# ON THE METHOD OF LOWER AND UPPER SOLUTIONS FOR THE HEAT EQUATION ON A POLYGONAL DOMAIN

COLLEGE OF NATURAL SCIENCES DEPARTMENT OF MATHEMATICS,
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# Contents

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The undersigned hereby certify that they have read and recommend to the school of graduate studies for acceptance of a project entitled on **ON THE METHOD OF LOWER AND UPPER SOLUTIONS FOR THE HEAT EQUATION ON A POLYGONAL DOMAIN** by Tesfaye sahile in partial fulfillment of the requirements for the degree of master of Science.

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# Notation

$\mathbb{R}^n$ = n-tuple of Real number
$\mathbb{C}$ = complex number
$\Omega$ = open set in $\mathbb{R}^n$
$\overline{\Omega}$ = the closure of $\Omega$
$\partial\Omega$ = the boundary of $\Omega$
$\Omega_1 \Subset \Omega_2 = \exists$ a compact set $K$ such that $\Omega_1 \subset K \subset \Omega_2$
$C(\Omega)$ = continuous functions on $\Omega$
$C(\overline{\Omega})$ = the subset of $C(\Omega)$ consisting of functions that extend continuously to $\partial\Omega$
$C_0(\Omega)$ = the subset of $C(\overline{\Omega})$ consisting of functions that vanish on $\partial\Omega$
$C^k(\Omega)$ = $k$ times differentiable functions
$C^\infty(\Omega)$ = smooth functions
$C_c^k(\Omega)$ = subset of $C^k(\Omega)$ with compact support.
$L^p(\Omega)$ = the space of p-integrable functions in $\Omega$
$L^p_{loc}(\Omega)$ = the space of locally integrable functions in $\Omega$
$W^{k,p}(\Omega)$ = Sobolev spaces in $\Omega$
$\|.\|$ = norm
$\langle , \rangle$ = Inner product
$supp$ = Support
$\partial$ = Partial derivatives
$\Delta = \sum_{i=1}^{n} \frac{\partial}{\partial x_i^2}$ $(i = 1, 2, 3, \cdots n)$
$\nabla u$ = grad $u = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3}, \cdots, \frac{\partial u}{\partial x_N})$

# Abstract

The purpose of this paper is to prove the existence of a solution in the presence of lower and upper solutions for the nonlinear periodic-Dirichlet heat equation on a polygonal domain $\Omega$ of the plane in weighted $L^p$-Sobolev spaces.
Consider the problem;

$$\begin{array}{rcl} \partial_t u - \Delta u &=& f(x,t,u), \quad in \;\; \Omega \times (-\pi, \pi), \\ u &=& 0, \quad on \;\; \partial\Omega \times (-\pi, \pi), \\ u(\cdot, -\pi) &=& u(\cdot, \pi) \;\; in \;\; \Omega, \end{array}$$

where $f$ is $L^p(0,T;L^p_\mu(\Omega))$-Carathèodory,
$L^p_\mu(\Omega) = \{v \in L^p_{loc}(\Omega) : r^\mu v \in L^p_{loc}(\Omega)\}$, with a real parameter $\mu$ and $r(x)$ the distance from $x$ to the set of corners of $\Omega$. We prove some existence results of this problem in presence of lower and upper solutions well-ordered or not. We first give existence results in an abstract setting obtained using degree theory. We secondly apply them for polygonal domains of the plane under geometrical constraints.

# Introduction

A solution of a PDE in some region $\mathbf{R}$ of the space of independent variables is a function $u$, which has all the derivatives that appear on the equation, and satisfies the equation everywhere in $\mathbf{R}$. Solving Boundary Value Problem mean finding a function that satisfies both the PDE and the Boundary Condition. In many cases specially for nonlinear, we can't find an explicit representation for the solution. So solving such problem mean showing that the solution exists.

In this paper we consider the following nonlinear periodic-Dirichlet heat equation:

$$\begin{aligned} \partial_t u - \Delta u &= f(x,t,u), \quad in \ \mathbf{Q}_{2\pi}, \\ u &= 0, \quad on \ \Sigma_{2\pi}, \\ u(\cdot - \pi) &= u(\cdot, \pi) \ \ in \ \Omega \end{aligned} \tag{1}$$

where $\mathbf{Q}_{2\pi} = \Omega \times I, \Sigma_{2\pi} = \partial\Omega \times I$ and $I := (-\pi, \pi)$.
It is a fact that, under regularity assumptions on $f$ and on the domain $\Omega$, if $\alpha$ and $\beta$ are, respectively, a lower and an upper solution of (1), with

$$\alpha(x,t) \leq \beta(x,t), \quad in \ \mathbf{Q}_{2\pi}, \tag{2}$$

then there exists a solution $u$ of (1) which satisfies

$$\alpha(x,t) \leq u(x,t) \leq \beta(x,t), \ in \ \mathbf{Q}_{2\pi}.$$

This was established in the sixties by Ju.S. Kolesov [8,13]. More recently, the study of the solvability of (1) in the presence of a pair of lower and upper solutions $\alpha$ and $\beta$ for which condition (2) fails, that is, $\alpha$ and $\beta$ satisfy

$$\alpha(x_0, t_0) > \beta(x_0, t_0), \tag{3}$$

for some $(x_0, t_0) \in \mathbf{Q}_{2\pi}$,has received much interest. This question, explicitly posed in the early seventies by D.H. Sattinger in [14], has been studied for some classes of boundary-value problems for elliptic partial differential equations in [3, 4,9, 7] .

All these results concern the case of a domain with a smooth boundary, and the proofs of these results use deeply the fact that the solution of the problem in the elliptic case, with $f(.,u(.)) \in L^p(\Omega)$ is in $W^{2,p}$ with $p > N(\Omega \subset \mathbb{R}^N$,with $N \geq 1)$ in such a way that $W^{2,p} \hookrightarrow C^1(\overline{\Omega})$, which is no more true without regularity of $\partial\Omega$. A first extension of this kind of result to polygonal domain has been made by the authors in [10] in the elliptic framework. In that case, we observe that a good way to solve the loss of regularity of the solutions is to work in $C_{\varphi 1}$ , the space of functions which are comparable to the first eigenfunction of the linear problem.

In this work we perform the same kind of extension for the problem (1) and we restrict ourselves to the study of (1) in case where is a polygonal domain of $\mathbb{R}^2$.

This paper is organized as follows. The first section is devoted to preliminary definitions. In section 2 we state the main theorem and we recall the first eigenfunction, and well-known results about the first eigenpair which play a fundamental role in this paper and we give the abstract formulation of the result. After these,in Section 3 the main result will be given.

# Chapter 1

# preliminaries

In this section,we recall some definitions and notations.

## 1.1 Partial differential equations and Dirichlet boundary condition

### 1.1.1 Partial differential equations (PDEs)

**Definition 1.1.** *Equations involving one or more partial derivatives of a function of two or more independent variables are called partial differential equations (PDEs).*

***Notation.*** The order of the highest derivative is the order of the equation.

For example,

- a first order PDE has the general form

$$F(u_x, u_y, u, x, y) = 0$$

- a second order PDE has the general form

$$F(u_{xx}, u_{yy}, u_{xy}, u_x, u_y, x, y) = 0$$

**Definition 1.2.** *A second-order PDE, of the form:*

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = F(x, y, u, u_x, u_y)$$

*is said to be:*

- *parabolic, if* $B^2 - AC = 0$
- *elliptic, if* $B^2 - AC < 0$

- *hyperbolic, if* $B^2 - AC > 0$

For all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$ the heat equation is:

$$u_t = \triangle u$$

where $\triangle u = u_{x_1x_1} + u_{x_2x_2} + . \quad . \quad . + u_{x_nx_n}$. It is second-order PDE.

**Definition 1.3.** *A PDE is homogeneous if each term in the equation contains either the dependent variable or one of its derivatives. Otherwise, the equation is said to be non-homogeneous.*

**Definition 1.4.** *A PDE is linear if the dependent variable and its functions are all of first order.*

### 1.1.2 Dirichlet boundary condition

There should be as many boundary conditions and the type of the condition used in an application will depend on modeling assumption.

Consider the following boundary value problem,

$$\begin{array}{rcl} u_t & = & \Delta u, \quad in \;\; \Omega, \quad (PDE) \\ A(x)u + B(x)\dfrac{\partial u}{\partial v} & = & g(x), \quad on \;\; \partial\Omega, \quad (BounderyConditions) \\ u(x,0) & = & u(x) \quad in \;\; \Omega \quad (InitioalCondition) \end{array} \tag{1.1}$$

for the domain $\Omega \in \mathbb{R}^n$ is a general expiration of linear boundary value problem for heat equation.
The boundary condition $A(x)u + B(x)\frac{\partial u}{\partial v} = g(x)$ has a different interpretation,
depending on the choice of $A, \quad B, \quad and \quad g$ .
It is said to be a **Dirichlet boundary condition** if $A \equiv 1$ and $B \equiv 0$ and equation (1.1) rewritten as

$$\begin{array}{rcl} u_t & = & \Delta u, \quad in \;\; \Omega, \\ u(x,t) & = & g(x), \quad on \;\; \partial\Omega, \\ u(x,0) & = & u(x) \quad in \;\; \Omega \end{array} \tag{1.2}$$

A common Dirichlet condition is $g \equiv 0$. In fact, if $g \neq 0$, we may transform the problem so that the boundary condition becomes $g \equiv 0$, as follows. Suppose that u satisfies (1.2), and that $h(x) \in C^2(\overline{\Omega})$ is any function such that $h(x) = g(x)$ for all $x \in \partial\Omega$. Then, the function $w(x,t) = u(x,t) - h(x)$ solves

$$\begin{array}{rcl} w_t & = & \Delta u + f, \quad in \;\; \Omega, \\ w(x,t) & = & 0, \quad on \;\; \partial\Omega, \\ w(x,0) & = & u(x) - h(x) \quad in \;\; \Omega \end{array} \tag{1.3}$$

## 1.2 Polygonal Domain

**Definition 1.5.** *An open connected subset $\Omega$ of n-dimensional Euclidean space $\mathbb{R}^n$ is called domain.*

The notations $\partial\Omega$ and $\overline{\Omega}$ stand for the boundary and the closure of $\Omega$, respectively.

**Definition 1.6.** *The domain $\Omega' \in \Omega \in \mathbb{R}^n$ is strictly interior sub domain of $\Omega$, written as $\Omega' \subset\subset \Omega$ if there exist a compact set $\boldsymbol{K}$ such that $\Omega' = K \subset \Omega$*

**Definition 1.7.** *let $\Omega$ be a bounded domain of $\mathbb{R}^2$. we say that $\Omega$ is a polygonal domain if its boundary is the union of a finite number of line segments $\overline{\Gamma}_J, j \in \{1, ..., J\}$($\Gamma_j$ being supposed to be open). Hence, we do not assume that $\Omega$ is a Lipschitz domain; that is, we include the presence of cracks.*

Denoted by $S_j, j = 1, ..., J$ the vertices of $\partial\Omega$ enumerated clockwise. With out loss of generality we may assume that $B(S_j, 1) \cap \Omega$ does not contain any other vertex of $\Omega$. For $j \in \{1, ..., J\}$, let $\psi$ be the interior angle of $\Omega$ at the vertices $S_j$, $\lambda_j = \frac{\pi}{\psi_j}$, and $(r_j, \theta)$ the polar coordinates centered at $S_j$ such that $B(S_j, 1) \cap \Omega = \{(r_j cos\theta_j, r_j sin\theta_j) :$
$0 < r_j < 1, 0 < \theta_j < \psi_j\} =: D_j$. For $\vec{\mu} = (\mu_j)_{j=1}^J$, we define the space $L^p_{\vec{\mu}}(\Omega) = \{f \in L^p_{loc}(\Omega) : wf \in L^p(\Omega)\}$ with

$$w = 1 + \sum_{j=1}^{J} \eta_j (r_j^{\mu_j} - 1) \tag{1.4}$$

where $r_j(x)$ is the distance from $x$ to the vertex $S_j$ and $\eta_j \in D(\mathbb{R}^2)$ are such that

$$\eta_j \equiv 1 \text{ in } D_j(1/2), \eta j \equiv 0 \text{ on } \Omega \backslash D_j,$$

where $D_j(r)$ is the truncated cone $D_j(r) = \Omega \cap B(S_j, r)$. This means that $w$ behaves like $r_j^{\mu_j}$ in a neighborhood of $S_j$ while it is equal to 1 far from the corners.

## 1.3 Sobolev space

### 1.3.1 Lebesgue spaces

**Definition 1.8.** $L^p(\Omega)$,$1 \leq p < \infty$ *is the set of p-integrable or all measurable functions $f(x)$ in $\Omega$, with norm*

$$\|f\|_{p,\Omega} = (\int_\Omega |f(x)|^p)^{\frac{1}{p}} < \infty.$$

**Definition 1.9.** *$L_{p,loc}(\Omega)$,$1 \leq p < \infty$ is the set of all measurable functions $f(x)$ in $\Omega$ such that $\int_{\Omega'} |f(x)|^p < \infty$ for any strictly bounded interior sub domain of $\Omega' \subset\subset \Omega$.*

**Definition 1.10.** *$L_\infty(\Omega)$ denotes the essentially bounded function and the norm $\|f\|_{\infty,\Omega} = ess \sup_{x\in\Omega} |f(x)|$.*

**Definition 1.11.** *$C_0^\infty$ is the class of infinitely smooth functions in $\Omega$ with compact support:*
$u \in C_0^\infty(\Omega) \Leftrightarrow u \in C^\infty(\overline{\Omega})$ *and* $\operatorname{supp} u \subset \Omega$.

### 1.3.2 Sobolev spaces

**Definition 1.12.** *The notation $W^{k,p}(\Omega)$ is the Sobolev space of differentiability $k$ and integrability $p$. It consists of functions $u$ which are $k$-weakly differentiable, such $D^\alpha u \in L^p(\Omega)$ for all $|\alpha| \leq k$, with the norm*

$$\|u\|_{p,k,\Omega} = (\int_\Omega \sum_{|\alpha|\leq k} |D^\alpha u|^p dx)^{\frac{1}{p}}. \tag{1.5}$$

Observe that the space $W^{0,p}(W)$ is just $L^p(\Omega)$. In the case that $p = 2$, we also introduce the notation $H^k(\Omega) = W^{k,2}(W)$. These $L^2$-Sobolev spaces are Hilbert spaces under the inner product

$$\langle u, v\rangle_{k,\Omega} = \int_\Omega \sum_{|\alpha|\leq k} D^\alpha u D^\alpha v dx. \tag{1.6}$$

***Notation.***

1. The spaces $W_0^{k,p}(\Omega)$ (similarly $H_0^k(\Omega)$) are the closure of $C_0^k(\Omega)$ under the Sobolev norm (1.5).

2. The spaces $W^{k,p}(\Omega) = W_0^{k,p}(\Omega)$ if $\Omega = \mathbb{R}^n$.

**Proposition 1.1.** *The subspace $C^\infty(\Omega) \cap W^{k,p}(\Omega)$is dense in $W^{k,p}(\Omega)$, for $p < \infty$.*

*Proof.* Let $\Omega_i$ be an approximation of $\Omega$ by compactly included subsets; that is, $\Omega_i \subset\subset \Omega_{i+1}, U_i\Omega_i = \Omega$ and $\Omega_j = \varnothing$ for $j \leq 0$. Let $\eta_j$ be a partition of unity subordinate to the covering $\{\Omega_{j+1}\backslash\Omega_{j-1}\}$. For $u \in W^{k,p}(\Omega)$, and for any $\epsilon > 0$, using the result mentioned before the statement of the theorem, we can choose a sequence of $\delta_j$ such that the following are satisfied:

$$\delta_j < d\{\Omega_{j+1}, \Omega_{j+3}\}$$

$$\|(\eta_j u)\delta_j - \eta_j u\|_{k,p;\Omega} \leq \frac{\epsilon}{2^{j+1}}$$

Let $v = \sum(\eta_j u)\delta_j$. By the definition of partition of unity, and the first condition above, we have that at each $x \in \Omega$, only finitely many terms in the infinite sum is non-zero. So $v \in C^\infty(\Omega)$ by construction. Furthermore, using the triangle inequality

$$\|u - v\|_{k,p;\Omega} \leq \|(\eta_j u)\delta_j - \eta_j u\|_{k,p;\Omega} \leq \epsilon$$

and we obtain the approximation. □

# Chapter 2

# Abstract formulation

Our main result is then the following (we refer to the next sections for the precise definitions of the involved notions).

**Theorem 2.1.** *Let* $\Omega$ *be a polygonal domain of* $\mathbb{R}^2$ *such that, in the above notation, for all* $j = 1, ..., J, \lambda_j < \frac{2}{2-\sqrt{3}}$. *Let* $p > 4$ *and* $\vec{\mu} = (\mu_j)_{1 \leq j \leq J}$ *with, for all* $j = 1, ..., J$, *either*
*case1:* $\max\{\frac{2p-2}{3p+2\sqrt{p-1}}, \frac{p-2}{p+2\sqrt{p-1}}\} < \lambda_j < 1 - \frac{2}{p}$
and
$\max\ \{1 - \frac{2}{p} - \lambda_j, -\frac{2\lambda_j\sqrt{p-1}}{p}\} < \min\{\frac{2\lambda_j\sqrt{p-1}}{p}, 2 - \frac{4}{p} - 2\lambda_j, 1 - \lambda_j\}$;
or
*case2:* $\max\ \{\frac{p-1}{p+\sqrt{p-1}}, 1 - \frac{2}{p}\} < \lambda_j < \frac{p}{p-2\sqrt{p-1}}$,
and
$\max\{1 - \frac{2}{p} - \lambda_j, 2 - \frac{2}{p} - 2\lambda_j, -\frac{2\lambda_j\sqrt{p-1}}{p}\} < \min\{\frac{2\lambda_j\sqrt{p-1}}{p}, 1 - \lambda_j\}$.
*Let* $\mathcal{A} = L^p(I; L^p_{\vec{\mu}}(\Omega))$. *Assume that* $f$ *is a* $\mathcal{A}$*-Carathéodory function and that there exist* $\alpha$ *and* $\beta \in W^{2,1}_{p,loc}(Q_{2\pi}) \cap C(\overline{\Omega})$*,respectively lower and upper solutions of (1).*

*(i) If* $\alpha \leq \beta$ *and there exists* $h \in \mathcal{A}$ *such that, for all* $u \in [\alpha, \beta]$,

$$|f(x, t, u(x, t))| \leq h(x, t) \ a.e \ in \ Q_{2\pi}$$

*then the problem (1) has at least one (weak) solution*
$u \in C_{\varphi_1} \cap W^{2,1}_{p,loc}(Q_{2\pi})$ *such that* $\alpha \leq u \leq \beta$*;*

*(ii) Assume that* $\alpha \nleq \beta$ *and, for some* $C > 0, \alpha \leq C\varphi_1$ *and* $-C\varphi_1 \leq \beta$. *If more over , for every* $R > C$*,there exist* $h_R \in \mathcal{A}$ *such that, for all*
$u \in [\alpha, R\varphi_1] \cup [-R\varphi_1, \beta] \cup [-R\varphi_1, R\varphi_1]$,

$$|f(x,t,u(x,t))| \leq h_R(x,t) \ a.e \ in \ Q_{2\pi}$$

*and there exists* $\gamma \in \mathcal{A}$ *such that, for all* $(x,t,u) \in Q_{2\pi} \times \mathbb{R}$,

$$|f(x,t,u) - \lambda_1 u| \leq \gamma(x,t)$$

*then the problem (1) has at least one (weak) solution* $u \in C_{\varphi_1} \cap W^{2,1}_{p,loc}(Q_{2\pi})$ *such that* $u \in \overline{\mathcal{O}}$,
*where* $\mathcal{O} = \{u \in C_{\varphi_1} : \min(u-\alpha) < 0 < \max(u-\beta)\}$.

***Remark*** 2.1. If $\vec{\mu_1}$ and $\vec{\mu_2}$ are such that $\mu_{1j} \leq \mu_{2j}$ for all $j = 1, ..., J$ and $p_1 \geq p_2$, then we have $L^{p1}_{\vec{\mu_1}} \subset L^{p2}_{\vec{\mu_2}}$. Hence, we try to take the $\mu_j$ as large as possible and the $p$ as small as possible. This means also that the "real" assumptions on $\mu_j$ are the upper bounds.

***Remark*** 2.2. Observe that, due to the condition $\lambda_j < \frac{2}{2-\sqrt{3}}$, we cannot consider small angles $\psi_j$ (but the non-convex case is fully covered). On the other hand, in the first case, the function
$f(p) = max\{\frac{2p-2}{3p+2\sqrt{p-1}}\}$ is increasing and smaller than $\frac{1}{2}$ for $p \leq 10$. The function

$$\begin{aligned} \max\{\frac{p-1}{p+\sqrt{p-1}, 1-\frac{2}{p}}\} &= \frac{p-1}{p+\sqrt{p-1}}, \quad if 4 < p < p_0 \\ &= 1 - \frac{2}{p} \quad if \quad p > p_0 \end{aligned}$$

where $p_0 \in (4,5)$ is the unique root of $(p-2)\sqrt{p-1} - p$, is also increasing, but its value in $p = 4$ is $\frac{3}{4+\sqrt{3}} > \frac{1}{2}$. Hence the first case is convenient for the small value of $\lambda_j$( i.e for angle $\psi_j$ close to $2\pi$) while the second case is suitable for large value of $\lambda_j$. Moreover, the two cases cover all the $\lambda_j$ between $\frac{1}{2}$ and $\frac{3}{4+\sqrt{3}}$. Observe also that in the second case, the two lower bounds are increasing functions and the two upper bounds are non increasing functions.

Hence, the best choice, in this case, is to choose p as small as possible.

to prove our main result Theorem 2.1, it remains only to prove that the following assumption is satisfied.

**Assumption(H.1)**There exist $p > 4$ and $\vec{\mu} \in \mathbb{R}^{\mathbb{J}}$ such that, for every $h \in \mathcal{A} = L^p(I; L^p_{\vec{\mu}}(\Omega))$, the problem

$$\begin{aligned} \partial_t u - \Delta u &= h(x,t), \quad in \ \mathbf{Q}_{2\pi}, \\ u &= 0, \quad on \ \Sigma_{2\pi}, \\ u(\cdot, -\pi) &= u(\cdot, \pi) \ in \ \Omega \end{aligned} \tag{2.1}$$

admits a unique weak solution $u$ with $u \in C_{\varphi 1}$.
Moreover, we ask that

- $C_{\varphi 1}$ is continuously imbedded in $A$
- the operator $T : A \to \mathcal{C}_{\varphi 1} : h \mapsto u$, with u the unique solution of (2.1),is compact

The verification of this assumption (H-1) is the main difficulty of the paper and is the content of the last section of this paper. Theorem 2.1 will then follow easily.

## 2.1 Strong maximum principle

The theory of lower and upper solutions is based on the maximum principle. Here we use it in the following form.

**Proposition 2.1.** *(Strong maximum principle). Let* $\Omega_1 \subset \omega$ *be a bounded domain with a Lipschitz boundary,* $t_1, t_2 \in \mathbb{R}$ *whit* $t_1 < t_2$ *be fixed, and let* $q \in L^\infty(\Omega_1 \times (t_1, t_2))$. *Assume that* $u \in W_p^{2,1}(\Omega_1 \times (t_1, t_2))$ *with* $p > 4$ *satisfies*

$$\partial_t u - \Delta u + q(x,t)u \leq 0, a.e. \ \ in \ \ \Omega_1 \times (t_1, t_2).$$

*moreover, suppose*
$M = \sup_{\overline{\Omega}_1 \times [t_1,t_2]} u(x_0, t_0)$ *for some* $(x_0, t_0) \in \Omega_1 \times [t_1, t_2]$ *and that one of the following holds:*
$q \equiv 0$, *or* $M = 0$, *or* $q \geq 0$ *and* $M \geq 0$.

*Then* $u = M$ *in* $\overline{\Omega}_1 \times [t_1, t_0]$

*Proof.* Suppose there is $(x_0, t_0) \in \Omega_1 \times [t_1, t_2]$ such that $u(x_0, t_0) = \sup_{\overline{\Omega}_1} u$, then by picking $r$ small enough so that $E(x_0, t_0; r) \subset \Omega_1 \times [t_1, t_2]$, and using the mean value property, we conclude that $u$ is constant inside $E(x_0, t_0; r)$. Next for any $(x_1, t_1) \in \Omega_1 \times [t_1, t_2]$ such that the line segment connecting $x_0$,$x_1$ is in $\Omega_1$, we can show that $(x_0, t_0) = (x_1, t_1)$ whenever $t_1 < t_0$ by covering the line segment connecting $(x_1, t_1)$ and $(x_0, t_0)$ wit h the heat balls. Finally, since is $\Omega$ connected, any $x_1$ can be connected from $x_o$ via finitely many line segments. And therefore $(x_0, t_0) = (x, t)$ for all $(x, t) \in \Omega$ and $t < t_0$.which implies
$u = M$ *in* $\overline{\Omega}_1 \times [t_1, t_0]$ □

We deduce easily from the strong maximum principle the following result.

**Corollary 2.1.** *Let* $\Omega \subset \mathbb{R}^2$ *be a bounded domain. Let* $q \in L^\infty(Q_{2\pi})$ *be such that* $q(x,t) \geq 0$ *almost everywhere in* $Q_{2\pi}$. *Assume that* $u \in W^{2,1}_{p,loc}(Q_{2\pi}) \cap C(\overline{Q}_{2\pi})$, *with* $p > 4$, *satisfies*

$$\begin{aligned} \partial_t u - \Delta u + q(x,t)u &\geq 0, \ \text{in} \ \mathbf{Q}_{2\pi}, \\ u &\geq 0, \ \text{on} \ \Sigma_{2\pi}, \\ u(\cdot,-\pi) &\geq u(\cdot,\pi) \ \text{in} \ \Omega \end{aligned}$$

*Then* $u > 0$ *in* $\overline{Q}_{2\pi}$

*Proof.* Assume that for the sake of contradiction that $min_{\overline{Q}_{2\pi}} u < 0$. Since $u(\cdot,-\pi) \geq u(\cdot,\pi)$ *in* $\Omega$ and $u \geq 0$ on $\Sigma_{2\pi}$, there exist $(x_0,t_0) \in \Omega \times [-\pi,\pi]$ such that $(x_0,t_0) = min_{\overline{Q}_{2\pi}} u$ and $u$ is not constant on $\overline{\Omega} \times [-\pi, t_0]$. This contradicts Proposition 2.1. ☐

If E is a Banach space and $1 \leq p \leq \infty$, let $L^p(I;E)$ be the Banach space of the measurable function $f : I \to E$ such that $\int_I \|f\|^p_E < \infty$. We set
$W = \{u \in L^2(I; H^1_0(\Omega)) : \frac{du}{dt} \in L^2(I; H^{-1}(\Omega)\}$,
where $H^{-1}(\Omega) = H^1_0(\Omega)'$ and the derivative $\frac{du}{dt}$ is taken in the sense of distributions taking values in $H^{-1}(\Omega)$. This space W is an Hilbert space equipped with the norm

$$\|u\|_W = (\int_I \|u(\tau)\|^2_{H^1_0(\Omega)} d\tau) + \int_I \|\frac{du}{dt}(\tau)\|^2_{H^{-1}(\Omega)} d\tau)^{\frac{1}{2}}$$

It is also known that, if we denote by $BC(\overline{I}; H^1_0(\Omega))$ the Banach space of the bounded and continuous functions $u : \overline{I} \to H^1_0(\Omega)$ provided with the norm

$$\|u\|_{BC} = \sup_{t \in I} \|u(t)\|_{H^1_0(\Omega)}$$

then the space W is continuously embedded in $BC(\overline{I}; H^1_0(\Omega))$. So for $u \in W$, $u(t)$ makes sense for each $t \in \overline{I}$.

**Definition 2.1.** *We say that* $u \in W$ *is a weak solution of (1) if* $u(.,-\pi) = u(.,\pi)$ *in* $\Omega$ *and if*

$$\int_{Q_{2\pi}} [-u\frac{\partial \varphi}{\partial t} + \langle \nabla u, \nabla \varphi \rangle] = \int_{Q_{2\pi}} f(x,t,u)\varphi, \quad for \quad all \quad \varphi \in C^\infty_c(\Omega \times I).$$

## 2.2 Principal eigenvalue and ordering

For our purposes, the first eigenfunction plays a crucial role. Let us recall known facts about it. We consider the problem

$$\begin{aligned} \partial_t u - \Delta u &= \lambda m(x,t)u, \quad in \;\; \mathbf{Q}_{2\pi}, \\ u &= 0, \quad on \;\; \Sigma_{2\pi}, \\ u(\cdot,-\pi) &= u(\cdot,\pi), \quad in \;\; \Omega \end{aligned} \tag{2.2}$$

where $m \in L^\infty(Q_{2\pi})$ is a weight function.

**Definition 2.2.** *(i) We say that $\lambda_1$ is an eigenvalue of (2.2) if the problem (2.2) with $\lambda = \lambda_1$* has a nontrivial weak solution*;*
*(ii) If (2.2) with $\lambda = \lambda_1$ has a nontrivial nonnegative weak solution, we say that $\lambda_1$ is a principal eigenvalue of (2.2).*

**Proposition 2.2.** *Let $\Omega \in \mathbb{R}^2$ be a bounded domain. Let $m \in L^\infty(Q_{2\pi})$ and define*

$$p(m) = \int_{-\pi}^{\pi} ess \sup_{x\in\Omega} m(x,t)dt$$

*Then the following assertions are equivalent:*
*(i) $P(m) > 0$;*
*(ii) Problem (2.2) has a positive principal eigenvalue $\lambda_1(m)$;*
*(iii) Problem (2.2) has an eigenvalue with positive real part. Moreover, in that case, we have the following: (a) $\lambda_1(m)$ is the only principal eigenvalue with positive real part.*
*(b) $\lambda_1(m)$ is algebraically simple.*
*(c) $\lambda_1(m) = inf\{Re\lambda : \lambda$ is an eigenvalue of (2.2) with $Re\lambda > 0\}$.*
*(d) If $m_1, m_2 \in L^\infty(Q_{2\pi})$ are such that $m_1 \leq m_2$ and $P(m_1 > 0$, then $\lambda_1(m_1) \geq \lambda_1(m_2)$.*
*If, in addition, $m_1 \neq m_2$ in a subset of positive measure, then $m_1 > m_2$*

*Proof.* See [16] and [5]. □

***Remark*** 2.3. Observe that, by uniqueness of the first eigenvalue, in case $m$ is independent of t, the principal eigenpair $(\lambda_1, \varphi_1)$ of (2.2) is the principal eigenpair of the elliptic problem

$$\begin{aligned} \Delta u &= \lambda m(x,t)u, \quad in \;\; \Omega, \\ u &= 0, \quad on \;\; \partial\Omega, \end{aligned} \tag{2.3}$$

In the future we will denote by $\varphi_1$ the principal eigenfunction of the problem

$$\begin{aligned} \Delta\varphi_1 &= \lambda\varphi_1, \quad in \;\; \Omega, \\ \varphi_1 &= 0, \quad on \;\; \partial\Omega, \end{aligned} \tag{2.4}$$

$\varphi_1$ has the following properties.

**Lemma 2.1.** *Let $\Omega$ be a polygonal domain of $\mathbb{R}^2$, and let $S$ be one of its vertices. Denote by $\psi$ the interior angle of $\Omega$ at the vertex $S$ and $\lambda = \frac{\pi}{\psi}$. Then there exists $C_1 > 0$ such that*

$$\varphi_1(x) \geq C_1 r^\lambda \sin(\lambda\theta), \quad in \quad B(S,1) \cap \Omega, \tag{2.5}$$

*where $(r,\theta)$ is the polar coordinates centered in $S$.*

*Proof.* Let us denote $D := B(S,1) \cap \Omega$. here we know that $\varphi_1(r,\theta) > 0$ for $(r,\theta) \in (0,1] \times (0,\psi)$ and by Hopf Boundary Point Lemma we know that $\partial_v \varphi_1(1,\theta) < 1$ for $\theta \in 0,\psi$. Hence, there exists $C_1 > 0$ such that $\varphi_1(1,\theta) \geq C_1 \sin(\lambda\theta)$ for $\theta \in [0,\psi]$. As

$$-\Delta\varphi_1 = \lambda_1\varphi_1 \geq 0 = -\Delta(C_1 r^\lambda \sin(\lambda\theta)), \quad in \quad D$$

$$\varphi_1 \geq C_1 r^\lambda \sin(\lambda\theta), \quad on \quad \partial D,$$

$\square$

***Remark*** 2.4. For all $\gamma < \lambda$ there exists $C_2 > 0$ such that
$\varphi_1(x) \leq C_2 r^\gamma, \quad in \quad B(S,1) \cap \Omega.$

**Definition 2.3.** *We define the space*

$$C_{\varphi_1} = \{u \in C(\overline{Q}_{2\pi}) : \exists a > 0, \forall (x,t) \in Q_{2\pi}, |u(x,t)| \leq a\varphi_1(x,t)\}$$

*This space is a Banach space endowed with the norm*

$$\|u\|_{\varphi_1} = \inf\{a > 0 : \forall (x,t) \in Q_{2\pi}, |u(x,t)| \leq a\varphi_1(x,t)\}.$$

*We denote the open ball in that space $B_{\varphi_1}(0,R) := \{u \in C_{\varphi_1} : \|u\|_{\varphi_1} < R\}$*

**Definition 2.4.** *Given continuous functions $u, v : \overline{Q}_{2\pi} \to \mathbb{R}$ we write*

- $u \leq v$ *if, for all $(x,t) \in Q_{2\pi}$, we have $u(x,t) \leq v(x,t)$;*
- $u < v$ *if $u \leq v$ and $u \neq v$;*
- $u \ll v$ *if there exist $\epsilon > 0$ such that $u + \epsilon\varphi_1 \leq v$;*
- $[u,v] = \{w \in C_{\varphi_1} : u \leq w \leq v\}$

***Remark*** 2.5. Defining in $C_{\varphi_1}$ the order cone $K_{\varphi_1} = \{v \in C_{\varphi_1} : v \geq 0\}$ we observe that, for $u, v \in C_{\varphi_1}, u \ll v$ if and only if $v - u \in int(K_{\varphi_1})$

## 2.3 Abstract formulation

In this section we give abstract results which are following this proposition

**Proposition 2.3.** *Let $\Omega$ be a polygonal domain of $\mathbb{R}^2$. Assume that condition* ***(H-1)*** *is satisfied. Let $(\lambda_1, \varphi_1)$ be the principal eigenpair of (2.2). Let $p > 4, \gamma \in \mathcal{A}$, and $d \in \mathcal{A} \cap L^\infty(Q_{2\pi})$ with $d > 0$. Then there exists a unique $w \in C_{\varphi_1} \cap W^{2,1}_{p,loc}(Q_{2\pi})$ weak solution of*

$$\begin{aligned} \partial_t w - \Delta w &= (\lambda_1 - d)w + \gamma, \quad \text{in } \mathbf{Q}_{2\pi}, \\ w &= 0, \quad \text{on } \Sigma_{2\pi}, \\ w(\cdot, -\pi) &= w(\cdot, \pi), \quad \text{in } \Omega \end{aligned} \tag{2.6}$$

*Proof.* We know that $w \in C_{\varphi_1} \cap W^{2,1}_{p,loc}(Q_{2\pi})$ weak solution of (2.6) if and only if $w$ is a solution of $w = T((\lambda_1 - d)w) + T(\gamma)$ Hence, by the Fredholm alternative, the result will be proved if we show that $w = T((\lambda_1 - d)w)$ has only the trivial solution in $C_{\varphi_1}$.
Let $w \in C_{\varphi_1}$ be a nontrivial solution of $w = T((\lambda_1 - d)w)$; i.e,$w \in C_{\varphi_1} \cap W^{2,1}_{p,loc}(Q_{2\pi})$ is a nontrivial weak solution of

$$\begin{aligned} \partial_t w - \Delta w &= (\lambda_1 - d)w + \gamma, \quad \textit{in } \mathbf{Q}_{2\pi}, \\ w &= 0, \quad \textit{on } \Sigma_{2\pi}, \\ w(\cdot, -\pi) &= w(\cdot, \pi), \quad \textit{in } \Omega \end{aligned}$$

This means that the problem

$$\begin{aligned} \partial_t w - \Delta w &= \mu(\lambda_1 - d)w + \gamma, \quad \textit{in } \mathbf{Q}_{2\pi}, \\ w &= 0, \quad \textit{on } \Sigma_{2\pi}, \\ w(\cdot, -\pi) &= w(\cdot, \pi), \quad \textit{in } \Omega \end{aligned} \tag{2.7}$$

has $\mu = 1$ as eigenvalue. Hence, by Proposition 2.2, this problem has a principal eigenvalue $\mu_1(\lambda_1 - d)$ and $P(\lambda_1 - d) > 0$ Moreover, by Proposition 2.2(c), $\mu_1(\lambda_1 - d) \leq 1$. As $\lambda_1 - d < \lambda_1$ we obtain by Proposition 2.2(d) $\mu_1(\lambda_1 - d) > \mu_1(\lambda_1) = 1$, which contradicts the previous inequality. □

On the nonlinearity $f$ we assume the following regularity.

**Assumption (H-2)**. For the space $\mathcal{A}$ of Assumption (H-1), we assume that the Nemytskii operator $N : C_{\varphi_1} \to \mathcal{A} : u \mapsto f(x, t, u)$ is continuous.

***Remark*** 2.6. Observe that as $\mathcal{A} = L^p(I; L^p_{\overline{\mu}}(\Omega))$,the assumption (H-2) will be satisfied in particular if $f : Q_{2\pi} \times \mathbb{R} \to \mathbb{R}$ is $\mathcal{A}-$*Carathéodory* according to the following definition.

**Definition 2.5.** *A function $f : Q_{2\pi} \times \mathbb{R} \to \mathbb{R}$ is said to be $\mathcal{A}-$Carathéodory if;*

***(i)*** *for a.e* $(x, t) \in Q_{2\pi}, the function f(x, t, .)$ *is continuous;*

***(ii)*** *for all* $z \in \mathbb{R}$, *the function* $f(., ., z)$ *is measurable;*

***(ii)*** *for all* $R > 0$, *there exists* $h_R \in \mathcal{A}$ *such that, for all* $u \in B_{\varphi_{11}}(0, R)$, $|f(x, t, u(x, t))| \leq h_R(x, t)$ *almost everywhere in* $Q_{2\pi}$.

Now we pass to our abstract results concerning the lower- and -upper solutions method.

**Definition 2.6.** *A lower solution of (1) is a function* $\alpha \in W^{2,1}_{p,loc}(Q_{2\pi}) \cap C(\overline{Q}_{2\pi})$ *such that*

$$\begin{aligned} \partial_t \alpha - \Delta \alpha &\leq f(x, t, \alpha), \quad in \quad Q_{2\pi}, \\ \alpha &\leq 0, on \quad \Sigma_{2\pi}, \\ \alpha(\cdot, -\pi) &\leq \alpha(\cdot, \pi), in \quad \Omega \end{aligned}$$

**Definition 2.7.** *An upper solution of (1) is a function* $\beta \in W^{2,1}_{p,loc}(Q_{2\pi}) \cap C(\overline{Q}_{2\pi})$ *such that*

$$\begin{aligned} \partial_t \beta - \Delta \beta &\geq f(x, t, \beta), \text{in} \quad \mathbf{Q}_{2\pi}, \\ \beta &\geq 0, \text{on} \quad \Sigma_{2\pi}, \\ \beta(\cdot, -\pi) &\geq \beta(\cdot, \pi), \text{in} \quad \Omega \end{aligned}$$

**Definition 2.8.** *A lower solution* $\alpha$ *of (1) is strict if every solution* $u$ *of (1) with* $u \geq \alpha$ *is such that* $u \gg \alpha$. *Similarly, an upper solution* $\beta$ *of (1) is strict if every solution* $u$ *of (1) with* $u \leq \beta$ *is such* $u \ll \beta$.

Our first result concerns the well-ordered case, i.e., the case $\alpha \ll \beta$.

**Theorem 2.2.** *Let* $\Omega$ *be a polygonal domain of* $\mathbb{R}^2$ *and assume that the assumptions (H-1) and (H-2) are satisfied. Suppose that* $\alpha$ *is a lower solution and* $\beta$ *is an upper solution of (1) satisfying* $\alpha \leq \beta$. *Assume moreover that there exists* $h \in \mathcal{A}$ *such that, for all* $u \in [\alpha, \beta]$.

$$|f(t, x, u(t, x))| \leq h(x, t) \quad a.e \quad in \quad Q_{2\pi}$$

. *Then the problem (1) has at least one weak solution* $u \in C_{\varphi_1}$ *such that* $\alpha \leq u \leq \beta$

*Proof.* Let $\gamma : Q_{2\pi}\times\mathbb{R} \to \mathbb{R}$ be defined by $\gamma(x,t,u) = \max\{\alpha(x,t), \min\{u, \beta(x,t)\}\}$. Observe that $\Gamma : \mathcal{C}_{\varphi_1} \to \mathcal{C}_{\varphi_1} : u \mapsto \gamma(.,.,u)$ is continuous.
We study the modified problem

$$\begin{array}{rcl} -\Delta u &=& f(x,t,\gamma(x,t,u)), in \;\; \mathbf{Q}_{2\pi}, \\ u &=& 0, on \;\; \Sigma_{2\pi}, \\ u(\cdot,-\pi) &=& u(\cdot,\pi), in \;\; \Omega \end{array} \tag{2.8}$$

*Claim 1*: Every solution $u \in \mathcal{C}_{\varphi_1}$ of (2.8) is such that $\alpha \leq u \leq \beta$. We prove that $\alpha \leq u$ the other part is proved in a similar way . By contradiction, assume that $\max_{(x,t)\in\overline{Q}_{2\pi}}(\alpha(x,t) - u(x,t)) = M > 0$. As $\alpha - u \leq 0$ *on* $\Sigma_{2\pi}$, we can find
$\Omega_1 \times (-\pi,\pi) \subset Q_{2\pi}$ of the class $C^{1,1}$ and $(x_0,t_0), (x_1,t_1) \in \Omega_1 \times (-\pi,\pi)$ such that
$\alpha(x_0,t_0) - u(x_0,t_0) = M$, $\alpha(x_1,t_1) - u(x_1,t_1) < M$ and $\alpha(x,t) - u(x,t) \geq 0$ on $\Omega_1 \times (-\pi,\pi)$. This contradicts the maximum principle of proposition (2.1). As for a. e $(x,t) \in \Omega_1 \times (-\pi,\pi)$

$$-\Delta(\alpha - u)(x,t,) \leq f(x,t,\alpha(x,t)) - f(x,t,\alpha(x,t)) = 0$$

*Claim 2:* The problem (2.8) has at least one solution $u \in \mathcal{C}_{\varphi_1}$. By the Assumptions (H-1) and (H-2), the operator $ToNo\Gamma : \mathcal{C}_{\varphi_1} \to \mathcal{C}_{\varphi_1}$ is completely continuous. Moreover, by assumptions, there exists $R > 0$ such that, for every $u \in \mathcal{C}_{\varphi_1}$, $\|ToNo\Gamma(x,t)\|_{\varphi_1} < R$. Hence, for all $\lambda \in [0,1]$

$$deg(I - ToNo\Gamma, B_{\varphi_1}(0,R) = deg(I - \lambda ToNo\Gamma, B_{\varphi_1}(0,R) = deg(I - B_{\varphi_1}(0,R) = 1$$

and (2.8) has at least one solution.
*Claim 3*: The problem (1) has at least one solution $u \in \mathcal{C}_{\varphi_1}$ satisfying $\alpha \leq u \leq \beta$. By Claim 2, (2.8) has at least one solution u. By Claim 1, this solution satisfies $\alpha \leq u \leq \beta$ and hence, is a solution of (1). □

***Remark*** 2.7. If $\alpha$ and $\beta$ are strict, then there exists $\epsilon > 0$ such that $\beta - \alpha \geq \epsilon\varphi_1$

Our next result extends the Amann Kolesov three-solutions theorem and gives the existence of three solutions in the presence of two pairs of lower and upper solutions with order relations.

**Theorem 2.3.** *Let be a polygonal domain of* $\mathbb{R}^2$ *, and assume that the assumptions (H-1) and (H-2) are satisfied. Assume that there exist* $\alpha_1$ *and* $\alpha_2$ *lower solutions and* $\beta_1$ *and* $\beta_2$ *upper solutions of (1) such that* $\alpha_1 \leq \beta_1$,$\alpha_1 \leq \beta_2$,$\alpha_2 \leq \beta_2$ *and* $\alpha_2 \not\leq \beta_1$. *Suppose further that* $\beta_1$ *and* $\beta_2$ *are strict. Moreover, assume that there exists* $h \in \mathcal{A}$ *such that, for all* $u \in [\alpha_1, \beta_1] \cup [\alpha_2, \beta_2] \cup [\alpha_1, \beta_2]$

$$|f(t,x,u(t,x))| \leq h(x,t) \;\; a.e \;\; in \;\; Q_{2\pi}.$$

*Then the problem (1) has at least three weak solutions* $u_1, u_2, u_3 \in \mathcal{C}_{\varphi_1}$ *such that* $\alpha_1 \leq u_1 \ll \beta_1, \alpha_2 \ll u_2 \leq \beta_2$, *and there exist* $(x_1, t_1), (x_2, t_2) \in Q_{2\pi}$ *with*

$$u_3(x_1, t_1) > \beta_1(x_1, t_1), \quad u_3(x_2, t_2) < \alpha_2(x_2, t_2).$$

*Proof.* Define, for $i, j \in \{1, 2\}$, $\gamma_{i,j}(x, t, u) = \max\{\alpha_i(x, t), \min\{u, \beta_j(x, t)\}\}$ and $\Gamma_{ij} : \mathcal{C}_{\varphi_1} \to \mathcal{C}_{\varphi_1} : u \mapsto \gamma_{ij}(., ., u)$. Observe that $\Gamma_{ij}$ is continuous and consider the modified problem

$$\begin{aligned} -\Delta u &= f(x, t, \gamma_{1,2}(x, t, u)), \quad in \ \mathbf{Q}_{2\pi}, \\ u &= 0, on \ \Sigma_{2\pi}, \\ u(\cdot, -\pi) &= u(\cdot, \pi), in \ \Omega \end{aligned} \tag{2.9}$$

Let us choose $k$ so that $\beta_1 \leq \beta_2 + k$ and $\alpha_1 - k \leq \alpha_2$ and let $R$ such that, for every $k \in \mathcal{A}$ with $|k| \leq h$, $\|T(k)\|_{\varphi_1} < R$.
*Step 1:* Computation of $deg(I - ToNo\Gamma_{1,2}, S_{1,1} \cap B_{\varphi_1}(0, R))$, where

$$S_{1,1} = \{u \in \mathcal{C}_{\varphi_1} | \alpha_1 - k \ll u \ll \beta_1\}$$

Define the alternative modified problem

$$\begin{aligned} -\Delta u &= \bar{f}(x, t, u), \quad in \ \mathbf{Q}_{2\pi}, \\ u &= 0, on \ \Sigma_{2\pi}, \\ u(\cdot, -\pi) &= u(\cdot, \pi), in \ \Omega \end{aligned} \tag{2.10}$$

where

$$\bar{f}(x, t, u) = \max\{f(x, t, \gamma_{1,1}(x, t, u)), f(x, t, \gamma_{1,2}(x, t, u))\}.$$

Observe that

$$\bar{N} : \mathcal{C}_{\varphi_1} \to A : u \mapsto \bar{f}(., ., u)$$

is continuous. For any $\lambda \in [0, 1]$, we consider then the
homotopy $\lambda To\bar{N}(1 - \lambda)ToNo\Gamma_{1,2}$.
*Claim 1*: If $\lambda \in [0, 1]$ and $u$ is a fixed point of $\lambda To\bar{N}(1 - \lambda)ToNo\Gamma_{1,2}$. we have $\alpha_1 \leq u \leq \beta_2$. This result follows from the usual maximum principle argument as in Claim 1 of the proof of Theorem 2.2.
*Claim 2*: If $\lambda \in [0, 1]$ and $u$ is a fixed point of $\lambda To\bar{N}(1 - \lambda)ToNo\Gamma_{1,2}$. we have $u \ll \beta_1$. Assume there exists $(x_0, t_0) \in \mathbf{Q}_{2\pi}$ such that $u(x_0, t_0) = \beta_1(x_0, t_0)$. We deduce from Claim 1 that $\alpha_1 \leq u \leq \beta_2$ so that u solves (1). As further $\beta_1$ is a strict upper solution, the claim follows.
*Claim 3*: $deg(I - ToNo\Gamma_{1,2}, S_{1,1}) = 1$. It follows from the above claims that $\alpha_1 - k$ and $\beta_1$ are strict lower and upper solutions of (2.10) and we deduce from Theorem 2.2 and the properties of the degree that

$$deg(I - ToNo\Gamma_{1,2}, S_{1,1} \cap B_{\varphi_1}(0, R))$$

$$= deg(\lambda To\bar{N}(1-\lambda)ToNo\Gamma_{1,2}), S_{1,1}\cap B_{\varphi_1}(0,R))$$
$$= deg(I - To\bar{N}o\Gamma_{1,2}, S_{1,1}\cap B_{\varphi_1}(0,R)) = 1$$

*Step 2:* Computation of $deg(I - ToNo\Gamma_{1,2}, S_{2,2}\cap B_{\varphi_1}(0,R)) = 1$, where

$$S_{2,2} = \{u \in \mathcal{C}_{\varphi_1} | \alpha_2 - k \ll u \ll \beta_2\}$$

The proof of this result parallels the proof of Step 1.
*Step 3*: There exist three solutions $u_i (i = 1,2,3)$ of (1) such that

$$\alpha_1 \leq u_1 \ll \beta_1, \quad \alpha_2 \ll u_2 \leq \beta_2, \quad \alpha_1 \leq u_i \ll \beta_2, \quad for \quad i = 1,2,3$$

and there exist $(x_1,t_1),(x_2,t_2) \in \mathbf{Q}_{2\pi}$ with

$$u_3(x_1,t_1) > \beta_3(x_1,t_1), \quad u_3(x_2,t_2) < \alpha_3(x_2,t_2)$$

The first two solutions are obtained from the fact that

$$deg(I - ToNo\Gamma_{1,2}, S_{1,1}\cap B_{\varphi_1}(0,R)) = 1$$

and

$$deg(I - ToNo\Gamma_{1,2}, S_{2,2}\cap B_{\varphi_1}(0,R)) = 1$$

Define

$$S_{1,2} = \{u \in \mathcal{C}_{\varphi_1} | \alpha_1 - k \ll u \ll \beta_2\}$$

we have
$1 = deg(I - ToNo\Gamma_{1,2}, S_{1,2}\cap B_{\varphi_1}(0,R))$
$= deg(\lambda ToN\Gamma_{1,2}), S_{1,1}\cap B_{\varphi_1}(0,R))$
$+deg(I - ToNo\Gamma_{1,2}, S_{2,2}\cap B_{\varphi_1}(0,R))$
$+deg(I - ToNo\Gamma_{1,2}, (S_{1,2}\backslash(\bar{S}_{1,1}\cup\bar{S}_{2,2}))\cap B_{\varphi_1}(0,R))$

which implies

$$deg(I - ToNo\Gamma_{1,2}, (S_{1,2}\backslash(\bar{S}_{1,1}\cup\bar{S}_{2,2}))\cap B_{\varphi_1}(0,R)) = -1$$

and the existence $u_3 \in (S_{1,2}\backslash(\bar{S}_{1,1}\cup\bar{S}_{2,2}))$ follows.
As we know from Claim 1, that the solutions u of (2.8) are such that

$$\alpha_1 \leq u \leq \beta_1$$

which are solutions of (1).

□

***Notation***. Notice that the condition $u_3(x_1,t_1) > \beta_1(x_1,t_1)$ and $u_3(x_2,t_2) < \alpha_2(x_2,t_2)$ is a localization condition that implies that $u_3 \neq u_1$ and $u_3 \neq u_2$ We have also that $u_1 \leq min\{\beta_1,\beta_2\}$ and $u_2 \geq max\{\alpha_1,\alpha_2\}$.

As a third result we obtain an existence result in case of existence of non-well-ordered lower and upper solutions.

**Theorem 2.4.** *Let $\Omega$ be a polygonal domain of $\mathbb{R}^2$, and let the assumptions (H-1) and (H-2) be satisfied. Assume that there exist $\alpha$ and $\beta$ lower and upper solutions of (1) such that, for some $C > 0$ $\alpha \ll C\varphi_1, \quad -C\varphi_1 \ll \beta$, and there exists $(x_0, t_0) \in Q_{2\pi}$ with $\alpha(x_0, t_0) > \beta(x_0, t_0)$ Moreover, suppose that, for every $R > C$, there exists $h_R \in A$ such that, for all $u \in [\alpha, R\varphi_1] \cup [-R\varphi_1, \beta] \cup [-R\varphi_1, R\varphi 1]$,*

$$|f(t, x, u(t, x))| \leq h_R(x, t) a.e \quad in \quad Q_{2\pi}.$$

*Assume further that there exists $\gamma \in \mathcal{A}$ such that, for all $(x, t, u) \quad in \quad Q_{2\pi} \times \mathbb{R}$,*

$$|f(x, t, u) - \lambda_1 u| \leq \gamma(x, t).$$

*Then the problem (1) has at least one weak solution $u \in C_{\varphi_1}$ such that $u \in \overline{\mathcal{O}}$, where $\mathcal{O} = \{u \in C_{\varphi_1} : \min(u - \alpha) < 0 < \max(u - \beta)\}$.*

*Proof.* In the course of this proof, we relabel $\alpha$ and $\beta$ as $\alpha = \alpha_1$ and $\beta = \beta_1$ in order to apply Theorem 2.3. For every $r > 1$, define the function $f_r : Q_{2\pi} \times \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned} f_r(x, t, u) &= f(x, t, u), \quad if \quad |u| \leq r, \\ &= (|u| - r)(\lambda_1 - \frac{1}{r} d)u + (r + 1 - |u|) f(x, t, u), \quad if \quad r \leq |u| \leq r + 1, \\ &= (\lambda_1 - \frac{1}{r} d)u, \quad if \quad r + 1 < |u| \end{aligned}$$

where $d \in \mathcal{A} \cap L^\infty(Q_{2\pi})$, $d > 0$ For every $r > 1$, consider the modified problem

$$\begin{aligned} -\Delta u &= f_r(x, t, u), in \ \mathbf{Q}_{2\pi}, \\ u &= 0, on \ \Sigma_{2\pi}, \\ u(\cdot, -\pi) &= u(\cdot, \pi), \quad in \ \Omega \end{aligned} \tag{2.11}$$

Observe that we can decompose $f_r(x.t, u) = p_r(x, t, u)u + q_r(x.t, u)$ such that, for all $(x, u) \quad \in \quad Q_{2\pi} \times \mathbb{R}$,

$$\begin{aligned} \lambda_1 - \frac{1}{r} d) \leq p_r(x, t, u) \leq \lambda_1, \\ q_r(x.t, u) \leq \gamma(x). \end{aligned}$$

*Claim:* There exists $K > 1$ such that, for all $r > K$ and for all $u \in \overline{\mathcal{O}}$, solution of (2.8), we have $\|u\|_{\varphi_1} < K$. Otherwise, for all $n \geq 1$, there exist

$r_n > n$ and $u_n \in \mathcal{O}$ solution of (2.8) for $r = r_n$ with $\|u\|_{\varphi_1} \geq n$.Then $u_n = u_n/\|u\|_{\varphi_1}$ satisfies

$$\begin{aligned} -\Delta v_n = p_{r_n}(x,t,u_n)v_n + \frac{q_{r_n}(x,t,u_n)}{\|u_n\|_{\varphi_1}}, \quad in \quad Q_{2\pi}, \\ v_n = 0, \quad on \quad \Sigma_{2\pi} v_n(\cdot,-\pi) &= v_n(\cdot,\pi), \quad in \quad \Omega \end{aligned}$$

As $\{p_{r_n}(x,t,u_n)v_n + \frac{q_{r_n}(x,t,u_n)}{\|u_n\|_{\varphi_1}} | n \in \mathbb{N}\}$ is bounded in $\mathcal{A}$, we deduce from the Assumption (H-1) that, up to a subsequence, $v_{nk} \to v$ in $C_{\varphi_1}$. It is then easy to see that $p_{r_{n_k}}(x,t,u_{n_k})v_{n_k} \to \lambda_1 v$ in $\mathcal{A}$ and $\frac{q_{r_{n_k}}(x,t,u_{n_k})}{\|u_{n_k}\|_{\varphi_1}} \to 0$ in $\mathcal{A}$. Passing to the limit in

$$v_{n_k} = T(p_{r_{n_k}}(x,t,u_{n_k})v_{n_k} + \frac{q_{r_{n_k}}(x,t,u_{n_k})}{\|u_{n_k}\|}$$

we obtain

$$v = \lambda_1 T v$$

i.e. $v$ is a solution of

$$\begin{aligned} -\Delta v_n = \lambda_1 v, \quad in \quad Q_{2\pi}, \\ v_n = 0, \quad on \quad \Sigma_{2\pi} v_n(\cdot,-\pi) &= v_n(\cdot,\pi), \quad in \quad \Omega \end{aligned}$$

By Proposition 2.3, we deduce that $v = \pm\varphi_1$. Hence for k large enough, either

$$u_{n_k} \geq \frac{1}{2}\|un_k\|_{\varphi_1}\varphi_1 \geq C\varphi_1 \gg \alpha$$

or

$$u_{n_k} \leq \frac{1}{2}\|un_k\|_{\varphi_1}\varphi_1 \leq C\varphi_1 \ll \beta$$

which contradicts the localization $u_{n_k} \in \overline{\mathcal{O}}$ and the claim is proved
**Conclusion**: We apply Theorem 2.3 to the problem (2.7) with $r = R := 1 + \max\{K, \|\alpha\|_\infty, \|\beta\|_\infty\}$. Let $w \in C_{\varphi_1}$ be the solution of

$$\begin{aligned} -\Delta w = (\lambda_1 - \frac{1}{R}d)w + d + \gamma, \quad in \quad Q_{2\pi}, \\ w = 0, \quad on \quad \Sigma_{2\pi} \\ w(\cdot,-\pi) &= w(\cdot,\pi), \quad in \quad \Omega \end{aligned}$$

which exists by Proposition 2.3. Choose $a > 0$ large enough such that $\beta_2 := w + a\varphi_1 \geq C\varphi_1 \geq \alpha_1$. It is then easy to see that $\beta_2$ is an upper solution of (2.11) and in the same way , for $b > 0$ large enough, $\alpha_1 := -w - b\varphi_1 \leq -C\varphi_1 \leq \beta_1$, is a lower solution of (2.8). Hence we have the two pairs of lower and upper solutions required by Theorem 2.3.
Assume $\alpha_2$ is not a strict lower solution. Then there exists a solution $u$ of

(2.8) with $u \geq \alpha_2$ and $u \not> \alpha_2$ As further $u(x_0, t_0) \geq \alpha_2(x_0, t_0) > \beta_1(x_0, t_0)$ we see that $u \in \overline{\mathcal{O}}$ and, by the Claim,$\|u\|_{\varphi_1} < R$. Hence $u$ is a solution of (1) in $\overline{\mathcal{O}}$ The same argument holds in case $\beta_1$ is not a strict upper solution. It remains to consider the case where $\alpha_2$ and $\beta_1$ are strict. In that case, we deduce from Theorem 2.3 the existence of three solutions of (2.8), one of them $u$ beeing in $\mathcal{O}$. Hence, from the Claim, we have $\|u\|_{\varphi_1} < R$ and u is a solution of (1) which concludes the proof. □

**Proposition 2.4.** *Let $\Omega$ be a polygonal domain of $\mathbb{R}^2$ and suppose that the assumptions (H-1) and (H-2) are satisfied. Assume that there exists $\alpha$ a lower solution of (1). Moreover, suppose that there exists $h \in \mathcal{A}$ such that,for all $u \in [\alpha, \|\alpha\|_\infty + 1]$,*

$$|f(t, x, u(t, x))| \leq h(x, t) \quad a.e \quad in \quad Q_{2\pi}.$$

*Then there exists $C > 0$ such that $\alpha \leq C\varphi_1$.*

*Proof.* Let $R = \|\alpha\|_\infty$ and consider the function

$$\begin{aligned} \bar{f}(x, t, u) &= f(x, t, u), \quad if \quad u \leq R, \\ &= (R + 1 - u) f(x, t, u), \quad if \quad R < u \leq R + 1, \\ &= 0, \quad if \quad R + 1 < u \end{aligned}$$

and the modified problem

$$\begin{aligned} -\Delta u &= \bar{f}(x, t, u), in \ \mathbf{Q}_{2\pi}, \\ u &= 0, on \ \Sigma_{2\pi}, \\ u(\cdot, -\pi) &= u(\cdot, \pi), in \ \Omega \end{aligned} \tag{2.12}$$

Observe that $\beta = R + 1$ is an upper solution of (2.8) with $\alpha \leq \beta$. Hence we conclude by Theorem 2.4 the existence of $u \in C_{\varphi_1}$ with $\alpha \leq u \leq \beta$ and hence, there exists $C > 0$ such that $\alpha \leq u \leq C\varphi_1$. □

***Remark*** 2.8. The same type of result holds true for $\beta$.

# Chapter 3

# Existence of strong solution

## 3.1 Proof of Theorem 1

In order to proceed to the verification of (H-1), we use the following result.

**Theorem 3.1.** *Let* $p \geq 2$, $\Omega$ *be a bounded polygonal domain of* $\mathbb{R}^2$, *and denote* $\vec{\lambda} = (\lambda)_{1 \leq j \leq J}$. *Let* $\bar{\mu} = (\mu)_{1 \leq j \leq J}$ *satisfy, for all* $j = 1, ..., J$,

$$-\lambda_j < \mu_j < 2 - \frac{2}{p}, 4(p-1)\lambda_j^2 - \mu_j^2 p^2 > 0 \tag{3.1}$$

*and, for all* $k \in \mathbb{N}^*$ *and all* $j \in \{1, 2, ..., J\}, 2 - \frac{2}{p} - \mu_j \neq k\lambda_j$ *and* $\mu_j + k\lambda_j \neq 1$. *Then, for all* $h \in L^p(I; L^p_{\vec{\mu}}(\Omega))$, *there exist a unique strong solution* $u \in L^p(I; L^p_{\vec{\mu}}(\Omega))$ *of (2.1). Moreover,* $u$ *admits the decomposition*

$$u = u_R + \sum_{j=1}^{J} \eta_j \sum_{0 < \lambda'_j < 2 - \frac{2}{p} - \mu_j \exists k \in \mathbb{N}, \lambda'_j = k\lambda_j} u\lambda'_j,$$

*with* $u_R \in L^p(I; V^{2,p}_{\vec{\mu}}(\Omega) \cap W^p_{2\pi}(I; L^p_{\vec{\mu}}(\Omega)$ *and* $u\lambda'_j = (E_{\lambda_{j'}*t} q_{\lambda_{j'}}) r^{\lambda_{j'}} sin(\lambda_{j'}\theta)$, *where* $q_{\lambda_{j'}} \in W^{\sigma_j, \lambda_{j'}, p}_{2\pi}(I)$, *with* $\sigma_j, \lambda_{j'} = -\frac{\sigma_j + \lambda_{j'}}{2} + 1 - \frac{1}{p}$, *and*

$$E_{\lambda_{j'}}(x,t) = \frac{2}{2\pi} \sum_{l \in \mathbb{Z}} e^{ilt} P_{j,\lambda_{j'}}(r\sqrt{il}) e^{-r\sqrt{il}}$$

*with* $P_{j,\lambda_{j'}}(s) = \sum_{i=0}^{l_j, \lambda_{j'} - 1} \frac{s^i}{i!}$, *where* $l_j, \lambda_{j'} > 2 - \mu_j - \frac{2}{p} - \lambda_{j'}$ *and* $\sqrt{il}$ *is chosen such that* $\Re\sqrt{il} \geq 0$ *for all* $l \in \mathbb{Z}$.
*Moreover, the applications*$L^p(I; L^p_{\vec{\mu}}(\Omega)) \rightarrow W^{\sigma_j, \lambda_{j'}, p}_{2\pi}(I) : h \mapsto q_{\lambda_{j'}}$ *and* $L^p(I; L^p_{\vec{\mu}}) \rightarrow L^p(I; V^{2,p}_{\vec{\mu}}(\Omega) \cap W^{1,p}_{2\pi}(I; L^p_{\vec{\mu}}(\Omega)) : h \mapsto u_R$ *are continuous.*

**Definition 3.1.** *We say that* $u$ *is a strong solution of (2.1) if there exist* $(u_n)_n \subset W^{1,p}_{2\pi}(I; L^p_{\vec{\mu}}(\Omega)) \cap L^p(I; D(\triangle_{p,\vec{\mu}}))$*, where* $D(\triangle_{p,\vec{\mu}}) = \{u \in H^1_0(\Omega) : \triangle u \in L^p_{\vec{\mu}}(\Omega)\}$ *and* $h_n \in L^p(I; L^p_{\vec{\mu}}(\Omega))$ *such that* $\partial_t u_n - \triangle u_n = h_n, \quad u_n \to u, \quad$ *and*
$h_n \to h \quad \in L^p(I; L^p_{\vec{\mu}}(\Omega))$.

In that result, for X a Banach space, we denote by $W^{s,p}_{2\pi}(I, X)$ the space of $2\pi$-periodic functions of $W^{s,p}_{2\pi}(I, X)$ $(for \quad s > \frac{1}{p)}$ as usually, $W^{s,p}_{2\pi}(I) = W^{s,p}_{2\pi}(I, \mathbb{R})$, and $V^{k,p}_{\vec{\mu}}(\Omega)$ is defined as the closure of

$$C^{\infty}_{S}(\Omega) = \{v \in C^{\infty}(\overline{\Omega}) : S_j \notin \sup v\}$$

with respect to the norm

$$\|u\|_{V^{k,p}_{\vec{\mu}}(\Omega)} = (\sum_{|\gamma| \leq k} \int_{\Omega} |D^{\gamma} u(x)|^p w^p(x) r^{(|\gamma|-k)p}(x) dx)^{\frac{1}{p}}.$$

In the course of this proof we will use the following notation: $a \lesssim b$ means the existence of a positive constant $C$, which is independent of the quantities $a$ and $b$ under consideration, such that $a \leqslant Cb$, and $a \backsim b$ means $a \lesssim b$ and $b \lesssim a$.

We divide this proof into two parts. In the first one, we explain the consequences of the assumptions on $\vec{\lambda}, p$ and $\vec{\mu}$ which shed some light on them, and in the second part, we show how to apply these consequences to obtain our result.

**Part 1: Consequences of the assumptions.**
**Step 1:** The assumptions of Theorem 2.1 imply that the assumptions of Theorem 3.1 are satisfied. We first have to verify that the conditions (3.1) are satisfied. Observe that, as $p > 4$, we have

$$2 - \frac{2}{p} > 1 > 1 - \lambda_j > \mu_j > 1 - \frac{2}{p} - \lambda_j > -\lambda_j.$$

This means that the conditions (2.1) are satisfied as the second part of (3.1) is explicit in the assumptions of Theorem 2.1.
We now show that for all $k \in \mathbb{N}^*$ and all $j \in \{1, 2, ..., J\}$, $2 - \frac{2}{p} > 1?1 - \mu_j \neq k\lambda_j$ and $\mu_j + k\lambda_j \neq 1$.Indeed, we observe that, as $\lambda_j \geq \frac{1}{2}$ and $p > 4$,

$$2 - \frac{2}{p} - \mu_j < 2 - \frac{2}{p} - (1 - \frac{2}{p} - \lambda_j) = 1 - \lambda_j \leq k\lambda_j, \quad for \quad k \geq 3,$$

$$0 < \lambda_j < 1 - \mu_j < 2 - \frac{2}{p} - \mu_j,$$

$$\mu_j + k\lambda_j > 1 - \frac{2}{p} + (k-1)\lambda_j \geq 1 - \frac{2}{p} + \frac{1}{2} > 1, \quad for \;\; k \geq 2,$$

$$\mu_j + \lambda_j < 1.$$

Moreover, in *Case 1*, $0 < 2\lambda_j < 1 - \mu_j < 2 - \frac{2}{p} - \mu_j$, and in *Case 2*, $2 - \frac{2}{p} - \mu_j < 2\lambda_j$.
Hence, all the assumptions of Theorem 3.1 are satisfied.

**Step 2:** The solution u of (2,1) admits the decomposition

$$u = u_R + \sum_{j=1}^{J} \eta_j \sum_{k \in N_j} u_k \lambda_j,$$

with $N_j = \{1, 2\}$ in *Case 1*, and $N_j = \{1\}$ in *Case 2*, and

$$E_{\lambda_{j'}}(x,t) = E(x,t) = \frac{1}{2\pi} \sum_{l \in \mathbb{Z}} e^{ilt} e^{-r\sqrt{il}}.$$

In order to prove this step we need to have, in *Case 1*, $2\lambda_j < 2 - \frac{2}{p} - \mu_j < 3\lambda_j$, and $1 > 2 - \mu_j - \frac{2}{p}\lambda_j$, which are true by assumption as $\lambda_j \geq \frac{1}{2}$; and, in *Case 2*, we need $\lambda_j < 2 - \frac{2}{p} - \mu_j < 2\lambda_j$, and $1 > 1 - \mu_j - \frac{2}{p} - \lambda_j$, which are part of the assumptions.

**Step 3:** The identity operator from $L^p(I; V^{2,p}_{\vec{\mu}}(\Omega)) \cap W^{1,p}_{2\pi}(I; L^p_{\vec{\mu}}(\Omega))$ to $C_{\varphi_1}$ is compact. Observe that, the application

$$L^p(I; V^{2,p}_{\vec{\mu}}(\Omega) \cap W^{1,p}_{2\pi}(I; L^p_{\vec{\mu}}(\Omega) \to W^{2,1}_p(\Omega \times I) : u \to wu$$

is continuous. By the compact embedding of $W^{2,1}_p(\Omega \times I)$ in $C^{1,0}(\overline{\Omega} \times \overline{I}$, for a bounded sequence $(v_n)_n \in L^p(I; V^{2,p}_{\vec{\mu}}(\Omega) \cap W^{1,p}_{2\pi}(I; L^p_{\vec{\mu}}(\Omega)$ we have, passing to a subsequence still denoted $(v_n)_n$, that $wv_n \to V$ in $C^{1,0}(\overline{\Omega} \times \overline{I}$, and hence

$$\sup_{\Omega \times I} \left| \frac{v_n(r,\theta) - w^{-1}(r)V(r,\theta)}{w^{-1}(r) r \sin(\lambda\theta)} \right| \to 0, \quad as \;\; n \to \infty \tag{3.2}$$

Recall that, by Lemma 3.2 and, as $\mu_j \leq 1 - \lambda_j$, we have $w^{-1}(r) r \sin(\lambda\theta) \lesssim \varphi_1$;
hence, the conclusion follows.

**Step 4:** The identity operator from $W^{\sigma_j, k\lambda_j, p}(I)$ ot $C(I)$ is compact. To this aim, we just have to verify that $\sigma_j, k\lambda_j > \frac{1}{p}$. This means, in the first case,$-\frac{\mu_j + 2\lambda_j}{2} + 1 - \frac{1}{p} > \frac{1}{p}$ and $-\frac{\mu_j + \lambda_j}{2} + 1 - \frac{1}{p} > \frac{1}{p}$, and in the second Case $-\frac{\mu_j + \lambda_j}{2} + 1 - \frac{1}{p} > \frac{1}{p}$. The first case is obviously true by the assumptions;

and in the second one, it is true as
$-\frac{\mu_j+\lambda_j}{2}+1-\frac{1}{p}>-\frac{1}{2}+1-\frac{1}{p}>\frac{1}{p}$ (recalling that $p>4$).

**Part 2:**

*Proof.* To prove the result, we apply Theorems 2.2 and 2.4. Hence we just have to verify Assumptions (H-1) and (H-2). The verification of Assumption (H-2) is easy via the $A-Carathéodory$ condition. Moreover, using the fact that $-\lambda_j-\frac{2}{p}<\mu_j$, we easily prove that $C_{\varphi_1}$ is continuously embedded in $L^p(I;L^p_{\vec{\mu}}(\Omega))$.
It remains to verify that $T:L^p(I;L^p_{\vec{\mu}}(\Omega))\to C_{\varphi_1}$ is well defined and compact.

**Step 1:** For all $h\in L^p(I;L^p_{\vec{\mu}}(\Omega))$, there exists a weak solution $u$ of (2.1). By Step 1 of Part 1, we can apply Theorem 3.1, and hence, for every $h\in L^p(I;L^p_{\vec{\mu}}(\Omega))$ , there exists a unique strong solution $u\in L^p(I;L^p_{\vec{\mu}}(\Omega))$ of (2.1); i.e., there exist sequences $(h_n)_n\in L^p(I;L^p_{\vec{\mu}}(\Omega))$ and $(u_n)_n\in W^{1,p}_{2\pi}(I;L^p_{\vec{\mu}}(\Omega))\cap L^p(I;D(\triangle_{p,\vec{\mu}}))$, with $h_n\to h$ and $u_n\to u$ in $L^p(I;L^p_{\vec{\mu}}(\Omega))$, and

$$\begin{aligned}\partial_t u_n-\Delta u_n &= h_n, in\mathbf{Q}_{2\pi},\\ u_n &= 0, on\Sigma_{2\pi},\\ u_n(\cdot,-\pi) &= u_n(\cdot,\pi), in\Omega\end{aligned}$$

Multiplying the equation by $u_n$ and integrating, we obtain by periodicity and using [11, Lemma 2.1],

$$\|u_n\|^2_{L^2(I,H^1_0(\Omega))}=\int_{Q_{2\pi}}|\nabla u_n|^2=\langle h_n,u_n\rangle$$

$$\leq\|h_n\|_{L^2(I,L^p_{\vec{\mu}}(\Omega))}\|u_n\|_{L^2(I,L^q{}_-\overline{\mu}(\Omega))}\leq\|h_n\|_{L^2(I,L^p_{\vec{\mu}}(\Omega))}\|u_n\|_{L^2(I,H^1_0(\Omega))},$$

and hence

$$\|u_n\|_{L^2(I,H^1_0(\Omega))}\leq\|h_n\|_{L^2(I,L^p_{\vec{\mu}}(\Omega))}\lesssim\|h_n\|_{L^p(I,L^p_{\vec{\mu}}(\Omega))}.$$

Moreover, using [11, Lemma 2.1], we have a continuous embedding from $L^p(I,L^p_{\vec{\mu}}(\Omega))$ to $L^p(I,H^{-1}(\Omega))$ and by [18], we have that
$\Delta u_n\in L^2(I,H^{-1}(\Omega))$ with $\|\Delta u_n\|_{L^2(I,H^{-1}(\Omega))}\lesssim\|u_n\|_{L^2(I,H^1_0(\Omega))}$. This implies that

$$\|\frac{\partial u_n}{\partial t}\|_{L^2(I,H^{-1}(\Omega))}\lesssim\|\Delta u_n\|_{L^2(I,H^{-1}(\Omega))}+\|h_n\|_{L^2(I,H^{-1}(\Omega))}\lesssim\|h_n\|_{L^p(I,H^{-1}(\Omega))}.$$

Hence we have $u_n\in W$ and $\|u_n\|_W\lesssim\|h_n\|_{L^p(I,H^{-1}(\Omega))}$. As $h_n$ converges in $L^p(I,L^p_{\vec{\mu}}(\Omega))$,this implies that $(u_n)_n$ is a Cauchy sequence in $W$ , and

then $u_n \to v$ with $v \in W$. By [11, Lemmas 2.1 and 2.2] we deduce that $u_n \to v$ in $L^2(I, L^2_{\bar{1}}(\Omega))$ and that $u_n \to u$ in $L^2(I, L^2_{\bar{1}}(\Omega))$. This implies that $u = v \in W$ .
Moreover, we have for all $\varphi \in C_c^\infty(\Omega \times I)$,

$$\int_{Q_{2\pi}} [-u_n \frac{\partial \varphi}{\partial t} + \langle \nabla u_n, \nabla \varphi \rangle] = \int_{Q_{2\pi}} h_n \varphi.$$

Passing to the limit, we conclude that a strong solution is a weak solution.

**Step 2:** The problem (2.1) has at most one solution in $C_{\varphi_1} \cap W^{2,1}_{p,loc}(Q_{2\pi})$. This is a consequence of Corollary 2.1.

**Step 3:** The operator $T : L^p(I, L^p_{\bar{\mu}}(\Omega)) \to C_{\varphi_1}$ is well defined and compact. Let $(h_n)_n$ be a bounded sequence in $L^p(I, L^p_{\bar{\mu}}(\Omega))$. Denote by $u_n$ the unique strong solution of (2.1) with datum $h_n$.
By Theorem 3.1 and Step 2 of Part 1, we know that

$$u = u_{n,R} + \sum_{j=1}^{J} \eta_j \sum_{k \in N_j} u_k \lambda_j,$$

with $u_{n,R} \in L^p(I; V^{2,p}_{\bar{\mu}}(\Omega)) \cap W^{1,p}_{2\pi}(I; L^p_{\bar{\mu}}(\Omega))$ and the operator $L^p(I; L^p_{\bar{\mu}}(\Omega)) \to L^p(I; V^{2,p}_{\bar{\mu}}(\Omega)) \cap W^{1,p}_{2\pi}(I; L^p_{\bar{\mu}}(\Omega)) : h \to u_R$ is continuous. Moreover, by Step 3 of Part 1, the identity operator from $L^p(I; V^{2,p}_{\bar{\mu}}(\Omega)) \cap W^{1,p}_{2\pi}(I; L^p_{\bar{\mu}}(\Omega))$ to $C_{\varphi_1}$ is compact. Hence, the application $L^p(I; L^p_{\bar{\mu}}(\Omega)) \to C_{\varphi_1} : h \to u_R$ is compact. This implies that, up to a subsequence, $u_{n,R} \to w^{-1}U$ in $C_{\varphi_1}$ On the other hand, we know from Theorem 3.1 and Step 2 of Part 1 that, for all $j \in \{1, ..., J\}$,

$$u_{n,k\lambda_j} = (E *_t q_n, k\lambda_j) r^{k\lambda_j} \sin(k\lambda_j \theta) \quad D_j$$

where $u_{n,k\lambda_j} \in W^{\sigma_j, k\lambda_j}_{2\pi}(I)$,and

$$E(x,t) = \frac{1}{2\pi} \sum_{l \in \mathbb{Z}} e^{ilt} e^{-r\sqrt{il}},$$

and the application $L^p(I; L^p_{\bar{\mu}}(\Omega)) \to W^{\sigma_j, k\lambda_j}_{2\pi}(I) : h \to q_{k\lambda_j}$ is continuous.
Moreover, by Step 4 of Part 1, we know that the identity operator is compact from $W^{\sigma_j, k\lambda_j}_{2\pi}(I)$ to $C(I)$. This implies that, up to a subsequence,$q_{k\lambda_j} \to q_{k\lambda}$ in $C(\overline{I})$.
Let us fix $j \in \{1, ..., J\}$ and, for brevity, let us drop the index $j$ . It remains to prove that, if $q_{k\lambda_j} \to q_{k\lambda}$ in $C(\overline{I})$, then $w_n = E *_t q_{n,k\lambda_j} \to w = E *_t q_{k\lambda}$ in $C(\overline{D} \times \overline{I})$.

Observe that, if we decompose $q \in L^2(I)$ into its Fourier series

$$q(t) = \frac{1}{2\pi} \sum_{l \in \mathbb{Z}} ql e^{ilt},$$

then

$$w(x,t) := (E *_t q)(r,t) = \frac{1}{2\pi} \sum_{l \in \mathbb{Z}} ql e^{ilt} e^{-r\sqrt{il}}$$

and

$$\|E *_t q\|^2_{L^2((0,1)\times I)} = \int_0^1 \|w\|^2_{L^2(I)} dr = \frac{1}{2\pi} \sum_{l \in \mathbb{Z}} |ql|^2 |e^{-r\sqrt{il}}|^2 dr \leq \frac{1}{2\pi} \|q\|^2_{L^2(I)}.$$

Hence, if $q_n \to q$ in $C(\overline{I})$ we have

$$E *_t q_n \to E *_t q \quad in \quad L^2((0,1)\times I) \tag{3.3}$$

Moreover, by the Stone-Weierstrass theorem, we know that

$$A = \{\sum_{l=n_0}^{n_1} a_l e^{ilt} : n_0, n_1 \in \mathbb{Z}, a_l \in \mathbb{R}\}$$

is dense in $C_{2\pi}(\overline{I})$. Now observe that, for $q \in A$, the function

$$w = E *_t q = \frac{1}{2\pi} \sum_{l=n_0}^{n_1} ql e^{ilt} e^{-r\sqrt{il}} \in C^\infty([0,+\infty) \times \bar{I})$$

and satisfies, for $R > 0$ large enough,

$$\begin{array}{rcl} \partial_t w - \partial_{rr} w &=& 0, \quad in \ (0,R) \times I, \\ w(0,.) &=& q(.), \quad in \ I, \\ |w(R,.)| &\leq& \|q\|_{C(\overline{I})}, \quad in \ I, \\ w(.,-\pi) &=& w(.,\pi), \quad in \ (0,R). \end{array}$$

Hence, by the maximum principle we prove that

$$\|w\|_{C([0,R]\times\bar{I})} \leq \|q\|_{C(\bar{I})}.$$

Otherwise, by the boundary conditions, there exists $(x_0,t_0) \in (0,R) \times (-\pi,\pi]$ such that $w(x_0,t_0) > \|q\|_{C(\overline{I})}$ and $\epsilon > 0$ such that $w(x,t) > \|q\|_{C(\overline{I})}$ on $(x_0-\epsilon, x_0+\epsilon) \times (t_0-\epsilon, t_0)$ and $w$ is not constant on $(x_0-\epsilon, x_0+\epsilon) \times (t_0-\epsilon, t_0)$.
This contradicts the strong maximum principle (Proposition 2.1).

Hence, for $q \in C_{2\pi}(\overline{I})$ we have $(q_k)_k \subset A$ with $q_k \to q$ in $C(\overline{I})$. This implies that $(q_k)_k$ is a Cauchy sequence and by the above inequality

$$\|E *_t q_k - E *_t ql\|_{C([0,1]\times \bar{I}} \leq \|q_k - ql\|_{C(\overline{I})}.$$

This proves that $(E *_t q_k)_k$ is a Cauchy sequence in $C([0,1] \times \overline{I})$ and hence converges in $C([0,1] \times \overline{I})$ to $E *_t q$ by(5.3).
Now as $q_{n,k\lambda} \to q_{k\lambda}$ in $C_{2\pi}(\overline{I})$, for all $\epsilon > 0$, there exist $N > 0$ such that, for all $n \geq N$, $\|q_{n,k\lambda} \to q_{k\lambda}\|_{C_{2\pi}(\overline{I})} \leq \frac{\epsilon}{6}$. Let $n \geq N$. By the above result, there exist $q_{n,1}, q_1 \in A$ such that

$$\|q_{n,1} - q_{n,k\lambda}\|_{C_{2\pi}(\overline{I})} \leq \frac{\epsilon}{6},$$

$$\|E *_t q_{n,k\lambda} - E *_t q_{n,1}\|_{C([0,1]\times\overline{I})} \leq \frac{\epsilon}{6},$$

$$\|q_1 - q_{k\lambda}\|_{C_{2\pi}(\overline{I})} \leq \frac{\epsilon}{6},$$

$$\|E *_t q_1 - E *_t q_{k\lambda}\|_{C([0,1]\times\overline{I})} \leq \frac{\epsilon}{6},$$

Hence, we have

$$\|q_{n,1} - q_1\|_{C_{2\pi}(\overline{I})} \leq \frac{\epsilon}{2},$$

and by the above result

$$\|E *_t q_{n,1} - E *_t q_1\|_{C([0,1]\times\overline{I})} \leq \frac{\epsilon}{2},$$

This implies that

$$\|E *_t q_{n,k\lambda} - E *_t q_{k\lambda}\|_{C([0,1]\times\overline{I})} \leq \|E *_t q_{n,k\lambda} - E *_t q_{n,1}\|_{C([0,1]\times\overline{I})}$$

$$+\|E *_t q_{n,1} - E *_t q_1\|_{C([0,1]\times\overline{I})} + \|E *_t q_1 - E *_t q_{k\lambda}\|_{C([0,1]\times\overline{I})} \leq \epsilon$$

This proves that $E *_t q_{n,k\lambda} - E *_t q_{k\lambda}$ in $C([0,1] \times \overline{I})$, which permits us to conclude the compactness of T .

**Conclusion.**

We have proved the existence of a subsequence $(u_{n_k})_n$ of $(u_n)_n$ such that

$$\sup_{\Omega\times I}|\frac{v_n(r,\theta)-w^{-1}(r)V(r,\theta)}{w^{-1}(r)r\sin(\lambda\theta)}|\to 0,\quad as\quad n\to\infty,$$

where $u=w^{-1}U+\sum_{j=1}^{J}\eta_j\sum_{k\in N_j}(E*_t q_{k\lambda_j})r^{k\lambda_j}sin(k\lambda_j\theta)$, which gives the result.

$\square$

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