), \tag{2.1.11}$$

where $\omega_c = \frac{eB}{m}$, $x_o = \frac{\hbar k_y}{eB} = k_y l_B^2, l_B = \sqrt{\frac{\hbar}{eB}}$
The resulting equation is one dimensional Schrodinger equation for simple Harmonic oscillator frequency, ω_c (the cyclotron frequency) and displacement $(x - x_o)$. The effective hamiltonian operating on the x variable is harmonic oscillator.

$$-\frac{\hbar^2}{2m}\frac{d^2\Psi(x)}{dx^2} + \frac{m\omega_c^2}{2}(x - x_o)^2\Psi(x) = \varepsilon\Psi(x), \tag{2.1.12}$$

Let us substitute $k = m\omega_c^2$, $X = x - x_o$ into the harmonic oscilltator Eqn. (2.1.12).

$$-\frac{\hbar^2}{2m}\frac{d^2\Psi(x)}{dx^2} + \frac{k}{2}X^2\Psi(x) = \varepsilon\Psi(x), \tag{2.1.13}$$

$$\frac{d^2\Psi(x)}{dx^2} - \frac{2m}{\hbar^2}\frac{k}{2}X^2\Psi(x) = -\frac{2m}{\hbar^2}\varepsilon\Psi(x),$$

$$\frac{d^2\Psi(x)}{dx^2} + \frac{2m}{\hbar^2}[\varepsilon - \frac{k}{2}X^2]\Psi(x) = 0$$

$$\frac{d^2\Psi(x)}{dx^2} + 2\sqrt{\frac{m}{\hbar^2}}\sqrt{\frac{mk}{\hbar^2}}[\frac{\varepsilon}{\sqrt{k}} - \frac{1}{2}\sqrt{k}X^2]\Psi(x) = 0$$

$$\frac{1}{\sqrt{\frac{mk}{\hbar^2}}}\frac{d^2\Psi(x)}{dx^2} + 2\sqrt{\frac{m}{\hbar^2}}[\frac{\varepsilon}{\sqrt{k}} - \frac{1}{2}\sqrt{k}X^2]\Psi(x) = 0$$

$$\frac{1}{\sqrt{\frac{mk}{\hbar^2}}}\frac{d^2\Psi(x)}{dx^2} + [2\varepsilon\sqrt{\frac{m}{k\hbar^2}} - \sqrt{\frac{mk}{\hbar^2}}\frac{1}{2}X^2]\Psi(x) = 0 \tag{2.1.14}$$

For simplicity let us substitute,

$$\alpha^2 = \sqrt{\frac{mk}{\hbar^2}} \tag{2.1.15}$$

$$\lambda = 2\varepsilon\sqrt{\frac{m}{k\hbar^2}}. \tag{2.1.16}$$

From Eqn.(2.1.14), then we have,

$$\frac{1}{\alpha^2}\frac{d^2\Psi(x)}{dx^2} + (\lambda - \alpha^2 X^2)\Psi(x) = 0. \tag{2.1.17}$$

Let us make change of variable,

$$q = \alpha X, \tag{2.1.18}$$

where α is constant given by Eqn. (2.1.15). Then

$$\frac{\partial\Psi(x)}{\partial X} = \frac{\partial\Psi(x)}{\partial q}\frac{\partial q}{\partial X} = \alpha\frac{\partial\Psi(x)}{\partial q}, \tag{2.1.19}$$

and

$$\frac{d^2\Psi(x)}{\partial X^2} = \alpha^2\frac{\partial^2\Psi(x)}{\partial q^2}. \tag{2.1.20}$$

Substituting these values in Eqn. (2.1.17), we get

$$\frac{\partial^2\Psi(x)}{\partial q^2} + (\lambda - q^2)\Psi(x) = 0. \tag{2.1.21}$$

To solve Eqn. (2.1.21), let us first make asymptotic solution for the case of $q^2 > \lambda$, in which Eqn. (2.1.21) becomes,

$$\frac{\partial^2\Psi(x)}{\partial q^2} - q^2\Psi(x) = 0. \tag{2.1.22}$$

The solution of this equation is

$$\Psi(x) = e^{\frac{q^2}{2}}. \tag{2.1.23}$$

By differentiating Eqn. (2.1.23) twice with respect to q; thus

$$\frac{\partial\Psi(x)}{\partial q} = qe^{q^2}, \quad \frac{\partial^2\Psi(x)}{\partial q^2} = (q^2+1)e^{q^2}. \tag{2.1.24}$$

As q is very large and so Eqn. (2.1.24) becomes

$$\frac{\partial\Psi^2(x)}{\partial q^2} = q^2 e^{q^2}, \tag{2.1.25}$$

which is the same as Eqn. (2.1.22). The solution of Eqn. (2.1.21) will contain the term $e^{\frac{-q^2}{2}}$ as a factor, the possible solution written as;

$$\Psi(x) = \chi(q)e^{\frac{-q^2}{2}}, \tag{2.1.26}$$

where $\chi(q)$ is a function of q and hence x differentiating Eqn. (2.1.26) twice with respect to q we get

$$\frac{\partial\Psi(x)}{\partial q} = -qe^{\frac{-q^2}{2}}\chi(q) + e^{\frac{-q^2}{2}}, \frac{\partial\chi(q)}{\partial q} \tag{2.1.27}$$

$$\frac{\partial^2\Psi(x)}{\partial q^2} = -e^{\frac{-q^2}{2}}[\frac{\partial^2\chi(q)}{\partial q^2} - 2q\frac{\partial\chi(q)}{\partial q} + \chi(q)(q^2-1)]\chi(q)e^{\frac{-q^2}{2}}. \tag{2.1.28}$$

By substituting value of $\Psi(x)$ and $\frac{\partial^2\Psi(x)}{\partial q^2}$ from Eqn. (2.1.26) and (2.1.28), Eqn. (2.1.21) is

$$e^{\frac{-q^2}{2}}[\frac{\partial^2\chi(q)}{\partial q^2} - 2q\frac{\partial\chi(q)}{\partial q} + \chi(q)(q^2-1)] + (\lambda - q^2)\chi(q)e^{\frac{-q^2}{2}} = 0,$$

$$\frac{\partial^2\chi(q)}{\partial q^2} - 2q\frac{\partial}{\partial q}\chi(q) + (\lambda - 1)\chi(q) = 0. \tag{2.1.29}$$

Now let us assume that the function $\chi(q)$ expressed in the form of power series in q, that is

$$\chi(q) = q^s \sum_{r=0}^{\infty} a_r q^r \equiv \sum_{r=0}^{\infty} a_r q^{r+s}, \tag{2.1.30}$$

differentiating we get

$$\frac{\partial\chi(q)}{\partial q} = \sum_r a_r (s+r) q^{r+s-1},$$

$$\frac{\partial^2\chi(q)}{\partial q^2} = \sum_r a_r (s+r)(s+r-1) q^{r+s-2}.$$

Substituting this values in Eqn. (2.1.29)

$$\sum_r a_r (s+r)(s+r-1) q^{r+s-2} - 2q \sum_r a_r (s+r) q^{r+s-1} + (\lambda - 1) \sum_r a_r q^{r+s} = 0,$$

$$\sum_r a_r (s+r)(s+r-1) q^{r+s-2} - \sum_r a_r [2(s+r) - (\lambda - 1)] q^{r+s} = 0. \tag{2.1.31}$$

This equation is the power series and it is satisfied only if coefficient of each in q must be separately equal to zero. Equating equal to zero coefficient of lowest power q (that is (q^{s-2})) we get,

$$a_{r+2}(s+r+2)(s+r+1) - a_r(2(s+r)) - (\lambda - 1) = 0,$$

$$a_{r+2} = \frac{a_r(2(s+r)) - (\lambda - 1))}{(s+r+2)(s+r+1}. \tag{2.1.32}$$

If we put numerator in recursion formula equal to zero.

$$2s + 2r + 1 - \lambda = 0, \quad \lambda = 2s + 2r + 1. \tag{2.1.33}$$

This equation may be written more generally as

$$\lambda = 2n + 1, \tag{2.1.34}$$

where $n = s + r$. Substituting this value of λ in Eqn. (2.1.16)

$$2\varepsilon\sqrt{\frac{m}{k\hbar^2}} = 2n + 1, \tag{2.1.35}$$

$$\varepsilon = \hbar\omega_c(n + \frac{1}{2}),$$

$$E - \frac{p_z^2}{2m} = \hbar\omega_c(n + \frac{1}{2}). \tag{2.1.36}$$

The quantized energy of the eigen value of free electron Hamilton in uniform magnetic field ($\mathbf{B}$) to the z-direction (Landau level) is obtained as a solution of the the above equation as

$$E = \hbar\omega_c(n + \frac{1}{2}) + \frac{\hat{p}_z^2}{2m} \quad (n = 0, 1, 2, 3, ...), \tag{2.1.37}$$

where $\hat{p}_z = \hbar k_z$. The above equation (2.1.37) are called Landau Energy Levels. Since they are independent of k_y , they have a degeneracy equal to the number of allowed values of k_y, such that x_0 lies within the constraint of the system. We have seen that in order to have satisfactory solutions of wave Eqn. (2.1.14), χ must break off after a finite number of terms that is χ should be restricted in such a manner so as to make it polynomial rather than a power of series. Then the eigen function Ψ can be set equal to product of this polynomial and the factor $e^{\frac{-q^2}{2}}$ is

$$\Psi_n(q) = N_n e^{\frac{-q^2}{2}} H_n(q), \tag{2.1.38}$$

where N_n is the normalization factor and $H_n(q)$ is the Hermite polynomial of degree n defined by

$$H_n(q) = (-1)^n e^{q^2} \frac{d^n}{dq^n} e^{-q^2}. \tag{2.1.39}$$

The orthogonality condition of Hermite polynomial is

$$\int_{-\infty}^{\infty} H_n(q) H_m(q) e^{-q^2} dX = 2^n n! \sqrt{\pi} \delta_{mn}. \tag{2.1.40}$$

Normalization condition is

$$\int_{-\infty}^{\infty} |\Psi(x)|^2 \, dX = 1, \tag{2.1.41}$$

$$\frac{N_n^2}{\alpha} \int_{-\infty}^{\infty} e^{-q^2} (H_n(q))^2 dq = 1. \tag{2.1.42}$$

Using orthogonality condition of hermite polynomial;

$$N_n = (\frac{\alpha}{2^n n! \sqrt{\pi}})^{\frac{1}{2}}. \tag{2.1.43}$$

Thus the corresponding wave function is

$$\Psi(x, y, z) = (\frac{m\omega_c}{\pi\hbar})^{1/4} (\frac{1}{2^n n!})^{\frac{1}{2}} \exp(i(k_y y + k_z z)) \exp(\frac{m\omega_c}{2\hbar}(x - x_0)^2) H_n[\sqrt{\frac{m\omega_c}{\hbar}}(x - x_0)]. \tag{2.1.44}$$

In Eqn. (2.1.44) the wave function , $\Psi(x, y, z)$, depend on the parameters n, k_y, k_z, where as in Eqn. (2.1.37) the Landau Level energy, E, depends only on the parameters n, p_z. This implies degeneracy.

2.1.1 Degeneracy of Landau level

The Landau levels are states with different values of n. The lowest Landau level has $n = 0$. The eigen value of Landau levels does not depend upon quantum number k_y. The state has a high degeneracy, in that many different value of k_y have the same eigenvalue. Let us put the system in a large cube of size L and impose periodic boundary conditions. The allowed values of k_y are of the form $\frac{2\pi}{L}n_y$, where $n_y = 0,1,2,3....$ The constant x_0 is the center of cyclotron orbit. The orbit center must be in the area, which provides the constraint [14].

$$0 \leq x_o = \frac{\hbar k_y}{eB} \leq L_x, \tag{2.1.45}$$

$$0 \leq \frac{\hbar k_y}{eB} = (\frac{\hbar}{eB})\frac{2\pi n_y}{L_y} \leq L_x,$$

For n_{max}, we have,

$$(\frac{\hbar}{eB})\frac{2\pi n_{max}}{L_y} = L_x,$$

$$n_{max} = \frac{eBL_xL_y}{\hbar 2\pi}. \tag{2.1.46}$$

At high magnetic field, we have maximum degeneracy which is

$$n_{max} = \frac{\phi}{\phi_o} \equiv g(B), \tag{2.1.47}$$

where $\phi = BA$ is the maximum magnetic flux, $\phi_o = \frac{h}{e}$ is the quanta of elementary flux and $g(B)$ is the degeneracy of Landau level.

From Eqn. (2.1.47) follows the magnetic flux is quantized and can not be smaller than ϕ_o. Degeneracy of the Landau Levels (LLs), $g(B)$, depends on the external magnetic field, $\vec{B}$, in high magnetic fields LLs are highly degenerate and degeneracy can be even larger than a number of electrons in the sample. An important concept is also the filling ratio ν. It is equal to the number of filled Landau levels

$$\nu = \frac{N_e}{N_\phi} = \frac{n_s h}{eB}, \tag{2.1.48}$$

where n_s is the surface charge density of number per unit area, $N_\phi(= g(B)/A)$ is the degeneracy of the Landau levels per unit area, A and $N_e(= n_s A)$ is the total number of electrons .

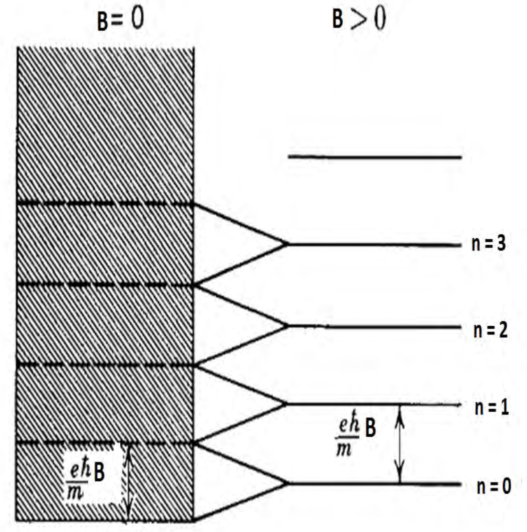

Figure 2.1: Comparison of energy spectra of a charged particle with and without magnetic field [10].

2.2 Integral quantized Hall resistance

The classical result of the Hall resistivity is $\rho_{xy} = \frac{B}{en_s} = \rho_H$. However, the density number of the current carrying states in each Landau level is $N_\phi = \frac{g(B)}{A} = \frac{eB}{h}$. Therefore, there are ν Landau levels at energy below the Fermi energy completely filled with $n_s = \nu N_\phi = \frac{\nu eB}{h}$ electrons. The Hall resistivity using filling factor can be obtained.

$$\rho_H = \frac{B}{\nu(\frac{eB}{h})e} = \frac{h}{\nu e^2} \quad (\nu = 1, 2, 3...). \tag{2.2.1}$$

The value of the Hall resistivity Eqn. (2.2.1) only depends on the fundamental constants of physics: h is plank's constant. The quantum Hall effect observed in two dimensional electron systems subjected to low temperatures(< 1K) and strong magnetic fields, in which value of the Hall resistivity is quantized in units of $\frac{h}{e^2}(= 25812.807\Omega)$ divided by consecutive integers [3]. This is the integral quantum Hall effect for an inversion layer in Si-MOSFET. The quantization of Hall resistance now used world wide in national metrology institutes as a primary standard of resistance. The experimental Hall resistance exhibits several plateaus. Figure 2.2 shows the integral quantum Hall effect in GaAs/GaAlAs heterojunction recorded at 30mK [9].

It also included in longitudinal resistivity. The experiments show that between two adjacent Landau levels the Hall resistance has fixed values (plateau) and the longitudinal

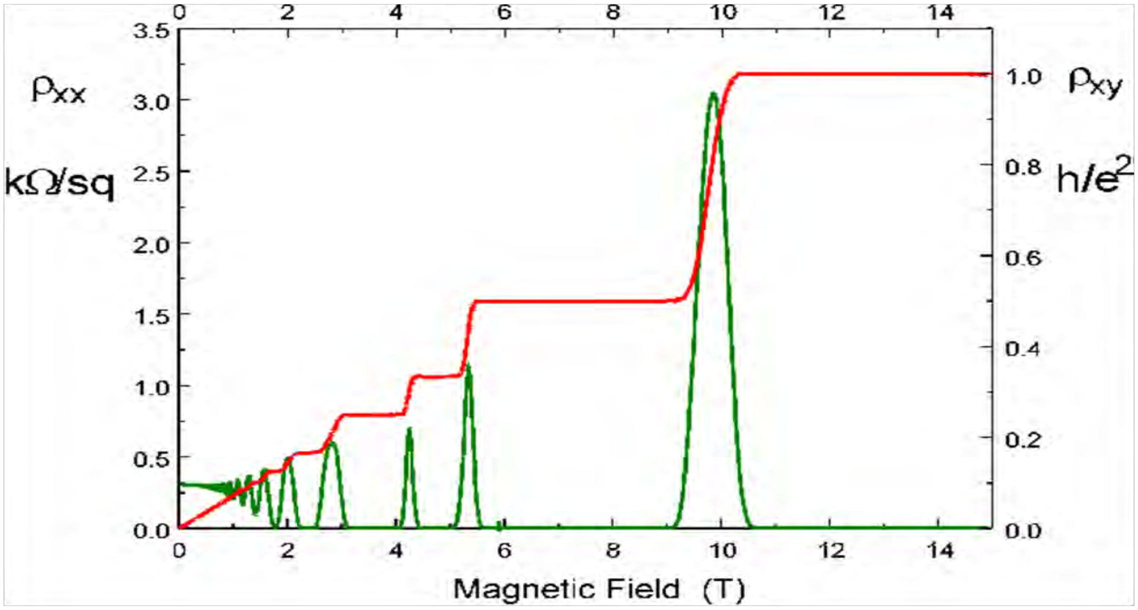

Figure 2.2: Hall resistance and longitudinal resistivity data as a function of the magnetic field for a GaAs/AlGaAs heterostructure at 30mK [9].

resistivity vanishes. Furthermore, corresponding to sharp rises in between quantized Hall plateau, there are sharp picks in the longitudinal resistivity. The zeros and plateaus in the two components of the resistivity tensor are intimately connected and can be understood in terms of Landau levels formed in the magnetic field. In the absence of magnetic field the density of states in 2D is constant as a function of energy, but in the applied magnetic field, the available states clump in to Landau levels (Fig. 2.1) separated by cyclotron energy, with region of energy between Landau levels where there are no allowed states. As the magnetic field swept the LLs move relative to the Fermi energy. When the Fermi energy lies in a gap between LLs electrons can not move to a new states and so there is no scattering. Thus the transport is dissipation less and the resistance falls to zero. Where as the Hall resistance can not change from the quantized value for the whole time the Fermi energy is in the gap between LLs , and so plateau results. This picture has assumed a fixed Fermi energy, i.e fixed carrier density, and changing magnetic field.

2.3 Fractional quantized Hall resistance

At sufficiently high magnetic fields, where the lowest landau level is occupied, the integral quantum Hall effect does vanish in this regime. However the Hall resistivity is found still

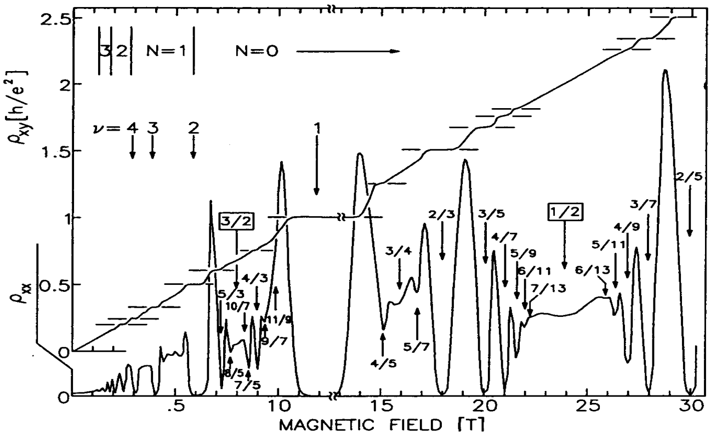

Figure 2.3: The figure shows the stepwise behavior of the transverse resistivity, superimposed with the longitudinal resistance, as a function of magnetic field. The same behavior as in figure 2.2 is seen except now at fractions of ν [15].

to be quantized but now the quantization factor is not integer but fraction, $\nu = n/m$, where $n(< m)$ and m are integers. This is the fractional quantum Hall effect.

$$\rho_H = \frac{B}{\nu(\frac{eB}{h})e} = \frac{h}{\nu e^2} \quad (\nu = 1/3, 2/3, 2/5...). \tag{2.3.1}$$

The quantized Hall resistance is occurred only at odd denominator of the filling factor [11]. Laughlin(1982) has formulated a theory for the fractional quantum Hall effect where in, because of electron-electron interactions, the 2D electron gas becomes an incompressible quantum fluid in which, for a fractional Landau level filling factor, say $\nu = 1/m$, the quasiparticle excitation have a charge Q that is the fractions of electronic charge i.e. $Q = e/m$.

2.3.1 Wave function for fractional quantum Hall effect

When the fractional quantum Hall effect was discovered, Laughlin realized that one could explain a many-body variational wave function at filling factor $\nu = 1/m$[16]. The free-electron Hamiltonian for the symmetric guage the third part of Eqn. (2.1.6) in the field

$B\hat{z}$ has the simple form:

$$\hat{H} = \frac{1}{2m}(\hat{p} - e\mathbf{A})^2$$

$$\hat{H} = \frac{1}{2m}[(\hat{p} + \frac{y}{2}eB)^2 + (\hat{p} - \frac{x}{2}eB)^2]. \tag{2.3.2}$$

The space variables are put in the dimensionless using $x = x'/l_B$, $y = y'/l_B$. First let us take the hamiltonian with prime variables,

$$\hat{H} = \frac{1}{2m}[(\frac{\hbar}{i}\frac{\partial}{\partial x'} + \frac{y^{'}}{2}eB)^2 + (\frac{\hbar}{i}\frac{\partial}{\partial y'} - \frac{x^{'}}{2}eB)^2], \tag{2.3.3}$$

To drop the primes we substitute $x^{'} = xl_B, y^{'} = yl_B, \partial x^{'} = l_B\partial x, \partial y^{'} = l_B\partial y$

$$\begin{aligned}
\hat{H} &= \frac{1}{2m}[(\frac{\hbar}{i}\frac{\partial}{l_B\partial x} + l_B\frac{y}{2}eB)^2 + (\frac{\hbar}{i}\frac{\partial}{l_B\partial y} - l_B\frac{x}{2}eB)^2], \\
&= \frac{1}{2m}[\frac{\hbar}{i}\sqrt{\frac{eB}{\hbar}}\frac{\partial}{\partial x} + eB\sqrt{\frac{\hbar}{eB}}\frac{y}{2}]^2 + \frac{1}{2m}[\frac{\hbar}{i}\sqrt{\frac{eB}{\hbar}}\frac{\partial}{\partial y} - eB\sqrt{\frac{\hbar}{eB}}\frac{x}{2}]^2 \\
&= \frac{1}{2m}[\sqrt{\hbar eB}\frac{1}{i}\frac{\partial}{\partial x} + \sqrt{eB\hbar}\frac{y}{2}]^2 + \frac{1}{2m}[\sqrt{eB\hbar}\frac{1}{i}\frac{\partial}{\partial y} - \sqrt{eB\hbar}\frac{x}{2}]^2.
\end{aligned}$$

And the hamiltonian is rewritten as

$$H = \frac{\hbar\omega_c}{2}[(\frac{1}{i}\frac{\partial}{\partial x} + \frac{y}{2})^2 + (\frac{1}{i}\frac{\partial}{\partial y} - \frac{x}{2})^2], \tag{2.3.4}$$

$$\begin{aligned}
H &= \frac{\hbar\omega_c}{2}[-\frac{\partial^2}{\partial x^2} + \frac{1}{i}y\frac{\partial}{\partial x} + \frac{y^2}{4} - \frac{\partial^2}{\partial y^2} - \frac{1}{i}x\frac{\partial}{\partial y} + \frac{x^2}{4}], \\
&= \frac{\hbar\omega_c}{2}[-(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) + \frac{1}{4}(x^2 + y^2) + i(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x})],
\end{aligned}$$

$$H = \frac{\hbar\omega_c}{2}[-\nabla^2 + \frac{1}{4}\rho^2 - L_z]. \tag{2.3.5}$$

Where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, $L_z = i(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}) = -i\frac{\partial}{\partial_\phi}$, $\tan(\phi) = y/x$ and $\rho^2 = x^2 + y^2$ in polar coordinates. Consider the eigen function $\Psi_m(\rho, \phi)$

$$\Psi_m(\rho, \phi) = N_m\rho^m e^{im\phi}e^{-\rho^2/4}, \tag{2.3.6}$$

where $N_m = \frac{1}{\sqrt{2\pi!2^m}}$ is normalzation constant, $m \geq 0$ is an integer
In FQHE the high value of magnetic field at low temperature ensures that $\hbar\omega_c \gg k_BT$. All electrons in the lowest landau level for the case that the fractional filling is $\nu < 1$. To construct a many electron wave function, it is useful describing the position vector (x, y) by the complex number $z = x + iy = \rho e^{i\phi}$. The eigen function is now written as

$$\Psi_m = N_m z^m e^{-|z|^2/4}, \tag{2.3.7}$$

where $z = (x + iy)/l_B$ is a dimensionless complex number representing the position vector $\mathbf{r} \equiv (x, y)$. Here we replace $N_m z^m$ as some function $f(z)$
By restricting our discussion to extreme quantum limit in which the landau level degeneracy is large enough that electrons can be accommodated within the lowest Landau level, the Laughlin's wave functions in the lowest Landau level can be written as [16].

$$\Psi_m(z) = f(z)\exp(\frac{|z_i|^2}{4}), \tag{2.3.8}$$

where f is a polynomial representing the Slater determinant with all states occupied. The FQHE is similar to the IQHE, except that it occurs at fraction values of ν, such that 1/3, 2/3, 2/5. Electron-electron interaction plays an essential roll in the FQHE. In a strong magnetic field such that all electrons accommodated in the lowest Landau levels, the kinetic energy of the electrons is quenched so that electron-electron interaction becomes important.
If the degeneracy is for N non-interacting electron in all the LLL has the form

$$\Psi_m(z_1, z_2 z_3, ..., z) = f(z_1, z_2 z_3, ..., z)\exp(\frac{-\sum_i |z_i|^2}{4}). \tag{2.3.9}$$

It was observed $f(z)$ is odd power of z.

$$f[z] = [z]^m, \tag{2.3.10}$$

$[z]$ stands for $(z_1, z_2, z_3, ..., z_N)$
If we consider the possible order of two particles. We get one particle Ψ_o and the other Ψ_1.

$$f(z_1, z_2) = det\begin{bmatrix} z_1^0 & z_2^0 \\ z_1^1 & z_2^1 \end{bmatrix} = z_2 - z_1. \tag{2.3.11}$$

This is the lowest possible order for two particles. For the case of three particles we have,

$$\begin{aligned} f(z_1, z_2, z_3) &= det\begin{bmatrix} z_1^0 & z_2^0 & z_3^0 \\ z_1^1 & z_2^1 & z_3^1 \\ z_1^2 & z_2^2 & z_3^2 \end{bmatrix}, \\ &= z_2 z_3^2 - z_2^2 z_3 - z_1 z_3^2 + z_1^2 z_3 + z_1 z_2^2 - z_1^2 z_2, \\ &= -(z_1 - z_2)(z_1 - z_3)(z_2 - z_3), \end{aligned}$$

$$= -\prod_{i<j}^{3}(z_i - z_j). \tag{2.3.12}$$

This form of the Slater determinant is known as the Vandermonde polynomial [17]. To simplify the positions of the particles, we will drop the overall minus sign. The single Slater determinant to fill the first N angular momentum states is a generalization of Eqn. (2.3.12)

$$f[z] = \prod_{i<j}^{N} (z_i - z_j). \tag{2.3.13}$$

Since the argument of Eqn. (2.3.10) is accepted, Eqn. (2.3.13) has the form

$$f[z] = \prod_{i<j}^{N} (z_i - z_j)^m. \tag{2.3.14}$$

The general Laughlin function for the ground state wave function (GSWF) of FQHE is

$$\Psi_{\frac{1}{m}}[z] = \prod_{i<j}^{N} (z_i - z_j)^m \exp(\frac{-\sum_k |z_k|^2}{4}). \tag{2.3.15}$$

From this function Laughlin get many electron wave function for $\nu = 1$ as

$$\Psi_{\nu=1}[z] = \prod_{i<j}^{N} (z_i - z_j) \exp(\frac{-\sum_k |z_k|^2}{4}). \tag{2.3.16}$$

The polynomial term tries to keep the particles away from each other and gets larger as the particles spread out. And also the exponential term is small if the particles spread

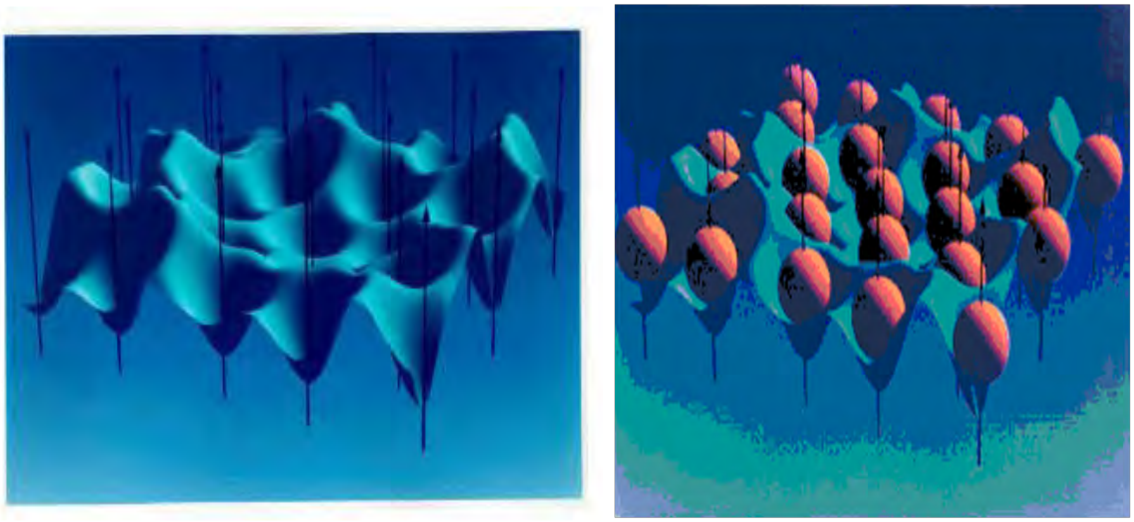

Figure 2.4: (Left) single electron in the lowest Landau Level (Right) Filled lowest Landau levels

out too much [18]. The probability distribution function in the LLL is

$$|\Psi_{\nu=1}[z]|^2 = \prod_{i<j}^{N}(z_i - z_j)^2 \exp(\frac{-\sum_k |z_k|^2}{2}). \tag{2.3.17}$$

We know that from classical statistical mechanics, the partition function Z is given by the norm of wave function [19,20]

$$Z = \int \prod_i d^2 z_i \, |\Psi_{\nu=1}[z]|^2 \,, \tag{2.3.18}$$

and

$$|\Psi_{\nu=1}[z]|^2 = \exp(-\beta U), \tag{2.3.19}$$

Since this is a formal analogy, the inverse temperature β which appears here is arbitrary. we choose here $\beta = 1/m$, so that we will get the potential energy U of the particle by using Eqn. (5.3.17) and (5.3.19)

$$\exp(-\beta U) = \prod_{i<j}^{N}(z_i - z_j)^2 \exp(\frac{-\sum_k |z_k|^2}{2}),$$

$$\ln[\exp(-\beta U)] = \ln[\prod_{i<j}^{N}(z_i - z_j)^2 \exp(\frac{-\sum_k |z_k|^2}{2})],$$

$$-\beta U = 2\sum_{i<i}^{N}(ln\,|z_i - z_j| - \frac{1}{2}\sum_k |z_k|^2),$$

$$U[z] = -2m(\sum_{i<j}^{N} ln\,|z_i - z_j| + \frac{m}{2}\sum_k |z_k|^2). \tag{2.3.20}$$

Laughlin remarked that U is the potential energy of a 2D classical one component gas of interacting particles with charge m in a uniform neutralizing background. In the equivalent classical problem, particles have logarithmic interactions, which are 2D Coulomb interactions.

Chapter 3

Graphene

3.1 Inroduction

Graphene is the name given to a flat monolayer of carbon atoms tightly packed into a two-dimensional (2D) honeycomb lattice, with a carbon-carbon distance of $1.42\mathring{A}$ [21]. The fact that charge carriers in graphene are described by a Dirac-like spectrum, rather than the usual Schrodinger equation for non-relativistic quantum particles, can be seen as a consequence of graphenes crystal structure. This consists of two equivalent carbon sublattices A and B (see Fig. 3.1).

The hexagonal arrangement of carbon atoms in graphene and the corresponding hexagonal Brillouin zone are shown in Figure 3.1.

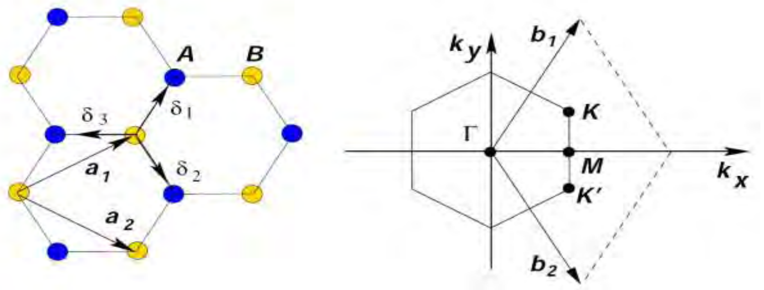

Figure 3.1: Honeycomb lattice and its Brillouin zone of graphene.

3.2 Properties of graphene

Graphene has number of properties which makes it for several different applications. It is an ultimately thin, mechanically very strong, transparent and flexible conductor. Carbon is arguably one of the most flexible elements in the whole periodic table. It forms many allotropes, some known are fullerenes (0D) or buckyballs, carbon nanotubes (1D), and graphite (3D). Interestingly, the two-dimensional form graphene (2D) was only obtained very recently, immediately attracting a great deal of attention. All these materials can be characterized by their dimensionality and the nature of the chemical bonding that keeps the atoms together [22]. Carbon nanotubes is rolling up direction of graphene. Graphite is essentially a stack of graphene layers (top right of Fig.3.2) whose properties depend on the stacking order.

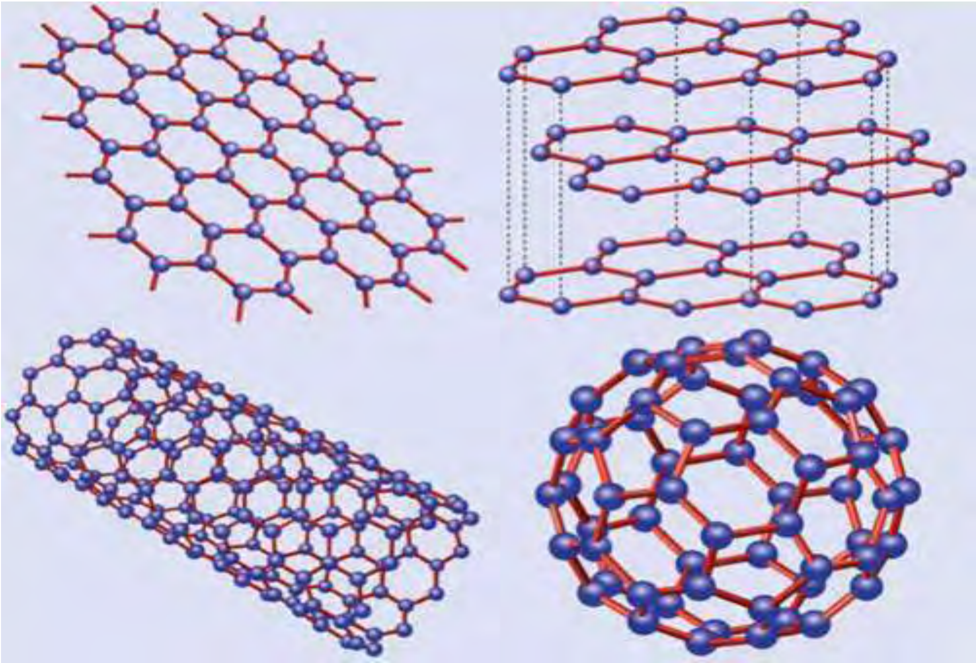

Figure 3.2: Graphene (top left) is a honeycomb lattice of carbon atoms. Graphite (top right) can be viewed as a stack of graphene layers. Carbon nanotubes are rolled-up cylinders of graphene (bottom left). Fullerenes (c_{60}) are molecules consisting of wrapped graphene by the introduction of pentagons on the hexagonal lattice (bottom right) [23].

3.3 Tight binding model of graphene

Tight binding is an approach to the calculation of electronic band structure using an approximate set of wave functions for isolated atoms located at each atomic site. The wave functions of the lattice, Ψ, should satisfy Bloch's function theorem [24].

$$T_{a_i}\Psi = e^{i\vec{k}.a_i}\Psi, \tag{3.3.1}$$

where T_{a_i} is the translational operator along the lattice vector a_i, and $\vec{k}$ is a wave vector. Another functional form which satisfies Eq. (3.3.1) is based on the j^{th} atomic orbital in the unit cell. A tight binding, Bloch function $\phi_j(\vec{k},\vec{r})$ (label j=1 is for A site and j=2 is for B site) is given by [24],

$$\phi_j(\vec{k},\vec{r}) = \frac{1}{\sqrt{N}}\sum_{\vec{R}_j}^{N}\Psi e^{i\vec{k}.a_i}\varphi_j(\vec{r}-\vec{R}) \quad (j=1,2), \tag{3.3.2}$$

where, $\vec{R}_j$ is the position vector of atom in state j, $\sum_{R_j}$ is sum over all type j atomic sites, $\varphi_j(\vec{r})$ is the atomic wave function in state j. The wave functions in solid $\Psi_j(\vec{k},\vec{r})$ are expressed by linear combination of Bloch functions $\phi_{j'}(\vec{k},\vec{r})$ as follow

$$\begin{gathered}\Psi_j(\vec{k},\vec{r}) = \sum_{j'=1}^{2} c_{jj'}(\vec{k})\phi_{j'}(\vec{k},\vec{r}),\\ \Psi_j(\vec{k},\vec{r}) = c_{jA}(\vec{k})\phi_A(\vec{k},\vec{r}) + c_{jB}(\vec{k})\phi_B(\vec{k},\vec{r}),\end{gathered} \tag{3.3.3}$$

where $c_{jj'}$ is coefficient to be determined.
Schrodinger equation is

$$H\Psi_j(\vec{k},\vec{r}) = E_j\Psi_j(\vec{k},\vec{r}), \tag{3.3.4}$$

with eigen value E_j is written as

$$\begin{aligned}<\Psi_j \mid H \mid \Psi_j> &= <\Psi_j \mid E_j \mid \Psi_j>,\\ E_j(k) &= \frac{<\Psi_j \mid H \mid \Psi_j>}{<\Psi_j \mid \Psi_j>},\\ E_j(k) &= \frac{\int \Psi_j^* \mid H \mid \Psi_j dr}{\int \Psi_j^*\Psi_j dr}.\end{aligned} \tag{3.3.5}$$

In terms of Bloch functions by substituting the first of Eqn. (3.3.3) into the second of Eqn. (3.3.5) and making change of subscripts, we will obtain the following equation.

$$\begin{aligned}E_j(k) &= \frac{\sum_{i,l} c_{ji}^* c_{jl} <\Psi_j \mid H \mid \Psi_j>}{\sum_{i,l} c_{ji}^* c_{jl} <\Psi_j \mid \Psi_j>},\\ E_j(k) &= \frac{\sum_{i,l} c_{ji}^* c_{jl} H_{il}}{\sum_{i,l} c_{ji}^* c_{jl} S_{il}},\end{aligned} \tag{3.3.6}$$

where

$$H_{il} =< \Psi_j \mid H \mid \Psi_j >, \qquad S_{il} =< \Psi_j \mid \Psi_j > \tag{3.3.7}$$

It is called transfer matrix elements and overlap matrix elements, respectively. If the H_{il} and S_{il} are known, the partial derivative of energy with respect to c^*_{ji} while fixing the other coefficients, we obtain zero for the local minimum condition as follow,

$$\frac{\partial E_j(k)}{\partial c^*_{ji}} = 0,$$

$$\frac{(\sum_{i,l} c^*_{ji}c_{jl}H_{il})'(\sum_{i,l} c^*_{ji}c_{jl}S_{il}) - (\sum_{i,l} c^*_{ji}c_{jl}H_{il})(\sum_{i,l} c^*_{ji}c_{jl}S_{il})'}{(\sum_{i,l} c^*_{ji}c_{jl}S_{il})^2} = 0,$$

$$\frac{\sum_{i,l} c_{jl}H_{il}}{\sum_{i,l} c^*_{ji}c_{jl}S_{il}} - \frac{(\sum_{i,l} c^*_{ji}c_{jl}H_{il})(\sum_{i,l} c_{jl}S_{il})}{(\sum_{i,l} c^*_{ji}c_{jl}S_{il})^2} = 0. \tag{3.3.8}$$

When we multiply both sides of Eqn. (3.3.8) by $\sum_{i,l} c^*_{ji}c_{jl}S_{il}$ and substituting $E_j(k)$ the second of Eqn. (3.3.6) into second term of Eqn. (3.3.8), we obtain;

$$\frac{(\sum_{i,l} c^*_{ji}c_{jl}S_{il})(\sum_{i,l} c_{jl}H_{il})}{\sum_{i,l} c^*_{ji}c_{jl}S_{il}} - \frac{(\sum_{i,l} c^*_{ji}c_{jl}S_{il})(\sum_{i,l} c^*_{ji}c_{jl}H_{il})(\sum_{i,l} c_{jl}S_{il})}{(\sum_{i,l} c^*_{ji}c_{jl}S_{il})^2} = 0,$$

$$\sum_{i,l} c_{jl}H_{il} - \frac{(\sum_{i,l} c^*_{ji}c_{jl}H_{il})(\sum_{i,l} c_{jl}S_{il})}{\sum_{i,l} c^*_{ji}c_{jl}S_{il})} = 0,$$

$$\sum_{l=1}^{2} c_{jl}H_{il} - E_j(k)\sum_{l=1}^{2} c_{jl}S_{il} = 0,$$

$$\sum_{l=1}^{2} c_{jl}H_{il} = E_j(k)\sum_{l=1}^{2} c_{jl}S_{il}. \tag{3.3.9}$$

Defining the coefficient c_{jl} as a matrix equation $c_j = (c_{jA}, c_{jB})$

$$c_j H = E_j(k)c_j S,$$

$$(H - E_j(k)S)c_j = 0 \tag{3.3.10}$$

If the inverse matrix of Eqn.(3.3.10) exists, we can multiply both sides by $\frac{1}{H-E_j(k)S}$ to obtain c_j=0 (where zero denotes the null vector), which wave function is obtained [25]. The eigen function is given only when the inverse matrix doen't exist, consistent with the condition given by

$$det(H - E_j(k)S)c_j = 0. \tag{3.3.11}$$

This is called secular equation, and it is used to determine energy dispersion $E_j(k)$, which will be calculated in the following section.

3.4 Calculation of transfer and overlap integrals of hexagonal lattice (H,S)

The tight binding approximation in Bloch functions of the diagonal matrix elements are

$$\begin{aligned} H_{AA} &=< \phi_A \mid H \mid \phi_A >, \\ &= \int \phi_A^*(H)\phi_A dr \end{aligned} \tag{3.4.1}$$

But,ϕ_A^* and ϕ_A is given by

$$\begin{aligned} \phi_A^* &= \frac{1}{\sqrt{N}} \sum_{R_{A'}} e^{-i(R_{A'})} \varphi_A(r - R_{A'}) \\ \phi_A &= \frac{1}{\sqrt{N}} \sum_{R_A} e^{i(R_A)} \varphi_A(r - R_A). \end{aligned} \tag{3.4.2}$$

Therefore, Eqn. (3.4.1) is

$$H_{AA} = \frac{1}{N} \sum_{R_{AA'}} e^{ik(R_A - R_{A'})} < \varphi_A(r - R_{A'}) \mid H \mid \varphi_A(r - R_{A'}) >,$$

$$H_{AA} = \frac{1}{N} \sum_{R_A = R_{A'}} \epsilon_{2p} + \frac{1}{N} \sum_{R_A = R_{A'} \pm a} e^{\pm ika} < \varphi_A(r - R_{A'}) \mid H \mid \varphi_A(r - R_{A'}) > +.... \tag{3.4.3}$$

After the order contribution to H_{AA} at $R_A = R_{A'} \pm a$ is neglected for simplicity

$$H_{AA} = \frac{1}{N} \sum_{R_A = R_{A'}} \epsilon_{2p} = \epsilon_{2p} \tag{3.4.4}$$

This is the orbital energy of $2p$ level. Since H_{AA} is real, we have,

$$H_{AA} = H_{BB} = \epsilon_{2p}. \tag{3.4.5}$$

If we assume the atomic wave function is normalized,

$$S_{AA} = S_{BB} = 1. \tag{3.4.6}$$

The tight binding approximation in Bloch functions of the off-diagonal matrix elements are

$$\begin{aligned} \phi_A &= \frac{1}{\sqrt{N}} \sum_{R_A} e^{ikR_A} \varphi_A(r - R_A), \\ \phi_B &= \frac{1}{\sqrt{N}} \sum_{R_B} e^{ikR_B} \varphi_B(r - R_B) \end{aligned} \tag{3.4.7}$$

The off-diagonal matrix is

$$H_{AB} =< \phi_A \mid H \mid \phi_B > \tag{3.4.8}$$

$$\begin{aligned} H_{AB} &= \frac{1}{N} < \sum_{R_A} e^{ikR_A} \varphi_A(r - R_A) \mid H \mid \sum_{R_B} e^{ikR_B} \varphi_B(r - R_B) >, \\ &= \frac{1}{N} \sum_{R_A} \sum_{R_B} e^{ik(R_B - R_A)} < \varphi_A(r - R_A) \mid H \mid \varphi_B(r - R_B) >, \\ &= \frac{1}{N} < \sum_{R_A - R} e^{-ik\frac{a}{2}} < \varphi_A(r - R - a/2) \mid H \mid \sum_{R_B - R} e^{ik\frac{a}{2}} \varphi_B(r - R + a/2) > \\ &= \frac{1}{N} (e^{-ik\frac{a}{2}} + e^{ik\frac{a}{2}}) < \varphi_A(r - R) \mid H \mid \varphi_B(r - R \pm a/2) >, \\ &= \frac{1}{N} 2\cos(k\frac{a}{2}) t, \end{aligned}$$

$$H_{AB} = 2t\cos(k\frac{a}{2}), \tag{3.4.9}$$

where,

$$t =< \varphi_A(r - R) \mid H \mid \varphi_B(r - R \pm a/2). \tag{3.4.10}$$

is hopping parameter for off-diagonal matrix.

The matrix element H_{BA} is obtained from H_{AB} through the hermitian conjugation relation H_{BA}=H^*_{AB}. But H_{AB} is real;

$$H_{AB} = H_{BA} = 2t\cos(k\frac{a}{2}). \tag{3.4.11}$$

The overlap matrix S_{ij} can be calculated by similar method H_{ij}.

$$S_{AB} = S_{BA} = 2s\cos(k\frac{a}{2}) \tag{3.4.12}$$

where s is the hopping parameter for diagonal matrix given by

$$s =< \varphi_A(r - R) \mid \varphi_B(r - R \pm a/2) \tag{3.4.13}$$

3.5 Electronic band structure of graphene

The electronic structure of graphene has two atoms per unit cell as shown in Fig. (3.1). The lattice vectors in Figure 3.1 (left) $\vec{a_1}$ and $\vec{a_2}$, are given by

$$\vec{a}_1 = \frac{3a}{2}\hat{x} + \frac{\sqrt{3}a}{2}\hat{y}, \qquad \vec{a}_2 = \frac{3a}{2}\hat{x} - \frac{\sqrt{3}a}{2}\hat{y}. \tag{3.5.1}$$

The reciprocal lattice vectors $\vec{b_1}$ and $\vec{b_2}$, indicated in figure 3.1 (right) are given by

$$\vec{b_1} = \frac{2\pi}{3a}\hat{k}_x + \frac{2\sqrt{3}}{3a}\hat{k}_y, \qquad \vec{b_2} = \frac{2\pi}{3a}\hat{k}_x - \frac{2\sqrt{3}}{3a}\hat{k}_y. \tag{3.5.2}$$

The three nearest-neighbor in real space are given by

$$\delta_1 = \frac{a}{2}\hat{x} + \frac{\sqrt{3}a}{2}\hat{y}, \quad \delta_2 = \frac{a}{2}\hat{x} - \frac{\sqrt{3}a}{2}\hat{y}, \quad \delta_3 = a\hat{x}. \tag{3.5.3}$$

The Dirac points in graphene are K and K' at the corners of Brillouin zone. Their position in momentum space is given by

$$K = \frac{2\pi}{3a}\hat{x} + \frac{2\pi}{3\sqrt{3}a}\hat{y}, \qquad K' = \frac{2\pi}{3a}\hat{x} - \frac{2\pi}{3\sqrt{3}a}\hat{y}. \tag{3.5.4}$$

To calculate the energy dispersion relation of graphene we must consider the nearest neighbor [26] B atoms related to A atom, which is given by Eqn. (3.5.3), then the Hamiltonian becomes

$$H_{AB} = t\sum_{i=1}^{3} e^{ik\delta_i} = tf(k), \tag{3.5.5}$$

where $f(k)$ is

$$\begin{aligned} f(k) &= \sum_{i=1}^{3} e^{ik\delta_i} \\ &= e^{ik\delta_1} + e^{ik\delta_2} + e^{ik\delta_3} \\ &= e^{ik_x\frac{1}{2}a}e^{ik_y\frac{\sqrt{3}}{2}a} + e^{ik_x\frac{1}{2}a}e^{-ik_y\frac{\sqrt{3}}{2}a} + e^{-ik_xa} \\ &= e^{ik_x\frac{1}{2}a}\left[e^{ik_y\frac{\sqrt{3}}{2}a} + e^{-ik_y\frac{\sqrt{3}}{2}k_ya}\right] + e^{-ik_xa} \\ &= e^{-ik_xa}2\cos(\frac{\sqrt{3}}{2}k_ya) + e^{-ik_xa} \end{aligned}$$

$$f(k) = e^{-ik_xa}\left[e^{ik_x\frac{3}{2}}2\cos(\frac{\sqrt{3}}{2}k_ya) + 1\right]. \tag{3.5.6}$$

Since $f(k)$ is the complex function, Hamiltonian forms hermitian matrix, $H_{AB} = H^*_{BA}$, and similarly the overlap matrix is

$$S_{AB} = sf(k). \tag{3.5.7}$$

The explicit forms of H and S can be written as

$$H = \begin{bmatrix} H_{AA} & H_{AB} \\ H^*_{BA} & H_{BB} \end{bmatrix}, \qquad S = \begin{bmatrix} S_{AA} & S_{AB} \\ S^*_{BA} & S_{BB} \end{bmatrix}. \tag{3.5.8}$$

Substitute the value of each matrix; for diagonal matrix Eqn. (3.4.5)and Eqn. (3.4.6), for off-diagonal matrix Eqn. (3.5.5) and Eqn. (3.5.7) are

$$H = \begin{bmatrix} \epsilon_{2p} & tf(k) \\ tf^*(k) & \epsilon_{2p} \end{bmatrix}, \quad S = \begin{bmatrix} 1 & sf(k) \\ sf^*(k) & 1 \end{bmatrix}. \tag{3.5.9}$$

Now to calculate the energy dispersion relation of band structure, we have to use the secular equation we have shown in section (3.3) as Eqn. (3.3.11)

$$\left| \begin{bmatrix} \epsilon_{2p} & tf(k) \\ tf^*(k) & \epsilon_{2p} \end{bmatrix} - \begin{bmatrix} E(k) & E(k)sf(k) \\ E(k)sf^*(k) & E(k) \end{bmatrix} \right| = 0, \tag{3.5.10}$$

performing the necessary calculations,

$$\left| \begin{bmatrix} \epsilon_{2p} - E(k) & tf(k) - E(k)sf(k) \\ tf^*(k) - E(k)sf^*(k) & \epsilon_{2p} - E(k) \end{bmatrix} \right| = 0 \tag{3.5.11}$$

$$(\epsilon_{2p} - E(k))^2 - [(tf^*(k) - E(k)sf^*(k))(tf(k) - E(k)sf(k))] = 0 \tag{3.5.12}$$

$$\begin{aligned} (\epsilon_{2p} - E(k))^2 &= [(tf^*(k) - E(k)sf^*(k))(tf(k) - E(k)sf(k))] \\ \epsilon_{2p} - E(k) &= t^2 \left|f(k)\right|^2 - 2tE(k)s\left|f(k)\right|^2 + E^2(k)s^2 f(k)\left|f(k)\right|^2 \\ \epsilon_{2p} - E(k) &= \pm\sqrt{t^2 \left|f^{(k)}\right|^2 - 2tE(k)s\left|f(k)\right|^2 + E^2(k)s^2 f(k)\left|f(k)\right|^2} \end{aligned}$$

$$E(k) = \frac{\epsilon_{2p} \pm t\sqrt{\left|f(k)\right|^2}}{1 + s\left|f(k)\right|^2} \tag{3.5.13}$$

Notice that we considered only the nearest neighbor, therefore the overlap integral matrix (S) and energy level at 2p are vanishes. Then Eqn. (3.5.13) is reduced to

$$E(k) = \pm t\sqrt{\left|f(k)\right|^2} = \pm t\left|f(k)\right| \tag{3.5.14}$$

By using Eqn. (3.5.6), the term under square root of Eqn. (3.5.14) is

$$\left|f(k)\right|^2 = e^{-ik_x a}\left(e^{ik_x \frac{3}{2}a} 2\cos(k_y \frac{\sqrt{3}}{2}a) + 1\right) e^{ik_x a}\left(e^{-ik_x \frac{3}{2}a} 2\cos(k_y \frac{\sqrt{3}}{2}a) + 1\right)$$

$$\left|f(k)\right|^2 = 1 + 2\cos(k_y \frac{\sqrt{3}}{2}a)\left(e^{-ik_x \frac{3}{2}a} + e^{ik_x \frac{3}{2}a}\right) + 4\cos^2(k_y \frac{\sqrt{3}}{2}a)$$

$$\left|f(k)\right|^2 = 1 + 4\cos(k_y \frac{\sqrt{3}}{2}a)\cos(k_x \frac{3}{2}a) + 4\cos^2(k_y \frac{\sqrt{3}}{2}a) \tag{3.5.15}$$

Substitute Eqn. (3.5.16) into Eqn. (3.5.14).

$$E(k) = \pm t\sqrt{1 + 4\cos(k_y\frac{\sqrt{3}}{2}a)\cos(k_x\frac{3}{2}a) + 4\cos^2(k_y\frac{\sqrt{3}}{2}a)} \tag{3.5.16}$$

This is the energy dispersion relation which is commonly used as a simple approximation for the electronic structure of graphene layer. Quantum mechanical hopping between sublattices leads to formations of two cosine-like energy bands [27].

After we vanish ϵ_{2p} because we consider only the nearest-neighbor the Hamiltonian of Eqn. (3.5.9) is

$$H = \begin{bmatrix} 0 & H_{AB} \\ H^*_{BA} & 0 \end{bmatrix} \tag{3.5.17}$$

Now let us expand the wave vector k near the K-point as follow

$$k = K + \mathbb{k}, \tag{3.5.18}$$

where $\mathbb{k}$ is some small ($|\mathbb{k}| < |K|$) vector having origin at K. Substitute Eqn. (3.5.18) into $f(k)$ as follow

$$\begin{aligned} f(K+\mathbb{k}) &= \sum_{i=1}^{3} e^{i(K+\mathbb{k})\delta_i}, \\ &= e^{iK\delta_1}e^{i\mathbb{k}\delta_1} + e^{iK\delta_2}e^{i\mathbb{k}\delta_2} + e^{iK\delta_3}e^{i\mathbb{k}\delta_3}. \end{aligned}$$

Since we considered only K dirac point, δ_2 is described by the same lattice vectors as site A and yields a zero phase to the hopping parameter. Therefore δ_3 correspond to δ_2 lattice vector. It is interesting to assume $K = \frac{4\pi}{3\sqrt{3}a}\hat{y}$ because of expanding.

$$f(K+\mathbb{k}) = e^{i\frac{2\pi}{3}}e^{i\mathbb{k}\delta_1} + e^{i\mathbb{k}\delta_2} + e^{-i\frac{2\pi}{3}}e^{i\mathbb{k}\delta_3}. \tag{3.5.19}$$

Considering the continuum (low energy) limit ($a \to 0$), one can expand exponents in Taylor series

$$\lim_{a\to 0} e^{i\delta_i} \simeq 1 + i\delta_i. \tag{3.5.20}$$

After substituting Eqn. (3.5.20) into Eqn. (3.5.19), we arrive at

$$f(\ Bbbk) = -\frac{3a}{2}(\ Bbbk_x + i\ Bbbk_y). \tag{3.5.21}$$

Use Eqn. (3.5.21) into Eqn. (3.5.14)

$$E(k) = \pm t(\frac{3a\,|\mathbb{k}|}{2}) = \pm\frac{3\hbar a t\,|\mathbb{k}|}{2\hbar} = \hbar v_f(\mathbb{k} - i\mathbb{k}), \tag{3.5.22}$$

where $v_f = \frac{3at}{2\hbar} \approx 10^6 m/s$ is fermi velocity, $|\mathbb{k}| = \mathbb{k} - i\mathbb{k}$ is imaginary
In order to find 2D Hamiltonian Dirc-like equation, we have to apply schrodinger equation for graphene.

$$H\Psi = E\Psi = \hbar v_f(\mathbb{k} - i\mathbb{k})\Psi, \tag{3.5.23}$$

where

$$H = \hbar v_f(\mathbb{k} - i\mathbb{k}). \tag{3.5.24}$$

Substitute Eqn. (3.5.24) into Eqn. (3.5.17)

$$H = \hbar v_f \begin{bmatrix} 0 & \mathbb{k} - i\mathbb{k} \\ \mathbb{k} + i\mathbb{k} & 0 \end{bmatrix} \tag{3.5.25}$$

Using Eqn. (3.5.25), Eqn. (3.5.23) can be expanded to

$$\hbar v_f \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbb{k}_x + \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \mathbb{k}_y \right] \begin{pmatrix} \Psi_A \\ \Psi_B \end{pmatrix} = E \begin{pmatrix} \Psi_A \\ \Psi_B \end{pmatrix}, \tag{3.5.26}$$

where $\Psi = \begin{pmatrix} \Psi_A \\ \Psi_B \end{pmatrix}$ is wave function for only K-point.
With the use of the Pauli matrices, $\vec{\sigma} = (\sigma_x, \sigma_y)$,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{3.5.27}$$

The 2D Dirac Hamiltonian written as

$$\hat{H} = v_f \hbar \sigma \mathbb{k}. \tag{3.5.28}$$

To find the eigen energy of Hamiltonian we used Eqn. (3.5.26)

$$\left[\hbar v_f \begin{pmatrix} 0 & \mathbb{k}_x - i\mathbb{k}_y \\ \mathbb{k}_x + i\mathbb{k}_y & 0 \end{pmatrix} - E \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} \Psi_A \\ \Psi_B \end{pmatrix} = 0,$$

$$\hbar v_f \begin{pmatrix} -\frac{E}{\hbar v_f} & \mathbb{k}_x - i\mathbb{k}_y \\ \mathbb{k}_x + i\mathbb{k}_y & -\frac{E}{\hbar v_f} \end{pmatrix} \begin{pmatrix} \Psi_A \\ \Psi_B \end{pmatrix} = 0, \tag{3.5.29}$$

$\hbar v_f \begin{pmatrix} \Psi_A \\ \Psi_B \end{pmatrix}$ is different from zero if

$$det \begin{vmatrix} -\frac{E}{\hbar v_f} & \mathbb{k}_x - i\mathbb{k}_y \\ \mathbb{k}_x + i\mathbb{k}_y & -\frac{E}{\hbar v_f} \end{vmatrix} = 0 \tag{3.5.30}$$

The massless Dirac electronic energy is

$$E = \pm\hbar v_f \mathbb{k} \tag{3.5.31}$$

Electrons in graphene, obeying a linear dispersion relation, behave like massless relativistic particles [28].
In order to find the eigen functions we use Eqn. (3.5.29).

$$\begin{pmatrix} -\lambda & \mathbb{k}_x - i\mathbb{k}_y \\ \mathbb{k}_x + i\mathbb{k}_y & -\lambda \end{pmatrix} \begin{pmatrix} \Psi_A \\ \Psi_B \end{pmatrix} = 0, \tag{3.5.32}$$

where $\lambda = -\frac{E}{\hbar v_f}$. Let us derive Ψ_A from Eqn. (3.5.32).

$$\Psi_A = \frac{\mathbb{k}_x - i\mathbb{k}_y}{\mathbb{k}} \Psi_B \tag{3.5.33}$$

Since we are doing with polar coordinate $\mathbb{k}_x = \mathbb{k}\cos\theta, \mathbb{k}_y = \mathbb{k}\sin\theta$ and $\theta = \arctan(\mathbb{k}_y/\mathbb{k}_x)$. Then Eqn. (3.5.33) is

$$\Psi_A = e^{-i\theta}\Psi_B \tag{3.5.34}$$

The eigen function

$$\Psi = \begin{pmatrix} \Psi_A \\ \Psi_B \end{pmatrix} = \Psi_B \begin{pmatrix} e^{-i\theta} \\ 1 \end{pmatrix} \tag{3.5.35}$$

Let us set Ψ_B in Eqn. (3.5.35) is equal to one and the normalized wave function will be

$$\Psi = \frac{\Psi}{|\Psi|} = \frac{1}{\sqrt{e^{-i2\theta} + 1}} \begin{pmatrix} e^{-i\theta} \\ 1 \end{pmatrix} \tag{3.5.36}$$

The term under square root is

$$\begin{aligned} e^{-i2\theta} + 1 &= e^{-i(\theta+\theta)} + 1 \\ &= (\cos\theta - i\sin\theta)^2 + 1 \\ &= 2\cos^2\theta - 2i\sin\theta\cos\theta \end{aligned}$$

$$e^{-i2\theta} + 1 = 2cos\theta e^{-i\theta} \tag{3.5.37}$$

Substitute Eqn. (3.5.37) into Eqn. (3.5.36)

$$\Psi = \frac{1}{\sqrt{2cos\theta e^{-i\theta}}} \begin{pmatrix} e^{-i\theta} \\ 1 \end{pmatrix} \tag{3.5.38}$$

If θ is rotates with $n\pi$ ($\theta = n\pi$), then $|\cos\theta|$ is always equal to one.

$$\Psi = \frac{1}{\sqrt{2e^{-i\theta}}} \begin{pmatrix} e^{-i\theta} \\ 1 \end{pmatrix} = \frac{e^{i\theta/2}}{\sqrt{2}} \begin{pmatrix} e^{-i\theta} \\ 1 \end{pmatrix}. \tag{3.5.39}$$

The normalized eigen function is

$$\Psi = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\theta/2} \\ e^{i\theta/2} \end{pmatrix}, \tag{3.5.40}$$

where θ is the polar angle of vector k. Therefore, we have to use this normalized wave function to the Hamiltonian equation given by Eqn. (23) for graphene.

Chapter 4

Quantum Hall effect in graphene

4.1 Landau level in graphene

In convectional electron gas (non-relativistic) Landau quantization produces equidistant energy level which is due to parabolic dispersion law of free electrons. Similarly the motion of relativistic charge in graphene in strong magnetic field is also quantized. In graphene the electrons have relativistic dispersion law, which strongly modifies the Landau quantization of energy and the positions levels.

From Landau guages given by Eqn. (2.1.6) we choce $A_x = -By$ and in chapter three, we got Dirac- like Hamiltonian $\hat{H} = v_f \hat{\sigma}.\hat{p}$

The Dirac-like Hamiltonian for field is

$$\hat{H} = v_f \hat{\sigma}.(\hat{p} - eA). \tag{4.1.1}$$

The Dirac-like Hamiltonian by using schrodinger equation is

$$v_f[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} p_x - eA_x + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} p_y] \begin{pmatrix} \Psi_A \\ \Psi_B \end{pmatrix} = E \begin{pmatrix} \Psi_A \\ \Psi_B \end{pmatrix}. \tag{4.1.2}$$

After addition the matrix

$$v_f \begin{pmatrix} 0 & (\ p_x - ip_y + eBy)^2 \\ (\ p_x + ip_y + eBy)^2 & 0 \end{pmatrix} \begin{pmatrix} \Psi_A \\ \Psi_B \end{pmatrix} = E \begin{pmatrix} \Psi_A \\ \Psi_B \end{pmatrix}, \tag{4.1.3}$$

$$v_f(p_x - ip_y + eBy)\Psi_B = E\Psi_A,$$
$$v_f(p_x + ip_y + eBy)\Psi_A = E\Psi_B \tag{4.1.4}$$

solving for Ψ_B from Eqn. (4.1.4). Then it gives

$$v_f^2(p_x - ip_y + eBy)(p_x + ip_y + eBy)\Psi_A = E^2\Psi_A \tag{4.1.5}$$
$$(p_x^2 + (eBy)^2 + i[p_x + eBy, p_y] + p_y^2)\Psi_A = \frac{E^2}{v_f^2}\Psi_A$$
$$(p_x^2 + (eBy)^2 - e\hbar B + p_y^2)\Psi_A = \frac{E^2}{v_f^2}\Psi_A. \tag{4.1.6}$$

Since $[p_x, x]$ doesn't commute, we have some function with the function of y and with plane wave along x.

$$\Psi_A = \chi(y)e^{ik_x x}. \tag{4.1.7}$$

Substitute Eqn. (4.1.7) into Eqn. (4.1.6)

$$[(p_x + eBy)^2 - e\hbar B + p_y^2]\chi(y)e^{ik_x x} = \frac{E^2}{v_f^2}\chi(y)e^{ik_x x}. \tag{4.1.8}$$

$$(-i\hbar\frac{\partial}{\partial x} + eBy)^2\chi(y)e^{ik_x x} - (e\hbar B)\chi(y)e^{ik_x x} - \hbar^2\frac{\partial^2\chi(y)}{\partial y^2}e^{ik_x x} = \frac{E^2}{v_f^2}\chi(y)e^{ik_x x},$$
$$(\hbar k_x + eBy)^2\chi(y)e^{ik_x x} - (e\hbar B)\chi(y)e^{ik_x x} - \hbar^2\frac{\partial^2\chi(y)}{\partial y^2}e^{ik_x x} = \frac{E^2}{v_f^2}\chi(y)e^{ik_x x},$$

The exponential term cancel out and $\hbar^2$ factor out.

$$\hbar^2\left\{(k_x + eBy)^2 - \frac{(eB)}{\hbar} - \frac{\partial^2}{\partial y^2}\right\}\chi(y) = \frac{E^2}{v_f^2}\chi(y), \tag{4.1.9}$$

$$-\frac{\partial^2\chi(y)}{\partial y^2} + (\frac{eB}{\hbar})^2(\frac{\hbar k_x}{eB} + y)^2\chi(y) - \frac{(eB)}{\hbar}\chi(y) = \frac{E^2}{\hbar^2 v_f^2}\chi(y),$$
$$-\frac{\partial^2\chi(y)}{\partial y^2} + (\frac{eB}{\hbar})^2(y - y_0)^2\chi(y) - \frac{(eB)}{\hbar}\chi(y) = \frac{E^2}{\hbar^2 v_f^2}\chi(y),$$

where $y_0 = -\frac{\hbar k_x}{eB}$

$$\frac{\partial^2\chi(y)}{\partial y^2} + [\frac{E^2}{\hbar^2 v_f^2} + \frac{eB}{\hbar}]\chi(y) - (\frac{eB}{\hbar})^2 Y^2\chi(y) = 0, \tag{4.1.10}$$

For simplicity we used $y - y_0 = Y$

$$\frac{\partial^2\chi(y)}{\partial y^2} + \frac{eB}{\hbar}[\frac{E^2}{\hbar^2 v_f^2}(\frac{\hbar}{eB}) + 1 - \frac{eB}{\hbar}Y^2]\chi(y) = 0. \tag{4.1.11}$$

Let us take α^2 is

$$\alpha^2 = eB/\hbar. \tag{4.1.12}$$

We can write as

$$\frac{\partial^2\chi(y)}{\partial y^2} + \alpha^2[\frac{E^2}{\hbar^2 v_f^2}(\frac{\hbar}{eB}) + 1 - \alpha^2 Y^2]\chi(y) = 0. \tag{4.1.13}$$

Substitute λ instead of some terms in square bracket of Eqn. (4.1.13) as follow

$$\lambda = \frac{E^2}{\hbar^2 v_f^2 \alpha^2} + 1. \tag{4.1.14}$$

$$\frac{1}{\alpha^2}\frac{\partial^2\chi(y)}{\partial y^2} + (\lambda - \alpha^2 Y^2)\chi(y) = 0. \tag{4.1.15}$$

Let us introduce new variable

$$q = \alpha Y, \quad \frac{\partial^2\chi(y)}{\partial y^2} = \alpha^2\frac{\partial^2\chi(y)}{\partial q^2}. \tag{4.1.16}$$

Substitute Eqn. (4.1.16) into Eqn. (4.1.15)

$$\frac{\partial^2\chi(y)}{\partial q^2} + (\lambda - q^2)\chi(y) = 0. \tag{4.1.17}$$

In chapter two we obtained the expression for λ as

$$\lambda = 2n + 1, \quad \frac{E^2}{\hbar^2 v_f^2 \alpha^2} + 1 = 2n + 1 \tag{4.1.18}$$

The eigen value is

$$E_{\pm} = \pm v_f \hbar \omega_D \sqrt{n}, \tag{4.1.19}$$

where $\omega_D = \sqrt{2eB/\hbar}$

The eigen function can be set equal to the product of polynomial.

$$\chi_n(q) = N_n e^{\frac{-q^2}{2}} H_n(q) \tag{4.1.20}$$

Normalization condition is

$$\int_{-\infty}^{\infty} |\Psi(x)|^2 \, dy = 1 \tag{4.1.21}$$

$$\frac{N_n^2}{\alpha}\int_{-\infty}^{\infty} e^{-q^2}(H_n(q))^2 dq = 1. \tag{4.1.22}$$

Using orthogonality condition of hermite polynomial;

$$N_n = (\frac{\alpha}{2^n n! \sqrt{\pi}})^{\frac{1}{2}} \tag{4.1.23}$$

Thus the corresponding wave function is

$$\Psi_n(x,y) = (\frac{eB}{\pi\hbar})^{\frac{1}{4}}(\frac{1}{2^n n!})^{\frac{1}{2}} \exp ik_x x \exp(\frac{eB}{\hbar}(y-y_o)^2)H_n[\sqrt{\frac{eB}{\hbar}}(y-y_o)]. \qquad (4.1.24)$$

The wave function in terms of cyclotron frequency is

$$\Psi_n(x,y) = (\frac{m\omega_c}{\pi\hbar})^{1/4}(\frac{1}{2^n n!})^{\frac{1}{2}} \exp(i(k_x x \exp(\frac{m\omega_c}{2\hbar}(y-y_o)^2)H_n[\sqrt{\frac{m\omega_c}{\hbar}}(y-y_o)]. \qquad (4.1.25)$$

The energy does not depend on the quantum number k_x but eigenfunction depends on k_x this means Landau levels have a degeneracy because of different wave functions with different k_x values.

4.2 Integer quantum Hall effect in graphene

In chapter two, we have seen that the QHE can be explained as quantum mechanical version of hall effect, especially observed in two-dimension system under low temperature and strong magnetic field. So, 2D graphene can be fit into this condition. But the result is quite different from usual QHE. The quantum Hall effect observed in graphene distinctively different from usual quantum Hall effect since the quantization condition Eqn (4.2.1) below is shifted by half-integer in case of monolayer graphene. This is so-called half-integer QHE which is unique to graphene. Due to the mass-less, chiral Dirac fermions, the quantum Hall effect in graphene displays a shift of the Hall resistance by a half-integer compared to the conventional integer QHE.

$$\sigma_{xy} = \pm 4(n+\frac{1}{2})\frac{e^2}{h} = \nu\frac{e^2}{h} \qquad \nu = ..., -10, -6, -2, 2, 6, 10, ... \qquad (4.2.1)$$

where n is integer, ± is for holes and electrons, the prefactor 4 reflects the two-fold spin and two-fold valley degeneracy in the graphene band structure [29]. This quantization condition can be translated into the quantized filling factor $\nu = \pm 4(n+\frac{1}{2})$. Here what we have seen in chapter two quantum Hall effect allow to seen the quantization of conductivity ($\sigma_{xy=\nu e^2/h}$) for thin materials exposed to a perpendicular magnetic field in 2D electron gases have an integer relationship to conductivity while graphene has a half integer (Eqn.(4.2.1)) relation.

Here we note that at $\nu = -2$ and $\nu = 2$ correspond to the emptying and filling of the eigen value $E_n = 0$ LL. Thus for $n = 0$ LL, the valley degeneracy (isospin) is the same as the sublattice degeneracy (pseudospin).

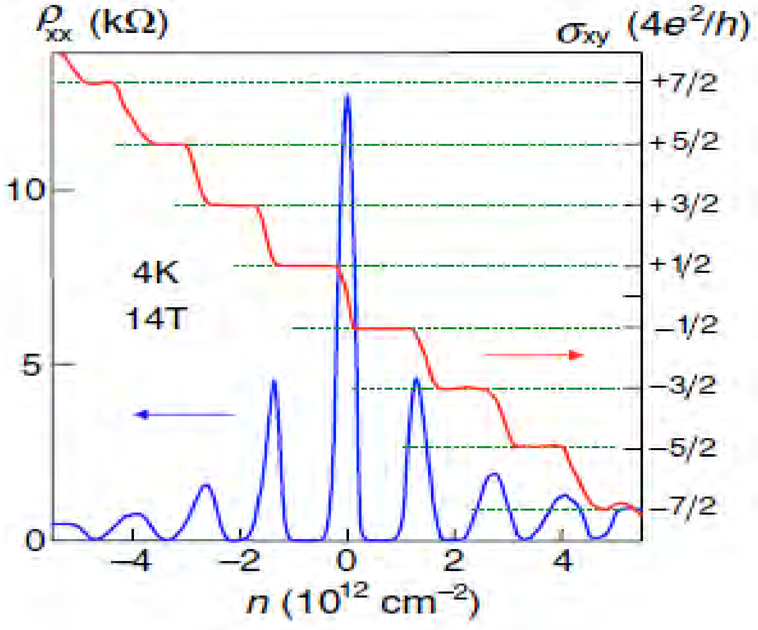

Figure 4.1: Quantum hall effect in graphene [30].

4.3 Fractional quantum Hall effect in graphene

Theoretical scientists expect that electrons in graphene are strongly interacting and hence exhibit FQHE. The FQHE depends upon the combined effect of the magnetic field and coulomb interaction between electrons. The FQHE occurs when the magnetic field is so strong, that the plateaus occurred at fractional values of the filling factor, because fractional values of the filling factor refers to partially filled Landau states. In this case, for non interacting electrons the ground state is macroscopically degenerate. It is the Coulomb interaction between the electrons that lifts the degeneracy and opens a gap [31]. Therefore the origin of the FQHE cannot be understood based on the behavior of individual electrons in a magnetic field; it is the behavior of all the electrons .

FQHE is essentially restricted to the lowest LL: very few fractions are seen in n=0 graphene LL, and almost none in higher LLs [32].

4.4 Comparison of Landau levels for conventional 2DEG and graphene

For conventional semiconductor 2DEGs, separate electron and hole LL form, that are equally spaced. For graphene the LL energy follows a square-root behavior and a zero-energy level appears that is shared equally by electrons and holes. Graphene not exhibits LLs of equal spacing as conventional semiconductors. The difference between the conventional LL structure and graphene is a direct a result of the chirality of the charge carriers. Carrier movement on cyclotron orbits in a magnetic field results in an accumulation of a phase shift the Berry's phase π for the wavefunction of the chiral mass-less quasi-particles in graphene. This explains the unusual LL at zero energy. Furthermore, backscattering is suppressed for the chiral charge carriers, which leads to protected edge states and the possibility to observe the quantum Hall effect up to room temperature.

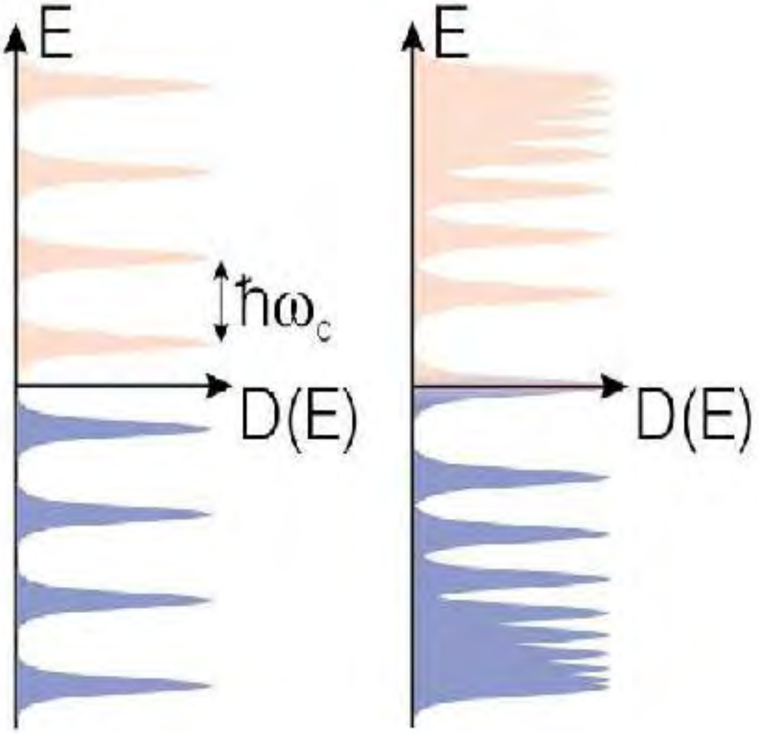

Figure 4.2: (Left) Landau levels for Schrodinger electrons with two parabolic bands touching each other at zero energy. (Right) Landau levels for Dirac electrons [26].

Chapter 5

Conclusion

Classical Hall Effect is a phenomena observed when an external magnetic field is applied perpendicularly to a current carrying conductor because of the charges experienced the Lorentz force and equal but opposite charges accumulate on the opposite side of conductor. The result is an asymmetric distribution of charge carriers on the conductor's surface. This separation of charges establishes an electric field that opposes further separation of charge and the resistivity of the conductor has linear depends on the magnetic field strength. The difference between integral quantized Hall effect, fractional quantized Hall effect and Hall effect is Hall resistance.

At very low temperatures and high magnetic fields, the thermal excitations between Landau levels are negligible. If the magnetic field is increased, the capacity of each Landau level will increase, and electrons from higher levels will drop to lower Landau levels until they are filled again. If one continues to increase the magnetic field, the highest Landau level will be depleted, while all the levels below are exactly filled. The Fermi energy of the system will drop suddenly and QHE observed.

The basic QHE experimental observation is the nearly vanishing longitudinal resistance $R_{xx} \to 0$ and the quantization of the Hall resistance $R_{xy} = \frac{h}{\nu e^2}$ of a two dimensional electron gas subjected to a strong magnetic field, the filing factor could take on integer value for IQHE and fractional values for FQHE. Fractional Quantum Hall Effect is the result of involving strong Coulomb interactions and correlations among the electrons.

The integral quantum Hall effect can be well understood using the concept of Landau levels which are completely filled by electrons subject to the Pauli exclusion principle. The fractional quantum Hall effect needs a more sophisticated theory, like Laughlin's wave function theory to understand its mechanism.

QHE in graphene behavior reveals the existence of Dirac fermions with massless electrons. The existence of a degenerate Landau level explains the unconventional QHE found in graphene. The sequence of Landau levels is described by $E_n \propto \sqrt{n}$ for massless Dirac fermions in graphene. This Landau level lies at the border between electron and hole gases and, taking into account the two degeneracy.

In graphene the QHE can be observed even at room temperature. The survival of the QHE to such high temperatures can be attributed to the large cyclotron gaps, $\hbar\omega_c$, characteristic to Dirac fermions in graphene. This is due to the highly unusual nature of charge carriers in graphene, which behave as massless relativistic particles (Dirac fermions).

FQHE in graphene is the Coulomb interaction between the electrons that lifts the degeneracy and opens a gap. The FQHE occurs when the magnetic field is so strong, that the plateaus occurred at fractional values of the filling factor, because fractional values of the filling factor refers to partially filled Landau states, and non-interacting electrons the ground state is macroscopically degenerate.

Bibliography

[1] Hall, E. H., "On a new action of the magnet on electric currents", American Journal of Mathematics 2, **No. 3** (1879).

[2] S.R.Elloit, *The physics and Chemistry of solids*, John Wiley and Sons Ltd.,England (1998).

[3] Charles Kittle, *Introduction to Solid State Physics*, 8th edition, USA (2005).

[4] R. B. Laughlin: Fractional quantization, Rev. Mod. Phys. **71** , pp. 863-874 (1999).

[5] B.I., Halperin, "Composite fermions and the Fermion-Chern-Simons Theory", Physica E **20**, 71-78, (2003).

[6] Jan G.K. Andreas Greiner, *Semiconductors for Micro and Nanotechnology*, Wiley- VCH Vernheim GMBH, Weinheim (2002).

[7] Tsui, D. C., Stormer, H. L., and Gossard, A. C. Phys. Rev. Lett. **48**, PP. 1559-1630 (1982).

[8] Adrian P. Suttan, *Electronic structure of materials*, Oxford University press (1993).

[9] Thomas Heinzel, *Mesoscopic Electronics in solid state Nanostructure*, 2nd edition, WILEY-VCH (1807).

[10] Richard E. Prange, Steven M. Girvin *The Quantum Hall Effect* 2nd edition,Maryland University (1990).

[11] V. Klitzing, K., Dorda, G., and Pepper, M. Phys. Rev. Lett **45**, 494 (1980).

[12] J.P Harrang et al, Physical Review B, **32**, (1985).

[13] R. B. Laughlin: The quantized Hall conductivity in two dimensions, Phys. Rev., **23**, PP. 5632-5633 (1981).

[14] M. Jalochowski et al, **76** (1996).

[15] Horst L. Stormer. Nobel Lecture: Fractional Quantum Hall Effect.Rev Mod Phys, **71**, pp. 875-889 (1999).

[16] V. K. Bharadwaj. Fractional quantum Hall effect, Rev. Mod. Phys. **6** (2014).

[17] Gerald D.Mahan, *Many-particle physics*, Springer Science+Business Media, LLC, New York (2011).

[18] A.J.Leggett. The fractional quantum Hall effect: Laughlin wave function, fractional charge and statistics, PHYS598PTD, **18**: pp. 1-12 (2013).

[19] R. B. Laughlin: Fractional quantization, Rev. Mod. Phys. **70** , pp. 263-286 (1998).

[20] R. B. Laughlin: Fractional quantization, Rev. Mod. Phys. **71**, pp. 863-874 (1999).

[21] Mikhail I. Katsnelson:Graphene Carbon in Two Dimensions, Cambridge University Press (2012).

[22] www.epljournal.org

[23] Castro Neto, Electronic properties of graphene. Rev.Mod.Phy.**81** (2009).

[24] R. SAITO,G. DRESSLHAUS and M. S. DRESSLHAUS, Physical Properties of Carbon Nanotubes, Imperial College Press, London ,(1998).

[25] Wallace P. R. The band theory of graphite, Phys. Rev. **71**, 622-634 (1947).

[26] Mikhail I. Katsnelson: Graphene: carbon in two dimensions, Elsevier Ltd. **10**, (2007).

[27] K. Vetenskaps-Akadamien: Graphene, The Royal Swedish academy of science, (2010).

[28] V.I. Fal'ko and A.K. Geim:Graphene: Emerging matter in two dimensions, Eur.phys.J.special topic, **148**, (2007)

[29] A.K. Geim and K.S. Novoselov: The rise of graphene, Manchister University

[30] Guohong Li and Eva Y.Andrel,Observation of Landau levels of Dirac fermions in graphite,Letters,nature physics VOL 3 (2007).

[31] T. Chakraborty and P. Pietilainen, *The Quantum Hall Effects Integral and Fractional* , 2^{nd} edition, Springer-Verlag, Heidelberg (1995).

[32] Csaba Toke, Paul E. Lammert, Vincent H. Crespi, and Jainendra K. Jain; Fractional quantum Hall effect in graphene.Phys. Rev.B **74**, (2006).

Declaration

This thesis is my original work, has not been presented for a degree in any other University and that all the sources of material used for the thesis have been dully acknowledged.

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