ADDIS ABABA UNIVERSITY
College of Natural and Computational Science

Department of Mathematics

Thesis Title

Analysis of Boundary Integral Equations for Laplace Dirichlet BVP in 2D

In partial fulfillment for the Master's degree program

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Abstract

Using an appropriate fundamental solution, Dirichlet boundary value problem is reduced to some direct Boundary Integral Equations (BIEs). Although the theory of BIEs in 3D is well developed, the BIEs in 2D need a special consideration due to their different equivalence properties. Consequently, we need to set conditions on the domain for the invertibility of corresponding fundamental based integral layer potentials and hence the unique solvability of BIEs. The properties of corresponding potential operators are investigated. The equivalence of the original BVP and the obtained BIEs are analyzed and the invertibility of the BIE operators is proved.

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Nomenclature

$\Gamma, \partial\Omega$	Boundary of Ω
$\overline{\Omega}$	$\Omega \cup \partial\Omega$
$\Omega = \Omega^+$	Interior domain
$\Omega^- = \mathbb{R}^n \setminus (\Omega \cup \Gamma)$	Exterior domain
n^+	Outward normal to Ω^+
n^-	Inward normal w.r. t.Ω^+
∇	Gradient
∇^2	Laplace operator
$\mathbb{R}$	Set of real numbers
$\mathbb{C}$	Set of complex numbers
$\mathcal{D}(\Omega)$	The set of all infinitely differentiable functions on Ω with compact support
$\mathcal{D}'(\Omega)$	All continuous linear functionals over $\mathcal{D}(\Omega)$
$W_p^s(\mathbb{R}^2)$	Sobolev space of order s based on L_p
$H^s(\mathbb{R}^2)$	Sobolev space on $\mathbb{R}^2$ definition via Bessel potential
$H^s(\Gamma)$	Sobolev space on Γ
$\tilde{H}^s(\mathbb{R}^2)$	The closure of $\mathcal{D}(\Omega)$ with respect to the norm of $H^s(\mathbb{R}^2)$

Acronyms

BIE	Boundary Integral Equation
BEM	Boundary Integral Element Method
BVP	Boundary Value Problem
ODE	Orginary Differential Equation
PDE	Partial Differential Equation

Acknowledgment

First and foremost, I would like to thank the Almighty God for his guidance and help in giving me the strength to complete this thesis. A special thanks to my supervisor, Dr. Tsegaye G., for his constructive advice throughout the period of this research project. I would also like to thank my Mom(Mebea Atske) and my brothers, Mathewos Fekade and Girum Fekade. Their support, encouragement, patience, and belief in me ultimately made this thesis possible.

Finally, I would like to express my sincere appreciation to all who has helped me in one way or another, but whose names are not mentioned.

To my mom,

Thanks for an unfailing love and the continuous prayers.

Introduction

Many equations of physics are second-order PDEs e.g. wave equation, diffusion (heat) equation, Helmholtz equation, equation of fluid-dynamics, Maxwell equations, Schrodinger equation. These PDEs can describe a wide variety of phenomena such as sound, heat, electrostatics, electrodynamics, fluid flow and elasticity.

There are various methods for solving BVPs for PDE analytically e.g. methods of separation of variables, Fourier and Laplace transforms, integral transforms and variation of parameters. However, many problems encountered in applications cannot be solved using analytical methods. Therefore, it is necessary to resort to approximate solution methods. Many methods have been developed for the numerical solution of partial differential equations and amongst the commonly used are volume-discretization methods e.g. finite difference method, finite volume method and finite element method.

Another numerical method that can give a comparable efficiency to volume-discretization methods is the Boundary Element Method (BEM). In order to use the BEM, we need to have representation formulas. Such representation formulas are well known for the classical boundary value problems of mathematical physics, e.g. Green's third identity for potential theory, Betti's formula for elasticity theory, and the Stratton-Chu formula for electrodynamic (Costabel (1988b)).

Boundary integral equations are a classical tool for boundary value problems for partial differential equations. Many boundary value problems of mathematical physics and engineering can be reduced to integral equations over the boundary of the domain of interest. Particularly, boundary integral equations are often used to solve numerically the Dirichlet and Neumann problems and also the mixed boundary value problem (Dirichlet-Neumann). One of the methods for the approximate numerical solution of these boundary integral equations is called "boundary element method" (BEM). The approximate solution of the boundary value problem obtained by using BEM has the distinguishing feature that it is an exact solution of the differential equation in the domain and is parametrized by a finite set of parameters living on the boundary. Thus, the problem dimensionality is reduced by one which requires only a line mesh around the boundary of the domain in 2-D and a surface mesh for 3-D geometries. This implies huge reduction in mesh generation efforts.

There are two approaches to derive BIEs of BVPs for PDE with constant coefficients. The first integral formulation is often named as a direct method and the integral equations are derived through the application of the second Greens identity. The second integral formulation known as an indirect method is founded on single or double layer potentials. The method of boundary integral equations has always had two important applications in the theory of boundary value problems for partial differential equations: as a theoretical tool for proving the existence of solutions and as a practical tool for the construction of solutions.

The main part of this paper is divided into three chapters. In Chapter 1 and 2, we go through some preliminary results concerning Laplace's Equation, formulation of Fundamental Solution, Poisson's Equation, Divergence Theorem, Reciprocal relation, Green's Identities and Representation Formula. These results are supposed to help to get into the subject and later on, also convince from the validity of some rather complicated proofs. In Chapter 2, we discuss the main result of this thesis, which is the Analysis of Boundary Integral Equations for constant coefficient (the case of Dirichlet boundary value problem in 2D). First, we formulate the boundary value problem, construct the fundamental solution and potential operators, then investigate the invertibility of these operators as well as analyze the corresponding BIEs.

Chapter 1

Laplace's Equation

1.1 Laplace's Equation

The n-dimensional **Laplace's Equation**:

$$\triangle u = 0 \tag{1.1}$$

and its inhomogeneous version, **Poisson's Equation**

$$\triangle u = f \tag{1.2}$$

A function u satisfying Laplace's equation is a **harmonic functions**.

$$\begin{aligned} &\triangle u = 0 \\ &\triangle u = u_{x_1x_1} + u_{x_2x_2} + \cdots + u_{x_nx_n} \qquad x = x(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n \end{aligned} \tag{1.3}$$

Laplace equation is used to calculate the density of the fluid in equilibrium. To look for the solution this equation, just like 2^{nd} order ODEs for example e^x or e^{rx} that played a certain role, it is useful to find may be easier kind of solution.

Fact 1 *$\triangle u$(Laplacian) is invariant under rotation.*

If $\triangle u = 0, V = u(Rx)$, where R is a rotation in $\mathbb{R}^n$(you may say orthogonal matrices), then $\triangle V = 0$ as well, that is, suppose you have a solution and rotate it and then get another solution which is in fact the same function. One kind of function which, when rotated and get another solution, which is in fact the same function, is a **Radial function**. Radial function only depends on how far is from the origin and it is invariant under rotation.

Goal 1 *To look for a solution of the form* $u(x) = V(|x|)$*(only depends on* $|x|$*), where* $V : \mathbb{R}^+ \to \mathbb{R}$.

Step 1. To turn the pde into ODE, that is, to an equation of one variable which is actual have a hope of solving.
Let $r = |x|$, since V depends on r only, we have an ODE and find a differential that V satisfies. That is, once we find the differential equation, it is easy to find the solution since $u(x) = V(|x|)$, but V is only one solution that satisfies the Laplacian equation

$$\begin{aligned} \triangle u &= 0 \\ u_{x_1x_1} + u_{x_2x_2} + \cdots + u_{x_nx_n} &= 0 \end{aligned}$$

Note that

$$|x| = \sqrt{{x_1}^2 + {x_2}^2 + \cdots + {x_n}^2}$$

$$\begin{aligned} \frac{\partial |x|}{\partial x_i} &= \frac{1}{2}[{x_1}^2 + {x_2}^2 + \cdots + {x_n}^2]^{-\frac{1}{2}} \cdot 2x_i \qquad \text{for } i \in \{1, 2, \cdots, n\} \\ &= \frac{1}{[{x_1}^2 + {x_2}^2 + \cdots + {x_n}^2]^{\frac{1}{2}}} \cdot x_i \\ &= \frac{x_i}{|x|} \end{aligned}$$

By chain rule,

$$u_{x_i} = \frac{\partial u}{\partial x_i} = \frac{\partial}{\partial x_i}(V(|x|) = V'(|x|) \cdot \frac{x_i}{|x|}$$

$$\implies u_{x_i} = V'(|x|) \cdot \frac{x_i}{|x|}$$

Solving for $u_{x_ix_i}$

$$\begin{aligned} u_{x_ix_i} &= \frac{\partial u_{x_i}}{\partial x_i} \\ &= V''(|x|) \cdot \frac{\partial |x|}{\partial x_i} \cdot \frac{x_i}{|x|} + V'(|x|) \cdot \frac{\partial}{\partial x_i}(\frac{x_i}{|x|}) \\ &= V''(|x|) \cdot \frac{x_i}{|x|} \cdot \frac{x_i}{|x|} + V'(|x|)(\frac{|x| - \frac{x_i}{|x|} \cdot x_i}{|x|^2}) \\ &= V''(|x|) \cdot \frac{{x_i}^2}{|x|^2} + V'(|x|)(\frac{|x|^2 - {x_i}^2}{|x|^3}) \\ &= V''(|x|) \cdot (\frac{x_i}{|x|})^2 + V'(|x|)(\frac{1}{|x|} - \frac{{x_i}^2}{|x|^3}) \end{aligned}$$

For Laplacian, we need to sum up all for $i = 1, 2, ..., n$

$$
\begin{aligned}
\triangle u &= 0 \\
&= \sum_{i=1}^{n} u_{x_i x_i} \\
&= \sum_{i=1}^{n} (V''(|x|) \cdot (\frac{x_i}{|x|})^2 + V'(|x|)(\frac{1}{|x|} - \frac{{x_i}^2}{|x|^3})) \\
&= \sum_{i=1}^{n} [(V''(|x|) \cdot (\frac{x_i}{|x|})^2 + V'(|x|) \cdot \frac{1}{|x|} - V'(|x|) \cdot \frac{{x_i}^2}{|x|^3})] \\
&= \frac{V''(|x|)}{|x|^2} \sum_{i=1}^{n} {x_i}^2 + \frac{V'(|x|)}{|x|} \sum_{i=1}^{n} 1 - \frac{V'(|x|)}{|x|^3} \sum_{i=1}^{n} {x_i}^2 \\
&= \frac{V''(|x|)}{|x|^2} \cdot |x|^2 + \frac{V'(|x|)}{|x|} \cdot n - \frac{V'}{|x|^3} \cdot |x|^2 \\
&= V''(|x|) + \frac{V'(|x|)}{|x|} \cdot n - \frac{V'}{|x|} \\
&= V''(|x|) + \frac{V'(|x|)}{|x|}(n-1)
\end{aligned}
$$

$$
0 = \triangle u = V''(|x|) + \frac{V'(|x|)}{|x|}(n-1)
$$

$$
V''(r) + \frac{V'(r)}{r}(n-1) = 0, \qquad r = |x|
$$

So far we have turned the pde into ODE(an equation of one variable which is actual have a hope of solving).

Step 2:Solving the above ODE

$$
\begin{aligned}
V''(r) + \frac{V'(r)}{r}(n-1) = 0& \\
V''(r) &= -\frac{V'(r)}{r}(n-1) \\
\frac{V''(r)}{V'(r)} &= -\frac{n-1}{r} \\
(\ln(|x|))' &= -(\frac{n-1}{r}) \\
\ln|V'| &= -(n-1)\ln|r| + A \\
|V'| &= e^{\ln r^{-(n+1)}} \cdot e^{A} \\
|V'| &= e^{A} \cdot r^{-(n-1)} \\
V' &= \underbrace{\pm e^{A}}_{\text{a generic constant, say A}} \cdot r^{-(n-1)} \\
V' &= A \cdot r^{-(n-1)}
\end{aligned}
$$

When anti-differentiating V', we have 2 cases.

For $n = 2$,

$$\begin{aligned} V(r) &= \int Ar^{-(n-1)}\,dr \\ &= \int \frac{A}{r}\,dr \\ &= A\ln r + B \end{aligned}$$

$$V(r) = A\ln r + B \tag{1.4}$$

For $n > 2$,

$$\begin{aligned} V(r) &= \int Ar^{-(n-1)}\,dr \\ &= A\frac{r^{-(n-1)+1}}{-(n-1)+1} + B \\ &= A\frac{r^{-(n-2)}}{-(n-2)} + B \\ &= \underbrace{\frac{A}{-(n-2)}} r^{-(n-2)} + B \\ &= \frac{-A}{(n-2)} r^{-(n-2)} + B \end{aligned}$$

$$\implies V(r) = \frac{-A}{(n-2)} r^{-(n-2)} + B \tag{1.5}$$

Therefore, the anti-differentiation of V' is given by:

$$V(r) = \left\{ \begin{array}{rl} A\ln r + B & \text{for} \quad n = 2 \\ \frac{-Ar^{(2-n)}}{(n-2)} + B & \text{for} \quad n \geq 3 \end{array} \right. \tag{1.6}$$

Summary:

$$V(r) = \left\{ \begin{array}{rl} A\ln r + B & \text{for} \quad n = 2 \\ \frac{-A}{(n-2)r^{(n-2)}} + B & \text{for} \quad n \geq 3 \end{array} \right. \tag{1.7}$$

for $x \in \mathbb{R}^n, x \neq 0$ is a solution of Laplace's equation in $\mathbb{R}^n - \{0\}$. We notice that the function u defined in (1.7) satisfies $\triangle u = 0$ for $x \neq 0$, but $\triangle u$ is undefined for $x = 0$.

Definition 1 ***Fundamental Solution***
Let $\mathcal{L}$ be a linear partial differential operator with constant coefficients in $\mathbb{R}^n$. $\Phi(p)$, $p \in \mathbb{R}^n$ is called a fundamental solution of $\mathcal{L}$ if Φ satisfies the equation

$$\mathcal{L}(\Phi) = \delta(p) \qquad p \in \mathbb{R}^n \tag{1.8}$$

where the operator $\mathcal{L}$ *has a form* $\sum_{|\alpha|\le m} a_\alpha\ D^\alpha,\ \alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$*, is a multi-index,* α_n*'s are non-negative integers,* $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n,\ a_{\alpha_1,\alpha_2,\cdots,\alpha_n}$ *are constant coefficients, and* $D^\alpha = (\frac{\partial}{\partial x_1}), (\frac{\partial}{\partial x_2}), \cdots, (\frac{\partial}{\partial x_n}) = \frac{\partial^{|\alpha|}}{\partial {x_1}^{\alpha_1} \cdots \partial {x_n}^{\alpha_n}}$ *and* $\delta(p)$ *is the dirac delta function in* $\mathbb{R}^n$

1.2 Formulation of Fundamental Solution

1.2.1 2D Laplace's Equation

According to the definition, the fundamental solution Φ of the 2D Laplace's equation

$$\triangle u = \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1.9}$$

should satisfy

$$\nabla^2 \Phi = \delta_p(q) \tag{1.10}$$

where $\delta_p(q)$ is the Dirac delta function centered at a point $p \in \Omega$ in a 2D domain, see figure 1.1.

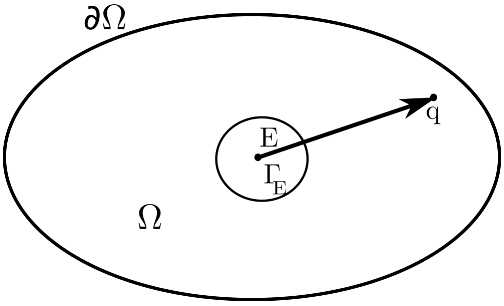


Figure 1.1: A 2D domain Ω with boundary $\partial\Omega$

We can take a fundamental solution which we proved already, i.e,

$$V(r) = \begin{cases} A\ln r + B & \text{for} \quad n = 2 \\ \frac{-A}{(n-2)r^{(n-2)}} + B & \text{for} \quad n \ge 3 \end{cases}$$

It is actually the same to the following derivation mode.

Taking p as the origin and expanding the Laplacian equation in polar coordinates, we analyze the circularity symmetric solution $\varphi = \varphi(r)$.

Thus, with φ only dependent of r and $\delta_p(q) = 0$ if $r \neq 0$, the equation (1.9) can be written in the form

$$\begin{aligned}
\nabla^2(\varphi) = \frac{1}{r}\cdot\frac{d}{dr}(r\cdot\frac{d\varphi}{dr}) &= 0, \qquad \text{for } r \neq 0 \\
\frac{1}{r}\cdot\frac{d}{dr}(r\cdot\frac{d\varphi}{dr}) &= 0 \\
\frac{d}{dr}(r\cdot\frac{d\varphi}{dr}) &= 0 \\
\int \frac{d}{dr}(r\cdot\frac{d\varphi}{dr})\,dr &= 0, \\
r\cdot\frac{d\varphi}{dr} &= A \\
\frac{d\varphi}{dr} &= \frac{A}{r} \\
\int \frac{d\varphi}{dr}\,dr &= \int \frac{A}{r}\,dr \\
\varphi(r) &= A\ln r + B \qquad \text{for } r \neq 0
\end{aligned} \tag{1.11}$$

where A and B are integration constants

Clearly φ satisfies

$$\nabla^2\varphi = \infty \text{ for } r = 0.$$

The surface integral of $\nabla^2\varphi$ should also satisfy

$$\int_\Omega \nabla^2\varphi\,d\Omega = \int_\Omega \delta_p(q)\,d\Omega = 1 \tag{1.12}$$

where Ω is an arbitrary 2D domain surrounding p.

Again ,let Γ_ε be a small disk centered at p(see figure 1.1)of area $A_\varepsilon = \varepsilon$. Then using the divergence theorem the integral in (1.12) is transformed into a line integral over the boundary of the disk Γ_ε, which we denote by Γ.
Thus,

$$\begin{aligned}
\iint_\Omega \nabla^2\varphi\,d\Omega &= \iint_\Omega \nabla\cdot\nabla\varphi\,d\Omega \\
&= \iint_\Omega div.(\nabla\varphi)\,d\Omega \\
&= \int_{\Gamma_\varepsilon} (\nabla\varphi)\cdot\vec{n}\,d\Gamma \\
&= \int_{\Gamma_\varepsilon} \frac{A}{r}\cdot\,d\Gamma \\
&= \frac{A}{r}\int_{\Gamma_\varepsilon} 1\,d\Gamma \\
&= \frac{A}{r}(2\pi r) \\
\iint_\Omega \nabla^2\varphi\,d\Omega &= 2\pi A
\end{aligned} \tag{1.13}$$

where $(\nabla\varphi)\cdot\vec{n}$ is the projection of the normal component of the gradient of φ along the contour $\partial\Gamma_\varepsilon$.
Thus substituting equation (1.13) into equation (1.12) yields

$$\iint_\Omega \nabla^2\varphi\, d\Omega = 2\pi A = 1$$

$$\implies A = \frac{1}{2\pi}$$

Setting $B = 0$ in equation(1.11), we obtain the fundamental solution to the 2-D Laplace equation,

$$\varphi(r) = A\ln r + B \qquad \text{with } A = \frac{1}{2\pi} \text{ and } B = 0$$

$$\implies \varphi(r) = \frac{1}{2\pi}\ln r \tag{1.14}$$

Note: 2-D Divergence Theorem

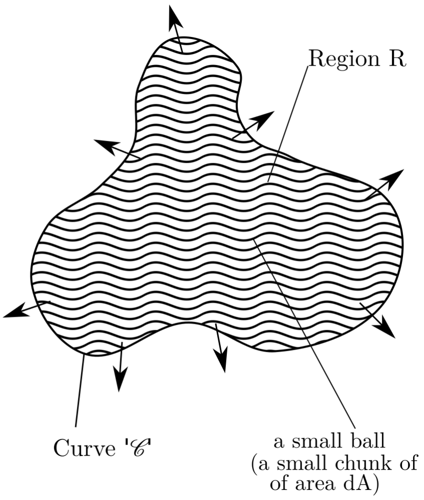


Figure 1.2: A 2D domain Ω with boundary $\partial\Omega$

$$\int_C (\vec{F}\cdot\vec{n})\, dr = \iint_R div\cdot\vec{F}\, dA \tag{1.15}$$

1.2.2 3-D Laplace's Equation

According to the definition , the fundamental solution of φ of the 3-D Laplace's Equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \tag{1.16}$$

should satisfy $\nabla^2\varphi = \delta_p(q)$ where $\delta_p(q)$ is th Dirac delta function centered at a point $p \in \Omega$ in a 3-D domain. Thus, the volume integral of $\nabla^2\varphi$ should satisfy

$$\iiint_\Omega \nabla^2\varphi \, d\Omega = \iiint_\Omega \delta_p(q)\,\Omega = 1 \tag{1.17}$$

where Ω is arbitrary 3-D domain surrounding p. Again, let Γ_ε be a small ball centered at p of volume $V_\varepsilon = \varepsilon$. Then using the divergence theorem the integral(1.17) is transformed into surface integral over the boundary of a small ball Γ_ε, which we denote by Γ. Thus, Note: from previous formula of fundamental solution of n-D.

$$V(r) = \begin{cases} A\ln r + B & \text{for} \quad n = 2 \\ \frac{-A}{(n-2)r^{(n-2)}} + B & \text{for} \quad n \geq 3 \end{cases}$$

And hence, for $n = 3$,

$$\varphi(r) = \frac{-A}{r} + B \quad \text{and} \tag{1.18}$$

$$\nabla\varphi = \frac{\partial\varphi(r)}{\partial r} = \frac{A}{r^2} \tag{1.19}$$

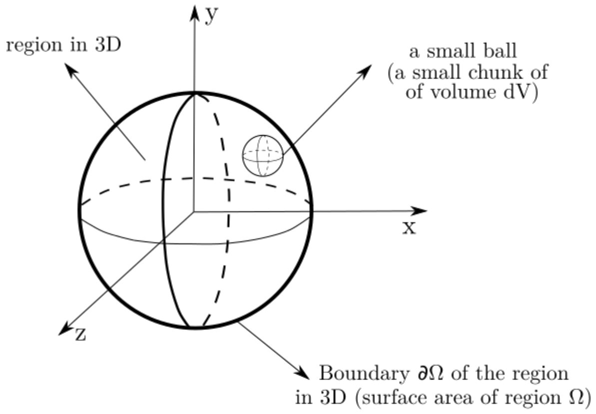


Figure 1.3: A 3D domain Ω with boundary $\partial\Omega$

Thus,

$$\begin{aligned} \iiint_\Omega \nabla^2\varphi \, d\Omega &= \iiint_\Omega \nabla \cdot \nabla\varphi \, d\Omega \\ &= \iint_\Omega div \cdot (\nabla\varphi) \, d\Omega \\ &= \iint_{\Gamma_\epsilon} \nabla\varphi \cdot \hat{n} \, d\Gamma \\ &= \iint_{\Gamma_\epsilon} \frac{A}{r^2} \cdot \, d\Gamma \\ &= \frac{A}{r^2}(4\pi r^2) \end{aligned}$$

$$\iiint_\Omega \nabla^2\varphi \, d\Omega = 4\pi A \tag{1.20}$$

Thus substituting (1.20) into equation (1.17) yields

$$\iiint_\Omega \nabla^2 \varphi \, d\Omega = 4\pi A = 1 \implies 4\pi A = 1 \implies A = \frac{1}{4\pi}$$

Setting $B = 0$ in equation (1.18), we obtain the fundamental solution to the 3-D Laplace's Equation.

$$\varphi(r) = \frac{-A}{r} + B \qquad \text{with } A = \frac{1}{4\pi} \text{ and } B = 0$$

$$\implies \varphi(r) = -\frac{1}{4\pi r} \qquad for\ n = 3$$

Note: 3-D Divergence Theorem

$$\iint_S (\vec{F} \cdot \hat{n}) \, dS = \iiint_R div \cdot \vec{F} \, dv$$

1.2.3 n-D Laplace's Equation

According to the definition, the fundamental solution φ of the n-D Laplace equation.

$$\nabla^2 u = \frac{\partial^2 u}{\partial {x_1}^2} + \frac{\partial^2 u}{\partial {x_2}^2} + \cdots + \frac{\partial^2 u}{\partial {x_n}^2} = 0 \tag{1.21}$$

should satisfy

$$\nabla^2 \varphi = \delta_p(q) \tag{1.22}$$

where $\delta_p(q)$ is th Dirac delta function centered at a point $p \in \Omega$ in a n-D domain. Thus, the volume integral of $\nabla^2 \varphi$ should satisfy

$$\int_\Omega \cdots \int_\Omega \nabla^2 \varphi \, d\Omega = \int_\Omega \cdots \int_\Omega \delta_p(q) \, d\Omega = 1 \tag{1.23}$$

where $\Omega$is arbitrary n-D domain surrounding p. Again, let Γ_ϵ be a small ball centered at p of volume $V_\epsilon = \epsilon$. Then using the divergence theorem the integral(1.23) is transformed into surface integral over the boundary of a small ball Γ_ϵ, which we denote by Γ.
Note: from previous formula of fundamental solution of n-D.

$$V(r) = \begin{cases} A \ln r + B & \text{for} \quad n = 2 \\ \frac{-A}{(n-2)r^{(n-2)}} + B & \text{for} \quad n \geq 3 \end{cases}$$

And hence, for $n = 2$ and $n = 3$, we have shown above.

Let's now develop for any $n \geq 3$. To do this we need to know the surface area of the volume with radius r in n-D. Let's now see the volume of the ball in n-D with radius r.

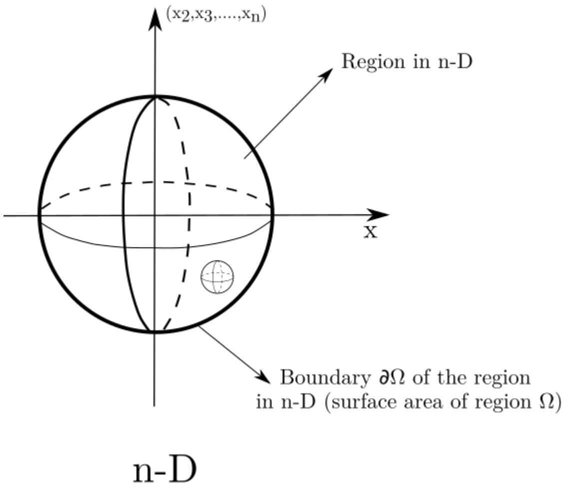


Figure 1.4: A n-D domain Ω with boundary $\partial\Omega$

Remark 1 ***Volume of Ball in*** $\mathbb{R}^n$
The volume of the ball with radius r in $\mathbb{R}^n$ *is given by*

$$V_n(r) = r^n \ V_n(1) \tag{1.24}$$

where $V_n(1)$ *is the volume of the ball with radius* $r = 1$ *in* $\mathbf{R^n}$ *(called unit volume).*

If we differentiate it with respect to r we get the surface area from the volume of the ball with radius r in $\mathbb{R}^n$. That is,

$$\begin{aligned} V_n(r) &= r^n \ V_n(1) \\ \frac{d}{dr}(V_n(r)) &= \frac{d}{dr}(r^n \ V_n(1)) = n \ r^{n-1} \ V_n(1) \\ \implies A_n(r) &= n \ r^{n-1} \ V_n(1) \end{aligned} \tag{1.25}$$

where $A_n(r)$ is the surface area of the ball in $\mathbb{R}^n$ with radius r.

Thus,

$$\varphi(r) = \frac{-A}{(n-2)r^{n-2}} + B \quad \text{and} \tag{1.26}$$

$$\nabla\varphi = \frac{\partial}{\partial r}(\varphi(r)) = \frac{\partial}{\partial r}(\frac{-A}{(n-2)r^{n-2}} + B) = \frac{-A}{(n-2)} \ (2-n) \ r^{1-n} = \frac{A}{r^{n-1}} \tag{1.27}$$

Thus,

$$\begin{aligned}
\underbrace{\int_\Omega \cdots \int_\Omega}_{n-integrals} \nabla^2 \varphi \, d\Omega &= \int_\Omega \cdots \int_\Omega \nabla \cdot \nabla \varphi \, d\Omega \\
&= \int_\Omega \cdots \int_\Omega div \cdot (\nabla \varphi) \, d\Omega \\
&= \underbrace{\int_{\Gamma_\epsilon} \cdots \int_{\Gamma_\epsilon}}_{(n-1) \ integrals} \nabla \varphi \cdot \hat{n} \, d\Gamma \\
&= \int_{\Gamma_\epsilon} \cdots \int_{\Gamma_\epsilon} \frac{A}{r^{n-1}} \cdot \, d\Gamma \\
&= \frac{A}{r^{n-1}} \int_{\Gamma_\epsilon} \cdots \int_{\Gamma_\epsilon} 1 \cdot \, d\Gamma \\
&= \frac{A}{r^{n-1}} \cdot A_n(r) \\
&= \frac{A}{r^{n-1}} \ n \ r^{n-1} \ V_n(1) \\
&= n \ V_n(1) \ A \\
\int_\Omega \cdots \int_\Omega \nabla^2 \varphi \, d\Omega &= n \ V_n(1) \ A \qquad (1.28)
\end{aligned}$$

Thus, substituting (1.28) into equation (1.23) yields

$$\int_\Omega \cdots \int_\Omega \nabla^2 \varphi \, d\Omega = n \ V_n(1) \ A = 1 \implies n \ V_n(1) \ A = 1 \implies A = \frac{1}{n \ V_n(1)}$$

Setting $B = 0$ in equation (1.26), we obtain the fundamental solution to the n-D Laplace's Equation.

$$\begin{gathered}
\varphi(r) = \frac{-A}{(n-2)r^{n-2}} + B \qquad \text{with } A = \frac{1}{n \ V_n(1)} \text{ and } B = 0 \\
\implies \varphi(r) = \frac{-1}{n \ V_n(1)} \frac{1}{(n-2)r^{n-2}} \\
\implies \varphi(r) = \frac{-1}{n \ (n-2) \ V_n(1) \ r^{n-2}} \qquad \text{for } n \geq 3 \qquad (1.29)
\end{gathered}$$

Note: n-D Divergence Theorem

$$\int \cdots \int_S (\vec{F} \cdot \hat{n}) \, dS = \int \cdots \int_R div \cdot \vec{F} \, dv$$

1.3 Poison's Equation

Definition 2 *A fundamental solution $\Phi(x)$ to Laplacian equation can then be defines by*

$$\Phi(x)=\left\{\begin{array}{rl}\frac{1}{2\pi}\ln|x| & for \quad n=2\\ \frac{-1}{n(n-2)\alpha(x)}\cdot\frac{1}{|x|^{n-2}} & for \quad n\geq 3\end{array}\right. \quad where\ \alpha(n) is\ a\ unit\ volume\ in\ \mathbb{R}^n \tag{1.30}$$

Thus, this fundamental solution gives a non-trivial Laplace's equation and also with this fundamental solution, we can actually solve an equation called **Poisson's Equation.**

Here is a fundamental fact,

Fact 2 *If f is given and $u(x)=\Phi * f=\int_{\mathbb{R}^n}\Phi(x-y)f(y)\,dy$ (Phi convolved with f), then $u(x)$ solves $\triangle u=f$.*

If we take this fundamental function $u(x)$ which involves f and a fundamental solution $\Phi(x)$, it turns out that it solves equation $\triangle u=f$ called Poisson's equation. In other word this process of convolution allows us to go from the solution of Laplace's equation to the solution of more general equation called a Poisson's equation.

Chapter 2

Boundary Value Problem Governed by 2-D Laplace's Equation

2.1 Introduction

As a starting point consider solving Equation (2.1)

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0. \tag{2.1}$$

in the two-dimensional region R (on the xy plane) bounded by a simple closed curve C subject to the boundary condition

$$\phi = f_1(x,\ y) \qquad \text{for } (x,\ y) \in C, \tag{2.2}$$

where f_1 is prescribed functions. See Figure 2.1 for a geometrical sketch of the problem.

The normal derivative $\partial\phi/\partial n$ in Eq. (2.2) is defined by

$$\frac{\partial \phi}{\partial n} = n_x \frac{\partial \phi}{\partial x} + n_y \frac{\partial \phi}{\partial y}, \tag{2.3}$$

where n_x and n_y are respectively the x and y components of a unit normal vector to the curve C. Here the unit normal vector $[n_x, n_y]$ on C is taken to be pointing away from the region R. Note that the normal vector may vary from point to point on C. Thus, $[n_x,\ n_y]$ is a function of x and y.
In the following section we see how a boundary integral solution can be derived for the boundary value problem under consideration Eq. (2.1) .

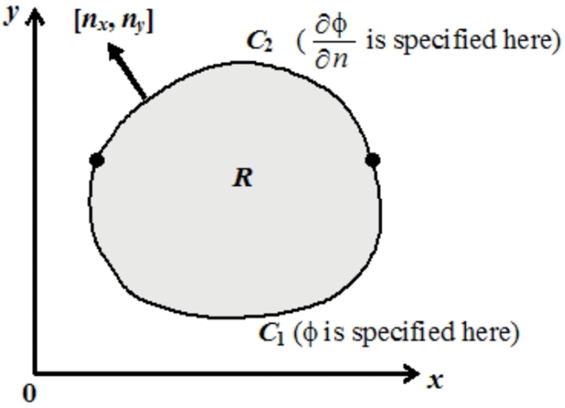


Figure 2.1: Region 'R' bounded by a curve 'C'

Fundamental Solution

If we use polar coordinates r and θ centered about $(\xi,\ \eta)$, as defined by $x = r\cos\theta$ and $y = r\sin\theta$, and introduce $\psi(r,\ \theta) = \phi(r\cos\theta, r\sin\theta)$, we can rewrite Eq.(2.1) as

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial\psi}{\partial r}) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} = 0. \tag{2.4}$$

For the case in which ψ is independent of θ, that is, if ψ is a function of r alone, Eq. (2.4) reduces to the ordinary differential equation

$$\frac{d}{dr}(r\frac{d}{dr}[\psi(r)]) = 0 \qquad \text{for } r \neq 0. \tag{2.5}$$

The ordinary differential equation in Eq. (2.5) can be easily integrated twice to yield the general solution

$$\psi(r) = A\ln(r) + B, \tag{2.6}$$

where A and B are arbitrary constants.

From (2.6), it is obvious that the two-dimensional Laplace's equation in Eq.(2.1) admits a class of particular solutions given by

$$\phi(x,\ y) = A\ln\sqrt{x^2+y^2} + B \qquad \text{for } (x, y) \neq (\xi,\ \eta). \tag{2.7}$$

The particular solution of Eq.(2.1) which we call it fundamental solution for n-D was formulated and given in previous chapter. In particular for 2-D the fundamental solution of Eq.(2.1) is

$$\Phi(x, y; \xi,\ \eta) = \frac{1}{2\pi}\ln\sqrt{(x-\xi)^2 + (y-\eta)^2} \qquad for\ (x,\ y) \neq (\xi,\ \eta) \tag{2.8}$$

Note that $\Phi(x,\ y; \xi,\ \eta)$ satisfies Eq.(2.1) everywhere except at $(\xi,\ \eta)$ where it is not well defined.

2.2 Divergence Theorem

According to the two-dimensional version of **the Gauss-Ostrogradskii (divergence) theorem**, if $\Gamma = u(x,y)\mathrm{i} + v(x,\ y)\mathrm{j}$ is a well defined vector function such that $\nabla \cdot F = \partial u/\partial x + \partial v/\partial y$ exists in the region R bounded by the simple closed curve C then

$$\int_C \vec{F} \cdot \hat{n}\, ds(x,y) = \iint_R \nabla \cdot \vec{F}\, dx\, dy,$$

that is,

$$\int_C [u\hat{n}_x + v\hat{n}_y]\, ds(x,y) = \iint_R [\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}]\, dx\, dy,$$

where $\hat{n} = [\hat{n}_x,\ \hat{n}_y]$ is the unit normal vector to the curve C, pointing away from the region R.
And therefore application of this theorem converts the double integral over R into a line integral in 2-D.

2.3 Reciprocal Relation

If ϕ_1 and ϕ_2 are any two solutions of Eq.(2.1) in the region R bounded by the simple closed curve C then it can be shown that

$$\int_C (\phi_2 \frac{\partial \phi_1}{\partial n} - \phi_1 \frac{\partial \phi_2}{\partial n})\, ds(x,\ y) = 0. \tag{2.9}$$

Eq.(2.9) provides a reciprocal relation between any two solutions of the Laplace's equation in the region R bounded by the curve C.
The derivation of the above reciprocal relation is as follows:

Since ϕ_1 and ϕ_2 are solutions of Eq.(2.1), we may write

$$\begin{cases} \frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial y^2} &= 0 \\ \frac{\partial^2 \phi_2}{\partial x^2} + \frac{\partial^2 \phi_2}{\partial y^2} &= 0 \end{cases}$$

$$\begin{cases} \phi^1_{xx} + \phi^1_{yy} = 0 \\ \phi^2_{xx} + \phi^2_{yy} = 0 \end{cases}$$

Multiplying the first equation by ϕ^2 and the second by ϕ^1, we have

$$\begin{cases} \phi^2\ \phi^1_{xx} + \phi^2\ \phi^1_{yy} = 0 \\ \phi^1\ \phi^2_{xx} + \phi^1\ \phi^2_{yy} = 0 \end{cases}$$

Subtracting them yields

$$[\phi^2\ \phi^1_{xx} + \phi^2\ \phi^1_{yy}] - [\phi^1\ \phi^2_{xx} + \phi^1\ \phi^2_{yy}] = 0$$
$$[\phi^2\ \phi^1_{xx} - \phi^1\ \phi^2_{xx}] + [\phi^2\ \phi^1_{yy} - \phi^1\ \phi^2_{yy}] = 0 \tag{2.10}$$

Note that:

$$\begin{aligned}\frac{\partial}{\partial x}(\phi^2\ \phi^1_x - \phi^1\ \phi^2_x) &= \phi^2_x\ \phi^1_x + \phi^2\ \phi^1_{xx} - \phi^1_x\ \phi^2_x - \phi^1\ \phi^2_{xx}\\ &= \phi^2\ \phi^1_{xx} - \phi^1\ \phi^2_{xx}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial x}(\phi^2\ \phi^1_y - \phi^1\ \phi^2_y) &= \phi^2_y\ \phi^1_y + \phi^2\ \phi^1_{yy} - \phi^1_y\ \phi^2_y - \phi^1\ \phi^2_{yy}\\ &= \phi^2\ \phi^1_{yy} - \phi^1\ \phi^2_{yy}\end{aligned}$$

Thus, equation (2.10) can be written in the following form

$$\begin{aligned}[\phi^2\ \phi^1_{xx} - \phi^1\ \phi^2_{xx}] + [\phi^2\ \phi^1_{yy} - \phi^1\ \phi^2_{yy}] &= 0\\ \frac{\partial}{\partial x}(\phi^2\ \phi^1_x - \phi^1\ \phi^2_x) + \frac{\partial}{\partial x}(\phi^2\ \phi^1_y - \phi^1\ \phi^2_y) &= 0\end{aligned}$$

This can be integrated over R to give

$$\iint_R [\frac{\partial}{\partial x}(\phi^2\ \phi^1_x - \phi^1\ \phi^2_x) + \frac{\partial}{\partial x}(\phi^2\ \phi^1_y - \phi^1\ \phi^2_y)]\, ds(x,y) = 0 \tag{2.11}$$

Recall Divergence Theorem

$$\int_C \vec{F}\cdot\hat{n}\, ds(x,y) = \iint_R \nabla\cdot\vec{F}\, dx\, dy,$$

that is,

$$\int_C [un_x + vn_y]\, ds(x,y) = \iint_R (u_x + v_y)\, dx\, dy,$$

where

$$\begin{aligned}\hat{n} &= [\hat{n}_x,\ \hat{n}_y]\\ \vec{F} &= u\ \hat{i} + v\ \hat{j}\\ \nabla\cdot\vec{F} &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = u_x + v_y\end{aligned}$$

From equation (2.11) above, let

$$\begin{aligned}u &= \phi^2\ \phi^1_x - \phi^1\ \phi^2_x\\ v &= \phi^2\ \phi^1_y - \phi^1\ \phi^2_y\end{aligned}$$

$$\iint_R [\frac{\partial}{\partial x}\underbrace{(\phi^2\ \phi^1_x - \phi^1\ \phi^2_x)}_{u} + \frac{\partial}{\partial x}\underbrace{(\phi^2\ \phi^1_y - \phi^1\ \phi^2_y)}_{v}]\, ds(x,y) = 0$$

$$\iint_R [\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v]\, ds(x,y) = 0$$

$$\int_C [un_x + vn_y]\, ds(x,y) = 0$$

$$\int_C [(\phi^2\ \phi^1_x - \phi^1\ \phi^2_x)n_x + (\phi^2\ \phi^1_y - \phi^1\ \phi^2_y)n_y]\, ds(x,y) = 0 \tag{2.12}$$

Note that:

$$\begin{aligned}
(\phi^2\ \phi^1_x - \phi^1\ \phi^2_x)\hat{n}_x + (\phi^2\ \phi^1_y - \phi^1\ \phi^2_y)\hat{n}_y &= \phi^2\ \phi^1_x\ n_x - \phi^1\ \phi^2_x\ n_x + \phi^2\ \phi^1_y\ n_y - \phi^1\ \phi^2_y\ n_y \\
&= (\phi^2\ \phi^1_x\ \hat{n}_x + \phi^2\ \phi^1_y\ \hat{n}_y) - (\phi^1\ \phi^2_x\ \hat{n}_x + \phi^1\ \phi^2_y\ \hat{n}_y) \\
&= \phi^2[\underbrace{\hat{n}_x\ \phi^1_x + \hat{n}_y\ \phi^1_y}_{\frac{\partial \phi^1}{\partial \hat{n}}}] - \phi^1[\underbrace{\hat{n}_x\ \phi^2_x + \hat{n}_y\ \phi^2_y}_{\frac{\partial \phi^2}{\partial \hat{n}}}] \\
&= \phi^2\ \frac{\partial \phi^1}{\partial \hat{n}} - \phi^1\ \frac{\partial \phi^2}{\partial \hat{n}}
\end{aligned}$$

Therefore substituting this in equation (2.12),

$$\int_C \underbrace{[(\phi^2\ \phi^1_x - \phi^1\ \phi^2_x)\hat{n}_x + (\phi^2\ \phi^1_y - \phi^1\ \phi^2_y)\hat{n}_y]}_{\phi^2\ \frac{\partial \phi^1}{\partial \hat{n}} - \phi^1\ \frac{\partial \phi^2}{\partial \hat{n}}}\, ds(x,y) = 0$$

$$\int_C (\phi^2\ \frac{\partial \phi^1}{\partial \hat{n}} - \phi^1\ \frac{\partial \phi^2}{\partial \hat{n}})\, ds(x,y) = 0$$

which is the reciprocal relation.

2.4 Green's Identity

In the theorey of boundary value problems for the Laplace and other elliptic equations a fundamental role is played by the so-called Green's identities.
Since

$$\begin{aligned}
&\nabla \cdot (u\ \nabla v) = (\nabla u\ \nabla v) + u\ \Delta v \\
&u\ (\nabla v) \cdot \hat{n} = u\ \nabla_n v
\end{aligned}$$

we apply the divergence theorem to $u\ (\nabla v)$ and establish the first identity

$$
\begin{aligned}
\nabla \cdot (u\ \nabla v) &= (\nabla u\ \nabla v) + u\ \Delta v \\
\int_\Omega \nabla \cdot (u\ \nabla v)\, d\Omega &= \int_\Omega [(\nabla u\ \nabla v) + u\ \Delta v]\, d\Omega \\
\int_\Gamma (u\ \nabla v) \cdot \hat{n}\, d\Gamma &= \int_\Omega [(\nabla u\ \nabla v) + u\ \Delta v]\, d\Omega \\
\int_\Gamma u\ \nabla_n v\, d\Gamma &= \int_\Omega [(\nabla u\ \nabla v) + u\ \Delta v]\, d\Omega
\end{aligned}
$$

We obtain

$$
\int_\Omega (\nabla u\ \nabla v + u\ \triangle v)\, d\Omega = \int_\Gamma u\ \nabla_n v\, \mathrm{d}\Gamma \tag{2.13}
$$

This expression is called **Green's first identity**.

Interchanging u and v in it, we obtain

$$
\int_\Omega (\nabla v\ \nabla u + v\ \triangle u)\, d\Omega = \int_\Gamma v\ \nabla_n u\, d\Gamma \tag{2.14}
$$

and subtracting this from the first identity, we obtain **Green's second identity**

$$
\int_\Omega [u\ \triangle v - \triangle u\ v]\, d\Omega = \int_\Gamma [u\ \nabla_n v - \nabla_n u\ v]\, d\Gamma. \tag{2.15}
$$

2.5 Green's Representation Formula

Formulation for Laplace's equation

$$
\triangle u = 0 \qquad \text{in} \quad \Omega \tag{2.16}
$$

using the generic variable u for the potential
First Weighting Laplace's equation (2.16) with a test function w, we obtain

$$
\int_\Omega \triangle u\ w\, d\Omega = 0 \tag{2.17}
$$

Next, we eliminate the partial derivatives of the potential function u from the domain integral. This is achieved as follows: integration by parts of (2.17) leads to

$$
\int_\Omega \triangle u\ w\, d\Omega = \int_\Omega \frac{\partial}{\partial \xi_i}(\frac{\partial u}{\partial \xi_i}\ w)\, d\Omega - \int_\Omega \frac{\partial u}{\partial \xi_i}\ \frac{\partial w}{\partial \xi_i}\, d\Omega \tag{2.18}
$$

and by applying the Gaus theorem to the first term on the right-hand side of (2.18) to replace the domain integral by a boundary integral, we obtain **Green's first identity**

$$
\begin{aligned}
\int_\Omega \triangle u\ w\, d\Omega &= \int_\Gamma \frac{\partial u}{\partial \xi_i}\ w\ n_i\, d\Gamma - \int_\Omega \frac{\partial u}{\partial \xi_i}\ \frac{\partial w}{\partial \xi_i}\, d\Omega \\
\int_\Omega \triangle u\ w\, d\Omega &= \int_\Gamma \nabla_n u\ w\ \ d\Gamma - \int_\Omega \nabla u \nabla w\, d\Omega
\end{aligned} \tag{2.19}
$$

To eliminate the remaining partial derivative u_i in the domain integral on the right-hand side, we have to again apply integration by parts and Gaus' theorem. This yields

$$\int_\Omega \triangle u \; w \, d\Omega = \int_\Gamma (\frac{\partial u}{\partial \xi_i} \; w - u \; \frac{\partial w}{\partial \xi_i}) n_i \, d\Gamma + \int_\Omega u \; \triangle w \, d\Omega \tag{2.20}$$

which is known as **Green second identity**.
By substituting the differential equation (2.17) into (2.20), we eliminate the first domain integral and obtain

$$\begin{aligned}
-\int_\Omega u \triangle w \, d\Omega &= \int_\Gamma (\frac{\partial u}{\partial \xi_i} \; w - u \; \frac{\partial w}{\partial \xi_i}) n_i \, d\Gamma \\
\int_\Omega u \triangle w \, d\Omega &= -\int_\Gamma (\frac{\partial u}{\partial \xi_i} \; w - u \; \frac{\partial w}{\partial \xi_i}) n_i \, d\Gamma \\
\int_\Omega u \triangle w \, d\Omega &= \int_\Gamma (u \; \frac{\partial w}{\partial \xi_i} - \frac{\partial u}{\partial \xi_i} \; w) n_i \, d\Gamma \\
\int_\Omega u \triangle w \, d\Omega &= \int_\Gamma (u \; \nabla_{n_i} w - \nabla_{n_i} u \; w) \, d\Gamma
\end{aligned} \tag{2.21}$$

The Dirac distribution $\delta(x, \; \xi)$ by its sifting property $\int f(x) \delta(x, \; \xi) \, dx = f(\xi)$. This allows us to filter out a specific functional value $f(\xi)$ from an integral, thereby eliminating this integral.

In general, a fundamental solution w of the differential operator $\mathcal{L}$ is defined as a solution of the equation

$$\mathcal{L} w = \delta(x, \; \xi) \tag{2.22}$$

in the full space Ω^∞, where the minus sign in front of the Dirac distribution is used for convenience. We will now employ this property to eliminate the domain integral in (2.21) by choosing

$$\triangle w := \delta(x, \; \xi) \tag{2.23}$$

which yields

$$\int_\Omega u \; \triangle w \, d\Omega = u(\xi) \tag{2.24}$$

Employing now as test function w the fundamental solution u, we obtain from (2.21)

$$u(\xi) = \int_\Gamma [u(x) \; \nabla_n w(x, \; \xi) - \nabla_n u(x) \; w(x, \; \xi)] \; d\Gamma, \tag{2.25}$$

This equation is called **representation formula**, which in this particular case is also known as **Green's representation formula**. The representation formula allows us to calculate unknown values of the potential u inside the domain ($\xi \in \Omega$) when the boundary solution of the problem (potential u and flux $\nabla_n u$) is known.

2.6 Fundamental Solutions

In 2-D, the fundamental solution of the Laplace operator as defined in previous section, is given by

$$w(x,\ \xi) = -\frac{1}{2\pi}\ln|x_i - \xi_i| = -\frac{1}{2\pi}\ln r, \tag{2.26}$$

$$\nabla_{n_i} w(x,\ \xi) = \nabla_{x_i} w\ n_i = -\frac{1}{2\pi r} r_i n_i = -\frac{1}{2\pi|x_i - \xi_i|^2}(x_i - \xi_i)n_i, \tag{2.27}$$

where r denotes the Euclidean distance $|x_i - \xi_i| = \sqrt{(x_i - \xi_i)^2}$ between the load point ξ_i and field point x_i.
In the 3-D case, we have

$$w(x,\ \xi) = \frac{1}{4\pi|x_i - \xi_i|} = \frac{1}{4\pi r}, \tag{2.28}$$

$$\nabla_{n_i} w(x,\ \xi) = -\frac{1}{4\pi r^2} r_i n_i = -\frac{1}{4\pi|\mathrm{x}_i - \xi_i|^3}(x_i - \xi_i)n_i. \tag{2.29}$$

The fundamental solution w of the Laplace operator is well known from electrostatics, where it describes the electric potential due to a point charge at the load point ξ_i.

2.7 Preparative Example for the Limiting Process

If we solve the integral

$$\int_{-a}^{b} \frac{1}{x}\mathrm{d}x = \ln|b| - \ln|-a|, \qquad a, b > 0 \tag{2.30}$$

as shown, the correct result is obtained. However, on second thoughts, we note that this must have been by chance, since the presence of the $1/x$-singularity in the integration interval has not been taken into account property. The problem at $x = 0$ becomes apparent when trying to calculate the improper integral $\displaystyle\int_{x=0}^{b} \frac{1}{x}\,\mathrm{d}x$, which is undefined.
If we approach the singularity in (2.30) by a limiting process, we obtain

$$\begin{aligned}\int_{-a}^{b} \frac{1}{x}\mathrm{d}x &= \lim_{\epsilon_1,\epsilon_2\to 0}\Big(\int_{-a}^{-\varepsilon_1} \frac{1}{x}\mathrm{d}x + \int_{\epsilon_2}^{b} \frac{1}{x}dx\Big) \\ &= \lim_{\epsilon_1,\epsilon_2\to 0}(\ln\epsilon_1 - \ln\mathrm{a} + \ln b - \ln\varepsilon_2) \\ &= \lim_{\varepsilon_1,\varepsilon_2\to 0}\Big(\ln\frac{\varepsilon_1}{\varepsilon_2}\Big)\ + \ln b - \ln a.\end{aligned} \tag{2.31}$$

We see that the result depends on ε_1 and ε_2 if they approach zero with different values. However, by choosing $\varepsilon_1 = \varepsilon_2 = \varepsilon$, we obtain

$$\ln\frac{\varepsilon}{\varepsilon} = \ln 1 = 0$$

,

and thus (2.31) yields the correct result

$$\int_{-a}^{b} \frac{1}{x}\, dx = \ln \frac{b}{a} \tag{2.32}$$

The integrand $f(x) = 1/x$ in (2.30) is **strongly singular** at $x = 0$, which means that its integral $F(x) = \int f(x)\mathrm{d}x$ is singular at $x = 0$, too. The value of the integral as calculated with the limiting process in (2.31) is called a **Cauchy principal value(CPV)** of the strongly singular integral and is denoted by

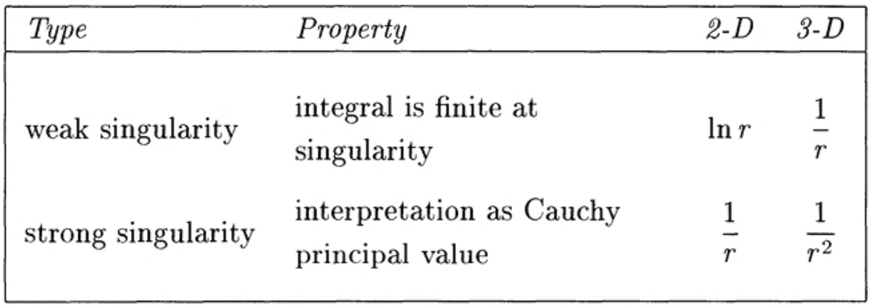
$$\oint_{-a}^{b} \frac{1}{x}\mathrm{d}x \tag{2.33}$$

In addition to the strongly singular integrands, we also encounter **weakly singular** integrands in the Boundary Element Method. In contrast to the strong singularity, the integral over a weakly singular integrand exists and is continuous at the singularity point. An example for this is the $\ln|x|$-function. At $x = 0$ the function is singular but the integral

$$\int \ln|x|\, dx = x\ln|x| - x + c \tag{2.34}$$

is continuous, which can be confirmed by applying the rule of **L'Hospital**:

$$\lim_{x\to 0} x\ln|x| = \lim_{x\to 0} \frac{\ln|x|}{\frac{1}{x}} = \lim_{x\to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = 0. \tag{2.35}$$



Type	*Property*	*2-D*	*3-D*
weak singularity	integral is finite at singularity	$\ln r$	$\frac{1}{r}$
strong singularity	interpretation as Cauchy principal value	$\frac{1}{r}$	$\frac{1}{r^2}$

Figure 2.2: Classification of singularities in the Boundary Element Method

2.8 Boundary Integral Solution of the 2-D Problem

Method 1. Formulation using reciprocal relation

Consider the above reciprocal relation defined with $\phi_1 = \Phi(x,\ y;\xi,\ \eta)$ (the fundamental solution as defined in Eq.(2.8) and $\phi_2 = \phi$, where ϕ is the required solution of the interior boundary value problem defined by Equ.(2.1) – (2.2) .

Since $\Phi(x,\ y;\xi,\ \eta)$ is not well defined at the point (ξ,η), the reciprocal relation in Eq. (2.9) is valid for $\phi_1 = \Phi(x,\ y;\xi,\ \eta)$ and $\phi_2 = \phi$ only if $(\xi,\ \eta)$ does not lie in the region $R \cup C$. Thus,

$$\int_C [\phi(x,\ y)\frac{\partial}{\partial n}(\Phi(x,\ y;\xi,\ \eta)) - \Phi(x,\ y;\xi,\ \eta)\frac{\partial}{\partial n}(\phi(x,\ y))]ds(x,\ y) = 0 \qquad \text{for } (\xi,\ \eta) \notin R \cup C. \tag{2.36}$$

For the case in which $(\xi,\ \eta)$ lies in the interior of R, Eq. (2.36) is valid if we replace C by $C \cup C_{\epsilon:}$, where $C_{\epsilon:}$ is a circle of center $(\xi,\ \eta)$ and radius ε as shown in Figure 2.8. This is because $\Phi(x,\ y;\xi,\ \eta)$ and its first order partial derivatives (with respect to x or y) are well defined in the region between C and C_ε. Thus, for C and $C_{\epsilon:}$ in Figure 2.8, we can write

$$\int_{C \cup C_\epsilon} [\phi(x,\ y)\frac{\partial}{\partial n}(\Phi(x,\ y;\xi,\ \eta)) - \Phi(x,\ y;\xi,\ \eta)\frac{\partial}{\partial n}(\phi(x,\ y))]\,ds(x,\ y) = 0,$$

that is,

$$\begin{aligned} &\int_C [\phi(x,\ y)\frac{\partial}{\partial n}(\Phi(x,y;\xi,\ \eta)) - \Phi(x,\ y;\xi,\ \eta)\frac{\partial}{\partial n}(\phi(x,\ y))]ds(x,\ y) \\ = &-\int_{C_\epsilon} [\phi(x,\ y)\frac{\partial}{\partial n}(\Phi(x,\ y;\xi,\ \eta)) - \Phi(x,\ y;\xi,\ \eta)\frac{\partial}{\partial n}(\phi(x,\ y))]ds(x,\ y) \end{aligned} \tag{2.37}$$

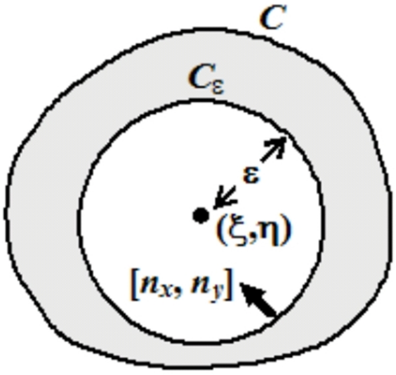


Figure 2.3: $(\xi,\ \eta)$ lies in the interior of R

Eq. (2.37) holds for any radius $\varepsilon > 0$, so long as the circle C_ε (in Figure 2.8) lies completely inside the region bounded by C. Thus, we may let $\varepsilon \to 0^+$ in Eq.(2.37). This gives

$$\begin{aligned} &\int_c [\phi(x,\ y)\frac{\partial}{\partial n}(\Phi(x,\ y;\xi,\ \eta)) - \Phi(x,\ y;\xi,\ \eta)\frac{\partial}{\partial n}(\phi(x,\ y))]\,ds(x,\ y) \\ = &-\lim_{\varepsilon \to 0^+} \int_{c_\varepsilon} [\phi(x,\ y)\frac{\partial}{\partial n}(\Phi(x,\ y;\xi,\ \eta)) - \Phi(x,\ y;\xi,\ \eta)\frac{\partial}{\partial n}(\phi_(x,\ y))]\,ds(x,\ y) \end{aligned} \tag{2.38}$$

Using polar coordinates r and θ centered about $(\xi,\ \eta)$ as defined by $x - \xi = r\cos\theta$ and $y - \eta = r\sin\theta$, we may write the fundamental solution and its normal derivative as follows

$$\begin{aligned}\Phi(x,\ y;\xi,\ \eta) &= \frac{1}{2\pi}\ln\sqrt{(x-\xi)^2+(y-\eta)^2} \ = \ \frac{1}{2\pi}\ln(r),\\ \frac{\partial}{\partial n}[\Phi(x,\ y;\xi,\ \eta)] &= n_x\frac{\partial}{\partial x}[\Phi(x,\ y;\xi,\ \eta)]+n_y\frac{\partial}{\partial y}[\Phi(x,y;\xi,\ \eta)]\\ &= \frac{n_x\cos\theta+n_y\sin\theta}{2\pi r}\end{aligned}$$

Note that:

$$\begin{aligned}\frac{\partial}{\partial x}\Phi(x,\ y;\xi,\eta) &= \frac{1}{2\pi}\ \frac{1}{\sqrt{(x-\xi)^2+(y-\eta)^2}}\ \frac{1}{2}\ \frac{1}{\sqrt{(x-\xi)^2+(y-\eta)^2}}\ 2(x-\xi)\\ &= \frac{1}{2\pi}\ \frac{1}{r^2}\ r\cos\theta\\ &= \frac{\cos\theta}{2\pi r}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial y}\Phi(x,\ y;\xi,\eta) &= \frac{1}{2\pi}\ \frac{1}{\sqrt{(x-\xi)^2+(y-\eta)^2}}\ \frac{1}{2}\ \frac{1}{\sqrt{(x-\xi)^2+(y-\eta)^2}}\ 2(x-\xi)\\ &= \frac{1}{2\pi}\ \frac{1}{r^2}\ r\sin\theta\\ &= \frac{\sin\theta}{2\pi r}\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\partial}{\partial n}\Phi(x,\ y;\xi,\ \eta) &= n_x\frac{\partial}{\partial x}\Phi(x,\ y;\xi,\eta)+n_y\frac{\partial}{\partial y}\Phi(x,\ y;\xi,\eta)\\ &= n_x\frac{\cos\theta}{2\pi r}+n_y\frac{\sin\theta}{2\pi r}\\ &= \frac{n_x\cos\theta+n_y\sin\theta}{2\pi r}\end{aligned} \tag{2.39}$$

The Taylor's series of $\phi(x,\ y)$ and $\frac{\partial}{\partial n}[\phi_(x,\ y)]$ about the point $(\xi,\ \eta)$ are given by

$$\begin{aligned}\phi_(x,\ y) &= \sum_{m=0}^{\infty}\sum_{k=0}^{m}(\frac{\partial^m\phi}{\partial x^k\partial y^{m-k}})|_{(x,y)=(\xi,\eta)}\frac{(x-\xi)^k(y-\eta)^{m-k}}{k!(m-k)!}\\ \frac{\partial}{\partial n}[\phi_(x,\ y)] &= \sum_{m=0}^{\infty}\sum_{k=0}^{m}(\frac{\partial^m\phi}{\partial x^k\partial y^{m-k}})[\frac{\partial}{\partial n}[\phi_(x,\ y)]|_{(x,y)=(\xi,\eta)}\frac{(x-\xi)^k(y-\eta)^{m-k}}{k!(m-k)!}\end{aligned}$$

On the circle $C_\epsilon,\ r=\varepsilon$. Since

$$
\begin{aligned}
(x-\xi)^k(y-\eta)^{m-k} &= (\varepsilon\cos\theta)^k(\varepsilon\sin\theta)^{m-k}\\
&= \varepsilon^k\cos^k\theta\ \varepsilon^{m-k}\sin^{m-k}\theta\\
&= \varepsilon^m\cos^k\theta\ \ \sin^{m-k}\theta
\end{aligned}
$$

and

$$
\begin{aligned}
(x-y) &= \varepsilon\cos\theta\\
(x-y) &= \varepsilon\sin\theta
\end{aligned}
$$

And hence, $\phi(x,\ y)$ and $\frac{\partial}{\partial n}[\phi(x,\ y)]$ are given as follows for $(x,\ y)\in C_\epsilon$

$$
\phi(x,\ y) = \sum_{m=0}^{\infty}\sum_{k=0}^{m}\Big(\frac{\partial^m}{\partial x^k\partial y^{m-k}}[\phi(x,\ y)]\Big)\ |_{(x,y)=(\xi,\eta)}\frac{\varepsilon^m\cos^k\theta\sin^{m-k}\theta}{k!(m-k)!} \tag{2.40}
$$

$$
\frac{\partial}{\partial n}[\phi_(x,\ y)]\ = \sum_{m=0}^{\infty}\sum_{k=0}^{m}\Big(\frac{\partial^m}{\partial x^k\partial y^{m-k}}\Big(\frac{\partial}{\partial n}[\phi(x,\ y)]\Big)|_{(x,y)=(\xi,\eta)}\frac{\varepsilon^m\cos^k\theta\sin^{m-k}\theta}{k!(m-k)!} \tag{2.41}
$$

Using the folowing already formulated Eqs.(2.39), (2.40) and (2.41) and writing $ds(x,\ y)=\varepsilon d\theta$ with θ ranging from 0 to 2π, we may now attempt to evaluate the limit on the right hand side of Eq.(2.38). On C_ε, the normal vector $[n_x,\ n_y]$ is given by $[-\cos\theta,\ -\sin\theta]$. Thus,

Note:

$$
\int_c\Big(\phi\frac{\partial}{\partial n}\Phi-\Phi\frac{\partial}{\partial n}\phi\Big)\,ds(x,\ y) = -\lim_{\varepsilon\to 0^+}\int_{c_\varepsilon}\Big(\frac{\partial}{\partial n}(\Phi-\Phi\frac{\partial}{\partial n}\phi\Big)\,ds(x,\ y)
$$

$$
\begin{aligned}
\Phi(x,\ y;\xi,\ \eta) &= \frac{1}{2\pi}\ln\sqrt{(x-\xi)^2+(y-\eta)^2} = \frac{1}{2\pi}\ln(r),\\
\frac{\partial}{\partial n}[\Phi(x,\ y;\xi,\ \eta)] &= \frac{n_x\cos\theta+n_y\sin\theta}{2\pi r}
\end{aligned} \tag{2.42}
$$

$$
\begin{cases}
\phi(x,\ y) & = \sum_{m=0}^{\infty}\sum_{k=0}^{m}\big(\frac{\partial^m}{\partial x^k\partial y^{m-k}}[\phi(x,\ y)]\big)\ |_{(x,y)=(\xi,\eta)}\frac{\varepsilon^m\cos^k\theta\sin^{m-k}\theta}{k!(m-k)!}\\
\frac{\partial}{\partial n}[\phi_(x,\ y)] & = \sum_{m=0}^{\infty}\sum_{k=0}^{m}\big(\frac{\partial^m}{\partial x^k\partial y^{m-k}}\big(\frac{\partial}{\partial n}[\phi(x,\ y)]\big)|_{(x,y)=(\xi,\eta)}\dfrac{\varepsilon^m\cos^k\theta\sin^{m-k}\theta}{k!(m-k)!}
\end{cases}
$$

$$\int_{c_\varepsilon} \phi_{(}x,\ y)\frac{\partial}{\partial n}[\Phi(x,\ y;\xi,\ \eta]\,ds(x,\ y) =$$

$$= \int_{c_\varepsilon}[\frac{n_x\cos\theta+n_y\sin\theta}{2\pi r}][\sum_{m=0}^{\infty}\sum_{k=0}^{m}(\frac{\partial^m}{\partial x^k\partial y^{m-k}}[\phi(x,\ y)])\mid_{(x,y)=(\xi,\eta)}\frac{\varepsilon^m\cos^k\theta\sin^{m-k}\theta}{k!(m-k)!}]\,dS$$

$$= \int_0^{2\pi}[\frac{-\cos\theta\cos\theta-\sin\theta\sin\theta}{2\pi\varepsilon}][\sum_{m=0}^{\infty}\sum_{k=0}^{m}(\frac{\partial^m}{\partial x^k\partial y^{m-k}}[\phi(x,\ y)])\mid_{(x,y)=(\xi,\eta)}\frac{\varepsilon^m\cos^k\theta\sin^{m-k}\theta}{k!(m-k)!}]\varepsilon\,d\theta$$

$$= \int_0^{2\pi} -\frac{1}{2\pi}[\underbrace{\sum_{m=0}^{\infty}\sum_{k=0}^{m}}_{\sum_{k=0}^{0}+\sum_{k=0}^{1}+\sum_{k=0}^{2}+\cdots}\underbrace{(\frac{\partial^m}{\partial x^k\partial y^{m-k}}[\phi(x,\ y)])\mid_{(x,y)=(\xi,\eta)}\frac{\varepsilon^m\cos^k\theta\sin^{m-k}\theta}{k!(m-k)!}]}_{=\ say\ H}\,d\theta$$

$$= \int_0^{2\pi} -\frac{1}{2\pi}(\sum_{k=0}^{0}+\sum_{k=0}^{1}+\sum_{k=0}^{2}+\cdots)(H)\,d\theta$$

$$= \int_0^{2\pi} -\frac{1}{2\pi}(\sum_{k=0}^{0}+\underbrace{\sum_{k=0}^{1}+\sum_{k=0}^{2}+\cdots}_{\sum_{m=1}^{\infty}\sum_{k=0}^{m}})(H)\,d\theta$$

$$= \int_0^{2\pi} -\frac{1}{2\pi}(\sum_{k=0}^{0}+\sum_{m=1}^{\infty}\sum_{k=0}^{m})(H)\,d\theta$$

$$= \int_0^{2\pi} -\frac{1}{2\pi}\underbrace{\sum_{k=0}^{0}(H)}_{\phi(\xi,\eta)}\ d\theta + \int_0^{2\pi} -\frac{1}{2\pi}\sum_{m=1}^{\infty}\sum_{k=0}^{m}(H)\,d\theta$$

$$= \int_0^{2\pi} -\frac{1}{2\pi}\phi(\xi,\eta)\,d\theta + \int_0^{2\pi} -\frac{1}{2\pi}\sum_{m=1}^{\infty}\sum_{k=0}^{m}(H)\,d\theta$$

$$= -\frac{1}{2\pi}\phi(\xi,\eta)\int_0^{2\pi}\ d\theta - \frac{1}{2\pi}\sum_{m=1}^{\infty}\sum_{k=0}^{m}(\frac{\varepsilon^m}{k!(m-k)!})(\frac{\partial^m}{\partial x^k\partial y^{m-k}}\mid_{(x,y)=(\xi,\eta)}\int_0^{2\pi}\cos^k\theta\sin^{m-k}\theta\,d\theta$$

$$= -\frac{1}{2\pi}\phi(\xi,\eta)\underbrace{\int_0^{2\pi}\ d\theta}_{2\pi} - \underbrace{\frac{1}{2\pi}\sum_{m=1}^{\infty}\sum_{k=0}^{m}(\frac{\varepsilon^m}{k!(m-k)!})(\frac{\partial^m}{\partial x^k\partial y^{m-k}}\mid_{(x,y)=(\xi,\eta)}\int_0^{2\pi}\cos^k\theta\sin^{m-k}\theta\,d\theta}_{\to\ 0\ as\ \varepsilon\to 0^+}$$

$$\to -\phi(\xi,\eta)\ as\ \varepsilon\to 0^+$$

Therefore,

$$\int_{c_\varepsilon} \phi_{(}x,\ y)\frac{\partial}{\partial n}[\Phi(x,\ y;\xi,\ \eta]\,ds(x,\ y) = -\phi(\xi,\eta)\ as\ \ as\ \varepsilon\to 0^+$$

and

$$\int_{c_\varepsilon} \Phi(x,\ y;\xi,\ \eta)\frac{\partial}{\partial n}[\phi(x,\ y)]\,ds(x,\ y)$$

$$=\ \frac{1}{2\pi}\sum_{m=0}^{\infty}\sum_{k=0}^{m}(\frac{\partial^m}{\partial x^k\partial y^{m-k}}(\frac{\partial}{\partial n}[\phi(x,\ y)]))\mid_{(x,y)=(\xi,\eta)}\times\frac{\varepsilon^{m+1}\ln(\varepsilon)}{k!(m-k)!}\int_0^{2\pi}\cos^k\theta\sin^{m-k}\theta d\theta$$

$$\to 0 \qquad as\ \varepsilon \to 0^+, \tag{2.43}$$

(since $\varepsilon^{m+1}\ln(\varepsilon) \to 0$ as $\varepsilon \to 0^+$ for $m = 0,\ 1,\ 2,\ \cdots$)
Consequently, as $\varepsilon \to 0^+$, Eq.(2.38) yields

$$\phi(\xi,\ \eta) = \int_c [\phi(x,\ y)\frac{\partial}{\partial n}(\Phi(x, y; \xi,\ \eta)) - \Phi(x,\ y; \xi,\ \eta)\frac{\partial}{\partial n}(\phi(x,\ y))]\, ds(x,\ y)$$
$$for\ (\xi,\ \eta) \in R. \tag{2.44}$$

For the case in which the point $(\xi,\ \eta)$ lies on C, Eq.(2.36) holds if we replace the curve C by $D \cup D_{\epsilon:}$, where the curves D and $D_{\epsilon:}$ are as shown in Figure 2.7. (If C_ε is the circle of center (ξ, η) and radius ε, then D is the part of C that lies outside $C_{\epsilon:}$ and $D_{\epsilon:}$ is the part of $C_{\epsilon:}$ that is inside R.). Thus,

$$\begin{aligned}
&\int_D [\phi(x,\ y)\frac{\partial}{\partial n}(\Phi(x,\ y; \xi,\ \eta)) - \Phi(x,\ y; \xi,\ \eta)\frac{\partial}{\partial n}(\phi(x,\ y))]\, ds(x,\ y) \\
=\ & -\int_{D_\varepsilon} [\phi(x,\ y)\frac{\partial}{\partial n}(\Phi(x,\ y; \xi,\ \eta)) - \Phi(x,\ y; \xi,\ \eta)\frac{\partial}{\partial n}(\phi(x,\ y))]\, ds(x,\ y).
\end{aligned} \tag{2.45}$$

Let us examine what happens to Eq.(2.45) when we let $\varepsilon \to 0^+$.
As $\varepsilon \to 0^+$, the curve D tends to C. Thus, we may write

$$\begin{aligned}
&\int_c [\phi(x,\ y)\frac{\partial}{\partial n}(\Phi(x,\ y; \xi,\ \eta)) - \Phi(x,\ y; \xi,\ \eta)\frac{\partial}{\partial n}(\phi(x,\ y))]\, ds(x,\ y) \\
=\ & -\lim_{\varepsilon \to 0+} \int_{D_\varepsilon} [\phi(x,\ y)\frac{\partial}{\partial n}(\Phi(x,\ y; \xi,\ \eta)) - \Phi(x,\ y; \xi,\ \eta)\frac{\partial}{\partial n}(\phi(x,\ y))] ds(x,\ y)
\end{aligned} \tag{2.46}$$

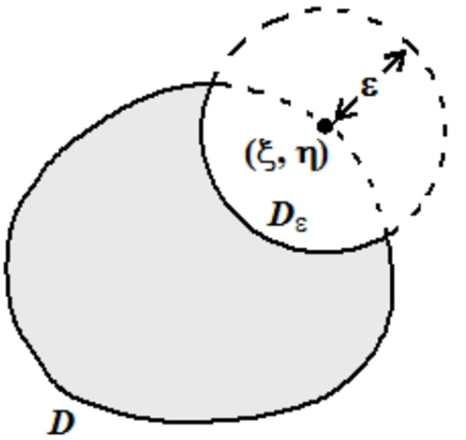


Figure 2.4: $(\xi,\ \eta)$ lying on a smooth part of C

To evaluate the limit on the right hand side of Eq. (2.46), we need to know what happens to $D_{\epsilon:}$ when we let $\varepsilon \to 0^+$. Now if $(\xi,\ \eta)$ lies on a smooth part of C (not at where the

gradient of the curve changes abruptly, that is, not at a corner point, if there is any), one can intuitively see that the part of C inside $C_{\epsilon:}$ approaches an infinitesimal straight line as $\varepsilon \to 0^+$. Thus, we expect $D_{\epsilon:}$ to tend to a semi-circle as $\varepsilon \to 0^+$, if $(\xi,\ \eta)$ lies on a smooth part of C. It follows that in attempting to evaluate the limit on the right hand side of Eq.(2.46) we have to integrate over only half a circle (instead of a full circle as in the case of Eq. (2.38)).
Modifying Eqs.(2.42) and (2.43), we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0+}\int_{D_\varepsilon}\phi(x,\ y)\frac{\partial}{\partial n}[\Phi(x,\ y;\xi,\ \eta)]ds(x,\ y) &= -\frac{1}{2}\phi(\xi,\ \eta)\ ,\\ \lim_{\varepsilon\to 0+}\int_{D_\varepsilon}\Phi(x,\ y;\xi,\ \eta)\frac{\partial}{\partial n}[\phi(x,\ y)]ds(x,\ y) &= 0.\end{aligned}$$

Hence Eq.(2.46) gives

$$\frac{1}{2}\phi(\xi,\eta)=\int_c[\phi(x,\ y)\frac{\partial}{\partial n}(\Phi(x,y;\xi,\ \eta))-\Phi(x,\ y;\xi,\ \eta)\frac{\partial}{\partial n}(\phi(x,\ y))]ds(x,\ y) \qquad (2.47)$$

for $(\xi,\ \eta)$ lying on a smooth part of C.
We may write Eqs.(2.36), (2.44) and (2.47) as a single equation given by

$$\lambda(\xi,\ \eta)\phi(\xi,\eta)=\int_c[\phi(x,\ y)\frac{\partial}{\partial n}(\Phi(x,y;\xi,\ \eta))-\Phi(x,\ y;\xi,\ \eta)\frac{\partial}{\partial n}(\phi(x,\ y))]ds(x,\ y) \quad (2.48)$$

if we define

$$\lambda(\xi,\ \eta)=\begin{cases}0 & \text{if } (\xi,\ \eta)\notin R\cup C,\\ 1/2 & \text{if } (\xi,\ \eta) \text{ lies on a smooth part of } C,\\ 1 & \text{if } (\xi,\ \eta)\in R.\end{cases} \qquad (2.49)$$

Method 2. Formulation
Note that:
In 2-D, the fundamental solution of the Laplace operator is given by

$$w(x,\ \xi)=\frac{1}{2\pi}\ln|x_i-\xi_i|=\frac{1}{2\pi}\ln r, \qquad (2.50)$$

$$\nabla_{n_i}w(x,\ \xi)=\nabla_{x_i}w\ n_i=\frac{1}{2\pi r}r_in_i=\frac{1}{2\pi|x_i-\xi_i|^2}(x_i-\xi_i)n_i, \qquad (2.51)$$

where r denotes the Euclidean distance $|x_i-\xi_i|=\sqrt{(x_i-\xi_i)^2}$ between the load point ξ_i and field point x_i.

Using Green's second replacing v by the fundamental solution w, we have the following:

$$\begin{aligned}
\int_\Omega [u \underbrace{\Delta v}_{\Delta w} - \underbrace{v}_{w} \Delta u] \mathrm{d}\Omega &= \int_{\partial\Omega} [u \underbrace{\nabla_n v}_{\nabla_n w} - \underbrace{v}_{w} \nabla_n u] \ \mathrm{d}\Gamma. \\
\int_\Omega [u \underbrace{\Delta w}_{\delta} - w \underbrace{\Delta u}_{f}] \mathrm{d}\Omega &= \int_{\partial\Omega} [u \nabla_n w - w \nabla_n u] \ \mathrm{d}\Gamma. \\
\int_\Omega [u \ \delta - w \ f] \mathrm{d}\Omega &= \int_{\partial\Omega} [u \nabla_n w - w \nabla_n u] \ \mathrm{d}\Gamma. \\
\int_\Omega u \ \delta \ \mathrm{d}\Omega &= \int_{\partial\Omega} [u \nabla_n w - w \nabla_n u] \ \mathrm{d}\Gamma + \int_\Omega w \ f \ \mathrm{d}\Omega
\end{aligned}$$

Note that the sifting property of the Dirac distribution is not defined when the source/load point lies on the boundary

$$\int f(x)\delta(x, \ \xi)\mathrm{d}\Omega = \begin{cases} f(\xi) & \text{for } \xi \in \Omega \\ 0 & \text{for } \xi \notin \Omega, \ \xi \notin \Gamma \ . \\ \text{undef.} & \text{for } \xi \in \Gamma. \end{cases} \tag{2.52}$$

Therefore,
1. For $\xi \in \Omega^+$

$$\begin{aligned}
\int_\Omega u \ \delta \ \mathrm{d}\Omega &= \int_{\partial\Omega} [u \nabla_n w - w \nabla_n u] \ \mathrm{d}\Gamma + \int_\Omega w \ f \ \mathrm{d}\Omega \\
u(\xi) &= \int_{\partial\Omega} [u \nabla_n w - w \nabla_n u] \ \mathrm{d}\Gamma + \int_\Omega w \ f \ \mathrm{d}\Omega
\end{aligned} \tag{2.53}$$

2. For $\xi \in \Omega^-$

$$\begin{aligned}
\int_\Omega u \ \delta \ \mathrm{d}\Omega &= \int_{\partial\Omega} [u \nabla_n w - w \nabla_n u] \ \mathrm{d}\Gamma + \int_\Omega w \ f \ \mathrm{d}\Omega \\
0 &= \int_{\partial\Omega} [u \nabla_n w - w \nabla_n u] \ \mathrm{d}\Gamma + \int_\Omega w \ f \ \mathrm{d}\Omega
\end{aligned} \tag{2.54}$$

3. For $\xi \in \partial\Omega$ and $\Omega \subset \mathbb{R}^2$
We need to start first by considering $\xi \in \Omega$

$$\begin{aligned}
\int_\Omega u \ \delta \ \mathrm{d}\Omega &= \int_{\partial\Omega} [u \nabla_n w - w \nabla_n u] \ \mathrm{d}\Gamma + \int_\Omega w \ f \ \mathrm{d}\Omega \\
u(\xi) &= \int_{\partial\Omega} [u \nabla_n w - w \nabla_n u] \ \mathrm{d}\Gamma + \int_\Omega w \ f \ \mathrm{d}\Omega \\
u(\xi) &= \underbrace{\int_{\partial\Omega} u \nabla_n w \ \mathrm{d}\Gamma}_{\textbf{Strongly Singular Integral}} - \underbrace{\int_{\Gamma} w \nabla_n u \ \mathrm{d}\Gamma}_{\textbf{Weakly Singular Integral}} + \int_\Omega w \ f \ \mathrm{d}\Omega
\end{aligned} \tag{2.55}$$

And then we need to have some modification so as to calculate the limit when $\xi \to \Gamma$

Calculation of the Limit $\xi \to \Gamma$

To move the source/load point to the boundary, we first modify the original boundary Γ, augmenting it by a small circular region with radius ϵ around the load point $\xi \in \Gamma$ as shown in Figure 2.5. The modified boundary Γ' is then given by

$$\Gamma' = \Gamma - \Gamma_\epsilon^* + \Gamma_\epsilon$$

so that

$$\Gamma = \lim_{\epsilon \to 0} \Gamma'$$

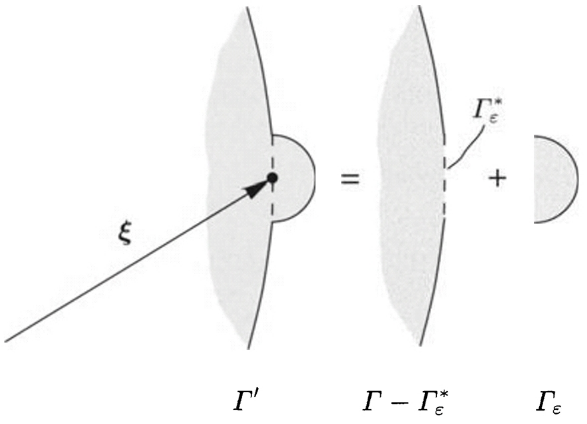


Figure 2.5: Boundary extension around load point ξ

By this process, the load point ξ again comes to lie inside the domain and the representation formula (2.53) remains valid.
As shown in Figure 2.6, the line element $d\Gamma_\epsilon$ along the boundary extension can be parametrized by

$$d\Gamma_\epsilon = \epsilon \, d\theta, \tag{2.56}$$

where

$$\epsilon = |x_i - \xi_i| \tag{2.57}$$

is the Euclidean distance between the load point ξ and the field point x. With this, we can perform the limiting process when moving the load point to the boundary.

Weakly Singular Integral

Using the 2-D fundamental solution $P_\triangle = w$ and $T = \nabla_n$ given in (2.49), we obtain for the first term in (2.55)

$$\begin{aligned}\int_\Gamma \nabla_n u w \, d\Gamma &= \lim_{\epsilon \to 0} \int_{\Gamma'} \nabla_n u \frac{\ln |x_i - \xi_i|}{2\pi} \, d\Gamma \\ &= \underbrace{\lim_{\epsilon \to 0} \int_{\Gamma - \Gamma_\epsilon^*} \nabla_n u \frac{\ln |x_i - \xi_i|}{2\pi} \, d\Gamma}_{*} + \underbrace{\lim_{\epsilon \to 0} \int_{\Gamma_\epsilon} \nabla_n u \frac{\ln |x_i - \xi_i|}{2\pi} \, d\Gamma}_{**} .\end{aligned} \tag{2.58}$$

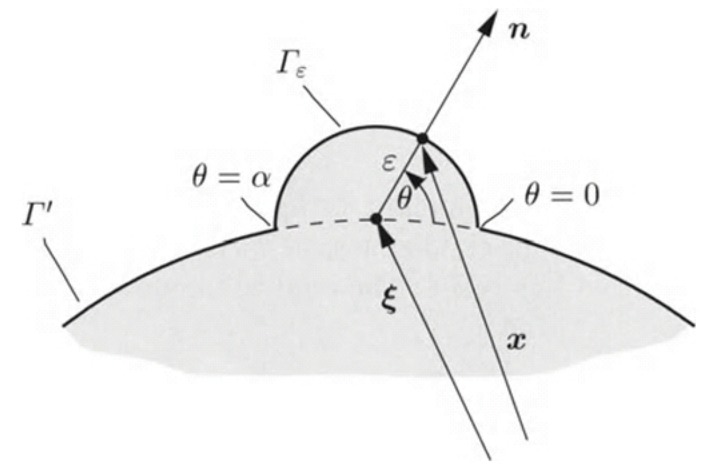


Figure 2.6: Geometry of the augmented boundary

The first integral(*) in (2.58) is weakly singular, so its calculation requires no special care. For the second integral(**), we obtain with (2.56), (2.57) and l'Hospital's rule

$$\begin{aligned}
\lim_{\epsilon\to 0}\int_{\Gamma_\epsilon}\nabla_n u\frac{\ln|x_i-\xi_i|}{2\pi}\,d\Gamma &= \lim_{\epsilon\to 0}\frac{1}{2\pi}\int_{\theta=0}^{\alpha}\nabla_n u(\ln\epsilon)\epsilon\,d\theta\\
&= \frac{1}{2\pi}\lim_{\epsilon\to 0}\frac{(\ln\epsilon)'}{(\frac{1}{\epsilon})'}\int_{\theta=0}^{\alpha}\nabla_n u\,d\theta\\
&= \frac{1}{2\pi}\lim_{\epsilon\to 0}\frac{(\frac{1}{\epsilon})}{(-\frac{1}{\epsilon^2})}\int_{\theta=0}^{\alpha}\nabla_n u\,d\theta\\
&= -\frac{1}{2\pi}\lim_{\epsilon\to 0}\epsilon\int_{\theta=0}^{\alpha}\nabla_n u\,d\theta\\
&= 0
\end{aligned} \tag{2.59}$$

Since the integral $\int_\Gamma \nabla_n uw\,d\Gamma$ is continuous at the (weak) singularity of w, the term in (2.59) over the boundary extension Γ, becomes zero for $\epsilon\to 0$.

Strongly Singular Integral

For the strongly singular integral in (2.55), we have

$$\begin{aligned}
\int_\Gamma u\nabla_n w\,d\Gamma &= \lim_{\epsilon\to 0}\int_{\Gamma'} u\frac{(x_i-\xi_i)n_i}{2\pi|x_i-\xi_i|^2}\,d\Gamma\\
&= \underbrace{\lim_{\epsilon\to 0}\int_{\Gamma-\Gamma^*_\epsilon} u\,\frac{(x_i-\xi_i)n_i}{2\pi|x_i-\xi_i|^2}\,d\Gamma}_{***} + \underbrace{\lim_{\epsilon\to 0}\int_{\Gamma_\epsilon} u\,\frac{(x_i-\xi_i)n_i}{2\pi|x_i-\xi_i|^2}\,d\Gamma}_{****}
\end{aligned} \tag{2.60}$$

Since $\int_\Gamma u\nabla_n w\,d\Gamma$ is strongly singular, the integral(***) over the modified boundary $\Gamma-\Gamma^*_\in$ represents its Cauchy principal value:

$$\lim_{\epsilon \to 0} \int_{\Gamma - \Gamma_\epsilon^*} u \ \nabla_n w \, d\Gamma = \int_\Gamma u \ \nabla_n w \, d\Gamma. \tag{2.61}$$

For the second integral(****) in (2.60), we obtain

$$\begin{aligned}
\lim_{\epsilon \to 0} \int_{\Gamma_\epsilon} u \frac{(x_i - \xi_i) n_i}{2\pi |x_i - \xi_i|^2} ?\Gamma &= \lim_{\epsilon \to 0} \int_{\theta=0}^{\beta} u \ \frac{\varepsilon}{2\pi\varepsilon^2} \varepsilon \, d\theta \\
&= \lim_{\epsilon \to 0} \int_{\theta=0}^{\beta} u \ \frac{1}{2\pi} \, d\theta \\
&= \frac{1}{2\pi} \lim_{\epsilon \to 0} \int_{\theta=0}^{\beta} u \, d\theta \\
&= \frac{1}{2\pi} u(\xi) \int_{\theta=0}^{\beta} d\theta \\
&= \frac{1}{2\pi} u(\xi) \theta \Big|_0^\beta \\
&= \frac{\beta}{2\pi} u(\xi) \qquad (2.62)
\end{aligned}$$

In contrast to (2.60), this term does *not* vanish but remains finite, since the integrand is singular at $\varepsilon = 0$. The strongly singular integral $\int_\Gamma u \nabla_n w \mathrm{d}\Gamma$ is therefore given by the sum of its Cauchy principal value and the contribution from (2.62).

By inserting the results of the weakly and strongly singular integrals into (2.55), we obtain the *boundary integral equation*

$$u(\xi) = \frac{\beta}{2\pi} u(\xi) + \oint_\Gamma u(x) \frac{(x_i - \xi_i) n_i}{2\pi |x_i - \xi_i|^2} \, d\Gamma - \int_\Gamma \nabla_n u(x) \frac{\ln |x_i - \xi_i|}{2\pi} \, d\Gamma + \int_\Omega w \ f \ \mathrm{d}\Omega, \tag{2.63}$$

or

$$\begin{aligned}
\left(1 - \frac{\beta}{2\pi}\right) u(\xi) &= \oint_\Gamma \nabla_n(x, \ \xi) u(x) \, d\Gamma - \int_\Gamma \nabla_n u(x) w(x, \ \xi) \, d\Gamma + \int_\Omega w \ f \ \mathrm{d}\Omega. \\
\alpha(\xi) u(\xi) &= \oint_\Gamma \nabla_n(x, \ \xi) u(x) \, d\Gamma - \int_\Gamma \nabla_n u(x) w(x, \ \xi) \, d\Gamma + \int_\Omega w \ f \ \mathrm{d}\Omega. \qquad (2.64)
\end{aligned}$$

The factor $c(\xi)$ is the *free term* coefficient and can be interpreted as the fraction of $u(\xi)$ that lies inside D:

$$c(\xi) = \begin{cases} \alpha(\xi) = 1 - \frac{\beta(\xi)}{2\pi} & \text{for } \xi \in \Gamma \\ 1 & \text{for } \xi \in \ \Omega \\ 0 & \text{for } \xi \notin \Gamma, \ \xi \notin \Omega \end{cases}$$

2.9 Boundary Potentials and Boundary Integral Operators

Consider the following representation formula:

$$C(y)u(y) = \int_\Gamma u(x) T_x P_\triangle(x,\ y)\, d\Gamma - \int_\Gamma T_x u(x) P_\triangle(x,\ y)\, d\Gamma + \int_\Omega P_\triangle(x,\ y)\ f\ \mathrm{d}\Omega. \quad (2.65)$$

where

$$c(y) = \begin{cases} \alpha(y) = 1 - \frac{\beta(y)}{2\pi} & \text{for } y \in \Gamma \\ 1 & \text{for } y \in\ \Omega \\ 0 & \text{for } y \notin \Gamma,\ y \notin \Omega \end{cases} \quad (2.66)$$

If u is the solution that satisfies the Laplace equation as in (2.1) with $f = 0$, then the third Green formula in (2.65) becomes

$$c(y)u(y) = \int_{\partial\Omega} [u(x) T_x P_\triangle(x,\ y) - Tu(x) P_\triangle(x,\ y)] \mathrm{d}\Gamma(x) \quad (2.67)$$

The representation formula (2.67) consists of two boundary potentials,

the single layer potential

$$V_\Delta \varphi(y)\ := - \int_{\partial\Omega} P_\triangle(x,\ y)\varphi(x)\mathrm{d}\Gamma(x)\ ,\ y \notin \partial\Omega, \quad (2.68)$$

and

the double layer potential

$$W_\Delta \psi(y)\ := - \int_{\partial\Omega} [T_x P_\Delta(x,\ y)_\triangle]\psi(x)\mathrm{d}\Gamma(x)\ ,\ y \notin \partial\Omega. \quad (2.69)$$

The corresponding boundary integral (pseudo differential) operators of the direct values of the single layer potential $\mathcal{V}_\Delta$ and the double layer potential $\mathcal{W}_\Delta$, the adjoint double layer potential $\mathcal{W}^*_\Delta$ and the double layer potential $\mathcal{L}^\pm_\Delta$ are

$$\begin{aligned} \mathcal{V}_\Delta \varphi(y_0) &:= - \int_{\partial\Omega} P_\triangle(x,\ y_0)\varphi(x)\mathrm{d}\Gamma(x)\ , & (2.70) \\ \mathcal{W}_\Delta \psi(y_0) &:= - \int_{\partial\Omega} [T_x P_\triangle(x,\ y_0)]\psi(x)\mathrm{d}\Gamma(x)\ , & (2.71) \\ \mathcal{W}^*_\Delta \varphi(y_0) &:= - \int_{\partial\Omega} [T_y P_\triangle(x,\ y_0)]\varphi(x)\mathrm{d}\Gamma(x)\ , & (2.72) \\ \mathcal{L}^\pm_\Delta \psi(y_0) &:= [T_y W_\Delta \psi(y_0)]^\pm, & (2.73) \end{aligned}$$

where $y_0 \in \partial\Omega$.
Therefore,
1. For $y \notin \partial\Omega$, **the representation formula**(2.67) can written in terms of V and W, i.e.,

$$
\begin{aligned}
c(y)u(y) &= \int_{\partial\Omega} [u(x) T_x P_{\triangle}(x,\ y) - P_{\triangle}(x,\ y) T u(x)] \mathrm{d}\Gamma(x) \\
&= \int_{\partial\Omega} u(x) T_x P_{\triangle}(x,\ y) \mathrm{d}\Gamma(x) - \int_{\partial\Omega} P_{\triangle}(x,\ y) T u(x) \mathrm{d}\Gamma(x) \\
&= \underbrace{\int_{\partial\Omega} u(x) T_x P_{\triangle}(x,\ y) \mathrm{d}\Gamma(x)}_{\textit{Double layer potential}(-W_{\Delta} u(y))} - \underbrace{\int_{\partial\Omega} P_{\triangle}(x,\ y) T u(x) \mathrm{d}\Gamma(x)}_{\textit{Single layer potential}(-V_{\Delta} T u(y))} \\
&= -W_{\Delta} u(y) - (-V_{\Delta} T u(y)) \\
&= -W_{\Delta} u(y) + V_{\Delta} T u(y)
\end{aligned}
$$

$$
c(y)u(y) = -W_{\Delta} u(y) + V_{\Delta} T u(y). \tag{2.74}
$$

2. For $y \in \partial\Omega$ (by extension), (2.67) can be written in terms of $\mathcal{V}_{\Delta}$ and $\mathcal{W}_{\Delta}$, i.e.,

$$
\begin{aligned}
c(y)u(y) &= \int_{\partial\Omega} [u(x) T_x P_{\triangle}(x,\ y) - P_{\triangle}(x,\ y) T u(x)] \mathrm{d}\Gamma(x) \\
&= \int_{\partial\Omega} u(x) T_x P_{\triangle}(x,\ y) \mathrm{d}\Gamma(x) - \int_{\partial\Omega} P_{\triangle}(x,\ y) T u(x) \mathrm{d}\Gamma(x) \\
&= \underbrace{\int_{\partial\Omega} u(x) T_x P_{\triangle}(x,\ y) \mathrm{d}\Gamma(x)}_{-\mathcal{W}_{\Delta} u(y)} - \underbrace{\int_{\partial\Omega} P_{\triangle}(x,\ y) T u(x) \mathrm{d}\Gamma(x)}_{\mathcal{V}_{\Delta} T u(y)} \\
&= -\mathcal{W}_{\Delta} u(y) + \mathcal{V}_{\Delta} T u(y)
\end{aligned}
$$

$$
c(y)u(y) \ = -\mathcal{W}_{\Delta} u(y) + \mathcal{V}_{\Delta} T u(y). \tag{2.75}
$$

Here $c(y)$ is defined as in (2.66).

Any boundary value problem can be reduced to Boundary Integral Equation by using several different approaches. The main classifications would fall into two broad categories: **direct method** and **indirect method.** The direct method is based on Green formula while the indirect method is based on the single or double layer potentials. However, the density function obtained from the indirect method, in general, has no physical meaning.

2.10 Formulation of Direct boundary integral equations

The following well-known jump relations might be useful for further discussions, see in e.g. (Hsiao and Wendland (2008)):

Theorem 1 ***(Jump Relations)***
a) Let $\varphi \in C(\partial\Omega)$ and $y_0 \in \partial\Omega$. Then

$$
\begin{aligned}
[\mathcal{V}_{\Delta}\varphi](y_0) \ &= \ [V_{\Delta}\varphi]^{+}(y_0) = \ [V_{\Delta}\varphi]^{-}(y_0), &(2.76)\\
\partial_n^{+}[V_{\Delta}\varphi](y_0) \ &= \ \mathcal{W}_{\Delta}^{*}\varphi(y_0) + \frac{1}{2}\varphi(y_0), &(2.77)\\
\partial_n^{-}[V_{\Delta}\varphi](y_0) \ &= \ \mathcal{W}_{\Delta}^{*}\varphi(y_0) - \frac{1}{2}\varphi(y_0). &(2.78)
\end{aligned}
$$

b) Let $\varphi \in C(\partial\Omega)$ *and* $y_0 \in \partial\Omega$*. Then*

$$[W_\Delta \psi]^+(y_0) = \mathcal{W}_\Delta \psi(y_0) - \frac{1}{2}\psi(y_0), \tag{2.79}$$

$$[W_\Delta \psi]^-(y_0) = \mathcal{W}_\Delta \psi(y_0) + \frac{1}{2}\psi(y_0), \tag{2.80}$$

$$-\mathcal{L}_\Delta \psi(y_0) := \partial_n^+[W_\Delta \psi](y_0) = \partial_n^- W_\Delta \psi(y_0), \tag{2.81}$$

where $+$ *and* $-$ *are the the limiting boundary values on* $\partial\Omega$ *from* Ω^+ *and* Ω^-*, respectively.*

Derivation of the above theorem

1. **Single layer potential(Approaching Γ from both interior and exterior)**

$$[\mathcal{V}_\Delta \varphi](y_0) \; = \; [V_\Delta \varphi]^+(y_0) = \; [V_\Delta \varphi]^-(y_0)$$

$$\begin{aligned}
[V_\Delta \varphi]^\pm(y_0) &= \lim_{\Omega^\pm \ni y \to y_0 \in \partial\Omega} [V_\Delta \varphi](y) \qquad\qquad\qquad y \in \Omega \\
[V_\Delta \nabla_n u]^\pm(y_0) &= -\lim_{\Omega^\pm \ni y \to y_0 \in \partial\Omega} \int_{\Gamma'} \nabla_n u P_\triangle \, d\Gamma \\
[V_\Delta \nabla_n u]^\pm(\xi) &= -\lim_{\epsilon \to 0} \int_{\Gamma'} \nabla_n u P_\triangle(x,\ \xi) \, d\Gamma \\
&= -\lim_{\epsilon \to 0} \int_{\Gamma'} \nabla_n u \frac{\ln|x_i - \xi_i|}{2\pi} \, d\Gamma \\
&= -\int_{\Gamma} \nabla_n u P_\triangle \, d\Gamma \\
&= -\lim_{\epsilon \to 0} \int_{\Gamma'} \nabla_n u \frac{\ln|x_i - \xi_i|}{2\pi} \, d\Gamma \\
&= -[\lim_{\epsilon \to 0} \int_{\Gamma - \Gamma_\epsilon^*} \nabla_n u \frac{\ln|x_i - \xi_i|}{2\pi} \, d\Gamma + \lim_{\epsilon \to 0} \int_{\Gamma_\epsilon} \nabla_n u \frac{\ln|x_i - \xi_i|}{2\pi} \, d\Gamma] \\
&= \underbrace{-\lim_{\epsilon \to 0} \int_{\Gamma - \Gamma_\epsilon^*} \nabla_n u \frac{\ln|x_i - \xi_i|}{2\pi} \, d\Gamma}_{*} \underbrace{- \lim_{\epsilon \to 0} \int_{\Gamma_\epsilon} \nabla_n u \frac{\ln|x_i - \xi_i|}{2\pi} \, d\Gamma}_{**} \\
&= -\lim_{\epsilon \to 0} \int_{\Gamma - \Gamma_\epsilon^*} \nabla_n u \frac{\ln|x_i - \xi_i|}{2\pi} \, d\Gamma \\
&= -\lim_{\epsilon \to 0} \int_{\Gamma - \Gamma_\epsilon^*} \nabla_n u P_\triangle \, d\Gamma + \ 0 \\
&= -\lim_{\epsilon \to 0} \int_{\Gamma - \Gamma_\epsilon^*} \nabla_n u P_\triangle \, d\Gamma \\
&= [\mathcal{V}_\Delta \varphi](\xi)
\end{aligned}$$

$$\Longrightarrow \; [\mathcal{V}_\Delta \varphi](y_0) \; = \; [V_\Delta \varphi]^+(y_0) = \; [V_\Delta \varphi]^-(y_0),$$

Note that The first integral(*) is weakly singular, so its calculation requires no special care. For the second integral(**), we obtain with (2.56), (2.57) and l'Hospital's rule

$$\begin{aligned}\lim_{\epsilon\to 0}\int_{\Gamma_\epsilon}\nabla_n u\frac{\ln|x_i-\xi_i|}{2\pi}\,d\Gamma &= \lim_{\epsilon\to 0}\frac{1}{2\pi}\int_{\theta=0}^{\alpha}\nabla_n u(\ln\epsilon)\epsilon\,d\theta\\ &= \frac{1}{2\pi}\lim_{\epsilon\to 0}\frac{(\ln\epsilon)'}{(\frac{1}{\epsilon})'}\int_{\theta=0}^{\alpha}\nabla_n u\,d\theta\\ &= \frac{1}{2\pi}\lim_{\epsilon\to 0}-\epsilon\int_{\theta=0}^{\alpha}\nabla_n u\,d\theta\\ &= 0\end{aligned}$$

Since the integral $\int_\Gamma \nabla_n u P_\triangle\,d\Gamma$ is continuous at the (weak) singularity of $P_\triangle$, the term in (2.59) over the boundary extension Γ, becomes zero for $\epsilon\to 0$.

2. Double layer potential(Approaching Γ from interior)

$$[W_\Delta\psi]^+(y_0) = \mathcal{W}_\Delta\psi(y_0) - \frac{1}{2}\psi(y_0),$$

$$\begin{aligned}[W_\Delta\psi]^+(y_0) &= \lim_{\Omega^+\ni y\to y_0\in\partial\Omega}[W_\Delta\varphi](y) \qquad\qquad\qquad y\in\Omega\\ [W_\Delta\psi]^+(y_0) &= -\lim_{\Omega^+\ni y\to y_0\in\partial\Omega}\int_\Gamma \psi\nabla_n P_\triangle\,d\Gamma\\ [W_\Delta\psi]^+(\xi) &= -\lim_{\epsilon\to 0}\int_{\Gamma'}\psi\nabla_n P_\triangle\,d\Gamma\\ &= -\lim_{\epsilon\to 0}\int_{\Gamma'}\psi\frac{(x_i-\xi_i)n_i}{2\pi|x_i-\xi_i|^2}\,d\Gamma\\ &= \underbrace{-\lim_{\epsilon\to 0}\int_{\Gamma-\Gamma^*_\epsilon}\psi\;\frac{(x_i-\xi_i)n_i}{2\pi|x_i-\xi_i|^2}\,d\Gamma}_{(***)}\underbrace{-\lim_{\epsilon\to 0}\int_{\Gamma_\epsilon}\psi\;\frac{(x_i-\xi_i)n_i}{2\pi|x_i-\xi_i|^2}\,d\Gamma}_{(****)}\\ &= -\lim_{\epsilon\to 0}\int_{\Gamma-\Gamma^*_\epsilon}\psi\;\nabla_n P_\triangle\,d\Gamma - \frac{1}{2}u(\xi)\\ &= \mathcal{W}_\Delta\psi(\xi) - \frac{1}{2}\psi(\xi)\\ [W_\Delta\psi]^+(y_0) &= \mathcal{W}_\Delta\psi(y_0) - \frac{1}{2}\psi(y_0)\end{aligned}$$

Since $\int_\Gamma \psi\nabla_n w\,d\Gamma$ is strongly singular, the integral(***) over the modified boundary $\Gamma-\Gamma^*_{\in}$ represents its Cauchy principal value:

$$\lim_{\epsilon\to 0}\int_{\Gamma-\Gamma^*_\epsilon}\psi\;\nabla_n P_\triangle\,d\Gamma = \int_\Gamma\psi\;\nabla_n P_\triangle\,d\Gamma.$$

For the second integral(****) in (2.60), we obtain

$$
\begin{aligned}
\lim_{\epsilon\to 0}\int_{\Gamma_\epsilon}\psi\frac{(x_i-\xi_i)n_i}{2\pi|x_i-\xi_i|^2}\,d\Gamma &= \lim_{\epsilon\to 0}\int_{\theta=0}^{\beta}\psi\ \frac{\varepsilon}{2\pi\varepsilon^2}\varepsilon\,d\theta\\
&= \psi(\xi)\int_{\theta=0}^{\beta}\frac{1}{2\pi}\,d\theta\\
&= \frac{1}{2\pi}\psi(\xi)\int_{\theta=0}^{\beta}d\theta\\
&= \frac{1}{2\pi}\psi(\xi)\ \theta|_{\theta=0}^{\beta}\\
&= \frac{1}{2\pi}\psi(\xi)\ \beta\\
&= \frac{\beta}{2\pi}\psi(\xi)
\end{aligned}
$$

If $\partial\Omega$ is a smooth boundary, then we have $\beta=\pi$

$$
\begin{aligned}
\lim_{\epsilon\to 0}\int_{\Gamma_\epsilon}\psi\frac{(x_i-\xi_i)n_i}{2\pi|x_i-\xi_i|^2}\,d\Gamma &= \frac{\pi}{2\pi}\psi(\xi)\\
&= \frac{1}{2}\psi(\xi)
\end{aligned}
$$

In contrast to (2.59), this term does *not* vanish but remains finite, since the integrand is singular at $\varepsilon=0$. The strongly singular integral $\int_\Gamma\psi\nabla_n P_\triangle \mathrm{d}\Gamma$ is therefore given by the sum of its Cauchy principal value and the contribution from (2.63).

3. Double layer potential(Approaching Γ from exterior)

$$
[W_\Delta\psi]^-(y_0) = \mathcal{W}_\Delta\psi(y_0)+\frac{1}{2}\psi(y_0),
$$

$$
\begin{aligned}
[W_\Delta\psi]^-(y_0) &= \lim_{\Omega^-\ni y\to y_0\in\partial\Omega}[W_\Delta\varphi](y) \qquad\qquad y\in\Omega\\
[W_\Delta\psi]^-(y_0) &= -\lim_{\Omega^-\ni y\to y_0\in\partial\Omega}\int_\Gamma\psi\nabla_n P_\triangle\,d\Gamma\\
[W_\Delta\psi]^-(\xi) &= -\lim_{\epsilon\to 0}\int_{\Gamma'}\psi\nabla_n P_\triangle\,d\Gamma\\
&= -\lim_{\epsilon\to 0}\int_{\Gamma'}\psi\frac{(x_i-\xi_i)n_i}{2\pi|x_i-\xi_i|^2}\,d\Gamma\\
&= \underbrace{-\lim_{\epsilon\to 0}\int_{\Gamma-\Gamma_\epsilon^*}\psi\ \frac{(x_i-\xi_i)n_i}{2\pi|x_i-\xi_i|^2}\,d\Gamma}_{(***)}\underbrace{-\lim_{\epsilon\to 0}\int_{\Gamma_\epsilon}\psi\ \frac{(x_i-\xi_i)n_i}{2\pi|x_i-\xi_i|^2}\,d\Gamma}_{(****)}\\
&= -\lim_{\epsilon\to 0}\int_{\Gamma-\Gamma_\epsilon^*}\psi\ \nabla_n P_\triangle\,d\Gamma+\frac{1}{2}u(\xi)\\
&= \mathcal{W}_\Delta\psi(\xi)+\frac{1}{2}\psi(\xi)\\
[W_\Delta\psi]^-(y_0) &= \mathcal{W}_\Delta\psi(y_0)+\frac{1}{2}\psi(y_0)
\end{aligned}
$$

Since $\int_{\Gamma} \psi \nabla_n w \, d\Gamma$ is strongly singular, the integral(***) over the modified boundary $\Gamma - \Gamma^*_{\in}$ represents its Cauchy principal value:

$$\lim_{\epsilon \to 0} \int_{\Gamma - \Gamma^*_{\epsilon}} \psi \ \nabla_n P_{\triangle} \, d\Gamma = \int_{\Gamma} \psi \ \nabla_n P_{\triangle} \, d\Gamma.$$

For the second integral(****) in (2.60), we obtain

$$\begin{aligned}
\lim_{\epsilon \to 0} \int_{\Gamma_\epsilon} \psi \frac{(x_i - \xi_i) n_i}{2\pi |x_i - \xi_i|^2} ;\Gamma &= \lim_{\epsilon \to 0} \int_{\theta=0}^{\beta} \psi \ \frac{\varepsilon}{2\pi \varepsilon^2} (-1) \ \varepsilon \, d\theta \\
&= -\psi(\xi) \int_{\theta=0}^{\beta} \frac{1}{2\pi} \, d\theta \\
&= -\frac{1}{2\pi} \psi(\xi) \int_{\theta=0}^{\beta} \, d\theta \\
&= -\frac{1}{2\pi} \psi(\xi) \ \theta |_{\theta=0}^{\beta} \\
&= -\frac{1}{2\pi} \psi(\xi) \ \beta \\
&= -\frac{\beta}{2\pi} \psi(\xi)
\end{aligned}$$

If $\partial\Omega$ is a smooth boundary, then we have $\beta = \pi$

$$\begin{aligned}
\lim_{\epsilon \to 0} \int_{\Gamma_\epsilon} \psi \frac{(x_i - \xi_i) n_i}{2\pi |x_i - \xi_i|^2} ;\Gamma &= -\frac{\pi}{2\pi} u(\xi) \\
&= -\frac{1}{2} \psi(\xi)
\end{aligned}$$

In contrast to (2.59), this term does *not* vanish but remains finite, since the integrand is singular at $\varepsilon = 0$. The strongly singular integral $\int_{\Gamma} \psi \nabla_n P_{\triangle} \mathrm{d}\Gamma$ is therefore given by the sum of its Cauchy principal value and the contribution from (2.62).

4. Normal Derivative a Single layer potential(Approaching Γ from interior)

$$\partial_n^+ [V_{\Delta} \varphi](y_0) \ = \ \mathcal{W}^*_{\Delta} \varphi(y_0) + \frac{1}{2} \varphi(y_0),$$

$$\begin{aligned}
\partial_n^+[V_\Delta\varphi](y_0) &= \lim_{\Omega^+\ni y\to y_0\in\partial\Omega} \partial_n[V_\Delta\varphi](y) \qquad\qquad\qquad\qquad y\in\Omega \\
\partial_n^+[V_\Delta\varphi](y_0) &= -\lim_{\Omega^+\ni y\to y_0\in\partial\Omega} \partial_{\nu_\xi}\int_\Gamma \varphi\ P_\triangle\, d\Gamma \\
\partial_n^+[V_\Delta\varphi](\xi) &= -\lim_{\epsilon\to 0}\int_{\Gamma'} \varphi\ \nabla_{n_\xi}P_\triangle(x,\ \xi)\, d\Gamma \\
&= -\lim_{\epsilon\to 0}\int_{\Gamma'} \varphi\ \nabla_{n_\xi}P_\triangle(x,\ \xi)\, d\Gamma \\
&= -\lim_{\epsilon\to 0}\int_{\Gamma'} \varphi\ \nabla_{n_\xi}P_\triangle\, d\Gamma \\
&= -\lim_{\epsilon\to 0}\int_{\Gamma'} \varphi\frac{(x_i-\xi_i)n_i}{2\pi|x_i-\xi_i|^2}\, d\Gamma \\
&= \underbrace{-\lim_{\epsilon\to 0}\int_{\Gamma-\Gamma_\epsilon^*} \varphi\ \frac{(x_i-\xi_i)n_i}{2\pi|x_i-\xi_i|^2}\, d\Gamma}_{(***)} \underbrace{-\lim_{\epsilon\to 0}\int_{\Gamma_\epsilon} \varphi\ \frac{(x_i-\xi_i)n_i}{2\pi|x_i-\xi_i|^2}\, d\Gamma}_{(****)} \\
&= -\lim_{\epsilon\to 0}\int_{\Gamma-\Gamma_\epsilon^*} \varphi\ \nabla_{n_\xi}P_\triangle\, d\Gamma - (-\frac{1}{2}u(\xi)) \\
&= \mathcal{W}_\Delta^*\varphi(y_0) + \frac{1}{2}\varphi(y_0)
\end{aligned}$$

Since $\int_\Gamma \varphi\nabla_{n_\xi}P_\triangle\, d\Gamma$ is strongly singular, the integral(***) over the modified boundary $\Gamma - \Gamma_\in^*$ represents its Cauchy principal value:

$$\lim_{\epsilon\to 0}\int_{\Gamma-\Gamma_\epsilon^*} \varphi\ \nabla_{n_\xi}P_\triangle\, d\Gamma = \int_\Gamma \varphi\ \nabla_{n_\xi}P_\triangle\, d\Gamma.$$

For the second integral(****) in (2.60), we obtain

$$\begin{aligned}
\lim_{\epsilon\to 0}\int_{\Gamma_\epsilon} \varphi\frac{-(x_i-\xi_i)n_i}{2\pi|x_i-\xi_i|^2};\Gamma &= \lim_{\epsilon\to 0}\int_{\theta=0}^{\beta} \varphi\ \frac{-\varepsilon}{2\pi\varepsilon^2}\varepsilon\, d\theta \\
&= \varphi(\xi)\int_{\theta=0}^{\beta} \frac{-1}{2\pi}\, d\theta \\
&= -\frac{1}{2\pi}\varphi(\xi)\int_{\theta=0}^{\beta} d\theta \\
&= -\frac{1}{2\pi}\varphi(\xi)\ \theta|_{\theta=0}^{\beta} \\
&= -\frac{1}{2\pi}\varphi(\xi)\ \beta \\
&= -\frac{\beta}{2\pi}\varphi(\xi)
\end{aligned}$$

If $\partial\Omega$ is a smooth boundary, then we have $\beta = \pi$

$$\begin{aligned}\lim_{\epsilon\to 0}\int_{\Gamma_\epsilon}\varphi\frac{-(x_i-\xi_i)n_i}{2\pi|x_i-\xi_i|^2}\,d\Gamma &= -\frac{\pi}{2\pi}\psi(\xi)\\ &= -\frac{1}{2}\psi(\xi)\end{aligned}$$

In contrast to (2.59), this term does *not* vanish but remains finite, since the integrand is singular at $\varepsilon = 0$. The strongly singular integral $\int_\Gamma \psi\nabla_n P_\triangle \mathrm{d}\Gamma$ is therefore given by the sum of its Cauchy principal value and the contribution from (2.62).

5. Normal Derivative a Single layer potential(Approaching Γ from exterior)

$$\partial_n^-[V_\Delta\varphi](y_0) \;=\; \mathcal{W}_\Delta^*\varphi(y_0)-\frac{1}{2}\varphi(y_0)$$

$$\begin{aligned}\partial_{n_\xi}^-[V_\Delta\varphi](y_0) &= \lim_{\Omega^+\ni y\to y_0\in\partial\Omega}\partial_{n_\xi}[V_\Delta\varphi](y) && y\in\Omega\\ \partial_{n_\xi}^-[V_\Delta\varphi](y_0) &= -\lim_{\Omega^-\ni y\to y_0\in\partial\Omega}\partial_{n_\xi}\int_\Gamma\varphi\;P_\triangle\,d\Gamma\\ \partial_{n_\xi}^-[V_\Delta\varphi](\xi) &= -\lim_{\epsilon\to 0}\int_{\Gamma'}\varphi\frac{-(x_i-\xi_i)n_i}{2\pi|x_i-\xi_i|^2}\,d\Gamma\\ &= \underbrace{-\lim_{\epsilon\to 0}\int_{\Gamma-\Gamma_\epsilon^*}\varphi\;\frac{-(x_i-\xi_i)n_i}{2\pi|x_i-\xi_i|^2}\,d\Gamma}_{(***)}\underbrace{-\lim_{\epsilon\to 0}\int_{\Gamma_\epsilon}\varphi\;\frac{-(x_i-\xi_i)n_i}{2\pi|x_i-\xi_i|^2}\,d\Gamma}_{(****)}\\ &= -\lim_{\epsilon\to 0}\int_{\Gamma-\Gamma_\epsilon^*}\varphi\;\nabla_{n_\xi}P_\triangle\,d\Gamma-\frac{1}{2}u(\xi)\\ &= \mathcal{W}_\Delta^*\varphi(y_0)-\frac{1}{2}\varphi(y_0)\end{aligned}$$

Since $\int_\Gamma \varphi\nabla_{n_\xi}P_\triangle\,d\Gamma$ is strongly singular, the integral(***) over the modified boundary $\Gamma-\Gamma_\in^*$ represents its Cauchy principal value:

$$\lim_{\epsilon\to 0}\int_{\Gamma-\Gamma_\epsilon^*}\varphi\;\nabla_{n_\xi}P_\triangle\,d\Gamma = \int_\Gamma\varphi\;\nabla_{n_\xi}P_\triangle\,d\Gamma.$$

For the second integral(****) in (2.60), we obtain

$$
\begin{aligned}
\lim_{\epsilon\to 0}\int_{\Gamma_\epsilon}\varphi\frac{-(x_i-\xi_i)n_i}{2\pi|x_i-\xi_i|^2}?\Gamma &= \lim_{\epsilon\to 0}\int_{\theta=0}^{\beta}\varphi\ \frac{-\varepsilon}{2\pi\varepsilon^2}(-1)\varepsilon\,d\theta\\
&= \varphi(\xi)\int_{\theta=0}^{\beta}\frac{1}{2\pi}\,d\theta\\
&= \frac{1}{2\pi}\varphi(\xi)\int_{\theta=0}^{\beta}d\theta\\
&= \frac{1}{2\pi}\varphi(\xi)\ \theta|_{\theta=0}^{\beta}\\
&= \frac{1}{2\pi}\varphi(\xi)\ \beta\\
&= \frac{\beta}{2\pi}\varphi(\xi)
\end{aligned}
$$

If $\partial\Omega$ is a smooth boundary, then we have $\beta=\pi$

$$
\begin{aligned}
\lim_{\epsilon\to 0}\int_{\Gamma_\epsilon}\varphi\frac{-(x_i-\xi_i)n_i}{2\pi|x_i-\xi_i|^2}\,d\Gamma &= \frac{\pi}{2\pi}\varphi(\xi)\\
&= \frac{1}{2}\varphi(\xi)
\end{aligned}
$$

In contrast to (2.59), this term does *not* vanish but remains finite, since the integrand is singular at $\varepsilon=0$. The strongly singular integral $\int_\Gamma \varphi\nabla_n P_\triangle \mathrm{d}\Gamma$ is therefore given by the sum of its Cauchy principal value and the contribution from (2.62).

4. Normal Derivative a Double layer potential(Approaching Γ from both interior and exterior)

$$
\mathcal{L}_\Delta^{\pm}\psi(y_0) := \partial_n^{+}[W_\Delta\psi](y_0) = \partial_n^{-}W_\Delta\psi(y_0),
$$

$$
\begin{aligned}
\partial_n^{\pm}[W_\Delta\psi](y_0) &= \lim_{\Omega^{\pm}\ni y\to y_0\in\partial\Omega}\partial_{n_\xi}[W_\Delta\psi](y) && y\in\Omega\\
&= -\lim_{\Omega^{\pm}\ni y\to y_0\in\partial\Omega}\partial_{n_\xi}\int_\Gamma\psi\ \nabla_{n_\xi}P_\triangle(x,\ \xi)\,d\Gamma\\
&:= \mathcal{L}_\Delta^{\pm}\psi(y_0)\\
\partial_n^{\pm}[W_\Delta\psi](y_0) &:= \mathcal{L}_\Delta^{\pm}\psi(y_0)
\end{aligned}
$$

The formulation of the boundary integral equation by using Green formula (2.67) which is known as **"direct method"** admits two ways of approaching the boundary based on the traces $|^{\pm}$ and on the normal derivative $\partial_\nu^{\pm}$.

First way is by considering traces.

For the interior Dirichlet, $c(y) = 1$. Therefore equation (2.67) gives

$$u = -W_{n+}u^+ + V(\partial^+_{n+}u) \qquad \textit{on } \Omega. \tag{2.82}$$

The notation n^+ in (2.82) denotes that the direction of n is outward to Ω.
Substituting (2.76) and (2.79) into the trace of (2.82), we obtain

$$u^+ = \frac{1}{2}u^+ - \mathcal{W}_{n+}u^+ + \mathcal{V}(\partial^+_{n+}u) \qquad \textit{on } \partial\Omega. \tag{2.83}$$

For the exterior region Ω^-, equation (2.67) gives

$$u = -W_{n-}u^- + V(\partial^-_{n-}u) \qquad \text{on } \Omega^-. \tag{2.84}$$

Here the notation n^- indicates that the direction of n is inward with respect to Ω. Equation (2.84) is true if u satisfies the additional condition at infinity.
Since $n^- = -n^+$, we can write (2.84) as follows:

$$u = W_{n^+}u^- - V(\partial^-_{n^+}u) \qquad \textit{on } \Omega^-. \tag{2.85}$$

Substituting (2.76) and (2.80) into the trace of (2.85), we obtain

$$u^- = \frac{1}{2}u^- + \mathcal{W}_{n^+}u^- - \mathcal{V}(\partial^-_{n+}u) \qquad \textit{on } \partial\Omega. \tag{2.86}$$

Second way based on the normal derivative $\partial^\pm_n$

For the interior problem, taking normal derivative (2.82) from Ω, we obtain

$$\partial^+_{n^+}u = -\partial^+_{n^+}[W_{n^+}u^+] + \partial^+_{n^+}[V(\partial^+_{n^+}u)] \qquad \text{on } \partial\Omega. \tag{2.87}$$

Therefore, taking into account the jump relations in (2.77) and (2.81) and substituting them into (2.87), we obtain

$$\partial^+_{n^+}u(y_0) = -\mathcal{L}u^+(y_0) + \mathcal{W}^*_{n^+}(\partial^+_{n^+}u(y_0)) + \frac{1}{2}\partial^+_{n^+}u(y_0), \qquad y_0 \in \partial\Omega. \tag{2.88}$$

For the exterior Dirichlet problem, taking the normal derivative of (2.85) from Ω^-, we arrive at the following equation:

$$\partial^-_{n^+}u = \partial^-_{n^+}[W_{n+}(u^-)] - \partial^-_{n^+}[V(\partial^-_{n^+}u)] \qquad \text{on } \partial\Omega. \tag{2.89}$$

Substitution (2.78) and (2.81) into (2.89), gives

$$\partial^-_{n+}u(y_0) = \mathcal{L}u^-(y_0) - \mathcal{W}^*(\partial^-_{n+}u(y_0)) + \frac{1}{2}\partial^-_{n+}u(y_0), \qquad y_0 \in \partial\Omega. \tag{2.90}$$

2.10.1 Interior Dirichlet Problem

We will look for the solution u which satisfies the Laplace equation (2.1) in $y \in \Omega$ with Dirichlet boundary condition (2.2). Equation (2.83) can be written as

$$(\frac{1}{2} + \mathcal{W}_{n+})u^{+} = \mathcal{V}(\partial_{n+}^{+} u) \qquad \text{on } \partial\Omega. \tag{2.91}$$

Substituting the boundary condition (2.2) into (2.91), we obtain an integral equation of the first kind w.r.t. $\partial_{n+}^{+} u$:

$$\mathcal{V}(\partial_{n^+}^{+} u(y_0)) = \ (\frac{1}{2} + \mathcal{W}_{n+})\mathrm{u}(y_0)\ ,\ y_0 \in \partial\Omega. \tag{2.92}$$

Besides, we can also reduce the interior Dirichlet problem to another integral equation by using (2.87). Substituting the boundary condition (2.2), equation (2.87) can also be written as the following integral equation of the second kind w.r.t. $\partial_{n^+}^{+} u$:

$$(\frac{1}{2} - \mathcal{W}_{n^+}^{*})\ (\partial_{n^+}^{+} u(y_0)) = \mathcal{L}\mathrm{u}(y_0)\ ,\ y_0 \in \partial\Omega. \tag{2.93}$$

2.10.2 Exterior Dirichlet Problem

Now we consider the exterior Dirichlet problem which consists of finding u which satisfies the Laplace equation (2.1) in $y \in \Omega^{-}$ with Dirichlet boundary condition (2.2) and condition at infinity.

Rearranging, equation (2.86) can be written as

$$(\frac{1}{2} - \mathcal{W}_{n^+})u^{-} = -\mathcal{V}(\partial_{n^+}^{-} u) \qquad \textit{on } \partial\Omega. \tag{2.94}$$

Substituting the boundary condition (2.2) into (2.94), we obtain the following integral equation of first kind w.r. t.$\partial_{n^+}^{-} u$:

$$-\mathcal{V}(\partial_{n^+}^{-} u(y_0)) = \ (\frac{1}{2} - \mathcal{W}_{n^+})\mathrm{u}(y_0)\ ,\ y_0 \in \partial\Omega. \tag{2.95}$$

As in interior Dirichlet case, we can also reduce the exterior Dirichlet problem to another integral equation by taking normal derivative ∂_{n}^{-}. Substituting the boundary condition (2.2) into equation (2.90) yields the following integral equation of the second kind w.r.t. $\partial_{n^+}^{-} u$:

$$(\frac{1}{2} + \mathcal{W}_{n^+}^{*})\ (\partial_{n^+}^{-} u(y_0)) = -\mathcal{L}\mathrm{u}(y_0)\ ,\ y_0 \in \partial\Omega. \tag{2.96}$$

Chapter 3

Analysis of Boundary Integral Equation in a sense of Distribution

3.1 Introduction

In this chapter, the Dirichlet boundary value problem for the second order "stationary heat transfer" elliptic partial differential equation with constant coefficient is considered in 2D. Using an appropriate fundamental solution which was stated and proved in chapter 1, the problem is reduced to some direct systems of Boundary Integral Equations (BIEs). Although the theory of BIEs in 3D is well developed, the BIEs in 2D need a special consideration due to their different equivalence properties. Consequently, we need to set conditions on the domain for the invertibility of corresponding fundamental based integral layer potentials and hence the unique solvability of BIEs. The properties of corresponding potential operators are investigated. The equivalence of the original BVP and the obtained BIEs are analyzed and the invertibility of the BIE operators is proved.

Let Ω be a domain in $\mathbb{R}^2$ bounded by simple closed infinitely differentiable curve $\partial\Omega$, the set of all infinitely differentiable function on Ω with compact support is denoted by $\mathcal{D}(\Omega)$. The function space $\mathcal{D}'(\Omega)$ consists of all continuous linear functionals over $\mathcal{D}(\Omega)$. For $s \in \mathbb{R}$, we denote by $H^s(\mathbb{R}^2)$ the Bessel potential space. Note that the space $H^1(\mathbb{R}^2)$ coincides with the Sobolev space $W_2^1(\mathbb{R}^2)$ with equivalent norm and $H^{-s}(\mathbb{R}^2)$ is the dual space to $H^s(\mathbb{R}^2)$. For any non-empty open set $\Omega \in \mathbb{R}^n$ we define $H^s(\Omega)$ $=\{u \in \mathcal{D}'(\Omega) : u = U|_\Omega$ for some $U \in H^s(\mathbb{R}^n)\}$. The space $\tilde{H}^s(\Omega)$ is defined to be the closure of $\mathcal{D}(\Omega)$ with respect to the norm of $H^s(\mathbb{R}^n)$. For $s \in (-\frac{1}{2}, \frac{1}{2})$, $H^s(\Omega)$ can be identified with $\tilde{H}^s(\Omega)$, see e.g.[8].

3.2 Formulation of Boundary Value Problem

Let Ω be a bounded open domain of $\mathbb{R}^2$. For simplicity, we assume that the boundary $\partial\Omega$ is a simply connected, closed, infinitely smooth curve.
Let us denote $\partial x_j = \dfrac{\partial}{\partial x_j}(j = 1,\, 2)$, $\partial x = (\partial x_1,\ \partial x_2)$.

For a linear operator L_* (either L_a or $\triangle$), we introduce the subspace of $H^1(\Omega)$
$(H^1(\Omega)\ := \{u \in L_2(\Omega)\ :\ \dfrac{\partial}{\partial x_i}u(x)\ \in L_2(\Omega)\})$

$$H^{1,0}(\Omega; L_*)\ := \{g : g \in H^1(\Omega),\ L_*g \in L_2(\Omega)\},$$

endowed with the norm

$$||g||^2_{H^{1,0}(\Omega;L_*)}\ :=\ ||g||^2_{H^1(\Omega)} + ||L_*g||^2_{L_2(\Omega)}.$$

Definition 3 *For any* $u \in C^\infty(\Omega)$, *define* ***the trace operator*** $\gamma^+_{|\partial\Omega}$ *by* $\gamma^+_{|\partial\Omega}u(x) = u(x)$, $x \in \partial\Omega$.

Theorem 2 ***Trace Theorem***
If $s > \dfrac{1}{2}$, *then* γ^+ *has a unique extension to a bounded linear operator*

$$\gamma^+ :\ H^s(\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega)$$

and we have

$$||\gamma^+u||_{H^{s-\frac{1}{2}}(\partial\Omega)}\ \le C||u||_{H^s(\Omega)},$$

$$H^{s-\frac{1}{2}}(\partial\Omega) = \{\gamma^+u : u \in H^s(\Omega)\}.$$

Note that the trace operator γ^+ has a bounded right inverse

$$\gamma^+_{-1}\ : H^{s-\frac{1}{2}}(\partial\Omega) \to H^s(\Omega)\ ,$$

i.e. $\gamma^+\gamma^+_{-1}w = w, w \in H^{s-\frac{1}{2}}(\partial\Omega)$ and

$$||\gamma^+_{-1}w||_{H^s(\Omega)}\ \le C||w||_{H^{s-\frac{1}{2}}(\partial\Omega)}$$

From the trace theorem for $u \in H^1(\Omega)$, it follows that $u^+ := \gamma^+_{\partial\Omega}u \in H^{\frac{1}{2}}(\partial\Omega)$, where $\gamma^+_{\partial\Omega}$ is the trace operator on $\partial\Omega$ from Ω.

Definition 4 *For* $u \in H^2(\Omega)$, ***the normal derivative*** *is defined as*

$$\begin{aligned} T^+(x,\ n^+(x),\ \partial_x)u(x)\ &:= \sum_{i=1}^{2} n_i^+(x)\ [\frac{\partial u(x)}{\partial x_i}]^+ \\ &=\ [\frac{\partial u(x)}{\partial n^+(x)}]^+ \end{aligned}$$

where $n^+(x)$ *is the exterior (to* Ω*) unit normal vectors at the point* $x \in \partial\Omega$.

In general the boundary differential operator T^+ is continuous mapping from $H^s(\Omega)$ to $H^{s-\frac{3}{2}}(\partial\Omega)$. That is

$$T^+ : H^s(\Omega) \to H^{s-\frac{3}{2}}(\partial\Omega)\ ,\ s > \frac{3}{2}.$$

We shall consider the scalar elliptic differential equation with constant coefficient 1.

$$\triangle u(x) = \sum_{i=1}^{2} \frac{\partial}{\partial x_i}[\frac{\partial u(x)}{\partial x_i}] = f(x) \qquad in\ \Omega \tag{3.1}$$

Given functions $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ and $f \in L_2(\Omega)$, we will consider the Dirichlet boundary value problem for function $u \in H^1(\Omega)$,

$$\triangle u = f \qquad in\ \ \Omega, \tag{3.2}$$

$$\gamma^+ u = \varphi_0 \qquad on\ \partial\Omega. \tag{3.3}$$

Here equation (3.2) is understood in the distributional sense and (3.3) in the trace sense.

For $u \in H^1(\Omega)$ the normal derivative operators on $\partial\Omega$ do not generally exist in the trace sense. However if $u \in H^{1,0}(\Omega;\triangle)$ one can correctly define the generalized (canonical) normal derivative $T^+u \in H^{-1/2}(\partial\Omega)$ with the help of the first Green's identity.

We define as in [5],[6],[11], the subspace

$$H^{1,0}(\Omega; A)\ := \{g \in H^1(\Omega)\ : \triangle g \in L_2(\Omega)\}$$

endowed with the norm $\|g\|^2_{H^{1,0}(\Omega;A)} := \|g\|^2_{H^1(\Omega)} + \|\triangle g\|^2_{L_2(\Omega)}$. For $u \in H^{1.0}(\Omega;\triangle)$ we can define the (canonical) normal derivative $T^+u \in H^{-\frac{1}{2}}(\partial\Omega)$ in the weak form (see, e.g.[6],[11] and the references therein),

$$(T^+u,\ w) := \int_\Omega [(\gamma^+_{-1}w)\triangle u + E(u,\ \gamma^+_{-1}w)]\,dx \qquad \forall w \in H^{\frac{1}{2}}(\partial\Omega), \tag{3.4}$$

where $\gamma^+_{-1}\ :\ H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega)$ is a continuous right inverse of the continuous interior trace operator $\gamma^+ :\ H^1(\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$, while $E(u,\ v) := \nabla u(x)\ .\ \nabla v(x)$ is the symmetric bilinear form.

For $u \in H^s(\Omega)$, $s > 3/2$, the canonical normal derivative defined by (3.4) coincides with the classical one, defined in the trace sense, i.e.,

$$T^+u = n \cdot \gamma^+ \nabla u, \tag{3.5}$$

where $n(x)$ is the exterior unit normal vector.

Remark 2 . *The first Green identity holds for any $u \in H^{1,0}(\Omega;\triangle)$ and $v \in H^1(\Omega)$ ([6],[11]) i.e,*

$$\int_\Omega E(u,\ v)dx = -\langle \triangle u,\ v\rangle_\Omega + \langle T^+u,\ \gamma^+ v\rangle_{\partial\Omega}$$

and the second Green identity holds for any $u, v \in H^{1,0}(\Omega; A)$,

$$\int_\Omega (v\triangle u - u\triangle v)dx = \langle T^+u,\ \gamma^+ v\rangle_{\partial\Omega} - \langle T^+v,\ \gamma^+ u\rangle_{\partial\Omega}.$$

3.3 Fundamental Based Potential-type Operators

A function $P_\triangle(x,\ y)$ is a fundamental solution for the operator $\triangle$ if

$$\triangle_x P_\triangle(x,\ y) = \delta(x-y),$$

where δ is the Dirac-delta distribution.
In particular, see e.g. [9], the function

$$P_\triangle(x,\ y) = \frac{1}{2\pi}\log|x-y|,\ x,\ y \in \mathbb{R}^2$$

is a fundamental solution for the Laplacian operator $\triangle$.

If $u \in H^{1,0}(\Omega; \triangle)$, then from the second Green's identity, we have the following fundamental-based third Green identity for $y \in \Omega$, [9],

$$u(y) \ = \int_{\partial\Omega} [\gamma^+ u(x) T_x^+ P_\triangle(x,\ y) - P_\triangle(x,\ y) T^+ u(x)]\, dx + \int_\Omega P_\triangle(x,\ y) f(x)\, dx, \qquad y \in \Omega. \tag{3.6}$$

Note that the direct substitution of $v(x)$ by $P(x,\ y)$ in the Second Green Identity is not possible as it has singularity at $x = y$. This difficulty is avoided by replacing Ω by $\Omega \backslash B(y,\ \varepsilon)$, where $B(y,\ \epsilon)$ is a disc of radius ε centered at y; taking the limit $\varepsilon \to 0$ we then arrive at (15.6), cf. e.g. [12].

The fundamental-based logarithmic operator is defined, similar to [1] in the 3D case, as

$$\mathcal{P}_\triangle g(y) \ := \int_\Omega P_\triangle(x,\ y) g(x)\, dx, \tag{3.7}$$

The single and double layer potential operators, corresponding to the fundamental $P_\triangle(x,\ y)$, are defined for $y \notin \partial\Omega$ as

$$\begin{aligned} V_\triangle g(y) &:= -\int_{\partial\Omega} P_\triangle(x,\ y) g(x)\, ds_x, \\ W_\triangle g(y) &:= -\int_{\partial\Omega} T_x^+ P_\triangle(x,\ y) g(x)\, ds_x. \end{aligned}$$

The following boundary integral (pseudo-differential) operators are also defined for $y \in \partial\Omega$,

$$\begin{aligned} \mathcal{V}_\triangle g(y) \ &:= -\int_{\partial\Omega} P_\triangle(x,\ y) g(x)\, ds_x, \\ \mathcal{W}_\triangle g(y) \ &:= -\int_{\partial\Omega} T_x^+ P_\triangle(x,\ y) g(x)\, ds_x, \\ \mathcal{W}'_\triangle g(y) \ &:= -\int_{\partial\Omega} T_y^+ P_\triangle(x,\ y) g(x)\, ds_x, \\ \mathcal{L}_\triangle^+ g(y) \ &:= \ T_y^+ W_\triangle g(y) = -T_y^+ \int_{\partial\Omega} T_x^+ P_\triangle(x,\ y) g(x)\, ds_x. \end{aligned}$$

Theorem 3 *For $s \in \mathbb{R}$, the following operators are continuous,*

$$\begin{aligned}
V_\Delta &: H^s(\partial\Omega) \to H^{s+\frac{3}{2}}(\Omega)\\
W_\Delta &: H^s(\partial\Omega) \to H^{s+\frac{1}{2}}(\Omega)\\
\mathcal{V}_\Delta &: H^s(\partial\Omega) \to H^{s+1}(\partial\Omega)\\
\mathcal{W}_\Delta,\ \mathcal{W'}_\Delta &: H^s(\partial\Omega) \to H^{s+1}(\partial\Omega)\\
\mathcal{L}^+{}_\Delta &: H^s(\partial\Omega) \to H^{s-1}(\partial\Omega)\ .
\end{aligned}$$

Proof: We have the corresponding mappings for the corresponding constant- coefficient operators.

Theorem 4 *Let $u \in H^{-\frac{1}{2}}(\partial\Omega)$ and $v \in H^{\frac{1}{2}}(\partial\Omega)$. Then the following jump relation hold on $\partial\Omega$*

$$\gamma^+ V_\Delta u(y) = \mathcal{V}_\Delta u(y) \tag{3.8}$$

$$\gamma^+ W_\Delta v(y) = -\frac{1}{2}v(y) + \mathcal{W}_\Delta v(y) \tag{3.9}$$

$$T^+ V_\Delta u(y) = \frac{1}{2}u(y) + \mathcal{W'}_\Delta u(y) \tag{3.10}$$

Proof: For the constant coefficient case, this theorem is well known.

Theorem 5 *Let Ω be a bounded open domain in $\mathbb{R}^2$ with closed, infinitely smooth boundary $\partial\Omega$. The following operators are continuous.*

$$\mathcal{P}_\Delta : \tilde{H}^s(\Omega) \to H^{s+2}(\Omega), \qquad s \in \mathbb{R}; \tag{3.11}$$

$$: H^s(\Omega) \to H^{s+2}(\Omega), \qquad s > -\frac{1}{2}; \tag{3.12}$$

$$\gamma^+\mathcal{P}_\Delta : \tilde{H}^s(\Omega) \to H^{s+\frac{3}{2}}(\partial\Omega), \qquad s > -\frac{3}{2}; \tag{3.13}$$

$$: H^s(\Omega) \to H^{s+\frac{3}{2}}(\partial\Omega), \qquad s > -\frac{1}{2}; \tag{3.14}$$

$$T^+\mathcal{P}_\Delta : \tilde{H}^s(\Omega) \to H^{s+\frac{1}{2}}(\partial\Omega), \qquad s > -\frac{1}{2}; \tag{3.15}$$

$$: H^s(\Omega) \to H^{s+\frac{1}{2}}(\partial\Omega)\ , \qquad s > -\frac{1}{2}; \tag{3.16}$$

Proof: The operator $\mathcal{P}_\Delta$ is a homogeneous pseudo-differential operator of order -2 on $\mathbb{R}^2$, mapping $\mathcal{P}_\Delta : H^s_{\mathrm{comp}}(\mathbb{R}^2) \to H^{s+2}_{\mathrm{loc}}(\mathbb{R}^2)$ continuously for any $s \in \mathbb{R}$. Hence the application of trace theorem, the operators (3.11), (3.13) and (3.15) are continuous. For $s \in (-\frac{1}{2}, \frac{1}{2})$, $\tilde{H}^s(\Omega)$ is identified with $H^s(\Omega)$, and (3.12) directly follows from (3.11). To prove the case $s \in (\frac{1}{2}, \frac{3}{2})$, we implement the Gauss divergence theorem and the fact that $\frac{\partial}{\partial x_j}\log|x-y| = -\frac{\partial}{\partial y_j}\log|x-y|$ and obtain

$$\begin{aligned}\frac{\partial}{\partial y_j}(\mathcal{P}_\Delta g)(y) &= -\frac{1}{2\pi}\int_\Omega g(x)\frac{\partial}{\partial x_j}\log|x-y|\,dx\\ &= \frac{1}{2\pi}\int_\Omega \log|x-y|\frac{\partial}{\partial x_j}g(x)dx - \frac{1}{2\pi}\int_{\partial\Omega}\log|x-y|n_j\gamma^+g(x)ds_x\\ &= \mathcal{P}_\Delta(\partial_j g)(y) + V_\Delta(n_j\gamma^+g)(y). \qquad (3.17)\end{aligned}$$

Now for $s \in (\frac{1}{2},\ \frac{3}{2})$, since $\partial_j : H^s(\Omega) \to H^{s-1}(\Omega)$ is continuous, we have $\mathcal{P}_\Delta\partial_j : H^s(\Omega) \to H^{s+1}(\Omega)$ is continuous, and from trace theorem $\gamma^+g \in H^{s-\frac{1}{2}}(\partial\Omega)$ and the properties of the single layer potential, we conclude that $\nabla\mathcal{P}_\Delta : H^s(\Omega) \to H^{s+1}(\Omega)$ is continuous. This implies that $\mathcal{P}_\Delta : H^s(\Omega) \to H^{s+2}(\Omega)$ is continuous for $s \in (\frac{1}{2},\ \frac{3}{2})$.

Further, with the help of these results and the relation (3.17), we can verify by induction that the operator (3.12) is continuous for $s \in (k-\frac{1}{2},\ k+\frac{1}{2})$, where k is an arbitrary nonnegative integer. For the values $s = k+\frac{1}{2}$ the continuity of the operator (3.12) then follows due to the complex interpolation properties of Bessel potential spaces.

The trace theorem will give the continuity proof for the operators (3.13) and (3.14). The continuity of the operators (3.15) and (3.16) follows if we remark that for the chosen s the normal derivative can be understood in the classical sense (15.5).□

By the Rellich compact embedding theorem (see e.g [8], Theorem 3.27]), Theorems 3 and 5 imply the following assertion.

Corollary 1 *Let $s \in \mathbb{R}$. The following operators are compact,*

$$\mathcal{V}_\Delta : H^s(\partial\Omega) \to H^s(\partial\Omega) \qquad (3.18)$$

$$\mathcal{W}_\Delta : H^s(\partial\Omega) \to H^s(\partial\Omega) \qquad (3.19)$$

$$\mathcal{W}'_\Delta : H^s(\partial\Omega) \to H^s(\partial\Omega) \qquad (3.20)$$

3.4 Invertibility of the Single Layer Potential Operator

The boundary integral operator $\mathcal{V}_\Delta : H^{-1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$ is Fredholm operator of index zero ([4], theorem 7.6). For the three dimensional case, the following holds. For $\psi^* \in H^{-1/2}(\partial\Omega)$, if $V_\Delta\psi^*(y) = 0, y \in \Omega$, then $\psi^* = 0$, which implies the invertibilty of single layer potential operator mapping from $H^{-1/2}(\partial\Omega)$ to $H^{1/2}(\partial\Omega)$. But it is not true the two dimensional case. It is well know ([4], Remark 1.42(ii), [13], proof of theorem 6.22) for some $2D$ domains the kernel of the operator $\mathcal{V}_\Delta$ is non-zero for the same domains. The following example illustrates this fact.

Example 1. Take the density function $\phi \equiv 1$ and $\Omega = B(0,\ R)$ *to be a disc of radius* R centered at the origin and $\partial\Omega = \partial B(0,\ R)$ be the circular boundary of the disc. We can show that

$$V_\Delta\phi(y) = \begin{cases} R\ln|y|, & for\ |y|\ > R,\\ \text{Rln } R, & for\ |y|\ \le R.\end{cases}$$

In the above example, if we take the value of $R = 1$, then $V_\triangle \phi(y) = 0$ in $\tilde{\Omega}$. This shows that the kernel of the operator $\mathcal{V}_\triangle : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ contains non zero element for a unit ball, i.e., Ker$\mathcal{V}_\triangle \neq \{0\}$ for $\Omega = B(0;1)$, which means, the operators $\mathcal{V}_\triangle$ is not one to one for this particular domain. In order to have invertibility for the single layer potential operator in 2D, we define the following subspace of the space $H^{-\frac{1}{2}}(\partial\Omega)$, see e.g. [13], Eq.(6.30),

$$H_*^{-\frac{1}{2}}(\partial\Omega) \ := \{\phi \in H^{-\frac{1}{2}}(\partial\Omega) \ : \ \langle\phi,\ 1\rangle_{\partial\Omega} = 0\},$$

where the norm in $H_*^{-\frac{1}{2}}(\partial\Omega)$ is the induced by the norm in $H^{-\frac{1}{2}}(\partial\Omega)$.

Theorem 6 *If $\psi \in H_*^{-\frac{1}{2}}(\partial\Omega)$ satisfies $\mathcal{V}_\triangle\psi = 0$ on $\partial\Omega$, then $\psi = 0$.*

Proof: (see e.g. [8], Corollary 8.11 ii]).□

Theorem 7 *Let $\Omega \subset \mathbb{R}^2$ have the diameter diam $(\Omega) < 1$. Then the single layer potential $\mathcal{V}_\triangle : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ is invertible.*

Proof: By [[13], Theorem 6.23], for *diam* $(\Omega) < 1$ the operator $\mathcal{V}_\triangle : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ is $H^{-\frac{1}{2}}(\partial\Omega)$-elliptic and since it is also bounded, c.f. Theorem 1 for $s = -1/2$, the Lax-Milgram theorem implies its invertibility.

3.5 The Third Green Identity

For $u \in H^{1,0}(A;\Omega)$, let us write the third Green's identity (3.6) using the surface and volume potential operator notations,

$$u - V_\triangle T^+ u + W_\triangle \gamma^+ u = P_\triangle \triangle u \qquad \textit{in } \Omega. \tag{3.21}$$

Applying the **trace operator** to equation (3.21) and using the jump relations from Theorem 4, we have

$$\begin{aligned}
\gamma^+[u - V_\triangle T^+ u + W_\triangle \gamma^+ u = \mathcal{P}_\triangle \triangle u] & \qquad \textit{in } \Omega. \\
\gamma^+ u - \underbrace{\gamma^+ V_\triangle T^+ u}_{\mathcal{V}_\triangle T^+ u} + \underbrace{\gamma^+ W_\triangle \gamma^+ u}_{-\frac{1}{2}\gamma^+ u \ +\mathcal{W}_\triangle \gamma^+ u} = \gamma^+ \mathcal{P}_\triangle \triangle u & \qquad \textit{on } \partial\Omega \\
\gamma^+ u - \mathcal{V}_\triangle T^+ u - \frac{1}{2}\gamma^+ u + \mathcal{W}_\triangle \gamma^+ u = \gamma^+ \mathcal{P}_\triangle \triangle u & \\
\frac{1}{2}\gamma^+ u - \mathcal{V}_\triangle T^+ u + \mathcal{W}_\triangle \gamma^+ u = \gamma^+ \mathcal{P}_\triangle \triangle u. &
\end{aligned}$$

$$\frac{1}{2}\gamma^+ u - \mathcal{V}_\triangle T^+ u + \mathcal{W}_\triangle \gamma^+ u = \gamma^+ \mathcal{P}_\triangle \triangle u \qquad \textit{on } \partial\Omega. \tag{3.22}$$

Similarly, applying the **normal derivative operator** to equation (3.21), and using again the jump relation, we obtain

$$
\begin{aligned}
u - V_\Delta T^+u + W_\Delta \gamma^+ u &= P_\Delta \triangle u \qquad\qquad \mathit{in}\ \Omega.\\
T^+[u - V_\Delta T^+u + W_\Delta \gamma^+ u &= P_\Delta \triangle u] \qquad\quad \mathit{on}\ \partial\Omega\\
T^+u - \underbrace{T^+V_\Delta T^+u}_{\frac{1}{2}T^+u + \mathcal{W}'{}_\Delta T^+u} + \underbrace{T^+W_\Delta \gamma^+ u}_{\mathcal{L}^+_\Delta \gamma^+ u} &= T^+\mathcal{P}_\Delta \triangle u\\
T^+u - (\frac{1}{2}T^+u + \mathcal{W}'{}_\Delta T^+u) + \mathcal{L}^+_\Delta \gamma^+ u &= T^+\mathcal{P}_\Delta \triangle u\\
T^+u - \frac{1}{2}T^+u - \mathcal{W}'{}_\Delta T^+u + \mathcal{L}^+_\Delta \gamma^+ u &= T^+\mathcal{P}_\Delta \triangle u\\
\frac{1}{2}T^+u - \mathcal{W}'{}_\Delta T^+u + \mathcal{L}^+_\Delta \gamma^+ u &= T^+\mathcal{P}_\Delta \triangle u.
\end{aligned}
$$

$$
\frac{1}{2}T^+u - \mathcal{W}'{}_\Delta T^+u + \mathcal{L}^+_\Delta \gamma^+ u = T^+\mathcal{P}_\Delta \triangle u, \qquad \mathit{on}\ \partial\Omega. \tag{3.23}
$$

For some functions f, Ψ and Φ let us consider a more general indirect integral relation associated with equation (3.21) .

$$
u - V_\Delta \Psi + W_\Delta \Phi = \mathcal{P}_\Delta f, \qquad \mathit{in}\ \Omega. \tag{3.24}
$$

Lemma 1 *Let $u \in H^1(\Omega)$, $f \in L_2(\Omega)$, $\Psi \in H^{-\frac{1}{2}}(\partial\Omega)$, and $\Phi \in H^{\frac{1}{2}}(\partial\Omega)$ satisfy equation (3.24). Then u belongs to $H^{1,0}(\Omega; A)$ and is a solution of PDE $\triangle u = f$ in Ω and*

$$
V_\Delta(\Psi - T^+u)(y) - W_\Delta(\Phi - \gamma^+ u)(y) = 0, \qquad y \in \ \Omega
$$

Proof. The proof follows word by word the corresponding proof in 3D case in [[1], Theorem 4.1].□

Lemma 2
(i) Let either $\Psi^* \in H^{-\frac{1}{2}}(\partial\Omega)$ and diam $(\Omega) < 1$, or $\Psi^* \in H_*^{-\frac{1}{2}}(\partial\Omega)$. If $V_\Delta \Psi^* = 0$ in Ω, then $\Psi^* = 0$ on $\partial\Omega$.
(ii) Let $\Phi^* \in H^{\frac{1}{2}}(\partial\Omega)$. If $W_\Delta \Phi^* = 0$ in Ω, then $\Phi^* = 0$ on $\partial\Omega$.

Proof:
(i) Taking the trace of equation in Lemma 2(i) on $\partial\Omega$, by the jump relation (3.8) we have $\mathcal{V}\Psi^*(y) = 0$ on $\partial\Omega$. If $\Psi^* \in H^{-\frac{1}{2}}(\partial\Omega)$ and diam $(\Omega) < 1$, then the result follows from the invertibility of the single layer potential given by Theorem 7. On the other hand, if $\Psi^* \in H_*^{-\frac{1}{2}}(\partial\Omega)$, then the result is implied by Theorem 5.
(ii) Let us take the trace of equation in Lemma 2(ii) on $\partial\Omega$, and use the jump relation (3.9) to obtain,

$$
-\frac{1}{2}\Phi^* + \mathcal{W}_\Delta \Phi^* = 0 \qquad on \partial\Omega. \tag{3.25}
$$

It is well known that this equation has only the trivial solution. It is particularly due to the contraction property of the operator $\frac{1}{2}I + \mathcal{W}_\Delta$, see [[14], Theorem 3.1].□

3.6 Boundary Integral Equations (BIEs)

To reduce the Dirichlet BVP (3.2) – (3.3) to a boundary integral equation system, let us denote the unknown normal derivative as $\psi := T^+u \in H^{-\frac{1}{2}}(\partial\Omega)$ and will further consider ψ as formally independent on u.

Assuming that the function u satisfies PDE $\triangle u = f$, by substituting the Dirichlet condition into the third Green identity (3.21) and either into its trace (3.22) or into its normal derivative (3.23) on $\partial\Omega$, we can reduce the BVP (3.2) – (3.3) to two different systems of Boundary Integral Equations for the unknown function $u \in H^{1,0}(\Omega;\triangle)$ and $\psi := T^+u \in H^{-\frac{1}{2}}(\partial\Omega)$.

BIE system (D1)obtained from equations (3.21) and (3.22) is

$$\begin{aligned} u - V_\Delta\psi &= F_0 \qquad in\ \Omega, \\ -\mathcal{V}_\Delta\psi &= \gamma^+F_0 - \varphi_0 \qquad on\ \partial\Omega, \end{aligned}$$

where

$$F_0 := \mathcal{P}_\Delta f - W_\Delta\varphi_0 \qquad in\Omega. \tag{3.26}$$

The system can be written in matrix form as $\mathcal{A}^1\mathcal{U} = \mathcal{F}^1$, where $\mathcal{U} := [u,\ \psi]^{\mathrm{T}} \in H^{1,0}(\Omega;A) \times H^{-\frac{1}{2}}(\partial\Omega)$ and

$$\mathcal{A}^1 = \begin{bmatrix} I & -V_\Delta \\ 0 & -\mathcal{V}_\Delta \end{bmatrix}, \qquad \mathcal{F}^1 = \begin{bmatrix} F_0 \\ \gamma^+F_0 - \varphi_0 \end{bmatrix}$$

From the mapping properties of W_Δ in Theorem 3 and $\mathcal{P}_\Delta$ in Theorem 5, we get the inclusion $F_0 \in H^{1,0}(\Omega;\Delta)$, and the trace theorem implies $\gamma^+F_0 \in H^{\frac{1}{2}}(\partial\Omega)$. Therefore, $\mathcal{F}^1 \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$. Due to the mapping properties of the operators involved in $\mathcal{A}^1$, the operator $\mathcal{A}^1 : H^{1,0}(\Omega;\Delta) \times H^{-\frac{1}{2}}(\partial\Omega) \to H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ is bounded.

BIE system (D2) obtained from equations (3.21) and (3.23) is

$$\begin{aligned} u - V_\Delta\psi &= F_0 \qquad in\Omega, \\ \frac{1}{2}\psi - \mathcal{W}'_\Delta\psi &= T^+F_0 \qquad on\partial\Omega, \end{aligned}$$

where F_0 is given by (3.26) . In matrix form it can be written as $\mathcal{A}^2\mathcal{U} = \mathcal{F}^2$, where

$$\mathcal{A}^2 = \begin{bmatrix} I & -V_\Delta \\ 0 & \frac{1}{2}I - \mathcal{W}'_\Delta \end{bmatrix}, \qquad \mathcal{F}^2 = \begin{bmatrix} F_0 \\ T^+F_0 \end{bmatrix}$$

Note that the operator $\mathcal{A}^2 : H^{1,0}(\Omega;\Delta) \times H^{-\frac{1}{2}}(\partial\Omega) \to H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ is bounded.

3.7 Equivalence and Invertibility Theorems

In the following theorem we shall see the equivalence of the original Dirichlet boundary value problem to the boundary integral equation systems.

Theorem 8 *Let* $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ *and* $f \in L_2(\Omega)$ *.*
(i) If some $u \in H^1(\Omega)$ *solves the BVP(3.2)-(3.3), then the pair* $(u,\ \psi)$ *, where*

$$\psi = T^+u \in H^{-\frac{1}{2}}(\partial\Omega), \tag{3.27}$$

solves BIE systems (D1) and (D2).
(ii) If a pair $(u,\ \psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ *solves BIE system (D1), and diam* $(\Omega) < 1$*, then u solves BIE system (D2) and BVP (3.2)-(3.3), this solution is unique, and* ψ *satisfies(3.27).*
(iii) If a pair $(u,\ \psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ *solves BIE system (D2), then u solves BIE system (D1) and BVP(3.2)-(3.3), this solution is unique, and* ψ *satisfies (3.27).*

Proof:
(i) Let $u \in H^1(\Omega)$ be solution of the BVP$(3.2) - (3.3)$. Since $f \in L_2(\Omega)$, we have that $u \in H^{1,0}(\Omega;\Delta)$. Setting ψ by (3.27) and recalling how BIE systems (D1) and (D2) were constructed, we obtain that $(u,\ \psi)$ solve them. That is,

Since $(u,\ \psi) = (u,\ T^+u)$ solves the BVP (3.2)-(3.3) implies

$$\triangle u = f$$

and

$$\varphi_0 = \gamma^+ u$$

And now for $u \in H^{1,0}(\Omega;\Delta)$ and from the second Green's identity, we have the fundamental based Green's third identity for $y \in \Omega$ using surface and volume potential operators notation:

$$\begin{aligned}
u - V_\Delta T^+u + W_\Delta \gamma^+ u &= \mathcal{P}_\Delta \triangle u \qquad\qquad\qquad in\ \Omega. \\
u - V_\Delta \underbrace{T^+u}_{\psi} &= \mathcal{P}_\Delta \underbrace{\triangle u}_{f} - W_\Delta \underbrace{\gamma^+ u}_{\varphi_o} \\
u - V_\Delta \psi &= \underbrace{\mathcal{P}_\Delta f - W_\Delta \varphi_o}_{F_0} \\
u - V_\Delta \psi &= F_0 \quad \text{ where } F_0 = P_\Delta f - W_\Delta \varphi_o
\end{aligned}$$

$\implies$ $(u,\ \psi) = (u,\ T^+u)$ solves the first equation of the BIE systems D_1 and D_2.

Applying trace simply to the above equation or applying trace to this third Green's identity and using the jump relations, we have

$$\begin{aligned}
u - V_\Delta \psi &= F_0 \qquad\qquad\qquad on\ \partial\Omega \\
\gamma^+[u - V_\Delta \psi &= F_0] \qquad\qquad\qquad on\ \partial\Omega \\
\underbrace{\gamma^+ u}_{\varphi_o} - \underbrace{\gamma^+ V_\Delta \psi}_{\mathcal{V}_\Delta \psi} &= \gamma^+ F_0 \\
\varphi_o - \mathcal{V}_\Delta \psi &= \gamma^+ F_0 \\
-\mathcal{V}_\Delta \psi &= \gamma^+ F_0 - \varphi_o
\end{aligned}$$

Or

$$\begin{aligned}
u - V_\Delta T^+ u + W_\Delta \gamma^+ u &= \mathcal{P}_\Delta \triangle u && \textit{in } \Omega. \\
\gamma^+ u - \gamma^+ V_\Delta T^+ u + \gamma^+ W_\Delta \gamma^+ u &= \gamma^+ \mathcal{P}_\Delta \triangle u && \textit{on } \partial\Omega \\
\gamma^+ u - \mathcal{V}_\Delta T^+ u - \frac{1}{2}\gamma^+ u + \mathcal{W}_\Delta \gamma^+ u &= \gamma^+ \mathcal{P}_\Delta \triangle u \\
\frac{1}{2}\gamma^+ u - \mathcal{V}_\Delta T^+ u + \mathcal{W}_\Delta \gamma^+ u &= \gamma^+ \mathcal{P}_\Delta \triangle u \\
\frac{1}{2}\varphi_o - \mathcal{V}_\Delta \psi + \mathcal{W}_\Delta \varphi_o &= \gamma^+ \mathcal{P}_\Delta f \\
-\mathcal{V}_\Delta \psi &= \gamma^+ \mathcal{P}_\Delta f - \frac{1}{2}\varphi_o - \mathcal{W}_\Delta \varphi_o \\
&= \gamma^+ \mathcal{P}_\Delta f + \frac{1}{2}\varphi_o - \mathcal{W}_\Delta \varphi_o - \varphi_o \\
&= \gamma^+ \mathcal{P}_\Delta f - (-\frac{1}{2}\varphi_o + \mathcal{W}_\Delta \varphi_o) - \varphi_o \\
&= \gamma^+ \mathcal{P}_\Delta f - \gamma^+ W_\Delta \varphi_o - \varphi_o \\
&= \gamma^+ (\mathcal{P}_\Delta f - W_\Delta \varphi_o) - \varphi_o \\
&= \gamma^+ (\mathcal{P}_\Delta f - W_\Delta \varphi_o) - \varphi_o \\
&= \gamma^+ F_o - \varphi_o
\end{aligned}$$

$\implies (u,\ \psi) = (u,\ T^+ u)$ solves the second equation of the BIE of system D_1.

Applying normal derivative simply to the first equation of BIE or to the third Green's identity and using jump relation, we have

$$\begin{aligned}
u - V_\Delta \psi &= F_0 && \textit{on } \Omega \\
T^+ [u - V_\Delta \psi &= F_0] && \textit{on } \partial\Omega \\
\underbrace{T^+ u}_{\psi} - \underbrace{T^+ V_\Delta \psi}_{\frac{1}{2}\psi + \mathcal{W}'_\Delta \psi} &= T^+ F_0 \\
\psi - (\frac{1}{2}\psi + \mathcal{W}'_\Delta \psi) &= T^+ F_0 \\
\psi - \frac{1}{2}\psi - \mathcal{W}'_\Delta \psi &= T^+ F_0 \\
\frac{1}{2}\psi - \mathcal{W}'_\Delta \psi &= T^+ F_0
\end{aligned}$$

Or

$$\begin{aligned}
u - V_\Delta T^+u + W_\Delta \gamma^+ u &= \mathcal{P}_\Delta \triangle u && \textit{in } \Omega. \\
T^+[u - V_\Delta T^+u + W_\Delta \gamma^+ u &= \mathcal{P}_\Delta \triangle u] && \textit{on } \partial\Omega. \\
\underbrace{T^+}_{\psi} u - T^+V_\Delta \underbrace{T^+u}_{\psi} + T^+W_\Delta \underbrace{\gamma^+u}_{\varphi_o} &= T^+\mathcal{P}_\Delta \underbrace{\triangle u}_{f} \\
\psi + T^+V_\Delta \psi - W_\Delta \varphi_o &= T^+\mathcal{P}_\Delta f \\
\psi - (\frac{1}{2}\psi + \mathcal{W}'{}_\Delta \psi) + T^+W_\Delta \varphi_o &= T^+\mathcal{P}_\Delta f \\
\psi - \frac{1}{2}\psi - \mathcal{W}'{}_\Delta \psi + T^+W_\Delta \varphi_o &= T^+\mathcal{P}_\Delta f \\
\frac{1}{2}\psi - \mathcal{W}'{}_\Delta \psi &= T^+\mathcal{P}_\Delta f - T^+W_\Delta \varphi_o \\
\frac{1}{2}\psi - \mathcal{W}'{}_\Delta \psi &= T^+(\underbrace{\mathcal{P}_\Delta f - W_\Delta \varphi_o}_{F_o}) \\
\frac{1}{2}\psi - \mathcal{W}'{}_\Delta \psi &= T^+F_o
\end{aligned}$$

$\implies (u,\ \psi) = (u,\ T^+u)$ solves the second equation of the BIE of system D_2.

Or
If $(u,\ \psi) = (u,\ T^+u)$ solves BVP (3.2) and (3.3), that is, $\triangle u = f$ and $\gamma^+u = \varphi_o$
First we want to show that $(u,\ \psi) = (u,\ T^+u)$ solves the first equation of both systems (D_1 and D_2)
Using Green's third identity ($u - V_\Delta T^+u + W_\Delta \gamma^+ u = \mathcal{P}_\Delta \triangle u$) and jump relations, we have

$$\begin{aligned}
u - V_\Delta \psi &= u - V_\Delta T^+u \\
&= \mathcal{P}_\Delta \triangle u - W_\Delta \gamma^+ u \\
&= \mathcal{P}_\Delta f - W_\Delta \varphi_o \\
&= F_o
\end{aligned}$$

$\implies (u,\ \psi) = (u,\ T^+u)$ solves the second equation of both systems D_1 and D_2

Second we want to show that $(u,\ \psi) = (u,\ T^+u)$ solves the second equation of BIE system D_1

$$
\begin{aligned}
-\mathcal{V}_\Delta\psi &= -\mathcal{V}_\Delta T^+u \\
&= -\gamma^+V_\Delta T^+u \\
&= -\gamma^+[-\mathcal{P}_\Delta\triangle u + u + W_\Delta\gamma^+u] \\
&= \gamma^+[\mathcal{P}_\Delta\triangle u - u - W_\Delta\gamma^+u] \\
&= \gamma^+[\mathcal{P}_\Delta\triangle u - u - W_\Delta\gamma^+u] \\
&= \gamma^+\mathcal{P}_\Delta\underbrace{\triangle u}_{f} - \underbrace{\gamma^+u}_{\varphi_o} - \gamma^+W_\Delta\underbrace{\gamma^+u}_{\varphi_o} \\
&= \gamma^+\mathcal{P}_\Delta f - \varphi_o - \gamma^+W_\Delta\varphi_o \\
&= \gamma^+\mathcal{P}_\Delta f - \gamma^+W_\Delta\varphi_o - \varphi_o \\
&= \gamma^+[\underbrace{\mathcal{P}_\Delta f - W_\Delta\varphi_o}_{F_o}] - \varphi_o \\
&= \gamma^+F_o - \varphi_o
\end{aligned}
$$

$\implies (u,\ \psi) = (u,\ T^+u)$ solves the second equation of system D_1

Third we need to show that $(u,\ \psi) = (u,\ T^+u)$ solves the second equation of BIE system D_2

$$
\begin{aligned}
\frac{1}{2}\psi - \mathcal{W}'_\Delta\psi &= \frac{1}{2}T^+u - \mathcal{W}'_\Delta T^+u \\
&= T^+u - \frac{1}{2}T^+u - \mathcal{W}'_\Delta T^+u \\
&= T^+u - [\frac{1}{2}T^+u + \mathcal{W}'_\Delta T^+u] \\
&= T^+u - T^+V_\Delta T^+u \\
&= T^+[u - V_\Delta T^+u] \\
&= T^+[u - V_\Delta T^+u] \\
&= T^+[\mathcal{P}_\Delta\underbrace{\triangle u}_{f} - W_\Delta\underbrace{\gamma^+u}_{\varphi_o}] \\
&= T^+[\mathcal{P}_\Delta f - W_\Delta\varphi_o] \\
&= T^+F_o
\end{aligned}
$$

$\implies (u,\ \psi) = (u,\ T^+u)$ solves the second equation of system D_2
And therefore, $\implies (u,\ \psi) = (u,\ T^+u)$ solves BIE system D_1 and D_2

For (ii) and (iii),
Let now a pair $(u,\ \psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ solves system (D1) or (D2). Due to the first equations in the BIE systems, the hypotheses of Lemma (1) are satisfied implying that u belongs to $H^{1,0}(\Omega;\Delta)$ and solves PDE (3.2) in Ω, while the following equation also holds,

$$V_\Delta(\psi - T^+u)(y) - W_\Delta(\varphi_0 - \gamma^+u)(y) = 0, \qquad y \in \Omega. \tag{3.28}$$

(ii) Let $(u,\ \psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ solves system (D1). Taking the trace of the first equation in (D1) and subtracting the second equation from it, we get $\gamma^+u = \varphi_0$ on $\partial\Omega$. Thus, the Dirichlet boundary condition is satisfied, and using it in (3.28) , we have $V(\psi - T^+u)(y) = 0,\ y \in \Omega$. Lemma 2(i) then implies $\psi = T^+u$. That is,

Let $(u,\ \psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ solves system (D1). That is,

$$\begin{aligned} u - V_\Delta\psi &= F_o && in\ \Omega \\ -\mathcal{V}_\Delta\psi &= \gamma^+F_o - \varphi_o && in\ \partial\Omega \end{aligned}$$

Taking the trace of the first equation in (D1) and applying the jump relation, we have

$$\begin{aligned} \gamma^+u - \gamma^+V_\Delta\psi &= \gamma^+F_o \\ \gamma^+u - \mathcal{V}_\Delta\psi &= \gamma^+F_o \end{aligned}$$

and subtracting the second equation from it, we get

$$-\left\{\begin{array}{rl} \gamma^+u - \mathcal{V}_\Delta\psi & = \gamma^+F_o \\ -\mathcal{V}_\Delta\psi & = \gamma^+F_o - \varphi_o \end{array}\right.$$

$$\begin{aligned} \implies \gamma^+u &= \varphi_o \qquad \text{on } \partial\Omega \\ \implies \gamma^+u - \varphi_o &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

Thus, the Dirichlet boundary is satisfied.
And using this in (3.28), we have

$$V_\Delta(\psi - T^+u)(y) - \underbrace{W_\Delta(\underbrace{\varphi_0 - \gamma^+u}_{0})(y)}_{0} = 0, \qquad y \in \Omega.$$

$$V_\Delta(\psi - T^+u)(y) = 0$$

And then by Lemma 2(i), since $\psi \in H^{-\frac{1}{2}}$ and $diam(\Omega) < 1$,
$V_\Delta(\psi - T^+u)(y) = 0,$ for $y \in \Omega \implies \psi - T^+u = 0,\ on\ \partial\Omega \implies \psi = T^+u\ on\ \partial\Omega$
$\implies T^+u = \psi \in H^{-\frac{1}{2}}\ on\ \partial\Omega$ and hence (3.27) satisfied.
And also if (u, ψ) solves BIE system (D_1), then (u, ψ) solves BIE system (D_2) since they are identical in their 1^{st} equation, and 2^{nd} equation of (D_2) comes from the 1^{st} equation of (D_1) by taking the normal derivative.

(iii) Let now $(u,\ \psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ solve system (D2). Taking the normal derivative of the first equation in (D2) and subtracting the second equation from it, we get $\psi = T^+u$ on $\partial\Omega$. Then inserting this in (3.28) gives $W_\Delta(\varphi_0 - \gamma^+u)(y) = 0, y \in \Omega$ and Lemma 2(ii) implies $\varphi_0 = \gamma^+u$ on $\partial\Omega$. That is,

Let now $(u,\ \psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ solve system (D2).
Taking the normal derivative of the first equation in (D2) and applying the jump relation, we have

$$\begin{aligned} u - V_\Delta \psi &= F_o && \textit{in } \Omega \\ T^+ u - T^+_y V_\Delta \psi &= T^+ F_o && \textit{on } \partial\Omega \\ T^+ u - \frac{1}{2}\psi - \mathcal{W}'_\Delta \psi &= T^+ F_o \end{aligned}$$

and subtracting the second equation from it, we get

$$-\left\{ \begin{array}{rl} T^+ u - \frac{1}{2}\psi - \mathcal{W}'_\Delta \psi & = T^+ F_o \\ \frac{1}{2}\psi - \mathcal{W}'_\Delta \psi & = T^+ F_o \end{array} \right.$$

$$\begin{aligned} &\Longrightarrow\ T^+ u - \psi = 0 \\ &\Longrightarrow\ T^+ u = \psi \qquad \textit{on } \partial\Omega \\ &\Longrightarrow\ T^+ u = \psi \in H^{-\frac{1}{2}}(\partial\Omega), \qquad \text{(3.27) satisfied.} \end{aligned}$$

Inserting this in (3.28), we have

$$V_\Delta(\psi - T^+ u)(y) - W_\Delta(\varphi_0 - \gamma^+ u)(y) = 0, \qquad y \in \Omega.$$

Since $\psi - T^+ u = 0$, we have

$$W_\Delta(\varphi_0 - \gamma^+ u)(y) = 0, \qquad y \in \Omega$$

And hence by Lemma 2(ii), since $\varphi_0 \in H^{\frac{1}{2}}$ and $W_\Delta(\varphi_0 - \gamma^+ u)(y) = 0, \quad y \in \Omega$

$$\Longrightarrow\ \varphi_o = \gamma^+ u \textit{ on } \partial\Omega$$

The uniqueness of the BIE system solutions follows from the fact that the corresponding homogeneous BIE systems can be associated with the homogeneous Dirichlet problem, which has only the trivial solution. Then paragraphs (ii) and (iii) above imply that the homogeneous BIE systems also have only the trivial solutions. □

Theorem 9 *If diam* $(\Omega) < 1$*, then the following operators are invertible,*

$$\mathcal{A}^1 : H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \to H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega), \tag{3.29}$$

$$\mathcal{A}^1 : H^{1,0}(\Omega;\Delta) \times H^{-\frac{1}{2}}(\partial\Omega) \to H^{1,0}(\Omega;\Delta) \times H^{\frac{1}{2}}(\partial\Omega). \tag{3.30}$$

Proof. Theorem 8(ii) implies that operators (3.29) and (3.30) are injective. Let us denote $\mathcal{A}^1 := \begin{bmatrix} I & -V_\Delta \\ 0 & -\mathcal{V}_\Delta \end{bmatrix}$. Then $\mathcal{A}^1 : H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \to H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ is bounded. It is invertible due to its triangular structure and invertibility of its diagonal operators I :

$H^1(\Omega) \to H^1(\Omega)$ and $-\mathcal{V} : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ (see Theorem 7) .

To prove invertibility of operator (3.30) , we remark that for any $\mathcal{F}^1 \in H^{1,0}(\Omega;\Delta) \times H^{\frac{1}{2}}(\partial\Omega)$, a solution of the equation $\mathcal{A}^1\mathcal{U} = \mathcal{F}^1$ can be written as $\mathcal{U} = (\mathcal{A}^1)^{-1}\mathcal{F}^1$, where $(\mathcal{A}^1)^{-1} : H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ is the continuous inverse to operator (3.29). But due to Lemma 1, the first equation of system (D1) implies that $\mathcal{U} = (\mathcal{A}^1)^{-1}\mathcal{F}^1 \in H^{1,0}(\Omega;\Delta) \times H^{-\frac{1}{2}}(\partial\Omega)$ and moreover, the operator $(\mathcal{A}^1)^{-1} : H^{1,0}(\Omega;\Delta) \times H^{\frac{1}{2}}(\partial\Omega) \to H^{1,0}(\Omega;\Delta) \times H^{-\frac{1}{2}}(\partial\Omega)$ is continuous, which implies invertibility of operator (3.30).□

The following similar assertion for the operator $\mathcal{A}^2$ holds without the limitation on the diameter of Ω.

Theorem 10 *The following operators are invertible.*

$$\mathcal{A}^2 : H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \to H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega), \tag{3.31}$$

$$\mathcal{A}^2 : H^{1,0}(\Omega;\Delta) \times H^{-\frac{1}{2}}(\partial\Omega) \to H^{1,0}(\Omega;\Delta) \times H^{-\frac{1}{2}}(\partial\Omega). \tag{3.32}$$

Proof: Theorem 6(iii) implies that operators (3.31) and (3.32) are injective. Since $\mathcal{A}^2 = \begin{bmatrix} I & -V_\Delta \\ 0 & \frac{1}{2}\, I - \mathcal{W}'_\Delta \end{bmatrix}$ Then $\mathcal{A}^2 : H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \to H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ is bounded. It is invertible due to its triangular structure and invertibility of its diagonal operators $I : H^1(\Omega) \to H^1(\Omega)$ and $\frac{1}{2}\, I - \mathcal{W}'_{.} : H^{-\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$(Since $\mathcal{W}'_{.}$ is compact, $\frac{1}{2}\, I - \mathcal{W}'_{.}$ is Fredholm with zero index). The invertibility of operator (3.32) is then proved similar to the last paragraph of the proof of Theorem 7. That is,

To prove invertibility of operator (3.32), we remark that for any $\mathcal{F}^2 \in H^{1,0}(\Omega;\Delta) \times H^{-\frac{1}{2}}(\partial\Omega)$, a solution of the equation $\mathcal{A}^2\mathcal{U} = \mathcal{F}^2$ can be written as $\mathcal{U} = (\mathcal{A}^2)^{-1}\mathcal{F}^2$, where $(\mathcal{A}^2)^{-1} : H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \to H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ is the continuous inverse to operator (3.31). But due to Lemma 1 the first equation of system (D2) implies that $\mathcal{U} = (\mathcal{A}^2)^{-1}\mathcal{F}^2 \in H^{1,0}(\Omega;\Delta) \times H^{-\frac{1}{2}}(\partial\Omega)$ and moreover, the operator $(\mathcal{A}^2)^{-1} : H^{1,0}(\Omega;\Delta) \times H^{-\frac{1}{2}}(\partial\Omega) \to H^{1,0}(\Omega;\Delta) \times H^{-\frac{1}{2}}(\partial\Omega)$ is continuous, which implies invertibility of operator (3.32).□

Chapter 4

Conclusion and Future Work

In this paper, we have considered the interior Dirichlet problem for constant coefficient PDE in a two-dimensional domain, where the right hand side function is from $L_2(\Omega)$ and the Dirichlet data from the space $H^{\frac{1}{2}}(\partial\Omega)$. The BVP was reduced to two systems of Boundary Integral Equations and their equivalence to the original BVP was shown. The invertibility of the associated operators in the corresponding Sobolev spaces was also proved.
In a similar way one can consider also the 2D versions of the BIEs for the Neumann problem, mixed problem in interior and exterior domains.

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