**ADDIS ABABA UNIVERSITY**

**COLLAGE OF NATURAL AND COMPUTATIONAL SCIENCES**

**DEPARTMENT OF MATHEMATICS**

# On Basic properties of *Harmonic Functions*

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A THESIS SUBMITTED TO THE DEPARTMENT OF MATHEMATICS, ADDIS ABABA UNIVERSITY IN PARTIAL FULFILLMENT OF THE REQUIREMENT OF THE DEGREE of MASTER OF SCIENCE IN MATHEMATICS

SEPT, 2017

ADDIS ABABA, ETHIOPIA

## Contents

# ACKNOWLEDGEMENTS

It is my pleasure to express my gratitude and appreciation to all those who provided me the possibility to complete this thesis.

I take this opportunity to express my profound gratitude and deep regards to my advisor Dr. Tsegaye Gedif ,for his exemplary guidance, monitoring and constant encouragement throughout the course of this thesis and preparation of the final manuscript of this thesis.

I would also like to thanks to Dr. Tadesse Abdi for his cordial support, valuable information and guidance, which helped me in completing this task through various stages.

I would like to express the deepest appreciation to members of Department of Mathematics for their co-operation.

Lastly, I thank almighty, my families, brother and friends for their constant encouragement, without which thesis would not be possible.

# ABSTRACT

This thesis provides an overview of harmonic functions of the following topic

a) Some basic results like, Maximum principle, uniqueness Theorem and Green's Formula.
b) The fundamental properties of harmonic functions, Results like Poisson's Integral Formula and its properties

C) The basic theory regarding harmonic functions and positive harmonic functions. .

Moreover, in this thesis we plan to focus on the advance theory of harmonic functions. Finally we give the detail prove of the Harnack's Inequality and Harnack's Theorem.

## NOTATION

$\mathbb{R}$ the set of real number.

$\mathbb{C}$ the complex plane.

$\mathbb{D}$ the unit disk $\{z \in \mathbb{C}: |z| < 1\}$ .

$\mathbb{R}^n$ n- dimensional real Euclidean space; the set of all n-tuples

$x=(x_1, x_2 \ldots)$

# CHAPTER 1

# INTRODUCTION

In this thesis, we plan to focus mainly on the theory of Harmonic functions.
The word “harmonic” is commonly used to describe a quality of sound. Harmonic functions derive their name from a roundabout connection they have with one source of sound.

Physicists label the movement of a point on a vibrating string “harmonic motion”. Such motion may be described using sine and cosine functions, and in this context the sine and cosine functions are sometimes called harmonics.

In classical Fourier analysis, functions on the unit circle are expanded in terms of sines and cosines. Analogous expansions exist on the sphere in $R^N, N > 2$, in terms of homogeneous harmonic polynomials. Because these polynomials play the same role on the sphere that the harmonics sine and cosine play on the circle, they are called spherical harmonics.

The term “spherical harmonic” was apparently first used in this context by William Thomson (Lord Kelvin) and Peter Tait. By the early 1900s, the word “harmonic” was applied not only to homogeneous polynomials with zero Laplacian, but to any solution of Laplace’s equation.

Harmonic function is the solution of Laplace’s equation and it plays a crucial role in many areas of mathematics, physics, and engineering. So it is necessary to extend harmonic functions to $R^n$, where $n$denotes a fixed positive integer greater than 1.

A function is harmonic if $u$ is $C^2$ and $x_0$ is a maximum of $u - v$ then, partials of $u - v$ at $x_0$ vanish and the second derivative is zero, so that

$$\Delta u = \Delta(u - v) = 0.$$

In fact this is enough to characterize harmonic functions whose second partial exist. Elementary properties of harmonic functions are often one-sided versions of properties of harmonic functions. For example, harmonic function $u$ on $D$ has a mean value property:

$$u(a) = \frac{1}{2\pi}\int_0^{2\pi} u\left(a + Re^{i\theta}\right)d\theta$$

This property characterizes harmonic functions.

This thesis organized into four chapters. In the *first* chapter there is introduction of the thesis which consist back ground of harmonic functions. The *second* chapter consist some basic and standard results on complex analysis and we study some basic results Lebeseque's covering lemma, Maximum modulus theorem, Schwarz lemma and Cauchy's integral formula.

In the beginning of chapter 3, we focus definition of harmonic function, basic properties of harmonic functions such as mean value property, maximum principle, uniqueness theorem and Greens formula. We prove the mean value property of harmonic function, and Poisons integral formula. The *fourth* chapter consists of the positive harmonic functions; we plan to study the harnacks inequality, and harnacks theorem.

CHAPTER 2

# REVIEW OF SOME IMPORTANT RESULTS IN COMPLEX ANALYSIS

In this chapter, we wish to revise some important results of complex analysis. We start with the connectedness and compactness of metric space. Next, we focus on some results regarding compactness and connectedness of metric space. We also study the following basic results: Heine-Borel theorem, Lebesgue covering lemma, Schwarz's lemma and Cauchy integral formula.

DEFINITION 2.1 (connectedness). A metric space (X, d) is connected if the only subsets of X which are both open and closed are $\emptyset$ and X. If A⊂X then A is connected subset of X if the metric space (A, d) is connected.

EXAMPLE 2.2: The set of real number $\mathbb{R}$ is connected.

PROPOSITION2.3. A set X⊂ $\mathbb{R}$ is connected if and only if X is an interval.

PROOF. Suppose X= [a, b], where a, b∈ $\mathbb{R}$. Let A⊂X be an open subset of X such that a∈ $A$ and A≠ $X$. We will show that A cannot also be closed and hence X must be connected. Since A is open and a∈ $A$ there is an $\epsilon > 0$ such that [a, a+$\epsilon$] $\subset A$.
Let r = sup $\{\epsilon : [a, a+\epsilon) \subset A\}$.

Claim: $[a, a+\epsilon) \subset A$. In fact, if $a \leq x < a + r$ then, putting h= $a + r - x > 0$,by the definition of supremum there is an $\epsilon$ $with\ r - h < \epsilon < r\ and\ [a, a+\epsilon) \subset A.$

But $a \leq x = a + (r - h) < a + \epsilon\ implies\ x \in A$ and our claim is established. However, $a + r \in A$: for if, on the contrary, $a + r \notin A$ then, by the openness of A, there is $\boldsymbol{\delta}$> 0 with $[a + r, a + r + \delta] \subset A,$ contradicting the definition of r. Now if A were also closed then $a + r \in B = x -$A which is open. Hence we could find a $\delta > 0$ such that $(a + r - \delta, a + r] \subset B,$ contradicting the above claim. Similarly we can prove this for other intervals. ■

DEFINITION 2.4 (Compactness) . A subset K of a metric space X is compact if for every collection $\mathbb{G}$ of open sets in X with property $K \subset (G: G \in \mathbb{G})$. A collection of set $\mathbb{G}$ satisfying above condition is called a cover of K. If each member of $\mathbb{G}$ is an open set it is called an open cover of K.

EXAMPLE 2.5

(1) The empty set and finite sets are compact.
(2) The set $D=\{z \in \mathbb{C}: |z| = 1\}$ is not compact.

PROPOSITION 2.6.Let K be a compact subset of X; then:

(a) K is closed;
(b) If F is closed and $F \subset K$ then F is compact.

PROOF. To prove part (a) we will show that $K=\bar{K}$. Let $x_o \in \bar{K}. so\ \exists B(x_o;\epsilon) \cap K \neq \emptyset for\ each\ \epsilon > 0.$

Let $G_n = X - \bar{B}\left(x_o;\frac{1}{n}\right)$ and suppose that $x_o \notin K$. Then each $G_n$ is open and $K \subset \bigcup_{n=1}^{\infty} G_n$ . since $K$ is compact there is an integer $m$ such that $K \subset \bigcup_{n=1}^{m} G_n$.

But $G_1 \subset G_2 \subset \ldots$ so that $K \subset G_m = X - \bar{B}\left(x_o;\frac{1}{m}\right)$. But this gives that

$B(x_o;\frac{1}{m}) \cap K = \emptyset$, a contradiction. Thus $K=\bar{K}$.

To prove part (b) let $\mathbb{G}$ be an open cover of F. Then, since F is closed. $\mathbb{G} \cup \{X-f\}$ is an open cover of K. Let $G_1,\ldots,G_n$ be sets in $\mathbb{G}$ such that $K \subset G_1 \cup \ldots \cup (X-F)$. Clearly, $F \subset G_1 \cup \ldots \cup G_n$ and so F is compact. ■

COROLLARY 2.7 Every compact metric space is complete.

DEFINITION 2.8 (sequentially compact) A metric space (X ,d) is sequentially compact if every sequence in X has a convergent subsequence.

LEMMA 2.9 (Lebesgue's covering Lemma). If (X, d) is sequentially compact and $\mathbb{G}$ is an open cover of X then there is an $\epsilon > 0$ such that if x is in X, there is a set G in $\mathbb{G}$ with $B(x;\epsilon) \subset G$.

PROOF  We will prove this lemma by method of contradiction. Suppose that $\mathbb{G}$ is an open cover of X and no such $\epsilon > 0$ can be found. In particularly, for every integer n there is a point $x_n$ *in X such that* $B\left(x_n;\frac{1}{n}\right)$ *is not contained* in any set G in $\mathbb{G}$. Since X is sequentially compact there is a point $x_0$ *in* $X$ and a subsequence $\{x_{n_k}\}$ such that $\lim_{n\to\infty} x_{n_k} = x_o$. Let $G_o \in \mathbb{G}$ such that $x_o \in G_o$ *and choose* $\epsilon > 0$

Such that $B(x_o;\epsilon) \subset G_o$. Now let N be such that $d(x_o;x_{n_k}) < \frac{\epsilon}{2} \, \forall_{n_k} \geq N$.

*Let* $n_k$ *be any integer larger than both N and* $\frac{2}{\epsilon}$ *and let* $y \in B\left(x_{n_k};\frac{1}{n_k}\right)$.Then by triangle inequality $d(x,y) = d(x_o;x_{n_k}) + d(x_{n_k};y) < \frac{\epsilon}{2} + \frac{1}{n_k} < \epsilon$. That is $B(x_{n_k};\frac{1}{n_k}) \subset B(x_o;\epsilon) \subset G_o$ which is a contradiction to fact the that $x_n$ has a convergence subsequence $x_o$ in X. So our choice of $x_{n_k}$ is wrong. ■

THEOREM 2.10 (Heine-Borel Theorem). A subset K of $\mathbb{R}^n (n \geq 1)$ is compact if and only if K is closed and bounded.

DEFINITION 2.11 Let $\Omega \subset \mathbb{C}$ and $f: \Omega \to \mathbb{C}$. then $f$ is said to be annalytic or holomorphic at $z_o \in \Omega$ if and only if $f$ is differentiable in some neieighborhocf $z_o$ including at $z_0$; i.e. there exist $\delta > 0$ such that $f$ is differntiable on $B(z_o; \delta)$.

REMARK 2.12 (1) $f$ is analytic at $z_0$ implies $f$ is differntable at $z_o$.

(2) Let $G \subset \mathbb{C}$ be a region and $f: G \to \mathbb{C}$ be continuously differentiable on G. Then $f$ is analytic on G.

THEOREM 2.13 (Cauchy's Integral Formula). Let $f$ be analytic on a positively oriented smooth simple closed curve C and inside C. If $z_0$ is any point in the interior of C, then $f(z_o) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz$

THEOREM 2.14 (The Maximum principle). Let $\Omega \subset \mathbb{C}$ and suppose $\alpha$ is in the interior of $\Omega$. We can therefore, choose a positive number N such that $B(\alpha, N) \subset \Omega$ it readily follows that there is a point N in $\Omega$ with $|N| > |\alpha|$ i.e. if $\alpha$ is a point in $\Omega$ with $|N| > |\alpha|$ for each N in the set $\Omega$ then $\alpha$ belongs to $\partial\Omega$.

THEOREM 2.15 (Maximum Modulus theorem) if $f$ is analytic in a region G $\alpha$ is a point in G with $|f(\alpha)| \geq |f(z)|$ $\forall z$ in G then $f$ must be a constant function.

THEOREM 2.16 (Schwarz's lemma) Let $\mathbb{D}$={z:|z|<1} and suppose $f$ is analytic on $\mathbb{D}$ with

$(a)|f(z)| \leq 1 \, for \, z \, in \, \mathbb{D}$ $\qquad$ $(b) \, f(0) = 0.$

Then $|f'(0)| \leq 1 \, and \, |f(z)| \leq |z| \forall z \in D. moreover \, if |f'(0)| = 0 or \, |f(z)| = |z|$ for some $z \neq 0$ $then \, there \, is \, a \, constant \, c, |c| < 1 \, such \, that$

$f(w) = cw \, \forall w \, in \, \mathbb{D}.$

PROOF Let define $g$:$\mathbb{D} \rightarrow \mathbb{C}$ by

$$g(z) = \frac{f(z)}{z} \Rightarrow f'(0) = g(0) \, for \, z \neq 0, then \, g \, is \, analytic \, in \, \mathbb{D}.$$

According to Maximum Modulus theorem for $|z| \leq r \, and \, 0 < r < 1,$

$we \, have \, |\, g(z)| = \frac{|f(z)|}{|z|} \leq r^{-1}, (\because \, |f(z)| \leq 1 \, \forall z \in \mathbb{D})$.As r→ 1 $we \, have \, |f(z)| \leq |z|$ $\forall z \in \mathbb{D}$and $|f'(0)| = |g(0)| \leq 1.$ If $|f(z)| \leq |z|$ for some z in $\mathbb{D}$, z=0 or $|f'(0)| = 1$, then $|g|$ assumes its maximum value inside $\mathbb{D}$. Then again by applying maximum modulus theorem,$|g(z)| \equiv c$ for some constant c with c=1, since $|g(z)| = \frac{|f)z)|}{|z|} = c, so \, we \, have \, f(z) = cz \, \forall z \in \mathbb{D}.$ ■

CHAPTER 3

# BASIC PROPERTIES OF HARMONIC FUNCTIONS

## 1. Preliminaries

DEFINATION 3.1

If G is an open subset of $\mathbb{C}$, then a function $U$: G→ $\mathbb{R}$ is harmonic

if it has continuous second order partial derivative and it satisfies Laplace's equation, that is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ or } u_{xx} + u_{yy} = 0.$$

Often written as $\nabla^2 u = 0$.

EXAMPLE 3.2

(1) The real and imaginary part of any holomorphic function is a harmonic function.

(2) The function $f(x, y) = e^x \cos y$ is a Harmonic function.

(3) $u(z) = \ln(|z|^2)$ *is harmonic on* $G = \mathbb{C} \setminus \{0\}$. *Indeed,*

$$ln|z| = ln\sqrt{x^2 + y^2} \text{ and } (u_{xx} + u_{yy})(z) = 0 \text{ if } z \neq 0.$$

LEMMA3.3: If $v$ is a conjugate harmonic function of $u$, then $u$ is a conjugate harmonic function of $v$.

PROOF, Given $v$ *is a conjugate harmonic function of* $u$

Claim: $-v+iu$ *is analytic. We know that* $f=u+iv$ *is analytic.*

$$\Rightarrow \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x} \quad \text{by Cauchy Riemann equation}$$

$$Now \frac{\partial}{\partial x}(-v)=-\frac{\partial v}{\partial x}=\frac{\partial u}{\partial y} \text{ and } \frac{\partial}{\partial y}(-v)=-\frac{\partial v}{\partial y}=-\frac{\partial u}{\partial x}$$

Hence $u$ *is a conjugate harmonic function of* $v$. ■

THEOREM 3.4 A function f on a region G is analytic if and only if $Ref=u$ and $Imf=v$ are harmonic functions.

Theorem 3.5 A region G is simply connected if and only if for each harmonic function u on G there is a harmonic function v on G such that $f=u+iv$ is analytic on G.

PROPSTION 3.6 If u: G→ ℝ is harmonic, then u is infinitely differentiable.

Proof: Fix $z_o=x_o+iy_o$ in G. Let $\delta>0$ chosen such that B $(z_o;\delta)\subset$ G.

As u is a harmonic conjugate v in B $(z_o;\delta)$ .That means $f=u+iv$ is analytic.

⇒ It is infinitely differentiable on B $(z_o;\delta)$.

So u is infinitely differentiable. ■

THEOREM 3.7 (Mean value Theorem):

Let. $u: G \to \mathbb{R}$ *be a harmonic function and* $\bar{B}(a, r)$ be a closed disk contained in G.

if $\gamma$ is a circle $|z - a| = r$ then

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + r\,e^{i\theta})\,d\theta$$

PROOF. Let D be a disk such that $\bar{B}(a, r) \subset D \subset G$ *and* $f$ *be a analytic function on* $D$ Such that $Ref = u$. By Cauchy's integral formula

$$f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - a} dz, \text{where } \gamma = B(z; r).$$

(3.1) $\quad$ Let $z - a = re^{i\theta} \Rightarrow dz = ire^{i\theta}\,d\theta.$

$$f(a) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(a + re^{i\theta})}{re^{i\theta}} ire^{i\theta}\,d\theta.$$

$$\Rightarrow f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta})\,d\theta.$$

So by taking the real part of equation (3.1), we get

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta})\,d\theta \qquad \blacksquare$$

THEOREM3.8: (Maximum principle{$first\ version$}).

Let G be a region and suppose that u is a continuous real valued function on G with the MVP. If there is a point $a$ in G such that $u(a) \geq u(z) \forall z \in G$. then u is a constant function.

PROOF. Set A be defined by A=$\{z \in G: u(z) = u(a)\}$. $As\ u\ is\ continuous\ on\ the$

$set\ A\ is$ Closed in G. If $z_o \in$ A, then we choose a r such that $\bar{B}(z_o; r) \subset G$. Suppose $\exists$ a point b$\in$ $B(z_{o;}r)$ $such\ that\ u(b) \neq u(a); then, u(b) < u(a). By\ continus$

$u(z) < u(a) = u(z_o) \forall$z in neighborhood of b. in particular $\rho = |z_o - b|$ and

b = $z_o + \rho e^{i\theta}$, $0 \leq \beta < 2\pi$.

So there is a proper interval I of $[0, 2\pi]$ such that $\beta \epsilon I\ and\ u(z_0 + \rho e^{i\theta}) < u(z_0)\ \forall \theta \in I$.

So by MVP

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + \rho e^{i\theta})d\theta < u(z_0),$$

Which is a contradiction. So $B(z_o; r) \subset A\ and\ A\ is\ open.\ so$ by definition 2.1 A=G.

THEOREM 3.9: (maximum principle{$second\ verstion$}).

Let G be a region and let u and v be two continuous real valued functions on G that have the MVP. If for each point a in the extend boundary $\partial_\infty G$ . $\lim\limits_{z \to a} sup u(z) \leq$ $\lim_{z \to a}$ inf$(z)$ $then\ either\ u(z) < \forall z \in G\ or\ u = v$.

PROOF: Fix a in$\partial_\infty G$ and for each $\delta > 0, let\ G_\delta \cap B(a;\delta)$. then by hypothesis,

$$0 \geq \lim_{\delta\to 0}[(sup\{u(z): z \in G_\delta\} - \inf\{v(z): z\epsilon G_\delta\}]$$

$$= \lim_{\delta\to 0}[(sup\{u(z): z \in G_\delta\} - \sup\{-v(z): z\epsilon G_\delta\}]$$

$$\geq \lim_{\delta\to 0}(sup\{u(z) - v(z): z\epsilon G_\delta\}$$

(3.2) So $\lim_{z\to a} sup\{[u(z) - v(z)]\} \leq 0$ for each$a \in \partial_\infty G$.Let $v(z)=0\forall z\in G$. that is, assume $\lim_{z\to a}. sup u(z) \leq 0 \forall a \in \partial_\infty G. claim: u(z) < 0 \forall z \in G or\ u = 0$.

If we show that u(z)$\leq 0\forall z \in G$,then by theorem 3.8 $u \equiv 0$. Suppose that u satisfies (3.2) and there is a point b in G with $u$(b)>0. Let $\epsilon$>0 be chosen so that $u$ (b)>$\epsilon$ and let B=$\{z \in G: u(z) \geq \epsilon\}$.

If $a \in \partial_\infty G$ then by propos ion 3.6, there is a $\delta = \delta(a) such\ that\ u(z) < \epsilon$

$\forall z \in G \cap B(a;\delta)$ . By lemma 2.9 a $\boldsymbol{\delta}$ can be found which is independent of $a$.

That means, there is a$\delta > o\ such\ that\ if\ z \in G\ and\ d(z, \partial_\infty G) < \delta\ then\ u(z) < \epsilon$. thus B$\subset \{z \in G: d(z, \partial_\infty G) \geq \delta\}$.

This gives that B is a bounded plane and closed. So B is compact. So B$\neq$ $\emptyset$, $there\ is\ a\ point\ z_0 \in B\ such\ that\ u(z_0) \geq u(z) \forall z \in B. since\ u(z) < \epsilon\ for\ z \in G - B$,it gives that u assumes a maximum value at a point in G.

So u must be constant, which is nothing but the $u(z_o)$ and positive. Which contradict (3.2). So it gives the prove of the theorem. ■

THEOREM 3.10(uniqueness Theorem): Let D be a domain bounded in ℂ and $h_1$ and $h_2$ be two harmonic functions that extend continuous to the boundary ∂D of D. if $h_1$=$h_2$ on ∂D. Then these two functions are equal throughout D.

PROOF: consider the function h=$h_1$-$h_2$. This function is harmonic and by construction, h=0 on ∂D. let h be a harmonic function on a domain D in ℂ.

If h attains a local maximum in D then h is constant. Suppose that D is bounded and h extends continuously to the boundary ∂D of D. if h≤0 an ∂D then h≤ 0 throughout D. Applying now the function –h, we conclude that h=0 on D.

Hence, $h_1$=$h_2$. ■

COROLLARY 3.11: Let D be a domain in ℂ and z∈D. If $h_1$ and $h_2$ are two functions harmonic on D, and such that $h_1$=$h_2$ on the boundary of a disk about z in D then these two functions are equal throughout D.

## 2. Green's formula and its properties

In this section, we summarize some useful formulas due to Green. These formulas are convenient in the computations related to Laplacian.

These are derived from Divergence Theorem (or, Gauss formula):

THEOREM 3.12 Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary ∂Ω. Let V be the unit outer normal of ∂Ω. For any smooth vector field $w \in C^1(\overline{\Omega})$, it holds that

$$\int_{\Omega} \nabla . w dx = \int_{\partial\Omega} w . V \, dS \tag{2.1}$$

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Let V be the unit outer normal of $\partial\Omega$. For $u \in C^2(\overline{\Omega})$, one derives from the divergence theorem

(Letting $w = Du$) that

$$\int_\Omega \Delta u dx = \int_{\partial\Omega} Du.V\ ds = \int_{\partial\Omega} \frac{\partial u}{\partial v} dS \qquad (2.2)$$

Now, for $u, v \in C^2(\overline{\Omega}) by\ choosing\ w = uDv\ or\ w = vDu$ respectively in the divergence theorem, we have

$$\int_\Omega (u\Delta v dx) = \int_{\partial\Omega} u \frac{\partial v}{\partial v} dS - \int_\Omega Du.Dv\ dx \qquad (First\ Green's formula) \qquad (2.3)$$

$$\int_\Omega (v\Delta u dx) = \int_{\partial\Omega} v \frac{\partial u}{\partial v} dS - \int_\Omega Dv.Du\ dx \qquad (2.4)$$

We subtract the above two equations to get

$$\int_\Omega (u\Delta v - v\Delta u) dx = \int_{\partial\Omega} \left( u \frac{\partial v}{\partial V} - v \frac{\partial u}{\partial V} \right) dS. \quad (Second\ Green's formula) \quad 2.5$$

**Remark.** *By* approximation, *(2.3) – (2.5)* continue to hold for function

$$u, v \in C^2(\Omega) \cap C^1(\overline{\Omega})$$

such that $\Delta u\ and\ \Delta v$ are essentially bounded $in\ \Omega$.

**Remark .** If u is harmonic in $\Omega$, then

$$\int_{\partial\Omega} \frac{\partial u}{\partial V} dS = 0$$

*and*

$$\int_\Omega |\Delta u|^2\ dx = \int_{\partial\Omega} u \frac{\partial u}{\partial V} dS.$$

## 3. POISSON INTEGRAL FORMULA AND IT'S PROPERTIES

In this section, we shall attempt to find a harmonic analog to Cauchy's integral formula. If $f$ is analytic inside and on a simple closed contour C, then

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}dz \qquad (3.1)$$

At all points $z_0$ inside C. we would like to find an expression for Re$f$ at points inside C in terms of the values of Re$f$ on C.

Unfortunately, the expression Re $\left\{\frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}dz\right\}$

Simplifies into one involving both Re$f$ and Im$f$ on C.

If, however, the integral of (3.1) is transformed into one of the form

$\int_a^b \phi(t)dt,$ where $\Phi(t)$ is a complex-valued function of a real variable t, then

$$Re\int_a^b \Phi(t)dt = \int_a^b \text{Re}\,\Phi(t)dt \; .$$

Recall that we performed this kind of transformation when proving the mean-value principle for harmonic functions. This enabled us to determine value of harmonic function at the center of a circle based on its values on the circumference.

By (3.1), we have the so-called mean value property for analytic functions:

$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a + re^{i\Phi})d\Phi, 0 < r < dis(a, C) = R$ For $a$ inside C.

The value $f(a)$ of $f$ at the center $a$ of the disk $|z_0 - a| < r$ is expressed by the integration of $f$ over the boundary circle $|z_0 - a| = r$ of this disk.

Note that $f(a)$ is the same for all $r$ in the interval (O, R). We wish to obtain similar expression for a point of the disk $|z_0 - a| < r$ other than the center.

But an analog to the Cauchy integral formula for the circle is an expression for the harmonic function at all points inside the circle in terms of its values on the circle.

LEMMA 3.1: (Poisson Integral Formula for analytic Function): Suppose $f(z_o)$ is analytic in a domain containing the closed unit disk $|z_o| \le 1$. then for $|z_o| < 1$,

We have $$f(z_o) = \frac{1}{2\pi} \int_{|Z|=1} \frac{1-|z_0|^2}{|z-z_0|^2} f(z) \frac{dz}{iz} \tag{3.2}$$

Or equivalently $$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1-|z_0|^2}{|e^{i\Phi}-z_0|^2} f(e^{i\Phi}) d\Phi \tag{3.3}$$

PROOF. By Cauchy's integral formula, we have

$$f(z_o) = \frac{1}{2\pi i} \int_{|Z|=1} \frac{f(z)}{z-z_0} dz$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{zf(z)}{z-z_0} d\Phi \qquad (|z| < 1) \tag{3.4}$$

If $z_o = 0$, the result follows from Gauss's Mean-value theorem.

So we may suppose that $z_o \ne 0$, and set $z^*_0 = \frac{1}{\bar{z}_o}$ which is the reflection of $z_o$ in the unit circle.

The point $z^*_o$, which lies on the ray from the origin through $z_o$, is outside the unit circle $|z| = 1$.

Hence (as $z^*_o = \frac{1}{\bar{z}_0}$), for $|z_o| < 1$.

$$0 = \frac{1}{2\pi i} \int_{|Z|=1} \frac{f(z)}{z-z^*_0} dz = -\frac{1}{2\pi i} \int_{|Z|=1} \frac{\overline{z_o} f(z)}{z - z\overline{z_0}} dz \tag{3.5}$$

Subtracting (3.5) from (3.4) we get

$$f(z_o) = \frac{1}{2\pi i} \int_{|Z|=1} \left[ \frac{1}{z-z_0} + \frac{\overline{z_0}}{1-z\overline{z_0}} \right] f(z) dz. \tag{3.6}$$

We can simplify (since $|z| = 1$) to,

$$\frac{1}{z-z_o}+\frac{\bar{z}_o}{1-z\bar{z}_o}=\frac{1-|z_0|^2}{(z-z_o)(1-z\bar{z}_o)}=\frac{1-|z_o|^2}{(z-z_o)(\bar{z}-\bar{z}_o)}=\frac{1-|z_o|^2 1}{|z-z_o|^2 z}$$

Using the last equality, (3.6) gives (3.2).

Equation (3.3) follows if we let z = $e^{i\theta}$ in (3.2)

The following general result is a consequence of the above lemma 3.1.

THEOREM 3.2:(Poisson Integral formula for analytic functions):

Suppose $f(z_o)$ is analytic in domain containing the closed disk

$|z_o - a| \le R. Then\ \ for\ \ |z_o - a| < R,$

We have $f(z_o) = \frac{1}{2\pi}\int_{|Z-a|=R}\frac{R^2-|z_0-a|^2}{|z-z_0|^2}f(z)\frac{dz}{i(z-a)}$,

Or equivalently,

$$f(z_o) = \frac{1}{2\pi}\int_0^{2®}\frac{R^2-|z_0-a|^2}{|\mathrm{Re}^{i¯}-(z_0-a)|^2}f(a+\mathrm{Re}^{i¯})\ d¤ \qquad (3.7)$$

PROOF By the change of variable $w = \frac{(z_o-a)}{R}$,

it reduces to the case where R=1 and $a$=0.

In particular, for $a$=0, the formula reduces to

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{\left|Re^{i\Phi} - re^{i\theta}\right|^2} f(Re^{i\Phi})d\Phi .$$

The expression (with z=Re$^{i\Phi}$, $z_o = re^{i\theta}$and $r <$ R).

P $(z_o, z) = \frac{|z|^2 - |z_o|^2}{|z-z_o|^2} =$ Re$\left(\frac{z+z_o}{z-z_o}\right) = \frac{R^2-r^2}{R^2-2rR\cos(\theta-\Phi)+r^2}$ is known as the Poisson kernel for the disk $|z_o| < R$.

Note that Poisson kernel is bounded above by, $\frac{R^2-r^2}{R^2-2rR+r^2} = \frac{R+r}{R-r}$, and

is bounded below by , $\frac{R^2-r^2}{R^2+2rR+r^2} = \frac{R-r}{R+r}$.

Let $a$=0 and let $f(z_o) = u(z_o) + iv(z_o)$ be analytic for $|z_o| \le R$. Then, from Theorem 3.2. It follows that $f(z_o) = \frac{1}{2\pi}\int_{|z|=R} p(z_o, z)f(z)d\phi$ and equating the real part gives

THEOREM3.3 : ( Poisson Integral Formula for harmonic functions)

Suppose $u(z_o)$ is harmonic in a domain containing the disk $|z_o| \le R$.

*Then for* $z_o = re^{i\theta}, r <$ R, we have $u(z_o) = \frac{1}{2\pi}\int_{|Z|=R} p(z_0, z)u(z)d\Phi$;

or equivalently,

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2rR\cos(\theta - \Phi) + r^2} u(Re^{i\Phi})d\Phi$$

A similar formula holds for the imaginary part $v(z_o)$ *of* $f(z_o)$.

COROLLAR3.4. For $r < R and\ \theta\ arbitrary,$

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{R^2-r^2}{R^2-2rR\cos(\theta-\Phi)+r^2}d\Phi=\frac{1}{2\pi}\int_0^{2\pi}p(re^{i\theta},\mathrm{Re}^{i\Phi})d\Phi=1.$$

PROOF .set $u(z_o)\equiv 1$ in theorem 3.3

THEOREM3.5 .Suppose $f(z_0)=u(z_0)+iv(z_0)$ is analytic in the disk $|z_0|\le 1$.then for $|z_0|<1$, we may express $f(z_0)$ as

$$f(z_0)=\frac{1}{2\pi}\int_0^{2\pi}\frac{z+z_0}{z-z_0}u(z)d\Phi+iv(0)(z=e^{i\theta}) \qquad (3.8)$$

PROOF to do this it suffices to recall (3.4) and (3.5):

$$f(z_0)=\frac{1}{2\pi}\int_0^{2\pi}\frac{z}{z-z_0}f(z)d\Phi, \qquad (3.9)$$

And (because $z\bar{z}=1$), $0=\frac{-1}{2\pi i}\int_{|z|=1}\frac{\overline{z_0}}{1-z\overline{z_0}}f(z)dz=\frac{-1}{2\pi}\int_0^{2\pi}\frac{\overline{z_0}}{\overline{z-z_0}}f(z)d\Phi.$

Since the integral on the right is a Riemann integral, taking conjugation on the right leads to:

$$0=\frac{-1}{2\pi}\int_0^{2\pi}\frac{z_0}{z-z_0}\overline{f(z)}d\Phi \qquad (3.10)$$

Writing $f(z)=u(z)-iv(z), and\ then\ adding\ (3.9) and\ (3.10) shows\ that:$

$$f(z_0)=\frac{1}{2\pi}\int_0^{2\pi}\frac{z+z_0}{z-z_0}u(z)d\Phi+i\frac{1}{2\pi}\int_0^{2\pi}v(z)d\Phi.$$

The desired formula (3.8) follows if we apply the mean-value property for the last integral to the harmonic function $v$.

THEOREM 3.6: Suppose $f(z_0) = u(z_0) + iv(z_0) is\ analytic\ in\ the\ disk\ |z_0| \leq R.$ then for $|z_0| < R$, we may express $f(z_0)$ as

$$f(z_0) = \frac{1}{2\pi}\int_0^{2®} \frac{z + z_0}{z - z_0} u(z) d\Phi + iv(0) \qquad (z = \mathrm{Re}^{i\Phi}); \qquad (3.11)$$

Or equivalently $$f(z_0) = \frac{1}{2®}\int_0^{2®} \frac{\mathrm{Re}^{i\Phi} + re^{i\Phi}}{\mathrm{Re}^{i\Phi} - re^{i\Phi}} u(\mathrm{Re}^{i\Phi}) d\Phi + iv(0).$$

(The integral on the right is called the complex Poisson integral)

Equating imaginary part on both sides of (3.11) gives.

COROLLARY 3.7: suppose $f(z_0) = u(z_0) + iv(z_0) is\ analytic\ in\ the\ disk\ |z_0| \leq R$. then

For $Z = Re^{i\Phi}$, $Z_0 = re^{i\theta}\ and\ r < R$ we may express $v(Z_0)$ as

$$v(z_0) = \frac{1}{2\pi}\int_{|Z|=R} \left(\frac{z + z_0}{z - z_0}\right) u(z) d\Phi + v(0)\ ) \qquad (Z = Re^{i\Phi});$$

Or equivalently, $v(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{2rR\sin(\bar{\ } - \Phi)}{R^2 - 2rR\cos(\overline{\theta} - \Phi) + r^2} u(\mathrm{Re}^{i\Phi}) d\Phi + v(o).$

REMARK: suppose $f(z_o)\ and\ g(z_o)$ are analytic inside and on a simple closed contour C, with $Ref(z_o) = Reg(z_o) on\ C. then$

$f(z_o) = g(z_o) + ik$ Inside C, where k is a real constant.

That is an analytic function is determined to within an imaginary constant by its real part. In the case that the function $f(z_O) + iv(z_o) isanalytic\ in\ the\ disk |z_0| \leq R$ . •

THEOREM 3.8: Suppose $u(z_0)$ is harmonic in the open disk $|z_0| < 1$ and continuous on the closed disk $|z_0| \leq 1$. Then for $z_0 = re^{i\theta}, r < 1, we\ have,$

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\bar{\ } - \Phi)+r^2} u(e^{i\Phi})d\Phi .$$

PROOF: Let $f(z_0) = u(z_0) + iv(z_0)$ be analytic for $|z_0| < 1$, and let $\{t_n\}$ be an increasing sequence of positive real numbers approaching 1.

Then for each n, define $f_n(z_0) = f(t_n, z_0)$, $u_n(z_0) = u(t_n z_0)$, and $v_n(z_0) = v(t_n z_0)$.

Clearly, $v_n(0) = v(0)$ for each n and,

$$u_n(z) = Ref(t_n z_0), \text{ and} v_n(z_0) = Imf(t_n z_0).$$

As $u(t_n z_0)$ is harmonic in the closed disk $|z_0| \leq 1$,

$we\ obtain\ that\ f(t_n z)$ is analytic in the closed $|z_0| \leq 1 \left(since\ {}^1/_{t_n} > 1\right)$ and so equation (3.8) is applicable for $f_n$.

Thus, for each fixed $z_0$ with $|z_0| < 1$,

$$f_n(z_0) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\Phi}+z_0}{e^{i\Phi}-z_0} u_n(e^{i\Phi})d\Phi + iv_n(0).$$

Since $f_n(z_0)$ is continuous at $z_0$ $(|z_o| < 1) and\ t_n z_0 \to z_o$ as $n \to \infty$,

$$\lim_{n\to\infty} f_n(z_0) = \lim_{n\to\infty} f(t_n z_o) = f(z_0); \quad |z_0| < 1.$$

The proof will be completed by verifying that

$$\int_0^{2®} \frac{e^{i\Phi}+z_0}{e^{i\Phi}-z_0} u_n(e^{i\Phi})d\Phi \to \int_0^{2®} \frac{e^{i\Phi}+z_0}{e^{i\Phi}-z_0} u(e^{i\Phi})d\Phi . \qquad (3.12)$$

(Recall that $v_n(0) = v(0)$).

It suffices to show that the difference

$$\left|\int_0^{2®} \frac{e^{i\Phi}+z_0}{e^{i\Phi}-z_0}(u_n(e^{i\Phi})-u(e^{i\Phi}))d\Phi\right| \le \frac{1+r}{1-r}\int_0^{2®}\left|u_n(e^{i\Phi})-u(e^{i\Phi})\right|d\Phi.$$

Can be made arbitrary small.

Note that $u(z_0)$, being continuous on the compact set $|z_0| \le 1$, is uniformly continuous on $|z_0| \le 1$. So $u_n\left(e^{i\Phi}\right) = u\left(t_n e^{i\Phi}\right) \to u\left(e^{i\Phi}\right)$; uniformly with respect $\Phi$, $0 \le \Phi \le 2\pi$.

Consequently, the expression on the last integral converges to zero as $n \to \infty$.

Thus, for $|z_0| < 1$, we have

$$f(z_0) = \frac{1}{2\pi}\int_0^{2®} \frac{e^{i\Phi}+z_0}{e^{i\Phi}-z_0}u(e^{i\Phi})d\Phi + iv(0).$$

Equating the real part on both sides, we have the desired result. ■

REMARK: The uniformly continuity of $u(z_o)$ ($|z_o| \le 1$) enabled us to show that the sequence $u_n(z_o) = u(t_n z_o)$

*converges uniformly to* $u(z_o)$ ($|z_o| \le 1$).

DEFINITION 3.9 The function

$$p_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}$$

*for* $0 \le r < 1$ *and* $-\infty < \theta < \infty$ *is called Poisson kernel.*

*Let* $z = re^{i\theta}, 0 \le r < 1;$ *then*

$$\frac{1+re^{i\theta}}{1-re^{i\theta}} = \frac{1+z}{1-z} = (1+z)(1-z)^{-1}$$

*by* exp*anding we get*

$$= (1+z)(1+z+z^2+\ldots) = 1+2\sum_{n=1}^{\infty} z^n = 1+2\sum_{n-1}^{\infty} r^n e^{in\theta}$$

*Hence,*

$$\operatorname{Re}\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right) = 1+2\sum_{n=1}^{\infty} r^n \cos n\theta$$

$$= 1+2\sum_{n=1}^{\infty} \frac{r^n (e^{in\theta} + e^{-in\theta})}{2}$$

$$= p_r(\theta)$$

And also

$$\frac{1+re^{i\theta}}{1-re^{i\theta}} = \frac{1+re^{i\theta} - re^{-i\theta} - r^2}{\left|1-re^{i\theta}\right|^2}$$

So that

$$p_r(\theta) = \frac{1-r^2}{1 = 2r\cos\theta + r^2} = \operatorname{Re}\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right) \qquad (3.13)$$

*PROPOSITION* 3.10 . *The Poisson kernel satisfies the follwings*:

$$(a) \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} p_r(\theta)d\theta = 1;$$

PROOF :

*for a fixed value of* $r\,; 0 \le r < 1$, *the series*

$$p_r(0) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} \quad \textit{converges uniformly in}\ \theta.$$

*so,*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} p_r(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} . \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}\, d\theta$$

$$= \sum_{n=-\infty}^{\infty} r^{|n|} \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\theta}\, d\theta$$

$$= \sum_{n=-\infty}^{\infty} r^{|n|} \frac{1}{2\pi}\left[e^{in\theta}\right]_{-\pi}^{\pi} \times \frac{1}{in}$$

$$= \sum_{n=-\infty}^{\infty} r^{|n|} \frac{1}{2\pi}\left[e^{in\pi} - e^{-in\pi}\right] \times \frac{1}{in}$$

$$= \sum_{n=-\infty}^{\infty} r^{|n|} \frac{1}{2\pi} \times \frac{1}{in} \times 2i \sin n\pi$$

$$= \sum_{n=-\infty}^{\infty} r^{|n|} \times \frac{\sin n\pi}{n\pi} = 1$$

CHAPTER 4

**POSITIVE HARMONIC FUNCTIONS**

The Poisson integral formula allows to obtain useful inequalities for positive harmonic functions. Note that if a non-negative harmonic function attains a minimum value zero on a domain, it is zero throughout the domain. So the class of non-negative harmonic functions on a domain consists of all positive functions and a zero function.

As an application of Poisson integral formula, we prove Harnack's inequality.

THEOREM 4.1.(Harnack's Inequality): Suppose u(z) is harmonic in the disk

$\Delta\ (w;R)=\{z:|z-w|<R\};$ *with* $u\ (z)\geq 0$ *for all* $z\in\Delta\ (w;R)$.

*Then for every z in this disk*;

We have, $u\,(w)\dfrac{R-|z-w\,|}{R+|z-w\,|}\leq u\,(z)\ \leq\ u\,(w)\dfrac{R+|z-w\,|}{R-|z-w\,|}.$

PROOF: Fix $z\in\Delta(w;R),$ , and let S<R. Then, for every s with S <R,

The Poisson integral formula given by theorem 3.2 leads to

$$u(z)=\frac{1}{2\pi}\int_0^{2\pi}\frac{s^2-|\ z-w\ |^2}{\left|se^{i\Phi}-(z-w)\right|^2}u\left(w+se^{i\Phi}\right)d\Phi\,. \tag{4.1}$$

For every $z\epsilon\Delta(w;s).\,using\ positivity\ of\ u(z) and\ the\ inequality$

$$\frac{s-|z-w|}{s+|z-w|}\leq\frac{s^2-|z-w|^2}{\left|se^{i\Phi}-(z-w)\right|^2}\leq\frac{s+|z-w|}{s-|z-w|}$$

We get from (4.1)

$$\frac{s-|z-w|}{s+|z-w|}u(w)\leq u(z)\leq\frac{s+|z-w|}{s-|z-w|}u(w)\,.$$

Because by the mean value property( i.e. from (4.1), for z = w)

$u(w)=\dfrac{1}{2®}\displaystyle\int_0^{2\pi}u(w+se^{i\Phi})d\Phi$ .Since the last inequality is valid

whenever $|z-w|\leq s<R$ , these inequalities continues to hold when s approaches R. ■

Recall that any function that is holomorphic on $\mathbb{C}$ and bounded in absolute value is constant. As an immediate corollary of Harnack's inequalities one obtains an analogue of this result for harmonic functions.

COROLLARY 4.2 ( *Liouville Theorem*) *if u is harmonic in $\mathbb{C}$ and is bounded above (or below), then u is constant.*

Proof: It suffices to prove $u(z) \geq 0$ in $\mathbb{C}$. Fix $z(|z| = r)$ and $R > r$.

$$By\ Harnack's\ inequality \quad \frac{R-r}{R+r}u(0) \leq u(re^{i\theta}) \leq \frac{R+r}{R-r}u(0).$$

*Letting* $R \to \infty$, *we see that* $u(z) \leq u(0)$ *so that u attains its* max*imum at* $z = 0$ *and therefore u is constant for* $|z| < r$ ■

THEOREM4.3. (Harnack's Theorem) Suppose $\{u_n(z)\}$ is a sequence of real-valued harmonic functions defined in a domain D, and that $u_{n+1}(z) \geq u_n(z)$ for each n.

If $\{u_n(z)\}$ converges for at least one point in D, then $\{u_n(z)\}$ converges for all points in D.

Furthermore , the convergence is uniform on compact subset of D, and the limit function is harmonic throughout D.

PROOF. We may assume that $u_n(z) \geq 0$; for if not, the theorem can be proved for the nonnegative sequence suppose$\{u_n(z) - u_1(z)\}$. By the monotonicity property, for each Z in D, either $\{u_n(z)\}$ converges or approaches $\infty$.

Let A $= \{Z \in D: u_n(z) \to \infty\}$, and

B $= \{Z \in D: u_n(z) coverges\}$.

Given that $w \in D$, *choose a disk* $|z - w| \leq R$ contained in D. then for all Z satisfying $|z - w| \leq \frac{R}{2}$; Harnack's inequality gives

$$\frac{1}{3}u_n = \frac{R-\frac{R}{2}}{R+\frac{R}{2}}u_n(w) \leq u_n(z) \leq \frac{R+\frac{R}{2}}{R-\frac{R}{2}}u_n(w) = 3u_n(w) \qquad (4.2)$$

If $u_n(w) \to \infty$, then the left hand inequality of (4.2) shows that $u_n(z) \to \infty for\ |z - w| \leq R/2$.

If $\{u_n(w)\}$ converges for $for\ |z - w| \leq R/2$.

Hence, A and B are both open sets, with AUB =D. Since the domain D is connected, either A=∅ $or\ B = \emptyset$.

By hypothesis, there is at least one point in D. Thus B=D, and $\{u_n(z)\}$ converges for all z in D.

Next we must show that $\{u_n(z)\}$ converges uniformly on compact subsets of D.

Applying Harnack's inequality to $u_{n+p}(z) - u_n(z)$, we get as in (4.2).

$$u_{n+p}(z) - u_n(z) \leq 3[u_{n+p}(w) - u_n(w)] \qquad (4.3)$$

For $|z - w| \leq R/2$ and p=1,2,. By the Cauchy criteria,

$$u_{n+p}(w) - u_n(w) < \in \qquad (n > N(\epsilon)).$$

Hence from (4.2), we see that $\{u_n(z)\}$ converges uniformly in some neighborhood w.

Since w was arbitrary to every point in D there corresponds a neighborhood in which the convergence of $\{u_n(z)\}$ is uniform.

Now let K be a compact subset of D.

For each point of K, construct a neighborhood in which $\{u_n(z)\}$ converges uniformly.

By the Heine-Borel theorem, finitely many such neighborhoods cover K.

But a sequence converging uniformly on finitely many different sets must converge uniformly on their union.

Therefore, $\{u_n(z)\}$ converges uniformly on K.

Hence, the limit function is harmonic throughout D.

## SUMMARY

Summarize the main points of harmonic function

DEFINATION 1 If G is an open subset of $\mathbb{C}$, then a function $U$: G→ $\mathbb{R}$ is harmonic

if it has continuous second order partial derivative and it satisfies Laplace's equation, that is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ or } u_{xx} + u_{yy} = 0. \text{ Often written as } \nabla^2 u = 0.$$

LEMMA 2 If $v$ is a conjugate harmonic function of $u$, then $u$ is a conjugate harmonic function of $v$.

PROPSTION 3 If u: G→ $\mathbb{R}$ is harmonic, then u is infinitely differentiable

THEOREM 4 (Mean value Theorem)

Let $u$: $G \to \mathbb{R}$ *be a harmonic function and* $\bar{B}(a,r)$ be a closed disk contained in G. if $\gamma$ is a circle $|z-a|=r$ then

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + r\, e^{i\theta}) d\theta$$

The maximum principle states that a non constant harmonic function cannot attain a maximum ( or minimum at an interior of its domain. This result implies that the values of a harmonic function in a bounded domain by its maximum and minimum values on the boundary.

THEOREM 5 (*Maximum princple*) *Let u be a harmonic function on*

*a domain G in* $\mathbb{C}$ .

(*a*) *If u attains a local maximum in G then u is constant.*

(*b*) *suppose that G is bounded and u extends continuously tto the boundary* $\partial G$ *o*$\partial G$ *of G.*

$If\ u \leq 0\ on\ \partial G$ then $u \leq 0$ throughout G.

THEOREM 6 (Uniqueness Theorem): Let G be a domain in $\mathbb{C}$ and $u_1$ and $u_2$ are two harmonic functions that extend continuous to the boundary $\partial G$ of G. if $u_1 = u_1$ on $\partial G$. Then these two functions are equal throughout G.

THEOREM7 :( Poisson Integral Formula for harmonic functions)

Suppose $u(z_o)$ is harmonic in a domain containing the disk $|z_o| \leq R$.

$Then\ for\ \ z_o = re^{i\theta}$, r<R, we have $u(z_o) = \frac{1}{2\pi}\int_{|Z|=R} p(z_0, z)u(z)d\Phi$;

or equivalently,

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2®} \frac{R^2 - r^2}{R^2 - 2rR\cos(\theta - \Phi) + r^2} u(\mathrm{Re}^{i\Phi})d\Phi$$

A similar formula holds for the imaginary part $v(z_o)\ of\ f(z_o)$.

COROLLAR 8 for $r < Rand\ \theta\ arbitrary,$

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2rR\cos(^- - \Phi) + r^2}\, d\Phi = \frac{1}{2®}\int_0^{2®} p(re^{i^-}, \mathrm{Re}^{i\Phi})d\Phi = 1.$$

THEOREM 9($harnack's$ Inequality): suppose u(z) is harmonic in the disk $\Delta(w;R) = \{z : |z - w| < R\};\ with\ u(z) \geq 0\ \ for\ \ all\ z \in \Delta(w;R).\ \ Then\ for\ every\ z\ in\ this\ disk;$

We have, $\quad u(w)\frac{R - |z - w|}{R + |z - w|} \leq u(z) \leq u(w)\frac{R + |z - w|}{R - |z - w|}.$

THEOREM 10 ($Harnack's$ Theorem) suppose $\{u_n(z)\}$ is a sequence of real-valued harmonic functions defined in a domain D, and that $u_{n+1}(z) \geq u_n(z)$ for each n.

If $\{u_n(z)\}$ converges for at least one point in D, then $\{u_n(z)\}$ converges for all points in D.

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