

Addis Ababa
University



(Since 1950)



Addis Ababa University
School of Graduate Studies
Faculty of computer and mathematical
Science
Department of Mathematics

Project

On

The Cauchy Kernel and the Transform of Cauchy-Bochner and Hilbert

By: Beza Zeleke

Advisor: Tsegaye G. (PhD)

A Project Submitted to the Office of Graduate Programs of
Addis Ababa University in Partial fulfillment of the requirements for the Degree of Master of
Science in mathematics

January, 2012

AA, Ethiopia

ABSTRACT

The Cauchy-Bochner integral representation is extended to holomorphic function of several complex variables belonging to the space $\mathcal{H}_s$ of the collection of generalized functions. The Cauchy-kernel is holomorphic in a tubular region T^c and also when $n=1$ the Cauchy-Bochner transform turns to classical Cauchy-Bochner integral.

ACKNOWLEDGEMENT

First, I would like to express my deepest gratitude and appreciation to my advisor Dr. Tsegaye G. for his unreserved help during my seminar report writing. I would also like to express my appreciation to my families for their continuous encouragement. Lastly, I would like to thanks my friends who comments and helping me in all activities.

BEZA ZELEKE

Table of content

Content	page
Introduction	1
CHAPTER 0	
0. Preliminary Concepts	2
CHAPTER ONE	
1. The Fourier transform of generalized function of slow growth	8
1.1 The Fourier transform of test function belongs to	8
1.1.1 Definitions and its properties	8
1.1.2 Examples	12
1.2 The Fourier transform of generalized functions belongs to	14
1.2.1 Definitions	14
1.2.2 Properties	18
1.2.3 Examples	20
CHAPTER TWO	
2. The Laplace transform of generalized function of slow growth	25
2.1 Definitions	25
2.2 properties.....	29
2.3 Examples	32
CHAPTER THREE	
3. The Cauchy kernel and the transform of Cauchy-Bochner and Hilbert	34
3.1 The Cauchy kernel	34
3.2 The Cauchy-Bochner transform	42
3.3 The Hilbert transform	45
3.4 Examples	47
4. REFERENCE	50

ABSTRACT

The Cauchy-Bochner integral representation is extended to holomorphic function of several complex variables belonging to the space $\mathcal{H}_s$ of the collection of generalized functions. The Cauchy-kernel is holomorphic in a tubular region T^c and also when $n=1$ the Cauchy-Bochner transform turns to classical Cauchy-Bochner integral.

ACKNOWLEDGEMENT

First, I would like to express my deepest gratitude and appreciation to my advisor Dr. Tsegaye G. for his unreserved help during my seminar report writing. I would also like to express my appreciation to my families for their continuous encouragement. Lastly, I would like to thanks my friends who comments and helping me in all activities.

BEZA ZELEKE

INTRODUCTION

One of the most effective means of solving problems in mathematical physics is the method of integral transformations. The Cauchy kernel transformation is considered in the setting of generalized function of slow growth and its representation is extended to holomorphic function and also the Cauchy-Bochner integral representation is extended to holomorphic function.

The Fourier transform is applied to construct the fundamental solution of the linear differential operators having constant coefficients. Naturally, fundamental solution of generalized function of slow growth can be obtained by this method.

Generally, in this seminar we can see that the Fourier transform and the Laplace transform that is closely linked with it; we also consider the transformation of the Cauchy-Bochner and Hilbert for the class of generalized function of slow growth.

CHAPTER 0

Preliminary Concepts

In this chapter, the required symbols and notation are briefly listed. Some basic notations and definitions which are use full for this seminar are also given.

Basic notations and Definitions

0.1 n-dimensional real and complex space

The points of n-dimensional real space $\mathbb{R}^n$ denote by $x, y, \dots$; where $x = (x_1, \dots, x_n)$. The points of n-dimensional complex space $\mathbb{C}^n$ are given by $z, \zeta, \dots$; where

$z = (z_1, z_2, \dots, z_n) = x + iy$; $x = \operatorname{Re} z$ is the real part of z and $y = \operatorname{Im} z$ is the imaginary part of z , $\bar{z} = x - iy$ is the complex conjugate of z .

The scalar (inner) products in $\mathbb{R}^n$ and $\mathbb{C}^n$ are defined as:

$$xy = (x, y) = x_1y_1 + x_2y_2 + \dots + x_ny_n = \sum_{i=1}^n x_iy_i$$

$$\langle z, \zeta \rangle = z_1\bar{\zeta}_1 + z_2\bar{\zeta}_2 + \dots + z_n\bar{\zeta}_n = \sum_{i=1}^n z_i\bar{\zeta}_i$$

And the norms (lengths) are defined as:

$$|x| = (x_1^2 + \dots + x_n^2)^{1/2}$$

$$|z| = (|z|_1^2 + \dots + |z|_n^2)^{1/2}.$$

0.2 multi-index

Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ be a multi-index that is its components α_j are nonnegative integers, where $j = 1, 2, \dots, n$. Then

$$|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n : \text{Length (order)}$$

$$\alpha! = \alpha_1! \alpha_2! \dots \alpha_n!$$

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}, \quad x \in \mathbb{R}^n$$

$$\binom{\beta}{\alpha} = \binom{\beta_1}{\alpha_1} \binom{\beta_2}{\alpha_2} \dots \binom{\beta_n}{\alpha_n} = \frac{\beta!}{\alpha! (\alpha - \beta)!}$$

Let $D = (D_1, D_2, \dots, D_n)$, $D_j = \frac{\partial}{\partial x_j}$, $j = 1, 2, \dots, n$. Then

$$D^\alpha f(x) = \frac{\partial^{|\alpha|} f(x)}{x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}}$$

0.3 The space $\mathcal{S}$ of basic function

- The space of basic (test) function $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ all functions infinitely differentiable in $\mathbb{R}^n$ that decreases together with all their derivatives, as $|x| \rightarrow \infty$, faster than any power of $|x|^{-1}$.

$$\mathcal{S}(\mathbb{R}^n) := \{ \varphi \in C^\infty(\mathbb{R}^n) : \|\varphi\|_{p,\alpha} < \infty, \text{ for all } p \in \mathbb{N}_0 \text{ and for all } \alpha \in \mathbb{N}_0^n \}$$

Example $\varphi(x) = e^{-c|x|^2}$, where $c > 0$

- Let introduce the norm in $\mathcal{S}$ via the formula

$$\|\varphi\|_p = \sup_{|\alpha| \leq p} (1 + |x|^2)^{p/2} |D^\alpha \varphi(x)|, \quad \varphi \in \mathcal{S}$$

- *Convergence in $\mathcal{S}$* : The sequence of the functions $\varphi_1, \varphi_2, \dots$ in $\mathcal{S}$ converges to $\varphi \in \mathcal{S}$, if for all $p \in \mathbb{N}_0$

$$\|\varphi_k - \varphi\|_p \rightarrow 0, \quad k \rightarrow \infty.$$

- Linear operator in $\mathcal{S}$: An operator $\mathcal{A}: \mathcal{S} \rightarrow \mathcal{S}$ is linear if

$$\checkmark \quad \varphi \in \mathcal{S} \Rightarrow \mathcal{A}\varphi \in \mathcal{S}$$

$$\checkmark \quad \mathcal{A}(\lambda\varphi + \mu\psi) = \lambda\mathcal{A}\varphi + \mu\mathcal{A}\psi, \quad \varphi, \psi \in \mathcal{S} \text{ and } \lambda, \mu \in \mathbb{C}$$

- Continuous operator in $\mathcal{S}$: $\mathcal{A}: \mathcal{S} \rightarrow \mathcal{S}$ is continuous if

$$\checkmark \quad \varphi_k \rightarrow \varphi \text{ in } \mathcal{S} \Rightarrow \mathcal{A}\varphi_k \rightarrow \mathcal{A}\varphi \text{ in } \mathcal{S}$$

0.4 The space $\mathcal{H}_s$

- Denoted by $\mathcal{L}_s^2$ the Hilbert space consisting of all function $g(\xi)$ with finite norm

$$\|g\|_{(s)} = \left(\int |g(\xi)|^2 (1 + |\xi|^2)^s d\xi \right)^{\frac{1}{2}} = \|g(\xi)(1 + |\xi|^2)^{\frac{s}{2}}\|$$

- The collection of all (generalized) functions $f(x)$ that are Fourier transform of the functions in $\mathcal{L}_s^2$, $f = \mathcal{F}[g]$ is denoted by

$$\mathcal{H}_s = \{ f \in \mathcal{S} : \mathcal{F}[g] \in \mathcal{L}_s^2(\mathbb{R}^n) \}, \text{ And with norm } \|f\|_s = \|\mathcal{F}^{-1}[f]\|_{(s)} = \|g\|_{(s)}$$

where the parameter s can assume any real values.

- Remark: when $s = 0$, then $\mathcal{H}_0 = \mathcal{L}^2 = \mathcal{L}_0^2$ and

$$\|g\|_{(0)} = \|g\| = \|\mathcal{F}^{-1}[f]\| = (2\pi)^{-n}\|f\| = \|f\|_0$$

By virtue of the parseval equation

0.5 The space $\mathcal{S}$ of generalized function of slow growth

- A generalized function of slow growth (tempered distribution) $f \in \mathcal{S}(\mathbb{R}^n)$ is any continuous linear functional on the space $\mathcal{S}$ of basic functions. The set all generalized function of slow growth is denoted by $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$.
- The generalized function f is a *functional* on the space of basic functions, that is, a (complex) number (f, φ) is associated with each $\varphi \in \mathcal{S}$.
- A generalized function $f \in \mathcal{S}(\mathbb{R}^n)$ is a *linear functional* on the space of basic function $\mathcal{S}$, that is, if $\varphi, \psi \in \mathcal{S}$ and $\lambda, \mu \in \mathbb{C}$, then

$$(f, \lambda\varphi + \mu\psi) = \lambda(f, \varphi) + \mu(f, \psi) \dots\dots\dots (1)$$

- A generalized function $f \in \mathcal{S}(\mathbb{R}^n)$ is a *continuous functional* on the space of basic function $\mathcal{S}$, that is if $\varphi_k \rightarrow \varphi$ as $k \rightarrow \infty$, then

$$(f, \varphi_k) \rightarrow (f, \varphi), \text{ as } k \rightarrow \infty \dots\dots\dots (2)$$

- The set $\mathcal{S}$ is linear if the linear combination $\lambda f + \mu g$ of generalized function of f and g is defined as a functional acting in accordance with the formula

$$(\lambda f + \mu g, \varphi) = \lambda(f, \varphi) + \mu(g, \varphi), \varphi \in \mathcal{S} \dots\dots\dots (3)$$

Hence, from equation (1) and (3) the operator $(.,.)$ is a bilinear.

- **Convergence in $\mathcal{S}$** : A sequence of generalized function $f_1, f_2, \dots$ taken from $\mathcal{S}$ converges to the generalized function $f \in \mathcal{S}$, $f_k \rightarrow f$ as $k \rightarrow \infty$ In $\mathcal{S}$, if for any $\varphi \in \mathcal{S}$

$$(f_k, \varphi) \rightarrow (f, \varphi), \text{ as } k \rightarrow \infty$$

- The linear set $\mathcal{S}$ equipped with convergence is called the space $\mathcal{S}$ of generalized function of slow growth. Example: If $f(x)$ is a function of slow growth in $\mathbb{R}^n$, that is, $\int |f(x)|(1 + |x|)^{-m} dx < \infty$, for some $m \geq 0$, then it determines a regular functional f in $\mathcal{S}$.
- Not every locally summable function defines a generalized function of slow growth. For instance, $x \notin \mathcal{S}$. On the other hand, not every locally summable function taken from $\mathcal{S}$ is of slow growth. For example $f(x) = x \cos(x)$. we set $f(x) = (\sin(x))$ and $g(x) = \sin(x)$. Then the function f is not a function of slow growth but it generates a generalized function of slow growth.

0.6 Cones in $\mathbb{R}^n$.

- A cone in $\mathbb{R}^n$ with vertex at 0 is a set with the properties that if $x \in$, and then λx also belongs to for all $\lambda > 0$.

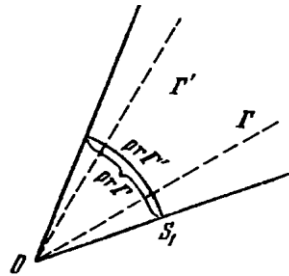


Figure 1

- The intersection of and the unit sphere s_1 is denoted by pr . The set of all limit point of convergent sequence $\{x_i\}_{i=1}^n, x_n \in$ is called the closure of and written as $\overline{}$.
- The cone is said to be compact (is strictly contained) in the cone if is bounded and $pr \subset pr$, then we write $\overline{} \in$.
- A set A is said to be convex if for any points x and x in A the line segment joining them

$$\lambda x + (1 - \lambda)x, \lambda \in [0,1], \text{ lies entirely in } A.$$

Let $A \subset$, then $chA = \cap \{K : A \subseteq K \subseteq, K \text{ is convex}\}$ is called the convex hull of A .

- The cone Γ is said to be acute if there exists a plane of support for $\overline{ch \Gamma}$ that has a unique common point 0 with $\overline{ch \Gamma^*}$, where $\overline{ch \Gamma} = (\Gamma^*)^*$.

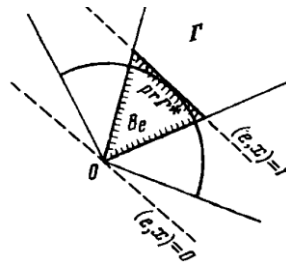


Figure 2

- Suppose Γ is a closed cone in $\mathbb{R}^n$. The collection of generalized functions of slow growth in $\mathcal{S}'(\Gamma)$ whose supports are bounded on the side of the cone Γ will be denoted by $\mathcal{S}'(\Gamma_+)$.
- A set A is bounded on the side of the cone Γ if $A \subset \Gamma + K$, where K is a certain compact. The space $\mathcal{S}'(\Gamma + K)$ is a generalized function of slow growth defined on the closed set $\Gamma + K \subset \mathbb{R}^n$. Then $\mathcal{S}'(\Gamma_+)$ is defined by

$$\mathcal{S}'(\Gamma_+) = \bigcup_{K \subset \mathbb{R}^n} \mathcal{S}'(\Gamma + K)$$

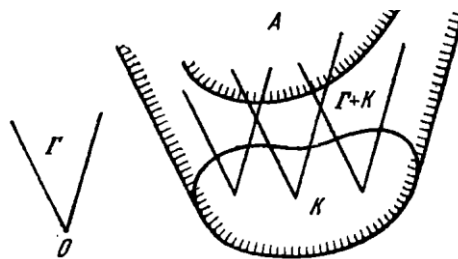


Figure 3

- The convergence in $\mathcal{S}'(\mathbb{R}^n)$: the sequence of the function f_k converges to 0 as $k \rightarrow \infty$ in $\mathcal{S}'(\mathbb{R}^n)$ if $f_k \rightarrow 0$ as $k \rightarrow \infty$ in $\mathcal{S}$ and $\text{supp} f_k \subset \{x \in \mathbb{R}^n : |x| \leq k\}$.

0.7 Fubini's theorem:

Let $f(x, y) \in \mathbb{R}^{n+m}$. Then the function

$\int f(x, y) dx$ and $\int f(x, y) dy$ exist almost everywhere with respect to x and y respectively and are integrables; furthermore

$$\int \int f(x, y) dx dy, \int dx \int f(x, y) dy, \int dy \int f(x, y) dx \dots (1)$$

also, if any one of the integrals

$$\int \int |f(x, y)| dx dy, \int dy \int |f(x, y)| dx, \int dx \int |f(x, y)| dy$$

is finite, then so are the other two and equation (1) holds.

CHAPTER ONE

The Fourier transform of generalized function of slow growth

1.1 The Fourier transform of basic functions belongs to $\mathcal{S}$

1.1.1 Definitions and its properties

Since, the basic functions $\varphi(x)$ in $\mathcal{S}$ are locally integrable over $\mathbb{R}^n$. Then the Fourier transform operation $\mathcal{F}$ defined over them:

$$\mathcal{F}[\varphi](\xi) = \int \varphi(x) e^{i(\xi, x)} dx, \quad \varphi \in \mathcal{S} \quad \dots\dots\dots (1.1)$$

In this case, the function $\mathcal{F}[\varphi](\xi)$, which is the Fourier transform of the function $\varphi(x)$, is bounded and continuous in $\mathbb{R}^n$.

From the general theory of Fourier transformations, it follows that the function $\varphi(x)$ is express in terms of its Fourier transform $\mathcal{F}[\varphi](\xi)$ by means of the operation of the inverse Fourier transform $\mathcal{F}^{-1}$:

$$\varphi = \mathcal{F}^{-1}[\mathcal{F}[\varphi]] = \mathcal{F}[\mathcal{F}^{-1}[\varphi]] \quad \dots\dots\dots (1.2)$$

Where,

$$\begin{aligned} \mathcal{F}^{-1}[\psi](x) &= (2\pi)^{-n} \int \psi(\xi) e^{-i(x, \xi)} d\xi \\ &= (2\pi)^{-n} \mathcal{F}[\psi](-x) \\ &= (2\pi)^{-n} \int \psi(-\xi) e^{i(x, \xi)} d\xi \\ &= (2\pi)^{-n} \mathcal{F}[\psi(-\xi)] \quad \dots\dots\dots (1.3) \end{aligned}$$

The operator of inversion $\mathcal{I}: \mathcal{S} \rightarrow \mathcal{S}$ operating by the formula $\mathcal{I}\varphi(x) = \varphi(-x)$ is linear and continuous.

Moreover,

$$\mathcal{F}^{-1}[\varphi](\xi) = (2\pi)^{-n} \mathcal{F}[\varphi(-x)] = (2\pi)^{-n} \mathcal{F}\mathcal{J}\varphi(x)$$

That is,

$$\mathcal{F}[\mathcal{F}^{-1}[\varphi]] = \mathcal{F}\mathcal{F}\mathcal{J}\varphi(x) \Leftrightarrow \mathcal{F}\mathcal{F}\mathcal{J} = I$$

Thus,

$$\mathcal{F}\mathcal{F} = \mathcal{J}, \text{ and consequently, } \mathcal{F}\mathcal{F}\varphi(x) = \varphi(-x)$$

Properties

Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and α be a multi-index. Then,

- a) 1. $D^\alpha \mathcal{F}[\varphi](\xi) = \mathcal{F}[(ix)^\alpha \varphi]$
- 2. $D^\alpha \mathcal{F}^{-1}[\varphi](\xi) = \mathcal{F}^{-1}[(-ix)^\alpha \varphi]$
- b) 1. $\mathcal{F}[D^\alpha \varphi] = (-i\xi)^\alpha \mathcal{F}[\varphi]$
- 2. $\mathcal{F}^{-1}[D^\alpha \varphi] = (i\xi)^\alpha \mathcal{F}^{-1}[\varphi]$

Proof:

- a) The test function $\varphi(x)$ decreases at infinity faster than any power of $|x|^{-1}$. Therefore, its Fourier transform may be differentiated under the integral sign any number of times.

$$\begin{aligned}
 1. \quad D^\alpha \mathcal{F}[\varphi](\xi) &= \frac{\partial^\alpha}{\partial \xi^\alpha} \int \varphi(x) e^{i(\xi, x)} dx \\
 &= \int \varphi(x) \frac{\partial^\alpha}{\partial \xi^\alpha} e^{i(\xi, x)} dx \\
 &= \int \varphi(x) (ix)^\alpha e^{i(\xi, x)} dx \\
 &= \mathcal{F}[(ix)^\alpha \varphi]
 \end{aligned}$$

$$\begin{aligned}
2. \quad D^\alpha \mathcal{F}^{-1}[\varphi](\xi) &= (2\pi)^{-n} \frac{\partial^\alpha}{\partial \xi^\alpha} \int \varphi(x) e^{-i(\xi, x)} dx \\
&= (2\pi)^{-n} \int \varphi(x) \frac{\partial^\alpha}{\partial \xi^\alpha} e^{-i(\xi, x)} dx \\
&= (2\pi)^{-n} \int \varphi(x) (-ix)^\alpha e^{-i(\xi, x)} dx \\
&= (2\pi)^{-n} \mathcal{F}^{-1}[(-ix)^\alpha \varphi]
\end{aligned}$$

b) Hence, every derivative $D^\alpha \varphi$ has the same properties with a) and therefore,

$$\begin{aligned}
1. \quad \mathcal{F}[D^\alpha \varphi](\xi) &= \int \frac{\partial^\alpha}{\partial x^\alpha} \varphi(x) e^{i(\xi, x)} dx \\
&= (-1)^{|\alpha|} \int \varphi(x) \frac{\partial^\alpha}{\partial x^\alpha} e^{i(\xi, x)} dx \\
&= (-1)^{|\alpha|} \int \varphi(x) (i\xi)^\alpha e^{i(\xi, x)} dx \\
&= (-1)^{|\alpha|} (i\xi)^\alpha \int \varphi(x) e^{i(\xi, x)} dx \\
&= (-i\xi)^\alpha \mathcal{F}[\varphi]
\end{aligned}$$

$$\begin{aligned}
2. \quad \mathcal{F}^{-1}[D^\alpha \varphi](\xi) &= (2\pi)^{-n} \int \frac{\partial^\alpha}{\partial x^\alpha} \varphi(x) e^{-i(\xi, x)} dx \\
&= (2\pi)^{-n} (-1)^{|\alpha|} \int \varphi(x) \frac{\partial^\alpha}{\partial x^\alpha} e^{-i(\xi, x)} dx \\
&= (2\pi)^{-n} (-1)^{|\alpha|} \int \varphi(x) (-i\xi)^\alpha e^{-i(\xi, x)} dx \\
&= (-1)^{|\alpha|} (-i\xi)^\alpha (2\pi)^{-n} \int \varphi(x) e^{-i(\xi, x)} dx \\
&= (i\xi)^\alpha \mathcal{F}^{-1}[\varphi]
\end{aligned}$$

Lemma 1: If $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then $x^\alpha \varphi \in \mathcal{S}(\mathbb{R}^n)$.

Proof:

Let $\alpha \in \mathbb{N}_0^n$ and $p \in \mathbb{N}_0$. Then from the definition of the space $\mathcal{S}$ of basic function, we have

$$(|x|^p + |x|^{p+1})|D^\alpha \varphi(x)| \leq \begin{cases} 2|x|^{p+1}|D^\alpha \varphi(x)| \leq 2(1 + |x|^{p+1})|D^\alpha \varphi(x)| \leq c_1, \text{ if } |x| \geq 1 \\ 2|x|^p|D^\alpha \varphi(x)| \leq 2(1 + |x|^p)|D^\alpha \varphi(x)| \leq c_2, \text{ if } |x| < 1 \end{cases}$$

Choose $C := \max\{c_1, c_2\}$, we get

$$|x|^p|D^\alpha \varphi(x)| \leq \frac{(|x|^p + |x|^{p+1})|D^\alpha \varphi(x)|}{1 + |x|} \leq \frac{C}{1 + |x|} \rightarrow 0, \quad |x| \rightarrow \infty$$

Therefore, $x^\alpha \varphi \in \mathcal{S}(\mathbb{R}^n)$

Theorem 1: If $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then

$$\text{a) } \mathcal{F}[\varphi], \mathcal{F}^{-1}[\psi] \in \mathcal{C}(\mathbb{R}^n)$$

$$\text{b) } \mathcal{F}[\varphi] \in \mathcal{S}(\mathbb{R}^n)$$

Proof:

a) From the above properties we get that $D^\alpha \mathcal{F}[\varphi](\xi)$ exists for all $\alpha \in \mathbb{N}_0^n$. Let $\xi^{(k)}$ be a sequence in $\mathbb{R}^n$

such that $\xi^{(k)} \rightarrow \xi$ as $k \rightarrow \infty$. Let $\alpha = (1, 0, \dots, 0)$. Then

$$\begin{aligned} |D^\alpha \mathcal{F}[\varphi](\xi^{(k)}) - D^\alpha \mathcal{F}[\varphi](\xi)| &= \left| \frac{\partial^\alpha}{\partial \xi_1} \int \varphi(x) e^{i(\xi^{(k)}, x)} dx - \frac{\partial^\alpha}{\partial \xi_1} \int \varphi(x) e^{i(\xi, x)} dx \right| \\ &= \left| \int \varphi(x) (ix_1) e^{i(\xi^{(k)}, x)} dx - \int \varphi(x) (ix_1) e^{i(\xi, x)} dx \right| \\ &= \left| \int \varphi(x) (ix_1) \left(e^{i(\xi^{(k)}, x)} - e^{i(\xi, x)} \right) dx \right| \\ &\leq \int |\varphi(x) (ix_1)| \left| e^{i(\xi^{(k)}, x)} - e^{i(\xi, x)} \right| dx < \epsilon, \quad k \rightarrow \infty. \end{aligned}$$

Thus, $\frac{\partial^\alpha}{\partial \xi_1^\alpha} \mathcal{F}[\varphi](\xi)$ is continuous in $\mathbb{R}^n$. Again, applying the same reasoning to the properties (a) holds for all second derivatives and so forth, and the same analogous to $\mathcal{F}^{-1}[\psi]$. Therefore, $\mathcal{F}[\varphi], \mathcal{F}^{-1}[\psi] \in C(\mathbb{R}^n)$.

b) Let $\alpha, \beta \in \mathbb{N}_0^n$. From (a) we get that $\mathcal{F}[\varphi] \in C(\mathbb{R}^n)$ and from properties, we have

$$\begin{aligned} \xi^\beta D^\alpha \mathcal{F}[\varphi](\xi) &= \xi^\beta \mathcal{F}[(ix)^\alpha \varphi] \\ &= i^{|\alpha|} \xi^\beta \int x^\alpha \varphi(x) e^{i(\xi, x)} dx \\ &= i^{|\alpha|} \int x^\alpha \varphi(x) \xi^\beta e^{i(\xi, x)} dx \\ &= i^{|\alpha|} (-i)^{|\beta|} \int x^\alpha \varphi(x) (i\xi)^\beta e^{i(\xi, x)} dx \\ &= i^{|\alpha|+|\beta|} \int D^\beta (x^\alpha \varphi(x)) e^{i(\xi, x)} dx \\ &= i^{|\alpha|+|\beta|} \mathcal{F}[D^\beta (x^\alpha \varphi)](\xi) \end{aligned}$$

From Lemma 1 it follows that for all α and β the magnitudes of $\xi^\beta D^\alpha \mathcal{F}[\varphi](\xi)$ are absolutely bounded with respect to $\xi \in \mathbb{R}^n$.

$$|\xi^\beta D^\alpha \mathcal{F}[\varphi](\xi)| \leq \int |D^\beta (x^\alpha \varphi)| dx$$

That is $\mathcal{F}[\varphi] \in \mathcal{S}(\mathbb{R}^n)$.

Theorem 2: The operation of Fourier transforms $\mathcal{F}$ and $\mathcal{F}^{-1}$ is continuous from $\mathcal{S}$ into $\mathcal{S}$.

Proof:

Let $\varphi_k \rightarrow \varphi$ as $k \rightarrow \infty$ in $\mathcal{S}$. Then applying the above inequality (b) of theorem 1, for all α and β be a multi-index, we obtain

$$\begin{aligned} |D^\alpha \mathcal{F}[\varphi_k - \varphi](\xi)| &= \left| \frac{\partial^\alpha}{\partial \xi^\alpha} \int D^\alpha (\varphi_k - \varphi)(x) e^{i(\xi, x)} dx \right| \\ &\leq \int |x^\alpha (\varphi_k - \varphi)(x) e^{i(\xi, x)}| dx \end{aligned}$$

$$\begin{aligned}
&= \int |x^\alpha (\varphi_k - \varphi)(x)| |i(\xi, x)| dx \\
&= \int |x^\alpha (\varphi_k - \varphi)(x)| dx \\
&\leq \int (1 + |x|^p) |(\varphi_k - \varphi)(x)| dx \\
&= \int \frac{1 + |x|^{p+1} + |x|^p + |x|^{2p+1}}{1 + |x|^{p+1}} |(\varphi_k - \varphi)(x)| dx \\
&\leq \int \frac{(1 + |x|)^{p+1}}{1 + |x|^{p+1}} |(\varphi_k - \varphi)(x)| dx \\
&= \int \left(\sup_{x \in \mathbb{R}^n} (1 + |x|)^{p+1} |(\varphi_k - \varphi)(x)| \right) \frac{dx}{1 + |x|^{p+1}} \\
&\leq \sup_{x \in \mathbb{R}^n} (1 + |x|)^{p+1} |(\varphi_k - \varphi)(x)| \underbrace{\int \frac{dx}{1 + |x|^{p+1}}}_{\leq \pi}
\end{aligned}$$

Hence, it follows that $|D^\alpha \mathcal{F}[\varphi_k](\xi) - D^\alpha \mathcal{F}[\varphi](\xi)| \Rightarrow 0, k \rightarrow \infty, \xi \in \mathbb{R}^n$.

That is, $\mathcal{F}[\varphi_k] \rightarrow \mathcal{F}[\varphi]$ as $k \rightarrow \infty$ in $\mathcal{S}$.

1.1.2 Examples

$$1. \quad \mathcal{F}\left[e^{-\frac{x^2}{2}}\right] = (2\pi)^{\frac{n}{2}} e^{-\frac{\xi^2}{2}}$$

Solution:

$$\begin{aligned}
\mathcal{F}\left[e^{-\frac{x^2}{2}}\right] &= \int e^{-\frac{x^2}{2}} e^{i(\xi, x)} dx = \int e^{-\frac{x^2}{2} + i\xi x} dx \\
&= \int e^{-2x^2 \left(\frac{1}{4} - \frac{i\xi}{2x}\right)} dx = \int e^{-2x^2 \left(\left(\frac{1}{2} - \frac{i\xi}{2x}\right)\right)^2 - \frac{x^2}{2}} dx \\
&= e^{-\frac{\xi^2}{2}} \int e^{-2x^2 \left(\left(\frac{1}{2} - \frac{i\xi}{2x}\right)\right)^2} dx = e^{-\frac{\xi^2}{2}} \int e^{-\left(\sqrt{2}x\left(\frac{1}{2} - \frac{i\xi}{2x}\right)\right)^2} dx
\end{aligned}$$

$$= e^{-\frac{\xi^2}{2}} \underbrace{2^{\frac{n}{2}} \int e^{-t^2} dt}_{=\pi^{\frac{n}{2}}} = (2\pi)^{\frac{n}{2}} e^{-\frac{\xi^2}{2}}$$

$$2. \quad \mathcal{F}[e^{-ax^2}] = \left(\frac{\sqrt{\pi}}{a}\right)^n e^{-\frac{\xi^2}{4a^2}}$$

Solution:

$$\begin{aligned} \mathcal{F}[e^{-ax^2}] &= \int e^{-ax^2} e^{i(\xi, x)} dx = \int e^{-ax^2 + i\xi x} dx \\ &= \int e^{-a^2 x^2 \left(1 - \frac{i\xi}{2a^2 x}\right)} dx = \int e^{-a^2 x^2 \left(1 - \frac{i\xi}{2a^2 x}\right)^2 - \frac{\xi^2}{4a^2}} dx \\ &= e^{-\frac{\xi^2}{4a^2}} \int e^{-\left(ax\left(1 - \frac{i\xi}{2a^2 x}\right)\right)^2} dx = \frac{e^{-\frac{\xi^2}{4a^2}}}{a^n} \underbrace{\int e^{-t^2} dt}_{(\sqrt{\pi})^n} \\ &= \left(\frac{\sqrt{\pi}}{a}\right)^n e^{-\frac{\xi^2}{4a^2}} \end{aligned}$$

1.2 The Fourier transform of generalized function of slow growth belongs to $\mathcal{S}'$

1.2.1 Definitions

Let $f(x)$ be a locally summable function on $\mathbb{R}^n$. Then its Fourier transform,

$$\mathcal{F}[f](\xi) = \int f(x) e^{i(\xi, x)} dx \quad \dots\dots\dots (1.4)$$

is bounded and continuous function on $\mathbb{R}^n$ and consequently, defines a regular generalized function of slow growth via the formula,

$$(\mathcal{F}[f], \varphi) = \int \mathcal{F}[f](\xi) \varphi(\xi) d\xi, \quad \varphi \in \mathcal{S}(\mathbb{R}^n) \quad \dots\dots\dots (1.5)$$

Using Fubini's theorem (see sec. 0.7), on changing the order of integration, we transform the last integral:

$$\begin{aligned}
\int \mathcal{F}[f](\xi) \varphi(\xi) d\xi &= \int \left[\int f(x) e^{i(\xi, x)} dx \right] \varphi(\xi) d\xi \\
&= \int f(x) \int \varphi(\xi) e^{i(x, \xi)} d\xi dx \\
&= \int f(x) \mathcal{F}[\varphi](x) dx
\end{aligned}$$

That is,

$$(\mathcal{F}[f], \varphi) = (f, \mathcal{F}[\varphi]), \quad \varphi \in \mathcal{S}(\mathbb{R}^n)$$

Definition: The Fourier transform $\mathcal{F}[f]$ of any generalized function of slow growth $f \in \mathcal{S}'(\mathbb{R}^n)$ is defined by

$$(\mathcal{F}[f], \varphi) = (f, \mathcal{F}[\varphi]), \quad \varphi \in \mathcal{S}(\mathbb{R}^n) \dots\dots\dots (1.6)$$

Let us now prove that the right hand side of equation (1.6) is a continuous linear functional over $\mathcal{S}(\mathbb{R}^n)$, that is, $\mathcal{F}[f] \in \mathcal{S}'(\mathbb{R}^n)$

Proof:

Since, $\mathcal{F}[\varphi] \in \mathcal{S}(\mathbb{R}^n)$ for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then $(f, \mathcal{F}[\varphi])$ is a functional over $\mathcal{S}(\mathbb{R}^n)$.

Linearity: let $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ and $\alpha, \beta \in \mathbb{C}$. Then

$$\begin{aligned}
(f, \mathcal{F}[\alpha\varphi + \beta\psi]) &= (f, \mathcal{F}[\alpha\varphi]) + (f, \mathcal{F}[\beta\psi]) \\
&= (f, \alpha\mathcal{F}[\varphi]) + (f, \beta\mathcal{F}[\psi]) \\
&= \alpha(f, \mathcal{F}[\varphi]) + \beta(f, \mathcal{F}[\psi])
\end{aligned}$$

Continuity: let $\varphi_k \rightarrow \varphi$ as $k \rightarrow \infty$ in $\mathcal{S}(\mathbb{R}^n)$. From theorem 1 of section 1.1, we have

$$\mathcal{F}[\varphi_k] \rightarrow \mathcal{F}[\varphi] \text{ as } k \rightarrow \infty \text{ in } \mathcal{S}(\mathbb{R}^n).$$

$$(f, \mathcal{F}[\varphi_k]) \rightarrow (f, \mathcal{F}[\varphi]) \text{ as } k \rightarrow \infty, \quad f \in \mathcal{S}'(\mathbb{R}^n)$$

So that the functional $(f, \mathcal{F}[\varphi])$ is continuous over $\mathcal{S}(\mathbb{R}^n)$.

Remark: The Fourier transform of generalized function of slow growth is a generalized function of slow growth.

Lemma 1: The operation of Fourier transform $\mathcal{F}$ is linear and continuous from $\mathcal{S}'$ into $\mathcal{S}'$.

Proof:

Linearity: Let $f, g \in \mathcal{S}'$ and for any $\alpha, \beta \in \mathbb{C}$

$$\begin{aligned} (\mathcal{F}[\alpha f + \beta g], \varphi) &= (\alpha f + \beta g, \mathcal{F}[\varphi]) \\ &= (\alpha f, \mathcal{F}[\varphi]) + (\beta g, \mathcal{F}[\varphi]) \\ &= \alpha (f, \mathcal{F}[\varphi]) + \beta (g, \mathcal{F}[\varphi]) \\ &= \alpha (\mathcal{F}[f], \varphi) + \beta (\mathcal{F}[g], \varphi) \end{aligned}$$

Continuity: Let $f_k \rightarrow f$ as $k \rightarrow \infty$, in $\mathcal{S}'$. then by definition of Fourier transform, for all $\varphi \in \mathcal{S}$, we obtain

$$(\mathcal{F}[f_k], \varphi) = (f_k, \mathcal{F}[\varphi]) \rightarrow (f, \mathcal{F}[\varphi]) = (\mathcal{F}[f], \varphi) \text{ as } k \rightarrow \infty$$

This show that $\mathcal{F}[f_k] \rightarrow \mathcal{F}[f]$ as $k \rightarrow \infty$, in $\mathcal{S}'$, that is, the operation $\mathcal{F}$ is continuous from $\mathcal{S}'$ into $\mathcal{S}'$.

Now, let us introduce into $\mathcal{S}'$ one more Fourier transform operation, which we will denote by $\mathcal{F}^{-1}$:

$$\mathcal{F}^{-1}[f] = (2\pi)^{-n} \mathcal{F}[f(-x)], f \in \mathcal{S}', \dots\dots\dots (1.7)$$

Lemma 2: The Fourier transform operation $\mathcal{F}^{-1}$ is the inverse operation of the Fourier transform $\mathcal{F}$, that is,

$$\mathcal{F}^{-1}[\mathcal{F}[f]] = f, \quad \mathcal{F}[\mathcal{F}^{-1}[f]] = f \dots\dots\dots (1.8)$$

Proof: Let $\varphi \in \mathcal{S}$. Then

$$\begin{aligned} (\mathcal{F}^{-1}[\mathcal{F}[f]], \varphi) &= (2\pi)^{-n} (\mathcal{F}[\mathcal{F}[f]](-\xi), \varphi) \\ &= (2\pi)^{-n} (\mathcal{F}[f](-\xi), \mathcal{F}[\varphi]) \\ &= (2\pi)^{-n} (\mathcal{F}[f], \mathcal{F}[\varphi](-\xi)) \end{aligned}$$

$$\begin{aligned}
 &= (\mathcal{F}[f], \mathcal{F}^{-1}[\varphi]) = (f, \mathcal{F}[\mathcal{F}^{-1}[\varphi]]) \\
 &= (f, \varphi) = (f, \mathcal{F}^{-1}[\mathcal{F}[\varphi]]) \\
 &= (\mathcal{F}^{-1}[f], \mathcal{F}[\varphi]) = (\mathcal{F}[\mathcal{F}^{-1}[f]], \varphi)
 \end{aligned}$$

Therefore, the operation $\mathcal{F}^{-1}$ is the inverse of the operation $\mathcal{F}$.

Let $f(x, y) \in \mathcal{S}'(\mathbb{R}^{n+m})$ where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$. We introduce the Fourier transform $\mathcal{F}_x[f]$ with respect to the variable $x = (x_1, x_2, \dots, x_n)$ for any basic function $\varphi(\xi, y) \in \mathcal{S}(\mathbb{R}^{n+m})$

$$(\mathcal{F}_x[f], \varphi) = (f, \mathcal{F}_\xi[\varphi]) \dots\dots\dots (1.9)$$

Let us now prove that the right hand side of equation (1.9) is a continuous linear functional over $\mathcal{S}(\mathbb{R}^{n+m})$.

Since $\mathcal{F}_\xi[\varphi] \in \mathcal{S}(\mathbb{R}^{n+m})$ for all $\varphi(\xi, y) \in \mathcal{S}(\mathbb{R}^{n+m})$, then $(f, \mathcal{F}_\xi[\varphi])$ is a functional over $\mathcal{S}(\mathbb{R}^{n+m})$.

Linearity: let $\varphi, \psi \in \mathcal{S}(\mathbb{R}^{n+m})$ and $\alpha, \beta \in \mathbb{C}$

$$\begin{aligned}
 (\mathcal{F}_x[f], \alpha\varphi + \beta\psi) &= (f, \mathcal{F}_\xi[\alpha\varphi + \beta\psi]) \\
 &= (f, \mathcal{F}_\xi[\alpha\varphi] + \mathcal{F}_\xi[\beta\psi]) \\
 &= (f, \alpha\mathcal{F}_\xi[\varphi]) + (f, \beta\mathcal{F}_\xi[\psi]) \\
 &= \alpha(f, \mathcal{F}_\xi[\varphi]) + \beta(f, \mathcal{F}_\xi[\psi]) \\
 &= \alpha(\mathcal{F}_x[f], \varphi) + \beta(\mathcal{F}_x[f], \psi)
 \end{aligned}$$

Continuity: let $\varphi_k \rightarrow \varphi$ as $k \rightarrow \infty$ in $\mathcal{S}(\mathbb{R}^{n+m})$ and hence, $\mathcal{F}_\xi[\varphi_k] \rightarrow \mathcal{F}_\xi[\varphi]$, $k \rightarrow \infty$ in $\mathcal{S}(\mathbb{R}^{n+m})$. Then

$$(\mathcal{F}_x[f], \varphi_k) = (f, \mathcal{F}_\xi[\varphi_k]) \rightarrow (f, \mathcal{F}_\xi[\varphi]) = (\mathcal{F}_x[f], \varphi)$$

Therefore, $\mathcal{F}_x$ is continuous linear functional over $\mathcal{S}(\mathbb{R}^{n+m})$, that is, $\mathcal{F}_x[f] \in \mathcal{S}'(\mathbb{R}^{n+m})$

1.2.2 Properties

1. Differentiation of the Fourier transform

$$D^\alpha \mathcal{F}[f](\xi) = \mathcal{F}[(ix)^\alpha f]$$

Proof:

For all $\varphi \in \mathcal{S}$ (from properties of section 1.1), we obtain

$$\begin{aligned} (D^\alpha \mathcal{F}[f], \varphi) &= (-1)^{|\alpha|} (\mathcal{F}[f], D^\alpha \varphi) \\ &= (-1)^{|\alpha|} (f, \mathcal{F}[D^\alpha \varphi]) \\ &= (-1)^{|\alpha|} (f, (-ix)^\alpha \mathcal{F}[\varphi]) \\ &= ((ix)^\alpha f, \mathcal{F}[\varphi]) \\ &= (\mathcal{F}[(ix)^\alpha f], \varphi) \end{aligned}$$

$$\text{Therefore, } D^\alpha \mathcal{F}[f](\xi) = \mathcal{F}[(ix)^\alpha f]$$

2. The Fourier transform of the derivatives

$$\mathcal{F}[D^\alpha f] = (-i\xi)^\alpha \mathcal{F}[f]$$

Proof:

For all $\varphi \in \mathcal{S}$ (from properties of section 1.1), we obtain

$$\begin{aligned} (\mathcal{F}[D^\alpha f], \varphi) &= (D^\alpha f, \mathcal{F}[\varphi]) \\ &= (-1)^{|\alpha|} (f, D^\alpha \mathcal{F}[\varphi]) \\ &= (-1)^{|\alpha|} (f, \mathcal{F}[(i\xi)^\alpha \varphi]) \\ &= (-1)^{|\alpha|} (\mathcal{F}[f], (i\xi)^\alpha \varphi) \\ &= ((-i\xi)^\alpha \mathcal{F}[f], \varphi) \end{aligned}$$

$$\text{Thus, } \mathcal{F}[D^\alpha f] = (-i\xi)^\alpha \mathcal{F}[f]$$

3. The Fourier transform of a translation

$$\mathcal{F}[f(x - x_0)] = e^{i(x_0, \xi)} \mathcal{F}[f]$$

Proof:

In fact, for all $\varphi \in \mathcal{S}$, we have

$$\begin{aligned}
 (\mathcal{F}[f(x - x_0)], \varphi) &= (f(x - x_0), \mathcal{F}[\varphi]) \\
 &= (f, \mathcal{F}[\varphi](x + x_0)) \\
 &= (f, \mathcal{F}[e^{i(x_0, \xi)} \varphi]) \\
 &= (\mathcal{F}[f], e^{i(x_0, \xi)} \varphi) \\
 &= (e^{i(x_0, \xi)} \mathcal{F}[f], \varphi)
 \end{aligned}$$

$$\text{Hence, } \mathcal{F}[f(x - x_0)] = e^{i(x_0, \xi)} \mathcal{F}[f]$$

4. *The translation of a Fourier transform*

$$\mathcal{F}[f](\xi + \xi_0) = \mathcal{F}[e^{i(\xi_0, x)} f]$$

Proof:

Using number (3) for all $\varphi \in \mathcal{S}$, we obtain

$$\begin{aligned}
 (\mathcal{F}[f](\xi + \xi_0), \varphi) &= (\mathcal{F}[f], \varphi(\xi - \xi_0)) \\
 &= (f, \mathcal{F}[\varphi(\xi - \xi_0)]) \\
 &= (f, e^{i(\xi_0, x)} \mathcal{F}[\varphi]) \\
 &= (e^{i(\xi_0, x)} f, \mathcal{F}[\varphi]) \\
 &= (\mathcal{F}[e^{i(\xi_0, x)} f], \varphi)
 \end{aligned}$$

$$\text{Therefore, } \mathcal{F}[f](\xi + \xi_0) = \mathcal{F}[e^{i(\xi_0, x)} f]$$

5. Analogous equations are valid for the Fourier transform $\mathcal{F}_x$.

$$\text{a) } D_x^\alpha \mathcal{F}_x[f] = \mathcal{F}_x[(ix)^\alpha f]$$

$$\text{b) } \mathcal{F}_x[D_x^\alpha f] = (-i\xi)^\alpha \mathcal{F}_x[f]$$

Proof:

Let $\varphi(\xi, y) \in \mathcal{S}(\mathbb{R}^{n+m})$. Then

$$\begin{aligned}
 (a) \quad (D_x^\alpha \mathcal{F}_x[f], \varphi) &= (-1)^{|\alpha|} (\mathcal{F}_x[f], D_\xi^\alpha \varphi) \\
 &= (-1)^{|\alpha|} (f, \mathcal{F}_\xi[D_\xi^\alpha \varphi]) \\
 &= (-1)^{|\alpha|} (f, (-ix)^\alpha \mathcal{F}_\xi[\varphi]) \\
 &= (-1)^{|\alpha|} ((-ix)^\alpha f, \mathcal{F}_\xi[\varphi]) \\
 &= (-1)^{|\alpha|} (\mathcal{F}_x[(-ix)^\alpha f], \varphi) \\
 &= (\mathcal{F}_x[(ix)^\alpha f], \varphi)
 \end{aligned}$$

Thus, $D_x^\alpha \mathcal{F}_x[f] = \mathcal{F}_x[(ix)^\alpha f]$

$$\begin{aligned}
 (b) \quad (\mathcal{F}_x[D_x^\alpha f], \varphi) &= (D_x^\alpha f, \mathcal{F}_\xi[\varphi]) \\
 &= (-1)^{|\alpha|} (f, D_\xi^\alpha \mathcal{F}_\xi[\varphi]) \\
 &= (-1)^{|\alpha|} (f, \mathcal{F}_\xi[(i\xi)^\alpha \varphi]) \\
 &= (-1)^{|\alpha|} (\mathcal{F}_x[f], (i\xi)^\alpha \varphi) \\
 &= ((-i\xi)^\alpha \mathcal{F}_x[f], \varphi)
 \end{aligned}$$

Thus, $\mathcal{F}_x[D_x^\alpha f] = (-i\xi)^\alpha \mathcal{F}_x[f]$

1.2.3 Examples

$$1. \quad \mathcal{F} \left[\frac{1}{|x|^2} \right] = \frac{2\pi^2}{|\xi|^2}, \quad n = 3$$

Solution:

The function $|x|^{-2}$ is locally integrable in $\mathbb{R}^3$ and for all $\varphi \in \mathcal{S}$

$$\begin{aligned}
\left(\mathcal{F}\left[\frac{1}{|x|^2}\right], \varphi\right) &= \left(\frac{1}{|x|^2}, \mathcal{F}[\varphi]\right) = \int \frac{1}{|x|^2} \mathcal{F}[\varphi](x) dx \\
&= \int \frac{1}{|x|^2} \int \varphi(\xi) e^{i(x, \xi)} d\xi dx \\
&= \int \varphi(\xi) \int \frac{1}{|x|^2} e^{i(\xi, x)} dx d\xi
\end{aligned}$$

Using polar coordinates:

$$x_1 = r \sin \alpha \cos \beta, \quad x_2 = r \sin \alpha \sin \beta, \quad x_3 = r \cos \alpha$$

Then the wronski determinant is

$$\begin{vmatrix}
\sin \alpha \cos \beta & r \cos \alpha \cos \beta & -r \sin \alpha \sin \beta \\
\sin \alpha \sin \beta & r \cos \alpha \sin \beta & r \sin \alpha \cos \beta \\
\cos \alpha & -r \sin \alpha & 0
\end{vmatrix} = r^2 \sin \alpha$$

Let $|x| = r$ and $(\xi, x) = |x||\xi| \cos \alpha = r|\xi| \cos \alpha$, then we get

$$\begin{aligned}
\int \varphi(\xi) \int \frac{1}{|x|^2} e^{i(\xi, x)} dx d\xi &= \int \varphi(\xi) \int_0^R \int_0^\pi \int_0^{2\pi} \frac{e^{ir|\xi| \cos \alpha}}{r^2} r^2 \sin \alpha d\beta d\alpha dr d\xi \\
&= \int \varphi(\xi) \int_0^R \int_0^\pi \int_0^{2\pi} e^{ir|\xi| \cos \alpha} \sin \alpha d\beta d\alpha dr d\xi \\
&= \int_0^{2\pi} d\beta \int \varphi(\xi) \int_0^R \int_0^\pi e^{ir|\xi| \cos \alpha} \sin \alpha d\alpha dr d\xi \\
&= 2\pi \int \varphi(\xi) \int_0^R \int_0^\pi e^{ir|\xi| \cos \alpha} \sin \alpha d\alpha dr d\xi \\
&= 2\pi \int \varphi(\xi) \int_0^R \int_{-1}^1 e^{ir|\xi| u} du dr d\xi
\end{aligned}$$

$$\begin{aligned}
&= 4\pi \int \frac{\varphi(\xi)}{|\xi|} \underbrace{\int_0^R \frac{\sin r|\xi|}{r} dr}_{=\pi/2} d\xi \\
&= 2\pi^2 \int \frac{\varphi(\xi)}{|\xi|} d\xi \\
&= \left(\frac{2\pi^2}{|\xi|}, \varphi \right)
\end{aligned}$$

$$\text{Therefore, } \mathcal{F} \left[\frac{1}{|x|^2} \right] = \frac{2\pi^2}{|\xi|^2}, \quad n = 3$$

2. Let $f(x) = 1, x \in \mathbb{R}^n$ and 1 = constant function. Then $\mathcal{F}[1] = \delta$

Solution:

Let $\varphi \in \mathcal{S}$

$$\begin{aligned}
(\mathcal{F}[1], \varphi) &= (1, \mathcal{F}[\varphi]) \\
&= \int 1 \cdot \int \varphi(x) e^{i(\xi, x)} dx d\xi = \int \underbrace{\left[\int \varphi(x) e^{i(\xi, x)} dx \right]}_{\text{fourier transform of } \varphi} e^{i(0, \xi)} d\xi \\
&= \int \mathcal{F}[\varphi](\xi) e^{i(0, \xi)} d\xi = \mathcal{F}[\mathcal{F}[\varphi]](0) \\
&= (\mathcal{F}\mathcal{F}\varphi)(0) = (\mathcal{I}\varphi)(0) = \varphi(0) = (\delta, \varphi)
\end{aligned}$$

$$\text{Therefore, } \mathcal{F}[1] = \delta$$

3. Find a fundamental solution of the heat conduction operator differential equation

$$L := \frac{\partial}{\partial t} - a^2 \Delta$$

Solution:

Let ε be a fundamental solution of the heat conduction operator differential equation. Then ε satisfy

$$\frac{\partial \varepsilon(x, t)}{\partial t} - a^2 \Delta \varepsilon(x, t) = \delta(x, t) \dots \dots \dots (\star)$$

Now, applying the Fourier transform $\mathcal{F}_x$ to $(\star)$, we obtain

$$\mathcal{F}_x \left[\frac{\partial \varepsilon(x, t)}{\partial t} - a^2 \Delta \varepsilon(x, t) \right] = \mathcal{F}_x[\delta(x, t)]$$

But,

$$\mathcal{F}_x \left[\frac{\partial \varepsilon(x, t)}{\partial t} \right] = \frac{\partial}{\partial t} \mathcal{F}_x[\varepsilon(x, t)]$$

$$\mathcal{F}_x[\Delta \varepsilon(x, t)] = \int \frac{\partial^2 \varepsilon(x, t)}{\partial x^2} e^{i(\xi, x)} dx = \int \varepsilon(x, t) \frac{\partial^2 e^{i(\xi, x)}}{\partial x^2} dx$$

$$= \int \varepsilon(x, t) (i\xi)^2 e^{i(\xi, x)} dx$$

$$= i^2 (\xi_1^2 + \xi_2^2 + \dots + \xi_n^2) \int \varepsilon(x, t) e^{i(\xi, x)} dx$$

$$= -|\xi|^2 \mathcal{F}_x[\varepsilon(x, t)]$$

$$(\mathcal{F}_x[\delta(x, t)], \varphi) = (\delta(x, t), \mathcal{F}_\xi[\varphi])$$

$$= \mathcal{F}_\xi[\varphi(x, t)](0) = \varphi(\xi, 0) = 1(\xi) \delta(t)$$

$$\text{Therefore, } \frac{\partial}{\partial t} \mathcal{F}_x[\varepsilon(x, t)] + a^2 |\xi|^2 \mathcal{F}_x[\varepsilon(x, t)] = 1(\xi) \delta(t)$$

$$\text{Let } E(\xi, t) = \mathcal{F}_x[\varepsilon(x, t)], \text{ then } \frac{\partial}{\partial t} E(\xi, t) + a^2 |\xi|^2 E(\xi, t) = 1(\xi) \delta(t)$$

Let $L E(\xi, t) := \frac{\partial}{\partial t} E(\xi, t) + a^2 |\xi|^2 E(\xi, t)$ is a linear differential operator with constant coefficients. The solution is a generalized function $\theta(t)z(t)$ where $z(t)$ is the solution of initial value problem

$$L z(t) = 0, \quad z(0) = 1$$

$$\text{Hence, } \frac{\partial}{\partial t} z(t) + a^2 |\xi|^2 z(t) = 0. \text{ this implies that } z(t) = e^{-a^2 |\xi|^2 t}$$

Therefore, $E(\xi, t) = \theta(t)z(t) = \theta(t)e^{-a^2|\xi|^2 t}$ and applying inverse Fourier transform $\mathcal{F}_x^{-1}$, we obtain

$$\begin{aligned}
 \varepsilon(x, t) &= \mathcal{F}_\xi^{-1}[\theta(t)e^{-a^2|\xi|^2 t}] = \theta(t)\mathcal{F}_\xi^{-1}[e^{-a^2|\xi|^2 t}] \\
 &= \frac{\theta(t)}{2\pi} \int e^{-a^2|\xi|^2 t} e^{-i(\xi, x)} d\xi \\
 &= \frac{\theta(t)}{2\pi} \int e^{-a^2 t \left(|\xi|^2 + \frac{i|x||\xi|}{a^2 t} \right)} d\xi \\
 &= \frac{\theta(t)}{2\pi} \int e^{-a^2 t \left(|\xi| + \frac{i|x|}{2a^2 t} \right)^2 - \frac{|x|^2}{4a^2 t}} d\xi \\
 &= \frac{\theta(t)e^{-\frac{|x|^2}{4a^2 t}}}{2\pi} \int e^{-a^2 t \left(|\xi| + \frac{i|x|}{2a^2 t} \right)^2} d\xi \\
 &= \frac{\theta(t)e^{-\frac{|x|^2}{4a^2 t}}}{2\pi} \int e^{-\left(a\sqrt{t} \left(|\xi| + \frac{i|x|}{2a^2 t} \right) \right)^2} d\xi \\
 &= \frac{\theta(t)e^{-\frac{|x|^2}{4a^2 t}}}{2\pi} a\sqrt{t} \underbrace{\int e^{-t^2} dt}_{=\sqrt{\pi}} = \frac{\theta(t)e^{-\frac{|x|^2}{4a^2 t}}}{2a\sqrt{\pi t}}
 \end{aligned}$$

Therefore, the fundamental solution is $\frac{\theta(t)e^{-\frac{|x|^2}{4a^2 t}}}{2a\sqrt{\pi t}}$.

CHAPTER TWO

The Laplace transform of generalized functions of slow growth

2.1 The Laplace transform defined

Let Γ be a closed convex acute cone in $\mathbb{R}^n$ with vertex at 0 and put $C = \text{int}\Gamma^*$. Let a tubular region in $\mathbb{C}^n$ with base C is denote by T^C :

$$T^C = \mathbb{R}^n + iC = \{Z = x + iy, x \in \mathbb{R}^n, y \in C\}$$

Definition: suppose $g \in \mathcal{S}'(\Gamma^+)$. Then the Laplace transform $\mathcal{L}[g]$ of g is defined by

$$\mathcal{L}[g](z) = \mathcal{F}[g(\xi)e^{-(y,\xi)}](x) \quad \dots\dots\dots (2.1)$$

where $\mathcal{F}$ is a Fourier transform operation.

Let us now prove that $g(\xi)e^{-(y,\xi)} \in \mathcal{S}'$ for all $y \in C$.

Suppose η is any C^∞ -function with the following properties:

$$\begin{aligned} |D^\alpha \eta(\xi)| &\leq C_\alpha; \\ \eta(\xi) &= \begin{cases} 1, & \xi \in (\text{supp } g)^\varepsilon \\ 0, & \xi \notin (\text{supp } g)^{2\varepsilon} \end{cases} \quad \dots\dots\dots (2.2) \end{aligned}$$

where ε is any number greater than zero.

First, let us now prove that

$$\eta(\xi)e^{-(y,\xi)} \in \mathcal{S}(\mathbb{R}^n), \text{ for all } y \in C$$

Since $e^{-(y,\xi)} \in \mathcal{S}$, then $\eta(\xi)e^{-(y,\xi)} \in C^\infty$ and by the properties of η :

$$\begin{aligned} \|\eta(\xi)e^{-(y,\xi)}\|_p &= \sup_{\xi \in \mathbb{R}^n, |\alpha| \leq p} (1 + |\xi|^2)^{p/2} |D^\alpha(\eta(\xi)e^{-(y,\xi)})| \\ &\leq \sup_{\xi \in \mathbb{R}^n, |\alpha| \leq p} (1 + |\xi|^2)^{p/2} \sum_{\beta \leq \alpha} \binom{\beta}{\alpha} |D^{\alpha-\beta}(\eta(\xi)D^\beta e^{-(y,\xi)})| \end{aligned}$$

$$\begin{aligned}
&\leq C_\alpha \sup_{\xi \in \mathbb{R}^n, |\alpha| \leq p} (1 + |\xi|^2)^{p/2} |D^\alpha e^{-(y, \xi)}| \\
&\leq C_\alpha \|e^{-(y, \xi)}\|_p
\end{aligned}$$

These inequality signify that $\eta(\xi)e^{-(y, \xi)} \in \mathcal{S}(\mathbb{R}^n)$ for all $y \in \mathbb{C}$ and therefore,

$$g(\xi)e^{-(y, \xi)} = g(\xi)\eta(\xi)e^{-(y, \xi)} \in \mathcal{S}'.$$

Remark: The Laplace transform of generalized functions of slow growth is a generalized function of slow growth with respect to x for all $y \in \mathbb{C}$.

Lemma 1: The operation of Laplace transform $\mathcal{L}$ is a linear.

Proof:

This follows from the appropriate properties of Fourier transform operation and let $f, g \in \mathcal{S}'$, let $\varphi \in \mathcal{S}$ and for any $\alpha, \beta \in \mathbb{C}$,

$$\begin{aligned}
(\mathcal{L}[\alpha f + \beta g], \varphi) &= (\mathcal{F}[(\alpha f(\xi) + \beta g(\xi))e^{-(y, \xi)}], \varphi) \\
&= ((\alpha f(\xi) + \beta g(\xi))e^{-(y, \xi)}, \mathcal{F}[\varphi]) \\
&= \alpha(f(\xi)e^{-(y, \xi)}, \mathcal{F}[\varphi]) + \beta(g(\xi)e^{-(y, \xi)}, \mathcal{F}[\varphi]) \\
&= \alpha(\mathcal{F}[f(\xi)e^{-(y, \xi)}], \varphi) + \beta(\mathcal{F}[g(\xi)e^{-(y, \xi)}], \varphi) \\
&= \alpha(\mathcal{L}[f], \varphi) + \beta(\mathcal{L}[g], \varphi) \\
&= (\alpha\mathcal{L}[f], \varphi) + (\beta\mathcal{L}[g], \varphi) \\
&= (\alpha\mathcal{L}[f] + \beta\mathcal{L}[g], \varphi)
\end{aligned}$$

Therefore, $\mathcal{L}[\alpha f + \beta g] = \alpha\mathcal{L}[f] + \beta\mathcal{L}[g]$

This show that the operation of Laplace transforms $\mathcal{L}$ is linear.

Lemma 2: Suppose $g \in \mathcal{S}'(\Gamma +)$, then the Laplace transform of g can be represented as

$$\mathcal{L}[g] = (g(\xi), \eta(\xi)e^{i(z, \xi)}) \quad z \in \mathbb{T}^{\mathbb{C}} \dots\dots\dots (2.3)$$

Proof:

Let $\varphi \in \mathcal{S}$ and $y \in \mathbb{C}$, then

$$\begin{aligned} (\mathcal{L}[g], \varphi) &= (\mathcal{F}[g(\xi)e^{-(y, \xi)}], \varphi) \\ &= (g(\xi)e^{-(y, \xi)}, \mathcal{F}[\varphi]) \\ &= (g(\xi), \eta(\xi)e^{-(y, \xi)} \int \varphi(x)e^{i(\xi, x)}) \\ &= \int (g(\xi), \eta(\xi)e^{-(y, \xi)} \varphi(x)e^{i(\xi, x)}) \varphi(x) dx \\ &= \int (g(\xi), \eta(\xi)e^{i(z, \xi)}) \varphi(x) dx \\ &= ((g(\xi), \eta(\xi)e^{i(z, \xi)}), \varphi) \end{aligned}$$

Therefore, $\mathcal{L}[g] = (g(\xi), \eta(\xi)e^{i(z, \xi)})$.

Remark: This representation does not depend on the auxiliary function η within the above-indicated properties.

Theorem 1: let $f(z) = \mathcal{L}[g]$. Then the function $f(z)$ is holomorphic in $\mathbb{T}^{\mathbb{C}}$ and the following differentiation formula holds true;

$$D^\alpha f(z) = ((i\xi)^\alpha g(\xi), \eta(\xi)e^{i(z, \xi)}) \dots\dots\dots (2.4)$$

Proof:

First let us now prove that $f(z)$ is continuous in $\mathbb{T}^{\mathbb{C}}$.

Let fix an arbitrary point $z_0 \in \mathbb{T}^{\mathbb{C}}$ and let $z \rightarrow z_0$. Then from the continuity of the function $\eta(\xi)e^{i(z, \xi)}$ with respect to $z \in \mathbb{T}^{\mathbb{C}}$ in the sense of convergence in $\mathcal{S}$,

$$\eta(\xi)e^{i(z,\xi)} \rightarrow \eta(\xi)e^{i(z_0,\xi)}, z \rightarrow z_0 \text{ in } \mathcal{S} \dots\dots\dots (2.5)$$

Taking into account the continuity of the functional g on $\mathcal{S}$, from lemma 2 and equation (2.5), as $z \rightarrow z_0$, we obtain

$$f(z) = (g(\xi), \eta(\xi)e^{i(z,\xi)}) \rightarrow (g(\xi), \eta(\xi)e^{i(z_0,\xi)}) = f(z_0),$$

which is the function $f(z)$ is continuous at arbitrary point z_0 .

Thus, $f(z)$ is continuous in T^C .

To prove the holomorphicity of the function $f(z)$ in T^C it suffices to establish that the existence of all first derivatives $\frac{\partial f}{\partial z_j}$, $j = 1, \dots, n$.

Suppose $e_1 = (1, 0, 0, \dots, 0)$. Then for every $z \in T^C$

$$\chi_h(\xi) = \frac{1}{h}(\eta(\xi)e^{i(z+he_1,\xi)} - \eta(\xi)e^{i(z,\xi)}) \rightarrow \eta(\xi)(i\xi)e^{i(z,\xi)}$$

as $h \rightarrow 0$ in $\mathcal{S}$.

Therefore, from lemma 2 and from the linearity and the continuity of the functional g on $\mathcal{S}$, (see sec. 1.1) as $h \rightarrow 0$. We have

$$\begin{aligned} \frac{f(z+he_1) - f(z)}{h} &= \frac{1}{h}[(g(\xi), \eta(\xi)e^{i(z+he_1,\xi)}) - (g(\xi), \eta(\xi)e^{i(z,\xi)})] \\ &= \frac{1}{h}[(g(\xi), \eta(\xi)e^{i(z+he_1,\xi)} - \eta(\xi)e^{i(z,\xi)})] \\ &= (g(\xi), \frac{1}{h}[\eta(\xi)e^{i(z+he_1,\xi)} - \eta(\xi)e^{i(z,\xi)}]) \\ &= (g(\xi), \chi_h(\xi)) \rightarrow (g(\xi), \eta(\xi)i\xi_1e^{i(z,\xi)}) \\ &= (i\xi_1g(\xi), \eta(\xi)e^{i(z,\xi)}) \end{aligned}$$

So that the derivatives $\frac{\partial f}{\partial z_1}$ exist and the differentiation formula (2.4) hold for $\alpha = (1, 0, \dots, 0)$ and hence, also for all first derivatives,

$$\frac{\partial f}{\partial z_j} = (i\xi_j g(\xi), \eta(\xi)e^{i(z, \xi)}) , j = 1, \dots, n \dots\dots\dots (2.6)$$

Again applying the same reasoning to the formula (2.6), we see that the formula (2.4) hold for all second derivatives and so forth.

Definition: The generalized function $g \in \mathcal{S}'(\Gamma^+)$, for which $f = \mathcal{L}[g]$, where the function f is called the transform of the function g and the function g is called *spectral function* of the function f . Hence, the spectral function is unique (if it exist) and by virtue of equation (2.1),

$$g(\xi) = e^{(y, \xi)} \mathcal{F}_x^{-1}[f(x + iy)](\xi) \dots\dots\dots (2.7)$$

And the right hand side of equation (2.7) is independent of $y \in \mathbb{C} = \text{int}\Gamma^*$

2.2 Properties of the Laplace transform

These Properties follows from the appropriate Properties of the Fourier transform.

1. Differentiation of the Laplace transform:

$$D^\alpha \mathcal{L}[g] = \mathcal{L}[(i\xi)^\alpha g]$$

Proof: let for all $\varphi \in \mathcal{S}$, then

$$\begin{aligned} (D^\alpha \mathcal{L}[g], \varphi) &= (D^\alpha \mathcal{F}[g(\xi)e^{-(y, \xi)}], \varphi) \\ &= (-1)^{|\alpha|} (\mathcal{F}[g(\xi)e^{-(y, \xi)}], D^\alpha[\varphi]) \\ &= (-1)^{|\alpha|} (g(\xi)e^{-(y, \xi)}, \mathcal{F}[D^\alpha[\varphi]]) \\ &= (-1)^{|\alpha|} (g(\xi)e^{-(y, \xi)}, (-i\xi)^\alpha \mathcal{F}[\varphi]) \\ &= (-1)^{|\alpha|} ((-i\xi)^\alpha g(\xi)e^{-(y, \xi)}, \mathcal{F}[\varphi]) \\ &= ((i\xi)^\alpha g(\xi)e^{-(y, \xi)}, \mathcal{F}[\varphi]) \end{aligned}$$

$$= (\mathcal{F}[(i\xi)^\alpha g(\xi)e^{-(y,\xi)}], \varphi) = (\mathcal{L}[(i\xi)^\alpha g(\xi)], \varphi)$$

$$\text{Therefore, } D^\alpha \mathcal{L}[g] = \mathcal{L}[(i\xi)^\alpha g]$$

2. *The Laplace transform of derivative:*

$$\mathcal{L}[D^\alpha g] = (-iz)^\alpha \mathcal{L}[g]$$

Proof:

Let $\varphi \in \mathcal{S}$. It suffices to prove the Laplace transform of a derivative for the first derivatives, we have

$$\begin{aligned} \mathcal{L}\left[\frac{\partial g}{\partial \xi_i}\right] &= \mathcal{F}\left[\frac{\partial g(\xi)}{\partial \xi_i} e^{-(y,\xi)}\right] \\ &= \mathcal{F}\left[\frac{\partial}{\partial \xi_i}(g(\xi))e^{-(y,\xi)}\right] + y_i \mathcal{F}[g(\xi)e^{-(y,\xi)}] \\ &= -ix_i \mathcal{F}[g(\xi)e^{-(y,\xi)}] + y_i \mathcal{F}[g(\xi)e^{-(y,\xi)}] \\ &= -ix_i \mathcal{L}[g] + y_i \mathcal{L}[g] \\ &= (-ix_i + y_i) \mathcal{L}[g] = -iz_i \mathcal{L}[g] \end{aligned}$$

$$\text{Therefore, } \mathcal{L}[D^\alpha g] = (-iz)^\alpha \mathcal{L}[g].$$

3. *The translation of Laplace transform:*

$$\text{If } \operatorname{Im} a \in \mathbb{C}, \text{ then } \mathcal{L}[g](z+a) = \mathcal{L}[g(\xi)e^{i(a,\xi)}]$$

Proof:

Let $\varphi \in \mathcal{S}$.

$$\begin{aligned} (\mathcal{L}[g](z+a), \varphi) &= (\mathcal{L}[g](x + \operatorname{Re} a + iy + i\operatorname{Im} a), \varphi) \\ &= (\mathcal{F}[g(\xi)e^{i(\operatorname{Re} a, \xi)}e^{-(y + i\operatorname{Im} a, \xi)}], \varphi) \\ &= (g(\xi)e^{i(\operatorname{Re} a, \xi)}e^{-(y + i\operatorname{Im} a, \xi)}, \mathcal{F}[\varphi]) \\ &= (g(\xi)e^{i(\operatorname{Re} a, \xi)}e^{-(\operatorname{Im} a, \xi)}e^{-(y, \xi)}, \mathcal{F}[\varphi]) \end{aligned}$$

$$\begin{aligned}
&= (g(\xi)e^{i(Rea+ilma,\xi)}e^{-(y,\xi)}, \mathcal{F}[\varphi]) \\
&= (g(\xi)e^{i(a,\xi)}e^{-(y,\xi)}, \mathcal{F}[\varphi]) \\
&= (\mathcal{F}[g(\xi)e^{i(a,\xi)}e^{-(y,\xi)}], \varphi) \\
&= (\mathcal{L}[g(\xi)e^{i(a,\xi)}], \varphi)
\end{aligned}$$

Therefore, $\mathcal{L}[g](z+a) = \mathcal{L}[g(\xi)e^{i(a,\xi)}]$

4. *The Laplace transform of a translation*

$$\mathcal{L}[g(\xi - \xi_0)] = e^{i(z,\xi_0)}\mathcal{L}[g](z)$$

Proof:

Let $\varphi \in \mathcal{S}$.

$$\begin{aligned}
(e^{i(z,\xi_0)}\mathcal{L}[g](z), \varphi) &= (e^{i(z,\xi_0)}\mathcal{F}[g(\xi)e^{-(y,\xi)}], \varphi) \\
&= (\mathcal{F}[g(\xi)e^{-(y,\xi)}], e^{i(z,\xi_0)}\varphi) \\
&= (g(\xi)e^{-(y,\xi)}, \mathcal{F}[e^{i(z,\xi_0)}\varphi]) \\
&= (g(\xi)e^{-(y,\xi)}, e^{-(y,\xi_0)}\mathcal{F}[\varphi](\xi + \xi_0)) \\
&= (g(\xi)e^{-(y,\xi+\xi_0)}, \mathcal{F}[\varphi](\xi + \xi_0)) \\
&= (g(\xi - \xi_0)e^{-(y,\xi)}, \mathcal{F}[\varphi]) \\
&= (\mathcal{F}[g(\xi - \xi_0)e^{-(y,\xi)}], \varphi) \\
&= (\mathcal{L}[g(\xi - \xi_0)], \varphi)
\end{aligned}$$

Therefore $\mathcal{L}[g(\xi - \xi_0)] = e^{i(z,\xi_0)}\mathcal{L}[g](z)$

2.3 Examples

1. Show that a) $\mathcal{L}[\theta(\xi)\sin(\omega\xi)] = \frac{\omega}{\omega^2 - z^2}$

b) $\mathcal{L}[\theta(\xi)\cos(\omega\xi)] = \frac{-iz}{\omega^2 - z^2}$

Solution: a) $\mathcal{L}[\theta(\xi)\sin(\omega\xi)] = \mathcal{F}[\theta(\xi)\sin(\omega\xi)e^{-(y,\xi)}]$

$$\begin{aligned}
 &= \int_0^\infty \sin(\omega\xi) e^{-(y,\xi)} e^{i(x,\xi)} d\xi \\
 &= \int_0^\infty \sin(\omega\xi) e^{i(z,\xi)} d\xi \\
 &= \frac{1}{2i} \int_0^\infty (e^{i\omega\xi} - e^{-i\omega\xi}) e^{i(z,\xi)} d\xi \\
 &= \frac{1}{2i} \int_0^\infty e^{i(\omega+z,\xi)} d\xi - \frac{1}{2i} \int_0^\infty e^{-i(\omega-z,\xi)} d\xi \\
 &= \frac{1}{2(\omega+z)} + \frac{1}{2(\omega-z)} \\
 &= \frac{\omega}{\omega^2 - z^2}
 \end{aligned}$$

b) $\mathcal{L}[\theta(\xi)\cos(\omega\xi)] = \mathcal{F}[\theta(\xi)\cos(\omega\xi)e^{-(y,\xi)}]$

$$\begin{aligned}
 &= \int_0^\infty \cos(\omega\xi) e^{-(y,\xi)} e^{i(x,\xi)} d\xi \\
 &= \int_0^\infty \cos(\omega\xi) e^{i(z,\xi)} d\xi \\
 &= \frac{1}{2} \int_0^\infty (e^{i\omega\xi} + e^{-i\omega\xi}) e^{i(z,\xi)} d\xi \\
 &= \frac{1}{2} \int_0^\infty e^{i(\omega+z,\xi)} d\xi + \frac{1}{2} \int_0^\infty e^{-i(\omega-z,\xi)} d\xi \\
 &= \frac{-1}{2i(\omega+z)} + \frac{1}{2i(\omega-z)} \\
 &= \frac{-iz}{\omega^2 - z^2}
 \end{aligned}$$

$$2 \quad a) \quad \mathcal{L}[\theta(\xi)e^{i\omega\xi}] = \frac{i}{z+\omega}$$

$$b) \quad \mathcal{L}[\theta(\xi)e^{-i\omega\xi}] = \frac{i}{z-\omega}$$

$$\text{Solutions: } a) \quad \mathcal{L}[\theta(\xi)e^{i\omega\xi}] = \mathcal{F}[\theta(\xi)e^{i\omega\xi}e^{-(y,\xi)}]$$

$$= \int_0^\infty e^{i\omega\xi} e^{-(y,\xi)} e^{i(x,\xi)} d\xi$$

$$= \int_0^\infty e^{i\omega\xi} e^{i(z,\xi)} d\xi$$

$$= \int_0^\infty e^{i(z+\omega,\xi)} d\xi = \frac{i}{z+\omega}$$

$$b) \quad \mathcal{L}[\theta(\xi)e^{-i\omega\xi}] = \mathcal{F}[\theta(\xi)e^{-i\omega\xi}e^{-(y,\xi)}]$$

$$= \int_0^\infty e^{-i\omega\xi} e^{-(y,\xi)} e^{i(x,\xi)} d\xi$$

$$= \int_0^\infty e^{-i\omega\xi} e^{i(z,\xi)} d\xi$$

$$= \int_0^\infty e^{i(z-\omega,\xi)} d\xi = \frac{i}{z-\omega}$$

CHAPTER THREE

The Cauchy kernel and the transform of Cauchy Bochner and Hilbert

3.1 The Cauchy kernel $\mathcal{K}_c(z)$

Let C be connected open cone in $\mathbb{R}^n$ with vertex at 0 and let C^* be the conjugate cone C .

Definition: The function

$$\mathcal{K}_c(z) = \int_{C^*} e^{i(z,\xi)} d\xi \equiv \mathcal{L}[\chi_{C^*}] = \mathcal{F}[\chi_{C^*}(\xi)e^{-(y,\xi)}] \dots\dots\dots (3.1)$$

is called the *Cauchy kernel* of a tubular region T^c and here, $\chi_{C^*}(\xi)$ is the characteristic function of the cone C^* .

From theorem 1 of sec. 2.1, the kernel $\mathcal{K}_c(z)$ is a holomorphic function in T^c ; and the integral in equation (3.1) converges uniformly with respect to z in any tubular region $T^k, k \in C$

($\bar{k}$ is compact).

Now we will show that the kernel $\mathcal{K}_c(z)$ can be represented by the integral

$$\mathcal{K}_c(z) = i^n \Gamma(n) \int_{prC^*} \frac{d\sigma}{(y,\sigma)^n}, \quad z \in T^c \dots\dots\dots (3.2)$$

Since, $(y, \sigma) > 0$ for all $y \in C, \sigma \in prC^*$, it follow that the denominator of the integrand on the right of equation (3.2) is equal to $[(x, \sigma) + (y, \sigma)]^n$ and does not vanish in T^c . Since, the kernel $\mathcal{K}_c(z)$ is holomorphic function in T^c . It suffices to prove equation (3.2) when $x = 0$ the formula follows readily from equation (3.1).

$$\begin{aligned} \mathcal{K}_c(iy) &= \int_{C^*} e^{-(y,\xi)} d\xi \\ &= \int_{prC^*} \int_0^\infty e^{-\rho(y,\sigma)} \rho^{n-1} d\rho d\sigma \\ &= \int_{prC^*} \frac{d\sigma}{(y,\sigma)^n} \int_0^\infty e^{-u} u^{n-1} du \end{aligned}$$

$$= i^n \Gamma(n) \int_{prC^*} \frac{d\sigma}{(iy, \sigma)^n}$$

Therefore, $\mathcal{K}_c(z) = i^n \Gamma(n) \int_{prC^*} \frac{d\sigma}{(y, \sigma)^n}$, $z \in T^c$.

From the representation (3.2) it follow that the kernel $\mathcal{K}_c(z)$ and the kernel $\mathcal{K}_{-c}(z)$ are holomorphic in the region

$$D = \mathbb{C}^n \setminus \bigcup_{\sigma \in prC^*} [z : (z, \sigma) = 0]$$

The region D contains the tubular region T^c and T^{-c} and also the real parts of the cones C and $-C$.

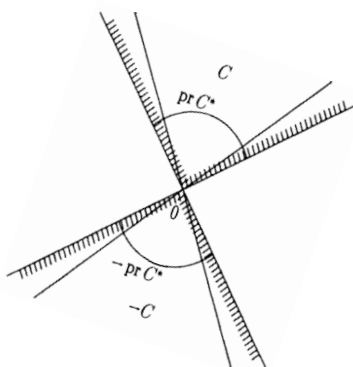


Figure 4

The kernels $\mathcal{K}_c$ and $\mathcal{K}_{-c}$ satisfy the relations

$$\mathcal{K}_{-c}(z) = (-1)^n \mathcal{K}_c(z) = \overline{\mathcal{K}_c(\bar{z})} = \mathcal{K}_c(-z)$$

$$\mathcal{K}_c(\lambda z) = \lambda^{-n} \mathcal{K}_c(z), \lambda \in \mathbb{C}^1 \setminus \{0\}, z \in T^c \cup T^{-c} \dots\dots\dots (3.3)$$

Proof: by definition of kernels $\mathcal{K}_c$ and representation (3.2) we have

$$\begin{aligned} \mathcal{K}_{-c}(iy) &= \int_{-C^*} e^{-(y, \xi)} d\xi = \int_{C^*} e^{(y, \xi)} d\xi \\ &= \int_{prC^*} \int_0^\infty e^{-\rho(y, \sigma)} \rho^{n-1} d\rho d\sigma \\ &= \int_{prC^*} \int_0^\infty e^{-u} \left(\frac{u}{\Delta(y)} \right)^{n-1} (-1)^n \frac{du}{\Delta(y)} \end{aligned}$$

$$\begin{aligned}
&= \int_{\text{prc}^*} \int_0^\infty \frac{e^{-u}(-1)^n u^{n-1}}{(y, \sigma)^n} du d\sigma \\
&= (-1)^n \int_{\text{prc}^*} \frac{d\sigma}{(y, \sigma)^n} \int_0^\infty e^{-u} u^{n-1} du \\
&= (-1)^n i^n \Gamma(n) \int_{\text{prc}^*} \frac{d\sigma}{(iy, \sigma)^n} = (-1)^n \mathcal{K}_c(iy)
\end{aligned}$$

Therefore, $\mathcal{K}_{-c}(z) = (-1)^n \mathcal{K}_c(z)$

$$\begin{aligned}
\overline{\mathcal{K}_c(\bar{z})} &= \overline{\int_{\mathbb{C}^*} e^{i(\bar{z}, \xi)} d\xi} = \overline{\int_{\mathbb{C}^*} e^{i(x-iy, \xi)} d\xi} \\
&= \overline{\int_{\mathbb{C}^*} e^{i(x, \xi)} e^{(y, \xi)} d\xi} \\
&= \int_{\mathbb{C}^*} e^{-i(x, \xi)} e^{(y, \xi)} d\xi \\
&= \int_{\mathbb{C}^*} e^{-i(x+iy, \xi)} d\xi = \int_{\mathbb{C}^*} e^{-i(z, \xi)} d\xi \\
&= \int_{\mathbb{C}^*} e^{i(-z, \xi)} d\xi = \mathcal{K}_c(-z)
\end{aligned}$$

Thus, $\mathcal{K}_{-c}(z) = (-1)^n \mathcal{K}_c(z) = \overline{\mathcal{K}_c(\bar{z})} = \mathcal{K}_c(-z)$

$$\begin{aligned}
\mathcal{K}_c(i\lambda y) &= \int_{\mathbb{C}^*} e^{-(\lambda y, \xi)} d\xi = \int_{\mathbb{C}^*} e^{-\lambda(y, \xi)} d\xi \\
&= \int_{\text{prC}^*} \int_0^\infty e^{-\rho(y, \sigma)} \rho^{n-1} d\rho d\sigma \\
&= \int_{\text{prC}^*} \int_0^\infty e^{-u} \left(\frac{u}{\lambda \Delta(y)} \right)^{n-1} \frac{du}{\lambda \Delta(y)} d\sigma \\
&= \int_{\text{prC}^*} \int_0^\infty e^{-u} \frac{u^{n-1}}{\lambda^n (y, \sigma)^n} du d\sigma
\end{aligned}$$

$$\begin{aligned}
 &= \lambda^{-n} \int_{\text{prC}^*} \frac{d\sigma}{(y, \sigma)^n} \int_0^\infty e^{-u} (u)^{n-1} du \\
 &= \lambda^{-n} i^n \Gamma(n) \int_{\text{prC}^*} \frac{d\sigma}{(iy, \sigma)^n} = \lambda^{-n} \mathcal{K}_c(iy)
 \end{aligned}$$

Therefore, $\mathcal{K}_c(\lambda z) = \lambda^{-n} \mathcal{K}_c(z)$, $\lambda \in \mathbb{C}^1 \setminus \{0\}$, $z \in T^c \cup T^{-c}$

Let us now prove that the estimate

$$|D^\alpha \mathcal{K}_c(z)| \leq M_\alpha \Delta^{-n-|\alpha|}(y), \quad z \in T^c \cup T^{-c} \quad \dots\dots\dots (3.4)$$

Where $\Delta(y) = \Delta(y, -\partial C \cup \partial C)$ is the distance from y to the boundary of the cone, $-C \cup C$;

$$\Delta(y) = \inf_{\sigma \in \text{prC}^*} (y, \sigma), \quad y \in C.$$

Proof: by using representation (3.2) we have, for $z \in T^c$

$$\begin{aligned}
 |D^\alpha \mathcal{K}_c(z)| &= \left| i^n \Gamma(n) D^\alpha \int_{\text{prC}^*} \frac{d\sigma}{(z, \sigma)^n} \right| \\
 &= \left| \int_{\text{prC}^*} \frac{\sigma^\alpha d\sigma}{(z, \sigma)^{n+|\alpha|}} \right| \\
 &\leq M_\alpha \int_{\text{prC}^*} \frac{|\sigma|^\alpha d\sigma}{(y, \sigma)^{n+|\alpha|}} \\
 &\leq M_\alpha \sup_{\sigma \in \text{prC}^*} (y, \sigma)^{-n-|\alpha|} \\
 &= M_\alpha \Delta^{-n-|\alpha|}(y)
 \end{aligned}$$

The case for $z \in T^{-c}$ can be considered with the use of the relation (3.3)

$$|D^\alpha \mathcal{K}_{-c}(z)| = |(-1)^n D^\alpha \mathcal{K}_c(z)| = |D^\alpha \mathcal{K}_c(z)|$$

Thus, $|D^\alpha \mathcal{K}_{-c}(z)| \leq M_\alpha \Delta^{-n-|\alpha|}(y)$

Therefore, $|D^\alpha \mathcal{K}_c(z)| \leq M_\alpha \Delta^{-n-|\alpha|}(y)$, $z \in T^c \cup T^{-c}$

We now prove that the estimate for all $s \geq 0$.

$$\|D^\alpha \mathcal{K}_c(x + iy)\|_s \leq K_{s,\alpha} (1 + \Delta^{-s}(y)) \Delta^{-n/2-|\alpha|}(y), \quad y \in -C \cup C \quad \dots\dots\dots (3.5)$$

Proof: from equation (3.1) and (3.2) we have

$$\begin{aligned} \|D^\alpha \mathcal{K}_c(x + iy)\|_s^2 &= \|\mathcal{F}^{-1}[D^\alpha \mathcal{K}_c(x + iy)]\|_{(s)}^2 \\ &= \|(-i\xi)^\alpha \theta_{C^*}(\xi) e^{-(y,\xi)}\|_{(s)}^2 \\ &= \int_{C^*} e^{-2(y,\xi)} (1 + |\xi|^2)^s |\xi^\alpha|^2 d\xi \\ &\leq \int_0^\infty (1 + \rho^2)^s \rho^{n-1+2|\alpha|} \int_{\text{pr}C^*} e^{-2(y,\sigma)} d\sigma d\rho \\ &\leq \frac{\sigma_n}{2} \int_0^\infty e^{-2\rho\Delta(y)} (1 + \rho^2)^s \rho^{n-1+2|\alpha|} d\rho \\ &\leq \frac{\sigma_n}{2^{n+1+2|\alpha|} \Delta^{n+2|\alpha|}(y)} \int_0^\infty e^{-u} \left(1 + \frac{u^2}{4\Delta^2(y)}\right)^s u^{n-1+2|\alpha|} du \\ &\leq K_{s,\alpha}^2 (1 + \Delta^{-s}(y))^2 \Delta^{-n-2|\alpha|}(y) \end{aligned}$$

Here, σ_n is the surface area of the unit sphere in $\mathbb{R}^n$.

Thus, $\|D^\alpha \mathcal{K}_c(x + iy)\|_s \leq K_{s,\alpha} (1 + \Delta^{-s}(y)) \Delta^{-n/2-|\alpha|}(y)$, $y \in C$

The case $y \in -C$ can be considered with the use of the relations (3)

$$\|D^\alpha \mathcal{K}_{-c}(x + iy)\|_s^2 = \|(-1)^n D^\alpha \mathcal{K}_c(x + iy)\|_s^2 = \|D^\alpha \mathcal{K}_c(x + iy)\|_s^2$$

Therefore, $\|D^\alpha \mathcal{K}_c(x + iy)\|_s \leq K_{s,\alpha} (1 + \Delta^{-s}(y)) \Delta^{-n/2-|\alpha|}(y)$, $y \in -C \cup C$

The boundary values of the kernel $\mathcal{K}_c(z)$ equal to $(\pm 1)^n \mathcal{F}[\chi_{\pm C^*}]$, respectively

$$\mathcal{K}_c(x + iy) \rightarrow (\pm 1)^n \mathcal{K}_c(\pm x) = (\pm 1)^n \mathcal{F}[\chi_{\pm C^*}] \dots\dots\dots (3.6)$$

as $y \rightarrow 0, y \in \pm C$ in norm in $\mathcal{H}_s$ for arbitrary $s < -n/2$.

Proof:

Since $\mathcal{K}_c(x + iy) \in \mathcal{H}_s$, for $y \in -C \cup C$ and for any s , while the generalized functions

$$\mathcal{F}[\chi_{\pm C^*}] \in \mathcal{H}_s \text{ for all } s < -n/2.$$

Therefore, when $s < -n/2$ and $y \in C, y \rightarrow 0$, we have

$$\begin{aligned} \|\mathcal{K}_c(x + iy) - \mathcal{F}[\chi_{C^*}]\|_s^2 &= \left\| \mathcal{F}^{-1} \left[\mathcal{F}[\chi_{C^*} e^{-(y, \xi)}] - \mathcal{F}[\chi_{C^*}] \right] \right\|_{(s)}^2 \\ &= \left\| \mathcal{F}^{-1} \left[\mathcal{F}[\chi_{C^*} e^{-(y, \xi)}] \right] - \mathcal{F}^{-1}[\mathcal{F}[\chi_{C^*}]] \right\|_{(s)}^2 \\ &= \left\| \chi_{C^*} e^{-(y, \xi)} - \chi_{C^*} \right\|_{(s)}^2 \\ &= \left\| \chi_{C^*} (e^{-(y, \xi)} - 1) \right\|_{(s)}^2 \\ &= \int_{C^*} (e^{-(y, \xi)} - 1)^2 (1 + |\xi|^2)^s d\xi \rightarrow 0 \end{aligned}$$

But, if $y \in -C, y \rightarrow 0$, then

$$\mathcal{K}_c(x + iy) = (-1)^n \mathcal{K}_c(-x - iy) \rightarrow (-1)^n \mathcal{K}_c(-x) = (-1)^n \mathcal{F}[\chi_{-C^*}].$$

From the formula (3.3) and (3.5), we have the following relations for the boundary values of the kernel $\mathcal{K}_c(z)$ and $\mathcal{K}_{-c}(z)$:

$$\left. \begin{aligned} \mathcal{K}_{-c}(x) &= \overline{\mathcal{K}_c(x)} = \mathcal{K}_c(-x), \quad x \in \mathbb{R}^n \\ \mathcal{K}_{-c}(x) &= (-1)^n \mathcal{K}_c(x), \quad x \in -C \cup C \end{aligned} \right\} \dots \dots \dots (3.7)$$

$$\left. \begin{aligned} \operatorname{Re} \mathcal{K}_c(x) &= \frac{1}{2} \mathcal{F}[\chi_{c^*} + \chi_{-c^*}] \\ &= \frac{1}{2} [\mathcal{K}_c(x) + \mathcal{K}_{-c}(x)] \\ \operatorname{Im} \mathcal{K}_c(x) &= \frac{1}{2i} \mathcal{F}[\chi_{c^*} - \chi_{-c^*}] \\ &= \frac{1}{2i} [\mathcal{K}_c(x) - \mathcal{K}_{-c}(x)] \end{aligned} \right\} \dots \dots \dots (3.8)$$

Proof: for $x \in \mathbb{R}^n$

$$\mathcal{K}_{-c}(x) = \int_{-c^*} e^{i(x,\xi)} d\xi = \int_{c^*} e^{-i(x,\xi)} d\xi = \int_{c^*} e^{i(-x,\xi)} d\xi = \mathcal{K}_c(-x)$$

$$\overline{\mathcal{K}_c(x)} = \overline{\int_{c^*} e^{i(x,\xi)} d\xi} = \int_{c^*} e^{-i(x,\xi)} d\xi = \int_{c^*} e^{i(-x,\xi)} d\xi = \mathcal{K}_c(-x)$$

Therefore, $\mathcal{K}_{-c}(x) = \overline{\mathcal{K}_c(x)} = \mathcal{K}_c(-x)$, $x \in \mathbb{R}^n$

$$\begin{aligned} \mathcal{K}_{-c}(x) &= \int_{-c^*} e^{i(x,\xi)} d\xi = \int_{c^*} e^{-i(x,\xi)} d\xi \\ &= \int_{\operatorname{pr} C^*} \int_0^\infty e^{-i\rho(x,\sigma)} \rho^{n-1} d\rho d\sigma \\ &= \int_{\operatorname{pr} C^*} \int_0^\infty e^{-u} \left(\frac{u}{i \Delta(x)} \right)^{n-1} \frac{du}{i \Delta(x)} d\sigma \\ &= \int_{\operatorname{pr} C^*} \int_0^\infty e^{-u} \frac{u^{n-1}}{i^n (x, \sigma)^n} du d\sigma \\ &= i^{-n} \int_{\operatorname{pr} C^*} \frac{d\sigma}{(x, \sigma)^n} \int_0^\infty e^{-u} (u)^{n-1} du \\ &= i^{-n} (-i)^n \Gamma(n) \int_{\operatorname{pr} C^*} \frac{d\sigma}{(ix, \sigma)^n} \\ &= (-1)^n i^n \Gamma(n) \int_{\operatorname{pr} C^*} \frac{d\sigma}{(ix, \sigma)^n} \\ &= (-1)^n \mathcal{K}_c(x) \end{aligned}$$

$$\begin{aligned}
\operatorname{Re} \mathcal{K}_c(x) &= \frac{\mathcal{K}_c(x) + \overline{\mathcal{K}_c(x)}}{2} = \frac{1}{2} \left[\int_{\mathbb{C}^*} e^{i(x,\xi)} d\xi + \overline{\int_{\mathbb{C}^*} e^{i(x,\xi)} d\xi} \right] \\
&= \frac{1}{2} [\mathcal{F}[\chi_{\mathbb{C}^*}] + \mathcal{F}[\chi_{-\mathbb{C}^*}]] \\
&= \frac{1}{2} \mathcal{F}[\chi_{\mathbb{C}^*} + \chi_{-\mathbb{C}^*}] \\
&= \frac{1}{2} [\mathcal{K}_c(x) + \mathcal{K}_{-c}(x)] \\
\operatorname{Im} \mathcal{K}_c(x) &= \frac{1}{2i} [\mathcal{K}_c(x) - \overline{\mathcal{K}_c(x)}] = \frac{1}{2i} \left[\int_{\mathbb{C}^*} e^{i(x,\xi)} d\xi - \overline{\int_{\mathbb{C}^*} e^{i(x,\xi)} d\xi} \right] \\
&= \frac{1}{2i} [\mathcal{F}[\chi_{\mathbb{C}^*}] - \mathcal{F}[\chi_{-\mathbb{C}^*}]] \\
&= \frac{1}{2i} \mathcal{F}[\chi_{\mathbb{C}^*} - \chi_{-\mathbb{C}^*}] \\
&= \frac{1}{2i} [\mathcal{K}_c(x) - \mathcal{K}_{-c}(x)]
\end{aligned}$$

From this, taking into account we have the following equalities

$$\begin{aligned}
(\chi_{\mathbb{C}^*} - \chi_{-\mathbb{C}^*})^2 &= (\chi_{\mathbb{C}^*} + \chi_{-\mathbb{C}^*})^2 = \chi_{\mathbb{C}^*} + \chi_{-\mathbb{C}^*} \\
(\chi_{\mathbb{C}^*} - \chi_{-\mathbb{C}^*})(\chi_{\mathbb{C}^*} + \chi_{-\mathbb{C}^*}) &= \chi_{\mathbb{C}^*} - \chi_{-\mathbb{C}^*} \quad \dots\dots\dots (3.9)
\end{aligned}$$

And we obtain the following relations between the generalized functions $\operatorname{Re} \mathcal{K}_c(x)$ and $\operatorname{Im} \mathcal{K}_{-c}(x)$

$$\begin{aligned}
-\operatorname{Im} \mathcal{K}_c(x) \star \operatorname{Re} \mathcal{K}_c(x) &= \operatorname{Re} \mathcal{K}_c(x) \star \operatorname{Re} \mathcal{K}_c(x) = \frac{(2\pi)^n}{2} \operatorname{Re} \mathcal{K}_c(x) \\
\operatorname{Im} \mathcal{K}_c(x) \star \operatorname{Re} \mathcal{K}_c(x) &= \frac{(2\pi)^n}{2} \operatorname{Im} \mathcal{K}_c(x)
\end{aligned}$$

3.2 The Cauchy-Bochner transforms

Definition: Suppose $f \in \mathcal{H}_s$. The function

$$f(z) = (2\pi)^{-n} (f(x'), \mathcal{K}_c(z - x')) , \quad z \in T^c \cup T^{-c} \quad \dots\dots\dots (3.10)$$

is called the Cauchy-Bochner transform (integral). It is assumed here, that the cone $\mathcal{C}$ is convex and acute.

Let us prove that the representation Cauchy-Bochner transform

$$f(z) = (2\pi)^{-n} f(x') * \mathcal{K}_c(x' + iy) , \quad z \in T^c \cup T^{-c} \quad \dots\dots\dots (3.11)$$

Since $\mathcal{K}_c(x + iy) \in \mathcal{H}_s$ for all s and $y \in -\mathcal{C} \cup \mathcal{C}$. Let $f \in \mathcal{H}_{-s}$ and let for all $\varphi \in \mathcal{S}$.

Here, $\mathcal{H}_{-s}$ is the conjugate of $\mathcal{H}_s$.

$$\begin{aligned} \text{Then } (f * \mathcal{K}_c(z), \varphi) &= (\mathcal{F}^{-1}[\mathcal{F}[f]\mathcal{F}[\mathcal{K}_c(z)]], \varphi) \\ &= (\mathcal{F}[f]\mathcal{F}[\mathcal{K}_c(z)], \mathcal{F}^{-1}[\varphi]) \\ &= \left(\mathcal{F}[f]\mathcal{F}[\mathcal{K}_c(z)], (2\pi)^{-n} \int \varphi(x) e^{-i(x\xi)} dx \right) \\ &= (2\pi)^{-n} \int (\mathcal{F}[f]\mathcal{F}[\mathcal{K}_c(z)], e^{-i(x\xi)}) \varphi(x) dx \end{aligned}$$

Since $\mathcal{F}[f]\mathcal{F}[\mathcal{K}_c(z)] \in \mathcal{L}^1$, Therefore,

$$\begin{aligned} (f * \mathcal{K}_c(z))(x) &= (2\pi)^{-n} (\mathcal{F}[f]\mathcal{F}[\mathcal{K}_c(z)], e^{-i(x\xi)}) \\ &= (2\pi)^{-n} (\mathcal{F}[f], \mathcal{F}[\mathcal{K}_c(z)] e^{-i(x\xi)}) \\ &= (2\pi)^{-n} (f, \mathcal{F}[\mathcal{F}[\mathcal{K}_c(z)] e^{-i(x\xi)}]) \\ &= (2\pi)^{-n} (f(x'), \int [\mathcal{F}[\mathcal{K}_c(z)](\xi) e^{-i(x+x',\xi)} d\xi) \\ &= (f(x'), \mathcal{F}^{-1}[\mathcal{F}[\mathcal{K}_c(z)]](x + x')) \\ &= (f(x'), \mathcal{K}_c(z - x')) \end{aligned}$$

The function $f(z)$ is holomorphic in $T^c \cup T^{-c}$, and

$$D^\alpha f(z) = (2\pi)^{-n} (f(x'), D^\alpha \mathcal{K}_c(z - x')), \dots \dots \dots (3.12)$$

$$|D^\alpha f(z)| \leq K_{-s,\alpha} (2\pi)^{-n} \|f\|_s (1 + \Delta^s(y)) \Delta^{-n/2 - |\alpha|}(y) \dots (3.12')$$

Where the numbers $K_{-s,\alpha}$ are the same as in the estimate (3.5) of section 3.1

Proof: The estimate (3.12') follows from equation (3.12) and also from equation (3.5) of section 3.1, we have:

$$\begin{aligned} |D^\alpha f(z)| &= (2\pi)^{-n} \left| \int f(x') D^\alpha \mathcal{K}_c(z - x') dx' \right| \\ &\leq (2\pi)^{-n} \int |f(x') D^\alpha \mathcal{K}_c(z - x')| dx' \\ &= (2\pi)^{-n} \int |f(x')| |D^\alpha \mathcal{K}_c(z - x')| dx' \\ &= (2\pi)^{-n} \|f\|_s \|D^\alpha \mathcal{K}_c(z - x')\|_{-s} \\ &= (2\pi)^{-n} \|f\|_s \|D^\alpha \mathcal{K}_c(x + iy)\|_{-s} \\ &\leq (2\pi)^{-n} K_{-s,\alpha} \|f\|_s (1 + \Delta^s(y)) \Delta^{-n/2 - |\alpha|}(y) \end{aligned}$$

Now let us prove that for all s and $y \in -C \cup C$, the estimates

$$\|f(x + iy)\|_s \leq \mathcal{N}_\alpha \|f\|_s \Delta^{-|\alpha|}(y) \dots \dots \dots (3.13)$$

$$\|f(x + iy)\|_{s-|\alpha|} \leq \|f\|_s \dots \dots \dots (3.14)$$

Proof: from the representation (3.11) and from the definition of the kernel $\mathcal{K}_c(z)$ it follows that

$$\begin{aligned} \mathcal{F}^{-1}[D^\alpha f(x + iy)] &= (2\pi)^{-n} \mathcal{F}^{-1}[f * D^\alpha \mathcal{K}_c] \\ &= \mathcal{F}^{-1}[f] \mathcal{F}^{-1}[D^\alpha \mathcal{K}_c] \\ &= \mathcal{F}^{-1}[f] (i\xi)^\alpha \mathcal{F}^{-1}[\mathcal{K}_c] \\ &= (i\xi)^\alpha \mathcal{F}^{-1}[f](\xi) \chi_{C^*}(\xi) e^{-(y,\xi)} \dots (3.15) \end{aligned}$$

Therefore, $\|D^\alpha f(x + iy)\|_s^2 = \int_{C^*} |\mathcal{F}^{-1}[f](\xi)|^2 |\xi^\alpha|^2 e^{-2(y,\xi)} (1 + \xi^2)^s d\xi$

$$\begin{aligned}
&\leq \|f\|_s^2 \sup_{\xi \in \mathbb{C}^*} |\xi|^{2|\alpha|} e^{-2(y, \xi)} \\
&= \|f\|_s^2 \sup_{\rho \geq 0} \rho^{2|\alpha|} \sup_{\sigma \in \text{pr}\mathbb{C}^*} e^{-2\rho(y, \sigma)} \\
&= \|f\|_s^2 \sup_{\rho \geq 0} \rho^{2|\alpha|} e^{-2\rho\Delta(y)} \\
&= \|f\|_s^2 2^{-2\alpha} \Delta^{-2\alpha}(y) \sup_{u \geq 0} e^{-u} u^{2\alpha} \\
&= \mathcal{N}_\alpha \|f\|_s^2 \Delta^{-|\alpha|}(y)
\end{aligned}$$

The function $f(z)$ assumes in $\mathcal{H}_s$, in norm, as $y \rightarrow 0$ for $y \in \pm C$, then the Boundary values of $f_\pm(x)$, which are respectively equal to

$$f_+ = (2\pi)^{-n} f * \mathcal{K}_c, \quad f_- = (-1)^n (2\pi)^{-n} f * \overline{\mathcal{K}_c}$$

Proof: taking into account equation (7) we have for $y \in C$

$$\begin{aligned}
\mathcal{F}^{-1}[f(x + iy) - f_+] &= \mathcal{F}^{-1}[f(x + iy) - (2\pi)^{-n} f * \mathcal{K}_c] \\
&= \mathcal{F}^{-1}[f(x + iy)] - (2\pi)^{-n} \mathcal{F}^{-1}[f * \mathcal{K}_c] \\
&= \mathcal{F}^{-1}[f(x + iy)] - \mathcal{F}^{-1}[f] \mathcal{F}^{-1}[\mathcal{K}_c] \\
&= \mathcal{F}^{-1}[f] \chi_{\mathbb{C}^*}(\xi) - \mathcal{F}^{-1}[f] e^{-(y, \xi)} \chi_{\mathbb{C}^*}(\xi) \\
&= \mathcal{F}^{-1}[f] (1 - e^{-(y, \xi)}) \chi_{\mathbb{C}^*}(\xi)
\end{aligned}$$

Therefore, when $y \rightarrow 0$ for $y \in C$ we obtain

$$\|f(x + iy) - f_+(x)\|_s^2 = \int_{\mathbb{C}^*} |\mathcal{F}^{-1}[f](\xi)|^2 (1 - e^{-(y, \xi)})^2 (1 + |\xi|^2)^s d\xi \rightarrow 0$$

3.3 The Hilbert Transform

Definition: Suppose $f \in \mathcal{H}_s$ for some s . The Hilbert transform f_1 of the generalized function f is the convolution

$$f_1 = \frac{-2}{(2\pi)^n} f \star \text{Im}\mathcal{K}_c \dots\dots\dots (3.16)$$

Applying the Fourier transform to the equation (1) and using equation (9) of sec.3.1, we obtain

$$\begin{aligned} \mathcal{F}[f_1] &= \mathcal{F}\left[\frac{-2}{(2\pi)^n} f \star \text{Im}\mathcal{K}_c\right] \\ &= \frac{-2}{(2\pi)^n} \mathcal{F}[f \star \text{Im}\mathcal{K}_c] \\ &= \mathcal{F}[f] \mathcal{F}[\text{Im}\mathcal{K}_c] \\ &= -i(\chi_{C^*} - \chi_{-C^*}) \mathcal{F}[f] \dots\dots\dots (3.17) \end{aligned}$$

Hence, it follows that $f_1 \in \mathcal{H}_s$ and $\text{supp}\mathcal{F}[f_1] \subset -C^* \cup C^*$.

Lemma 1: If $f \in \mathcal{H}_s$, then the following conditions are equivalent

1. $\text{supp}\mathcal{F}[f_1] \subset -C^* \cup C^*$
2. $f = \frac{-2}{(2\pi)^n} f_1 \star \text{Im}\mathcal{K}_c$
3. $f = \frac{-2}{(2\pi)^n} f \star \text{Re}\mathcal{K}_c$

Proof:

(1) $\rightarrow$ (2) From equation (3.17) and equation (3.8) of sec. 3.1, we obtain

$$\mathcal{F}[f_1] = i(\chi_{C^*} - \chi_{-C^*}) \mathcal{F}[f]$$

Multiplying by $i(\chi_{C^*} - \chi_{-C^*})$, we obtain

$$i(\chi_{C^*} - \chi_{-C^*}) \mathcal{F}[f_1] = (\chi_{C^*} - \chi_{-C^*})^2 \mathcal{F}[f]$$

However, from relation (3.9) of sec. 3.1, we have

$$i(\chi_{c^*} - \chi_{-c^*})(\chi_{c^*} + \chi_{-c^*})\mathcal{F}[f_1] = (\chi_{c^*} + \chi_{-c^*})\mathcal{F}[f]$$

$$\mathcal{F}[f] = i(\chi_{c^*} - \chi_{-c^*})\mathcal{F}[f_1]$$

Finally, by applying the Fourier transform $\mathcal{F}^{-1}$

$$\begin{aligned}\mathcal{F}^{-1}[\mathcal{F}[f]] &= \mathcal{F}^{-1}[i(\chi_{c^*} - \chi_{-c^*})\mathcal{F}[f_1]] \\ &= i\mathcal{F}^{-1}[(\chi_{c^*} - \chi_{-c^*})]\mathcal{F}^{-1}[\mathcal{F}[f_1]] \\ &= (2\pi)^{-n}i\mathcal{F}[\chi_{c^*} - \chi_{-c^*}] \star f_1 \\ &= \frac{2}{(2\pi)^n}f_1 \star \text{Im}\mathcal{K}_c\end{aligned}$$

(2) $\rightarrow$ (3) from equation (3.16) and equation (3.9) of sec. 3.1, we obtain

$$\begin{aligned}f &= \frac{-2}{(2\pi)^n}f_1 \star \text{Im}\mathcal{K}_c = \frac{-4}{(2\pi)^{2n}}(f \star \text{Im}\mathcal{K}_c) \star \text{Im}\mathcal{K}_c \\ &= \frac{-4}{(2\pi)^{2n}}f \star (\text{Im}\mathcal{K}_c \star \text{Im}\mathcal{K}_c) \\ &= \frac{-4}{(2\pi)^{2n}}f \star \left(\frac{-(2\pi)^n}{2}\text{Re}\mathcal{K}_c\right) \\ &= \frac{2}{(2\pi)^n}f \star \text{Re}\mathcal{K}_c\end{aligned}$$

Finally, from (3) $\rightarrow$ (1) by virtue of (3.8) of sec. 3.1 and applying Fourier transform, we obtain

$$\begin{aligned}\mathcal{F}[f] &= \mathcal{F}\left[\frac{2}{(2\pi)^n}f \star \text{Re}\mathcal{K}_c\right] \\ &= \frac{2}{(2\pi)^n}\mathcal{F}[f \star \text{Re}\mathcal{K}_c] \\ &= \frac{2}{(2\pi)^n}\mathcal{F}[f]\mathcal{F}[\text{Re}\mathcal{K}_c]\end{aligned}$$

3.4 Examples

1. When $n = 1$, $C = (0, \infty)$ then find the following

a) $Re\mathcal{K}_c(z) = \pi\delta(x)$

b) $Im\mathcal{K}_c(z) = \mathcal{P}\frac{1}{x}$

Solutions:

From equation (3.8) of sec. 3.1, we have

$$a) \quad Re\mathcal{K}_c(z) = \frac{1}{2}\mathcal{F}[\chi_{C^*} + \chi_{-C^*}] = \frac{1}{2}[\mathcal{F}[\chi_{C^*}] + \mathcal{F}[\chi_{-C^*}]]$$

Consider the equation $\chi_{C^*}(\xi)e^{-a\xi}$, $a > 0$

$$\begin{aligned} \mathcal{F}[\chi_{C^*}(\xi)e^{-a\xi}] &= \int_{C^*} e^{-a\xi} e^{ix\xi} d\xi \\ &= \int_0^\infty e^{-(a-ix)\xi} d\xi = \frac{1}{a-ix} \\ &= \frac{i}{x+ia} \end{aligned}$$

Since, $\chi_{C^*}(\xi)e^{-a\xi} \rightarrow \chi_{C^*}(\xi)$ as $a \rightarrow 0^+$.

Thus, $\mathcal{F}[\chi_{C^*}(\xi)e^{-a\xi}] \rightarrow \mathcal{F}[\chi_{C^*}(\xi)] = \mathcal{F}[\chi_{C^*}]$, $a \rightarrow 0^+$, then

$$\lim_{a \rightarrow 0^+} \frac{i}{x+ia} = \frac{i}{x+i0}$$

Recall that from Sochoski formula, we have

$$\lim_{a \rightarrow 0^+} \frac{1}{x \pm ia} = \mp i\pi\delta(x) + \mathcal{P}\frac{1}{x}$$

$$\text{But, } \frac{i}{x+i0} = i \left(\frac{1}{x+i0} \right) = i \left(-i\pi\delta(x) + \mathcal{P}\frac{1}{x} \right) = \pi\delta(x) + i\mathcal{P}\frac{1}{x}$$

$$\text{Hence, } \mathcal{F}[\chi_{C^*}] = \pi\delta(x) + i\mathcal{P}\frac{1}{x}$$

Again, consider $\chi_{C^*}(\xi)e^{a\xi}$, $a < 0$

$$\begin{aligned}\mathcal{F}[\chi_{-C^*}(\xi)e^{a\xi}] &= \int_{-\infty}^0 e^{a\xi} e^{ix\xi} d\xi \\ &= \int_{-\infty}^0 e^{(a+ix)\xi} d\xi = \frac{1}{a+ix} \\ &= \frac{-i}{x-ia}\end{aligned}$$

Since, $\chi_{-C^*}(\xi)e^{a\xi} \rightarrow \chi_{-C^*}(\xi)$ as $a \rightarrow 0^+$.

Thus, $\mathcal{F}[\chi_{-C^*}(\xi)e^{a\xi}] \rightarrow \mathcal{F}[\chi_{-C^*}(\xi)] = \mathcal{F}[\chi_{-C^*}]$, $a \rightarrow 0^+$, then

$$\lim_{a \rightarrow 0^+} \frac{-i}{x-ia} = \frac{-i}{x-i0}$$

But, $\frac{-i}{x-i0} = -i \left(\frac{1}{x-i0} \right) = -i \left(i\pi\delta(x) + \mathcal{P}\frac{1}{x} \right) = \pi\delta(x) - i\mathcal{P}\frac{1}{x}$

Therefore, $\mathcal{F}[\chi_{-C^*}] = \pi\delta(x) - i\mathcal{P}\frac{1}{x}$

Thus,

$$\begin{aligned}Re\mathcal{K}_c(z) &= \frac{1}{2}[\mathcal{F}[\chi_{C^*}] + \mathcal{F}[\chi_{-C^*}]] \\ &= \frac{1}{2}[\pi\delta(x) + i\mathcal{P}\frac{1}{x} + \pi\delta(x) - i\mathcal{P}\frac{1}{x}] = \pi\delta(x)\end{aligned}$$

b) From equation (3.8) of sec. 3.1, we have

$$Im\mathcal{K}_c(z) = \frac{1}{2i}\mathcal{F}[\chi_{C^*} - \chi_{-C^*}] = \frac{1}{2i}[\mathcal{F}[\chi_{C^*}] - \mathcal{F}[\chi_{-C^*}]]$$

But, from b) we obtain $\mathcal{F}[\chi_{C^*}] = \pi\delta(x) + i\mathcal{P}\frac{1}{x}$ and $\mathcal{F}[\chi_{-C^*}] = \pi\delta(x) - i\mathcal{P}\frac{1}{x}$

Therefore,

$$Im\mathcal{K}_c(z) = \frac{1}{2i}[\mathcal{F}[\chi_{C^*}] - \mathcal{F}[\chi_{-C^*}]]$$

$$= \frac{1}{2i} \left[\left(\pi \delta(x) + i\mathcal{P} \frac{1}{x} \right) - \left(\pi \delta(x) - i\mathcal{P} \frac{1}{x} \right) \right] = \mathcal{P} \frac{1}{x}$$

c) when $n = 1$, $C = (0, \infty)$ and $f \in \mathcal{L}^1$, Cauchy-Bochner integral (3.10) turns into the classical Cauchy integral:

$$f(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(x')}{x' - z} dx'.$$

Solution:

$$\begin{aligned} f(z) &= (2\pi)^{-1} (f(x'), \mathcal{K}_c(z - x')) \\ &= (2\pi)^{-1} \int_{-\infty}^{\infty} f(x') \mathcal{K}_c(z - x') dx' \\ &= (2\pi)^{-1} \int_{-\infty}^{\infty} f(x') \int_0^{\infty} e^{i(z-x', \xi)} d\xi dx' \\ &= (2\pi)^{-1} \int_{-\infty}^{\infty} f(x') \frac{-1}{i(z - x')} dx' \\ &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(x')}{(x' - z)} dx' \end{aligned}$$

REFERENCES

V.S.Vladimirov, Generalized functions in Mathematical physics, Mir publishers,

Moscow, 1979.

V.S. Vladimirov, Equations of Mathematical physics, 3rd ed, Nauka, Moscow, 1976.

V.S. Vladimirov, A generalization of the Cauchy-Bochner integral representation,

Izv. Akad. Nauk SSSR ser. [English transl.: Math. USSR-Izv, 1969].

F.G. Friedlander, Introduction to the theory of distributions, Cambridge university press,

London, 1982.

Zemanian, A. Distribution theory and transform analysis, McGraw Hill, New York, 1965.