College of Natural Science
Department of Mathematics

Graduate Project Report on
Difference Equations and Bifurcation Analysis of
Discrete Dynamical System
Submitted in partial fulfilment of the requirements for
the Degree of Master of Science in Mathematics

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Addis Ababa, Ethiopia

# Contents

Addis Ababa University
Department of Mathematics

The undersigned hereby certify that they have read and recommend to the school of graduate studies for acceptance of a project entitled **Difference Equations and Bifurcation Analysis of Discrete Dynamical System** by Yoseph Alemu in partial fulfillment of the requirements for the degree of master of Science.

Date: September, 2014

Advisor: ______________________
Semu Mitiku (Ph.D)

Examining committee: ______________________

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# Notation

| | |
|---|---|
| $\mathbf{N}$ | The set of natural numbers. |
| $\mathbf{Z}$ | The set of integers. |
| $\mathbf{R}$ | The set of all real numbers. |
| $\mathbb{C}$ | The set of complex numbers. |
| $\mathbf{f} : \mathbf{R^n} \to \mathbf{R^n}$ | The function on $\mathbf{R^n}$. |
| $\mathbf{R^n}$ | n-dimensional real vector space. |
| $\mathbf{f^k}(\mathbf{n})$ | The $\mathbf{k^{th}}$ iterate of $\mathbf{f}$. |

# Abstract

The discrete time predator prey model in two dimension is investigated. The fixed points are obtained and their stability is analyzed. Prey predator solution and bifurcation diagram are presented for selected range of growth parameter. It is observed that prey population exhibits chaotic dynamics. Numerical simulations are performed.

# Introduction

Dynamical system is means of describing how one state develops into another state over the course of time. It describes the evolution of some variable over time. Low of evolution is the rule which allows us, if we know the system at any other moment of time. The existence of this law is equivalent to the assumption that our process is deterministic in the past and in the future. The value of this variable in period $t$ is denoted by $x_t$ the time index $t$ takes on discrete value and typically runs over all integer numbers $Z$ by interpreting $t$ as the time index, we have automatically introduced the notion of past, present and future. A difference equation then nothing but a rule or a function which instructs how to compute the value of the variable of interest in period $t$ given past values of that variable and time.The general form of difference equation is $f(x_t, x_{t-1}, ..., x_{t-p}; t) = 0$ where $f$ is the given function, $x_t$ is dependent variable and is an n-vector i.e, $x_t \in R^n, n \geq 1.r$ is called the dimension of the system. The difference between the largest and the smallest time index of the dependent variable explicitly involved is called the order of the difference equation. In the formulation of above this is $p$ with $p \geq 1$. In discrete-time models, only the outcomes for discrete time periods are considered and no reference is made to the timing of events within a period. As an additional simplification, all events are often assumed to occur at only discrete points in time and the time intervals in between are not considered explicitly in the model. Thus an aggregation over time is applied, resulting in a loss of information about the event history within the time period. Basically, a discrete-time model considers the change in the state variables between the start and the end of a time period without references to the process in between. This can be done either by directly modeling the state of a unit at the end of the time period or by modeling the transitions between subsequent points in time. A discrete-time model represents the time as a sequence of integers. This type of model is good for systems where change occurs at isolated times, for instance modeling insects that v reproduce once per year. A number of discrete time predator-prey models possess three fixed points or equilibriums that correspond to (1) extinction of both species, (2) extinction of the predator with survival of the prey at its carrying capacity, and (3) coexistence of both species. The main questions of interest in this paper are:

- How to compute stability boundaries of equilibria and limit cycles in the parameter space?
- How to predict qualitative changes in systems behavior (bifurcations) occurring at this equilibrium points?

**Bifurcation theory** is the mathematical study of changes in the qualitative or topological structure of a given family. Most commonly applied to the mathematical study of dynamical systems. Bifurcation is a change in the equilibrium points or periodic orbits, or in their stability properties as a parameter is varied. Bifurcation shows how systems behave as control parameters are varied and the aim bifurcation analysis is to provide exact or approximate information about how the possible outcomes of a dynamical process depend on the system parameters. Bifurcation occurs in both continuous systems (described by ODEs, or PDEs), and discrete systems (described by maps). We are especially interested in the change nature of solutions as the parameters are varied, and to investigate such changes we shall use the bifurcation diagram plot. The bifurcation diagram shows many sudden qualitative changes in the attractor as well as in the periodic orbits. The name bifurcation was first introduced by Heneri Poincare in 1885.

# Chapter 1

# First Order Systems

In this chapter we will see that the general introductions, description of the problem and background knowledge are mentioned. A first order system of difference equations has the general form

$$
\begin{aligned}
x_{1,t+1} &= a_{11}x_{1,t} + a_{12}x_{2,t} + ... + a_{1n}x_{n,t} + b_1(t) \\
x_{2,t+1} &= a_{21}x_{1,t} + a_{22}x_{2,t} + ... + a_{2n}x_{n,t} + b_2(t) \\
&... \\
x_{n,t+1} &= a_{n1}x_{1,t} + a_{n2}x_{2,t} + ... + a_{nn}x_{n,t} + b_n(t) \qquad (1.1)
\end{aligned}
$$

where,for $t \geq 0$, $x_{1,t}, x_{2,t}, ..., x_{n,t}$ are unknown sequences, $a_{11}, a_{22}, ..., a_{nn}$ are constant coefficients and if $b_i(t) = 0$ for all $1 \leq i \leq n$ we call (1) a linear autonomous system. It is convenient to express (1) interms of vectors and matrices.

Indeed, let $X = (x_1, x_2, x_3, ..., x_n)$, $b = (b_1, b_2, b_3, ..., b_n)$ and as with systems of difference equation, we shall find it more convenient to use the matrix notation.

Denoting $A = \{a_{ij}\}_{1\leq i,j \leq j}$. That is

$$
A = \begin{pmatrix}
a_{11} & a_{12} & . & . & . & a_{1n} \\
a_{21} & a_{22} & . & . & . & a_{2n} \\
. & . & & & & . \\
. & . & & & & . \\
. & . & & & & . \\
a_{n1} & a_{n2} & . & . & . & a_{nn}
\end{pmatrix} \qquad (1.2)
$$

can be written as

$$X_{t+1} = AX_t + b_t \qquad (1.3)$$

If $b_t = 0$, then we have

$$X_{t+1} = AX_t \qquad (1.4)$$

## 1.1 Qualitative behavior of solutions to linear difference equations

Linear difference equations are characterized by the following properties

1. An $m^{th}-$order equation typically takes the form

$$a_0 x_n + a_1 x_{n-1} + ... + a_m x_{n-m} = b_n \tag{1.5}$$

2. The order $m$ of the equation is the number of previous generations that directly influence the value of $x$ in a given generation.

3. When $a_0, a_1, a_2, ..., a_m$ are constants and $b_n = 0$, solutions are composed of linear combinations of basic expressions of the form

$$x_n = c\lambda^n \tag{1.6}$$

4. Values of $\lambda$ appearing in equation (1.4) are obtained by finding the roots of the corresponding characteristic equation

$$a_0 \lambda^m + a_1 \lambda^{m-1} + ... + a_m = 0.$$

5. The number of (distinct) basic solutions to a difference equation is determined by its order. For example, a first order equation has one solution, and a second order equation has two solutions. In general, an $m^{th}$ - order equation, like a system of $m$ basic solutions.

6. The general solution is a linear superposition of the $m$ basic solutions of the equation (provide all values of $\lambda$ are distinct).

7. For real values of $\lambda$ the qualitative behavior of a basic solution (1.6) depends on whether $\lambda$ falls into one of four possible ranges:
$\lambda \geq 1, \lambda \leq -1, 0 < \lambda < 1, -1 < \lambda < 0$.

To observe how the nature of a basic solution is characterized by this broad classification scheme, note that

**a.** For $\lambda > 1, \lambda^n$ grows as $n$ increases; thus $x_n = c\lambda^n$ grows without bound.

**b.** For $0 < \lambda < 1, \lambda^n$ decreases to zero with increasing $n$, thus $x_n$ decreases to zero.

**c.** For $-1 < \lambda < 0, \lambda^n$ oscillates between positive and negative values while declining in magnitude to zero.

**d.** For $\lambda < -1, \lambda^n$ oscillates as in (c) but with increasing magnitude. The cases where $\lambda = 1, \lambda = 0$ or $\lambda = -1$ which are marginal points of demarcation between realms of behavior, correspond respectively to (1) the static (non-growing) solution where $x = c$, (2) $x = 0$ and (3) an oscillation between the value $x = c$ and $x = -c$.

**Definition 1.1.1.** *The polynomial* $p(\lambda) = \lambda^k - a_1\lambda^{k-1} - ... - a_{k-1}\lambda - a_k$ *is called the characteristic polynomial of the homogeneous difference equation. If* $\lambda$ *is a root of the characteristic polynomial* $p(x)$ *of the difference equation* $x_n = a_1x_{n-1} + a_2x_{n-2} + ... + a_kx_{n-k}$. *Then the solutions be*

**I.** $x_n = c_1\lambda_1^n + c_2\lambda_2^n + ... + c_k\lambda_k^n$ *if* $\lambda_i \neq \lambda_j, i \neq j$ *and* $c_i'$*s are constants.*

**II.** $x_n = c_1\lambda^n + c_2n\lambda^n + ... + c_nn^{k-1}\lambda^n$ *if* $\lambda_i = \lambda_j, i \neq j$.

**III.** $x_n = r^n[a_1\cos(n\Theta) + a_2\sin(n\Theta)]$ *if* $\lambda_1 = \overline{\lambda_2}$.

## 1.2 Stability of solutions of linear system

The system of difference equation $x_{n+1} = Ax_n$ has a solution $x(t) = 0$ this solution is represented by the origin of the coordinates in the solution space and the origin is called an equilibrium point. Equilibrium point is a point on a dynamical system at which there is no dynamics. An equilibrium solution of the system $x(n+1) = Ax(n)$ is a point $(x_1, x_2)$ where $x(n+1) = 0 = Ax(n)$. An equilibrium solution is a constant solution of the system, and is usually called a critical point. For a linear system, an equilibrium solution occurs at each solution of the system (of homogeneous algebraic equations) $Ax = 0$. As we have seen, such a system has exactly one solution, located at the origin, if $det(A) \neq 0$. If $det(A) = 0$. then there are infinitely many solutions.

**Definition 1.2.1.** *Stability Stability means that the trajectories do not change too much under small perturbations.*

**Phase portraits of linear systems**

Consider systems of linear difference equations $x(n+1) = Ax(n)$ . Its phase portrait is a representative set of its solutions, plotted as parametric curves (with $t$ as the parameter) on the Cartesian plane tracing the path of each particular solution $(x, y) = (x_1(t), x_2(t))$ Similar to a direction field, a phase portrait is a graphical tool to visualize how the solutions of a given system of differential equations would behave in the long run. In this context, the Cartesian plane where the phase portrait resides is called the phase plane.

The parametric curves traced by the solutions are sometimes also called their trajectories. The phase plane portrait for homogeneous system

$$x_{n+1} = ax_n + by_n$$

$$y_{n+1} = cx_n + dy_n, ad - bc \neq 0, a, b, c, d \in R$$

The characteristic polynomial

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0$$

| -Root of characteristic equation | -Phase portrait | -Figure |
|---|---|---|
| Real, different, negative | Stable node | 1.1 |
| Real, different, positive | Unstable node | 1.1 |
| Real, opposite sign | Saddle(Unstable) | 1.1 |
| Complex, negative real part | Stable spiral | 1.2 |
| Complex, positive real part | Unstable spiral | 1.2 |
| Complex, zero real part | Center(Neutrally stable) | 1.2 |
| Real, equal, negative | Stable(Improper node) | — |
| Real, equal, positive | Unstable(Improper node) | — |

**Definition 1.2.2.** *The phase plane portrait of a linear system is stable if the real parts of all roots of the characteristic equations are negative and unstable if the real part of all roots of the characteristic equations of at least one root is positive. That is the origin is said to be a stable equilibrium point for the system $x_{n+1} = Ax_n$ if $x_n \to 0$ as $n \to \infty$ for all initial points or complex numbers with real part negative.*

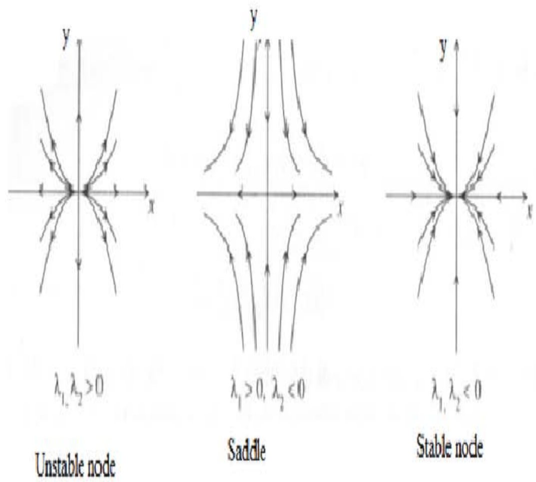


Figure 1.1: Saddle, Stable and Unstable node.

Repeated root case:
Suppose the characteristic polynomial

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0$$

has a repeated root $\lambda \neq 0$ this occurs if

$$(a + d)^2 - 4(ad - bc) = 0$$

in which the root is $\lambda = \frac{a+d}{2}$. If $b = c = 0$ and $a = d$ the repeated root $\lambda = a$ all orbits associated with the general solution $x_n = c_1\lambda^n + c^2 n\lambda^n$ then the orbit tend to origin as $n \to \infty$ if $\lambda \neq 0, 0 < \lambda < 1$ and tend away from origin as $n \to \infty$ if $\lambda > 1$ the phase point is called stable and unstable respectively.
Complex roots case: If the eigen values of a matrix are complex. Then the general solution is

$$x_n = r^n[a_1 \cos(n\Theta) + a_2 \sin(n\Theta)]$$

If $0 < r < 1$, then

$$\lim_{n\to\infty} x(n) = 0$$

hence stable focus. If $r = 1$ then, $x_n = [a_1 \cos(n\Theta) + a_2 \sin(n\Theta)]$ hence it is elliptic.

If $r > 1$, then

$$\lim_{n\to\infty} x(n) = \infty$$

hence it is unstable.

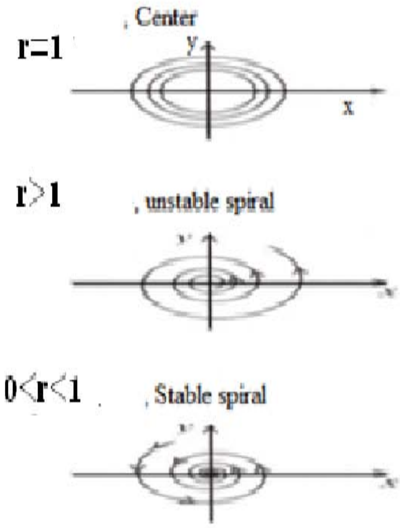


Figure 1.2: center, stable spiral and unstable spiral

**Definition 1.2.3.** *The linear system $x_{n+1} = Ax_n$ is globally asymptotic stable if*

$$\lim_{n\to\infty} x_n = 0.$$

**Theorem 1.2.1.** *The linear system $x_{n+1} = Ax_n$ is globally asymptotic stable if and only if all the eigen values $\lambda$ of $A$ are located inside the unit circle $|z| = 1$ in the complex plane.*

**Proof 1.2.1.** *In case of distinct eigen values the result follows immediately from (1.7) the general solution of the linear system may be cast in the form*

$$x_n = c_1\lambda_1^n u_1 + c_2\lambda_2^n u_2 + ... + c_n\lambda_n^n u_n \tag{1.7}$$

*where $\lambda_1, \lambda_2, ..., \lambda_m$ are distinct eigenvalues and the associated eigenvectors $u_1, u_2, ..., u_n$ are linearly independent. eigenvalues with multiplicity $m$ lead according to our previous discussion to terms in the solution of form $t^q\lambda^t$*

*where $q \leq m-1$.*
*Now, if $|\lambda| < 1$, let $|\lambda| = \frac{1}{s}$, where $s > 0$. Then by L'Hopltal's rule:*

$$\lim_{t \to \infty} \frac{t^q}{s^t} = 0.$$

*So here the result follows too.*

The following lemma is useful in the study of the nature of fixed points

**Lemma 1.2.1.** *Let $p(\lambda) = \lambda^2 - B\lambda + C$ and $\lambda_1, \lambda_2$ be the roots of $p(\lambda) = 0$. Suppose that $p(1) > 0$ then we have*

**I.** $|\lambda_1| < 1$ *and* $|\lambda_2| < 1$ *iff* $p(-1) > 0$ *and* $C < 1$.

**II.** $|\lambda_1| < 1$ *and* $|\lambda_2| > 1$ *(or* $|\lambda_1| > 1$ *and* $|\lambda_2| < 1$*)* *iff* $p(-1) < 0$.

**III.** $|\lambda_1| > 1$ *and* $|\lambda_2| > 1$ *iff* $|\lambda_1| = |\lambda_2|, p(-1) > 0$ *and* $C > 1$.

**IV.** $|\lambda_1| = -1$ *and* $|\lambda_2| \neq 1$ *iff* $p(-1) = 0$ *and* $B \neq 0, 2$.

**V.** $\lambda_1, \lambda_2$ *are complex and* $|\lambda_1| = |\lambda_2|$ *iff* $B^2 - 4AC < 0$ *and* $C = 0$. *The characteristic roots* $\lambda_1$ *and* $\lambda_2$ *of* $p(\lambda) = 0$ *are called eigen values of the fixed point* $(x^*, y^*)$. *The fixed point* $(x^*, y^*)$ *is a sink if* $|\lambda_{1,2}| < 1$, *hence the sink is locally asymptotically stable. The fixed point* $(x^*, y^*)$ *is a source if* $|\lambda_{1,2}| > 1$, *the source is locally unstable. The fixed point* $(x^*, y^*)$ *is a saddle if* $|\lambda_1| > 1$ *and* $|\lambda_2| < 1$ *(or* $|\lambda_1| < 1$ *and* $|\lambda_2| > 1$*). Finally* $(x^*, y^*)$ *is called non hyperbolic if either* $|\lambda_1| = 1$ *or* $|\lambda_2| = 1$.

## 1.3 Existence and uniqueness theorem

*For given $f : N_0 X R \to R$ and $a_0 \in R$ there exist exactly one sequence $y_n (n \in N_0)$ which satisfies the equation*

$$y(n+1) = f(n, y(n)), n \in N_0$$

*and the initial condition*

$$y(0) = a(0).$$

**Proof 1.3.1.** *Consider the following equation*

$$y(n+1) + py(n) = f(n)$$

*where* $p \neq 0$ *is a constant and* $f : N_0 \to R$ *is given.*
*Using the equation for* $n = 0, 1, 2, ...$ *we get*

$$\begin{aligned}
y(1) &= -py(0) + f(0) \\
y(2) &= -py(1) + f(1) \\
y(2) &= (-p)^2 y(0) - pf(0) + f(1) \\
y(3) &= -py(2) + f(2) \\
y(3) &= (-p)^3 y(0) + (-p)^2 f(0) - pf(1) + f(2)
\end{aligned}$$

*By induction we can show that*

$$y(n) = (-p)^n y(0) + \sum_{k=0}^{n-1} (-p)^{n-1-k} f(k)$$

*is the general solution of our equation where* $y(0)$ *is arbitrary.*

## 1.4 Nonlinear system

Nonlinear systems are much harder to analyze than linear systems since they rarely possess analytical solutions. One of the most useful and important technique for analyzing nonlinear systems qualitatively is the analysis of the behavior of the solutions near equilibrium points using linearization. The local stability analysis of the model can be carried out by computing the jacobian corresponding to each equilibrium point. We saw that any linear system $x_{n+1} = Ax_n$ has a unique solution through each point $x_0$ in the plane space $R^n$. Systems of first order Non-linear difference equations have the form

$$\begin{aligned}
x_{1,n+1} &= f_1(x_{1,n}, x_{2,n}, ..., x_{k,n}) \\
x_{2,n+1} &= f_2(x_{1,n}, x_{2,n}, ..., x_{k,n}) \\
&\cdots \\
x_{k,n+1} &= f_k(x_{1,n}, x_{2,n}, ..., x_{k,n}) \qquad (1.8)
\end{aligned}$$

Then change form is:

$$\begin{aligned}
\Delta x_1 &= x_{1,n+1} - x_{1,n} = F_1(x_{1,n}, x_{2,n}, ..., x_{k,n}) \\
\Delta x_2 &= x_{2,n+1} - x_{2,n} = F_2(x_{1,n}, x_{2,n}, ..., x_{k,n}) \\
&\cdots \\
\Delta x_k &= x_{k,n+1} - x_{k,n} = F_k(x_{1,n}, x_{2,n}, ..., x_{k,n}) \qquad (1.9)
\end{aligned}$$

In these models $x_{i,n}$ is the abundance of species $i$ at time $r$; $f_i$ is the absolute finite rate of increase of species $i$, a function of the abundances of all the species; and $F_i$ is the absolute change in the abundance of species $i$, again a function of the abundances of all the species. We don't solve non-linear difference equations. Therefore, the dynamics depend upon end behavior not solutions. To explore the end behavior, find steady state and stability. Equilibrium points of (1.8) are the sets of abundances $(x_1^*, x_2^*, ..., x_k^*)$ at which

$$\begin{aligned} F_1(x_1^*, x_2^*, ..., x_k^*) &= 0 \\ F_1(x_1^*, x_2^*, ..., x_k^*) &= 0 \\ &\cdots \\ F_k(x_1^*, x_2^*, ..., x_k^*) &= 0 \end{aligned} \tag{1.10}$$

There will always be the trivial equilibrium in which all species are absent, i.e. $N_i = 0$ for all $i$. There often also will be equilibria in which some species are absent ($N_i = 0$) and others are present ($N_i > 0$) we usually are most interested, though, in the equilibrium (or perhaps multiple equilibria) at

which all species are present (all $N_i > 0$). Nonlinear systems are much harder to analyze since in most cases they dont possess quantitative solutions are available, they are often too complicated to provide much insight. Hence mathematician prefers to analyze nonlinear systems qualitatively. One of the most useful techniques for analyzing nonlinear systems qualitatively is the linearized stability techniques. The stability of the system is investigated by obtaining the eigen values of the Jacobian matrix associated with fixed points.

## 1.5 Linearization techniques

Linearization is the process of replacing the nonlinear system model by its linear counterpart in a small region about its equilibrium point. Linearization by use of a Jacobian matrix is necessary to get information on the equilibrium point. Given a system of difference equations and a point of interest, the Jacobian matrix of this linearized system therefore is given by:

$$J = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & . & . & . & \frac{\partial F_1}{\partial x_k} \\ . & . & & & & . \\ . & . & & & & . \\ . & . & & & & . \\ \frac{\partial F_k}{\partial x_1} & \frac{\partial F_k}{\partial x_2} & . & . & . & \frac{\partial F_k}{\partial x_k} \end{pmatrix}_{x^*}$$

**Definition 1.5.1.** *Equilibrium point is a point on a dynamical system at which there is no dynamics. That is $x^* \in R^n$ is called an equilibrium point of $x_{n+1} = f(x_n)$ if $x_{n+1} = f(x^*) = 0$.*

**Definition 1.5.2.** *The equilibrium is said to be hyperbolic if all eigenvalues of the jacobian matrix have non-zero real parts. Hyperbolic equilibrium are robust (i.e; the system is structurally stable) or if $T : R^n \to R^n$ is $c^r$ map and $x^*$ is an equilibrium point when the jacobian matrix $DT(x^*)$ has no an eigen values on the unit circle. Small perturbations of order do not change qualitatively the phase portrait of a hyperbolic equilibrium of a nonlinear system is equivalent to that of its linearization. This statement has a mathematically precise form known as the Hartman-Grobman. This theorem guarantees that the stability of the steady state $(x^*, y^*)$ of the nonlinear system is the same as the stability of the trivial steady state $(0, 0)$ of the linearized system.*

**Definition 1.5.3.** *If at least one eigen value of the jacobian matrix is zero or has a zero real part, then the equilibrium is said to be non-hyperbolic. Non- hyperbolic equilibria are not robust(i.e, the system is not structurally stable).or if $T : R^n \to R^n$ is $c^r$ map and $x^*$ is an equilibrium point when the jacobian matrix $DT(x^*)$ has an eigen values on the unit circle. Small perturbations can result in a local bifurcation of a non-hyperbolic equilibrium, i.e; it can be change stability, disappear, or split into many equilibria. Some refer to such equilibrium by the name of bifurcation. $(x^*, y^*)$is called non hyperbolic if either $|\lambda_1| = 1$ or $|\lambda_1| = -1$.*

Two dynamic systems that have qualitatively similar orbit structures are called topologically equivalent. Formally we have the following definition.

**Definition 1.5.4.** *Let $f$ and $g$ be continuously differentiable maps from $X \subseteq R^n$ into $R^n$. Then we say that the discrete dynamical systems $x_{n+1} = f(x_n)$ and $x_{n+1} = g(x_n)$ are topological equivalent if there exists a homeomorphism $h : R^n \to R^n$ (that is, a continuous change of coordinates with a continuous inverse) that maps $f$ orbits into $g$ orbits while preserving the sense of direction in time. Topologically equivalent systems have many similar features: their phase diagrams, orbital structure, number of steady states, and other properties are closely related.*

Consider the Parameter dependent nonlinear difference equation

$$F(x) = x_{n+2} = \varepsilon(1 - x_n^2)x_{n+1} - x_n$$

Linearize the equation
Let $x_{n+1} = y_n$, then

$$y_{n+1} = x_{n+2} = \varepsilon(1 - x_n^2)x_{n+1} - x_n$$

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} y_n \\ \varepsilon(1-x_n^2)x_{n+1} - x_n \end{pmatrix} = \begin{pmatrix} y_n \\ \varepsilon(1-x_n^2)y_n - x_n \end{pmatrix}$$

The equilibrium point:

$$y^* = 0 \text{ and } \varepsilon(1-(x^*)^2)y^* - x^* = 0 \Rightarrow y^* = 0 \text{ and } x^* = 0.$$

Therefore $(0,0)$ is the only equilibrium point.

$$Df(x_0) = \begin{pmatrix} 0 & 1 \\ -2x_n\varepsilon y_n - 1 & \varepsilon(1-x_n^2) \end{pmatrix}$$

$$Df(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & \varepsilon \end{pmatrix} = A$$

The linear approximation

$$x_{n+1} = Ax_n = \begin{pmatrix} 0 & 1 \\ -1 & \varepsilon \end{pmatrix} x_n$$

The characteristics equation

$$\lambda^2 - \varepsilon\lambda + 1 = 0$$

The corresponding eigen values are

$$\lambda_1, \lambda_2 = \frac{\varepsilon \pm \sqrt{\varepsilon^2 - 4}}{2}$$

Case 1: If $\varepsilon < -2$ both $\lambda_{1,2}$ are negative and real therefore the equilibrium point is stable node.
Case2: If $\varepsilon = -2$ then the eigenvalue is $-1$ .Thus the equilibrium point is stable improper node.
Case 3: If $-2 < \varepsilon < 0$ then the eigenvalues are complex with negative real part. Thus the equilibrium point is a stable spiral/ focus.
Case 4:If $0 < \varepsilon < 2$, then $\lambda_1, \lambda_2$ are complex with positive real part. Therefore the origin is unstable focus.
Case 5: if $\varepsilon = 0$ then $\lambda_1, \lambda_2$ purely imaginary the equilibrium point is center.
Case 6: If $\varepsilon > 2$ then $\lambda_1, \lambda_2 \in R$ and positive the equilibrium point is unstable node.
Case 7: If $\varepsilon = 2$ then the eigenvalue is 1. Thus the equilibrium point is unstable improper node.

## 1.6 Center manifolds in discrete-time system

We are going to formulate without proof the main theorem that allows us to reduce the dimension of a given system near a local bifurcation. Let us start the critical case; we assume in this section that the parameters of the system are fixed at their bifurcation values, which are those values for which there is a non-hyperbolic equilibrium (fixed point). Consider now a discrete time dynamical system defined by $x \to f(x), x \in R^n$ where $f$ is sufficiently smooth, $f(0) = 0$ Let the eigen values of the jacobian matrix $A$ evaluated at the fixed point $x_0 = 0$ be $\mu_1, \mu_2, ..., \mu_n$. Recall that we call them multipliers. Suppose that the equilibrium is not hyperbolic and there are therefore multipliers on the unit circle (with absolute value one). Assume that there are $n_+$ multipliers outside the unit circle, $n_0$ multipliers on the unit circle, and $n_-$ multipliers inside the unit circle. Let $T^c$ denote the linear invariant (generalized) eigenspace of $A$ corresponding to the union of $n_0$ multipliers on the unit circle.

**Theorem 1.6.1.** *(Center Manifold Theorem)*
*There is locally defined smooth $n_0$-dimensional invariant manifold $W^c_{loc}(0)$ of $x \to f(x), x \in R^n$ that is tangent to $T^c$ at $x = 0$ Moreover, there is neighborhood $U$ of $x_0 = 0$ such that if the $k^{th}$ iteration of $f(f^k x) \in U$ for only integer time values, then $f^k x \to W^c_{loc}(0)$ for $t \to \infty (t \to -\infty)$.*

**Theorem 1.6.2.** *(Hartman-Grobman Theorem)*
*Let $f : R^k \to R^k$ be a $c^r$ diffeomorphism with hyperbolic fixed point $x^*$. Then there exists neighborhoods $V$ of $x^*$ and $W$ of $0$ and a homeomorphism $h : W \to V$ such that $f(h(x)) = h(Ax)$, where $A = Df(x^*)$. The theorem says, if $x*$ is a hyperbolic fixed point of a nonlinear system $f$ then the nonlinear system is equivalent up to a continuous change of coordinates, to a linear system with coefficient matrix $Df(x^*)$.*

**Theorem 1.6.3.** *The homogeneous two-dimensional system has an asymptotically stable solution if and only if*

$$|tr\tau| < 1 + det\tau < 2 .................. (*)$$

**Proof 1.6.1.** *The characteristic polynomial,$p(\lambda)$ of a $2x2$ matrix is*

$$p(\lambda) = \lambda^2 - tr\tau\lambda + det\tau$$

*The roots of this quadratic polynomial are thus*

$$\lambda_{1,2} = \frac{tr\tau \pm \sqrt{tr^2\tau - 4det\tau}}{2}$$

*Suppose that the zero point is asymptotically stable then* $|\lambda_{1,2}| < 1$ *But in the case of real roots, this equivalent to the following two inequalities.*

$$-2 - tr\tau < \sqrt{tr^2\tau - 4det\tau} < 2 - tr\tau$$

$$-2 - tr\tau - < \sqrt{tr^2\tau - 4det\tau} < 2 - tr\tau$$

*Squaring the second inequality in the first line and simplifying gives:*

$$tr\tau < 1 + det\tau$$

*Squaring the first inequality in the second line gives*

$$-1 - det\tau < tr\tau$$

*Combing both results gives the first part of the stability condition(*). The second part follows from the observation that*

$$det\tau = \lambda_1\lambda_2$$

*and the assumption that* $|\lambda_{1,2}| < 1$. *If the roots are complex, they are conjugate complex, so that the second part of the stability (*) result from*

$$det\tau = \lambda_1\lambda_2 = |\lambda_1||\lambda_2| < 1$$

*The first part follows from* $tr^2\tau - 4det\tau < 0$ *which is equivalent to* $0 < tr^2\tau < 4det\tau$ *can be used to show that*

$$4(1 + det\tau - tr\tau) > 4 + tr^2\tau - 4tr\tau = (2 - tr\tau)^2 > 0.$$

*This is the required inequality.*

*Conversely*

*If the stability condition (*) is satisfied and if the roots are real, we have*

$$\begin{aligned} -1 < \frac{-2 + \sqrt{tr^2\tau - 4det\tau}}{2} < \lambda_1 &= \frac{tr\tau + \sqrt{tr^2\tau - 4det\tau}}{2} \\ &< \frac{tr\tau + \sqrt{tr^2\tau - 4det\tau}}{2} \\ &= \frac{tr\tau + \sqrt{(2 - tr\tau)^2}}{2} < 1 \end{aligned}$$

*Similarly, for* $\lambda_2$

*If the roots are complex, they are conjugate complex and we have*

$$|\lambda_1|^2 = |\lambda_2|^2 = \lambda_1\lambda_2 = \frac{tr^2\tau - tr^2\tau + 4det\tau}{4} = det\tau < 1.$$

*This completes the proof.*

**Example 1.6.1.** *Determine the stability of the following nonlinear system:*

$$x_1(n+1) = x_2(n) - x_2(n)[x_1^2(n) + x_2^2(n)]$$

$$x_2(n+1) = x_1(n) - x_1(n)[x_1^2(n) + x_2^2(n)]$$

Solution

We start by finding the fixed points of the simultaneous system:

$$x_1(n+1) = x_2(n) - x_2(n)[x_1^2(n) + x_2^2(n)]$$

$$x_2(n+1) = x_1(n) - x_1(n)[x_1^2(n) + x_2^2(n)]$$

This system then has three equilibrium points which are:

$$(x_1^*, x_2^*) = (0,0), (x_1^*, x_2^*) = (1,-1), (x_1^*, x_2^*) = (-1,1)$$

To determine stability of those points, linearization using the Jacobian matrix is necessary since the system is not linear.

$$Df(x_1^*, x_2^*) = A = \begin{pmatrix} -2x_1x_2 & -3x_2^2 - x_1^2 + 1 \\ -3x_1^2 - x_2^2 + 1 & -2x_1x_2 \end{pmatrix}_{(x_1^*, x_2^*)}$$

For the equilibrium point $(0,0)$, the Jacobian matrix is:

$$Df(0,0) = A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Which gives eigen-values $\lambda = \pm 1$. Although the eigen-values $\lambda = \pm 1$ where obtained from a linearization of the system, they may suggest the existence of a periodic solution about the equilibrium $(0,0)$. For the equilibrium point $(1,-1)$, the Jacobian matrix is:

$$Df(1,-1) = A = \begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}$$

which gives eigen values $\lambda_1 = -1$ and $\lambda_2 = 5$. The eigen values $\lambda_1 = -1$ and $\lambda_2 = 5$ were obtained from a linearization of the system, which imply the equilibrium point $(1,-1)$ is unstable. For the equilibrium point $(-1,1)$, the Jacobian matrix is:

$$Df(-1,1) = A = \begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}$$

which gives eigen values $\lambda_1 = -1$ and $\lambda_2 = 5$. Which implies the equilibrium point $(1,-1)$ is unstable.

# Chapter 2

# Bifurcation Analysis of Dynamical System

In the first chapter we have seen that the basic concepts, so without difficulty in this chapter we can see that bifurcation analysis of dynamical systems in a discrete function and also types of bifurcations are the core idea of the report. In a dynamical system, a parameter is allowed to vary, and then the difference system may change. An equilibrium can become unstable and a periodic solution may appear making the previous equilibrium unstable. The value of parameter at which these changes occur is known as **bifurcation value** and the parameter that is varied is known as the **bifurcation parameter**. A bifurcation occurs when a small smooth change made to the parameter values(the bifurcation parameters) of a system causes a sudden **qualitative** change in its behavior. Generally, at a bifurcation the local stability properties of equilibriums, periodic orbits or other invariant sets changes.

## 2.1 Types of bifurcation

1. Local bifurcation

2. Global bifurcation

### 2.1.1 Local bifurcations

Local bifurcations which can be analyzed entirely through changes in the local stability properties of equilibria, periodic orbits or other invariant sets as parameters cross through critical threshholds. Local stability means that the system converges to the steady state from any point in neighborhood

of the steady state. Local bifurcation is a sudden change in the number or nature of the fixed and periodic points caused by a parameter change in system. Fixed points may appear or disappear, change their stability, or even break apart into periodic points. Such bifurcation can be analyzed entirely through changes in the local stability properties of equilibria, periodic orbits or other invariant sets.
Local bifurcation includes:

**a.** The saddle node bifurcation

**b.** Transcritical bifurcation

**c.** period doubling bifurcation

**d.** The pitchfork bifurcation

**e.** The hopf bifurcation

**The saddle node bifurcation**
A saddle node bifurcation occurs if, near the bifurcation point $(x^*, \mu^*)$, the map possesses a unique curve of fixed points in the $(\mu, x)$ plane which passes through the bifurcation point and lies on one side of the line $\mu = \mu^*$ A saddle node bifurcation has the normal form

$$x_{t+1} = \mu + x_t - x_t^2$$

At a saddle node bifurcation an equilibrium point bifurcates to a repeller (the saddle) and an attractor (the node).

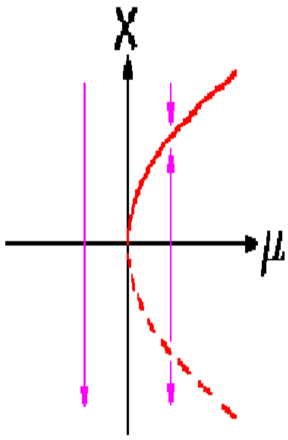


Figure 4: saddle-node bifurcation

The equilibrium points are $x^*_{1,2} = \pm\sqrt{\mu}$ hence, when $\mu > 0$ there are two equilibrium points which equals when $\mu = 0$. If $\mu < 0$ there are no equilibrium points. In case of $\mu > 0, \mu$ small, we have

$$f'_\mu(x^*_1 = \sqrt{\mu}) = 1 - 2\sqrt{\mu} < 1$$

hence $x^*_1 = \sqrt{\mu}$ is stable. On the other hand

$$f'_\mu(x^*_2 = -\sqrt{\mu}) = 1 + 2\sqrt{\mu} > 1$$

consequently $x^*_2$ is unstable. Thus a saddle-node bifurcation is characterized by that there is no equilibrium point when the parameter $\mu$ falls below a certain threshold $\mu_0$. When $\mu$ is increased to $\mu_0, \lambda = 1$, and two branches of equilibrium points are born, one stable and one unstable.

**Transcritical Bifurcation**

A transcritical bifurcation occurs if near the bifurcation point $(x^*, \mu^*)$, the map possesses two curves of fixed points in the $(\mu, x)$ plane both of which pass through the bifurcation point and lie on both sides of the line $\mu = \mu^*$ an exchange of stability takes place at the bifurcation point. A transcritical bifurcation has the normal form

$$x_{t+1} = (1 + \mu)x_t - x_t^2.$$

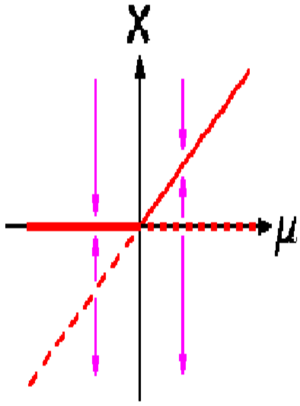

Figure 5: Transcritical bifurcation

The transcritical bifurcation occurs when in the combined space of phase space and control parameter space two different manifolds of fixed points cross each other. At the crossing point the fixed points exchange there stability property. That is, the unstable fixed point becomes stable and vice versa. Note that, beyond the bifurcation point the number of fixed points has not changed contrary to saddle node bifurcation where two fixed points appear or disappear. A transcritical bifurcation is not a generic bifurcation in a phase space with more than one dimension because it is unlikely that two lines cross each other in a space with more than two dimensions.

**Pitchfork bifurcation**

A pitchfork bifurcation occurs if near the bifurcation point $(x^*, \mu^*)$ the map possesses two curves of fixed points in the $(\mu, x)$ plane both of which pass through the bifurcation point and one of which lie on both sides of the line $\mu = \mu^*$. Apitchfork bifurcation has the normal form

**A.** Supercritical pitchfork bifurcation:

$$x_{t+1} = x_t + \mu x_t - x_t^3$$

**B.** Subcritical pitchfork bifurcation:

$$x_{t+1} = x_t + \mu x_t + x_t^3$$

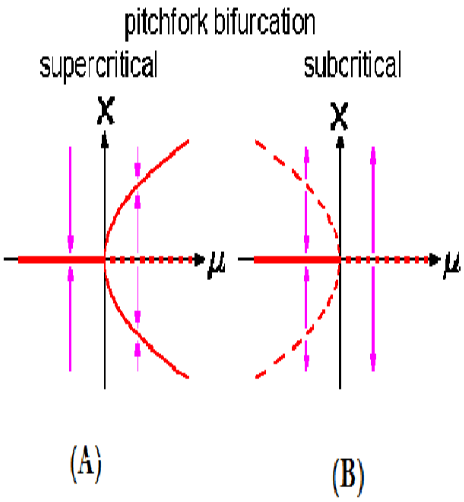

Figure 6: pitchfork bifurcation

Pitchfork bifurcations are possible in dynamical systems with inversion or reflection symmetry. That is, an equation of motion that remains unchanged if one changes the sign of all phase space variables (or at least for one). Pitchfork bifurcations are the generic bifurcations when such a symmetric solution changes its stability. The bifurcating solution does not have the full symmetry of the equation of motion. This phenomenon is called broken symmetry. A solution with a broken symmetry does not occur in isolation because the broken symmetry applied onto such a solution generates a new solution where the same symmetry is broken. All such solutions build a family. Therefore, always two fixed points with a broken symmetry bifurcate at once in a pitchfork bifurcation. Both are either stable (supercritical pitchfork bifurcation) or unstable (subcritical pitchfork bifurcation).

**A period doubling bifurcation**

Period-doubling is the change in dynamics in which a $N-$ point attractor is replaced by a $2N-$ point attractor. A period doubling bifurcation occurs if, near the bifurcation point $(x^*, \mu^*)$ the map possesses a single curve of fixed points in the $(\mu, x)$ plane, while the second iterate $f^2$ undergoes a pitchfork bifurcation point

$$x_{t+1} = -(1+\mu)x_t + x_t^3.$$

The map is invertible for small $|\mu|$ in a neighborhood of the origin. System has the fixed point $x^* = 0$ for all $\mu$ with multiplier $\lambda = -(1 + \mu)$. The point is linearly stable for small $\mu < 0$ and is linearly unstable for $\mu > 0$. At $\mu = 0$ the point is not hyperbolic, since the multiplier $\lambda = f_x(0, 0) = -1$, but is nevertheless (nonlinearly) stable. There are no other fixed points near the origin for small $|\mu|$ the bifurcation diagrams of a Hopf and a period doubling bifurcation are similar to the diagram of a pitchfork bifurcation. That is, the bifurcating periodic solution is either stable (supercritical bifurcation) or unstable (subcritical bifurcation).
**The Hopf bifurcation**
It is a local bifurcation in which a fixed point of a dynamical system loses stability as a pair of complex conjugate eigen values of linearization around the fixed point cross the imaginary axis of the complex plane. At a Hope bifurcation, a stable limit cycle is born while an equilibrium loses stability. This is the case if

- The dominant eigen-value is complex.
- The real part of the dominant eigen-value changes from negative to positive (with a non-zero slope).

There are two kinds of hopf bifurcation:

1. In a Supercritical hopf bifurcation, a stable equilibrium is replaced by stable limit cycle ( around an unstable equilibrium).
2. In a subcritical hopf bifurcation an unstable equilibrium is replaced by an unstable limit cycle( around a stable equilibrium).

**Theorem 2.1.1.** *(Saddle-node bifurcation)*
*Suppose that a one dimensional system*

$$x_{n+1} = f(x_n, \alpha), x_n \in R, \alpha \in R$$

*with smooth $f$. We say that there is a saddle node bifurcation at $(x^*, \alpha_c)$ if the following conditions are satisfied:*

**SN1.** $f(x^*, \alpha_c) = x^*$

**SN2.** $f_x(x^*, \alpha_c) = 1$

**SN3.** $f_{xx}(x^*, \alpha_c) \neq 0$

**SN4.** $f_\alpha(x^*, \alpha_c) \neq 0$

**Theorem 2.1.2.** *(Pitchfork bifurcation)*
*Suppose that a one dimensional system*

$$x_{n+1} = f(x_n, \alpha), x_n \in R, \alpha \in R$$

*with smooth $f$ We say that there is a pitchfork bifurcation at $(x^*, \alpha_c)$ if the following conditions are satisfied.*

**P1.** $f(x^*, \alpha) = x^*$ *and* $\frac{\partial^{2k}}{\partial x^{2k}} f(x^*, \alpha) = 0$ *for all* $\alpha$.

**P2.** $f_x(x^*, \alpha_c) = 1$.

**P3.** $f_{x\alpha}(x^*, \alpha_c) \neq 0$.

**P4.** $f_{xxx}(x^*, \alpha_c) \neq 0$.

**Theorem 2.1.3.** *(Transcritical bifurcation) Suppose that a one dimensional system*

$$x_{n+1} = f(x_n, \alpha), x_n \in R, \alpha \in R$$

*with smooth $f$. We say that there is a transcritical bifurcation at $(x^*, \alpha_c)$ if the following conditions are satisfied:*

**T1.** $f(x^*, \alpha) = x^*$ *for all* $\alpha$.

**T2.** $f_x(x^*, \alpha_c) = 1$.

**T3.** $f_{x\alpha}(x^*, \alpha_c) \neq 0$.

**T4.** $f_{xx}(x^*, \alpha_c) \neq 0$.

**Theorem 2.1.4.** *(Period-doubling bifurcation)*
*Suppose that a one dimensional system*

$$x_{n+1} = f(x_n, \alpha), x_n \in R, \alpha \in R$$

*with smooth $f$. We say that there is a transcritical bifurcation at $(x^*, \alpha_c)$ if the following conditions are satisfied:*

**PD1.** $f(x^*, \alpha_c) = x^*$

**PD2.** $f_x(x^*, \alpha_c) = -1$.

**PD3.** $f_{x\alpha}(x^*, \alpha_c) + \frac{1}{2} f_\alpha(x^*, \alpha_c) f_{xx}(x^*, \alpha_c) \neq 0$.

**PD4.** $\frac{1}{2}(f_{xx}(x^*, \alpha_c))^2 + \frac{1}{3} f_{xxx}(x^*, \alpha_c) \neq 0$.

### 2.1.2 Global bifurcations

Global bifurcation which often occur when larger invariant sets of the systems callide with each other, or with equilibria of the system. They cannot be detected purely by a stability analysis of the equilibria( fixed or equilibrium points). Global stability requires that the system converges to the steady from all points. The global bifurcation includes, for example, collision of a limit cycle with a saddle point, collision of a limit cycle with a node etc.

# Chapter 3

# Stability in a Discrete Prey-Predator Model

## 3.1 Discrete Lotka-Volterra model

Alfred James Lotka was a US mathematician, physical chemist, and statistician famous for his work in population dynamics and energetic. Vito Volterra was an Italian mathematician and physicist, best known for his contribution to mathematical biology. After world war1, Volterra turned his attention to the application of his mathematical ideas to biology, principally reiterating and developing the work of Pierre Francois Verhulst. The most famous outcome of this period is the Volterra-Lotka equations. One of the first models to incorporate interactions between predators and prey was proposed in 1925 by the American Biophysicist Alfred Lotka and the Italian mathematician Vito Voltrra. The Lotka-Volterra model describes interactions between two species in an ecosystem, a predator and a prey. This represents our first multi-species model. Since we are considering two species, the model will involve two equations, one which describes how the prey population changes and the second which describes how the predator population changes. A predator is an organism that eats another organism. The prey is the organism which the predator eats. Some examples of predator and prey are lion and zebra, bear and fish, and fox and rabbit. The words "predator" and "prey" are almost always used to mean only animals that eat animals, but the same concept also applies to plants: Bear and berry, rabbit and lettuce, grasshopper and leaf. Predator and prey evolve together. The prey is part of the predator's environment, and the predator dies if it does not get food.

 

The Lotka-Voltrra model is one of the earliest predator-prey models to be based on sound mathematical principles. It forms the basis of many models used today in the analysis of population dynamics. Unfortunately, in its original form Lotka-Volterra has some significant problems. Neither equilibrium point is Stable. Instead the predator and prey populations seem to cycle endlessly without setting down quickly. It can be shown that this behavior will be observed in nature, it is not overwhelmingly common. A small perturbation corresponding to the existence of a finite carrying capacity for the prey has qualitatively changed the phase portrait of the Lotka- Volterra model (a precise definition of what exactly is meant by small perturbation and qualitatively change of the phase portrait will be given when we study structural stability). A model whose qualitative properties dont change significantly when it is subjected to small perturbations is said to be structurally stable. Since a model is not a precise description of a system, qualitative predictions should not be altered by slight modifications. Satisfactory models should be structurally stable. We see that two species discrete-time predator-prey dynamical models have been extensively investigated. The main studied subjects are the dynamical behaviors, such as, the local and global stability of the equilibriums, the persistence, permanence and extinction.

## 3.2 Discrete predator-prey models on plant-herbivore interactions

Considering many research and studies in mathematical biology, we can say that discrete time models described by difference equations are more appropriate than the continuous time models when populations have non-overlapping generations. Especially, using discrete time models can also provide more efficient computational models for numerical simulations and these results reveal richer dynamics of the discrete models compared to the continuous ones. In this first attempt at modeling plant-herbivore interactions we will focus only on quantitative changes, i.e, changes in the biomass of the populations. To give structure to the problem, we make the following broad assumptions.

1. Herbivores have discrete generations that correspond to the seasonality of the vegetation.

2. The availability of vegetation and the current population density of herbivores are the main factors that determine fecundity and survivorship of the herbivores.

3. The abundance of the vegetation dependence on the extent of herbivore to which the plant was subjected in the previous season as well as on the previous biomass of the vegetation.

Consider the following discrete time predator prey system:

$$\begin{aligned} N_{t+1} &= N_t + rN_t(1 - N_t) - aN_tP_t \\ P_{t+1} &= P_t + aP_t(N_t - P_t) \end{aligned}$$

where $r$ and $a$ are positive constants and $N_t$ vegetation biomass in generation $t$ and $P_t$ number of herbivores in generation $t$.
Here the term $N_t + rN_t(1 - N_t)$ stands for the rate of the increase of the prey population in the absence of predator while the term $aN_tP_t$ represents the rate of decrease due to predation where the parameter $a$ is the predation parameter.
Finally the term $P_t+aP_t(N_t-P_t)$ stands for the variation of predator density with respect to the prey population. Notice that if the predator density disappears in this model, then the prey density satisfies the discrete logistic-type model.

### 3.2.1 Equilibrium points of the Model

In this section we obtain local stability conditions of the equilibrium points of model (*). Mainly, we obtain the conditions under which the equilibrium points of the predator-prey system are asymptotically stable. To get this, first observe that corresponding equilibrium points:

**I.** $E_0 = (0, 0)$ is extinction of both species.

**II.** $E_1 = (1, 0)$ is the axial steady state in the absence of predator.

**III.** $E_2 = (N_0^*, P_0^*)$ is the interior steady state, where

$$N_0^* = P_0^* = \frac{r}{a + r}$$

Thus $(N_0^*, P_0^*)$ is the unique positive equilibrium points of (*).

### 3.2.2 Dynamic behavior of the model

In this subsection, we investigate the local behavior of the model (*) around each steady state. The local stability analysis of the model (*) can be studied by computing the variation matrix corresponding to each steady state. The variation matrix of the model at state variable is given by

$$J(N_t, P_t) = \begin{pmatrix} 1 + r - 2rN_t - aP_t & -aN_t \\ aP_t & 1 + aN_t - 2aP_t \end{pmatrix}_{(N_0^*, P_0^*)}$$

For the equilibrium point $(0, 0)$, the corresponding characteristics equation is

$$\lambda^2 - 2\lambda + 1 = 0$$

and its roots are $\lambda_1 = \lambda^2 = 1$ that means $(0, 0)$ is not asymptotically stable. Usually, such a point is called a non-hyperbolic equilibrium point. Similarly, we have the characteristics equation at the point $(1, 0)$ as

$$\lambda^2(r - 2)\lambda + 1 - r = 0.$$

In this case since the roots are $\lambda_1 = 1$ and $\lambda_2 = 1 - r$, we see that $(1, 0)$ is also a non-hyperbolic equilibrium point of (*). We know focus on the positive equilibrium point $(N_0^*, P_0^*)$ where $N_0^* and P_0^*$ are given by (2.1). Linearizing the system (*) about $N_0^*, P_0^*$, we have the fallowing coefficient matrix:

$$\tau = \begin{pmatrix} 1 - \frac{r^2}{a+r} & -\frac{ar}{a+r} \\ \frac{ar}{a+r} & 1 - \frac{ar}{a+r} \end{pmatrix}$$

Obtaining the eigenvalues of the linearized system about $(N_0^*, P_0^*) = (\frac{r}{a+r}, \frac{r}{a+r})$ will not be practical. So we will be required to make the use of theorem 1.6.3 i.e $|tr\tau| < 1 + det\tau < 2$

So $tr\tau = 2 - r$ and $det\tau = 1 - r + \frac{ar^2}{a+r}$, then

$$|2 - r| < 2 - r + \frac{ar^2}{a + r} < 0$$

The positive equilibrium point $(N_0^*, P_0^*)$ of the predator prey system (1) is asymptotically stable if $2 - \frac{4}{r} < \frac{ar}{a+r} < 1$ holds and unstable if and only if $2 - \frac{4}{r} > \frac{ar}{a+r}$ or $\frac{ar}{a+r} > 1$ holds.

### 3.2.3 Numerical simulations

In this section, we give the numerical simulations to verify our theoretical results proved in the previous sections by using MATLAB programming. Mainly, we present the graphs of the solutions $N_t, P_t$ versus time. For the numerical simulations, we fix $a = 2$ then positive equilibrium points $(N_0^*, P_0^*)$ reduce to the following $N_0^* = P_0^* = \frac{2r}{2+r}$. Then $(N_0^*, P_0^*)$ is asymptotically stable if $0 < r < 2$ In the systems take $a = 2$ and initial conditions $N_0^* = 0.3, P_0^* 0.2$ we use 1.4 in fig. 3(a) and it shows the solution of predator prey system over time.fig. 3(b) shows bifurcation diagrams of predator and prey densities.

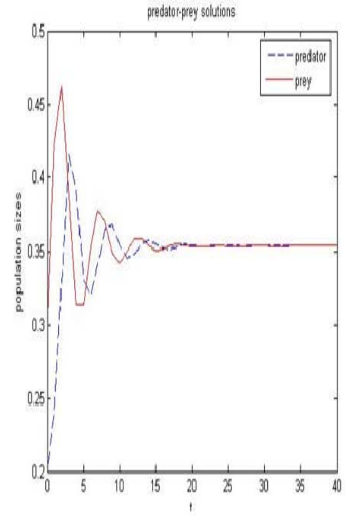


Figure 3(a): Solutions of predator-prey systems

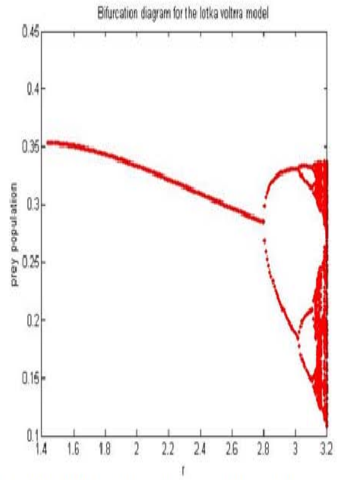


Fig. (3b): Bifurcation diagram of predator prey system

# Summary

In this paper, I consider a two dimensional nonlinear discrete-time prey-predator model and obtained equilibrium points. Numerical simulations are presented to show the dynamical behavior of the system. Finally, graphs of predator prey solutions and bifurcation diagrams are also presented.

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# Index

**Matlab code of figure 3(a)**

$r = 1.4$; $^0/_0$ input('input $r = prey$ reproductive rate:')
$a = 2$; $^0/_0$ input ('input $a =$ search efficiency of predator:')
$N0 = 0.3$; $^0/_0$ input ('input initial population $N0$ of prey :')
$P0 = 0.2$; $^0/_0$ input ('input initial population $P0$ of predator :')
$n = 40$; $^0/_0$ input ('input time period of run :')
$N = zeros(n + 1, 1)$;
$P = zeros(n + 1, 1)$;
$t = zeros(n + 1, 1)$;
$N(1) = N0$;
$P(1) = P0$;
for $i = 1 : n$
$t(i) = i - 1$;
$N(i + 1) = N(i) + r * N(i) * (1 - r * N(i)) - a * N(i) * P(i)$;
$P(i + 1) = P(i) + a * P(i) * (N(i) - P(i))$;
end
$t(n + 1) = n$;
plot $(t, P,' b - -', t, N,' r-')$
title ('predator-prey solutions');
xlabel($'t'$);
ylabel('population sizes')

**Matlab code of figure 3(b)**

$a = 2$; $^{0}/_{0}$ $a =$ search efficiency of predator

$rmin = 1.4$; $rmax = 3.2$; $^{0}/_{0}$ $r = prey$ reproductive rate

$N0 = 0.3$; $^{0}/_{0}$ initial population $N0$ of prey

$P0 = 0.2$; $^{0}/_{0}$ initial population $P0$ of predator

$n = 1000$;

$jmax = 200$;

$t = zeros(jmax + 1, 1)$;

$z = zeros(jmax + 1, 250)$;

$del = (rmax - rmin)/jmax$;

for $j = 1 : jmax + 1$

$N = zeros(n + 1, 1)$;

$P = zeros(n + 1, 1)$;

$N(1) = N0$;

$P(1) = P0$;

$t(j) = (j - 1) * del + rmin$;

$r = t(j)$;

for $i = 1 : n$

$N(i + 1) = N(i) + r * N(i) * (1 - r * N(i)) - a * N(i) * P(i)$;

$P(i + 1) = P(i) + a * P(i) * (N(i) - P(i))$;

if $(i > 750)$

$z(j, i - 750) = N(i + 1)$;

end;

plot $(t, z,' r.',' MarkerSize', 6)$

xlabel ('r','FontSize',10), ylabel('prey population','FontSize',10)

title ('Bifurcation diagram for the lotka-volterra model')