Interaction of Coherently Driven Cavity Mode with Three-Level Atom

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DECLARATION

I hereby declare that this PhD dissertation is my original work and has not been presented for a degree in any other university, and that all sources of material used for the dissertation have been duly acknowledged.

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by

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Abstract

In this PhD dissertation we have studied the quantum properties of the cavity mode driven by coherent light and interacting with a three-level atom available in an open cavity and coupled to a vacuum reservoir via a single-port mirror. We have carried out our analysis by putting the noise operators associated with the vacuum reservoir in normal order and by taking into consideration the interaction of the three-level atom with the vacuum reservoir outside the cavity. With the aid of the quantum Langevin equations, we have determined the equations of evolution for the cavity mode operators. In addition, employing the pertinent master equation, we have obtained the equations of evolution for the expectation values of the atomic operators. Then applying the steady-state solutions of the equations of evolution for the cavity mode operators and the atomic operators, we have calculated the global mean and variance of the photon number for the light modes emitted from the top and the intermediate levels and for the driven cavity mode. We have also determined the local mean photon number for the driven cavity mode and for the two-mode cavity light.

We have seen that the cavity modes a_1 and a_2 (for $\varepsilon \gg \gamma_c + \gamma$) are separately in a chaotic state . Moreover, the driven cavity mode exhibits subPoissonian photon statistics. We have also established that the local mean photon number for the driven cavity mode as well as for the two-mode cavity light approaches the global mean photon number in the absence or presence of spontaneous emission.

In addition, we have shown that the driven cavity mode is in a squeezed state and the squeezing occurs in both the plus and the minus quadratures with the maximum squeezing for the plus quadrature being 52.08% below the vacuum state level. The maximum squeezing occurs at $\varepsilon = 0.6$ (in the presence of spontaneous emission) and at $\varepsilon = 0.37$ (in the absence of spontaneous emission). On the other hand, the maximum squeezing for the minus quadrature is 33.32% and occurs for values of $\varepsilon \geq 15$ in the presence or absence of spontaneous emission. We consider the discovery of squeezing in both the plus and the minus quadratures to be the single most important result of this PhD dissertation. Furthermore, the two-mode cavity light is in a squeezed state and the squeezing occurs in the minus quadrature with the maximum squeezing being 43.42% below the vacuum state level and occurs at $\varepsilon = 0.22$ (in the absence of spontaneous emission) and at $\varepsilon = 0.35$ (in the presence of spontaneous emission). Moreover, we have found that the maximum local quadrature squeezing for the driven cavity mode is 65.32% (in the absence of spontaneous emission) and 62.11% (in the presence of spontaneous emission). And the two local maxima occur in the frequency interval $\lambda_{\pm} = 0.01$.

On applying the steady-state solution of the quantum Langevin equation for a pair of superposed driven cavity modes, we have calculated the mean and variance of the photon number as well as the quadrature squeezing. We have found that the mean photon number of the superposed driven cavity modes is twice the mean photon number of one of the constituent driven cavity mode. Finally, our result shows that the amount of squeezing in the plus or the minus quadrature of the superposed driven cavity modes is the average of the squeezing in the plus and the minus quadratures of one of the constituent driven cavity modes.

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Introduction

Many problems of interest in quantum optics involve the interaction of light with atoms [1-7]. In particular, the interaction of a three-level atom with light has been widely studied by different authors [8-20]. Such interaction generates squeezed light under certain conditions. Squeezing is one of the nonclassical features of light that has been extensively studied by several authors [20-36]. In a squeezed state the noise in one or the two quadratures may be below the vacuum-state level, with the product of the uncertainties in the two quadratures satisfying the uncertainty relation. Squeezed light has potential applications in the detection of weak signals and in low-noise communications [20, 30]. Squeezed light can be generated by various quantum optical processes such as subharmonic generation [20, 29, 37, 38], resonance fluorescence [1, 5, 25], second harmonic generation [1, 29, 39, 40, 41, 42], and three-level laser dynamics [20, 43, 44, 45, 46, 47].

A three-level laser is a quantum optical system in which light is generated by three-level atoms inside a cavity usually coupled to a vacuum reservoir via a single-port mirror. When a three-level atom in a cascade configuration makes a transition from the top to the bottom level via the intermediate level, a pair of highly correlated photons are generated. This correlation is responsible for the productions of squeezed light.

The squeezing and the statistical properties of the light generated by three-level lasers have been studied by several authors [20, 43, 44, 45, 46, 47]. Fesseha [20] has studied the statistical and the squeezing properties of the light generated by a three-level laser with the three-level atoms available in a closed cavity and pumped to the top level at a constant rate by means of electron bombardment. He has found a maximum global quadrature squeezing of the two-mode cavity light generated by the laser, operating far below threshold, to be 50% below the vacuum-state level. Moreover, he has found that the local quadrature squeezing is greater than the global quadrature squeezing and a large part of the mean photon number is confined in a relatively small frequency interval.

In addition, Fesseha [20] has studied the squeezing and the statistical properties of the light generated by a three-level laser with the three-level atoms available in a closed cavity and pumped to the top level by means of coherent light. He has established that the three-level laser under certain conditions generates squeezed light with a maximum global quadrature squeezing of 43.4%, which is slightly less than the result found with electron bombardment. However, under special condition the maximum global quadrature squeezing of the two-mode cavity light is found to be 50% below the vacuum state level.

This PhD dissertation mainly has two parts. In the first part of the dissertation we seek to study the quantum properties of a cavity mode driven by coherent light and interacting with a three-level atom available in an open cavity and coupled to a vacuum reservoir via a single-port mirror. We carry out our analysis by putting the noise operators associated with the vacuum reservoir in normal order and by taking into consideration the interaction of the three-level atom with the vacuum reservoir outside the cavity. We first obtain the quantum Langevin equations for the cavity mode operators and the pertinent master equation for the three-level atom interacting with the cavity modes and the vacuum reservoir outside the cavity. Employing the master equation we obtain equations of evolution for the atomic operators. Furthermore, we determine the steady-state solutions of the equations of evolution for the atomic operators and the cavity mode operators. Then applying the resulting solutions, we calculate the global mean and variances of the photon number for the light modes emitted from the top and the intermediate levels and for the driven cavity mode. In addition, we determine the local mean photon number for the driven cavity mode and the two-mode cavity light. Moreover, we obtain the global quadrature squeezing for the two-mode cavity light and the driven cavity mode. Finally, we calculate the local quadrature squeezing for the driven cavity mode.

In the second part of this dissertation, we wish to study the statistical and the squeezing properties of a pair of superposed cavity light modes each driven by coherent light and interacting with a three-level atom. We then determine the equation of evolution for the superposed cavity light modes. Applying the steady-state solution of the resulting equation, we calculate the global mean and variance of the photon number and the global quadrature squeezing.

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Operator Dynamics

In this chapter we wish to obtain the equations of evolution for the atomic and cavity mode operators. A light mode confined in a cavity usually formed by two mirrors is called a cavity mode. A commonly used cavity has a single-port mirror. One side of such a cavity is a mirror through which light can enter or leave the cavity, with the other side being a mirror through which light may enter but can not leave the cavity [20]. We consider here the case in which a three-level atom in a cascade configuration is available in an open cavity driven by coherent light and coupled to a vacuum reservoir via a single-port mirror. We denote the top, intermediate, and bottom levels of the three-level atom by $|a\rangle$, $|b\rangle$, and $|c\rangle$, respectively as shown in Figure 2.1.

The atom absorbs a photon from the cavity and makes a transition from the bottom level to the top level. Then it emits a photon and decays to the intermediate level. Eventually, the atom emits another photon and decays to the bottom level. We prefer to call the light emitted from the top level cavity mode a_1 and the one emitted from the intermediate level cavity mode a_2. In addition, we represent the driving coherent light by the operator $\hat{c}$ and the driven cavity mode by the operator $\hat{b}$.

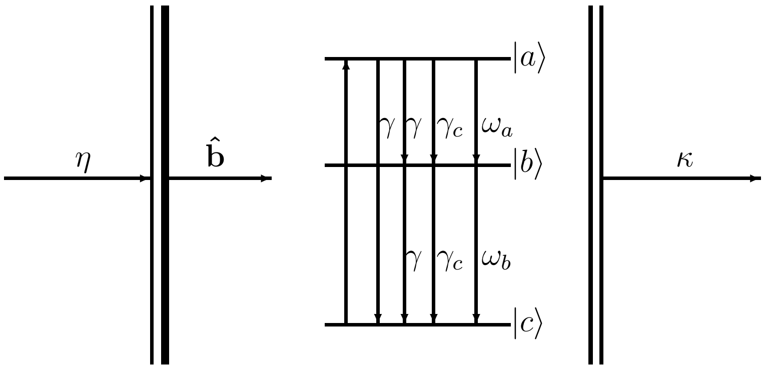

Figure 2.1: Schematic representation of a three-level atom in a cascade configuration .

We carry out our analysis with cavity modes a_1 and a_2 having the same or different frequencies. And we take into account the interaction of the three-level atom with the vacuum-reservoir outside the cavity. This interaction leads to spontaneous emission. Moreover, we assume that the cavity modes a_1, a_2, and the driven cavity mode b to be at resonance with the transitions $|a\rangle \rightarrow |b\rangle$, $|b\rangle \rightarrow |c\rangle$, and $|a\rangle \rightarrow |c\rangle$, respectively.

The interaction between the driving coherent light c and the driven cavity mode b can be described by the Hamiltonian

$$\hat{H}' = i\lambda(\hat{c}\hat{b}^\dagger - \hat{b}\hat{c}^\dagger), \tag{2.1}$$

where λ is the coupling constant between the coherent light and the driven cavity mode. In order to have a mathematically manageable analysis, we replace the operator $\hat{c}$ by a real and constant c-number μ. On account of this, we can write Eq. (2.1) as

$$\hat{H}' = i\eta(\hat{b}^\dagger - \hat{b}), \tag{2.2}$$

where $\eta = \lambda\mu$.

In addition, the interaction of the three-level atom with the cavity modes b, a_1, and a_2 can be described at resonance by the Hamiltonian

$$\hat{H}'' = ig\left(\hat{\sigma}_c^\dagger \hat{b} - \hat{b}^\dagger \hat{\sigma}_c + \hat{\sigma}_a^\dagger \hat{a}_1 - \hat{a}_1^\dagger \hat{\sigma}_a + \hat{\sigma}_b^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{\sigma}_b\right), \tag{2.3}$$

where

$$\hat{\sigma}_a = |b\rangle\langle a|, \tag{2.4}$$

$$\hat{\sigma}_b = |c\rangle\langle b|, \tag{2.5}$$

$$\hat{\sigma}_c = |c\rangle\langle a|, \tag{2.6}$$

are lowering atomic operators and g is the coupling constant between the atom and the cavity mode b or a_1 or a_2, and is considered to be the same for the three interactions. Thus upon combining Eqs. (2.2) and (2.3), the Hamiltonian describing the interaction of the driven cavity mode with the driving coherent light and the three-level atom as well as the interaction of the three-level atom with the the cavity light modes a_1 and a_2 has the form

$$\hat{H} = i\eta(\hat{b}^\dagger - \hat{b}) + ig\left(\hat{\sigma}_c^\dagger \hat{b} - \hat{b}^\dagger \hat{\sigma}_c + \hat{\sigma}_a^\dagger \hat{a}_1 - \hat{a}_1^\dagger \hat{\sigma}_a + \hat{\sigma}_b^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{\sigma}_b\right). \tag{2.7}$$

2.1 Dynamics of cavity mode operators

The dynamics of a cavity mode coupled to a reservoir can be described using the quantum Langevin equation. In this approach the time evolution of the cavity mode is carried by the operators. We seek here to obtain the quantum Langevin equations for the cavity mode operators $\hat{a}_1$, $\hat{a}_2$, and $\hat{b}$. We carry out our calculation by putting the noise operators associated with the vacuum-reservoir in normal order. It can be verified that the noise operators in this case will not have any effect on the dynamics of the cavity mode operators. We can thus drop the noise operator and write the quantum Langevin equation for the operator $\hat{a}_1$ as [20, 48]

$$\frac{d\hat{a}_1(t)}{dt} = -\frac{\kappa}{2}\hat{a}_1(t) - i[\hat{a}_1(t), \hat{H}(0)], \tag{2.8}$$

where κ is the cavity damping constant. In order to obtain the commutator of $\hat{a}_1(t)$ and $\hat{H}(0)$ following the discussion described in Ref. [49], we fix the Hamiltonian given by Eq. (2.7) at the initial time

$$\begin{aligned} \hat{H}(0) &= i\eta(\hat{b}^\dagger(0) - \hat{b}(0)) + ig\Big(\hat{\sigma}_c^\dagger(0)\hat{b}(0) - \hat{b}^\dagger(0)\hat{\sigma}_c(0) + \hat{\sigma}_a^\dagger(0)\hat{a}_1(0) - \hat{a}_1^\dagger(0)\hat{\sigma}_a(0) \\ &\quad + \hat{\sigma}_b^\dagger(0)\hat{a}_2(0) - \hat{a}_2^\dagger(0)\hat{\sigma}_b(0)\Big). \end{aligned} \tag{2.9}$$

We observe that $\hat{\rho}(0)[\hat{a}_1(t), \hat{H}(0)]$ is in the Heisenberg picture where $\hat{\rho}(0)$ is the density operator at the initial time. This can also be written in the Schrödinger picture as $\hat{\rho}(t)[\hat{a}_1(0), \hat{H}(0)]$. One can thus write

$$\hat{\rho}(0)[\hat{a}_1(t), \hat{H}(0)] = \hat{\rho}(t)[\hat{a}_1(0), \hat{H}(0)]. \tag{2.10}$$

Then with the aid of Eq. (2.9), the right hand side of Eq. (2.10) can be put in the form

$$\begin{aligned} \hat{\rho}(t)[\hat{a}_1(0), \hat{H}(0)] &= i\eta\hat{\rho}(t)\Big([\hat{a}_1(0), \hat{b}^\dagger(0)] - [\hat{a}_1(0), \hat{b}(0)]\Big) + ig\hat{\rho}(t)\Big(\hat{\sigma}_c^\dagger(0)[\hat{a}_1(0), \hat{b}(0)] \\ &\quad + [\hat{a}_1(0), \hat{\sigma}_c^\dagger(0)]\hat{b}(0) - \hat{b}^\dagger(0)[\hat{a}_1(0), \hat{\sigma}_c(0)] - [\hat{a}_1(0), \hat{b}^\dagger(0)]\hat{\sigma}_c(0) \\ &\quad + \hat{\sigma}_a^\dagger(0)[\hat{a}_1(0), \hat{a}_1(0)] + [\hat{a}_1(0), \hat{\sigma}_a^\dagger(0)]\hat{a}_1(0) - \hat{a}_1^\dagger(0)[\hat{a}_1(0), \hat{\sigma}_a(0)] \\ &\quad - [\hat{a}_1(0), \hat{a}_1^\dagger(0)]\hat{\sigma}_a(0) + \hat{\sigma}_b(0)[\hat{a}_1(0), \hat{a}_2(0)] + [\hat{a}_1(0), \hat{\sigma}_b^\dagger(0)]\hat{a}_2(0) \\ &\quad - \hat{a}_2^\dagger(0)[\hat{a}_1(0), \hat{\sigma}_b(0)] - [\hat{a}_1(0), \hat{a}_2^\dagger(0)]\hat{\sigma}_b(0)\Big). \end{aligned} \tag{2.11}$$

We assume that

$$[\hat{a}_1(0), \hat{a}_1(0)] = 0, [\hat{a}_1(0), \hat{a}_1^\dagger(0)] = 1. \tag{2.12}$$

We also assume that the cavity mode operator $\hat{a}_1$ commute with the atomic operators and the cavity modes b and a_2. We then readily obtain

$$\hat{\rho}(t)[\hat{a}_1(0), \hat{H}(0)] = -ig\hat{\rho}(t)\hat{\sigma}_a(0) = -ig\hat{\rho}(0)\hat{\sigma}_a(t). \tag{2.13}$$

Now on account of Eqs. (2.10) and (2.13), we easily find

$$[\hat{a}_1(t), \hat{H}(0)] = -ig\hat{\sigma}_a(t). \tag{2.14}$$

Upon substituting (2.14) into (2.8), we get

$$\frac{d\hat{a}_1(t)}{dt} = -\frac{\kappa}{2}\hat{a}_1(t) - g\hat{\sigma}_a(t). \tag{2.15}$$

Following the same procedure, it can be readily established that

$$\frac{d\hat{a}_2(t)}{dt} = -\frac{\kappa}{2}\hat{a}_2(t) - g\hat{\sigma}_b(t), \tag{2.16}$$

$$\frac{d\hat{b}(t)}{dt} = -\frac{\kappa}{2}\hat{b}(t) + \eta - g\hat{\sigma}_c(t). \tag{2.17}$$

2.2 Dynamics of atomic operators

We next wish to obtain the equations of evolution of the expectation values of the atomic operators, using the pertinent master equation. The master equation for the three-level atom interacting with the cavity modes and the vacuum-reservoir outside the cavity can be written as [20, 50]

$$\begin{aligned}\frac{d\hat{\rho}(t)}{dt} &= -i[\hat{H}'', \hat{\rho}] + \frac{\gamma}{2}\Big(2\hat{\sigma}_a\hat{\rho}\hat{\sigma}_a^\dagger - \hat{\sigma}_a^\dagger\hat{\sigma}_a\hat{\rho} - \hat{\rho}\hat{\sigma}_a^\dagger\hat{\sigma}_a\Big) + \frac{\gamma}{2}\Big(2\hat{\sigma}_b\hat{\rho}\hat{\sigma}_b^\dagger - \hat{\sigma}_b^\dagger\hat{\sigma}_b\hat{\rho} - \hat{\rho}\hat{\sigma}_b^\dagger\hat{\sigma}_b\Big) \\ &\quad + \frac{\gamma}{2}\Big(2\hat{\sigma}_c\hat{\rho}\hat{\sigma}_c^\dagger - \hat{\sigma}_c^\dagger\hat{\sigma}_c\hat{\rho} - \hat{\rho}\hat{\sigma}_c^\dagger\hat{\sigma}_c\Big),\end{aligned} \tag{2.18}$$

where, γ considered to be the same for the three levels, is the spontaneous emission decay constant. Hence on account of Eq. (2.3), the master equation for the three-level atom interacting with the cavity modes and the vacuum-reservoir outside the cavity can be expressed as

$$\begin{aligned}\frac{d\hat{\rho}(t)}{dt} &= g\Big(\hat{\sigma}_c^\dagger\hat{b}\hat{\rho} - \hat{b}^\dagger\hat{\sigma}_c\hat{\rho} + \hat{\sigma}_a^\dagger\hat{a}_1\hat{\rho} - \hat{a}_1^\dagger\hat{\sigma}_a\hat{\rho} + \hat{\sigma}_b^\dagger\hat{a}_2\hat{\rho} - \hat{a}_2^\dagger\hat{\sigma}_b\hat{\rho} - \hat{\rho}\hat{\sigma}_c^\dagger\hat{b} \\ &\quad +\hat{\rho}\hat{b}^\dagger\hat{\sigma}_c - \hat{\rho}\hat{\sigma}_a^\dagger\hat{a}_1 + \hat{\rho}\hat{a}_1^\dagger\hat{\sigma}_a - \hat{\rho}\hat{\sigma}_b^\dagger\hat{a}_2 + \hat{\rho}\hat{a}_2^\dagger\hat{\sigma}_b\Big) \\ &\quad +\frac{\gamma}{2}\Big(2\hat{\sigma}_a\hat{\rho}\hat{\sigma}_a^\dagger - \hat{\sigma}_a^\dagger\hat{\sigma}_a\hat{\rho} - \hat{\rho}\hat{\sigma}_a^\dagger\hat{\sigma}_a\Big) + \frac{\gamma}{2}\Big(2\hat{\sigma}_b\hat{\rho}\hat{\sigma}_b^\dagger - \hat{\sigma}_b^\dagger\hat{\sigma}_b\hat{\rho} - \hat{\rho}\hat{\sigma}_b^\dagger\hat{\sigma}_b\Big) \\ &\quad +\frac{\gamma}{2}\Big(2\sigma_c\hat{\rho}\sigma_c^\dagger - \sigma_c^\dagger\sigma_c\hat{\rho} - \hat{\rho}\sigma_c^\dagger\sigma_c\Big). \end{aligned} \tag{2.19}$$

Furthermore, applying the relation

$$\frac{d}{dt}\langle\hat{A}\rangle = Tr\left(\frac{d\hat{\rho}(t)}{dt}\hat{A}\right), \tag{2.20}$$

along with Eq. (2.19), we have

$$\begin{aligned}\frac{d}{dt}\langle\hat{\sigma}_a\rangle &= gTr\Big(\hat{\rho}\hat{\sigma}_a\hat{\sigma}_c^\dagger\hat{b} - \hat{\rho}\hat{\sigma}_a\hat{b}^\dagger\hat{\sigma}_c + \hat{\rho}\hat{\sigma}_a\hat{\sigma}_a^\dagger\hat{a}_1 - \hat{\rho}\hat{\sigma}_a\hat{a}_1^\dagger\hat{\sigma}_a + \hat{\rho}\hat{\sigma}_a\hat{\sigma}_b^\dagger\hat{a}_2 - \hat{\rho}\hat{\sigma}_a\hat{a}_2^\dagger\hat{\sigma}_b \\ &\quad -\hat{\rho}\hat{\sigma}_c^\dagger\hat{b}\hat{\sigma}_a + \hat{\rho}\hat{b}^\dagger\hat{\sigma}_c\hat{\sigma}_a - \hat{\rho}\hat{\sigma}_a^\dagger\hat{a}_1\hat{\sigma}_a + \hat{\rho}\hat{a}_1^\dagger\hat{\sigma}_a\hat{\sigma}_a - \hat{\rho}\hat{\sigma}_b^\dagger\hat{a}_2\hat{\sigma}_a + \hat{\rho}\hat{a}_2^\dagger\hat{\sigma}_b\hat{\sigma}_a\Big) \\ &\quad +\frac{\gamma}{2}Tr\Big(2\hat{\rho}\hat{\sigma}_a^\dagger\hat{\sigma}_a\hat{\sigma}_a - \hat{\rho}\hat{\sigma}_a\hat{\sigma}_a^\dagger\hat{\sigma}_a - \hat{\rho}\hat{\sigma}_a^\dagger\hat{\sigma}_a\hat{\sigma}_a\Big) \\ &\quad +\frac{\gamma}{2}Tr\Big(2\hat{\rho}\hat{\sigma}_b^\dagger\hat{\sigma}_a\hat{\sigma}_b - \hat{\rho}\hat{\sigma}_a\hat{\sigma}_b^\dagger\hat{\sigma}_b - \hat{\rho}\hat{\sigma}_b^\dagger\hat{\sigma}_b\hat{\sigma}_a\Big) \\ &\quad +\frac{\gamma}{2}Tr\Big(2\hat{\rho}\hat{\sigma}_c^\dagger\hat{\sigma}_a\hat{\sigma}_c - \hat{\rho}\hat{\sigma}_a\hat{\sigma}_c^\dagger\hat{\sigma}_c - \hat{\rho}\hat{\sigma}_c^\dagger\hat{\sigma}_c\hat{\sigma}_a\Big), \end{aligned} \tag{2.21}$$

where we have used the cyclic property of the trace operation. Then with the aid of Eqs. (2.4), (2.5), and (2.6), we arrive at

$$\frac{d}{dt}\langle\hat{\sigma}_a\rangle = -\frac{3}{2}\gamma\langle\hat{\sigma}_a\rangle + g\left(\langle\hat{\sigma}_b^\dagger\hat{b}\rangle + \langle(\hat{\eta}_b - \hat{\eta}_a)\hat{a}_1\rangle + \langle\hat{a}_2^\dagger\hat{\sigma}_c\rangle\right). \tag{2.22}$$

Following a similar procedure, one can readily obtain

$$\frac{d}{dt}\langle\hat{\sigma}_b\rangle = -\frac{1}{2}\gamma\langle\hat{\sigma}_b\rangle - g\left(\langle\hat{a}_1^\dagger\hat{\sigma}_c\rangle - \langle(\hat{\eta}_c - \hat{\eta}_b)\hat{a}_2\rangle + \langle\hat{\sigma}_a^\dagger\hat{b}\rangle\right), \tag{2.23}$$

$$\frac{d}{dt}\langle\hat{\sigma}_c\rangle = -\gamma\langle\hat{\sigma}_c\rangle + g\left(\langle\hat{\sigma}_b\hat{a}_1\rangle + \langle(\hat{\eta}_c - \hat{\eta}_a)\hat{b}\rangle - \langle\hat{\sigma}_a\hat{a}_2\rangle\right), \tag{2.24}$$

$$\frac{d}{dt}\langle\hat{\eta}_a\rangle = -2\gamma\langle\hat{\eta}_a\rangle + g\left(\langle\hat{\sigma}_c^\dagger\hat{b}\rangle + \langle\hat{\sigma}_a^\dagger\hat{a}_1\rangle + \langle\hat{b}^\dagger\hat{\sigma}_c\rangle + \langle\hat{a}_1^\dagger\hat{\sigma}_a\rangle\right), \tag{2.25}$$

$$\frac{d}{dt}\langle\hat{\eta}_b\rangle = -\gamma\langle\hat{\eta}_b\rangle + \gamma\langle\hat{\eta}_a\rangle + g\left(\langle\hat{\sigma}_b^\dagger\hat{a}_2\rangle - \langle\hat{\sigma}_a^\dagger\hat{a}_1\rangle + \langle\hat{a}_2^\dagger\hat{\sigma}_b\rangle - \langle\hat{a}_1^\dagger\hat{\sigma}_a\rangle\right), \tag{2.26}$$

where

$$\hat{\eta}_a = |a\rangle\langle a|, \tag{2.27}$$

$$\hat{\eta}_b = |b\rangle\langle b|, \tag{2.28}$$

$$\hat{\eta}_c = |c\rangle\langle c|. \tag{2.29}$$

We see that Eqs. (2.22) - (2.26), are nonlinear and coupled differential equations and hence it is not possible to obtain the exact time dependent solutions of these equations. We intend to overcome this problem by making use of the large-time approximation scheme [48]. Applying this approximation scheme, we obtain from Eqs. (2.15), (2.16), and (2.17) the approximately valid relations

$$\hat{a}_1(t) = -\frac{2g}{\kappa}\hat{\sigma}_a(t), \tag{2.30}$$

$$\hat{a}_2(t) = -\frac{2g}{\kappa}\hat{\sigma}_b(t), \tag{2.31}$$

$$\hat{b}(t) = \frac{2\eta}{\kappa} - \frac{2g}{\kappa}\hat{\sigma}_c(t). \tag{2.32}$$

Upon substituting Eqs. (2.30), (2.31), and (2.32) (with the time argument suppressed) and their adjoint into Eqs. (2.22), (2.23), (2.24), (2.25),and (2.26), we easily arrive at

$$\frac{d}{dt}\langle\hat{\sigma}_a\rangle = -\frac{3}{2}(\gamma_c+\gamma)\langle\hat{\sigma}_a\rangle + \varepsilon\langle\hat{\sigma}_b^\dagger\rangle, \tag{2.33}$$

$$\frac{d}{dt}\langle\hat{\sigma}_b\rangle = -\frac{1}{2}(\gamma_c+\gamma)\langle\hat{\sigma}_b\rangle - \varepsilon\langle\hat{\sigma}_a^\dagger\rangle, \tag{2.34}$$

$$\frac{d}{dt}\langle\hat{\sigma}_c\rangle = -(\gamma_c+\gamma)\langle\hat{\sigma}_c\rangle + \varepsilon(\langle\hat{\eta}_c\rangle - \langle\hat{\eta}_a\rangle), \tag{2.35}$$

$$\frac{d}{dt}\langle\hat{\eta}_a\rangle = -2(\gamma_c+\gamma)\langle\hat{\eta}_a\rangle + \varepsilon(\langle\hat{\sigma}_c^\dagger\rangle + \langle\hat{\sigma}_c\rangle), \tag{2.36}$$

$$\frac{d}{dt}\langle\hat{\eta}_b\rangle = -(\gamma_c+\gamma)\langle\hat{\eta}_b\rangle + (\gamma_c+\gamma)\langle\hat{\eta}_a\rangle, \tag{2.37}$$

where

$$\gamma_c = \frac{4g^2}{\kappa} \tag{2.38}$$

is the stimulated emission decay constant and ε is defined by

$$\varepsilon = \frac{2g\eta}{\kappa}. \tag{2.39}$$

We note that the steady-state solutions of Eqs. (2.33)-(2.37) are given by

$$\langle\hat{\sigma}_a\rangle = \frac{2\varepsilon}{3(\gamma_c+\gamma)}\langle\hat{\sigma}_b^\dagger\rangle, \tag{2.40}$$

$$\langle\hat{\sigma}_b\rangle = -\frac{2\varepsilon}{\gamma_c+\gamma}\langle\hat{\sigma}_a^\dagger\rangle, \tag{2.41}$$

$$\langle\hat{\sigma}_c\rangle = \frac{\varepsilon}{\gamma_c+\gamma}[\langle\hat{\eta}_c\rangle - \langle\hat{\eta}_a\rangle], \tag{2.42}$$

$$\langle\hat{\eta}_a\rangle = \frac{\varepsilon}{2(\gamma_c+\gamma)}[\langle\hat{\sigma}_c^\dagger\rangle + \langle\hat{\sigma}_c\rangle], \tag{2.43}$$

$$\langle\hat{\eta}_b\rangle = \langle\hat{\eta}_a\rangle. \tag{2.44}$$

Applying Eqs. (2.40) and (2.41), we easily find

$$\langle\hat{\sigma}_a\rangle = \langle\hat{\sigma}_b\rangle = 0. \tag{2.45}$$

We assume that the energy levels $|a\rangle$, $|b\rangle$, and $|c\rangle$ of the three-level atom form a complete set, so that

$$\hat{I} = \sum_\alpha |\alpha\rangle\langle\alpha|, \tag{2.46}$$

where $\alpha = a, b$, and c. It then follows that

$$\langle\hat{\eta}_a\rangle + \langle\hat{\eta}_b\rangle + \langle\hat{\eta}_c\rangle = 1. \tag{2.47}$$

We interpret $\langle\hat{\eta}_a\rangle$, $\langle\hat{\eta}_b\rangle$, and $\langle\hat{\eta}_c\rangle$ as the probability for the three-level atom to be in the upper, intermediate, or bottom level. In addition, using Eqs. (2.42), (2.43), (2.44) along with Eq. (2.47) and applying the fact that $\langle\hat{\sigma}_c^\dagger\rangle = \langle\hat{\sigma}_c\rangle$, we readily obtain

$$\langle\hat{\sigma}_c\rangle = \frac{\varepsilon(\gamma_c+\gamma)}{3\varepsilon^2+(\gamma_c+\gamma)^2}, \tag{2.48}$$

$$\langle \hat{\eta}_a \rangle = \langle \hat{\eta}_b \rangle = \frac{\varepsilon^2}{3\varepsilon^2 + (\gamma_c + \gamma)^2}, \tag{2.49}$$

$$\langle \hat{\eta}_c \rangle = \frac{\varepsilon^2 + (\gamma_c + \gamma)^2}{3\varepsilon^2 + (\gamma_c + \gamma)^2}. \tag{2.50}$$

Upon setting $\gamma = 0$ (in the absence of spontaneous emission) in Eqs. (2.48), (2.49), (2.50), we have

$$\langle \hat{\sigma}_c \rangle = \frac{\varepsilon \gamma_c}{3\varepsilon^2 + \gamma_c^2}, \tag{2.51}$$

$$\langle \hat{\eta}_a \rangle = \langle \hat{\eta}_b \rangle = \frac{\varepsilon^2}{3\varepsilon^2 + \gamma_c^2}, \tag{2.52}$$

$$\langle \hat{\eta}_c \rangle = \frac{\varepsilon^2 + \gamma_c^2}{3\varepsilon^2 + \gamma_c^2}. \tag{2.53}$$

Moreover, for $\varepsilon = 0$, we obtain

$$\langle \hat{\sigma}_c \rangle = 0, \tag{2.54}$$

$$\langle \hat{\eta}_a \rangle = \langle \hat{\eta}_b \rangle = 0, \tag{2.55}$$

$$\langle \hat{\eta}_c \rangle = 1. \tag{2.56}$$

This shows that the atom is initially in the bottom level. Furthermore, for $\gamma_c = \gamma = 0$, we have

$$\langle \hat{\sigma}_c \rangle = 0, \tag{2.57}$$

$$\langle \hat{\eta}_a \rangle = \langle \hat{\eta}_b \rangle = \frac{1}{3}, \tag{2.58}$$

$$\langle \hat{\eta}_c \rangle = \frac{1}{3}. \tag{2.59}$$

We thus see that the probabilities for the atom to be in the upper, intermediate , and bottom levels are equal.

On the other hand, the steady-state solutions of Eqs. (2.14), (2.15), and (2.16) are given by

$$\hat{a}_1 = -\frac{2g}{\kappa}\hat{\sigma}_a, \tag{2.60}$$

$$\hat{a}_2 = -\frac{2g}{\kappa}\hat{\sigma}_b, \tag{2.61}$$

$$\hat{b} = \frac{2\eta}{\kappa} - \frac{2g}{\kappa}\hat{\sigma}_c. \tag{2.62}$$

Employing Eqs. (2.60), (2.61), and (2.62) together with Eqs. (2.4), (2.5), and (2.6), we obtain

$$\hat{a}_1\hat{a}_1^\dagger = \frac{\gamma_c}{\kappa}\hat{\eta}_b, \tag{2.63}$$

$$\hat{a}_1^\dagger\hat{a}_1 = \frac{\gamma_c}{\kappa}\hat{\eta}_a, \tag{2.64}$$

$$\hat{a}_2\hat{a}_2^\dagger = \frac{\gamma_c}{\kappa}\hat{\eta}_c, \tag{2.65}$$

$$\hat{a}_2^\dagger\hat{a}_2 = \frac{\gamma_c}{\kappa}\hat{\eta}_b, \tag{2.66}$$

$$\hat{b}\hat{b}^\dagger = \frac{4\eta^2}{k^2} - \frac{4\eta g}{k^2}(\hat{\sigma}_c^\dagger + \hat{\sigma}_c) + \frac{4g^2}{\kappa^2}\hat{\eta}_c, \tag{2.67}$$

$$\hat{b}^\dagger\hat{b} = \frac{4\eta^2}{k^2} - \frac{4\eta g}{k^2}(\hat{\sigma}_c + \hat{\sigma}_c^\dagger) + \frac{4g^2}{\kappa^2}\hat{\eta}_a. \tag{2.68}$$

Moreover, adding Eqs. (2.15) and (2.16), we get

$$\frac{d\hat{c}_1}{dt} = -\frac{\kappa}{2}\hat{c}_1 - g\hat{\sigma}, \tag{2.69}$$

where

$$\hat{c}_1 = \hat{a}_1 + \hat{a}_2 \tag{2.70}$$

is the annihilation operator for the superposition of the cavity light modes a_1 and a_2, and

$$\hat{\sigma} = \hat{\sigma}_a + \hat{\sigma}_b. \tag{2.71}$$

The steady-state solution of Eq. (2.69) is given by

$$\hat{c}_1 = -\frac{2g}{\kappa}\hat{\sigma}. \tag{2.72}$$

Hence with the aid of (2.72), one can easily get

$$\hat{c}_1\hat{c}_1^\dagger = \frac{\gamma_c}{\kappa}(\hat{\eta}_b + \hat{\eta}_c), \tag{2.73}$$

$$\hat{c}_1^\dagger \hat{c}_1 = \frac{\gamma_c}{\kappa}(\hat{\eta}_a + \hat{\eta}_b), \tag{2.74}$$

$$\hat{c}_1^2 = \frac{\gamma_c}{\kappa}\hat{\sigma}_c. \tag{2.75}$$

Applying the results obtained in sections 2.1 and 2.2, we will calculate the photon statistics and quadrature squeezing.

3

Photon Statistics

In this chapter we seek to study the statistical properties of the cavity mode driven by coherent light and interacting with the three-level atom available in an open cavity and coupled to a vacuum reservoir via a single-port mirror. Applying the steady-state solutions of the equations of evolution of the cavity light modes and the expectation values of atomic operators, we calculate the global mean and variance of the photon number for the cavity light modes. In addition, we wish to obtain the local mean photon number for the two-mode cavity light and the driven cavity mode.

3.1 Single-mode photon statistics

In this section we wish to calculate the steady-state global mean and variance of the photon number for the cavity modes a_1, a_2, and b in the entire frequency interval.

3.1.1 The global mean photon number

The mean photon number for the cavity mode a_1 is defined by

$$\bar{n}_1 = \langle \hat{a}_1^\dagger \hat{a}_1 \rangle. \tag{3.1}$$

Then taking into account Eq. (2.64), we see that

$$\bar{n}_1 = \frac{\gamma_c}{\kappa} \langle \hat{\eta}_a \rangle, \tag{3.2}$$

so that upon substituting Eq. (2.49) into (3.2), we get

$$\bar{n}_1 = \frac{\gamma_c}{\kappa}\left(\frac{\varepsilon^2}{3\varepsilon^2 + (\gamma_c + \gamma)^2}\right). \tag{3.3}$$

In the absence of spontaneous emission ($\gamma = 0$), the mean photon number for the cavity mode a_1 takes the form

$$\bar{n}_1 = \frac{\gamma_c}{\kappa}\left(\frac{\varepsilon^2}{3\varepsilon^2 + \gamma_c^2}\right). \tag{3.4}$$

From Eqs. (3.3) and (3.4), we note that the presence of spontaneous emission decreases the global mean photon number. On the other hand, for $\varepsilon \gg \gamma_c + \gamma$, Eq. (3.3) reduces to

$$\bar{n}_1 = \frac{\gamma_c}{\kappa}\left(\frac{1}{3}\right). \tag{3.5}$$

Furthermore, employing (2.66), the mean photon number for the cavity mode a_2 can be written as

$$\bar{n}_2 = \frac{\gamma_c}{\kappa}\langle\hat{\eta}_b\rangle. \tag{3.6}$$

Therefore, in view of Eq. (2.49), we note that

$$\bar{n}_2 = \frac{\gamma_c}{\kappa}\left(\frac{\varepsilon^2}{3\varepsilon^2 + (\gamma_c + \gamma)^2}\right). \tag{3.7}$$

We observe from Eqs. (3.3) and (3.7) that the steady state global mean photon number for the cavity mode a_2 is equal to that of the cavity mode a_1.

In addition, with the aid of Eq. (2.68) the mean photon number for the cavity mode b is found to be

$$\bar{n}_b = \frac{4\eta^2}{\kappa^2} - \frac{8\eta g}{\kappa^2}\langle\hat{\sigma}_c\rangle + \frac{\gamma_c}{\kappa}\langle\hat{\eta}_a\rangle. \tag{3.8}$$

The first term represents the mean photon number for the cavity mode b due to the driving coherent light, the second term represents the mean number of photons absorbed by the atom, and the third term represents the mean photon number due

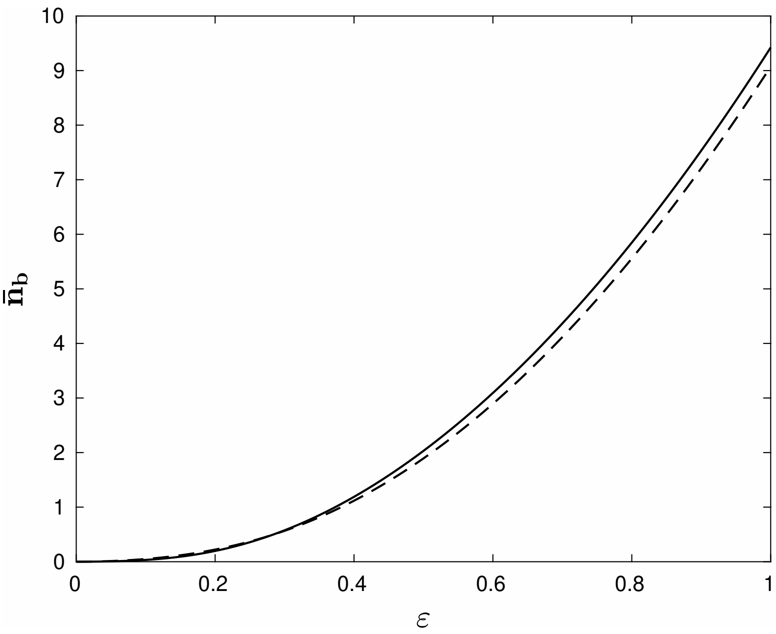

Figure 3.1: Plots of the mean photon number versus ε [Eq. (3.10)] for $\gamma_c = 0.5$, $\kappa = 0.8$, and for $\gamma = 0$ (solid curve) and for $\gamma = 0.3$ (dashed curve).

to stimulated emission. Now upon introducing Eqs. (2.48) and (2.49) into Eq. (3.8), we have

$$\bar{n}_b = \frac{4\eta^2}{\kappa^2} - \frac{\varepsilon}{3\varepsilon^2 + (\gamma_c + \gamma)^2}\left[\frac{8\eta g(\gamma_c + \gamma) - \kappa\varepsilon\gamma_c}{\kappa^2}\right]. \tag{3.9}$$

Then with the aid of Eq. (2.39), the mean photon number for the driven cavity mode can be put in the form

$$\bar{n}_b = \frac{4\varepsilon^2}{\kappa\gamma_c} - \frac{\varepsilon^2}{\kappa}\left[\frac{3\gamma_c + 4\gamma}{3\varepsilon^2 + (\gamma_c + \gamma)^2}\right]. \tag{3.10}$$

In the absence of spontaneous emission ($\gamma = 0$), Eq. (3.10) reduces to

$$\bar{n}_b = \frac{4\varepsilon^2}{\kappa\gamma_c} - \frac{\varepsilon^2}{\kappa}\left[\frac{3\gamma_c}{3\varepsilon^2 + \gamma_c^2}\right]. \tag{3.11}$$

In addition, we note that for $g = 0$, the mean photon number for the driven cavity mode is equal to the mean photon number of the driving coherent light. Moreover, the plots in Figure 3.1 show that the presence of spontaneous emission leads to a decrease in the global mean photon number of the driven cavity mode b for $\varepsilon > 0.36$. We also note that the global mean photon number increases with increasing ε.

3.1.2 The global photon number variance

We next seek to calculate the variances of the photon number for the cavity light modes a_1, a_2, and b. To this end, the formal solution of Eq. (2.15) can be written as

$$\hat{a}_1(t) = \hat{a}_1(0)e^{-\kappa t/2} - ge^{-\kappa t/2}\int_0^t e^{\kappa t'/2}\hat{\sigma}_a(t')dt'. \tag{3.12}$$

On taking the expectation value of this equation, we have

$$\langle\hat{a}_1(t)\rangle = \langle\hat{a}_1(0)\rangle e^{-\kappa t/2} - ge^{-\kappa t/2}\int_0^t e^{\kappa t'/2}\langle\hat{\sigma}_a(t')\rangle dt'. \tag{3.13}$$

We next proceed to determine $\langle\hat{\sigma}_a(t')\rangle$ that appears in Eq. (3.13). Thus applying the large-time approximation scheme to Eq. (2.34), we obtain

$$\langle\hat{\sigma}_b(t)\rangle = -\frac{2\varepsilon}{(\gamma_c+\gamma)}\langle\hat{\sigma}_a^\dagger(t)\rangle \tag{3.14}$$

and taking the complex conjugate of this equation, we see that

$$\langle\hat{\sigma}_b^\dagger(t)\rangle = -\frac{2\varepsilon}{(\gamma_c+\gamma)}\langle\hat{\sigma}_a(t)\rangle, \tag{3.15}$$

so that on account of (3.15), one can write Eq. (2.33) as

$$\frac{d}{dt}\langle\hat{\sigma}_a\rangle = -\frac{\mu_a}{2}\langle\hat{\sigma}_a\rangle, \tag{3.16}$$

where

$$\mu_a = \frac{4\varepsilon^2 + 3(\gamma_c+\gamma)^2}{(\gamma_c+\gamma)}. \tag{3.17}$$

We note that the solution of Eq.(3.16) is expressible as

$$\langle\hat{\sigma}_a(t)\rangle = \langle\hat{\sigma}_a(0)\rangle e^{-\mu_a t/2}. \tag{3.18}$$

Hence introducing Eq. (3.18) into Eq. (3.13) and carrying out the integration, we arrive at

$$\langle\hat{a}_1(t)\rangle = \langle\hat{a}_1(0)\rangle e^{-\kappa t/2} - \frac{2g\langle\hat{\sigma}_a(0)\rangle}{(\kappa-\mu_a)}\left[e^{-\mu_a t/2} - e^{-\kappa t/2}\right]. \tag{3.19}$$

We next wish to obtain the explicit form of $\langle\hat{\sigma}_a(0)\rangle$ that appears in Eq. (3.19). To this end, we consider the atom to be initially in the bottom level $|c\rangle$. Then we note that the density operator associated with this level has the form

$$\hat{\rho} = |c\rangle\langle c|. \tag{3.20}$$

Applying this equation, the expectation value of the atomic operator $\hat{\sigma}_a(0)$ is expressible as

$$\langle\hat{\sigma}_a(0)\rangle = Tr(\hat{\rho}\hat{\sigma}_a(0)) = Tr(|c\rangle\langle c|b\rangle\langle a|), \tag{3.21}$$

so that in view of the relation $\langle c|b\rangle = 0$, we find

$$\langle\hat{\sigma}_a(0)\rangle = 0. \tag{3.22}$$

Thus on combining this equation with Eq. (3.19), we have

$$\langle\hat{a}_1(t)\rangle = \langle\hat{a}_1(0)\rangle e^{-\kappa t/2}. \tag{3.23}$$

Furthermore, on assuming that the cavity mode is initially in a vacuum state, one can easily check that

$$\langle\hat{a}_1(0)\rangle = 0. \tag{3.24}$$

It then follows that

$$\langle\hat{a}_1(t)\rangle = 0. \tag{3.25}$$

In view of the linear differential equation described by (2.15) and the result given by Eq. (3.25), we claim that $\hat{a}_1(t)$ is a Gaussian variable with zero mean [20].

In addition, the expectation value of the solution of Eq. (2.16) can be written as

$$\langle\hat{a}_2(t)\rangle = \langle\hat{a}_2(0)\rangle e^{-\kappa t/2} - ge^{-\kappa t/2}\int_0^t e^{\kappa t'/2}\langle\hat{\sigma}_b(t')\rangle dt'. \tag{3.26}$$

Applying the large-time approximation scheme to Eq. (2.33), we have

$$\langle\hat{\sigma}_a(t)\rangle = \frac{2\varepsilon}{3(\gamma_c+\gamma)}\langle\hat{\sigma}_b^\dagger(t)\rangle \tag{3.27}$$

and the complex conjugate of this relation is

$$\langle\hat{\sigma}_a^\dagger(t)\rangle = \frac{2\varepsilon}{3(\gamma_c+\gamma)}\langle\hat{\sigma}_b(t)\rangle. \tag{3.28}$$

With this result substituted into Eq. (2.34), we obtain

$$\frac{d}{dt}\langle\hat{\sigma}_b\rangle = -\frac{\mu_b}{2}\langle\hat{\sigma}_b\rangle, \tag{3.29}$$

where

$$\mu_b = \frac{4\varepsilon^2+3(\gamma_c+\gamma)^2}{3(\gamma_c+\gamma)}. \tag{3.30}$$

The solution of (3.29) is expressible as

$$\langle\hat{\sigma}_b(t)\rangle = \langle\hat{\sigma}_b(0)\rangle e^{-\mu_b t/2}. \tag{3.31}$$

Assuming the atom to be initially in the bottom level, we easily find

$$\langle\hat{\sigma}_b(0)\rangle = 0. \tag{3.32}$$

Hence on account of Eq. (3.32) and the assumption that the cavity mode is initially in a vacuum state, one can easily arrive at

$$\langle\hat{a}_2(t)\rangle = 0. \tag{3.33}$$

In view of the linear differential equation described by (2.16) and the result given by Eq. (3.33), we claim that $\hat{a}_2(t)$ is a Gaussian variable with zero mean. Therefore, on account of Eq. (2.70), we see that

$$\langle\hat{c}_1(t)\rangle = 0. \tag{3.34}$$

Now taking into account the linear differential equation given by Eq. (2.69) and the result given by Eq. (3.34), we claim that $\hat{c}_1(t)$ is also a Gaussian variable with zero mean.

The photon number variance for the cavity mode a_1 is expressible as

$$(\Delta n_1)^2 = \langle \hat{n}_1^2 \rangle - \langle \hat{n}_1 \rangle^2. \tag{3.35}$$

Employing the definition $\hat{n}_1 = \hat{a}_1^\dagger \hat{a}_1$, we have

$$(\Delta n_1)^2 = \langle \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1^\dagger \hat{a}_1 \rangle - \langle \hat{a}_1^\dagger \hat{a}_1 \rangle^2 \tag{3.36}$$

and making use of the fact that $\hat{a}_1(t)$ is a Gaussian variable with zero mean, we get

$$\begin{aligned} (\Delta n_1)^2 &= \langle \hat{a}_1^\dagger \hat{a}_1 \rangle \langle \hat{a}_1^\dagger \hat{a}_1 \rangle + \langle \hat{a}_1^\dagger \hat{a}_1 \rangle \langle a_1 \hat{a}_1^\dagger \rangle + \langle \hat{a}_1^{\dagger 2} \rangle \langle \hat{a}_1^2 \rangle - \langle \hat{a}_1^\dagger \hat{a}_1 \rangle^2 \\ &= \langle \hat{a}_1^\dagger \hat{a}_1 \rangle \langle \hat{a}_1 \hat{a}_1^\dagger \rangle + \langle \hat{a}_1^{\dagger 2} \rangle \langle \hat{a}_1^2 \rangle. \end{aligned} \tag{3.37}$$

Then with the aid of Eq. (2.60), we readily find

$$\langle \hat{a}_1^2 \rangle = 0. \tag{3.38}$$

Therefore on account of Eqs. (2.63), (2.64), and (3.38) along with Eq. (2.44), we arrive at

$$(\Delta n_1)^2 = \bar{n}_1^2. \tag{3.39}$$

This represents the normally-ordered variance of the photon number for chaotic state.

Moreover, making use of the fact that the cavity mode a_2 is a Gaussian variable with zero mean, the photon number variance for the cavity mode a_2 can be put in the form

$$(\Delta n_2)^2 = \langle \hat{a}_2^\dagger \hat{a}_2 \rangle \langle \hat{a}_2 \hat{a}_2^\dagger \rangle + \langle \hat{a}_2^{\dagger 2} \rangle \langle \hat{a}_2^2 \rangle. \tag{3.40}$$

Now on account of Eq. (2.61), one easily gets

$$\langle \hat{a}_2^2 \rangle = 0. \tag{3.41}$$

Thus in view of Eqs. (2.65), (2.66), and (3.41), we obtain

$$(\Delta n_2)^2 = \left(\frac{\gamma_c}{\kappa}\right)^2 \langle\hat{\eta}_b\rangle\langle\hat{\eta}_c\rangle. \tag{3.42}$$

Upon substituting Eqs. (2.49) and (2.50) into Eq. (3.42), we readily get

$$(\Delta n_2)^2 = \left(\frac{\gamma_c}{\kappa}\right)^2 \left[\frac{\varepsilon^2(\varepsilon^2 + (\gamma_c + \gamma)^2)}{(3\varepsilon^2 + (\gamma_c + \gamma)^2)^2}\right]. \tag{3.43}$$

This equation reduces to

$$(\Delta n_2)^2 = \bar{n}_2^2, \tag{3.44}$$

for $\varepsilon \gg \gamma_c + \gamma$. This represents the normally-ordered variance of the photon number for chaotic state.

Finally, we recall that the photon number variance for the driven cavity mode b is given by

$$(\Delta n_b)^2 \;\; = \;\; \langle\hat{b}^\dagger\hat{b}\hat{b}^\dagger\hat{b}\rangle - \langle\hat{b}^\dagger\hat{b}\rangle^2. \tag{3.45}$$

With the aid of Eqs. (2.67) and (2.68), we have

$$\hat{b}\hat{b}^\dagger - \hat{b}^\dagger\hat{b} = \frac{\gamma_c}{\kappa}(\hat{\eta}_c - \hat{\eta}_a). \tag{3.46}$$

On the basis of this equation, the variance of the photon number can be expressed as

$$(\Delta n_b)^2 \;\; = \;\; \langle\hat{b}^{\dagger 2}\hat{b}^2\rangle + \frac{\gamma_c}{\kappa}[\langle\hat{b}^\dagger\hat{\eta}_c\hat{b}\rangle - \langle\hat{b}^\dagger\hat{\eta}_a\hat{b}\rangle] - \bar{n}_b^2. \tag{3.47}$$

Applying Eq. (2.62), one readily gets

$$\langle\hat{b}^{\dagger 2}\hat{b}^2\rangle = \frac{16\varepsilon^4}{\kappa^2\gamma_c^2} - \frac{32\varepsilon^3}{\kappa^2\gamma_c}\langle\hat{\sigma}_c\rangle + \frac{16\varepsilon^2}{\kappa^2}\langle\hat{\eta}_a\rangle, \tag{3.48}$$

$$\langle\hat{b}^\dagger\hat{\eta}_c\hat{b}\rangle - \langle\hat{b}^\dagger\hat{\eta}_a\hat{b}\rangle = \frac{4\varepsilon^2}{\kappa\gamma_c} - \frac{12\varepsilon^2}{\kappa\gamma_c}\langle\hat{\eta}_a\rangle - \frac{2\varepsilon}{\kappa}\langle\hat{\sigma}_c\rangle. \tag{3.49}$$

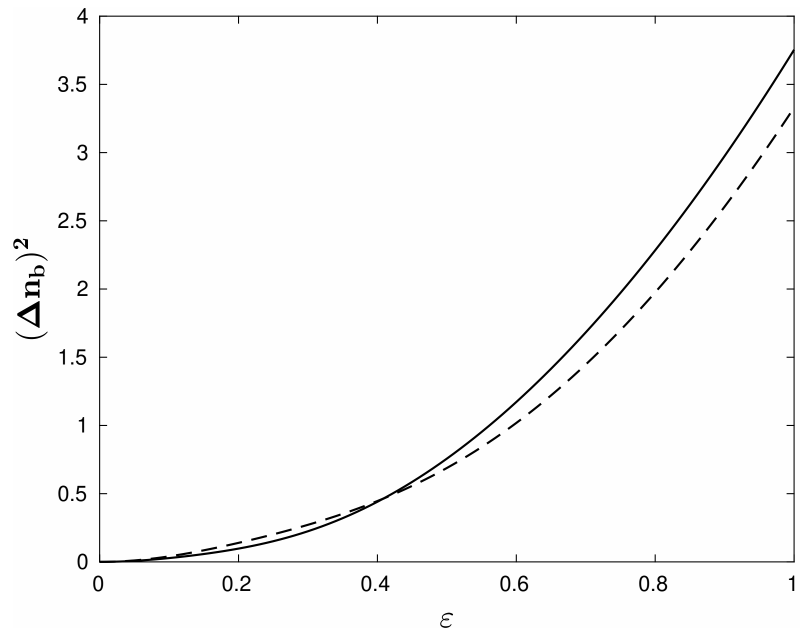

Figure 3.2: Plots of the photon number variance [Eq. (3.51)] versus ε for $\gamma_c = 0.5$, $\kappa = 0.8$, and for $\gamma = 0$ (solid curve) and for $\gamma = 0.3$ (dashed curve).

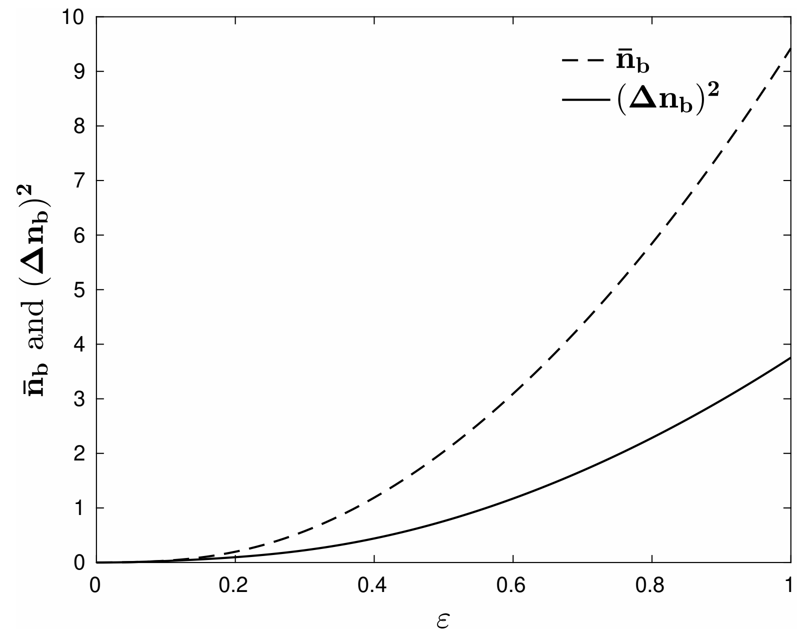

Figure 3.3: Plots of the mean photon number [Eq. (3.10)] (dashed curve) and photon number variance [Eq. (3.51)] (solid curve) versus ε for $\gamma_c = 0.5$, $\kappa = 0.8$, and $\gamma = 0$.

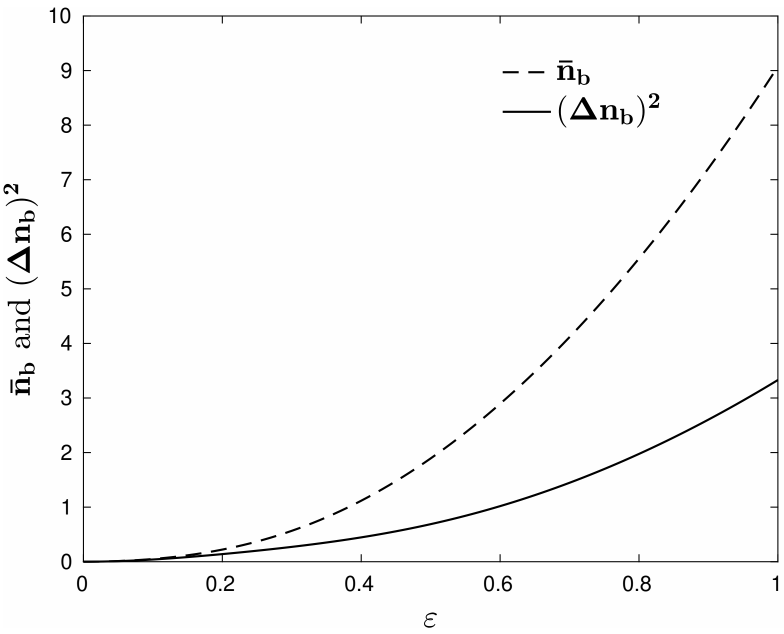

Figure 3.4: Plots of the mean photon number [Eq. (3.10)] (dashed curve) and photon number variance [Eq. (3.51)] (solid curve) versus ε for $\gamma_c = 0.5$, $\kappa = 0.8$, and $\gamma = 0.3$.

Upon introducing Eqs. (3.48) and (3.49) into Eq. (3.47), we get

$$(\Delta n_b)^2 = \frac{4\varepsilon^2}{\kappa^2}\left(\frac{4\varepsilon^2+\gamma_c^2}{\gamma_c^2}\right) - \left(\frac{32\varepsilon^3+2\varepsilon\gamma_c^2}{\kappa^2\gamma_c}\right)\langle\hat{\sigma}_c\rangle + \frac{4\varepsilon^2}{\kappa^2}\langle\hat{\eta}_a\rangle - \bar{n}_b^2. \tag{3.50}$$

Then on account of Eqs. (2.48) and (2.49), we arrive at

$$(\Delta n_b)^2 = \frac{\varepsilon^2}{\kappa^2\gamma_c}\left[\frac{(16\varepsilon^2+4\gamma_c^2)}{\gamma_c} - \left(\frac{(\gamma_c+\gamma)(32\varepsilon^2+2\gamma_c^2)-4\gamma_c\varepsilon^2}{3\varepsilon^2+(\gamma_c+\gamma)^2}\right)\right] - \bar{n}_b^2, \tag{3.51}$$

where $\bar{n}_b$ is given by Eq. (3.10). On the other hand, on setting $\gamma = 0$ in Eq. (3.51), there follows

$$(\Delta n_b)^2 = \frac{\varepsilon^2}{\kappa^2\gamma_c}\left[\frac{(16\varepsilon^2+4\gamma_c^2)}{\gamma_c} - \left(\frac{\gamma_c(32\varepsilon^2+2\gamma_c^2)-4\gamma_c\varepsilon^2}{3\varepsilon^2+\gamma_c^2}\right)\right] - \bar{n}_b^2, \tag{3.52}$$

where $\bar{n}_b$ is given by Eq. (3.11). The plots in Figure 3.2 show that the effect of the spontaneous emission is to slightly increase the photon number variance for $0 < \varepsilon \leq 0.4$ and to decrease the photon number variance for $\varepsilon > 0.4$. In addition, we observe that the photon number variance is less than the man photon number in the absence of spontaneous emission (Figure 3.3) as well as in the presence of spontaneous emission (Figure 3.4). We then realize that the driven cavity mode has subPoissonian photon statistics.

3.1.3 Local mean photon number for the driven cavity mode

We next wish to calculate the local mean photon number for the driven cavity mode. To this end, the power spectrum of the driven cavity mode with central frequency ω_0 is expressible as

$$P(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\tau e^{i(\omega-\omega_0)\tau}\langle \hat{b}^{\dagger}(t)\hat{b}(t+\tau)\rangle_{ss}. \tag{3.53}$$

Upon integrating both sides of Eq. (3.53) over ω, we readily get

$$\int_{-\infty}^{\infty} P(\omega)d\omega = \bar{n}_b, \tag{3.54}$$

where $\bar{n}_b$ is given by Eq. (3.10) [20]. From Eq. (3.54), we observe that $P(\omega)d\omega$ is the steady-state mean photon number in the interval between ω and $\omega + d\omega$.

We now proceed to determine the two-time correlation function that appears in Eq. (3.53). To this end, we realize that the solution of Eq. (2.17) can be written as

$$\hat{b}(t+\tau) = \hat{b}(t)e^{-\kappa\tau/2} + \eta e^{-\kappa\tau/2}\int_0^{\tau} e^{\kappa\tau'/2}d\tau' - g e^{-\kappa\tau/2}\int_0^{\tau} e^{\kappa\tau'/2}\hat{\sigma}_c(t+\tau')d\tau'. \tag{3.55}$$

On account of Eqs. (2.47) and (2.44), we have

$$\langle\hat{\eta}_c\rangle = 1 - 2\langle\hat{\eta}_a\rangle. \tag{3.56}$$

Now upon substituting this equation into Eq. (2.35), we readily get

$$\frac{d}{dt}\langle\hat{\sigma}_c\rangle = -(\gamma_c+\gamma)\langle\hat{\sigma}_c\rangle + \varepsilon(1 - 3\langle\hat{\eta}_a\rangle). \tag{3.57}$$

Moreover, using the large-time approximation scheme, we obtain from Eq. (2.36)

$$\langle\hat{\eta}_a(t)\rangle = \frac{\varepsilon}{2(\gamma_c+\gamma)}(\langle\hat{\sigma}_c^{\dagger}(t)\rangle + \langle\hat{\sigma}_c(t)\rangle) \tag{3.58}$$

and assuming that

$$\langle \hat{\sigma}_c^\dagger(t) \rangle = \langle \hat{\sigma}_c(t) \rangle, \tag{3.59}$$

we get

$$\langle \hat{\eta}_a(t) \rangle = \frac{\varepsilon}{\gamma_c + \gamma} \langle \hat{\sigma}_c(t) \rangle. \tag{3.60}$$

Hence combination of Eqs. (3.60) and (3.57) leads to

$$\frac{d}{dt} \langle \hat{\sigma}_c \rangle = -\alpha \langle \hat{\sigma}_c \rangle + \varepsilon, \tag{3.61}$$

where

$$\alpha = \frac{3\varepsilon^2 + (\gamma_c + \gamma)^2}{\gamma_c + \gamma}. \tag{3.62}$$

On the basis of (3.61), one can write

$$\frac{d}{dt} \hat{\sigma}_c = -\alpha \hat{\sigma}_c + \varepsilon + \hat{f}_c(t), \tag{3.63}$$

in which $\hat{f}_c(t)$ is a noise operator. We observe that Eqs. (3.61) and (3.63) will have the same form if

$$\langle \hat{f}_c(t) \rangle = 0. \tag{3.64}$$

The solution of Eq. (3.63) can be put in the form

$$\hat{\sigma}_c(t+\tau) = \hat{\sigma}_c(t) e^{-\alpha\tau} + \varepsilon e^{-\alpha\tau} \int_0^\tau e^{\alpha\tau'} d\tau' + e^{-\alpha\tau} \int_0^\tau e^{\alpha\tau'} \hat{f}_c(t+\tau') d\tau', \tag{3.65}$$

Thus combination of Eqs. (3.55) and (3.65) yields

$$\begin{aligned} \hat{b}(t+\tau) \quad = \quad & \hat{b}(t) e^{-\kappa\tau/2} + \left(\frac{2\eta\alpha - 2g\varepsilon}{\kappa\alpha} \right) \left[1 - e^{-\kappa\tau/2} \right] + \left(\frac{g\varepsilon - g\alpha\hat{\sigma}_c(t)}{\alpha(\alpha - \kappa/2)} \right) \\ & \times \left[e^{-\kappa\tau/2} - e^{-\alpha\tau} \right] - g e^{-\kappa\tau/2} \int_0^\tau d\tau' e^{-(\alpha-\kappa/2)\tau'} \\ & \times \int_0^{\tau'} d\tau'' e^{\alpha\tau''} \hat{f}_c(t+\tau''). \end{aligned} \tag{3.66}$$

On multiplying Eq. (3.66) by $\hat{b}^\dagger(t)$ from the left and taking the expectation value of the resulting equation, we have

$$\begin{aligned}\langle\hat{b}^\dagger(t)\hat{b}(t+\tau)\rangle &= \langle\hat{b}^\dagger(t)\hat{b}(t)\rangle e^{-\kappa\tau/2} + \left(\frac{2\eta\alpha - 2g\varepsilon}{\kappa\alpha}\right)\langle\hat{b}^\dagger(t)\rangle\left[1 - e^{-\kappa\tau/2}\right] \\ &+ \left(\frac{g\varepsilon\langle\hat{b}^\dagger(t)\rangle - g\alpha\langle\hat{b}^\dagger(t)\hat{\sigma}_c(t)\rangle}{\alpha(\alpha-\kappa/2)}\right)\left[e^{-\kappa\tau/2} - e^{-\alpha\tau}\right] \\ &- g e^{-\kappa\tau/2}\int_0^\tau d\tau' e^{-(\alpha-\kappa/2)\tau'}\int_0^{\tau'} d\tau'' e^{\alpha\tau''}\langle\hat{b}^\dagger(t)\hat{f}_c(t+\tau'')\rangle. \qquad (3.67)\end{aligned}$$

Since a noise operator at a certain time should not affect a cavity mode operator at earlier time [20], we have

$$\langle\hat{b}^\dagger(t)\hat{f}_c(t+\tau'')\rangle = 0, \qquad (3.68)$$

so that Eq. (3.67) reduces to

$$\begin{aligned}\langle\hat{b}^\dagger(t)\hat{b}(t+\tau)\rangle &= \langle\hat{b}^\dagger(t)\hat{b}(t)\rangle e^{-\kappa\tau/2} + \left(\frac{2\eta\alpha - 2g\varepsilon}{\kappa\alpha}\right)\langle\hat{b}^\dagger(t)\rangle\left[1 - e^{-\kappa\tau/2}\right] \\ &+ \left(\frac{g\varepsilon\langle\hat{b}^\dagger(t)\rangle - g\alpha\langle\hat{b}^\dagger(t)\hat{\sigma}_c(t)\rangle}{\alpha(\alpha-\kappa/2)}\right)\left[e^{-\kappa\tau/2} - e^{-\alpha\tau}\right]. \qquad (3.69)\end{aligned}$$

On account of Eq. (2.32), we see that

$$\hat{\sigma}_c(t) = -\frac{\kappa\hat{b}(t)}{2g} + \frac{\eta}{g}. \qquad (3.70)$$

With this substituted into Eq. (3.69), there emerges

$$\begin{aligned}\langle\hat{b}^\dagger(t)\hat{b}(t+\tau)\rangle &= \langle\hat{b}^\dagger(t)\hat{b}(t)\rangle\left[\frac{2\alpha}{2\alpha-\kappa}e^{-\kappa\tau/2} - \frac{\kappa}{2\alpha-\kappa}e^{-\alpha\tau}\right] \\ &+ \langle\hat{b}^\dagger(t)\rangle\left[\left(\frac{2\eta\alpha - 2g\varepsilon}{\kappa\alpha}\right) - \left(\frac{2\eta\alpha - 2g\varepsilon}{\kappa(\alpha-\kappa/2)}\right)e^{-\kappa\tau/2}\right. \\ &\left. + \left(\frac{\eta\alpha - g\varepsilon}{\alpha(\alpha-\kappa/2)}\right)e^{-\alpha\tau}\right]. \qquad (3.71)\end{aligned}$$

Upon substituting Eq. (3.71) into (3.53), we have

$$\begin{aligned} P(\omega) &= \bar{n}_b\left(\frac{2\alpha}{2\alpha-\kappa}\right)\frac{1}{2\pi}\int_{-\infty}^{\infty}d\tau e^{-(\kappa/2-i(\omega-\omega_0))\tau}-\bar{n}_b\left(\frac{\kappa}{2\alpha-\kappa}\right)\\ &\times\frac{1}{2\pi}\int_{-\infty}^{\infty}d\tau e^{-(\alpha-i(\omega-\omega_0))\tau}+\langle\hat{b}^\dagger\rangle_{ss}\Bigg[\left(\frac{2\eta\alpha-2g\varepsilon}{\kappa\alpha}\right)\frac{1}{2\pi}\int_{-\infty}^{\infty}d\tau e^{i(\omega-\omega_0)\tau}\\ &-\left(\frac{2\eta\alpha-2g\varepsilon}{\kappa(\alpha-\kappa/2)}\right)\frac{1}{2\pi}\int_{-\infty}^{\infty}d\tau e^{-(\kappa/2-i(\omega-\omega_0))\tau}+\left(\frac{\eta\alpha-g\varepsilon}{\alpha(\alpha-\kappa/2)}\right)\\ &\times\frac{1}{2\pi}\int_{-\infty}^{\infty}d\tau e^{-(\alpha-i(\omega-\omega_0))\tau}\Bigg]. \end{aligned} \tag{3.72}$$

Now carrying out the integration, applying the relations

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}dx e^{-(a-ib)x}=\frac{1}{\pi}Re\int_{0}^{\infty}dx e^{-(a-ib)x} \tag{3.73}$$

and

$$\int_0^\infty e^{-ax}dx=\frac{1}{a}, \tag{3.74}$$

where a and b are real and $a>0$, we find the power spectrum of the cavity light mode b to be

$$\begin{aligned} P(\omega) &= \bar{n}_b\left[\frac{2\alpha}{2\alpha-\kappa}\left(\frac{\kappa/2\pi}{(\kappa/2)^2+(\omega-\omega_0)^2}\right)-\frac{\kappa}{2\alpha-\kappa}\left(\frac{\alpha/\pi}{\alpha^2+(\omega-\omega_0)^2}\right)\right]\\ &+\langle\hat{b}^\dagger\rangle_{ss}\left[\left(\frac{2\eta\alpha-2g\varepsilon}{\kappa\alpha}\right)\delta(\omega-\omega_0)-\left(\frac{2\eta\alpha-2g\varepsilon}{\kappa(\alpha-\kappa/2)}\right)\left(\frac{\kappa/2\pi}{(\kappa/2)^2+(\omega-\omega_0)^2}\right)\right.\\ &\left.+\left(\frac{\eta\alpha-g\varepsilon}{\alpha(\alpha-\kappa/2)}\right)\left(\frac{\alpha/\pi}{\alpha^2+(\omega-\omega_0)^2}\right)\right]. \end{aligned} \tag{3.75}$$

We next proceed to calculate the local mean photon number for the cavity light mode b in the interval between $\omega'=-\lambda$ and $\omega'=+\lambda$. The local mean photon number in this interval is expressible as

$$\bar{n}_{b\pm\lambda}=\int_{-\lambda}^{+\lambda}P(\omega')d\omega', \tag{3.76}$$

in which $\omega'=\omega-\omega_0$ [20]. Upon substituting (3.75) into Eq. (3.76), we get

$$\begin{aligned} \bar{n}_{b\pm\lambda} &= \bar{n}_b\left[\left(\frac{2\alpha}{2\alpha-\kappa}\right)\frac{\kappa}{2\pi}\int_{-\lambda}^{+\lambda}\frac{d\omega'}{(\kappa/2)^2+\omega'^2}-\left(\frac{\kappa}{2\alpha-\kappa}\right)\frac{\alpha}{\pi}\int_{-\lambda}^{+\lambda}\frac{d\omega'}{\alpha^2+\omega'^2}\right]\\ &+\langle\hat{b}^\dagger\rangle_{ss}\Bigg[\left(\frac{2\eta\alpha-2g\varepsilon}{\kappa\alpha}\right)\int_{-\lambda}^{+\lambda}\delta(\omega')d\omega'-\left(\frac{2\eta\alpha-2g\varepsilon}{\kappa(\alpha-\kappa/2)}\right)\frac{\kappa}{2\pi}\\ &\times\int_{-\lambda}^{+\lambda}\frac{d\omega'}{(\kappa/2)^2+\omega'^2}+\left(\frac{\eta\alpha-g\varepsilon}{\alpha(\alpha-\kappa/2)}\right)\frac{\alpha}{\pi}\int_{-\lambda}^{+\lambda}\frac{d\omega'}{\alpha^2+\omega'^2}\Bigg]. \end{aligned} \tag{3.77}$$

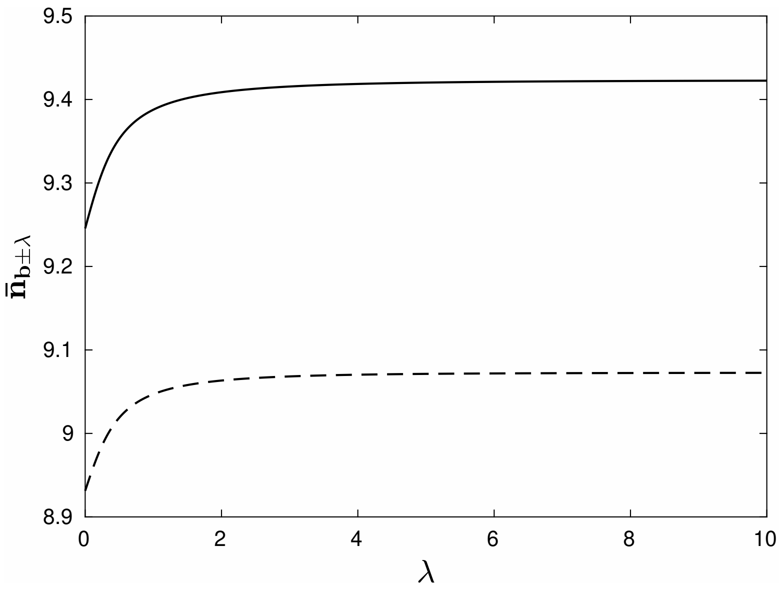

Figure 3.5: Plots of local mean photon number [Eq. (3.84)] versus λ for $\varepsilon = 1$, $\gamma_c = 0.5$, and $\kappa = 0.8$, for $\gamma = 0$ (solid curve), and for $\gamma = 0.3$ (dashed curve).

Therefore, carrying out the integration, applying the relation

$$\int_{-\lambda}^{+\lambda} \frac{dx}{a^2 + x^2} = \frac{2}{a} \tan^{-1}\left(\frac{\lambda}{a}\right), \tag{3.78}$$

we readily obtain

$$\begin{aligned} \bar{n}_{b\pm\lambda} &= \bar{n}_b \left[\left(\frac{2\alpha}{2\alpha-\kappa}\right)\frac{2}{\pi}\tan^{-1}\left(\frac{2\lambda}{\kappa}\right) - \left(\frac{\kappa}{2\alpha-\kappa}\right)\frac{2}{\pi}\tan^{-1}\left(\frac{\lambda}{\alpha}\right)\right] \\ &\quad + \langle\hat{b}^\dagger\rangle_{ss}\left[\left(\frac{2\eta\alpha-2g\varepsilon}{\kappa\alpha}\right) - \left(\frac{2\eta\alpha-2g\varepsilon}{\kappa(\alpha-\kappa/2)}\right)\frac{2}{\pi}\tan^{-1}\left(\frac{2\lambda}{\kappa}\right) \right. \\ &\quad \left. + \left(\frac{\eta\alpha-g\varepsilon}{\alpha(\alpha-\kappa/2)}\right)\frac{2}{\pi}\tan^{-1}\left(\frac{\lambda}{\alpha}\right)\right]. \end{aligned} \tag{3.79}$$

On account of Eqs. (2.62) along with Eqs. (2.48) and (3.62), we readily obtain

$$\langle\hat{b}^\dagger\rangle_{ss} = \frac{2\eta\alpha - 2g\varepsilon}{\kappa\alpha}. \tag{3.80}$$

Then using (3.80) in Eq. (3.79), we get

$$\begin{aligned} \bar{n}_{b\pm\lambda} &= \bar{n}_b \left[\left(\frac{2\alpha}{2\alpha-\kappa}\right)\frac{2}{\pi}\tan^{-1}\left(\frac{2\lambda}{\kappa}\right) - \left(\frac{\kappa}{2\alpha-\kappa}\right)\frac{2}{\pi}\tan^{-1}\left(\frac{\lambda}{\alpha}\right)\right] \\ &\quad + \left(\frac{2\eta\alpha-2g\varepsilon}{\kappa\alpha}\right)\left[\left(\frac{2\eta\alpha-2g\varepsilon}{\kappa\alpha}\right) - \left(\frac{2\eta\alpha-2g\varepsilon}{\kappa(\alpha-\kappa/2)}\right)\frac{2}{\pi}\tan^{-1}\left(\frac{2\lambda}{\kappa}\right) \right. \\ &\quad \left. + \left(\frac{\eta\alpha-g\varepsilon}{\alpha(\alpha-\kappa/2)}\right)\frac{2}{\pi}\tan^{-1}\left(\frac{\lambda}{\alpha}\right)\right]. \end{aligned} \tag{3.81}$$

Furthermore, in view of Eqs. (2.38) and (2.39), one can easily verify that

$$g = \frac{1}{2}\sqrt{\kappa\gamma_c}, \tag{3.82}$$

$$\eta = \frac{\kappa\varepsilon}{\sqrt{\kappa\gamma_c}}. \tag{3.83}$$

Thus on account of Eqs. (3.82) and (3.83), the local mean photon number for the driven cavity mode can be written in the form

$$\begin{aligned}
\bar{n}_{b\pm\lambda} &= \bar{n}_b\left[\left(\frac{2\alpha}{2\alpha-\kappa}\right)\frac{2}{\pi}\tan^{-1}\left(\frac{2\lambda}{\kappa}\right) - \left(\frac{\kappa}{2\alpha-\kappa}\right)\frac{2}{\pi}\tan^{-1}\left(\frac{\lambda}{\alpha}\right)\right] \\
&+ \left(\frac{2\kappa\varepsilon\alpha-\kappa\gamma_c\varepsilon}{\kappa\alpha\sqrt{\kappa\gamma_c}}\right)\left[\left(\frac{2\kappa\varepsilon\alpha-\kappa\gamma_c\varepsilon}{\kappa\alpha\sqrt{\kappa\gamma_c}}\right) - \left(\frac{2\kappa\varepsilon\alpha-\kappa\gamma_c\varepsilon}{\kappa\sqrt{\kappa\gamma_c}(\alpha-\kappa/2)}\right)\frac{2}{\pi}\tan^{-1}\left(\frac{2\lambda}{\kappa}\right)\right. \\
&\left.+ \left(\frac{2\kappa\varepsilon\alpha-\kappa\gamma_c\varepsilon}{2\alpha\sqrt{\kappa\gamma_c}(\alpha-\kappa/2)}\right)\frac{2}{\pi}\tan^{-1}\left(\frac{\lambda}{\alpha}\right)\right],
\end{aligned} \tag{3.84}$$

where $\bar{n}_b$ and α are given by Eqs. (3.10) and (3.62), respectively. From the plots in Figures 3.5 and 3.1, we see that as λ increases the local mean photon number for the driven cavity mode in the presence (dashed curve) or in the absence (solid curve) of spontaneous emission approaches to the global mean photon number for $\varepsilon = 1$.

3.2 Two-mode photon statistics

In this section, we wish to calculate the global mean and variance of the photon number for the superposition of the light modes emitted from the top and intermediate levels of the three-level atom.

3.2.1 The mean photon number

On account of Eq. (2.74), the steady-state mean photon number for the two-mode cavity light can be written as

$$\bar{n} = \langle \hat{c}_1^\dagger \hat{c}_1 \rangle = \frac{\gamma_c}{\kappa}\langle \hat{\eta}_a \rangle + \frac{\gamma_c}{\kappa}\langle \hat{\eta}_b \rangle \tag{3.85}$$

and this happens to be the sum of the mean photon numbers of the cavity modes

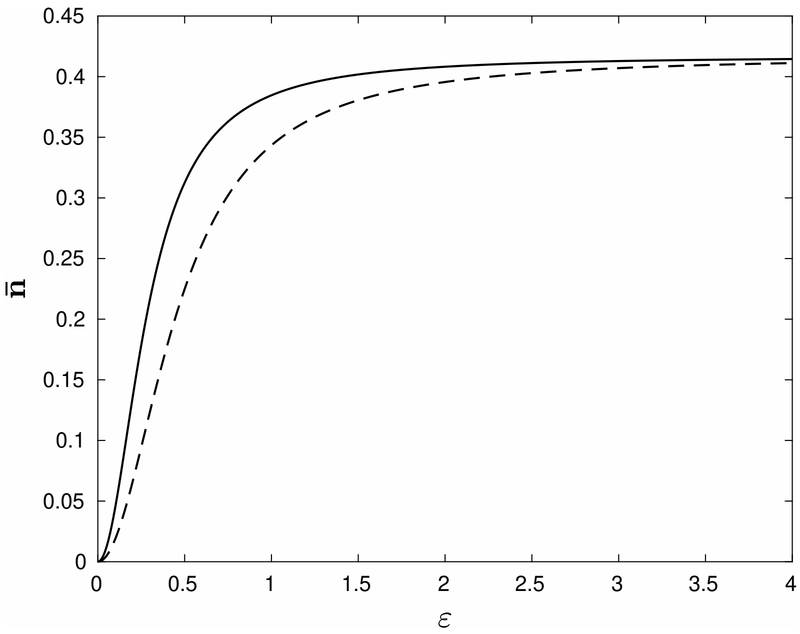

Figure 3.6: Plots of mean photon number [Eq. (3.87)] versus ε for $\gamma_c = 0.5$ and $\kappa = 0.8$, for $\gamma = 0$ (solid curve), and for $\gamma = 0.3$ (dashed curve).

a_1 and a_2. Thus employing Eq. (2.44), we get

$$\bar{n} = 2\frac{\gamma_c}{\kappa}\langle\hat{\eta}_a\rangle. \tag{3.86}$$

Finally, in view of Eq. (2.49), we see that

$$\bar{n} = 2\frac{\gamma_c}{\kappa}\left(\frac{\varepsilon^2}{3\varepsilon^2 + (\gamma_c + \gamma)^2}\right) \tag{3.87}$$

and for $\varepsilon \gg \gamma_c + \gamma$, Eq. (3.87) reduces to

$$\bar{n} = 2\frac{\gamma_c}{\kappa}\left(\frac{1}{3}\right). \tag{3.88}$$

In the absence of spontaneous emission ($\gamma = 0$), the mean photon number for the two-mode cavity light turns out to be

$$\bar{n} = 2\frac{\gamma_c}{\kappa}\left(\frac{\varepsilon^2}{3\varepsilon^2 + \gamma_c^2}\right) \tag{3.89}$$

The plots in Figure 3.6 show that the presence of spontaneous emission leads to a decrease in the global mean photon number for the two-mode cavity light. We also note that the mean photon number increases with increasing ε.

3.2.2 The photon number variance

The variance of the photon number for the two-mode cavity light c_1 can be written as

$$(\Delta n)^2 = \langle \hat{c}_1^\dagger \hat{c}_1 \hat{c}_1^\dagger \hat{c}_1 \rangle - \langle \hat{c}_1^\dagger \hat{c}_1 \rangle^2 \tag{3.90}$$

and using the fact that $\hat{c}_1$ is a Gaussian variable with zero mean, the photon number variance can be written as

$$(\Delta n)^2 = \langle \hat{c}_1^\dagger \hat{c}_1 \rangle \langle \hat{c}_1 \hat{c}_1^\dagger \rangle + \langle \hat{c}_1^{\dagger 2} \rangle \langle \hat{c}_1^2 \rangle. \tag{3.91}$$

Now on account of Eqs. (2.73), (2.74), and (2.75), one readily gets

$$(\Delta \hat{n})^2 = \left(\frac{\gamma_c}{\kappa}\right)^2 [2\langle \hat{\eta}_a \rangle^2 + 2\langle \hat{\eta}_a \rangle \langle \hat{\eta}_c \rangle + \langle \hat{\sigma}_c \rangle^2]. \tag{3.92}$$

On the other hand, with the aid of Eq. (2.48) along with Eq. (2.49), one can write $\langle \hat{\sigma}_c \rangle^2$ as

$$\langle \hat{\sigma}_c \rangle^2 = \left(\frac{(\gamma_c + \gamma)^2}{3\varepsilon^2 + (\gamma_c + \gamma)^2}\right) \langle \hat{\eta}_a \rangle. \tag{3.93}$$

Then using (3.93) in Eq. (3.92), we have

$$(\Delta \hat{n})^2 = \left(\frac{\gamma_c}{\kappa} \langle \hat{\eta}_a \rangle\right) \left[\frac{\gamma_c}{\kappa} \left(2\langle \hat{\eta}_a \rangle + 2\langle \hat{\eta}_c \rangle + \frac{(\gamma_c + \gamma)^2}{3\varepsilon^2 + (\gamma_c + \gamma)^2}\right)\right]. \tag{3.94}$$

Hence on account Eqs. (2.49) and (2.50) along with Eq. (3.86), we readily get

$$(\Delta n)^2 = \frac{\bar{n}}{2} \left(\frac{\gamma_c}{\kappa}\right) \left[\frac{4\varepsilon^2 + 3(\gamma_c + \gamma)^2}{3\varepsilon^2 + (\gamma_c + \gamma)^2}\right]. \tag{3.95}$$

We note that Eq. (3.95) reduces to

$$(\Delta \hat{n})^2 = \bar{n}^2, \tag{3.96}$$

for $\varepsilon \gg \gamma_c + \gamma$. This represents the normally-ordered variance of the photon number for a chaotic state. The plots in Figure 3.7 show that the photon number variance for

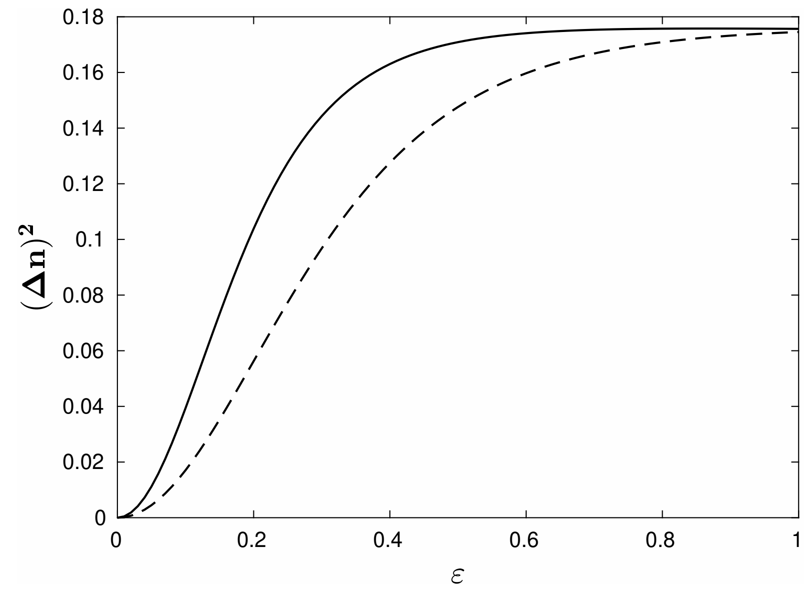

Figure 3.7: Plots of the photon number variance [Eq. (3.95)] versus $varepsilon$ for $\gamma_c = 0.5$, $\kappa = 0.8$, for $\gamma = 0$ (solid curve), and for $\gamma = 0.3$ (dashed curve).

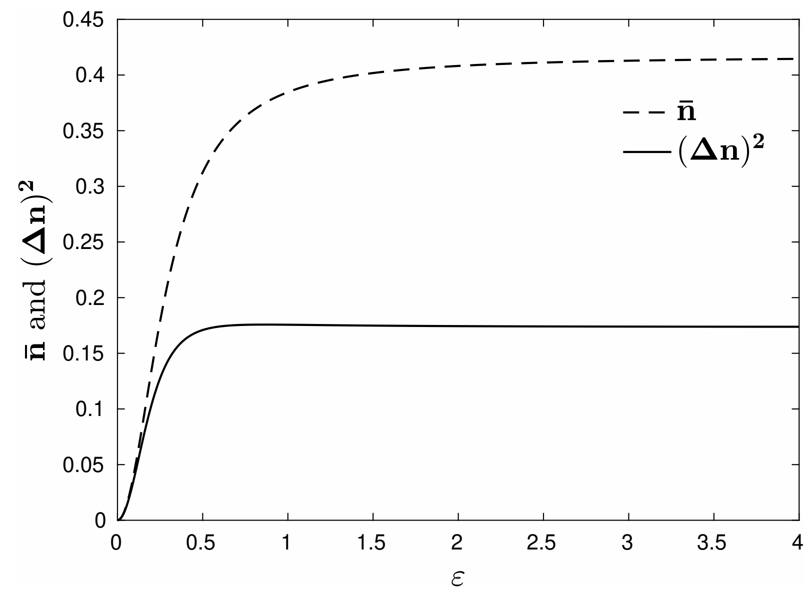

Figure 3.8: Plots of $\bar{n}$ (dashed curve) and $(\Delta n)^2$ (solid curve) for $\gamma_c = 0.5$, $\kappa = 0.8$, and $\gamma = 0$.

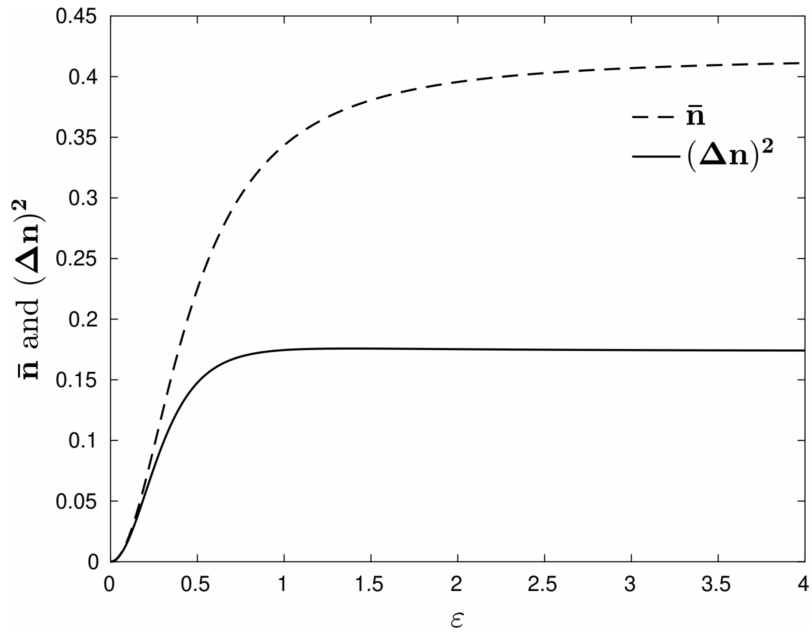

Figure 3.9: Plots of $\bar{n}$ (dashed curve) and $(\Delta n)^2$ (solid curve) for $\gamma_c = 0.5$, $\kappa = 0.8$, and $\gamma = 0.3$.

the two-mode cavity light in the presence of spontaneous emission is less than that in the absence of spontaneous emission. Furthermore, we observe that the photon number variance is less than the mean photon number in the absence (Figure 3.8) as well as in the presence (Figure 3.9) of spontaneous emission. Therefore, the photon statistics is subPoissonian. We also note that the photon number variance for the two-mode cavity light increases as ε increases.

3.2.3 Local mean photon number for the two-mode cavity light

We next seek to calculate the power spectrum of the two-mode cavity light when both light modes a_1 and a_2 have the same central frequency ω_0. The power spectrum of the two-mode cavity light is expressible as

$$P(\omega) = \frac{1}{\pi} Re \int_0^\infty d\tau e^{i(\omega-\omega_0)\tau} \langle \hat{c}_1^\dagger(t)\hat{c}_1(t+\tau)\rangle_{ss}. \tag{3.97}$$

Upon integrating both sides of Eq. (3.97) over ω, we readily get

$$\int_{-\infty}^{\infty} P(\omega) d\omega = \bar{n}, \tag{3.98}$$

From this result, we observe that $P(\omega)d\omega$ is the steady-state mean photon number in the interval between ω and $\omega + d\omega$.

We now proceed to determine the two-time correlation function that appears in Eq. (3.97). To this end, the solution of Eq. (2.69) can be written as

$$\hat{c}_1(t+\tau) = \hat{c}_1(t)e^{-\kappa\tau/2} - ge^{-\kappa\tau/2}\int_0^\tau e^{\kappa\tau'/2}\hat{\sigma}(t+\tau')d\tau'. \tag{3.99}$$

Now adding Eqs. (3.16) and (3.29), we get

$$\frac{d}{dt}\langle\hat{\sigma}\rangle = -\frac{\mu_a}{2}\langle\hat{\sigma}_a\rangle - \frac{\mu_b}{2}\langle\hat{\sigma}_b\rangle, \tag{3.100}$$

where

$$\langle\hat{\sigma}\rangle = \langle\hat{\sigma}_a\rangle + \langle\hat{\sigma}_b\rangle. \tag{3.101}$$

We then see that

$$\langle\hat{\sigma}_b\rangle = \langle\hat{\sigma}\rangle - \langle\hat{\sigma}_a\rangle. \tag{3.102}$$

Upon introducing this relation into Eq. (3.100), we have

$$\frac{d}{dt}\langle\hat{\sigma}\rangle = -\frac{\mu_b}{2}\langle\hat{\sigma}\rangle + \frac{1}{2}(\mu_b - \mu_a)\langle\hat{\sigma}_a\rangle, \tag{3.103}$$

On the basis of this equation, one can write

$$\frac{d}{dt}\hat{\sigma} = -\frac{\mu_b}{2}\hat{\sigma} + \frac{1}{2}(\mu_b - \mu_a)\hat{\sigma}_a + \hat{f}(t), \tag{3.104}$$

in which $\hat{f}(t)$ is a noise operator. We observe that Eqs. (3.103) and (3.104) have the same form if

$$\langle\hat{f}(t)\rangle = 0. \tag{3.105}$$

The formal solution of Eq. (3.104) can be put in the form

$$\begin{aligned}\hat{\sigma}(t+\tau) &= \hat{\sigma}(t)e^{-\mu_b\tau/2} + \frac{1}{2}(\mu_b - \mu_a)e^{-\mu_b\tau/2}\int_0^\tau e^{\mu_b\tau'/2}\hat{\sigma}_a(t+\tau')d\tau' \\ &\quad + e^{-\mu_b\tau/2}\int_0^\tau e^{\mu_b\tau'/2}\hat{f}(t+\tau')d\tau'. \end{aligned} \tag{3.106}$$

Thus combination of Eqs. (3.99) and (3.106) yields

$$\begin{aligned}\hat{c}_1(t+\tau) &= \hat{c}_1(t)e^{-\kappa\tau/2} - \frac{2g}{(\kappa-\mu_b)}\hat{\sigma}(t)[e^{-\mu_b\tau/2} - e^{-\kappa\tau/2}] \\ &-\frac{g(\mu_b-\mu_a)}{2}e^{-\kappa\tau/2}\int_0^{\tau} e^{(\kappa-\mu_b)\tau'/2}d\tau' \int_0^{\tau'} e^{\mu_b\tau''/2}\hat{\sigma}_a(t+\tau'')d\tau'' \\ &-ge^{-\kappa\tau/2}\int_0^{\tau} e^{(\kappa-\mu_b)\tau'/2}d\tau' \int_0^{\tau'} e^{\mu_b\tau''/2}\hat{f}(t+\tau'')d\tau''. \end{aligned} \tag{3.107}$$

On multiplying Eq. (3.107) by $\hat{c}_1^\dagger(t)$ from the left and taking the expectation value of the resulting equation, we get

$$\begin{aligned}\langle\hat{c}_1^\dagger(t)\hat{c}_1(t+\tau)\rangle &= \langle\hat{c}_1^\dagger(t)\hat{c}_1(t)\rangle e^{-\kappa\tau/2} - \frac{2g}{(\kappa-\mu_b)}\langle\hat{c}_1^\dagger(t)\hat{\sigma}(t)\rangle[e^{-\mu_b\tau/2} - e^{-\kappa\tau/2}] \\ &-\frac{g(\mu_b-\mu_a)}{2}e^{-\kappa\tau/2}\int_0^{\tau} e^{(\kappa-\mu_b)\tau'/2}d\tau' \int_0^{\tau'} e^{\mu_b\tau''/2}\langle\hat{c}_1^\dagger(t)\hat{\sigma}_a(t+\tau'')\rangle d\tau'' \\ &-ge^{-\kappa\tau/2}\int_0^{\tau} e^{(\kappa-\mu_b)\tau'/2}d\tau' \int_0^{\tau'} e^{\mu_b\tau''/2}\langle\hat{c}_1^\dagger(t)\hat{f}(t+\tau'')\rangle d\tau''. \end{aligned} \tag{3.108}$$

On the other hand, in view of Eq. (3.16), one can write

$$\frac{d}{dt}\hat{\sigma}_a = -\frac{\mu_a}{2}\hat{\sigma}_a + \hat{f}_a(t), \tag{3.109}$$

where $\hat{f}_a(t)$ is a noise operator with a vanishing mean. Applying the large-time approximation scheme to (3.109), we have

$$\hat{\sigma}_a(t) = \frac{2}{\mu_a}\hat{f}_a(t). \tag{3.110}$$

Then with the aid of this result, we can put Eq. (3.108) in the form

$$\begin{aligned}\langle\hat{c}_1^\dagger(t)\hat{c}_1(t+\tau)\rangle &= \langle\hat{c}_1^\dagger(t)\hat{c}_1(t)\rangle e^{-\kappa\tau/2} - \frac{2g}{(\kappa-\mu_b)}\langle\hat{c}_1^\dagger(t)\hat{\sigma}(t)\rangle[e^{-\mu_b\tau/2} - e^{-\kappa\tau/2}] \\ &-\frac{g(\mu_b-\mu_a)}{\mu_a}e^{-\kappa\tau/2}\int_0^{\tau} e^{(\kappa-\mu_b)\tau'/2}d\tau' \int_0^{\tau'} e^{\mu_b\tau''/2}\langle\hat{c}_1^\dagger(t)\hat{f}_a(t+\tau'')\rangle d\tau'' \\ &-ge^{-\kappa\tau/2}\int_0^{\tau} e^{(\kappa-\mu_b)\tau'/2}d\tau' \int_0^{\tau'} e^{\mu_b\tau''/2}\langle\hat{c}_1^\dagger(t)\hat{f}(t+\tau'')\rangle d\tau''. \end{aligned} \tag{3.111}$$

Since a noise operator at a certain time should not affect a cavity mode operator at earlier time, we readily get

$$\begin{aligned}\langle \hat{c}_1^\dagger(t)\hat{c}_1(t+\tau)\rangle &= \langle \hat{c}_1^\dagger(t)\hat{c}_1(t)\rangle e^{-\kappa\tau/2} - \frac{2g}{(\kappa-\mu_b)}\langle \hat{c}_1^\dagger(t)\hat{\sigma}(t)\rangle \\ &\quad \times\big[e^{-\mu_b\tau/2} - e^{-\kappa\tau/2}\big]. \end{aligned} \tag{3.112}$$

Applying the large-time approximation scheme once more to Eq. (2.69) and solving for $\hat{\sigma}$, we find

$$\hat{\sigma}(t) = -\frac{\kappa}{2g}\hat{c}_1(t) \tag{3.113}$$

and upon introducing this relation into Eq. (3.112), there emerges

$$\langle \hat{c}_1^\dagger(t)\hat{c}_1(t+\tau)\rangle_{ss} = \bar{n}\left[\frac{\kappa}{\kappa-\mu_b}e^{-\mu_b\tau/2} - \frac{\mu_b}{\kappa-\mu_b}e^{-\kappa\tau/2}\right]. \tag{3.114}$$

Hence on substituting (3.114) into Eq. (3.97) and carrying out the integration, applying the relation

$$\int_0^\infty e^{-ax}dx = \frac{1}{a}, \tag{3.115}$$

we arrive at

$$P(\omega) = \frac{\bar{n}\kappa}{\kappa-\mu_b}\left(\frac{\mu_b/2\pi}{(\mu_b/2)^2+(\omega-\omega_0)^2}\right) - \frac{\bar{n}\mu_b}{\kappa-\mu_b}\left(\frac{\kappa/2\pi}{(\kappa/2)^2+(\omega-\omega_0)^2}\right), \tag{3.116}$$

where $a > 0$.

We next proceed to calculate the local mean photon number in the interval between $\omega' = -\lambda$ and $\omega' = \lambda$. We recall that the local mean photon number in this interval is expressible as

$$\bar{n}_{\pm\lambda} = \int_{-\lambda}^{\lambda} P(\omega')d\omega', \tag{3.117}$$

in which $\omega' = \omega - \omega_0$. Thus on account of Eq. (3.116), we see that

$$\bar{n}_{\pm\lambda} = \left(\frac{\bar{n}\kappa}{\kappa-\mu_b}\right)\frac{\mu_b}{2\pi}\int_{-\lambda}^{\lambda}\frac{d\omega'}{(\mu_b/2)^2+\omega'^2} - \left(\frac{\bar{n}\mu_b}{\kappa-\mu_b}\right)\frac{\kappa}{2\pi}\int_{-\lambda}^{\lambda}\frac{d\omega'}{(\kappa/2)^2+\omega'^2} \tag{3.118}$$

and on carrying out the integration, applying the relation

$$\int_{-\lambda}^{\lambda}\frac{dx}{a^2+x^2} = \frac{2}{a}\tan^{-1}\left(\frac{\lambda}{a}\right), \tag{3.119}$$

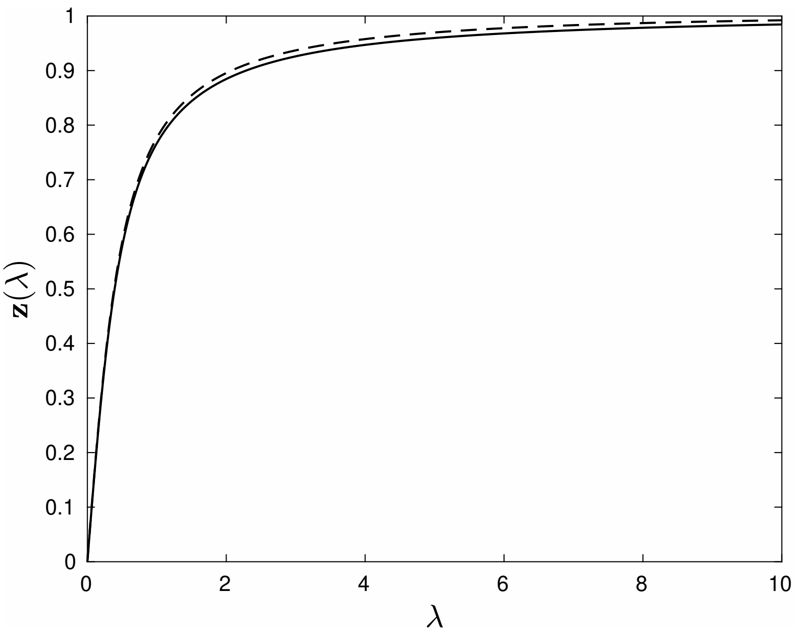

Figure 3.10: Plots of Eq. (3.122) for $\varepsilon = 5$, $\gamma_c = 0.5$, $\kappa = 0.8$, and for $\gamma = 0.6$ (dashed curve), and for $\gamma = 0$ (solid curve).

we easily get

$$\bar{n}_{\pm\lambda} = \bar{n} z(\lambda), \tag{3.120}$$

where

$$z(\lambda) = \frac{2\kappa/\pi}{\kappa - \mu_b} \tan^{-1}\left(\frac{2\lambda}{\mu_b}\right) - \frac{2\mu_b/\pi}{\kappa - \mu_b} \tan^{-1}\left(\frac{2\lambda}{\kappa}\right). \tag{3.121}$$

Thus with the aid of Eq. (3.30), we readily obtain

$$\begin{aligned} z(\lambda) \;\; = \;\; & \frac{6\kappa(\gamma_c + \gamma)/\pi}{3\kappa(\gamma_c + \gamma) - 4\varepsilon^2 + 3(\gamma_c + \gamma)^2} \tan^{-1}\left(\frac{6\kappa(\gamma_c + \gamma)\lambda}{4\varepsilon^2 + 3(\gamma_c + \gamma)^2}\right) \\ & - \frac{\left(8\varepsilon^2 + 6(\gamma_c + \gamma)^2\right)/\pi}{3\kappa(\gamma_c + \gamma) - 4\varepsilon^2 + 3(\gamma_c + \gamma)^2} \tan^{-1}\left(\frac{2\lambda}{\kappa}\right). \end{aligned} \tag{3.122}$$

From the plots in Figure 3.10 and Eq. (3.120), we see that the local mean photon number of the two-mode cavity light in the presence (dashed curve) and in the absence (solid curve) of spontaneous emission approaches the global mean photon number as λ increases.

4

Quadrature Squeezing

In this chapter, we seek to study the squeezing properties of the driven cavity mode and the two-mode cavity light. In a squeezed state the noise in one or the two quadratures may be below the vacuum-state level, with the product of the uncertainties in the two quadratures satisfying the uncertainty relation. We first calculate the global and local quadrature squeezing for the driven cavity mode. We next determine the global quadrature squeezing for the two-mode cavity light.

4.1 Global quadrature squeezing for the driven cavity mode

The plus and the minus quadrature operators for the driven cavity mode b are given by

$$\hat{b}_+ = \hat{b}^\dagger + \hat{b}, \tag{4.1}$$

$$\hat{b}_- = i(\hat{b}^\dagger - \hat{b}). \tag{4.2}$$

Then one can readily establish that

$$[\hat{b}_-, \hat{b}_+] = 2i\frac{\gamma_c}{\kappa}(\hat{\eta}_a - \hat{\eta}_c). \tag{4.3}$$

On account of this result, the quadrature uncertainty relation becomes [51]

$$\Delta b_+ \Delta b_- \geq \frac{\gamma_c}{\kappa}\left|\langle\hat{\eta}_a\rangle - \langle\hat{\eta}_c\rangle\right|. \tag{4.4}$$

Upon substituting Eqs. (2.49) and (2.50) into Eq. (4.4), we easily get

$$\Delta b_{+}\Delta b_{-} \geq f_a(\varepsilon), \tag{4.5}$$

where

$$f_a(\varepsilon) = \frac{\gamma_c}{\kappa}\left(\frac{(\gamma_c+\gamma)^2}{3\varepsilon^2+(\gamma_c+\gamma)^2}\right). \tag{4.6}$$

On setting $\varepsilon = 0$ in Eq. (4.6), we have

$$\Delta b_{+}\Delta b_{-} \geq \frac{\gamma_c}{\kappa}. \tag{4.7}$$

This represents the quadrature uncertainty relation for a vacuum state.

The variances of the plus and the minus quadrature operators are expressible as

$$(\Delta b_{+})^2 = \langle \hat{b}^{\dagger}\hat{b}\rangle + \langle \hat{b}\hat{b}^{\dagger}\rangle + \langle \hat{b}^{\dagger 2}\rangle + \langle \hat{b}^2\rangle - \langle \hat{b}^{\dagger}\rangle^2 - \langle \hat{b}\rangle^2 - 2\langle \hat{b}^{\dagger}\rangle\langle \hat{b}\rangle \tag{4.8}$$

and

$$(\Delta b_{-})^2 = \langle \hat{b}^{\dagger}\hat{b}\rangle + \langle \hat{b}\hat{b}^{\dagger}\rangle - \langle \hat{b}^{\dagger 2}\rangle - \langle \hat{b}^2\rangle + \langle \hat{b}^{\dagger}\rangle^2 + \langle \hat{b}\rangle^2 - 2\langle \hat{b}^{\dagger}\rangle\langle \hat{b}\rangle. \tag{4.9}$$

Now with the aid of Eq. (2.62), we find

$$\langle \hat{b}^2\rangle = \frac{4\eta^2}{\kappa^2} - \frac{8\eta g}{\kappa^2}\langle \hat{\sigma}_c\rangle, \tag{4.10}$$

$$\langle \hat{b}\rangle^2 = \frac{4\eta^2}{\kappa^2} - \frac{8\eta g}{\kappa^2}\langle \hat{\sigma}_c\rangle + \frac{4g^2}{\kappa^2}\langle \hat{\sigma}_c\rangle^2, \tag{4.11}$$

$$\langle \hat{b}^{\dagger}\rangle\langle \hat{b}\rangle = \frac{4\eta^2}{\kappa^2} - \frac{8\eta g}{\kappa^2}\langle \hat{\sigma}_c\rangle + \frac{4g^2}{\kappa^2}\langle \hat{\sigma}_c\rangle^2, \tag{4.12}$$

Therefore, on introducing Eqs. (2.67), (2.68), (4.10), (4.11), and (4.12) into Eqs. (4.8) and (4.9), we obtain

$$(\Delta b_{+})^2 = \frac{\gamma_c}{\kappa}[\langle \hat{\eta}_a\rangle + \langle \hat{\eta}_c\rangle - 4\langle \hat{\sigma}_c\rangle^2], \tag{4.13}$$

$$(\Delta b_{-})^2 = \frac{\gamma_c}{\kappa}[\langle \hat{\eta}_a\rangle + \langle \hat{\eta}_c\rangle]. \tag{4.14}$$

Then on account of Eqs. (2.48) and (2.49) along with (3.93), the plus and the minus quadrature variances are found to be

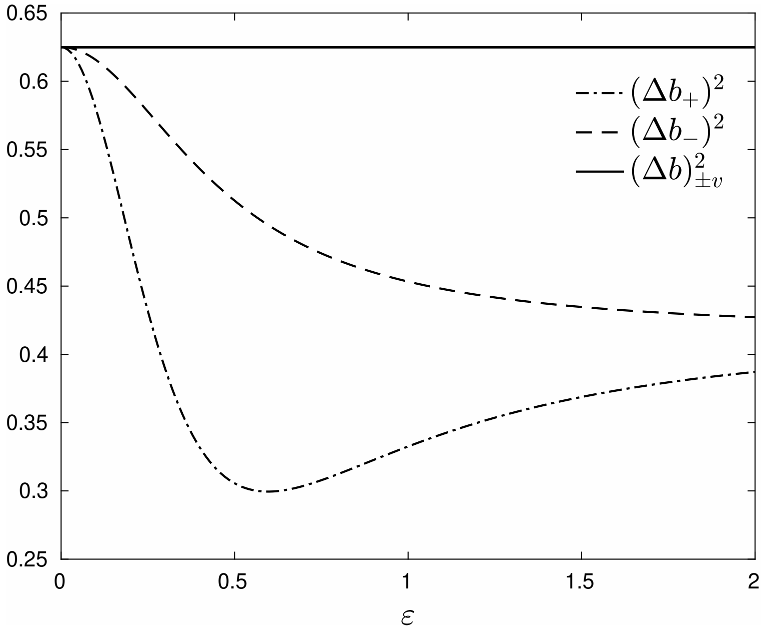

Figure 4.1: Plots of $(\Delta b_-)^2$, $(\Delta b_+)^2$, and $(\Delta b)^2_{\pm v}$ versus ε for $\gamma_c = 0.5$, $\gamma = 0.3$, and $\kappa = 0.8$.

$$(\Delta b_+)^2 = \frac{\gamma_c}{\kappa}\left[\frac{6\varepsilon^4 + \varepsilon^2(\gamma_c+\gamma)^2 + (\gamma_c+\gamma)^4}{(3\varepsilon^2 + (\gamma_c+\gamma)^2)^2}\right], \tag{4.15}$$

$$(\Delta b_-)^2 = \frac{\gamma_c}{\kappa}\left[\frac{2\varepsilon^2 + (\gamma_c+\gamma)^2}{3\varepsilon^2 + (\gamma_c+\gamma)^2}\right]. \tag{4.16}$$

Upon setting $\varepsilon = 0$ in Eqs. (4.15) and (4.16), we have

$$(\Delta b_+)^2_v = (\Delta b_-)^2_v = \frac{\gamma_c}{\kappa}. \tag{4.17}$$

This indeed represents the quadrature variance for the cavity vacuum state in which the uncertainties in the two quadratures are equal and satisfy the minimum uncertainty relation given by Eq. (4.7).

The product of the uncertainties in the plus and minus quadratures can be written as

$$\Delta b_+ \Delta b_- = f_b(\varepsilon), \tag{4.18}$$

where

$$f_b(\varepsilon) = \frac{\gamma_c}{\kappa}\sqrt{\frac{[6\varepsilon^4 + \varepsilon^2(\gamma_c+\gamma)^2 + (\gamma_c+\gamma)^4][2\varepsilon^2 + (\gamma_c+\gamma)^2]}{(3\varepsilon^2 + (\gamma_c+\gamma)^2)^3}}. \tag{4.19}$$

From the plots in Figure 4.1, we observe that both $(\Delta b_+)^2$ and $(\Delta b_-)^2$ are below

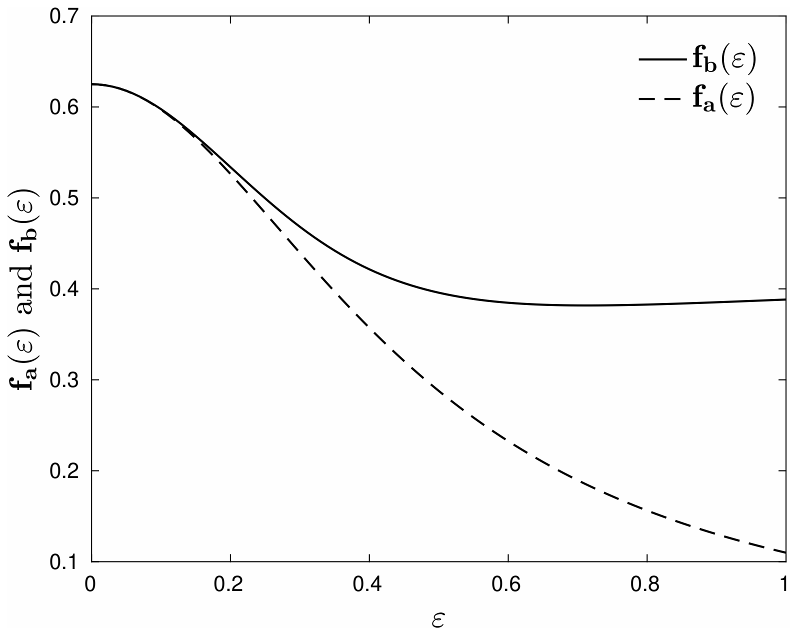

Figure 4.2: Plots of $f_a(\varepsilon)$ and $f_b(\varepsilon)$ versus ε for $\gamma_c = 0.5$, $\gamma = 0.3$, and $\kappa = 0.8$.

the quadrature variance for the vacuum state ($(\Delta b)^2_{\pm v}$). This is certainly unexpected result. We then see that the cavity mode b is in a squeezed state in both the plus and the minus quadratures. The occurrence of squeezing in both the plus and minus quadratures has been reported in Ref. [52]. In addition, we note from the plots in Figure 4.2 that $f_b(\varepsilon)$ (solid curve) and $f_a(\varepsilon)$ (dashed curve) have equal values for $0 < \varepsilon \leq 0.03$. However, the values of $f_b(\varepsilon)$ are greater than that of $f_a(\varepsilon)$ for $\varepsilon > 0.03$. On the basis of these results, we realize that the cavity mode b satisfies the uncertainty relation.

We next seek to calculate the global quadrature squeezing for the driven cavity mode relative to the quadrature variance of the cavity vacuum state. We then define the squeezing for the plus and the minus quadratures by

$$S_+ = \frac{(\Delta b_+)^2_v - (\Delta b_+)^2}{(\Delta b_+)^2_v}, \tag{4.20}$$

$$S_- = \frac{(\Delta b_-)^2_v - (\Delta b_-)^2}{(\Delta b_-)^2_v}. \tag{4.21}$$

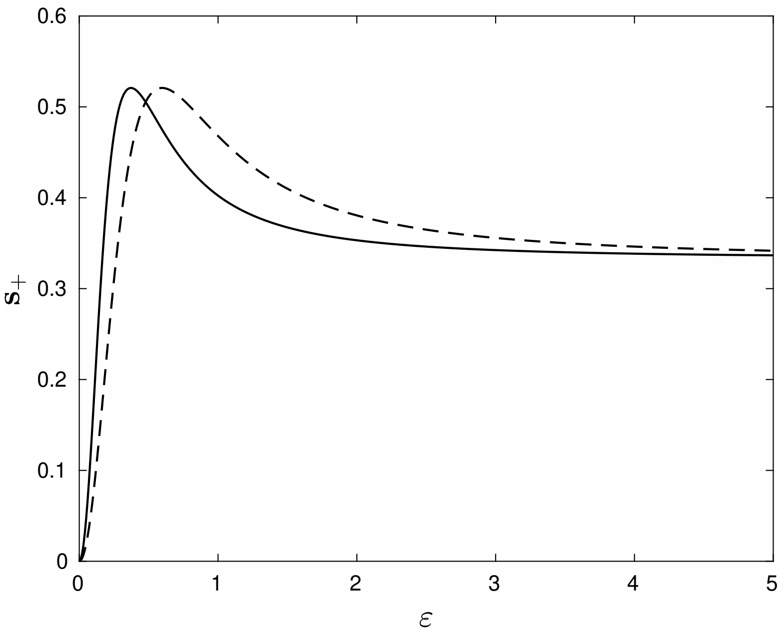

Figure 4.3: Plots of S_+ versus ε for $\gamma_c = 0.5$, $\kappa = 0.8$, and for $\gamma = 0.3$ (dashed curve), and $\gamma = 0$ (solid curve) .

Hence on account of Eqs. (4.15), (4.16), and (4.17), we readily obtain

$$S_+ = 1 - \left[\frac{6\varepsilon^4 + \varepsilon^2(\gamma_c+\gamma)^2 + (\gamma_c+\gamma)^4}{(3\varepsilon^2 + (\gamma_c+\gamma)^2)^2}\right], \tag{4.22}$$

$$S_- = 1 - \left[\frac{2\varepsilon^2 + (\gamma_c+\gamma)^2}{3\varepsilon^2 + (\gamma_c+\gamma)^2}\right]. \tag{4.23}$$

The maximum squeezing for the plus quadrature (Figure 4.3) is 52.08% in the presence of spontaneous emission (dashed curve) and occurs at $\varepsilon = 0.6$ and in the absence of spontaneous emission (solid curve) the squeezing occurs at $\varepsilon = 0.37$. In addition, the squeezing in the presence of spontaneous emission is less than the squeezing in the absence of spontaneous emission for $0 < \varepsilon \leq 0.48$ and vice versa for $\varepsilon > 0.48$. On the other hand, we see from the plots in Figure 4.4 that the maximum squeezing for the minus quadrature is 33.32% in the presence or absence of spontaneous emission and occurs for values of $\varepsilon \geq 15$.

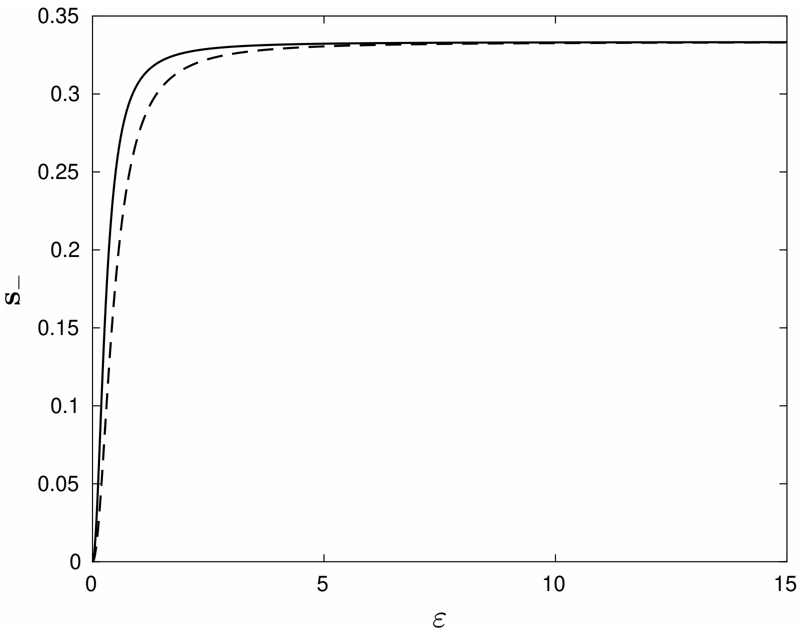

Figure 4.4: Plots of S_- versus ε for $\gamma_c = 0.5$, $\kappa = 0.8$, and for $\gamma = 0.3$ (dashed curve), and $\gamma = 0$ (solid curve).

4.2 Local quadrature squeezing for the driven cavity mode

We seek here to calculate the local squeezing for the plus quadrature for the driven cavity mode. The spectrum of the plus quadrature fluctuations for the cavity mode with central frequency ω_0 is defined by

$$S_+(\omega) = \frac{1}{\pi} Re \int_0^\infty \langle \hat{b}_+(t), \hat{b}_+(t+\tau)\rangle_{ss} e^{i(\omega-\omega_0)\tau} d\tau, \tag{4.24}$$

where ss stands for steady-state. Upon integrating both sides of Eq. (4.24) over ω, we get

$$\int_{-\infty}^{\infty} S_+(\omega) d\omega = (\Delta b_+)^2, \tag{4.25}$$

in which

$$(\Delta b_+)^2 = \langle \hat{b}_+(t), \hat{b}_+(t)\rangle_{ss} \tag{4.26}$$

is the global quadrature variance at steady-state. On the basis of the result given by Eq.(4.25), we assert that $S_+(\omega)d\omega$ is the steady-state quadrature variance in the interval between ω and $\omega + d\omega$. The expression $\langle \hat{b}_+(t), \hat{b}_+(t+\tau)\rangle_{ss}$ can be written as

$$\langle \hat{b}_+(t), \hat{b}_+(t+\tau)\rangle_{ss} = \langle \hat{b}_+(t)\hat{b}_+(t+\tau)\rangle - \langle \hat{b}_+(t)\rangle\langle \hat{b}_+(t+\tau)\rangle, \tag{4.27}$$

so that on account of Eq. (4.1), we have

$$\begin{aligned}\langle \hat{b}_+(t), \hat{b}_+(t+\tau)\rangle_{ss} &= \langle \hat{b}^\dagger(t)\hat{b}(t+\tau)\rangle + \langle \hat{b}(t)\hat{b}(t+\tau)\rangle + \langle \hat{b}(t)\hat{b}^\dagger(t+\tau)\rangle \\ &\quad + \langle \hat{b}^\dagger(t)\hat{b}(t+\tau)\rangle - \Big[\langle \hat{b}^\dagger(t)\rangle\langle \hat{b}^\dagger(t+\tau)\rangle + \langle \hat{b}(t)\rangle\langle \hat{b}(t+\tau)\rangle \\ &\quad + \langle \hat{b}^\dagger(t)\rangle\langle \hat{b}(t+\tau)\rangle + \langle \hat{b}(t)\rangle\langle \hat{b}^\dagger(t+\tau)\rangle\Big].\end{aligned} \tag{4.28}$$

Employing this result, one can write Eq. (4.24) as

$$\begin{aligned}S_+(\omega) &= \frac{1}{\pi}Re\int_0^\infty \Big(\langle \hat{b}^\dagger(t)\hat{b}(t+\tau)\rangle + \langle \hat{b}(t)\hat{b}(t+\tau)\rangle) + + \langle \hat{b}(t)\hat{b}^\dagger(t+\tau)\rangle \\ &\quad + \langle \hat{b}^\dagger(t)\hat{b}(t+\tau)\rangle - \Big[\langle \hat{b}^\dagger(t)\rangle\langle \hat{b}^\dagger(t+\tau)\rangle + \langle \hat{b}(t)\rangle\langle \hat{b}(t+\tau)\rangle \\ &\quad + \langle \hat{b}^\dagger(t)\rangle\langle \hat{b}(t+\tau)\rangle + \langle \hat{b}(t)\rangle\langle \hat{b}^\dagger(t+\tau)\rangle\Big]\Big) e^{i(\omega-\omega_0)\tau}d\tau.\end{aligned} \tag{4.29}$$

We next proceed to calculate the expectation values involved in Eq. (4.29). To this end, upon multiplying Eq. (3.66) by $\hat{b}(t)$ from the left and taking the expectation value of the resulting equation, we have

$$\begin{aligned}\langle \hat{b}(t)\hat{b}(t+\tau)\rangle &= \langle \hat{b}(t)\hat{b}(t)\rangle e^{-\kappa\tau/2} + \left(\frac{2\eta\alpha - 2g\varepsilon}{\kappa\alpha}\right)\langle \hat{b}(t)\rangle[1 - e^{-\kappa\tau/2}] \\ &\quad + \left(\frac{g\varepsilon\langle \hat{b}(t)\rangle - g\alpha\langle \hat{b}(t)\hat{\sigma}_c(t)\rangle}{\alpha(\alpha-\kappa/2)}\right)[e^{-\kappa\tau/2} - e^{-\alpha\tau}] \\ &\quad - ge^{-\kappa\tau/2}\int_0^\tau d\tau' e^{-(\alpha-\kappa/2)\tau'}\int_0^{\tau'} d\tau'' e^{\alpha\tau''}\langle \hat{b}(t)\hat{f}_c(t+\tau'')\rangle.\end{aligned} \tag{4.30}$$

Making use of the fact that

$$\langle \hat{b}(t)\hat{f}_c(t+\tau'')\rangle = 0, \tag{4.31}$$

there follows

$$\begin{aligned}\langle \hat{b}(t)\hat{b}(t+\tau)\rangle &= \langle \hat{b}(t)\hat{b}(t)\rangle e^{-\kappa\tau/2} + \left(\frac{2\eta\alpha - 2g\varepsilon}{\kappa\alpha}\right)\langle \hat{b}(t)\rangle[1 - e^{-\kappa\tau/2}] \\ &\quad + \left(\frac{g\varepsilon\langle \hat{b}(t)\rangle - g\alpha\langle \hat{b}(t)\hat{\sigma}_c(t)\rangle}{\alpha(\alpha-\kappa/2)}\right)[e^{-\kappa\tau/2} - e^{-\alpha\tau}].\end{aligned} \tag{4.32}$$

Now using Eq. (3.70) in Eq. (4.32), we readily obtain

$$\begin{aligned}\langle\hat{b}(t)\hat{b}(t+\tau)\rangle &= \langle\hat{b}^2(t)\rangle\left[\frac{2\alpha}{2\alpha-\kappa}e^{-\kappa\tau/2}-\frac{\kappa}{2\alpha-\kappa}e^{-\alpha\tau}\right]\\ &+\langle\hat{b}(t)\rangle\bigg[\left(\frac{2\eta\alpha-2g\varepsilon}{\kappa\alpha}\right)-\left(\frac{2\eta\alpha-2g\varepsilon}{\kappa(\alpha-\kappa/2)}\right)e^{-\kappa\tau/2}\\ &+\left(\frac{\eta\alpha-g\varepsilon}{\alpha(\alpha-\kappa/2)}\right)e^{-\alpha\tau}\bigg]. \quad (4.33)\end{aligned}$$

Following a similar procedure, one can readily establish that

$$\begin{aligned}\langle\hat{b}(t)\hat{b}^\dagger(t+\tau)\rangle &= \langle\hat{b}(t)\hat{b}^\dagger(t)\rangle\left[\frac{2\alpha}{2\alpha-\kappa}e^{-\kappa\tau/2}-\frac{\kappa}{2\alpha-\kappa}e^{-\alpha\tau}\right]\\ &+\langle\hat{b}(t)\rangle\bigg[\left(\frac{2\eta\alpha-2g\varepsilon}{\kappa\alpha}\right)-\left(\frac{2\eta\alpha-2g\varepsilon}{\kappa(\alpha-\kappa/2)}\right)e^{-\kappa\tau/2}\\ &+\left(\frac{\eta\alpha-g\varepsilon}{\alpha(\alpha-\kappa/2)}\right)e^{-\alpha\tau}\bigg], \quad (4.34)\end{aligned}$$

$$\begin{aligned}\langle\hat{b}^\dagger(t)\hat{b}^\dagger(t+\tau)\rangle &= \langle\hat{b}^{\dagger 2}(t)\rangle\left[\frac{2\alpha}{2\alpha-\kappa}e^{-\kappa\tau/2}-\frac{\kappa}{2\alpha-\kappa}e^{-\alpha\tau}\right]\\ &+\langle\hat{b}^\dagger(t)\rangle\bigg[\left(\frac{2\eta\alpha-2g\varepsilon}{\kappa\alpha}\right)-\left(\frac{2\eta\alpha-2g\varepsilon}{\kappa(\alpha-\kappa/2)}\right)e^{-\kappa\tau/2}\\ &+\left(\frac{\eta\alpha-g\varepsilon}{\alpha(\alpha-\kappa/2)}\right)e^{-\alpha\tau}\bigg], \quad (4.35)\end{aligned}$$

$$\begin{aligned}\langle\hat{b}(t)\rangle\langle\hat{b}(t+\tau)\rangle &= \langle\hat{b}(t)\rangle^2\left[\frac{2\alpha}{2\alpha-\kappa}e^{-\kappa\tau/2}-\frac{\kappa}{2\alpha-\kappa}e^{-\alpha\tau}\right]\\ &+\langle\hat{b}(t)\rangle\bigg[\left(\frac{2\eta\alpha-2g\varepsilon}{\kappa\alpha}\right)-\left(\frac{2\eta\alpha-2g\varepsilon}{\kappa(\alpha-\kappa/2)}\right)e^{-\kappa\tau/2}\\ &+\left(\frac{\eta\alpha-g\varepsilon}{\alpha(\alpha-\kappa/2)}\right)e^{-\alpha\tau}\bigg], \quad (4.36)\end{aligned}$$

$$\begin{aligned}\langle\hat{b}(t)\rangle\langle\hat{b}^\dagger(t+\tau)\rangle &= \langle\hat{b}(t)\rangle\langle\hat{b}^\dagger(t)\rangle\left[\frac{2\alpha}{2\alpha-\kappa}e^{-\kappa\tau/2}-\frac{\kappa}{2\alpha-\kappa}e^{-\alpha\tau}\right]\\ &+\langle\hat{b}(t)\rangle\bigg[\left(\frac{2\eta\alpha-2g\varepsilon}{\kappa\alpha}\right)-\left(\frac{2\eta\alpha-2g\varepsilon}{\kappa(\alpha-\kappa/2)}\right)e^{-\kappa\tau/2}\\ &+\left(\frac{\eta\alpha-g\varepsilon}{\alpha(\alpha-\kappa/2)}\right)e^{-\alpha\tau}\bigg], \quad (4.37)\end{aligned}$$

$$\begin{aligned}\langle\hat{b}^\dagger(t)\rangle\langle\hat{b}^\dagger(t+\tau)\rangle &= \langle\hat{b}^\dagger(t)\rangle^2\left[\frac{2\alpha}{2\alpha-\kappa}e^{-\kappa\tau/2}-\frac{\kappa}{2\alpha-\kappa}e^{-\alpha\tau}\right]\\ &+\langle\hat{b}^\dagger(t)\rangle\bigg[\left(\frac{2\eta\alpha-2g\varepsilon}{\kappa\alpha}\right)-\left(\frac{2\eta\alpha-2g\varepsilon}{\kappa(\alpha-\kappa/2)}\right)e^{-\kappa\tau/2}\\ &+\left(\frac{\eta\alpha-g\varepsilon}{\alpha(\alpha-\kappa/2)}\right)e^{-\alpha\tau}\bigg]. \quad (4.38)\end{aligned}$$

Upon substituting Eqs. (3.71), (4.33), (4.34), (4.35), (4.36), (4.37), and (4.38) into Eq. (4.29), we arrive at

$$\begin{aligned} S_+(\omega) &= (\Delta b_+)^2 \left(\frac{2\alpha}{2\alpha-\kappa}\right) \frac{Re}{\pi} \int_0^\infty d\tau e^{-(\kappa/2 - i(\omega-\omega_0))\tau} \\ &\quad -(\Delta b_+)^2 \left(\frac{\kappa}{2\alpha-\kappa}\right) \frac{Re}{\pi} \int_0^\infty d\tau e^{-(\alpha - i(\omega-\omega_0))\tau}. \end{aligned} \tag{4.39}$$

On carrying out the integration, applying the relation

$$\int_0^\infty e^{-ax} dx = \frac{1}{a}, Re(a) > 0, \tag{4.40}$$

we find the spectrum of the plus quadrature fluctuations for the cavity mode to be

$$\begin{aligned} S_+(\omega) &= (\Delta b_+)^2 \Bigg[\left(\frac{2\alpha}{2\alpha-\kappa}\right) \left(\frac{\kappa/2\pi}{(\kappa/2)^2 + (\omega-\omega_0)^2}\right) \\ &\quad - \left(\frac{\kappa}{2\alpha-\kappa}\right) \left(\frac{\alpha/\pi}{\alpha^2 + (\omega-\omega_0)^2}\right) \Bigg] \end{aligned} \tag{4.41}$$

in which $(\Delta b_+)^2$ is given by Eq. (4.15).

The variance of the plus quadrature in the interval between $\omega' = -\lambda$ and $\omega' = +\lambda$ can be expressed as

$$(\Delta b_+)^2_{\pm\lambda} = \int_{-\lambda}^{+\lambda} S_+(\omega') d\omega', \tag{4.42}$$

in which $\omega' = \omega - \omega_0$. Upon substituting Eq. (4.41) into (4.42), we see that

$$\begin{aligned} (\Delta b_+)^2_{\pm\lambda} &= (\Delta b_+)^2 \Bigg[\left(\frac{2\alpha}{2\alpha-\kappa}\right) \int_{-\lambda}^{+\lambda} \left(\frac{\kappa/2\pi}{(\kappa/2)^2 + \omega'^2}\right) d\omega' \\ &\quad - \left(\frac{\kappa}{2\alpha-\kappa}\right) \int_{-\lambda}^{+\lambda} \left(\frac{\alpha/\pi}{\alpha^2 + \omega'^2}\right) d\omega' \Bigg] \end{aligned} \tag{4.43}$$

and on carrying out the integration, applying the relation

$$\int_{-\lambda}^{+\lambda} \frac{d\omega'}{\omega'^2 + a^2} = \frac{2}{a} \tan^{-1}\left(\frac{\lambda}{a}\right), \tag{4.44}$$

we arrive at

$$(\Delta b_+)^2_{\pm\lambda} = (\Delta b_+)^2 \left[\frac{4\alpha/\pi}{2\alpha-\kappa} \tan^{-1}\left(\frac{2\lambda}{\kappa}\right) - \frac{2\kappa/\pi}{2\alpha-\kappa} \tan^{-1}\left(\frac{\lambda}{\alpha}\right) \right], \tag{4.45}$$

where α is given by Eq. (3.62). Now on account of Eqs. (4.15) and (3.62), we have

$$\begin{aligned}(\Delta b_+)^2_{\pm\lambda} &= \frac{\gamma_c}{\kappa}\left(\frac{6\varepsilon^4+(\gamma_c+\gamma)^2((\gamma_c+\gamma)^2+\varepsilon^2)}{(3\varepsilon^2+(\gamma_c+\gamma)^2)^2}\right)\Bigg[\left(\frac{4(3\varepsilon^2+(\gamma_c+\gamma)^2)/\pi}{2(3\varepsilon^2+(\gamma_c+\gamma)^2)-\kappa(\gamma_c+\gamma)}\right)\\ &\times\tan^{-1}\left(\frac{2\lambda}{\kappa}\right)-\left(\frac{2\kappa(\gamma_c+\gamma)/\pi}{2(3\varepsilon^2+(\gamma_c+\gamma)^2)-\kappa(\gamma_c+\gamma)}\right)\\ &\times\tan^{-1}\left(\frac{\lambda(\gamma_c+\gamma)}{3\varepsilon^2+(\gamma_c+\gamma)^2}\right)\Bigg].\end{aligned} \tag{4.46}$$

Furthermore, upon setting $\varepsilon = 0$ in Eq. (4.46), we find the local quadrature variance for the vacuum state to be

$$\begin{aligned}(\Delta b_+)^2_{v\pm\lambda} &= \frac{\gamma_c}{\kappa}\left[\left(\frac{4(\gamma_c+\gamma)/\pi}{2(\gamma_c+\gamma)-\kappa}\right)\tan^{-1}\left(\frac{2\lambda}{\kappa}\right)-\left(\frac{2\kappa/\pi}{2(\gamma_c+\gamma)-\kappa}\right)\right.\\ &\left.\times\tan^{-1}\left(\frac{\lambda}{\gamma_c+\gamma}\right)\right].\end{aligned} \tag{4.47}$$

The local quadrature squeezing for the driven cavity mode in the interval between $\lambda_\pm$ frequency interval is given by

$$S_{\pm\lambda} = \frac{(\Delta b_+)^2_{v\pm\lambda}-(\Delta b_+)^2_{\pm\lambda}}{(\Delta b_+)^2_{v\pm\lambda}}, \tag{4.48}$$

so that on account of Eqs. (4.46) and (4.47), there follows

$$S_{\pm\lambda} = 1-\left[\frac{\left(\frac{6\varepsilon^4+(\gamma_c+\gamma)^2((\gamma_c+\gamma)^2+\varepsilon^2)}{p^2}\right)\left[\left(\frac{4p/\pi}{2p-q}\right)\tan^{-1}\left(\frac{2\lambda}{\kappa}\right)-\left(\frac{2q/\pi}{2p-q}\right)\tan^{-1}\left(\frac{\lambda(\gamma_c+\gamma)}{p}\right)\right]}{\left[\left(\frac{4(\gamma_c+\gamma)/\pi}{2(\gamma_c+\gamma)-\kappa}\right)\tan^{-1}\left(\frac{2\lambda}{\kappa}\right)-\left(\frac{2\kappa/\pi}{2(\gamma_c+\gamma)-\kappa}\right)\tan^{-1}\left(\frac{\lambda}{\gamma_c+\gamma}\right)\right]}\right] \tag{4.49}$$

where

$$p = 3\varepsilon^2+(\gamma_c+\gamma)^2, \tag{4.50}$$

$$q = \kappa(\gamma_c+\gamma). \tag{4.51}$$

The plots in Figure 4.5 show that the maximum local quadrature squeezing is 65.32% below the vacuum-state level without spontaneous emission and occurs in the frequency interval $\lambda_\pm = 0.01$. On the other hand, the maximum local quadrature squeezing with spontaneous emission is 62.11% below the vacuum state level and

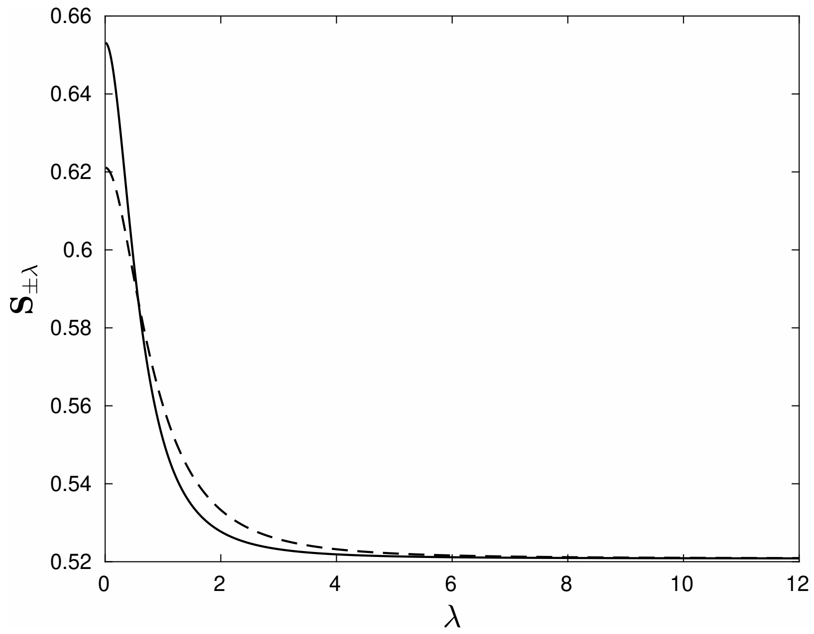

Figure 4.5: Plots of Eq. (4.49) for $\gamma_c = 0.5$, $\kappa = 0.8$, and for $\gamma = 0$, $\varepsilon = 0.37$ (solid curve) and for $\gamma = 0.3$, $\varepsilon = 0.6$ (dashed curve).

occurs at $\lambda_{\pm} = 0.01$. From this results we observe that the effect of the spontaneous emission is to decrease the maximum local quadrature squeezing. However, as the value of λ increases the local quadrature squeezing with or without spontaneous emission approaches to the global quadrature squeezing (Figure 4.3).

4.3 Global quadrature squeezing for the two-mode cavity light

We wish here to calculate the global quadrature squeezing (in the entire frequency interval) for the two-mode cavity light. To this end, the squeezing properties of the cavity mode c_1 are described by two quadrature operators defined by

$$\hat{c}_{1+} = \hat{c}_1^{\dagger} + \hat{c}_1, \tag{4.52}$$

$$\hat{c}_{1-} = i(\hat{c}_1^{\dagger} - \hat{c}_1), \tag{4.53}$$

where $\hat{c}_1$ is defined by Eq. (2.70). Employing Eqs. (4.52) and (4.53), the commutator of the minus and the plus quadrature operators can be put in the form

$$[\hat{c}_{1-}, \hat{c}_{1+}] = -2i[\hat{c}_1, \hat{c}_1^\dagger]. \tag{4.54}$$

Then on account of Eqs. (2.73) and (2.74), we readily get

$$[\hat{c}_{1-}, \hat{c}_{1+}] = 2i\frac{\gamma_c}{\kappa}(\hat{\eta}_a - \hat{\eta}_c). \tag{4.55}$$

Applying this result, one can write the uncertainty relation as

$$\Delta c_{1+}\Delta c_{1-} \geq \frac{\gamma_c}{\kappa}\left|\langle\hat{\eta}_a\rangle - \langle\hat{\eta}_c\rangle\right|, \tag{4.56}$$

so that in view of Eqs. (2.49) and (2.50), one easily gets

$$\Delta c_{1+}\Delta c_{1-} \geq \frac{\gamma_c}{\kappa}\left(\frac{(\gamma_c+\gamma)^2}{3\varepsilon^2 + (\gamma_c+\gamma)^2}\right). \tag{4.57}$$

Upon setting $\varepsilon = 0$, we find the uncertainty relation to be

$$\Delta c_{1+}\Delta c_{1-} \geq \frac{\gamma_c}{\kappa}. \tag{4.58}$$

This represents the uncertainty relation for the plus and the minus quadrature operators for the two-mode vacuum state.

We now proceed to calculate the quadrature variance for the two-mode cavity light. To this end, The variance of the quadrature operators $\hat{c}_{1+}$ and $\hat{c}_{1-}$ is expressible as

$$(\Delta c_{1\pm})^2 = \langle\hat{c}_{1\pm}^2\rangle - \langle\hat{c}_{1\pm}\rangle^2. \tag{4.59}$$

Now with the aid of Eq. (2.70), we find

$$\langle\hat{c}_{1\pm}\rangle = 0. \tag{4.60}$$

Therefore, $(\Delta c_{1\pm})^2$ can be put in the form

$$(\Delta c_{1\pm})^2 = \langle\hat{c}_1^\dagger\hat{c}_1\rangle + \langle\hat{c}_1\hat{c}_1^\dagger\rangle \pm (\langle\hat{c}_1^2\rangle + \langle\hat{c}_1^{\dagger 2}\rangle). \tag{4.61}$$

Then taking into account Eqs. (2.73), (2.74), and (2.75), we have

$$(\Delta c_{1\pm})^2 = \frac{\gamma_c}{\kappa}(3\langle\hat{\eta}_a\rangle + \langle\hat{\eta}_c\rangle \pm 2\langle\hat{\sigma}_c\rangle). \tag{4.62}$$

Upon substituting Eqs. (2.48), (2.49), and (2.50) into Eq. (4.62), we obtain

$$(\Delta c_{1+})^2 = \frac{\gamma_c}{\kappa}\left[1 + \frac{\varepsilon^2 + 2\varepsilon(\gamma_c+\gamma)}{3\varepsilon^2 + (\gamma_c+\gamma)^2}\right], \tag{4.63}$$

$$(\Delta c_{1-})^2 = \frac{\gamma_c}{\kappa}\left[1 + \frac{\varepsilon^2 - 2\varepsilon(\gamma_c+\gamma)}{3\varepsilon^2 + (\gamma_c+\gamma)^2}\right]. \tag{4.64}$$

Now on setting $\varepsilon = 0$ in Eqs. (4.63) and (4.64), there follows

$$(\Delta c_{1+})^2_v = (\Delta c_{1-})^2_v = \frac{\gamma_c}{\kappa}. \tag{4.65}$$

We thus see that for $\varepsilon = 0$, the cavity mode is in a two-mode vacuum state in which the uncertainties in the two quadratures are equal and satisfy the minimum uncertainty relation given by Eq. (4.58). The two-mode cavity light is said to be in a squeezed state if $(\Delta c_{1-})^2 < \frac{\gamma_c}{\kappa}$ or $(\Delta c_{1+})^2 < \frac{\gamma_c}{\kappa}$. Hence from Eq. (4.63), we see that squeezing cannot occur in the plus quadrature. However, squeezing can occur in the minus quadrature if

$$\varepsilon^2 - 2\varepsilon(\gamma_c+\gamma) < 0. \tag{4.66}$$

It then follows that

$$\varepsilon < 2(\gamma_c+\gamma). \tag{4.67}$$

We next proceed to calculate the global quadrature squeezing for the two-mode cavity light relative to the quadrature variance of the two-mode cavity vacuum state. We define the quadrature squeezing for the two-mode cavity light by

$$S = \frac{(\Delta c_{1-})^2_v - (\Delta c_{1-})^2}{(\Delta c_{1-})^2_v}. \tag{4.68}$$

Thus combination of Eqs. (4.64), (4.65), and (4.68), leads to

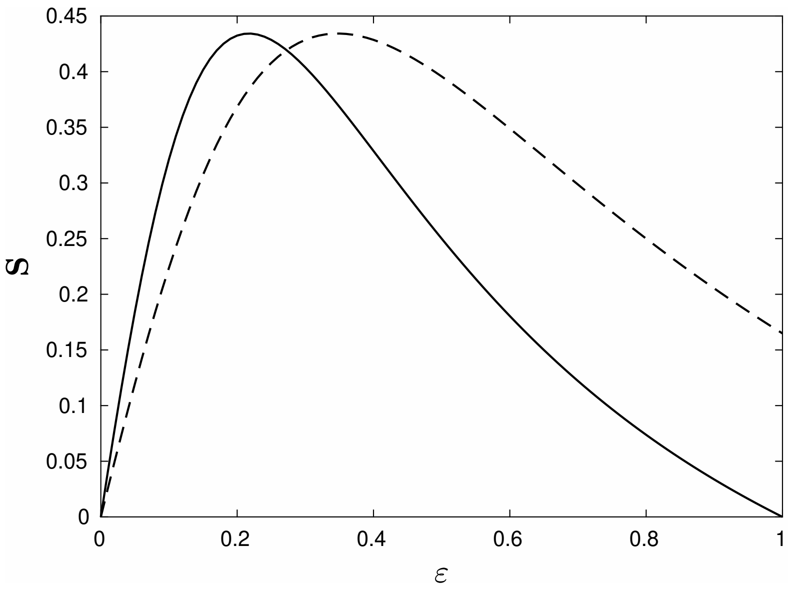

Figure 4.6: Plots of Eq. (4.69) for $\gamma = 0$ (solid curve), for $\gamma = 0.3$ (dashed curve), $\gamma_c = 0.5$, and $\kappa = 0.8$.

$$S = 1 - \left[\frac{(\gamma_c + \gamma)^2 + 4\varepsilon^2 - 2\varepsilon(\gamma_c + \gamma)}{3\varepsilon^2 + (\gamma_c + \gamma)^2}\right]. \tag{4.69}$$

The plots in Figure 4.6 indicate that the light produced by the three-level atom available in the cavity driven by coherent light and coupled to a vacuum reservoir via a single-port mirror is in a squeezed state, with the maximum global quadrature squeezing being 43.42% below the vacuum-state level and occurs at $\varepsilon = 0.22$ in the absence of spontaneous emission and occurs at $\varepsilon = 0.35$ in the presence of spontaneous emission. This result agrees with the one found in Ref. [20]. In addition, for $\varepsilon \leq 0.27$, we see that the squeezing for $\gamma = 0$ (solid curve) is greater than the squeezing for $\gamma = 0.5$ (dashed curve) . On the other hand, for $\varepsilon > 0.27$, the squeezing for $\gamma = 0.5$ (dashed curve) is greater than the squeezing for $\gamma = 0$ (solid curve).

5

Superposed Cavity Light Modes

Here we seek to study the statistical and the squeezing properties of a pair of superposed cavity light modes each driven by coherent light and interacting with a three-level atom. We wish to represent the two driven cavity light modes by the annihilation operators $\hat{a}$ and $\hat{b}$. Then we define the annihilation operator for the pair of superposed cavity light modes. We then determine the equation of evolution for the superposed cavity light modes. Applying the steady-state solution of the resulting equation, we calculate the mean and variance of the photon number and the quadrature squeezing.

We can define the annihilation operator representing the superposition of the driven cavity light modes a and b by [49, 52]

$$\hat{c} = \hat{a} + i\hat{b}. \tag{5.1}$$

Then differentiating this equation with respect to time, we have

$$\frac{d\hat{c}}{dt} = \frac{d\hat{a}}{dt} + i\frac{d\hat{b}}{dt}. \tag{5.2}$$

On the basis of Eq. (2.17), one can write the equations of evolution for the cavity modes a and b as

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} + \eta - g\hat{\sigma}_c \tag{5.3}$$

and

$$\frac{d\hat{b}}{dt} = -\frac{\kappa}{2}\hat{b} + \eta - g\hat{\sigma}'_c. \tag{5.4}$$

Now upon substituting Eqs. (5.3) and (5.4) into Eq. (5.2), we get

$$\frac{d\hat{c}}{dt} = -\frac{\kappa}{2}(\hat{a} + i\hat{b}) + \eta(1+i) - g(\hat{\sigma}_c + i\hat{\sigma}'_c). \tag{5.5}$$

Hence in view of Eq. (5.1), we see that

$$\frac{d\hat{c}}{dt} = -\frac{\kappa}{2}\hat{c} + \eta(1+i) - g\hat{\sigma}_s, \tag{5.6}$$

where

$$\hat{\sigma}_s = \hat{\sigma}_c + i\hat{\sigma}'_c, \tag{5.7}$$

with s indicating the superposed cavity light modes. In addition, $\hat{\sigma}_c$ and $\hat{\sigma}'_c$ are taken to be

$$\hat{\sigma}_c = |c\rangle\langle a|, \tag{5.8}$$

$$\hat{\sigma}'_c = |c'\rangle\langle a'|. \tag{5.9}$$

Now applying Eq. (5.7), one can readily verify that

$$\hat{\sigma}_s^\dagger\hat{\sigma}_s = \hat{\eta}_a + \hat{\eta}'_a, \tag{5.10}$$

$$\hat{\sigma}_s\hat{\sigma}_s^\dagger = \hat{\eta}_c + \hat{\eta}'_c, \tag{5.11}$$

$$\hat{\sigma}_s^2 = 0, \tag{5.12}$$

with $\hat{\eta}_a$ ($\hat{\eta}'_a$) and $\hat{\eta}_c(\hat{\eta}'_c)$ representing the probability for each atom to be in the upper and bottom levels, respectively.

5.1 Photon statistics

In this section we wish to calculate the mean photon number and the variance of the photon number for the superposed cavity light modes.

5.1.1 The mean photon number

We define the mean photon number for the superposed driven cavity light modes by $\bar{n}_s = \langle \hat{c}^\dagger \hat{c} \rangle$. The steady-state solution of Eq. (5.6) can be written as

$$\hat{c} = \frac{2\eta}{\kappa}(1+i) - \frac{2g}{\kappa}\hat{\sigma}_s. \tag{5.13}$$

Then using Eq. (5.13), we get

$$\langle \hat{c}^\dagger \hat{c} \rangle = \frac{8\eta^2}{\kappa^2} - \frac{4\eta g}{\kappa^2}[(1-i)\langle \hat{\sigma}_s \rangle + (1+i)\langle \hat{\sigma}_s^\dagger \rangle] + \frac{\gamma_c}{\kappa}\langle \hat{\sigma}_s^\dagger \hat{\sigma}_s \rangle, \tag{5.14}$$

so that in view of Eq. (5.10), the mean photon number for the superposed driven cavity light modes can be put in the form

$$\bar{n}_s = \frac{8\eta^2}{\kappa^2} - \frac{4\eta g}{\kappa^2}[(1-i)\langle \hat{\sigma}_s \rangle + (1+i)\langle \hat{\sigma}_s^\dagger \rangle] + \frac{\gamma_c}{\kappa}\langle \hat{\eta}_a + \hat{\eta}_a' \rangle. \tag{5.15}$$

In addition, on taking the expectation value of Eq. (5.7), we see that

$$\langle \hat{\sigma}_s \rangle = \langle \hat{\sigma}_c \rangle + i\langle \hat{\sigma}_c' \rangle \tag{5.16}$$

and the complex conjugate of this equation is

$$\langle \hat{\sigma}_s^\dagger \rangle = \langle \hat{\sigma}_c^\dagger \rangle - i\langle \hat{\sigma}_c'^\dagger \rangle. \tag{5.17}$$

Now on account of Eqs. (5.16) and (5.17), we get

$$\begin{aligned} \bar{n}_s &= \frac{8\eta^2}{\kappa^2} - \frac{4\eta g}{\kappa^2}[\langle \hat{\sigma}_c \rangle + \langle \hat{\sigma}_c^\dagger \rangle + \langle \hat{\sigma}_c' \rangle + \langle \hat{\sigma}_c'^\dagger \rangle - i(\langle \hat{\sigma}_c \rangle - \langle \hat{\sigma}_c^\dagger \rangle) + i(\langle \hat{\sigma}_c' \rangle - \langle \hat{\sigma}_c'^\dagger \rangle)] \\ &\quad + \frac{\gamma_c}{\kappa}(\langle \hat{\eta}_a \rangle + \langle \hat{\eta}_a' \rangle). \end{aligned} \tag{5.18}$$

On applying the fact that $\langle \hat{\sigma}_c \rangle = \langle \hat{\sigma}_c^\dagger \rangle$ and $\langle \hat{\sigma}_c' \rangle = \langle \hat{\sigma}_c'^\dagger \rangle$, Eq. (5.18) can be written as

$$\bar{n}_s = \frac{8\eta^2}{\kappa^2} - \frac{8\eta g}{\kappa^2}[\langle \hat{\sigma}_c \rangle + \langle \hat{\sigma}_c' \rangle] + \frac{\gamma_c}{\kappa}[\langle \hat{\eta}_a \rangle + \langle \hat{\eta}_a' \rangle]. \tag{5.19}$$

We now proceed to determine $\langle\hat{\sigma}_c\rangle$, $\langle\hat{\sigma}'_c\rangle$, $\langle\hat{\eta}_a\rangle$, and $\langle\hat{\eta}'_a\rangle$. Thus on the basis of Eqs. (2.48) and (2.49), we realize that

$$\langle\hat{\sigma}_c\rangle = \frac{\varepsilon(\gamma_c+\gamma)}{3\varepsilon^2+(\gamma_c+\gamma)^2}, \tag{5.20}$$

$$\langle\hat{\sigma}'_c\rangle = \frac{\varepsilon(\gamma_c+\gamma)}{3\varepsilon^2+(\gamma_c+\gamma)^2}, \tag{5.21}$$

$$\langle\hat{\eta}_a\rangle = \frac{\varepsilon^2}{3\varepsilon^2+(\gamma_c+\gamma)^2}, \tag{5.22}$$

$$\langle\hat{\eta}'_a\rangle = \frac{\varepsilon^2}{3\varepsilon^2+(\gamma_c+\gamma)^2}. \tag{5.23}$$

Hence on account of Eqs. (5.20), (5.21), (5.22), and (5.23), along with Eq. (2.39), we readily get

$$\bar{n}_s = 2\left[\frac{4\varepsilon^2}{\kappa\gamma_c} - \frac{\varepsilon^2}{\kappa}\left(\frac{(3\gamma_c+4\gamma)}{3\varepsilon^2+(\gamma_c+\gamma)^2}\right)\right]. \tag{5.24}$$

or

$$\bar{n}_s = 2\bar{n}, \tag{5.25}$$

with $\bar{n}$ representing the mean photon number for one of the driven cavity modes. We see from Eq.(5.25) that the mean photon number of the superposed cavity modes is twice the mean photon number of either of the driven cavity modes.

5.1.2 The photon number variance

We next wish to calculate the photon number variance for the superposed cavity light modes. The photon number variance for the superposed driven cavity modes is expressible as

$$(\Delta n_s)^2 = \langle(\hat{c}^\dagger\hat{c})^2\rangle - \langle\hat{c}^\dagger\hat{c}\rangle^2. \tag{5.26}$$

Applying Eq. (5.13) and its complex conjugate along with Eq. (5.10), we have

$$\hat{c}^\dagger\hat{c} = \frac{8\eta^2}{\kappa^2} - \frac{4\eta g}{\kappa^2}M + \frac{\gamma_c}{\kappa}(\hat{\eta}_a+\hat{\eta}'_a), \tag{5.27}$$

where

$$M = (1-i)\hat{\sigma}_s + (1+i)\hat{\sigma}_s^\dagger. \tag{5.28}$$

Therefore, squaring Eq. (5.27), yields

$$\begin{aligned}(\hat{c}^\dagger\hat{c})^2 &= \frac{64\eta^4}{\kappa^4} - \frac{64\eta^3 g}{\kappa^4}M + \frac{16\eta^2}{\kappa^2}\frac{\gamma_c}{\kappa}(\hat{\eta}_a + \hat{\eta}'_a) + \frac{16\eta^2 g^2}{\kappa^4}M^2 \\ &-\frac{4\eta g}{\kappa^2}\frac{\gamma_c}{\kappa}\Big[M(\hat{\eta}_a + \hat{\eta}'_a) + (\hat{\eta}_a + \hat{\eta}'_a)M\Big] + \Big(\frac{\gamma_c}{\kappa}(\hat{\eta}_a + \hat{\eta}'_a)\Big)^2.\end{aligned} \tag{5.29}$$

We then see that

$$\begin{aligned}\langle(\hat{c}^\dagger\hat{c})^2\rangle &= \frac{64\eta^4}{\kappa^4} - \frac{64\eta^3 g}{\kappa^4}\langle M\rangle + \frac{16\eta^2}{\kappa^2}\frac{\gamma_c}{\kappa}(\langle\hat{\eta}_a + \hat{\eta}'_a\rangle) + \frac{16\eta^2 g^2}{\kappa^4}\langle M^2\rangle \\ &-\frac{4\eta g}{\kappa^2}\frac{\gamma_c}{\kappa}\Big[\langle M(\hat{\eta}_a + \hat{\eta}'_a) + (\hat{\eta}_a + \hat{\eta}'_a)M\rangle\Big] + \Big(\frac{\gamma_c}{\kappa}\Big)^2\langle(\hat{\eta}_a + \hat{\eta}'_a)^2\rangle.\end{aligned} \tag{5.30}$$

With the aid of Eq. (5.28) along with Eqs. (5.8), (5.9), (5.10), (5.11), and (5.16) we readily get

$$\langle M\rangle = \langle\hat{\sigma}_c\rangle + \langle\hat{\sigma}'_c\rangle + \langle\hat{\sigma}_c^\dagger\rangle + \langle\hat{\sigma}_c'^\dagger\rangle, \tag{5.31}$$

$$\langle M^2\rangle = 2\langle\hat{\eta}_a\rangle + 2\langle\hat{\eta}'_a\rangle + 2\langle\hat{\eta}_c\rangle + 2\langle\hat{\eta}'_c\rangle, \tag{5.32}$$

$$\langle M(\hat{\eta}_a + \hat{\eta}'_a) + (\hat{\eta}_a + \hat{\eta}'_a)M\rangle = 2\langle\hat{\sigma}_c\rangle + 2\langle\hat{\sigma}'_c\rangle + 2\langle\hat{\sigma}_c^\dagger\rangle + 2\langle\hat{\sigma}_c'^\dagger\rangle, \tag{5.33}$$

$$\langle(\hat{\eta}_a + \hat{\eta}'_a)^2\rangle = 2\langle\hat{\eta}_a\rangle + 2\langle\hat{\eta}'_a\rangle. \tag{5.34}$$

Upon substituting Eqs. (5.31),(5.32), (5.33), and (5.34) into Eq. (5.30), we find

$$\begin{aligned}\langle(\hat{c}^\dagger\hat{c})^2\rangle &= \frac{64\eta^4}{\kappa^4} - \frac{64\eta^3 g}{\kappa^4}[\langle\hat{\sigma}_c\rangle + \langle\hat{\sigma}'_c\rangle + \langle\hat{\sigma}_c^\dagger\rangle + \langle\hat{\sigma}_c'^\dagger\rangle] \\ &+\frac{16\eta^2}{\kappa^2}\frac{\gamma_c}{\kappa}[\langle\hat{\eta}_a\rangle + \langle\hat{\eta}'_a\rangle] + \frac{32\eta^2 g^2}{\kappa^4}[\langle\hat{\eta}_a\rangle + \langle\hat{\eta}'_a\rangle + \langle\hat{\eta}_c\rangle + \langle\hat{\eta}'_c\rangle] \\ &-\frac{8\eta g}{\kappa^2}\frac{\gamma_c}{\kappa}[\langle\hat{\sigma}_c\rangle + \langle\hat{\sigma}'_c\rangle + \langle\hat{\sigma}_c^\dagger\rangle + \langle\hat{\sigma}_c'^\dagger\rangle] + 2\Big(\frac{\gamma_c}{\kappa}\Big)^2[\langle\hat{\eta}_a\rangle + \langle\hat{\eta}'_a\rangle]\end{aligned} \tag{5.35}$$

and making use of the fact that $\langle\hat{\sigma}_c\rangle = \langle\hat{\sigma}_c^\dagger\rangle$ and $\langle\hat{\sigma}'_c\rangle = \langle\hat{\sigma}_c'^\dagger\rangle$, we obtain

$$\begin{aligned}\langle(\hat{c}^\dagger\hat{c})^2\rangle &= \frac{64\eta^4}{\kappa^4} - \frac{128\eta^3 g}{\kappa^4}[\langle\hat{\sigma}_c\rangle + \langle\hat{\sigma}'_c\rangle] + \frac{16\eta^2}{\kappa^2}\frac{\gamma_c}{\kappa}[\langle\hat{\eta}_a\rangle + \langle\hat{\eta}'_a\rangle] \\ &+\frac{32\eta^2 g^2}{\kappa^4}[\langle\hat{\eta}_a\rangle + \langle\hat{\eta}'_a\rangle + \langle\hat{\eta}_c\rangle + \langle\hat{\eta}'_c\rangle] - \frac{16\eta g}{\kappa^2}\frac{\gamma_c}{\kappa}[\langle\hat{\sigma}_c\rangle + \langle\hat{\sigma}'_c\rangle] \\ &+2\Big(\frac{\gamma_c}{\kappa}\Big)^2[\langle\hat{\eta}_a\rangle + \langle\hat{\eta}'_a\rangle].\end{aligned} \tag{5.36}$$

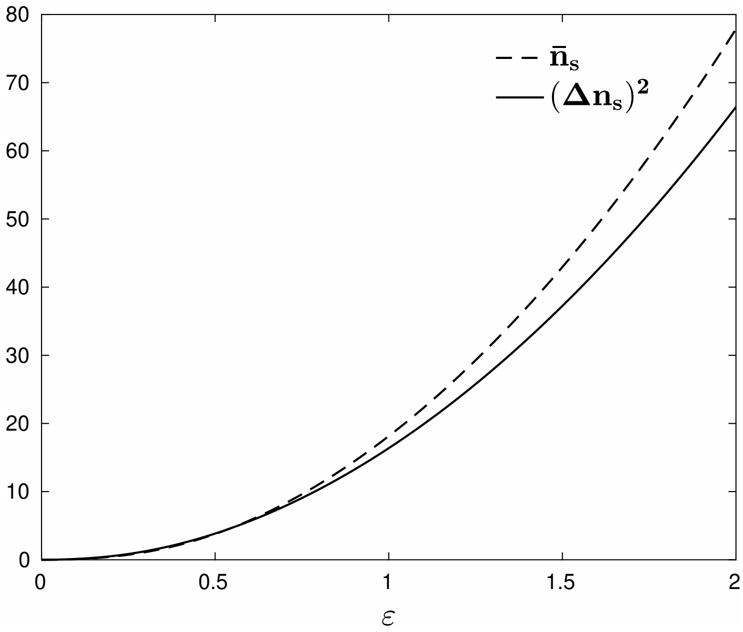

Figure 5.1: Plots of Eqs. (5.24) and (5.40) versus ε for $\gamma_c = 0.5$, $\kappa = 0.8$, and $\gamma = 0.3$.

On the other hand, squaring both sides Eq. (5.19), we have

$$\begin{aligned}\langle\hat{c}^\dagger\hat{c}\rangle^2 &= \frac{64\eta^4}{\kappa^4} - \frac{128\eta^3 g}{\kappa^4}[\langle\hat{\sigma}_c\rangle + \langle\hat{\sigma}'_c\rangle] + \frac{16\eta^2}{\kappa^2}\frac{\gamma_c}{\kappa}[\langle\hat{\eta}_a\rangle + \langle\hat{\eta}'_a\rangle] \\ &+ \frac{64\eta^2 g^2}{\kappa^4}[\langle\hat{\sigma}_c\rangle + \langle\hat{\sigma}'_c\rangle]^2 - \frac{16\eta g}{\kappa^2}\frac{\gamma_c}{\kappa}[\langle\hat{\eta}_a\rangle + \langle\hat{\eta}'_a\rangle][\langle\hat{\sigma}_c\rangle + \langle\hat{\sigma}'_c\rangle] \\ &+ \left(\frac{\gamma_c}{\kappa}\right)^2[\langle\hat{\eta}_a\rangle + \langle\hat{\eta}'_a\rangle]^2. \end{aligned} \tag{5.37}$$

Upon introducing Eqs. (5.36) and (5.37) into Eq. (5.26), we obtain

$$\begin{aligned}(\Delta n_s)^2 &= \frac{32\eta^2 g^2}{\kappa^4}[\langle\hat{\eta}_a\rangle + \langle\hat{\eta}'_a\rangle + \langle\hat{\eta}_c\rangle + \langle\hat{\eta}'_c\rangle] - \frac{16\eta g}{\kappa^2}\frac{\gamma_c}{\kappa}[\langle\hat{\sigma}_c\rangle + \langle\hat{\sigma}'_c\rangle] \\ &+ \frac{16\eta g}{\kappa^2}\frac{\gamma_c}{\kappa}[\langle\hat{\eta}_a\rangle + \langle\hat{\eta}'_a\rangle][\langle\hat{\sigma}_c\rangle + \langle\hat{\sigma}'_c\rangle] - \frac{64\eta^2 g^2}{\kappa^4}[\langle\hat{\sigma}_c\rangle + \langle\hat{\sigma}'_c\rangle]^2 \\ &+ 2\left(\frac{\gamma_c}{\kappa}\right)^2[\langle\hat{\eta}_a\rangle + \langle\hat{\eta}'_a\rangle] - \left(\frac{\gamma_c}{\kappa}\right)^2[\langle\hat{\eta}_a\rangle + \langle\hat{\eta}'_a\rangle]^2 \end{aligned} \tag{5.38}$$

Hence in view of Eqs. (5.20), (5.21), (5.22), (5.23), and (2.50), the variance of the photon number for the superposed driven cavity light modes takes the form

$$\begin{aligned}(\Delta n_s)^2 &= \left[\frac{32\eta^2 g^2(4\varepsilon^2 + 2(\gamma_c+\gamma)^2) - 32\eta g\kappa\varepsilon\gamma_c(\gamma_c+\gamma) + 4\kappa^2\gamma_c^2\varepsilon^2}{\kappa^4(3\varepsilon^2 + (\gamma_c+\gamma)^2)}\right] \\ &+ \left[\frac{64\eta g\gamma_c\varepsilon^3(\gamma_c+\gamma) - 64\eta^2 g^2\varepsilon^2(\gamma_c+\gamma)^2 - 4\kappa^2\gamma_c^2\varepsilon^4}{\kappa^4(3\varepsilon^2 + (\gamma_c+\gamma)^2)^2}\right]. \end{aligned} \tag{5.39}$$

Employing Eq. (2.39), one can write Eq. (5.39) as

$$(\Delta n_s)^2 = \left[\frac{4\varepsilon^2[4(2\varepsilon^2+(\gamma_c+\gamma)^2)-4\gamma_c\gamma-3\gamma_c^2]}{\kappa^2(3\varepsilon^2+(\gamma_c+\gamma)^2)}\right] + \left[\frac{4\varepsilon^4[(8-\kappa)\gamma_c^2-4\kappa(\gamma_c+\gamma)^2+8\gamma_c\gamma]}{\kappa^3(3\varepsilon^2+(\gamma_c+\gamma)^2)^2}\right]. \tag{5.40}$$

The plots in Figure 5.1 show that for $\varepsilon \geq 0.7$ the photon statistics for the superposed cavity light modes is subPoissonian.

5.2 Quadrature variance

The squeezing properties of the superposed driven cavity light modes are described by two quadrature operators defined by

$$\hat{c}_+ = \hat{c}^\dagger + \hat{c}, \tag{5.41}$$

$$\hat{c}_- = i(\hat{c}^\dagger - \hat{c}), \tag{5.42}$$

where $\hat{c}_+$ and $\hat{c}_-$ are Hermitian conjugate operators which represent physical quantities called the plus and the minus quadratures. It can be readily established that

$$[\hat{c}_-, \hat{c}_+] = -2i[\hat{c}, \hat{c}^\dagger]. \tag{5.43}$$

Applying Eq. (5.13), one can easily verified that

$$[\hat{c}, \hat{c}^\dagger] = \frac{\gamma_c}{\kappa}(\hat{\eta}_c + \hat{\eta}_c' - \hat{\eta}_a - \hat{\eta}_a'). \tag{5.44}$$

We then see that

$$[\hat{c}_-, \hat{c}_+] = 2i\frac{\gamma_c}{\kappa}[(\hat{\eta}_a - \hat{\eta}_c) + (\hat{\eta}_a' - \hat{\eta}_c')]. \tag{5.45}$$

In view of Eq. (4.4), one can write that

$$\Delta c_- \Delta c_+ \geq \frac{\gamma_c}{\kappa}\left|\langle\hat{\eta}_a\rangle - \langle\hat{\eta}_c\rangle + \langle\hat{\eta}_a'\rangle - \langle\hat{\eta}_c'\rangle\right|. \tag{5.46}$$

Now on account of Eqs. (2.50), (5.22), and (5.23), we readily obtain

$$\Delta c_{-}\Delta c_{+} \geq f_c(\varepsilon), \tag{5.47}$$

where

$$f_c(\varepsilon) = \frac{\gamma_c}{\kappa}\left(\frac{2(\gamma_c+\gamma)^2}{3\varepsilon^2+(\gamma_c+\gamma)^2}\right). \tag{5.48}$$

We note that Eq. (5.47) represents the uncertainty relation for the plus and the minus quadratures.

The quadrature variances of the plus and the minus quadrature operators of the superposed cavity light modes are expressible as

$$(\Delta c_{+})^2 = \langle\hat{c}^\dagger\hat{c}\rangle + \langle\hat{c}\hat{c}^\dagger\rangle + \langle\hat{c}^{\dagger 2}\rangle + \langle\hat{c}^2\rangle - \langle\hat{c}^\dagger\rangle^2 - \langle\hat{c}\rangle^2 - 2\langle\hat{c}^\dagger\rangle\langle\hat{c}\rangle \tag{5.49}$$

and

$$(\Delta c_{-})^2 = \langle\hat{c}^\dagger\hat{c}\rangle + \langle\hat{c}\hat{c}^\dagger\rangle - \langle\hat{c}^{\dagger 2}\rangle - \langle\hat{c}^2\rangle + \langle\hat{c}^\dagger\rangle^2 + \langle\hat{c}\rangle^2 - 2\langle\hat{c}^\dagger\rangle\langle\hat{c}\rangle. \tag{5.50}$$

Now with the aid of Eq. (5.13), one can readily establish that

$$\langle\hat{c}\hat{c}^\dagger\rangle = \frac{8\eta^2}{\kappa} - \frac{8\eta g}{\kappa^2}[\langle\sigma_c\rangle + \langle\hat{\sigma}_c'\rangle] + \frac{\gamma_c}{\kappa}[\langle\eta_c\rangle + \langle\eta_c'\rangle], \tag{5.51}$$

$$\langle\hat{c}^2\rangle = \frac{8\eta^2 i}{\kappa^2} - \frac{i8\eta g}{\kappa^2}[\langle\sigma_c\rangle + \langle\hat{\sigma}_c'\rangle], \tag{5.52}$$

$$\langle\hat{c}\rangle^2 = \frac{8\eta^2 i}{\kappa^2} - \frac{i8\eta g}{\kappa^2}[\langle\sigma_c\rangle + \langle\hat{\sigma}_c'\rangle] + 2i\frac{\gamma_c}{\kappa}\langle\hat{\sigma}_c\rangle\langle\hat{\sigma}_c'\rangle, \tag{5.53}$$

$$\langle\hat{c}^\dagger\rangle\langle\hat{c}\rangle = \frac{8\eta^2}{\kappa^2} - \frac{8\eta g}{\kappa^2}[\langle\sigma_c\rangle + \langle\hat{\sigma}_c'\rangle] + \frac{\gamma_c}{\kappa}[\langle\hat{\sigma}_c\rangle^2 + \langle\hat{\sigma}_c'\rangle^2]. \tag{5.54}$$

On account of Eqs. (5.19), (5.51), (5.52), (5.53), and (5.54), we find

$$(\Delta c_{+})^2 = \frac{\gamma_c}{\kappa}[\langle\hat{\eta}_a\rangle + \langle\hat{\eta}_a'\rangle + \langle\eta_c\rangle + \langle\eta_c'\rangle] - 2\frac{\gamma_c}{\kappa}[\langle\sigma_c\rangle^2 + \langle\hat{\sigma}_c'\rangle^2], \tag{5.55}$$

$$(\Delta c_{-})^2 = \frac{\gamma_c}{\kappa}[\langle\hat{\eta}_a\rangle + \langle\hat{\eta}_a'\rangle + \langle\eta_c\rangle + \langle\eta_c'\rangle] - 2\frac{\gamma_c}{\kappa}[\langle\sigma_c\rangle^2 + \langle\hat{\sigma}_c'\rangle^2]. \tag{5.56}$$

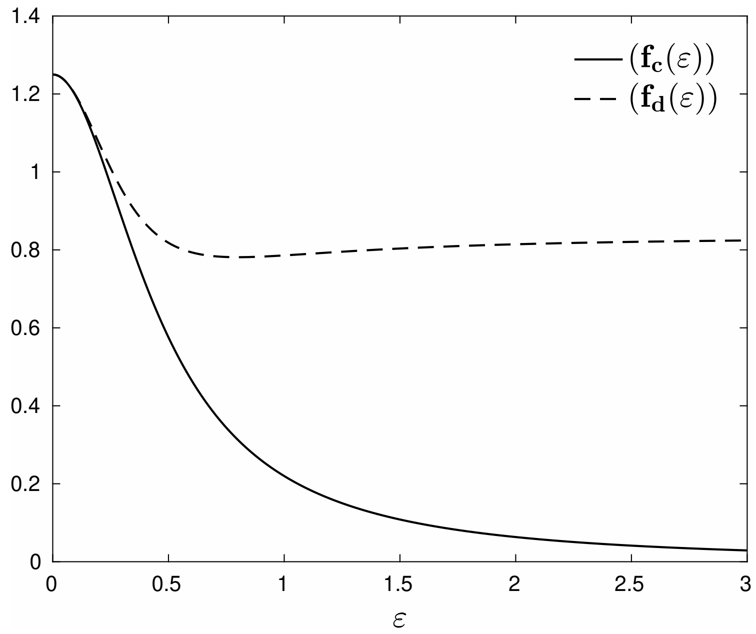

Figure 5.2: Plots of Eqs. (5.48) and (5.63) versus ε for $\gamma_c = 0.5$, $\kappa = 0.8$, and $\gamma = 0.3$.

Hence on substituting Eqs. (2.50), (5.20), (5.21), (5.22), (5.23) into Eqs. (5.55) and (5.56), the plus and the minus quadrature variances are found to be

$$(\Delta c_+)^2 = \frac{2\gamma_c}{\kappa}\left[1 - \left(\frac{3\varepsilon^4 + 3\varepsilon^2(\gamma_c+\gamma)^2}{(3\varepsilon^2 + (\gamma_c+\gamma)^2)^2}\right)\right], \tag{5.57}$$

$$(\Delta c_-)^2 = \frac{2\gamma_c}{\kappa}\left[1 - \left(\frac{3\varepsilon^4 + 3\varepsilon^2(\gamma_c+\gamma)^2}{(3\varepsilon^2 + (\gamma_c+\gamma)^2)^2}\right)\right]. \tag{5.58}$$

We note that the quadrature variances of the plus and the minus quadratures are equal. Now upon setting $\varepsilon = 0$ in Eqs. (5.57) and (5.58), the quadrature variance for a vacuum state turns out to be

$$(\Delta c_+)^2_v = (\Delta c_-)^2_v = 2\frac{\gamma_c}{\kappa}. \tag{5.59}$$

Furthermore, in view of Eqs. (5.55) and (5.56), we see that

$$\Delta c_+ = \sqrt{\frac{2\gamma_c}{\kappa}\left[1 - \left(\frac{3\varepsilon^4 + 3\varepsilon^2(\gamma_c+\gamma)^2}{(3\varepsilon^2 + (\gamma_c+\gamma)^2)^2}\right)\right]}, \tag{5.60}$$

$$\Delta c_- = \sqrt{\frac{2\gamma_c}{\kappa}\left[1 - \left(\frac{3\varepsilon^4 + 3\varepsilon^2(\gamma_c+\gamma)^2}{(3\varepsilon^2 + (\gamma_c+\gamma)^2)^2}\right)\right]}. \tag{5.61}$$

It then follows

$$\Delta c_{-}\Delta c_{+} = f_d(\varepsilon), \tag{5.62}$$

where

$$f_d(\varepsilon) = \frac{2\gamma_c}{\kappa}\left[1 - \left(\frac{3\varepsilon^4 + 3\varepsilon^2(\gamma_c+\gamma)^2}{(3\varepsilon^2 + (\gamma_c+\gamma)^2)^2}\right)\right]. \tag{5.63}$$

Comparison of Eq. (5.57) or (5.58) with Eq. (5.59) shows that the superposed cavity light modes are in a squeezed state and the squeezing occurs in both the plus and the minus quadratures. In addition, we note from the plots in Figure 5.2 that $f_d(\varepsilon) = f_c(\varepsilon)$ for ε $0 < \varepsilon < 0.08$ and $f_d(\varepsilon) > f_c(\varepsilon)$ for $\varepsilon > 0.08$. On the basis of these results, we observe that the quadrature operators satisfy the uncertainty relation.

5.3 Quadrature squeezing

The quadrature squeezing for the superposed driven cavity light modes is defined relative to the quadrature variance of the vacuum state as

$$S_{sup\pm} = \frac{(\Delta c_{\pm})^2_v - (\Delta c_{\pm})^2}{(\Delta c_{\pm})^2_v} \tag{5.64}$$

Now on account of Eqs. (5.57), (5.58), and (5.59) the plus and the minus quadrature squeezing take the form

$$S_{sup+} = 1 - \left[\frac{6\varepsilon^4 + (\gamma_c+\gamma)^2(3\varepsilon^2 + (\gamma_c+\gamma)^2)}{(3\varepsilon^2 + (\gamma_c+\gamma)^2)^2}\right], \tag{5.65}$$

$$S_{sup-} = 1 - \left[\frac{6\varepsilon^4 + (\gamma_c+\gamma)^2(3\varepsilon^2 + (\gamma_c+\gamma)^2)}{(3\varepsilon^2 + (\gamma_c+\gamma)^2)^2}\right]. \tag{5.66}$$

We note from Eqs. (5.65) and (5.66) that the global squeezing in the plus and minus quadratures are equal. Thus in view of Eqs. (4.22), (4.23), one can write

$$S_{sup+} = \frac{S_+ + S_-}{2}, \tag{5.67}$$

$$S_{sup-} = \frac{S_+ + S_-}{2}. \tag{5.68}$$

The amount of squeezing in the plus or the minus quadrature for the superposed driven cavity modes is the average of the squeezing in the plus and the minus quadratures of either driven cavity mode.

6

Conclusion

We have studied the quantum properties of the cavity mode driven by coherent light and interacting with a three-level atom available in an open cavity and coupled to a vacuum reservoir via a single-port mirror. We have carried out our analysis by putting the noise operators associated with the vacuum reservoir in normal order and by taking into consideration the interaction of the three-level atom with the vacuum reservoir outside the cavity. Employing the pertinent master equation, we have obtained the equations of evolution for the expectation values of the atomic operators that involve the cavity mode operators. Then applying the large-time approximation scheme to the quantum Langevin equations of the cavity mode operators, we have decoupled the equations of evolution for the expectation values of the atomic operators. Applying the steady-state solutions of the equations of evolution of the cavity mode operators and the atomic operators, we have calculated the global mean and variances of the photon number for the light modes emitted from the top and the intermediate levels and for the driven cavity mode. In addition, we have determined the local mean photon number for the driven cavity mode and for the two-mode cavity light. Moreover, we have obtained the global quadrature squeezing for the two-mode cavity light and for the driven cavity mode. Finally, we have calculated the local quadrature squeezing for the driven cavity mode. We have seen that the driven cavity mode exhibits subPoissonian photon statistics. We have also noted that the presence of spontaneous emission leads to a decrease of the mean photon number and the photon number variance. In addition, we have established that the local mean photon number for the driven cavity mode as well as the two-mode cavity light approaches the global mean photon number in the absence or presence of spontaneous emission as the frequency increases.

Moreover, we have shown that the driven cavity mode is in a squeezed state and the squeezing occurs in both the plus and the minus quadratures with the maximum squeezing for the plus quadrature being 52.08%below the vacuum state level in the presence or absence of spontaneous emission and occurs at $\varepsilon = 0.6$ and $\varepsilon = 0.37$, respectively. And the maximum squeezing for the minus quadrature is 33.32% and occurs for $\varepsilon \geq 15$ below the vacuum state level in the presence or absence of spontaneous emission. We consider the discovery of squeezing in both the plus and the minus quadratures to be the single most important result of this PhD dissertation. On the other hand, the two-mode cavity light is in a squeezed state and the squeezing occurs in the minus quadrature with the maximum squeezing being 43.42% below the vacuum state level and occurs at $\varepsilon = 0.22$ in the absence of spontaneous emission and at $\varepsilon = 0.35$ in the presence of spontaneous emission. Moreover, we have found that the maximum local quadrature squeezing for the driven cavity mode is 65.32% (absence of spontaneous emission) and 62.11% (presence of spontaneous emission), and both occur in the frequency interval $\lambda_{\pm} = 0.01$.

Furthermore, applying the steady-state solution of the quantum Langevin equation of a pair of superposed driven cavity modes, we have calculated the mean and variance of the photon number as well as the quadrature squeezing. We have found that the mean photon number of the superposed driven cavity mode is twice that of one of the driven cavity modes. In addition, our result shows that the amount of squeezing in the plus or the minus quadrature of the superposed driven cavity modes is the average of the squeezing in the plus and the minus quadratures of either driven cavity mode.

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