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SEMINAR REPORT ON CONSTRUCTION OF FUNDAMENTAL SOLUTION AND PARAMETRIX

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CONTENT

PREFACE

Analysis has been the dominant branch of mathematics for more than 300 years, and both ordinary and partial differential equations are the heart of analysis. Differential equations are the most important part of mathematics for understanding of physical science. The primary use of both ordinary and partial differential equation is to serve as a tool for the study of changes in physics world.

The main purpose of this paper is to deal with the construction of fundamental solution and Parametrix. In science, there are many phenomenon which are expressed by differential equation. For instance heat equation ($\frac{\partial\varepsilon(x,t)}{\partial t}-\frac{a^2\partial^2\varepsilon(x,t)}{\partial x^2}=\delta(x,t)$), wave equation ($\Box_a\varepsilon(x,t)=\delta(x,t)$), Laplace's equation ($\nabla^2\varepsilon(x)=\delta(x)$),Helmholtz's equation (($\nabla^2+k^2)\varepsilon(x)=\delta(x)$) , etc. These all differential equation require fundamental solution.

This seminar paper consists of three chapters: chapter one gives a brief discussion of preliminaries such as symbols, notation, basic concepts and definitions. Chapter two deals with fundamental solutions of linear differential operators with constant and variable coefficients. Lastly ,chapter three discusses about construction of fundamental solution to some differential operators. Here the concept of parametrix is discussed

INTRODUCTION

This paper commences with basic concepts and definition that are very important for my work. The concepts of generalized function, test function, generalized function of slow growth, convolution of generalized function, Fourier Transform are discussed in detail.

In this seminar, construction of fundamental solution and parametrix mainly focus on differential operators. In different field of sciences such as physics, chemistry, astronomy, engineering, etc., there are many differential operators which need fundamental solution. Thus construction of fundamental solutions and parametrix are carried out in this work.

CHAPTER -1-

PRELIMINARIS

In this chapter, the required symbols and notation are briefly listed. Some basic concepts and definitions which are use full for this paper are also given.

1.1 Symbols and notation

- $\Re^n$ – An n- dimensional real space
- $\mathcal{O}$ – open set in $\Re^n$
- $\mathcal{D}(\mathcal{O}) = \mathcal{D}$ The space of basic function
- $\mathcal{D}'$ ($\mathcal{O}$)= $\mathcal{D}'$The space of generalized function
- $\mathcal{S}$ - The space of test function
- $\mathcal{S}'$ - The space of generalized function of slow growth
- C^∞ ($\Re^n$)- The space of infinitely often continuously differentiable function on $\Re^n$
- F[φ]- Fourier Transform of function $\varphi\epsilon\mathcal{S}'$
- D^α f(x)-The derivative of the function f(x) of order $|\alpha|=\alpha_1, \alpha_2, \ldots, \alpha_n$ where $\alpha=(\alpha_1, \alpha_2, \ldots, \alpha_n)$is a vector with non-negative integral components . α_j's are called multi indices.

$$D^\alpha f(x)= \frac{\partial^{|\alpha_1|} f(x_1,x_2,\ldots,x_n)}{\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\ldots\partial_{x_n}^{\alpha_n}}$$

1.2 Basic concepts and definition

A. Heaviside unit function

The characteristic function of a set A is the function $\theta_A(x)$ which is equal to 1 when $X \epsilon A$ and equals to 0 when $x \notin A$.

Definition:-The characteristic function $\theta_{[0,\infty]}(x)$ of the semi axis x≥o is called ***Heaviside unit function*** ; denoted by $\theta(x)$; and is defined as :-

$$\theta(x) = \begin{cases} 1, x \geq 0 \\ 0, x < 0 \end{cases}$$

B.The delta function $\delta(x)$

Strongly peaked functions are common to all branches of Physics. For instance concentrated force acting on a beam is actually strongly peaked distribution of load. In electrical circuits, strongly peaked currents of extremely short duration often occur in switching processes, like the redistribution of charges between two capacitors when the switch is closed.

In order to facilitate a variety of operation in mathematical physics , and particularly in quantum mechanics, Dirac proposed the introduction of the so called ***delta function*** $\delta(x)$ which will be representative of an infinitely sharply peaked function given symbolically by :

$$\boldsymbol{\delta(x) = \begin{cases} 0, x \neq 0, \\ \infty, x = 0, \end{cases}}$$

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$$\boldsymbol{\delta(x) = \begin{cases} 0, x \neq 0, \\ \infty, x = 0, \end{cases}}$$

but such that the integral of δ(x) is ***normalized to unity*** :

$$\int_{-\infty}^{\infty} \delta(x) = 1$$

The first and basic operation to which Dirac sought to subject δ(x) is the integral $\int_{-\infty}^{\infty} \delta(x) f(x) dx$,where f(x) is any continuous function . This integral can be "evaluated" by the following argument:

Since δ(x) is zero for x≠0, the limit of integration may be changed to $-\varepsilon$ to $+\varepsilon$, where ε is a small positive number. More over, since f(x) is continuous at x=0, its value within the interval (-ε,+ ε) will not differ much f(0) and we can claim, approximately, that

$$\int_{-\infty}^{\infty} \delta(x) f(x) dx = \int_{-\varepsilon}^{\varepsilon} \delta(x) f(x) dx .$$

$$\Rightarrow \int_{-\infty}^{\infty} \delta(x) f(x) dx \cong f(0) \int_{-\varepsilon}^{\varepsilon} \delta(x) dx$$;with the approximation improving as ε approaches zero.

However, $\int_{-\varepsilon}^{\varepsilon} \delta(x) dx = 1$……for all values of ε , because δ(x) =0 for x≠0 ,and δ(x) is normalized.

Now, letting $\varepsilon \to \infty$, we have exactly:

$$\int_{-\infty}^{\infty} \delta(x) f(x) dx = f(0)$$

C. Linear space and functional

Definition Let M and N be linear sets. An operator L transforming the elements of set M in to elements of set N is called ***linear*** if for any elements f and g of M and complex number λ and μ , $L(\lambda f+\mu g)=\lambda Lf+\mu Lg$

Example $Lf=\int_G K(x,y)f(y)dy$ is linear integral operator

where K(x,y) is its kernel.

Definition If a linear operator L transforms a set of elements of M into a set of complex numbers Lf with $f\in M$, then L is called a ***linear functional*** on M.

D. The space of basic function [$\mathcal{D}$ ($\mathcal{O}$) = $\mathcal{D}$]

In the case of the delta function we have already seen that

$\int_{-\infty}^{\infty}\delta(x)f(x)dx=f(0)$ and the delta function is determined by

means of continuous functions as a linear(continuous) functional . Continuous functions are said to be ***basic function*** for delta function.

A sequence of function $\varphi_k \in \mathcal{D}(\mathcal{O})$ is said to be converge to the function $\varphi \in \mathcal{D}(\mathcal{O})$ if $\lim_{k\to\infty}\varphi_k=\varphi$ in $\mathcal{D}(\mathcal{O})$.A linear set $\mathcal{D}(\mathcal{O})$ equipped with

convergence is called the space of basic function $\mathcal{D}(\mathcal{O})$.

E. <u>The space of generalized functions</u> **[$\mathcal{D}'$ ($\mathcal{O}$)= $\mathcal{D}'$]**

A generalized (distribution) is a generalization of the classical notion of a function. On the one hand, this generalization permits expressing in a mathematically proper form such an idealized concepts as;

- the density of the material point
- the density of a point charge or dipole
- the intensity of an instantaneous point source
- the magnitude of an instantaneous force applied to a point etc.

On the other hand, the notion of generalized function can reflect the fact that in reality one can not measure the value of physical quantity at a point but can only measure the mean values within sufficiently small neighborhood of the point and claim that these mean values as the values of the physical quantity.

This can be explained by attempting to determine the density set up by a material point of mass 1. Assume that the point is the origin of co-ordinates. To determine the density, we distribute (or,as one says ,spread) the unit mass uniformly inside a ball U_ε . We then obtain the mean density:

$$f_\varepsilon(x)=\begin{cases}\frac{3}{4\pi\varepsilon^3}, |x|<\varepsilon\\ 0, |x|>\varepsilon\end{cases}$$

To begin with, for the desired density (which is denoted by $\delta(x)$)

We take the point of the sequence of mean density $f_\varepsilon(x)$ as $\varepsilon \to 0$. ie.,

$$\delta(x) = \lim_{\varepsilon \to 0} f_\varepsilon(x) = \begin{cases} +\infty, if x = 0 \\ 0, if x \neq 0 \end{cases}$$

It is this functional value that we take as the definition of the density $\delta(x)$ and this is the famous delta function of Dirac.

Thus, $f_\varepsilon(x) \to \delta(x)$ as $\varepsilon \to 0$ in the sense that for any continuous function $\varphi(x)$ the following limiting relationship is valid.

$$\int f_\varepsilon(x)\,\varphi(x)d(x) \to (\delta,\varphi),\ \varepsilon \to 0, \text{where } (\delta,\varphi)$$

Stands for $\varphi(0)$,the value of functional δ on the function φ.

<u>Definition</u> A generalized function specified on an open set $\mathcal{O}$ is a continuous linear functional on the space of basic functions $\mathcal{D}(\mathcal{O})$.

We will write the value of functional (generalized function) f on the basic function φ as (f, φ) . By analogy with ordinary functions, we sometimes write formally as $f(x)$ instead of f, and regard x as the argument of basic functions on which the function f operates. (f, φ) is bilinear.

A generalized function f is :

a) a ***functional*** on $\mathcal{D}(\mathcal{O})$, that is , with each $\varphi \in \mathcal{D}(\mathcal{O})$, there is associated a (complex valued) number (f, φ)

b) a ***linear functional*** on $\mathcal{D}(\mathcal{O})$, that is if φ and ψ belong

to $\mathcal{D}(\mathcal{O})$ and λ & μ are complex number,

$$(f, \lambda\varphi + \mu\psi) = \lambda(f, \varphi) + \mu(f, \psi).$$

c) a ***continuous functional*** on $\mathcal{D}(\mathcal{O})$, that is, if $\varphi_k \to \varphi$,

$k \to \infty$ in $\mathcal{D}(\mathcal{O})$, then $(f, \varphi_k) \to (f, \varphi)$, $k \to \infty$

The generalized function f and g specified in $\mathcal{O}$ are ***equal*** in $\mathcal{O}$ if they are equal as a functional on $\mathcal{D}(\mathcal{O})$. That is, if $\varphi \in \mathcal{D}(\mathcal{O})$ then $(f, \varphi) = (g, \varphi)$. We then write f=g in $\mathcal{O}$ or $f(x)=g(x)$, $x \in \mathcal{O}$

Let f(x) is locally integrable function in $\Re^n$. A generalized function which is generated by f(x) is said to be ***regular*** generalized function if $(f, \varphi) = \int f(x)\varphi(x)dx$, $\varphi \in \mathcal{D}(\mathcal{O})$. All other generalized functions are said to be ***singular***.

The derivative of generalized function f is:

$$(D^\alpha f, \varphi) = (-1)^\alpha (f, D^\alpha \varphi)\ ,\ \varphi \in \mathcal{D}(\mathcal{O}) \text{ and with } |\alpha| \le p$$

where $f \in C^p(\Re^n)$.

This can be shown using integration by parts and induction as follows:

Let $|\alpha|=1$ and $\varphi \epsilon C_0^\infty(\mathbb{R}^n)$

$(Df, \varphi) = \int_{-\infty}^{\infty} Df(x)\varphi(x)dx$ let $u=\varphi(x)$ $dv=Df(x)dx$

$du=D\varphi(x)dx$ $v=f(x)$

$=f(x)\varphi(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)D\varphi(x)dx$

$= - \int_{-\infty}^{\infty} f(x)D\varphi(x)dx$

$= (-1)^1 (f, D\varphi)$

Let $|\alpha|=2$,

$(D^2f, \varphi) = \int_{-\infty}^{\infty} D^2 f(x)\varphi(x)dx$ Let $u=\varphi(x)$ $dv=D^2f(x)dx$

$du=D\varphi(x)dx$ $v=Df(x)$

$=Df(x)\,\varphi(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} Df(x)D\varphi(x)dx$ Let $t=D\varphi(x)$ $dp=Df(x)dx$

$dt=D^2\varphi(x)dx$ $p=f(x)$

$= Df(x)\varphi(x)\Big|_{-\infty}^{\infty} - f(x)D\varphi(x)\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} f(x)D^2\varphi(x)dx$

$=(f, D^2\varphi(x))$

$=(-1)^2(f, D^2\varphi(x))$

Thus by induction principle, $(D^\alpha f, \varphi)=(-1)^\alpha (f, D^\alpha\varphi)$

For $f=\delta$, $(D^\alpha\delta, \varphi)=(-1)^\alpha(\delta, D^\alpha\varphi)$

Example Let $\theta(x)=\begin{cases}1, x\geq 0\\ 0, x<0\end{cases}$ be a Heaviside unit function. Show that

$\theta'(x)=\delta(x)$.

solution Let $\varphi\in \mathcal{D}(\mathcal{O}) = C_0^\infty(\Re^n)$.

Then, $(\theta', \varphi) = -(\theta, \varphi')$...... by definition of derivative of generalized function.

But, $-(\theta, \varphi') = -\int_{-\infty}^{\infty}\theta(x)\varphi'(x)dx$

$= -\int_{-\infty}^{0}\theta(x)\varphi'(x)dx - \int_{0}^{\infty}\theta(x)\varphi'(x)dx$

$= 0 - \int_{0}^{\infty}1.\varphi'(x)dx$

$= -\varphi(x)\Big|_0^\infty$

$= \varphi(0) - \varphi(\infty)$

$= \varphi$...as $\varphi(\infty)=0$

$= \int_{-\infty}^{\infty}\delta(x)\varphi(x)dx$

$= (\delta,\varphi)$

Thus $(\theta', \varphi) = -(\theta, \varphi') = (\delta,\varphi)$

$\Rightarrow \theta'(x) = \delta(x)$

Definition a) suppose $f\epsilon\mathcal{D}'$. The union of all neighborhoods where f=0 forms an open set $\mathcal{O}_f$ called the zero set of generalized function f.

b) The support of generalized function f is the complement of O_f with respect to $\Re^n$;it is symbolized as suppf; so that suppf=$\Re^n \setminus O_f$.

c) The generalized function $\mathcal{P}\frac{1}{x}$ that operates in accordance with the formula:

$$(\mathcal{P}\frac{1}{x}, \varphi) = PV \int_{-\infty}^{\infty} \frac{\varphi(x)}{x} dx = \lim_{\varepsilon \to +0} (\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty}) \frac{\varphi(x)}{x} dx, \varphi \epsilon \mathcal{D}(\Re^1)$$

is said to be the finite part or principal value (PV)of the integral of the function $\frac{1}{x}$

NOTE:- A simple layer on a surface is a generalization of the delta function. Suppose that S is a piecewise smooth surface and $\mu(x)$ is continuous function defined on $\boldsymbol{S}$.We introduce the generalized function $\mu\delta_s$ acting according to the rule:

$$(\mu\delta_s, \varphi) = \int_s \mu(x)\varphi(x)ds, \varphi\epsilon\mathcal{D}$$

Clearly $\mu\delta_s \epsilon \mathcal{D}'$ and $\mu\delta_s(x) = 0$ where $x \notin S$; so that supp $\mu\delta_s \subseteq S$. The generalized function $\mu\delta_s$ is a simple layer on the surface S with density μ.

Definition **(convolution of generalized function)**

Let f(x) and g(x) be function locally itegrable in $\Re^n$ with the function $h(x)=\int |g(y)f(x-y)| dy$ being also locally integrable in $\Re^n$. The function

$$(f\star g)(x) = \int |g(y)f(x-y)| dy = \int |f(y)g(x-y)| dy = (g\star f)(x)$$

is called the ***convolution*** f⋆g of the two function f and g.

F. The space of test function [$\mathcal{S}(\mathcal{O}) = \mathcal{S}$]

Definition ***Test function or rapidly diminishing functions*** are function of the class of $C^{\infty}(\Re^n)$ that decreases together with all their derivatives as $|x| \to \infty$ faster than any power of $|x|^{-1}$. The space of test function is denoted by $\mathcal{S}$.

The sequence (φ_k) in $\mathcal{S}$ converges to function $\varphi \in \mathcal{S}$ if $\lim_{k\to\infty}(\varphi_k) = \varphi$. It is clear that $\mathcal{S}$ is linear space; more over, $\mathcal{D}(\mathcal{O}) \subseteq \mathcal{S}(\mathcal{O})$ and that convergence in $\mathcal{D}$ implies convergence in $\mathcal{S}$

G. The space of generalized function of slow growth [$\mathcal{S}' = \mathcal{S}'(\mathcal{O})$]

Definition A generalized function of ***slow growth*** (tempered distribution) is any continuous linear functional on the space $\mathcal{S}$ of test functions.

$\mathcal{S}'$ is a linear set. A sequence of generalized function $\{f_k\} \in \mathcal{S}'$ converges to a generalized function $f \in \mathcal{S}'$, ie $f_k \to f$ as $k \to \infty$ in $\mathcal{S}'$ if $(f_k, \varphi) \to (f, \varphi)$ as $k \to \infty$, for any $\varphi \in \mathcal{S}$. The linear set $\mathcal{S}'$ equipped with convergence is termed the space $\mathcal{S}'$ of generalized function of slow growth.
$\mathcal{S}' \subseteq \mathcal{D}'$ that converges in $\mathcal{S}'$ implies convergence in $\mathcal{D}'$

H. Fourier transform of generalized function of slow growth

-I-The Fourier transform of test functions

Definition Let $\varphi(x) \in \mathcal{S}$ be absolutely integrable on $\Re^n$. The operation F of Fourier transform of φ is defined as:

$$F[\varphi](\xi) = \int \varphi(x) e^{i(\xi, x)} dx$$

The function $F[\varphi](\xi)$, which is the Fourier transform of the function $\varphi \in \mathcal{S}$, is bounded and continuous in $\Re^n$. A test function $\varphi(x)$ decreases at infinity faster than any power of $|x|^{-1}$.

Thus ,its Fourier transform may be differentiated under the integral sign any number of times:

$$D^{\alpha}F[\varphi](\xi)= \int (ix)^{\alpha}\varphi(x)e^{i(\xi,x)}dx = F[(ix)^{\alpha}\varphi](\xi)$$

Whence it follows that $F[\varphi]\in C^{\infty}$ .Further more, every derivative $D^{\alpha}\varphi(x)$ has the same properties and so

$$F[D^{\alpha}\varphi](\xi)= \int D^{\alpha}\varphi(x)e^{i(\xi,x)}dx$$

$$= (-i\xi)^{\alpha} F[\varphi](\xi).$$

From the general theory of the Fourier transformation , it follows that the function $\varphi(x)$ is expressed in terms of its Fourier Transform $F[\varphi](\xi)$ with the aid of the inverse Fourier transform,

$$F^{-1} : \varphi= F^{-1}[F[\varphi]], \text{ where } F^{-1}[\psi](x)=\frac{1}{(2\pi)^n}\int \psi(\xi)e^{-i(x,\xi)}d\xi$$

$$= \frac{1}{(2\pi)^n}F[\psi](-x)$$

$$= \frac{1}{(2\pi)^n}\int \psi(-\xi)e^{i(x,\xi)}d\xi$$

$$= \frac{1}{(2\pi)^n}F[\psi(-\xi)] \ .$$

-II- <u>The Fourier Transform of generalized function of slow growth</u>

Suppose f(x) is integrable function on $\Re^n$ .Then its Fourier transform ;

$F[f](\xi)= \int f(x)e^{i(\xi,x)}dx$; $|F[f](\xi)| \le \int |f(x)dx| < \infty$;is a (continuous) bounded function in $\Re^n$ and, hence determines the regular generalized function of slow growth by the formula:

$$(F[f], \varphi) = \int F[f](\xi)\varphi(\xi)d\xi \ , \varphi \in \boldsymbol{S}$$

Using the Fubini theorem on changing the order of integration; we transform the last integral:

$$\int F[f](\xi)\varphi(\xi)d\xi = \int[\int f(x)e^{i(\xi,x)}dx]\varphi(\xi)d\xi$$

$$= \int f(x)[\int \varphi(\xi)e^{i(\xi,x)}d\xi]dx$$

$$= \int f(x)F[\varphi](x)dx$$

$$= (f,F[\varphi])$$

Thus, $(F[f], \varphi)=(f,F[\varphi])$, $\varphi \in \boldsymbol{S}$. It is this equation that we take for definition of the Fourier transform F[f] of any generalized function of slow growth:

$(F[f], \varphi)=(f,F[\varphi])$,$f\in \boldsymbol{S}'$, $\varphi \in \boldsymbol{S}$.

The inverse Fourier transform of a function f is defined as:

$F^{-1}[f]\,(\xi) = \frac{1}{(2\pi)^n}F[f(-x)]$, $f\in \boldsymbol{S}'$, where f(-x) is the reflection of f(x).

Also, $F^{-1}[F(f)]=F[F^{-1}[f]]=f$, $f\in \boldsymbol{S}'$

Suppose $f(x,y) \in \boldsymbol{S}'(\Re^{n+m})$, where $x \in\Re^n$ and $y \in \Re^m$. We introduce the Fourier transform $F_x[f]$ with respect to the variable $X=(x_1,x_2,\dots,x_n)$ by putting for any basic function $\varphi(\xi,y) \in \boldsymbol{S}(\Re^{n+m})$,

$(F_x[f], \varphi)=(f,F_\xi[\varphi])$

Example Show that

a. $(\delta(x-x_0),f)=f(x_0)$

b. $F[\delta(x-x_0)]=e^{i(x_0,\xi)}$

c. $F[\delta]=1$

Solution a. $(\delta(x-x_0),f)= \int_{-\infty}^{\infty}\delta(x-x_0)f(x)dx$ Let $x-x_o=\xi$, $x=\xi+x_o$

$= \int_{-\infty}^{\infty}\delta(\xi)f(\xi+x_0)d\xi$ let $f(\xi+x_0)=g(\xi)$

$= \int\delta(\xi)g(\xi)d\xi$

$= g(0)$

$= f(0+x_0)$

$= f(x_0)$

b. $(F[\delta(x-x_0)], \varphi) =(\delta(x-x_0),F[\varphi])$, $\varphi \in \mathcal{S}$

$=F[\varphi](x_0)$ … by (a) above

$= \int\varphi(\xi)e^{i(x_0,\xi)}d\xi$

$=(e^{i(x_0,\xi)},\varphi)$, $\varphi \in \mathcal{S}$

$\Rightarrow F[\delta(x-x_0)]= e^{i(x_0,\xi)}$ … by equality of generalized function

c. Put $x_0=0$ in (b) above. Then $F[\delta(x)]=e^{i(0,\xi)}=1$

Note:- Here we consider that , $\delta=F^{-1}[1]= \frac{1}{(2\pi)^n}F[1]$

$\Rightarrow F[1]=(2\pi)^n\delta(\xi)$

-III-Properties of Fourier transform

a. Differentiating a Fourier transform

If $f \in \mathcal{S}'$, then $D^\alpha F[f]=F[(ix)^\alpha f]$. For $\varphi \in \mathcal{S}$, we have :

$$(D^{\alpha}F[f], \varphi) = (-1)^{|\alpha|}(F[f], D^{\alpha}\varphi)$$
$$= (-1)^{|\alpha|}(f, F[D^{\alpha}\varphi])$$
$$= (-1)^{|\alpha|}(f, (-ix)^{\alpha}F[\varphi])$$
$$= ((ix)^{\alpha}f, F[\varphi])$$
$$= \underline{(F[(ix)^{\alpha}f], \varphi)}$$

b. The Fourier transform of a derivative.

If $f \in \boldsymbol{S}'$, then $F[D^{\alpha}f] = (-i\xi)^{\alpha}F[f]$

In deed, for $\varphi \in \boldsymbol{S}$,we have

$$(F[D^{\alpha}f], \varphi) = (D^{\alpha}f, F[\varphi])$$
$$= (-1)^{|\alpha|}(f, D^{\alpha}F[\varphi])$$
$$= (-1)^{|\alpha|}(f, F[(i\xi)^{\alpha}\varphi])$$
$$= (-1)^{|\alpha|}(F[f], (i\xi)^{\alpha}\varphi)$$
$$= \underline{((-i\xi)^{\alpha}F[f], \varphi)}$$

REMARKS a) $F[e^{-a^2x^2}] = \frac{\sqrt{\pi}}{a}e^{\frac{-|\xi|^2}{4a^2}}$ for n=1

b) $F[e^{-(Ax,x)}] = \frac{\pi^{\frac{n}{2}}}{\sqrt{detA}}e^{\frac{-(A^{-1}\xi,\xi)}{4}}$ where A is a real positive matrix .

Example. Prove that $F[\frac{1}{|x|^2}] = \frac{2\pi^2}{|\xi|^2}$ in n=3.

Solution. $F\left[\frac{1}{|x|^2}\right] = \int_{|x|<R}\frac{e^{i(\xi,x)}}{|x|^2}dx$

$$= \int_0^R\int_0^{\pi}\int_0^{2\pi}\frac{e^{1|\xi|\rho\cos\theta}}{\rho^2}\rho^2 d\psi\sin\theta d\theta d\rho$$

$$= 2\pi\int_0^R\int_{-1}^{1} e^{i|\xi|\rho\mu}\, d\mu d\rho$$

$$= \frac{4\pi}{|\xi|}\int_0^R \frac{\sin(|\xi|\rho)}{\rho}d\rho$$

Since $\left|\int_R^\infty \frac{\sin|\xi|\rho}{\rho}d\rho\right| = \left|\frac{\cos(|\xi|\rho)}{|\rho|R} - \frac{1}{|\xi|}\int_R^\infty \frac{\cos(|\xi|\rho)}{\rho^2}d\rho\right| \le \frac{2}{|\xi|R}$,we have

$\int_0^R \frac{\sin(|\xi|\rho)}{\rho}d\rho = \frac{\pi}{2}$, $|\rho| \neq 0$

$$\Rightarrow \frac{4\pi}{|\xi|}\int_0^R \frac{\sin(|\xi|\rho}{\rho}d\rho \to \frac{2\pi^2}{|\xi|}$$

$$\Rightarrow F[\frac{1}{|x|^2}] = \frac{2\pi^2}{|\xi|}, \text{ for n=3}$$

REMARK A) For n=2,$\mathcal{P}(\frac{1}{|x|^2})\epsilon\mathcal{S}'$ is given by the formula:

$$\left(\mathcal{P}\frac{1}{|x|^2},\varphi\right) = \int_{|x|<1}\frac{\varphi(x)-\varphi(0)}{|x|^2}dx + \int_{|x|>1}\frac{\varphi(x)}{|x|^2}dx, \varphi\epsilon\mathcal{S}.$$

$F\left[\frac{1}{|x|^2}\right] = -2\pi\ln|\xi| - 2\pi c_0$, where $c_0 = \int_0^1 \frac{1-J_0(0)}{u}du - \int_1^\infty \frac{J_0(u)}{u}du$

and J_0 is the Bessel function of order zero.

B) Let $\delta_{s_r}(x)$ be a simple layer on sphere of radius r in $\mathfrak{R}^3$.

then $F[\delta_{s_r}(x)] = 4\pi r\frac{\sin r|\xi|}{|\xi|}$.

<u>The gamma function</u>

The ***gamma*** function; denoted by $\Gamma(n)$ is defined by

$\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}dx$; which is convergent for n>0

REMARK A) $\Gamma(n+1) = n\Gamma(n)$. This is shown as:

$$\Gamma(n+1) = \int_0^\infty x^n e^{-x}dx \qquad \text{Let} u = x^n \qquad dv = e^{-x}dx$$

$$du = nx^{n-1}dx \qquad v = -\mathbf{e}^{-x}$$

$$= \lim_{m\to\infty} -x^n e^{-x}\Big|_0^m + n\int_0^m x^n e^{-x}dx$$

$= n(\Gamma(n))$

B) $\Gamma(n+1) = n!$

C) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

D) For n≥ **3**,we get that $\nabla^2 \frac{1}{|x|^{n-2}} = -(n-2)\sigma_n \delta(x)$,where $\boldsymbol{\sigma_n}$ is the surface area of a unit sphere in $\mathfrak{R}^n$:

$\sigma_n = \int_{s_1} ds = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$,and **Γ** is the gama function.

CHAPTER -2-

FUNDAMENTAL SOLUTION OF LINEAR DIFFERNTIAL OPERATORS

To build the fundamental solution of linear differential operator with constant coefficients, we will use the Fourier transform method. Naturally, only fundamental solution of slow growth can be obtained in this way.

2.1 Generalized solution of linear differential equation

Let $\sum_{|\alpha|=0}^{m} a_{\alpha}(x)D^{\alpha}u = f(x)$, **f∈ $\mathcal{D}'$ ..... [1]** be a linear differential equation of order m with coefficients $a_\alpha \in c^\infty(\Re^n)$. By introducing the differential operator $L(x,D) = \sum_{|\alpha|=0}^{m} a_{\alpha}(x)D^{\alpha}$, we can rewrite equation [1] as:

L(x,D)u=f(x) ..... [2]

Definition The function U∈ $\mathcal{D}'$ which satisfies equation [2] in a region G ,in a generalized sense , i.e **(L(x,D)u, φ) = (f, φ)....[3]** for any φ∈ $\mathcal{D}$ (G) is said to be a ***generalized solution*** of equation [1] .

Lemma (2.1) (L(x,D)U, φ)= (U,L*(x,D) φ)= (f, φ) , φ∈ $\mathcal{D}$ (G) where

$$(L^*(x,D)\,\varphi)= \sum_{|x|=0}^{m}(-1)^{|\alpha|}D^{\alpha}(a_{\alpha}\varphi) .$$

Proof : $(L(x,D)U, \varphi)= (\sum_{|x|=0}^{m} a_{\alpha}D^{\alpha}u,\varphi)$

$$= \sum_{|x|=0}^{m}(a_{\alpha}D^{\alpha}u,\varphi)$$

$$= \sum_{|x|=0}^{m}(-1)^{\alpha}(u,D^{\alpha}(a_{\alpha}\varphi))$$

$$= (u, \sum_{|\alpha|=0}^{m}(-1)^{\alpha}D^{\alpha}(a_{\alpha}\varphi))$$

$= (u, L^*(x,D)\,\varphi)$

Thus, $(L(x,D)u, \varphi) = (u, L^*(x,D)\,\varphi) = (f, \varphi)$

Note that each classical solution is also a generalized solution.

2.2. The fundamental solution of linear differential operator with ordinary derivatives

THEOREM 2.1 Let L=: $\frac{d^n}{dt^n} + \frac{a_1 d^{n-1}}{dt^{n-1}} + \ldots + a_n$ be linear differential operator where the coefficients $a_1, a_2, a_3, \ldots, a_n$ are constants. Let $\mathcal{E} \in \mathcal{D}'$ such that

$$\mathbf{L}\,\boldsymbol{\varepsilon} = \frac{d^n \varepsilon}{dt^n} + a_1 \frac{d^{n-1}\varepsilon}{dt^{n-1}} + \ldots + a_n \varepsilon = \delta(t) \ldots.. \mathbf{[4]}$$

Also, let z(t) be the solution of the homogeneous differential equation $Lz = z^{(n)} + a_1 z^{(n-1)} + a_2 z^{(n-2)} + \ldots + a_n z = 0$; satisfying the condition $z(0) = z'(0) = z''(0) = \ldots = z^{n-2}(0) = 0$ and $z^{n-1}(0) = 1$

Then $\boldsymbol{\varepsilon}$ is the fundamental solution of L $\boldsymbol{\varepsilon} = \delta(x)$ and it is given by the formula:

$\boxed{\boldsymbol{\varepsilon}(\mathbf{t}) = \boldsymbol{\theta}(\mathbf{t})\mathbf{z}(\mathbf{t})}$..... **[5]** ,where $\theta(t) = \begin{cases} 1, t \geq 0 \\ 0, t < 0 \end{cases}$ is a Heaviside unit function.

Proof $\boldsymbol{\varepsilon}(t) = \theta(t)z(t)$

$\Rightarrow \boldsymbol{\varepsilon}'(t) = (\theta(t)z(t))' = \theta'(t)z(t) + \theta(t)z'(t)$

$\Rightarrow \boldsymbol{\varepsilon}'(t) = \delta(t)z(t) + \theta(t)z'(t)$.... Since $\theta'(t) = \delta(t)$

$\Rightarrow \boldsymbol{\varepsilon}'(t) = \theta(t)z'(t)$ .... Since $z(0)=0$

Also, $\boldsymbol{\varepsilon}''(t) = [\theta(t)z'(t)]' = \theta'(t)z'(t) + \theta(t)z''(t)$

$\Rightarrow \boldsymbol{\varepsilon}''(t) = \delta(t)z'(t) + \theta(t)z''(t) = 0 + \theta(t)z''(t)$ .... As $z'(0)=0$

$\Rightarrow \boldsymbol{\varepsilon}''(t) = \theta(t)z''(t)$

Thus, $\boldsymbol{\varepsilon}^n(t) = [\theta(t)z^{n-1}(t)]'$

$\Rightarrow \boldsymbol{\varepsilon}^n(t) = \theta'(t)\, z^{n-1}(t) + \theta(t)\, z^n(t)$

$\Rightarrow \varepsilon^n(t)= \delta(t)\, z^{n-1}(t)+ \theta(t)\, z^n(t)$

$\Rightarrow \varepsilon^n(t)= \delta(t)+ \theta(t)\, z^n(t)$ …… since $z^{n-1}(o)=1$

Here, we consider that $\varepsilon(t)= \delta(t)+ \theta(t)\, z(t)$ …. Since

$\varepsilon^n(t)= \delta(t)+ \theta(t)\, z^n(t)$ and for n=1

$\Rightarrow L\,\varepsilon(t)= \delta(t)+ \theta(t) L\, z(t)$

$\Rightarrow L\,\varepsilon(t)= \delta(t)$…..since $Lz(t)=0$

$\Rightarrow \varepsilon(t)$ is a fundamental solution and hence $\varepsilon(t) = \theta(t)\, z(t)$

Example Find the fundamental solution of the following linear differential operator with ordinary derivatives .

a) $L:=\frac{d}{dt}+a$ b) $L:=\frac{d^2}{dt^2}+a^2$

solution a) $L=\frac{d}{dt}+a$

Let $\varepsilon \in \mathcal{D}'$ be the fundamental solution .

$\Rightarrow \varepsilon(t) = \theta(t)\, z(t)$ where $z(0)=1$ and $Lz=0$

Now , $Lz=0$

$$\Rightarrow \frac{dz}{dt}+az=0$$

$$\Rightarrow \frac{dz}{z}=-adt$$

$$\Rightarrow \int\frac{dz}{z}=-a\int dt$$

$$\Rightarrow \ell n|z|=-at+c$$

$\Rightarrow z(t) = ke^{-at}$ But $z(0)= ke^0= k= 1$

$\Rightarrow z(t)= e^{-at}$

Thus, $\boxed{\varepsilon(t) = \theta(t)e^{-at}}$ is the fundamental solution.

b) $L= \frac{d^2}{dt^2}+a^2$

Let $\varepsilon \in D'$ be the fundamental solution

$\Rightarrow \varepsilon(t) = \theta(t)\, z(t)$ where $z(0)=0, z'(0)=1$ and $Lz=0$

$\Rightarrow \frac{d^2z}{dt^2}+a^2z=0$, this is 2nd order ODE with constant coefficient.

$\Rightarrow m^2+a^2=0$ is the characteristic equation

$\Rightarrow m=\pm ai$

$\Rightarrow z(t)=c_1\cos at+c_2\sin at$ is the homogenous solution

$z'(t)=-ac_1\sin at+ac_2\cos at$, $z(0)=0 \Rightarrow c_1=0, z'(0)=ac_2=1$

$\Rightarrow c_2=1/a$

$\Rightarrow z(t)=\frac{\sin at}{a}$

Thus, $\boxed{\varepsilon(t) = \theta(t)\frac{\sin at}{a}}$ is the fundamental solution

2.3 Fundamental solution of linear differential operator with constant coefficients

One of the basic and most profound results is the proof of existence of a fundamental solution $\boldsymbol{\varepsilon}(x)$in $\mathcal{D}'$of any linear differential operator L(D)$\neq$ **0** with constant coefficients. This result was first obtained independently by **L. Ehrenpreis (1954)**and **B.Malgrange (1953)**.

Definition Let L be a differential operator with constant Coefficients, $a_\alpha(x)= a_\alpha$:

$$\mathbf{L(D)}=\sum_{|\alpha|=0}^{m} a_\alpha D^\alpha,\ \mathbf{L^*(D)=L(-D)}..... [6]$$

. A generalized function $\boldsymbol{\varepsilon}\in D'$ that satisfies equation

$$\boxed{\mathbf{L(D)}\boldsymbol{\varepsilon} = \delta(\mathbf{x})} ..... [7]$$

in $\mathfrak{R}^n$ is said to be a ***fundamental solution (influence function)*** of operator L(D)

THEOREM 2.2 (Malgrange-Ehrenpreis)

Every differential operator with constant coefficients L(D)$\neq$ **0** has a fundamental solution in $\mathcal{D}'$

REMARK Fundamental solution $\boldsymbol{\varepsilon}$(x) of the operator L(D) is not generally unique; it is determined to which a term $\boldsymbol{\varepsilon}$(x) , which an arbitrary solution of the homogeneous equation : $\mathbf{L(D)}\ \boldsymbol{\varepsilon}_0 = 0$..... **[8]**

The general function $\boldsymbol{\varepsilon}(x)+\boldsymbol{\varepsilon}_0(x)$ is also the fundamental solution of the operator L(D):

$$L(D)(\boldsymbol{\varepsilon}+\boldsymbol{\varepsilon}_0)=L(D)\,\boldsymbol{\varepsilon}+L(D)\,\boldsymbol{\varepsilon}_0 \text{ .... Linearity of L}$$

$$= L(D)\,\boldsymbol{\varepsilon} = \delta(x)$$

Lemma (2.2) For generalized function $\boldsymbol{\varepsilon} \in \boldsymbol{S}'$ to be the fundamental solution of the operator L(D), it is necessary and sufficient that its Fourier transform $F[\boldsymbol{\varepsilon}]$ satisfies the equation :

$$\mathbf{L(-i\xi)F[\boldsymbol{\varepsilon}]=1} \ldots\ldots \mathbf{[9]}, \text{ where } L(\xi)=\sum_{|\alpha|=0}^{m} a_\alpha \xi^\alpha .$$

Proof $\Rightarrow$) Suppose $\boldsymbol{\varepsilon} \in \boldsymbol{S}'$ to be fundamental solution of L(D) .

Wts $L(-i\xi)F[\boldsymbol{\varepsilon}]=1$.

Since $\varepsilon \in \boldsymbol{S}'$ is fundamental solution, $L(D)\,\boldsymbol{\varepsilon} = \delta(x)$

$$\Rightarrow F[L(D)\,\boldsymbol{\varepsilon}]=F[\delta]$$

$$\Rightarrow F[\sum_{|\alpha|=0}^{m} a_\alpha D^\alpha \varepsilon]=1$$

$$\Rightarrow \sum_{|\alpha|=0}^{m} a_\alpha F[D^\alpha \varepsilon] =1$$

$$\Rightarrow \sum_{|\alpha|=0}^{m} a_\alpha (-i\xi)^\alpha F[\varepsilon]=1$$

$$\Rightarrow L(-i\xi)F[\boldsymbol{\varepsilon}]=1$$

$\Leftarrow$) Suppose $L(-i\xi)F[\boldsymbol{\varepsilon}] =1$

Wts $\boldsymbol{\varepsilon} \in \boldsymbol{S}'$ is a fundamental solution. i.e., $L(D)\,\boldsymbol{\varepsilon} =\delta(x)$.

$$L(-i\xi)F[\boldsymbol{\varepsilon}] =1$$

$$\Rightarrow \sum_{|\alpha|=0}^{m} a_\alpha (-i\xi)^\alpha F[\varepsilon]=1$$

$$\Rightarrow \sum_{|\alpha|=0}^{m} a_\alpha F[D^\alpha \varepsilon] =1$$

$$\Rightarrow F[\sum_{|\alpha|=0}^{m} a_\alpha D^\alpha \varepsilon] = 1 = F[\delta]$$

$$\Rightarrow F[L(D)\,\boldsymbol{\varepsilon}] = F[\delta]$$

$$\Rightarrow L(D)\,\boldsymbol{\varepsilon} = \delta(x)\,.$$

$\Rightarrow \boldsymbol{\varepsilon}$ is a fundamental solution of L(D) .

REMARK Lemma(2.2) reduces the problem of building fundamental solution of slow growth for linear differential operator with constant coefficients solving in $\boldsymbol{S}$.

$\boldsymbol{S'}$ algebraic equation of the type:

p(ξ)x=1 ….. [10] ,with p arbitrary polynomial

The solution of [8] belongs to $\boldsymbol{\mathcal{D}'}$ (if such solution) must coincide with the function $\frac{1}{p(\xi)}$ out side the set N_p of the zero of the polynomial p(ξ). i.e., $N_p=\{\xi: p(\xi)=0\}$. This implies that if $N_p \neq \varnothing$,then [10] has no unique solution.

If the solution $\frac{1}{p(\xi)}$ is locally integrable in $\Re^n$ then it (or, precisely, the regular function defined by this function)is a solution in $\boldsymbol{S'}$ of [10]. But if the function $\frac{1}{p(\xi)}$ is not locally integrabl in $\Re^n$, there arises the non-trivial problem of building a solution of [8] in $\boldsymbol{S'}$

We will denote any solution of [10] belonging to $\boldsymbol{S}'$ by Reg($\frac{1}{p(\xi)}$).

The construction of such solution depends to great extent on the construction of the set N_p and can be carried out for each concrete polynomial p. Thus equation [10] is always solvable in $\boldsymbol{S}'$ or F[$\boldsymbol{\varepsilon}$]=reg$\frac{1}{l(-i\xi)}$.

Each linear differential operator L(D) with constant coefficient has a fundamental solution of slow growth ,and their solution is given by the formula :

$$F[\boldsymbol{\varepsilon}] = \text{reg}\frac{1}{L(-i\xi)} \Leftrightarrow \boldsymbol{\varepsilon} = F^{-1}[\text{reg}\frac{1}{L(-i\xi)}]$$

$$\Leftrightarrow \boldsymbol{\varepsilon} = \frac{1}{(2\pi)^n} F[\text{reg}\frac{1}{L(i\xi)}]$$

THEOREM 2.3 Let $f \in \mathcal{D}'$ be such that $\boldsymbol{\varepsilon} \star f$ exists in $\mathcal{D}'$. Then the equation **L(D)u=f(x) ….. [11]** has a solution in $\mathcal{D}'$ that is given by the formula **u= $\boldsymbol{\varepsilon} \star$f ….. [12].** This solution is unique in a class of generalized function belonging to $\mathcal{D}'$ for which $\boldsymbol{\varepsilon}$ exist .

Proof Suppose $\boldsymbol{\varepsilon} \star$f exists in $\mathcal{D}'$

Wts. L(D)($\boldsymbol{\varepsilon} \star$f)= f

Now, L(D)($\boldsymbol{\varepsilon} \star$f)= $\sum_{|\alpha|=0}^{m} a_\alpha D^\alpha(\varepsilon * f)$

$$= (\sum_{|\alpha|=0}^{m} a_\alpha D^\alpha \varepsilon) \star f$$

$$= L(D)\, \boldsymbol{\varepsilon} \star f$$

$= \delta \star f$

$= \int \delta(y) f(x-y) dy$

$= f(x)$

Thus, u= $\boldsymbol{\varepsilon} \star f$ is indeed a solution of equation [11]. Next, we will prove the uniqueness of such a solution in the class of the generalized function $\mathcal{D}'$ whose convolution with $\boldsymbol{\varepsilon}$ exists in $\mathcal{D}'$. For this, it suffice only to show the corresponding homogeneous equation L(D)u=0 has only a zero solution in $\mathcal{D}'$.

Now, $u = u \star \delta$

$= u \star L(D)\,\boldsymbol{\varepsilon}$

$= L(D)u \star \boldsymbol{\varepsilon}$

$=0$

Thus, equation [11] has unique solution.

REMARK (Method of descent)

Consider linear differential equation with constant coefficient in the space of $\mathfrak{R}^{n+1}$ of variables $(x,t)=(x_1,x_2,x_3,\ldots,x_n,t)$:

$$\boldsymbol{L\left(D,\frac{\partial}{\partial t}\right)u = f(t)\delta(t), f\epsilon\mathcal{D}'(\mathfrak{R}^n)} \ \ldots\textbf{[13]}, \text{ where}$$

$\mathbf{L\left(D,\frac{\partial}{\partial t}\right)u = \sum \ \frac{\partial q}{\partial t^q}L_q + L_0}$, **and** $\mathbf{L_q}$(D) are differential operators in the variables x.

Let the generalized function $u \epsilon \mathcal{D}'(\mathfrak{R}^{n+1})$ allow continuation of function of the type $\varphi(x)1(t)$,where $\varphi \epsilon \mathcal{D}(\mathfrak{R}^n)$ in the following sense:

What ever the sequence of test function $\eta_k(t), k = 1,2,3, \ldots$belonging to $\mathcal{D}(\mathfrak{R}^1)$ and converging to 1 in $\mathfrak{R}^1$, there exists the limit:

$\lim_{k\to\infty}(u, \varphi(x)\eta_k(t)) = (u, \varphi(x)1(t)) = (u_0, \varphi)$, ...[14] and this limit does not depend on the sequence $\{\eta_k\}$.

If $\varepsilon(x,t)$ is a fundamental solution of the operator $L(D, \frac{\partial}{\partial t})$ and admits of the continuation ε_0 of the type [14],then the generalized function $(\varepsilon_0, \varphi) = (\varepsilon, \varphi(x)1(t), \varphi \epsilon \mathcal{D}(\mathfrak{R}^n)$,is the fundamental solution of the operator $L_0(\mathcal{D})$; In particular, if $\mathcal{E}(x,t)$is such that$\int |\mathcal{E}(x,t)|\, dt$is locally integerable in $\mathfrak{R}^n$,then

$$\varepsilon_0(x) = \int_{-\infty}^{\infty} \varepsilon(x,t)dt \ldots [15]$$

The physical meaning of formula [15] is that $\varepsilon_0(x)$ is a (t-independent) perturbation generated by the source $\delta(x)1(t)$ concentrated along the t-axis.

CHAPTER -3-

CONSTRUCTION OF FUNDAMENTAL SOLUTION OF DIFFERENTIAL OPERATORS ;AND PARAMETRIX

3.1 The fundamental solution of heat conduction operator

Let $L:=\frac{\partial}{\partial t}-a^2\nabla^2$ be heat conduction operator with $\nabla^2=\frac{\partial^2}{\partial x^2}$.

Let $\boldsymbol{\varepsilon}$(x,t)∈ $\boldsymbol{S}'$ and $\frac{\partial\varepsilon}{\partial t}-a^2\nabla^2\varepsilon=\delta(x,t)$ ….. **[16]** be heat conduction equation.

Then the solution of heat conduction equation [16] is given by:

$\varepsilon(x,t)=\frac{\theta(t)}{(2a\sqrt{\pi t})^n}e^{\frac{-|x|^2}{4a^2t}}$ ….. **[17]** and therefore ε(x,t) is the fundamental solution of heat conduction operator .

Now, we derive formula [17] using Fourier transform method as follows:

$$\frac{\partial\varepsilon}{\partial t}-a^2\nabla^2\varepsilon=\delta(x,t)$$

$$\Rightarrow F_x[\frac{\partial\varepsilon}{\partial t}-a^2\nabla^2\varepsilon]=F_x[\delta(x,t)]$$

$$\Rightarrow F_x[\frac{\partial\varepsilon}{\partial t}]-F_x[a^2\nabla^2\varepsilon]=F_x[\delta(x,t)]$$

Now, i) $F_x[\frac{\partial\varepsilon}{\partial t}]=\frac{\partial}{\partial t}F_x[\varepsilon](\xi,t)$ ii) $F_x[a^2\nabla^2\varepsilon=a^2F_x[\frac{\partial^2\varepsilon}{\partial x^2}]=a^2(-i\xi)^2F_x[\varepsilon](\xi,t)$

iii) $F_x[\delta(x,t)]=F_x[\delta(x).\ \delta(t)]=F_x[\delta(x)].\ \delta(t)=1(\xi).\delta(t)$

Then, $F_x[\frac{\partial\varepsilon}{\partial t}-a^2\nabla^2\varepsilon]=F_x[\delta(x,t)]$

$\Rightarrow \frac{\partial}{\partial t} F_x[\varepsilon](\xi,t) - a^2(-i\xi)^2 F_x[\varepsilon](\xi,t) = 1(\xi).\delta(t)$ .... [a] Now, let the

generalized function $\bar{\varepsilon}(\xi,t) = F_x[\varepsilon](\xi,t)$ ..... [b]

$\Rightarrow \frac{\partial}{\partial t}\bar{\varepsilon} + a^2|\xi|^2\,\bar{\varepsilon} = 1(\xi).\delta(t)$ .... [c] ; substitution of (b) in (a)

Let $z(\xi,t)$ be the homogeneous solution of [c] with $z(\xi,0)=1$

$$\Rightarrow \frac{\partial z}{\partial t} + a^2|\xi|^2 z = 0$$

$$\Rightarrow \int \frac{\partial z}{z} = \int -a^2|\xi|^2 dt$$

$$\Rightarrow \ln z = -a^2|\xi|^2 t + c$$

$$\Rightarrow z(\xi,t) = ke^{-a^2|\xi|^2 t} \text{ ,but } z(\xi,0) = k = 1$$

$$\Rightarrow z(\xi,t) = e^{-a^2|\xi|^2 t}$$

Thus, $\bar{\varepsilon}(\xi,t) = \theta(t)\, z(\xi,t)$ is the fundamental solution of [c] by [5]

$$\Rightarrow \bar{\varepsilon}(\xi,t) = \theta(t)\, e^{-a^2|\xi|^2 t}$$

Applying the inverse Fourier Transform, we have:

$$\bar{\varepsilon}(\xi,t) = F_x[\varepsilon](\xi,t)$$

$$\Rightarrow \varepsilon(x,t) = F_\xi^{-1}[\bar{\varepsilon}](\xi,t)$$

$$\Rightarrow \varepsilon(x,t) = \frac{1}{(2\pi)^n}\int \varepsilon(\xi,t)e^{-i(\xi,x)}d\xi$$

$$\Rightarrow \varepsilon(x,t) = \frac{1}{(2\pi)^n}\int \theta(t)e^{-a^2|\xi|^2 t}e^{-i(\xi,x)}d\xi$$

$$\Rightarrow \varepsilon(x,t)=\frac{1}{(2\pi)^n}\int \theta(t)e^{-a^2|\xi|^2 t-i(\xi,x)}d\xi$$

$$\Rightarrow \varepsilon(x,t)=\frac{\theta(t)}{(2\pi)^n}\int e^{-a^2|\xi|^2 t-i\xi x} \quad , \quad \text{Let } \sigma=a\xi\sqrt{t}$$

$$\Rightarrow \varepsilon(x,t)=\frac{\theta(t)}{(2\pi)^n}\int e^{-(\sigma^2+\frac{ix\sigma}{a\sqrt{t}})}\frac{1}{a\sqrt{t}}d\sigma$$

$$\Rightarrow \varepsilon(x,t)=\frac{\theta(t)}{a\sqrt{t}(2\pi)^n}\int e^{-(\sigma+\frac{ix}{2a\sqrt{t}})^2-\frac{|x|^2}{4a^2 t}}d\sigma \text{...completing the square}$$

$$\Rightarrow \varepsilon(x,t)=\frac{\theta(t)}{a\sqrt{t}(2\pi)^n}e^{-\frac{|x|^2}{4a^2 t}}\int e^{-(\sigma+\frac{ix}{2a\sqrt{t}})^2}d\sigma$$

$$\Rightarrow \varepsilon(x,t)=\frac{\theta(t)}{a\sqrt{t}(2\pi)^n}e^{-\frac{|x|^2}{4a^2 t}}\int_{\operatorname{Im}\beta=\frac{x}{2a\sqrt{t}}} e^{-\beta^2}d\beta \text{ … integration along imaginary axis}$$

$$\Rightarrow \varepsilon(x,t)=\frac{\theta(x)}{a\sqrt{t}(2\pi)^n}e^{-\frac{|x|^2}{4a^2 t}}\int_{-\infty}^{\infty} e^{-\beta^2}d\beta \text{ … shifting line of integration to the real axis}$$

$$\Rightarrow \varepsilon(x,t)=\frac{\theta(t)}{a\sqrt{t}(2\pi)^n}e^{-\frac{|x|^2}{4a^2t}}a\sqrt{t}\left(\frac{\sqrt{\pi}}{a\sqrt{t}}\right)^n$$

$$\Rightarrow \boxed{\varepsilon(x,t)=\frac{\theta(t)}{(2a\sqrt{\pi t})^n}e^{-\frac{|x|^2}{4a^2t}}}$$ is the fundamental solution of heat conduction equation.

3.2 The fundamental solution of the wave operator

Let $\Box_a$ is the wave operator (or D'Alembert's operator) where

$\Box_a = \frac{\partial^2}{\partial t^2} - a^2\nabla^2$; $\Box \equiv \Box_1$, with $\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2}$ called ***Laplace's*** operator.

Then $\Box_a \, \varepsilon_n(x,t) = \delta(x,t)$ ..... **[18]** is the wave equation. Thus,

$\frac{\partial^2}{\partial t^2}\varepsilon_n(x,t) - a^2\frac{\partial^2}{\partial x^2}\varepsilon_n(x,t) = \delta(x,t)$. Applying Fourier transform F_x

$$F_x[\frac{\partial^2}{\partial t^2}\varepsilon_n(x,t) - a^2\frac{\partial^2}{\partial x^2}\varepsilon_n(x,t)] = F_x[\delta(x,t)]$$

$$\Rightarrow F_x[\frac{\partial^2}{\partial t^2}\varepsilon_n] + F_x[-a^2\frac{\partial^2}{\partial x^2}\varepsilon_n] = F_x[\delta(x)].\delta(t)$$

Now, i) $F_x\,[\frac{\partial^2}{\partial t^2}\varepsilon_n] = \frac{\partial^2}{\partial t^2}F_x\varepsilon_n$

ii) $F_x[-a^2\frac{\partial^2}{\partial x^2}\varepsilon_n] = -a^2(-i\xi)^2F_x[\varepsilon_n] = a^2|\xi|^2F_x[\varepsilon_n]$ iii) $F[\delta(x)]\delta(t) = 1(\xi).\delta(t)$

Thus, $F_x[\frac{\partial^2}{\partial t^2}\varepsilon_n] + F_x[-a^2\frac{\partial^2}{\partial x^2}\varepsilon_n] = F_x[\delta(x)].\delta(t)$

$\Rightarrow \frac{\partial^2}{\partial t^2}F_x[\varepsilon_n] + a^2|\xi|^2F_x[\varepsilon_n] = (\xi).\delta(t)$ Let $\overline{\varepsilon_n}(\xi,t) = F_x[\varepsilon_n(x,t)]$.

$$\Rightarrow \frac{\partial^2}{\partial t^2}\overline{\varepsilon_n}(\xi,t) + a^2|\xi|^2\overline{\varepsilon_n}(\xi,t) = 1(\xi).\delta(t) \quad ..... [a]$$

Let z(ξ,t) be the fundamental solution of homogeneous equation [a], with $z(\xi,0)=0$, $z_t(\xi,0)=1$.

$$\Rightarrow \frac{\partial^2}{\partial t^2} z + a^2|\xi|^2 z = 0$$

$\Rightarrow m^2+a^2|\xi|^2 = 0$ is the characteristic equation.

$\Rightarrow m_{1,2}=\pm a|\xi|i$

$\Rightarrow z(\xi,t)=c_1\cos a|\xi|t+c_2\sin a|\xi|t$ is the solution and

$Z_t(\xi,t)=-c_1a|\xi|\sin a|\xi|t+c_2a|\xi|\cos a|\xi|t$, $z(\xi,0)=c_1=0$,

$Z_t(\xi,0)=c_2a|\xi| =1 \quad \Rightarrow c_2=\frac{1}{a|\xi|}$ Thus $z(\xi,t)=\frac{\sin a|\xi|t}{a|\xi|}$.

$\Rightarrow \overline{\varepsilon_n}(\xi,t) = \theta(t)z(\xi,t)$,where θ(t) is the Heaviside unit function.

$$\Rightarrow \overline{\varepsilon_n}(\xi,t) = \frac{\theta(t)\sin a|\xi|t}{a|\xi|} = F_x[\varepsilon_n(x,t)]$$

$\Rightarrow \boxed{\varepsilon_n(x,t) = \frac{\theta(t)}{a} F_\xi^{-1}\left[\frac{\sin a|\xi|t}{|\xi|}\right]}$ is the fundamental soluton to the

wave operator

For instance , let n=3. Then we know that

$F[\delta_{s_{at}}] = 4\pi at\frac{\sin a|\xi|t}{|\xi|}$, where $\boldsymbol{\delta_{S_{at}}}$ is the simple layer on the sphere of radius at.

$$\Rightarrow \delta_{s_{at}} = F_{\xi}^{-1}\left[4\pi at\frac{\sin at|\xi|}{|\xi|}\right] = 4\pi at F_{\xi}^{-1}\left[\frac{\sin at|\xi|}{|\xi|}\right]$$

$$\Rightarrow F_{\xi}^{-1}\left[\frac{\sin at|\xi|}{|\xi|}\right] = \frac{1}{4\pi at}\delta_{s_{at}}$$

$$\Rightarrow \varepsilon_3(x,t) = \frac{\theta(t)}{4\pi a^2 t}\delta_{s_{at}}$$

$\Rightarrow \varepsilon_3(x,t) = \frac{\theta(t)}{2\pi a}\delta(a^2t^2 - |x|^2)$, where the generalized function $\boldsymbol{\varepsilon_3}$ acts according to the rule:

$$(\varepsilon_3,\varphi) = \frac{1}{4\pi a^2}\int_0^{\infty}\frac{(\delta_{s_{at}},\varphi)dt}{t} = \frac{1}{4\pi a^2}\int_0^{\infty}\frac{1}{t}\int_{s_{at}}\varphi(x,t)ds_x\,dt \ ,\varphi\epsilon S(\Re^4)$$

Similarly, we can obtain that :

$\varepsilon_1(x,t) = \frac{1}{2a}\theta(at - |x|)$ and $\varepsilon_2(x,t) = \frac{\theta(at+|x|)}{2\pi\sqrt{a^2t^2-|x|^2}}$

3.3. The fundamental solution of Laplace's operator

Let L=:∇^2 be the Laplace's operator , where $\nabla^2=\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}$ (i=1,2,3,…). Let $\varepsilon_n(x)\epsilon\mathcal{S}'$ and $\nabla^2\varepsilon_n(x)=\delta(x)$….[19]; for $x=(x_1,x_2,x_3,\dots,x_n)$ be Laplace's operator. Then the fundamental solution of Laplace's operator is:

$$\varepsilon_n(x)=\begin{cases}\frac{1}{2\pi}\ln|x| & ,n=2\\ -\frac{1}{(n-2)\sigma_n}|x|^{-n+2} & ,n\geq 3\end{cases}\quad \text{…[20]}$$

We will compute the fundamental solutions using Fourier transform method. Applying Fourier transform to equation [16], we get:

$$F[\nabla^2\varepsilon_n(x)]=F[\delta(x)]$$

$$\Rightarrow(-i\xi)^2F[\varepsilon_n]=1$$

$$\Rightarrow F[\varepsilon_n]=-\frac{1}{|\xi|^2}$$

$$\Rightarrow\varepsilon_n(x)=-F^{-1}[\frac{1}{|\xi|^2}]$$

$$\Rightarrow\varepsilon_n(x)=-\frac{1}{(2\pi)^n}F[\frac{1}{|\xi|^2}]$$

Now, let us consider for n=2,n=3 and n>3.

CASE 1. Let n=2.

$$\Rightarrow \varepsilon_2 = -\frac{1}{4\pi^2}F[\frac{1}{|\xi|^2}] \Rightarrow \varepsilon_2 = -\frac{1}{4\pi^2}(-2ln|x| - 2\pi C_0)$$

$$\Rightarrow \varepsilon_2 = \frac{1}{2\pi}\ln|x| + \frac{C_0}{2\pi}$$

Since a constant satisfies the homogeneous Laplace's equation, disregarding the term $\frac{C_0}{2\pi}$,we see the fundamental solution for n=2 is:

$$\boxed{\varepsilon_2(x) = \frac{1}{2\pi}ln|x|}$$

CASE 2. Let n=3.

$$\Rightarrow \varepsilon_3(x) = -\frac{1}{8\pi^3}F[\frac{1}{|\xi|^2}]$$

$$\Rightarrow \varepsilon_3(x) = -\frac{1}{8\pi^3}\cdot\frac{2\pi^2}{|x|} \text{ since for n=3,} F\left[\frac{1}{|\xi|^2}\right] = \frac{2\pi^2}{|x|}$$

$\Rightarrow \varepsilon_3(x) = -\frac{1}{4\pi|x|} = -\frac{1}{\sigma_3|x|}$. But $\sigma_3 = \frac{2\pi^{\frac{3}{2}}}{\Gamma\left(\frac{3}{2}\right)}$,where Γ is a gama function and as $\sigma_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}$.

$\Rightarrow \sigma_3 = \frac{1}{4\pi}$, and thus for n=3, $\boxed{\varepsilon_3(x) = -\frac{1}{4\pi|x|}}$

CASE 3. Form [15] we know that $\boldsymbol{\varepsilon_0}\ (\mathbf{x}) = \int_{-\infty}^{\infty}\boldsymbol{\varepsilon}(\mathbf{x},\mathbf{t})\mathbf{dt}$. Suppose

$\varepsilon(x,t) = \frac{\theta(t)}{(2a\sqrt{\pi t})^n}e^{\frac{-|x|^2}{4a^2 t}}$ be the fundamental solution of heat conduction operator.

Since $\varepsilon(x,t)$ is locally integerable in$\Re^n$,letting $\boldsymbol{\theta}(\boldsymbol{t})$ =1 for t≥0,a=1, we have :

$$\varepsilon(x,t) = \frac{1}{(2\sqrt{\pi t})^{n}} e^{\frac{-|x|^2}{4t}}$$

$$\Rightarrow \varepsilon_n(x) = \int_{-\infty}^{\infty} \frac{1}{(2\sqrt{\pi t})^n} e^{-\frac{|x|^2}{4t}} dt$$

$$\Rightarrow \varepsilon_n(x) = \frac{1}{2^n \pi^{\frac{n}{2}}} \int_0^\infty \frac{1}{t^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} dt \qquad \text{Let } u = \frac{|x|^2}{4t}, t = \frac{|x|^2}{4u} \quad du = -\frac{4u^2}{|x|^2} dt$$

$$\Rightarrow \varepsilon_n(x) = -\frac{|x|^{-n+2}}{4\pi^{\frac{n}{2}}} \int_0^\infty e^{-u} u^{\frac{n}{2}-2} du$$

$$\Rightarrow \varepsilon_n(x) = -\frac{|x|^{-n+2}}{4\pi^{\frac{n}{2}}} \Gamma\left(\frac{n}{2} - 1\right)$$, where Γ is the gamma function

$$\Rightarrow \varepsilon_n(x) = -\frac{1}{(n-2)\sigma_n} |x|^{-n+2}$$ for $n \geq 3$.

Therefore, the fundamental solution of Laplace's operator is:

$$\varepsilon_n(x) = \begin{cases} \frac{1}{2\pi} \ln|x|, n = 2 \\ -\frac{1}{(n-2)\sigma_n} |x|^{-n+2}, n \geq 3 \end{cases}$$

3.4. PARAMETRIX

The fundamental solution is a useful tool for the construction and representation of more general solution of a partial differential equation. How ever, some of the problems in which it plays a role can be handled just as easily, or perhaps better ,by means of function possessing a singularity that is not annihilated but merely smoothed out by the differential operator in equation .The smoothing might even be so weak that the singularity actually becomes augmented but acquires a less rapid growth than to be anticipated from the order of the operator. Such a function is called a parametrix associated with relevant partial differential equation. In particular, any fundamental solution is automatically a parametrix.

Thus, parametrix is a generalization of the notion of a fundamental solution . We say that a function p(x,y) of two variables x,y$\epsilon\Omega$ is a parametrix for the operator L(x,∂_x) in$\Re^3$ if :

[A]. $\boxed{\boldsymbol{L(x,\partial_x)p(x,y) = \delta(x-y) + R(x,y)}}$ where $\delta(x,y)$ is the Dirac distribution and R(x,y) possesses a weak (integrable) singularity at x=y, i.e.,R(x,y)=$\mathcal{O}(|x-y|^{-k})$ with k<3.

It is easy to see that ,for the operator L(x,∂_x) given by:

Lu(x)=: L(x,∂_x)$u(x)$ =:$\sum_{i=1}^{3}\frac{\partial}{\partial_{x_i}}(a(x)\frac{\partial u(x)}{\partial_{x_i}}) = f(x), x \in \Omega$ the function;

[B]. p(x,y)=$-\frac{1}{4\pi a(y)|x-y|}$ $,\boldsymbol{x,y} \in \boldsymbol{\Re}^3$is a parametrix while the remainder R in [A] is:

[C]. R(x,y)=$\sum_{i=1}^{3} \frac{x_i - y_i}{4\pi a(y)|x-y|^3} \frac{\partial a(x)}{\partial_{x_i}}$ **,** $x, y \in \Re^3$ and thus is weakly singular, $O(|x-y|^{-2})$, due to the smoothness of the function a(x).

This can be shown as follows :-

Let $L(x, \partial_x)p(x, y) = \delta(x - y) + R(x, y)$

WTS p(x,y)=$\frac{-1}{4\pi a(y)|x-y|}$ $and\ R(x, y) = \sum_{i=1}^{3} \frac{x_i - y_i}{4\pi a(y)} \frac{\partial a(x)}{\partial x_i}$

Now, $L(x, \partial_x)p(x, y) = \sum_{i=1}^{3} \frac{\partial}{\partial x_i}(a(x) \frac{\partial}{\partial x_i} p(x, y)$

$\Rightarrow L(x, \partial_x)p(x, y) = \sum_{i=1}^{3} [a(x) \frac{\partial^2}{\partial x_i^2} p(x, y) + \frac{\partial}{\partial x_i} a(x) \frac{\partial}{\partial x_i} p(x, y)$

$\Rightarrow L(x, \partial_x)p(x, y) = \sum_{i=1}^{3} a(x) \frac{\partial^2}{\partial x_i^2} p(x, y) + \sum_{i=1}^{3} \quad \frac{\partial}{\partial x_i} a(x) \frac{\partial}{\partial x_i} p(x, y)$...[a]

But $L(x, \partial_x)\left(\frac{-1}{4\pi a(y)|x-y|}\right) = \sum_{i=1}^{3} \frac{\partial}{\partial x_i}(a(x) \frac{\partial}{\partial x_i}\left(\frac{-1}{4\pi a(y)}\right))$

$$= \frac{a(x)}{a(y)} \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}\left(\frac{-1}{4\pi|x-y|}\right) + \sum_{i=1}^{3} \frac{\partial}{\partial x_i} a(x) \frac{\partial}{\partial x_i}\left(\frac{-1}{4\pi a(y)|x-y|}\right)$$

$=\frac{a(x)}{a(y)} \delta(x - y) + \sum_{i=1}^{3} \frac{x_i - y_i}{4\pi a(y)|x-y|} \frac{\partial}{\partial x_i} a(x)$

$=1. \delta(x - y) + \sum_{i=1}^{3} \frac{x_i - y_i}{4\pi a(y)|x-y|} \frac{\partial}{\partial x_i} a(x)$ ... [b]

Thus, from [a] and [b] we have:

$L(x, \partial_x)p(x, y) = \delta(x - y) + R(x, y) = L(x, \partial_x)\left(\frac{-1}{4\pi a(y)|x-y|}\right) = \delta(x - y) + \sum_{i=1}^{3} \frac{x_i - y_i}{4\pi a(y)|x-y|} \frac{\partial}{\partial x_i} a(x)$

$$\Rightarrow P(x,y) = \frac{-1}{4\pi a(y)|x-y|} \; and R(x,y) = \sum_{i=1}^{3} \frac{x_i - y_i}{4\pi a(y)|x-y|} \frac{\partial}{\partial x_i} a(x)$$

We evidently have that the parametix p(x,y)given by [B] is a fundamental solution to operator L(y, ∂_x) =: $a(y)\Delta(\partial_x)$ with frozen coefficients a(x)=a(y);i.e.,

[D]. $\boxed{L(y, \partial_x) p(x,y) = \delta(x-y)}$

This can be illustrated as follows:

Suppose a(x)=a(y) and $L(y,\partial_x) =: a(y)\Delta(\partial_x)$

Now, $L(y,\partial_x)p(x,y) = a(y)\Delta(\partial_x)p(x,y)$

$$= a(y) \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2} p(x,y)$$

$$= \frac{a(y)}{a(y)} \frac{-1}{4\pi a|x-y|}$$

$$= \frac{-1}{4\pi a|x-y|}$$

$$= \delta(x-y)$$

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