MULTI- OBJECTIVE QUADRATIC PROGRAMMING PROBLEM: A PRIORITY BASED FUZZY GOAL PROGRAMMING

ADDIS ABABA UNIVERSITY
COLLEGE OF COMPUTATIONAL AND NATURAL SCIENCE DEPARTMENT OF MATHEMATICS

In partial fulfillment of the requirement of the degree of Master of Science in mathematics
By: Amsalu Ayeru
Stream: Optimization
Advisor: Semu Mitiku(PhD)

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Contents

ABSTRACT

This project presents priority based fuzzy goal programmingapproach to multi-objective quadratic programming problem. Inthe proposed approach, we construct the quadratic membershipfunctions by determining the individual best solution of theobjective functions subject to the system constraints. Thequadratic membership functions are then transformed intoequivalent linear membership functions at the individual bestsolution point by first order Taylor series approximation. Thenfuzzy goal programming approach is used for achieving highestdegree of each of the membership goals by minimizing negativedeviational variables. Then, sensitivity analysis with thevariations of the priority structure is performed to identify themost appropriate priority structure in the decision-makingcontext by using distance function. A numerical example issolved in order to show the efficiency of the proposed approach.

Keywords:Goal programming, Multi-objective quadratic programming,Priority based fuzzy goal programming, Quadraticprogramming.

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NOTATION

- *R the set of real numbers*
- S is the implicit set of constraints
- transpose of vector x
- partial ordering
- >>much greater than in ordering
- be a feasible solution of the objective function
- summation notation
- multi objective function
- membership function
- equivalence relation
- the sub differential of the function
- negative and positive deviational variables respectively

INTRODUCTION

Multi-objective optimization (also known as multi-objective programming, vector optimization, multicriteria optimization, multiattribute optimization or Pareto optimization) is an area of multiple criteria decision making, that is concerned with mathematical optimization problems involving more than one objective function to be optimized simultaneously. Multi-objective optimization has been applied in many fields of science, including engineering, economics and logistics where optimal decisions need to be taken in the presence of trade-offs between two or more conflicting objectives.For a nontrivial multi-objective optimization problem, there does not exist a single solution that simultaneously optimizes each objective. In that case, the objective functions are said to be conflicting, and there exists a (possibly infinite) number of Pareto optimal solutions. A solution is called nondominated, Pareto optimal, Pareto efficient or noninferior, if none of the objective functions can be improved in value without degrading some of the other objective values. Without additional subjective preference information, all Pareto optimal solutions are considered equally good (as vectors cannot be ordered completely).

Quadratic programming is a special type of mathematical optimizationproblem. It is the problem of optimizing (minimizing or maximizing) a quadratic function of several variables subject to linear constraints on these variables.The quadratic programming problem can be formulated as follows.Suppose n is a positive integer representing the number of variables and m is a positive integer representing the number of constraints. Suppose c is an n-dimensional vector, $D$is an nxn real symmetric matrix, $A$is an mxnreal matrix, and b is an m-dimensional real vector.The quadratic programming problem is

$$Min/max = + \frac{1}{2}$$

Subject to

Ax b and x 0

Where denotes the vector transpose of*x*. The notation *Ax* *b* means that every entry of the vector *Ax*is less than or equal to the corresponding entry of the vector*b*.The combination of Multi-objective optimization and quadratic programming problem is Multi-objective quadratic programming problem. Multi-objective quadratic programming problem consists of a decision-making unit with multiple objectives. Here, the objective functions are quadratic in nature and the system constraints are assumed to be linear functions. In this project, we have considered priority based fuzzy goal programming approach for solving multi-objective quadratic programming problem.

This project consists of three chapters

*Chapter 1*we present the basic concept and definition of multi-objective optimization and methods that helps to solve multi-objective optimization. In addition to this we describe convex, concave,linear and non linear multi-objective optimization.

*Chapter 2*describes about formulation of Multi-objective quadratic programming problem and how to solve a scalar quadratic programming problem.

In Chapter 3 We present formulation of fuzzy goal programming approach for solving multi-objective quadratic programming problem and explain linearization of membership functions by first order Taylor series approximation. The results of chapter 3 are as follows:

(i) a priority based fuzzy goal programming approach to Multi-objective quadratic programming problem is presented.

(ii) Transform the quadratic membership functions into an equivalent linear membership functions by first order Taylor series approximation at the individual best solution point and priority based fuzzy goal programming is used to solve the transformed multi-objective quadratic programming problem.

(iii) Sensitivity analysis with the variations of the priority structure is performed and distance function is used to identify the most appropriate priority structure in the decision-making context.

It is well known that the priority based goal programming is one of the powerful techniques in the field of multi criteria decision-making problems with multiple and conflicting objectives. Ijiri [1] introduced priority based goal programming at first in 1965. Ignizio [2], Lee [3], Steuer [4] and other researchers developed priority based goal programming and they successfully applied goal programming to various real-life decision-making problems. In the proposed priority based goal programming, the ranking of goals are grouped according to their Priorities for achieving the respective aspiration levels in the decision-making context. The goals, which are considered equally important, belong to the same priority level.

The goals at the first priority level are considered to be infinitely more important than the goals at the second priority level. The goals at the second priority level are considered to be infinitely more important than the goals at the third priority level and so on. Since the goals are generally conflicting in nature, differential weights are assigned to their relative importance for achieving the respective desired values. Multi-objective quadratic programming problem is a special case of nonlinear programming problem. However, Multi-objective quadratic programming problem has not been studied extensively in the literature. Korhonen and Yu [5] discussed a reference direction approach to multiple objective quadratic – linear programming problems. Ammar and Khalifa [6] studied quadratic programming to fuzzy portfolio optimization problems. Ammar [7] presented fuzzy random multi-objective quadratic programming with applications in portfolio problem. In this project, we have transformed multi-objective quadratic programming problem into multi-objective linear programming problem by first order Taylor series approximation. Then, fuzzy goal programming approach due to Pramanik and Roy [8, 9] is used for achieving highest degree of eachof membership goals by minimizing negative deviational variables. In the priority based fuzzy goal programming solution approach, the goals at the first priority level are considered first for achievement of their aspiration levels according to their relative importance of the weights at that priority level. Then, the achievement of goals of second priority level is taken into consideration and the process will continue until the last priority level is considered. In the solution process, sensitivity analysis with the variations of priority structure of the goals is performed to identify the most appropriate priority structure by using distance function.

CHAPTER 1

BASIC CONCEPTS AND DEFINITION

1.1 Multi-objective Optimization

A basic single-objective optimization problem can be formulated as follows:

Where f is a scalar function and S is the (implicit) set of constraints that can be defined as

Multi-objective optimization can be described in mathematical terms as follows:

Min/max ....(1.1)

A variable x is a vector of m decision variables: are called bounds restricting each decision to take a value with a lower and an upper bound. This bound constitute a decision variable space or simply the decision space. The term and)are constraint functions.Any satisfies all the constraints and a variable bound is known as feasible solution otherwise it is called infeasible solution.

Multi-objective optimization problem is referred as vector optimization because a vector objective instead of a single objective is optimized. If all objective function and constraints are linear then the multi objective optimization problem is called multi objective linear program. Thus the objective functions are

Where**.** Theconstraints are similarly written in matrix form and as equality constraints. As usual in linear programming we restrict the variables to the non negative orthant of a multi objective linear program is given by

Min

With a *pxn* objective or criteria matrix consisting of the rows, the feasible set in decision space is defined by the $m \times n$ constraint matrix and the right hand side vector.

The feasible set in decision space is

The space in which the objective vector belongs is called the objective space, and the image of the feasible set under *F*is called the attained set. Such a set will be denoted in the following with

Minimizing a vector valued objective function needs some more explanation since there is no canonical ordering defined in for *2*. Throughout this project we use the pareto concept of optimality for multi objective linear programs which is based on the following two

binary relations and . Let $y^1, y^2 \in$.Then

$y^1 y^2$,
$y^1 y^2$,

a point $y^2 \in$is called dominated by$y^1 \in$if $y^1 y^2$.

Definition: let X be a feasible solution of multi objective linear program and let = c

1. is called weak efficient if there is no such that is weakly non dominated.
2. is called efficient if there is no such that is non dominated.

3. is called properly efficient and if there exist a real number such that for all and with there is an index andsuch thatand

1.1.1 LINEAR AND NON LINEAR MULTI-OBJECTIVE OPTIMIZATION

If all objective function and constraints function are linear the resulting multi objective optimization problem is called a multi objective linear program like a linear programming problems, multi objective optimization problems have many theoretical properties. However if any of the objective or constraints are non linear the resulting problem are non linear problems the solution techniques often do not have convergence proofs since most real world multi objective optimization problems are non linear in nature.

1.1.2 CONVEX AND NON CONVEX FUNCTIONS

Before we discuss a convex multi objective optimization problem let us first define a convex set and a convex function.

Definition:1.1 A set is convex if the line segment between any two points in lies in,

i.e A set is called a convex set if and only ifand for all it holds that

*Definition:1.2*A functionwhere S is a non empty convex set is a convex function if

for all and for all *[0,1]*

the above definition gives rise to the following properties of convex function

1. the linear approximation of at any point in the interval always underestimates the actual function value
2. the Hessian matrix of is positive semi definite for all
3. For a convex function a local minimum is always a global minimum.

Equation with a sign is instead of sign is called a concave function. To test if the function is convex within the interval the Hessian matrix is calculated and checked for its positive definiteness at all points in the interval. One way to check the positive definiteness of a matrix is to compute the eigenvalues of a matrix and check to see if all eigenvalues are positive. To test if the function is non convex in an interval the Hessian matrix is checked for its positive definiteness. If it is not positive semi definite the function f is non convex.

1.1.3 METHODS TO SOLVE MULTI OBJECTIVE OPTIMIZATION

Approximating methods can have different goals: representing the solution set when the latter is numerically available (for convex multi-objective problems); approximating the solution set when some but not all the Pareto curve is numerically available; approximating the solution set when the whole efficient set is not numerically available (for discrete multi-objective problems).

We have different approach to solve multi-objective optimization problem some of them are Scalarization technique, ε -constraint method, Multi level programming, Goal programming.

THE SCALARIZATION TECHNIQUE

A multi-objective problem is often solved by combining its multiple objectives into one single-objective scalar function. This approach is in general known as the weighted-sum or scalarization method.

that represents a new optimization problem with a unique objective function. We denote It can be proved that the minimizer of this single-objective function =is an efficient solution for equation (1.1), i.e., its image belongs to the Pareto curve. In particular, we can say that if the weight vector is strictly greater than zero, then the minimizer is a strict Pareto optimum, while in the case of at least one

The minimizer is a weak Pareto optimum. Let us denote by. There is not an a-priori correspondence between a weight vector and a solution vector; it is up to the decision maker to choose appropriate weights, noting that weighting coefficients do not necessarily correspond directly to the relative importance of the objective functions. Besides the fact that the decision maker cannot be aware of which weights are the most appropriate to retrieve a satisfactory solution, he/she does not know in general how to change weights to consistently change the solution. This means also that it is not easy to develop heuristic algorithms that, starting from certain weights, are able to define iteratively weight vectors to reach a certain portion of the Pareto curve. Since setting a weight vector conducts to only one point on the Pareto curve, performing several optimizations with different weight values can produce a considerable computational burden; therefore, the decision maker needs to choose which different weight combinations have to be considered to reproduce a representative part of the Pareto front. Besides this possibly huge computation time, the scalarization method has two technical shortcomings, as explained in the following.

- The relationship between the objective function weights and the Pareto curve is such that a uniform spread of weight parameters, in general, does not produce a uniform spread of points on the Pareto curve. What can be observed about this fact is that all the points are grouped in certain parts of the Pareto front, while some (possibly significative) portions of the trade-off curve have not been produced.
- Non-convex parts of the Pareto set cannot be reached by minimizing convex combinations of the objective functions. An example can be made showing a geometrical interpretation of the weighted-sum method in two dimensions, i.e., when $n = 2$. In the two-dimensional space the objective function is a line

Where

= +

The minimization of $rf(x)$in the weight-sum approach can be interpreted as the attempt to find the y value for which, starting from the origin point, the line with slope is tangent to the region C. Obviously, changing the weight parameters leads to possibly different touching points of the line to the feasible region. If the Pareto curve is convex then there is room to calculate such points for different r vectors (see Fig.1).

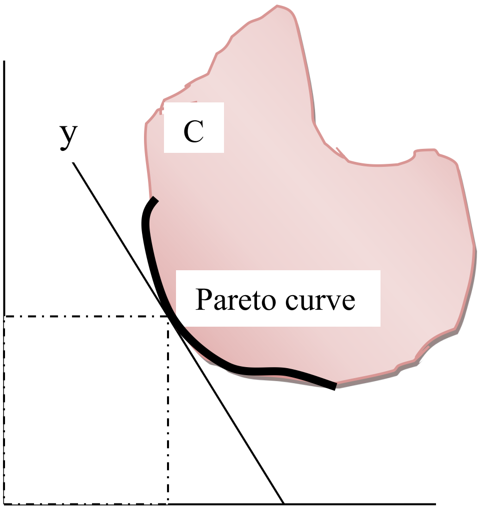

Fig.1Geometrical representation of the weight-sum approach in the convex Pareto curve case.

On the contrary, when the curve is non-convex, there is a set of points that cannot be reached for any combinations of the r weight vector (see Fig. 2)

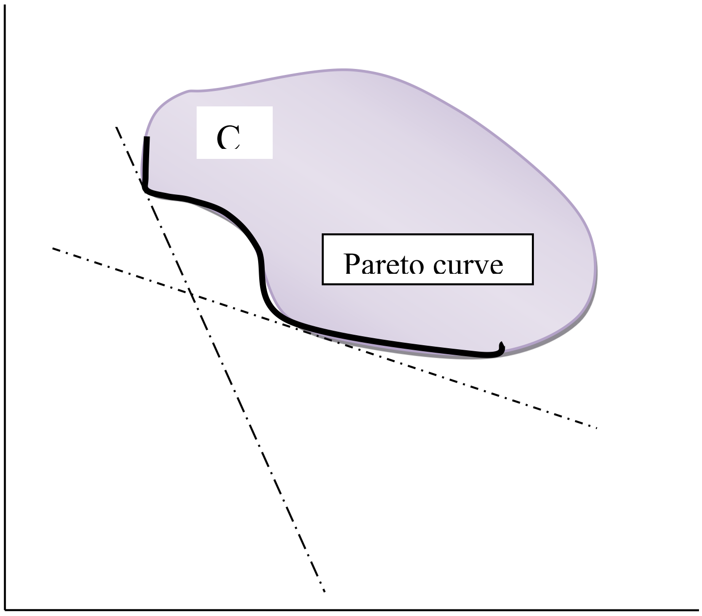

Fig. 2Geometrical representation of the weight-sum approach in the non-convex Pareto curve case

The following result by Geoffrion(1968) [15] states a necessary and sufficient condition in the case of convexity as follows:

Preposition:1.1 If the solution set *S* is convex and the *n*objectivesare

convex on *S,* is a strict Pareto optimum if and only if there exists *r* ∈,

such that is an optimal solution of problem *(r).*

Similarly: If the solution set S is convex and the n objectives are

conve*x on S,* is a weak Pareto optimum if and only if there exists *r* ∈*,*

such that is an optimal solution of problem *(r)*.If the convexity hypothesis does not hold, then only the necessary condition remainsvalid, i.e., the optimal solutions of *(r)*and *(r)*are strict and weak Paretooptima, respectively

- CONSTRAINTS METHOD

Besides the scalarization approach, another solution technique to multi-objective optimization is the ε -constraints method proposed by Chankong and Haimes in 1983[15]. Here, the decision maker chooses one objective out of n to be minimized; the remaining objectives are constrained to be less than or equal to given target values. In mathematical terms, if we let be the objective function chosen to be minimized, we have the following problem *p():*

Min

$, \forall i \in \{1, . . . ,n\}\backslash\{2\}$

$x \in S.$

We note that this formulation of the ε -constraints method can be derived by a more general result by Miettinen[15], that in 1994 proved that: If an objective *j* and a vector *ε=(,...,,, ...,)∈*exist, such thatis an

optimal solution to the following problem *p(ε):*
Min

, ∀i ∈ {1, . . . ,n}\{ j}

x ∈S,

then is a weak Pareto optimum. In turn, the Miettinen theorem is derived from a more general theorem by Yu (1974)[15]stating that:is a strict Pareto optimum if and only if for each objective j, with *j* = *1,..., n,*

there exists a vector *ε* = *(, . . . ,,, . . . ,)∈*such that *f()*is the unique

objective vector corresponding to the optimal solution to problem *p(ε).*Note that the Miettinen theorem is an easy implementable version of the resultby Yu (1974)[15]. Indeed, one of the difficulties of the result by Yu, stems from theuniqueness constraint. The weaker result by Miettinen[15] allows one to use a necessarycondition to calculate weak Pareto optima independently from the uniqueness of theoptimal solutions.

However, if the set S and the objectives are convex this resultbecomes a necessary and sufficient condition for weak Pareto optima. When, as inproblem $p()$, the objective is fixed, on the one hand, we have a more simplifiedversion, and therefore a version that can be more easily implemented in automateddecision-support systems; on the other hand, however, we cannot say that in thepresence of S convex and

convex, $\forall i = 1,\ldots ,n$, all the set of weak Pareto optimacan be calculated

by varying the ε vector.One advantage of the ε -constraints method is that it is able to achieve efficientpoints in a non-convex Pareto curve. For instance, assume we have two objective functions where objective function $(x)$is chosen to be minimized, i.e., the problem is

Min

$x \in S.$

we can be in the situation depicted in Fig. 3 where, when , is an efficient point of the non-convex Pareto curve.

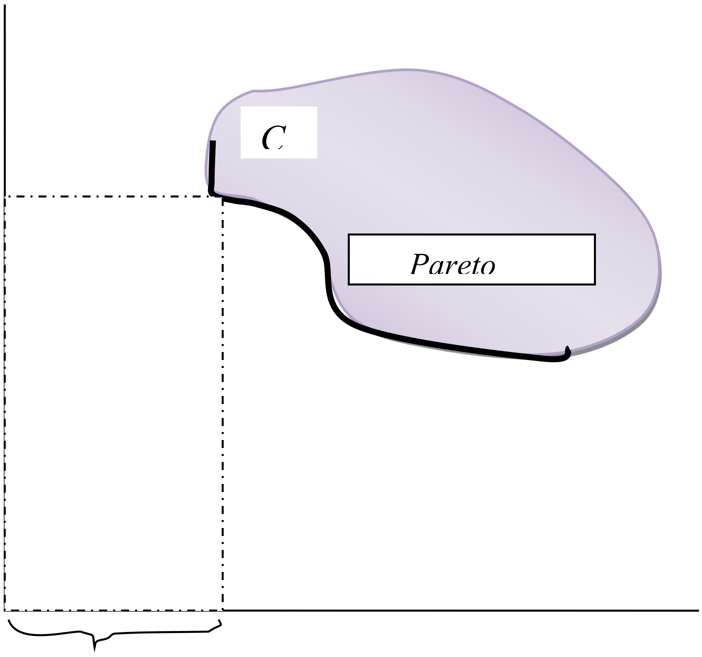

Fig. 3Geometrical representation of the ε -constraints approach in the non-convex Pareto curve case

Therefore, as proposed in Steurer (1986)[4] the decision maker can vary the upper bounds to obtain weak Pareto optima. Clearly, this is also a drawback of this method, i.e., the decision maker has to choose appropriate upper bounds for the constraints, i.e., the values. Moreover, the method is not particularly efficient if the number of the objective functions is greater than two. For these reasons, Erghott and Rusika in 2005[15], proposed two modifications to improve this method, with particular attention to the computational difficulties that the method generates.

MULTI-LEVEL PROGRAMMING

Multi-level programming is another approach to multi-objective optimization and aims to find one optimal point in the entire Pareto surface. Multi-level programming orders the n objectives according to a hierarchy. Firstly, the minimizers of the first objective function are found; secondly, the minimizers of the second most important objective are searched for, and so forth until all the objective function have been optimized on successively smaller sets.

Multi-level programming is a useful approach if the hierarchical order among the objectives is meaningful and the user is not interested in the continuous tradeoff among the functions. One drawback is those optimizations problems that are solved near the end of the hierarchy can be largely constrained and could become infeasible, meaning that the less important objective functions tend to have no influence on the overall optimal solution.

Bi-level programming is the scenario in which $n=2$ and has received several attention, also for the numerous applications in which it is involved. In a bi-level mathematical program one is concerned with two optimization problems where the feasible region of the first problem, called the upper-level (or leader) problem, is determined by the knowledge of the other optimization problem, called the lower-level (or follower) problem. Problems that naturally can be modelled by means of bi-level programming are those for which variables of the first problem are constrained to be the optimal solution of the lower-level problem.

In general, bi-level optimization is issued to cope with problems with two decision makers in which the optimal decision of one of them (the leader) is constrained by the decision of the second decision maker (the follower). The second-level decision maker optimizes his/her objective function under a feasible region that is defined by the first-level decision maker. The latter, with this setting, is in charge to define all the possible reactions of the second-level decision maker and selects those values for the variable controlled by the follower that produce the best outcome for his/her objective function.

A bi-level program can be formulated as follows:

Min f (,)

∈

∈argmin g,)

∈

The analyst should pay particular attention when using bi-level optimization (or multi-level optimization in general) in studying the uniqueness of the solutions of the follower problem. Assume, for instance, one has to calculate an optimal solution to the leader model. Let be an optimal solution of the follower problem associated with. If is not unique, i.e., | *argming(,)*| *>1*, we can have a situation in which the follower decision maker can be free, without violating the leader Constraints, to adopt for his problem another optimal solution

different from, i.e., *∈argmin g(,)*with =, possibly inducing a *f (,) >f*

*(,)*onthe leader, forcing the latter to carry out a sensitivity analysis on the values attained by his objective function in correspondence to all the optimal solutions in*argmin g(,)*.

GOAL PROGRAMMING

Goal Programming dates back to Charnes et al. (1955)[15] and Charnes and Cooper (1961)[15]. It does not pose the question of maximizing multiple objectives, but rather it attempts to find specific goal values of these objectives. An example can be given by the following program:

(x) ≥
(x) =
(x) ≤

x ∈*S.*

Clearly we have to distinguish two cases, i.e., if the intersection between the image set *C*and the Utopian set (ideal set), i.e., the image of the admissible solutions for the objectives, is empty or not. In the former case, the problem transforms into one in which we have to find a solution whose value is as close as possible to the utopian set. To do this, additional variables and constraints are introduced. In particular, for each constraint of the type

$(x) \geq$

We introduce a variable such that the above constraint becomes

$(x) + \geq$

For each constraint of the type

$(x) =$

We introduce a surplus two variables and such that the above constraint becomes

$(x) + =$

For each constraint of the type

$(x) \leq$

We introduce a variable such that the above constraint becomes

$(x) -$

Let us denote with $s$the vector of the additional variables. A solution $(x,s)$to the above problem is called a strict Pareto-slack optimum if and only if a solution $(,)$, for every $\in S$, such that $\leq$ with at least one strict inequality does not exist. There are different ways of optimizing the slack/surplus variables. Where the problem becomes that of minimizing a linear combination of the surplus and slack variables each one weighted by a positive coefficient as follows:

$Min \; + \; + +$

$(x) + \geq$

$(x) + =$

$(x) -$

$\geq 0 \geq 0 \geq 0 \geq 0$

$x \in S.$

For the above problem, the Geoffrion theorem says that the resolution of this problem offers strict or weak Pareto-slack optimum. Besides Archimedean goal programming, other approaches are the lexicographical goal programming, the interactive goal programming, the reference goal programming and the multi-criteria goal programming.

Goal Programming problems can be categorized according to the importance of each objective considered Nonpreemptive Goal Programming is the case in which all the goals are of roughly comparable importance. Preemptive Goal Programming has a hierarchy of priority levels for the goals, in which goal of greater importance receive greater attention in general Goal Programming models consist of three components: an objective function, a set of goal constraints, and non-negativity requirements. However, the target value associated with each goal could be fuzzy in the real-world application the fuzzy sets theory is recurrently used in recent project. A fuzzy set A can be characterized by a membership function, which assigns to each object of a domain its grade of membership in A (Zadeh, 1965)[16].

The more an element or object can be said to belong to a fuzzy set A, the closer to 1 is its grade of membership. Various types of membership functions can be used to support the fuzzy analytical framework although the fuzzy description is hypothetical and membership values are subjective. Membership functions, such as linear, piecewise linear, exponential, and hyperbolic functions, were used in different analysis.

In general, the non-increasing and non-decreasing linear membership functions are frequently applied for the inequalities with less than or equal to and greater than or equal to relationships, respectively. Since the solution procedure of the fuzzy mathematical programming is to satisfy the fuzzy objective, a decision in a fuzzy environment is thus defined as the intersection of those membership functions corresponding to fuzzy objectives. Hence, the optimal decision could be any alternative in such a decision space that can maximize the minimumattainable aspiration levels in decision Making, represented by those corresponding membership functions.

CHAPTER 2

FORMULATION OF MULTI- OBJECTIVE QUADRATIC PROGRAMMING PROBLEM

A linearly constrained optimization problem with a quadratic objective function is called a quadratic program. More importantly, though, it forms the basis of several general nonlinear programming algorithms. The general quadratic program can be written as

Max $=+\frac{1}{2}$,..............*(2.1)*

Subject to

*Axb*and*x 0*

Where *c*is an *n*-dimensional row vector describing the coefficients of the linear terms in the objective function, and *D*is an *(n x n)* symmetric matrix describing the coefficients of the quadratic terms. If a constant term exists it is dropped from the model. As in linear programming, the decision variables are denoted by the n-dimensional column vector *x*, and the constraints are defined by an *(m xn)* matrix *A* and an *m*-dimensional column vector *b*of right-hand-side coefficients. We assume that a feasible solution exists and that the constraint region is bounded. When the objective function *Z*is strictly convex for all feasible points the problem has a unique local minimum which is also the global minimum. A sufficient condition to guarantee strictly convexity is for *D* to be positive definite. For unconstrand objective function if $D$is symmetric positive definite, then Zhas a unique global minimum precisely when$Dx=c$.

Definition:2.1.A symmetric positive definite matrix is a matrix whose eigenvalues are strictly positive, anda symmetric positive semidefinite matrix is a matrixwhose eigenvalues are nonnegative.

Definition:2.2. Given any $n\times n$symmetric matrix Awe write $A\geq 0$ if A is positive semidefinite and we write$A\ 0$ if A is positive definite.

It should be noted that we can define the relation $A \geq B$ between any two $n\times n$matrices (symmetric or not) iff $A - B$is symmetric positive semidefinite. It is easy to check that this relation is actually a partial order on matrices, called the positive semidefinite coneordering.

If $D$is symmetric positive definite, it is easily checked that is also symmetric positive definite. Also, if $d$is a symmetric positive definite $m\times m$matrix and $p$is an $m \times n$matrix of rank n(and so $m \geq n$), then$dp$is symmetric positive definite.

$=+\frac{1}{2}$ has a global minimum when $D$is symmetric positive definite.

*Proposition:2.1.*Given a quadratic function$=+ \frac{1}{2}$,if $D$is symmetric positive definite, then $Z(x)$has a unique global minimum for the solution of the linear system $Dx = c$. The minimum value of $Z(x)$is$= -\frac{1}{2}$.

Where $D$is symmetric positive definite, finding the global minimum of $Z(x)$is equivalent to solving the linear system $Dx = c$.Sometimes, it is useful to recast a linear problem $Dx = c$as a variational problem.

However, very often, a minimization problem comes with extra constraints that must be satisfied for all admissible solutions.Thus, the constrained minimum of $Z$is located on the parabola that is the intersection of the paraboloid $Z$with the plane H.A nice way to solve constrained minimization problems of the above kind is to use the method of Lagrange multipliers.

*Definition: 2.3.*The quadratic constrained minimization problem consists of minimizing a quadratic function

$Min = \frac{1}{2} + cx$

Subject to $Ax=b$

Where is an $n \times n$symmetric positive definite matrix, A is an $n \times m$matrix

of rank m(so that $n \geq m$), and where c ∈(viewed as raw vector), x

∈(viewed as column vectors), and b ∈(viewed as a column vector).The

method of Lagrange consists in incorporating the $m$constraints

$Ax=b$into the quadratic function $Z(x)$, by introducing extra variables $\lambda=$

$(, . . . ,)$called Lagrange multipliers, one for each constraint. We form the Lagrangian

$$L(x, \lambda) = Z(x) + (Ax-b)$$

$$= \frac{1}{2}$$

We shall prove that our constrained minimization problem has a unique solution given by the system of linear equations

$Dx +=,$

$Ax=b$

This can be written in matrix form as

$=$

Note that the matrix of this system is symmetric. Eliminating xfrom the first equation

$Dx +=$,

We get $x = (-)$,and substituting into the second equation, we get

That is, $= -$ b

Since is symmetric positive definite and the columns of are linearly independent, is symmetric positive definite, and thus invertible.

Note that this way of solving the system requires solving for the Lagrange multipliers first.

Letting $e =$, we also note that the system

$=$

is equivalent to the system

$e =$

$x=e$

$Ax=b$

In order to prove that our constrained minimization problem has a unique solution, we proceed to prove that the constrained minimization of $Z(x)$subject to$Ax=b$is equivalent to the unconstrained maximization of another function $-P()$.We get $P()$by minimizing the Lagrangian $L(x,)$treated as a function of $x$alone. Since $D$is symmetric positive definite and

$$L(x,) = \frac{1}{2}$$

by Proposition2.1 the global minimum (with respect to x) of $L(x, \lambda)$ is obtained for the solution $x$of$=-$,and the minimum of $L(x, \lambda)$ is

$$= - \frac{1}{2} -b$$

Letting $P()=\frac{1}{2} + b$

We claim that the solution of the constrained minimization of $Z(x)$subject to is equivalent to the unconstrained maximization of $P(\lambda)$.In order to prove that the unique minimum of the constrainedproblem *Z(x)*subject to is the uniquemaximum of *P(λ),* we compute *Z(x)-P().*

Proposition: 2.3the quadratic constrained minimization problem of Definition 2.4 has a unique solution *(x, λ)* that satisfies the system

=

Furthermore, the component *λ*of the above solution is the unique value for which *P(λ)*is maximum.

We now specialize the general first-order necessary conditions to the quadratic program. These conditions are sufficient fora global minimum when *D*is positive definite; otherwise, the most we cansay is that they are necessary.Excluding the nonnegativity conditions, the Lagrangian functionfor the quadratic program is

$$L(x, \lambda) = \frac{1}{2}$$

Where *λ is an j*-dimensional row vector. The Karush-Kuhn-Tuckerconditions for a local minimum are given as follows.

$0, n=1,2,\ldots,N$ $c + D + 0$ (2.2)

$0, j=1,2,\ldots,J$ $Ax-b0$ (2.3)

$0, n=1,2,\ldots,N+Dx +)=0$ (2.4)

$(Ax-b)=0$ $(Ax-b)=0$ (2.5)

0 $x\ 0$ (2.6)

0 0 (2.7)

To put (2.2) – (2.7) into a more manageable form we introducenonnegative surplus variables yto the inequalities in (2.2) andnonnegative slack variables vto the inequalities in (2.3) to obtainthe equations

$+Dx + -y = 0$and $Ax - b + v = 0$.

The KKT conditions can now be written with the constants moved to theright-hand side.

$x + -y =$ (2.8)

$Ax + v = b$ (2.9)

$x\ 0,\ 0,\ y\ 0,\ v\ 0$(2.10)

$x = 0,\ v = 0$ (2.11)

The first two expressions are linear equalities, the third restricts all thevariables to be nonnegative, and the fourth prescribes complementaryslackness.

The simplex algorithm can be used to solve (2.8) – (2.11) by treating thecomplementary slackness conditions (2.11) implicitly with a restrictedbasis entry rule. The procedure for setting up the linear programmingmodel follows.

- Let the structural constraints be Equations. (2.8) and (2.9) defined by theKKT conditions.
- If any of the right-hand-side values are negative, multiply the corresponding equation by –1.
- Add an artificial variable to each equation.
- Let the objective function be the sum of the artificial variables.
- Put the resultant problem into simplex form.

The goal is to find the solution to the linear program thatminimizes the sum of the artificial variables with the additionalrequirement that the complementarity slackness conditions be satisfied at each iteration. If the sum is zero, the solution will satisfy (2.8) – (2.11). To accommodate (2.11), the rule for selecting the entering variable must bemodified with the following relationships in mind.

and are complementary for $j = 1, \ldots, n$

and are complementary for $i = 1, \ldots, m$

The entering variable will be the one whose reduced cost is most negativeprovided that its complementary variable is not in the basis or would leavethe basis on the same iteration. At the conclusion of the algorithm, thevector*x*defines the optimal solution and the vector defines the optimaldual variables.

This approach has been shown to work well when the objectivefunction is positive definite, and requires computational effort comparableto a linear programming problem with *m* + *n*constraints, where *m* is thenumber of constraints and *n* is the number of variables in the Quadratic programming. Positivesemi-definite forms of the objective function, though, can presentcomputational difficulties. The simplest practical approach to overcomeany difficulties caused by semi-definiteness is to add a small constant toeach of the diagonal elements of*D*in such a way that the modified *D*matrix becomes positive definite. Although the resultant solution will notbe exact, the difference will be insignificant if the alterations are keptsmall.

Example

Solve the following problem.

Minimize Z(x)=– 8 – 16+ + 4

Subject to +5, 3, 0, 0

Solution: The data and variable definitions are given below. As can beseen, the*D*matrix is positive definite so the KKT conditions are necessaryand sufficient for a global optimum.

= *D*= *A*= *b*=

x=() , y=() , =() , v=()

The linear constraints (2.8) and (2.9) take the following form.

2 +– = 8

8 + – = 16

+ + = 5

+ = 3

To create the appropriate linear program, we add artificial variables to each constraint and minimize their sum.

Minimize + + +

subject to 2 + – + = 8

8 + – += 16

+ + = 5

+ = 3

all variables are greater than zero and complementarily conditionsApplying the modified simplex technique to this example, yields thesequence of iterations given in Table 1. The optimal solution to theoriginal problem is ()=(3, 2).

Table1. Simplex iterations for Quadratic Programming example

Iteration	Basic variables	Solution	Objective value	Entering variable	Leaving variable
1	(,,,)	(8,16,5, 3)	32		
2	(,,,)	(8,2,3,3)	14		
3	(,,,)	(2,2,3,0)	2		
4	(,,,)	(2,2,3,0)	2		
5	(,,,)	(2,2,3,0)	0	-	-

So We can examine the Karush-Kuhn-Tucker conditions for the quadratic program and see that they turn out to be a set of linear equalities and complimentarily constraints. Much like in separable programming, a modified version of the simplex algorithm can be used to find solutions.

Based on the above two formulation multi objective optimization and quadratic optimization the general formulation of multi- objective quadratic programming problem can be written as:

Max $=+\frac{1}{2}$ *, (t=1,2,…,k)…………………..(2.12)*

Subject to

……………………….(2.13)

Here, t belongs number of objective functions,and are constant vectors. isconstant matrix. *(t = 1,2,…,k)* is constant symmetricmatrix and the symbol '*T*' denotes transposition. We assume that theobjective functions are concave and the system constraints areconvex. Here, assume *S* to be non-empty and bounded.

Definition 2.4 x* S is an efficient solution to the problem (2.12) if and only if there exist no other x S such that for all t= 1, 2,…, k and for at least one t. For our purpose we define ideal solution (ideal point) of single objective compromise efficient solution for multi objective programming problem.

Definition 2.4 For problem (2.12) a compromise optimal solution is an efficient solution selected by the decision maker as being the best solution where the selection is based on the decision explicit or implicit criteria.

Our fuzzy model for determining compromise optimal (efficient) solution based on finding subset of efficient solutions with the decision making, then choosing one best solution using distance formula.

CHAPTER 3

MULTI- OBJECTIVE QUADRATIC PROGRAMMING PROBLEM

3.1 FUZZY PROGRAMMING FORMULATION OF MULTI-OBJECTIVE QUADRATIC PROGRAMMING PROBLEM

In real world, the concept of decision-making takes place in an environment in which the objectives and constraints arenot known precisely. Such situations can be tackled efficiently with the help of fuzzy set theory. Fuzzy sets were first introducedby Zadeh (1965)[15]. These sets were used to introduce the concept of decision-making in a fuzzy environment by Bellman andZadeh. They have defined appropriate aggregation of fuzzy sets for the fuzzy decision. Zimmermann (1978) has usedfuzzy decision concept in the fuzzy linear programming for several objectives. Guu and Wu (1997)extended the Zimmermann'sapproach to two-phase approach for solving the multi-objective linear programming in the fuzzy environment. Several authorshave studied linear programming in the fuzzy environment and applied to real world problems like transportation, productionplanning etc.

A fuzzy decision is based on the intersection of membership functions of the goals and constraints. Most of theauthors have used linear membership functions. Leberling (1981)has defined the hyperbolicmembership function. Li and Lee (1991) have defined the exponential membership function and Yang, Ignizio and Kim (1991)have defined piecewise nonlinear membership function. In our project, we have used linear membership function.

A fuzzy decision is defined as the fuzzy set of alternatives resulting from the intersection of the goals or objectives and constraints.*Definition 3.1*(Fuzzy set). Let X be a classical set of elements which mustbe evaluated with regard to a fuzzy statement. Then the set of order pairs= {(x, (x/)): x X}, where: X [0; 1],is called a fuzzy set in X. The evaluation function (x/) is called the membershipfunction or the grade of membership of x in.

To formulate the fuzzy programming model of Multi-objective quadratic programming problem, the objective functions would be transformed into fuzzy goals by means of assigning an imprecise aspiration level to each of the objectives. The optimal solution of each objective function of the decision-making unit when calculated in isolation would be considered as the best solution and the associated objective value can be considered as the aspiration level of the corresponding fuzzy goal.

Let, = *()*, $(t = 1,2,\ldots,k)$ be the individual best solution of the objective function subject to the system constraints.

Also let, = = , $(t = 1,2,\ldots,k)$. It is quite natural that objective values which are equal to or larger than $(t = 1,2,\ldots,k)$ should be absolutely acceptable to the decision-making unit. Then the fuzzy goals appear in the form: $(t = 1,2,\ldots,k)$. To build the membership functions, upper tolerance limit and lower tolerance limit should be determined first. Using the individual best solutions, we find the value of all the objective functions at each best solution and construct a payoff matrix; the payoff matrix helps us to find the upper tolerance limit and lower tolerance limit and it is constructing as:

…………………(3.1)

The maximum value of each column of $(t = 1,2,\ldots,k)$ provides the upper tolerance limit or aspired level of achievement for the objective goal. The minimum value of each column provides lower tolerance limit or lowest acceptable level of achievement for the objective goal. The membership function of the objective function can be defined as:

$=,(t=1,2,\ldots,k)$…………..(3.2)

Here, and are respectively the upper and lower tolerance limit of the fuzzy objective goal.

3.2 LINEARIZATION OF MEMBERSHIP FUNCTIONS BY FIRST ORDER TAYLOR SERIES APPROXIMATION

Let, = (), $(t = 1,2,\ldots,k)$ be the individual best solution of$(t = 1, 2, \ldots, k)$ subject to the system constraints, N is the total number of variables. Next, we transform the quadratic membership function $(t = 1, 2, \ldots, k)$ at =() into equivalent linear membership function () by first order Taylor series approximation. The transformed linear membership function can be formulated as:

$()+()+()+$ ………..$+() =()$, $t=1:k$ ……………………………..(3.3)

3.3 PRIORITY BASED FUZZY GOAL PROGRAMMING MODEL OF MULTI-OBJECTIVE QUADRATIC PROGRAMMING PROBLEM

The problem discussed in equation (3.1) reduces to the following problem

Max () (t = 1, 2, …, k) ……………………………………………(3.4)

Subject to

Now, achievement of the highest degree (unity) of a membership function implies absolute achievement of the aspired level of the associated fuzzy goal. So membership goal corresponding to the t^{th} membership function with unity as the aspiration level can be presented as:

() + =1 (t=1,2,…..,k)………………………………………(3.5)

Here, and *(t = 1, 2, …, k)* represent the negative and positive deviational variables respectively. It may be noted that any over deviation from a fuzzy goal indicates the full achievement of the membership value. Then according to Pramanik and Roy *[8, 9],* negative deviational variables are to be minimized in order to get satisfying solution. Therefore, *(3.5)* can be formulated as:

() + 1 (t = 1, 2, …, k) ………………………………………… (3.6)

Therefore, under the framework of min-sum Goal Programming, the priority based Fuzzy Goal Programming model of the problem can be explicitly formulated as:

Find so as to

Minimize *=[,,…,,…,]…………..(3.7)*

Subject to

()+()+()+……..+() + 1,

(t = 1, 2, …, k)

0 (t = 1, 2, …, k)

Here, represents the vector of priority achievement functions. is a linear function of the weighted negative deviational variables, where is of the form:

Here, is renamed for to represent it at the prioritylevel.

is the numerical weight correspondingto . represents the weight of importance of achieving theaspired level of the goal relative to the other goals, whichare grouped together at the priority level. Decision-making unit may providethe numerical weight or normalized weight according to theneeds, desires and practical situation of the decision-making situation.

Here, priority factor is assigned to the set of commensurable goals that are grouped together in the problem formulation. In the preemptive priority fuzzy goal programming, priority is preferred to the next priority regardless of any weightassociated with . The priority factors have the relationships

>>>> .. >> .. >> .... *(3.9)*

Here, *'>>'* denotes much greater than i.e. the membership goals at the first priority level *()* are achieved to the maximum possible before the set of membership goals at second priority Level *()* is considered. The process will continue until the last priority level () is considered. It is to be noted that if all the fuzzy goals are considered as equally important in a decision-making context, the priority based fuzzy goal programming model *(3.7)* will be transformed into the weighted fuzzy goal programming model. It is to be noted that "too many" different priority structure can increase the computational burden to the decision-making unit. If *J* be the total priority levels, then *J!* Priority structure may be involved there. However, in practice two to five priority levels are important to the decision-making unit in the decision-making situation and the conflict of assigning priorities arises at the most three priority levels *[2].*

3.4 SELECTION OF APPROPRIATE PRIORITY STRUCTURE

In the priority based fuzzy goal programming approach, priorities are assigned to the goals based on the importance of achieving of the aspired levels of the goals in the decision-making situation. However, it is to be noted that in the highly conflicting decision making situation, the decision-making unit feels confused with assigning appropriate priority structure for achievement of the aspired goals.

In order to deal with such situation, we use the concept of distance function in the proposed Multi-objective quadratic programming problem. In the present fuzzy goal programming formulation of the Multi-objective quadratic programming problem, since the aspired level of each of the membership goals is unity, the point comprising of the highest membership value of each of the goals would represent the ideal point. For different multi-objective decision making problems such as transportation problems and quality control problems, Pramanik and Roy *[10]* used distance functions for identifying best compromise solution. Let, J be the total number of different possible priority structure. The family of distance functions is defined as follows

= .... *(3.10)*

Where (t =1, 2, ..., k) represents the degrees of closeness of the preferred compromise solution to the optimal solution vector with respect to the objective function under the priority structure.

w = (, ,...,) is a vector of attribute attention levels .

We assume that =*1*.If all the attributes are equal, then

= (t = 1, 2,......,k). The power p represents the distance parameter $1 \le p \le \infty$.

=.................. *(3.11)*

For maximization problem, is defined by

=

For minimization problem, is defined by

=

The solution for which will be minimum, would be the most satisfying solution.

Let, , $1 \le q \le J$ ....*(3.12)*

Then, the q^{th} priority structure can be identified as the appropriate priority structure for decision-making unit.

3.5 PRIORITY BASED FUZZY GOAL PROGRAMMING ALGORITHM TO MULTI-OBJECTIVE QUADRATIC PROGRAMMING PROBLEM

The proposed priority based fuzzy goal programming algorithm for solving Multi-objective quadratic programming problem can be presented as follows:

Step 1: Determine the individual best solution of each objective function subject to the system constraints.

Step 2: Construct the payoff matrix as given by Then we define upper tolerance limit and lower tolerance limit of each objective function of the decision-making unit.

Step 3: Construct the membership function of the objective function as given by

Step 4: Find the individual best solution of the quadratic membership function (t = 1, 2, …, k) subject to the system constraints.

Step 5: Transform the quadratic membership function *(t=1,2,…,k)* into equivalent linear membership function at *=()* by first order Taylor series approximation as given by *(3.3).* Step 5: Form the priority based fuzzy goal programming model as given by *(3.7).*

Step 6: Solve the problem *(3.7)* for priority based fuzzy goal programming model.

Step 7: Distance function as given by *(3.11)* is used to identify the appropriate priority structure.

Step 8: End.

EXAMPLE

Consider the following Multi Objective Quadratic Programming Problem to illustrate the proposed priority based Fuzzy Goal Programming approach:

$max\ (x) = 7+5 - 2- 3$

$max\ (x) = +-$

$max\ (x) =6+$

Subject to

$+ \leq 3,$

$4 + \leq 7,$

$\geq 0, \geq 0.$

The individual best solution subject to the system constraints is

$= 8.125$ at $(1.55, 0.8);$ $= 4.485$ at $(1.382, 1.47);$ $=7.898$ at $(1.653, 0.388).$

Then the fuzzy goals appear as $8.125, 4.485, 7.898.$

Payoff matrix

Here,

$= 8.125, = 6.721; = 4.485, = 3.24; =7.898, = 4.309.$

The quadratic membership functions corresponding to the objective functions are as follows:

$(x) = =$

$(x) = =$

$(x) = =$

The quadratic membership functions $(x),(x)$ and(x) are maximal at the points $(1.55, 0.8)$, $(1.382, 1.47)$ and $(1.653, 0.388)$ respectively subject to the system constraints. Then, the quadratic membership functions are transformed into equivalent linear membership functions at the individual best solution point by first order Taylor polynomial series as follows:

$(x) = (1.55, 0.8) + (-1.55) + (- 0.8)$

$=1 + (- 1.55) x\ 0.57 + (- 0.8) x\ 0.142,$

$(x) = (1.382, 1.47)+(-1.382) + (- 1.47)$

$=1 + (-1.382) x\ 0.19 + (-1.47) x\ 0.048,$

$(x) =(1.653,0.388)+(-1.653)+(- 0.388)$

$=1 + (-1.635)x\ 0.751 + (-0.388)x\ 0.187$

Then priority based Fuzzy Goal Programming approach for Multi Objective Quadratic Programming Problem can written as

Find so as to

Minimize $=[,,\ldots,,\ldots,]$

Subject to

$1 + (- - 1.55) x\ 0.57 + (- 0.8) x\ 0.142 +$

$1 + (-1.382) x\ 0.19 + (-1.47) x\ 0.048 +$

$1 + (-1.635)x\ 0.751 + (-0.388)x\ 0.187 +$

$+ \le 3,$

$4 + \le 7,$

$\ge 0, \ge 0.$

$, (t=1,2,3)$

The results, obtained by different priority structure are shown in the Table 2

Note 1: All solutions of the problem are obtained by using Lingo version 6.0.

Note 2: It is to be noted that some priority level may provide infeasible solutions. In that case, that priority level should be discarded by decision making unit. Now from the Table 2, we observe that the minimum distance value is *0.041.* The results show that the priority structure under the serial *2&4* are appropriate for the decision making unit to get the most satisfactory solution. The optimal solution set corresponding to the appropriate priority structure is given by *=8.125, = 4.008, = 7.378 at = 1.55, = 0.8.* The resulting membership values are*(x) = 1, (x) = 0.616, (x) =0.855.*

SUMMARY

In this paper, priority based fuzzy goal programming approach for solving Multi-objective quadratic programming problem is presented. The proposed approach is simple and easy to implement. In the proposed approach, we transform Multi-objective quadratic programming problem into multi-objective linear programming problem by first order Taylor series approximation and then priority based fuzzy goal programming is used to solve the problem.

Distance function is then applied to obtain a proper priority structure to reach the most satisfactory solution of the decision-making unit in the decision-making context. We hope that the proposed approach can be applied to portfolio problems and other real world Multi-objective quadratic programming problems.

Serial number	*priority structure*	*Solution point*	*Objective values*	*Membership values*	*Distance values*
1	*[]*	*1.653,0388*	*7.594, 3.24, 7.9*	*0.622, 0, 0.999*	*0.095*
2		*1.55, 0.8*	*8.125,4.008,7.378*	*1, 0.616, 0.855*	*0.041*
3		*1.43, 1.28*	*7.405, 4.447, 5.46*	*0.487,0.969,0.321*	*0.107*
4		*1.55, 0.8*	*8.125,4.008,7.378*	*1, 0.616, 0.855*	*0.041*
5	*[]*	*1.43, 1.28*	*7.405, 4.447, 5.46*	*0.487,0.969,0.321*	*0.107*

Table 2

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