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APPLICATION OF MAXIMUM PRINCIPLES IN MICROECONOMICS

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The undersigned here by certify that they have read and recommend to the school of graduate studies for acceptance of a project entitled **Application of Maximum Principles in Microeconomics** by Fikadu Tekle in partial fulfillment of the requirements for the Master of Science degree in Mathematics.

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Table of contents

Contents **page**

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Abstract

In this paper we use Discrete and Analytic maximum principle to show uniqueness of solution of Dirichlet problem. We also show application of maximum principle in showing uniqueness and stability of solution of heat equation. Finally, we also apply it in finding optimum utility function under a given budget constraint in Microeconomics.

Introduction

The maximum principle is one of the most useful and best known tools employed in the study of differential equation in n-dimensions ($n \geq 1$). It enables us to obtain information about uniqueness and approximation of the solution of Initial value or boundary value problems like Dirichlet problem and heat equation.

The sheer mathematical settings of economic theory of the maximum principle can be regarded as an economic counterpart of formulations in the physical sciences, as for instance theoretical mechanics. Detail understanding of the economics content of mathematical techniques like the maximum principle rests on the understanding of basic methods of calculus (at least) and methods of calculus of variations (at best). A prototype of this is the wealth and income version of the maximum principle. In this avenue, the pioneering work of Pontryagin and his co-researchers, namely, V. G. Boltyanskii, R. V. Gamkrelidze and E.F. Mishchenko takes the central place.

Roughly, the Pontryagin's Maximum Principle can be paraphrased as follows,

Let $(x;u)$ be a trajectory controlled by u over the time interval $[t_0 , t_1]$. If $(x;u)$ is optimal then there exists a constant ª_0 and a convector $\text{ª}:[t_0, t_1] \to R^n$ such that the following conditions are satisfied.

- **i)** $(\text{ª}_0, \text{ª}(t)) \neq O$; $\forall t \in [t_0 , t_1]$
- **ii)** $H(\text{ª}_0, \text{ª}(t), x(t); u(t)) = \min_{v \in U} H(\text{ª}_0, \text{ª}(t), x(t); v(t))$

where $h : t \mapsto H(\text{ª}_0, \text{ª}(t), x(t), u(t)$ is a continuously differentiable function with derivative,

$$\frac{dh}{dt} = \frac{\partial H}{\partial t}(\text{ª}_0, \text{ª}(t), x(t), u(t)$$

Here, we recall a couple of things pertaining to derivatives of a scalar field and a vector field,

1) If $h: R^n \to R^1$ is differentiable then the gradient of h is given by

$$\nabla h(x) = \frac{\partial h}{\partial x} = (\tfrac{\partial h}{\partial x_1}, \dots, \tfrac{\partial h}{\partial x_n})$$

2) If $H: R^n \to R^p$ is differentiable, then the Jacobian of H is given by

$$DH(x) = \frac{\partial H}{\partial x} = \begin{pmatrix} \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \vdots & \frac{\partial h_i}{\partial x_i} & \vdots \\ \frac{\partial h_p}{\partial x_1} & \cdots & \frac{\partial h_p}{\partial x_n} \end{pmatrix}_{1 \le i \le n}$$

3) If $h: R^n \to R^1$ is twice differentiable then the Hessian is simply,

$$D^2 h(x) = \frac{\partial^2 h}{\partial x^2}(x)$$

This thesis is organized as follows, in chapter I, we shall refresh basic facts and notions that are useful for understanding the Maximum Principle in the general setting. In chapter II we overview variants of the maximum and shade light on its use by building the classical solution of the Dirichlet problem as a limit of discrete ones. The Dirichlet problem is:

$$\begin{aligned} \Delta u(x) &= 0, x \in \Omega \\ u(x) &= f(x), x \in \partial\Omega \end{aligned} \qquad (1.1)$$

Where $\Omega \in \Re^2$ is open and bounded, and $f \in C(\partial\Omega, \Re)$. A function satisfying $\Delta u = 0$ on some Ω is called harmonic on that set. The solution of problem (1.1) describes the distribution of temperature for a steady heat flow on a plate of shape Ω, when the temperature at the boundary of the plane is constrained to be given by f. In chapter III we discuss heat transfer problem in 1-dimension along with the pertinent maximum principle and associated properties. Finally, in chapter IV we discuss economics perspective of the maximum principle.

Chapter 1

1. Preliminaries

1.1. Definitions

Definition 1.1.1

Let M be a set in $\Re^n$, M is open if for all elements ¡ in M one can find a ball around it which lies completely in M. This implies that an open set M has no points on its boundary, since we cannot make a ball around a point on the boundary without getting points out side of the set. No matter how close an element is to the boundary, however, a ball can always be made around it with all its points inside the set.

Definition 1.1.2 Let M be a set in $\Re^n$, M is called closed if the complement to M is open.

Definition 1.1.3 A partial differential equation is an equation that involve unknown function

$$u : \Omega \to \Re$$

along with its derivatives. Where Ω is an open sub set of $\Re^n$, $n \geq 2$ (or, more generally, of a differentiable manifold of dimension $n \geq 2$).

Example The Laplace equation

$$\Delta u = \sum_{i=1}^{n} u_{x_i x_i} = 0$$

(Δ is called the Laplace operator) or, more generally, the Poisson equation

$$\Delta u = f$$

for a given function $f : \Omega \to \Re$.

1.2. Harmonic Functions

Definition 1.2.1

A function $u \in C^2(\Omega)$ is called harmonic in Ω if $\Delta u = 0$ in Ω.

1.2.2 Property of harmonic function

Roughly speaking, the mean–value property states that the value of a harmonic function u at a point x_0 equals the average of u on a sphere centered at x_0. Let's take a closer look through examples. If we take the function

$$y = ax + b$$

as an example in one dimension, we have

$$y'' = 0 .$$

Obviously, y is harmonic. Since the domain is one dimensional a "sphere" around a point x_0 is just two points at equally spaced distances from x_0 on the $x-$axis. Let h be the distance between x_0 and one of these points. If the mean-value property holds for y the following equality should then be true, namely

$$y(x) = \frac{y(x+h) - y(x-h)}{2}$$

Indeed, the mean-value property holds for balls as well, so that the value of function u at a Point x_0 equals to the mean-value of u over a ball centered at x_0. This would in our example means that

$$\frac{1}{2h}\int_{x_0-h}^{x_0+h} y(z)dz = y(x_0) .$$

In what follows, we shall formally state and sketch the proof of mean-value theorem in n-dimensions.

Theorem (The mean-value property)

Let M be an open set in $\mathfrak{R}^n$ and let u be harmonic in M. Also let $V(r)$ be the volume of a ball with radius r and $B(x,r)$ be a ball with center at a point x and radius r. The boundary of $B(x,r)$ denoted $\partial B(x,r)$ is the sphere of radius r centered at x. But then,

$$u(x) = \frac{1}{A(B)}\int_{\partial B(x,r)} u(y)dS(y) = \frac{1}{V(B)}\int_{\partial B(x,r)} u(y)du$$

Where, dS is surface measure and du is the volume measure.

<u>**Proof**</u> The proof is divided in to two parts where the property is shown for spheres in step one and for balls in step two. Keeping the earlier notations, let ¤ be the mean value of u on a sphere of radius r centered at x,

$$\text{¤}(x) = \frac{1}{A(B)}\int_{\partial B(x,r)} u(y)dS(y)$$

We want to show that

$$\text{¤}(r) = u(x).$$

This is done by first showing that

$$\lim_{x\to 0} \text{¤}(x) = u(x)$$

To do this we take advantage of u being harmonic and using one of Green's formulas, for which a definition is needed.

Definition 1.2.2 Let ∇ denote an operator with $\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_n} \right)$ for a function u of n variables.

We start by differentiating ¤ but in order to do so we do not want r in the limits of the integral, and we also have that y depends on r. We, therefore do the substitution

$$y = x + rz$$

for a vector z varies in the same extent as y which also makes u a function of z. The mean value of u over these spheres will be the same. Observe that the new model has a new area which has to divide by to get the mean value.

$$\frac{1}{A(B)} \int_{\partial B(x,r)} u(y) dS(y) = \frac{1}{A(B)} \int_{\partial B(x,r)} u(x + rz) dS(z)$$

We can differentiate ¤ under the integral sign and according to the chain rule we differentiate

$$u(x + rz)$$

with r as variable as follows,

$$\frac{d}{dr} u(x + rz) = \left(\frac{\partial u}{\partial (x + rz_1)}, \frac{\partial u}{\partial (x + rz_2)}, ... \frac{\partial u}{\partial (x + rz_n)} \right) . (z_1, z_2, ... z_n)$$

$$= \nabla u(x + rz) . Z$$

That means we have,

$$¤'(x) = \frac{1}{A(B)} \int_{\partial B(x,r)} \nabla u(x + rz) z dS(y).$$

Since $y = x + rz$, returning to our original ball $B(x, r)$ leads to,

$$¤'(r) = \frac{1}{A(B)} \int_{\partial B(x,r)} \nabla u(y) . \frac{y - x}{r} dS(y)$$

The outer unit normal vector v to B is Z. The scalar product between ∇u and the normal v to the domain equals the normal derivative (see [3], Evans),

$$\nabla u . v = \frac{\partial u}{\partial v}.$$

Thus, $$¤'(r)=\frac{1}{A(B)}\int_{\partial B(x,r)}\frac{\partial u(y)}{\partial v}dS(y) \qquad (1.1)$$

According to Green's formula, see $[3]$, the following applies for M ,

$$\int_{\partial M}\frac{\partial u(y)}{\partial v}dS(y)=\int_{M}\Delta u(y)dy\ . \qquad (1.2)$$

Inserting (1.2) into (1.1) gives, $$¤'(x)=\frac{1}{A(B)}\int_{B(x,r)}\Delta u(y)dy \qquad (1.3)$$

However, $\Delta u = 0$ in M (by assumption)

And as a result $¤(r)$ is constant, this implies that the mean value of the ball is the same no matter the radius. We can therefore imagine a new radius t which approaches 0. But the mean value of a continuous function over sphere with center at x will approach the value at x when the radius goes toward 0, i.e.

$$¤(r)=\lim_{t\to 0}^{¤(t)}=\lim_{t\to 0}^{\frac{1}{A(t)}}\int_{\partial B(x,t)}u(y)dS(y)=u(x)$$

We have proved that the mean value of a harmonic function over every sphere around any given point equals the value at the center. What remains to be shown is that the same principles applies for ball,

$$u(x)=\frac{1}{v(r)}\int_{B(x,r)}u(y)dy\ .$$

A ball with radius r can be seen as the sum of spheres with radius r and less. This means that if we let s be a new radius that grows from 0 to r and sum the value of our function u over all of these spheres we will have a sum of the value of u over a whole ball with radius *r*,

$$\int_{B(x,r)}u(x)dy=\int_0^r\left(\int_{\partial B(x,s)}u(y)dS(y)\right)dy \qquad (1.4)$$

But we had

$$u(x)=¤(s)=\frac{1}{A(s)}\int_{\partial B(x,s)}u(y)dS(y)$$

$$\Rightarrow \int_{\partial B(x,s)}u(y)dS(y)=u(x)A(s).$$

Inserting this in to (1.2) gives

$$\int_{B(x,r)} u(y)dy = \int_0^r u(x)A(s)ds \ ,$$

Which since $u(x)$ does not depend on S equals

$$\int_{B(x,r)} u(y)dy = \int_0^r u(x)A(x)ds = u(x)\int_0^r A(s)ds = u(x)(v(r) - v(0)) = u(x)(v(r)),$$

dividing by $v(r)$ finally gives

$$u(x) = \frac{1}{v(r)} \int_{B(x,r)} u(y)dy \, .$$

And thus proving the fact that mean-value property holds for harmonic function is done.

1.3 Discretization of Dirichlet problem

Consider the Dirichlet Problem for the Laplace equation in the unit square, $\Omega = (0,1)^2$ **,**

$$\begin{cases} \Delta u(x) = 0 \, , x \in \Omega \\ u(x) = f(x) \, , x \in \partial\Omega \end{cases} \qquad (1.5)$$

Where $f \in C(\partial\Omega, R)$. If we divide the unit square using equally spaced gird lines, where the distance from each other equal to

$$h = \frac{1}{n}, n \in N \setminus \{0\},$$

then we obtain the following discretization of $\bar{\Omega}$:

$$\bar{\Omega} = \{0, h, 2h, ..., (n-1)h, 1\}^2$$

The discrete analogue of the boundary and interior Ω is given by:

$$\partial\Omega_h = \bar{\Omega}_h \cap \partial\Omega \quad \textbf{and} \quad \partial\bar{\Omega}_h = \bar{\Omega}_h - \partial\Omega_{h.} \, .$$

To discretize Δ, just replace the second derivatives with the corresponding center.

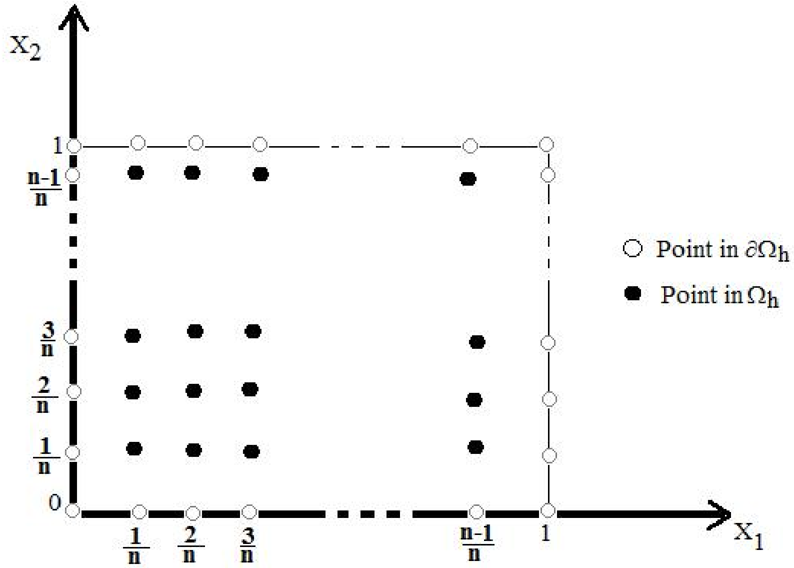


Figure 1 : Discretization of $(0,1)^2$

Difference quotients, and get:

$$\Delta_h u_h(x) = \frac{1}{h^2}\left(u_h(x_1+h,x_2)+u_h(x_1-h,x_2)+u_h(x_1,x_2+h)+u_h(x_1,x_2-h)-4u_h(x_1,x_2)\right).$$

This leads to the discritized problem:

$$\Delta_h u_h(x) = 0 \text{ if } x \in \Omega_h,$$

$$u_h(x) = f(x) \text{ if } x \in \partial\Omega_h \qquad (1.6)$$

Which is equally a family of discretized, indexed in $h = \frac{1}{n}, n \in N \setminus \{0\}$,each one leading to a unique solution, u_h. Since problem (1.6) is a system of linear equations. It follows from the discrete maximum principle on page 13 that the problem has a unique solution. Nonstandard analysis provides a very appropriate frame work and deal with convergence of the $u_h(as \quad n \to \infty, or\, h \to 0)$. By considering u_h,where h is positive infinitesimal, we are already looking at problem (1.6) “in the limit “ that means, in a situation where u_h is infinitely close to the solution, u ,of (1.5).

Chapter 2
Overview of the Maximum Principles

2.1 Approximate Maximum Principles

Maximum principles will constitute a key tool in this chapter. We begin by the discrete version.

Theorem 2.1 (Discrete maximum principles)

Let $h>0$. If $u;\bar{\Omega} \to \Re$ satisfies $\Delta_h u(x) \geq 0 \ \ \forall x \in \Omega_h$, then

$$\max_{x \in \bar{\Omega}_h} u(x) = \max_{x \in \Omega_h} u(x)$$

<u>Proof</u> Let $M = \max_{x \in \partial\Omega_h} u(x)$. Assume the maximum of u occurs at some interior points, say $(x_1, x_2) \in \Omega_h$, for all elements of $\bar{\Omega}$ of the form $(x_1 + nh, x_2), n \in N$, such that all points of the form $(x_1 + ih, x_2)$, $0 \leq i \leq n$, are in Ω_h, we show by induction on n that $u(x_1 + nh, x_2) = M$. For $n = o$,the result is given by hypothesis. Now, suppose $u(x_1 + nh, x_2) = M$, If $(x_1 + nh, x_2)$ $\in \partial\Omega_h$, we are done. If not, that is, if $(x_1 + nh, x_2) \in \Omega_h$, then, using the induction hypothesis

$$0 \leq h^2 \Delta_h U(x_1 + nh, x_2)$$
$$= U(x_1 + (n-1)h, x_2) + U(x_1 + (n+1)h, x_2) + U(x_1 nh, x_2 + h) + U(x_1 + nh, x_2 - h) - 4U(x_1 + nh, x_2)$$

$$= U(x_1 + (n-1)h, x_2) + U(x_1 + (n+1)h, x_2) + U(x_1 nh, x_2 + h) + U(x_1 + nh, x_2 - h) - 4M$$

$$\leq M + U(x_1 + (n+1)h, x_2) + M + M - 4M$$

$$= U(x_1 + (n+1)h, x_2) - M$$

Therefore, $U(x_1 + (n+1)h, x_2) = M$. Now, for the least $n \in N$ such that $(x_1 + nh, x_2) \in \partial\Omega_h$ (which exists because Ω is bounded), $U(x_1 + (n+1)h, x_2) = M$. Hence, we have just shown that, if maximum of U occurs in Ω_h, then it must occur also in $\partial\Omega_h$.

Corollary 2.2 (Discrete minimum principle)

Let $h > 0$. If $u; \bar{\Omega}_h \to \Re$ satisfies $\Delta_h u(x) \leq 0 \ \forall x \in \Omega_h$ then

$$\min_{x \in \bar{\Omega}_h} u(x) = \min_{x \in \partial\Omega_h} u(x)$$

<u>Proof:</u> Apply discrete maximum principle to the function $u = -v$

Corollary 2.3 Let $h. > 0$. If $u : \bar{\Omega} \to \Re$ satisfies $\Delta_h u(x) = 0 \ \forall x \in \Omega_h$, then

$$\max_{x \in \bar{\Omega}_h} u(x) = \max_{x \in \partial\Omega_h} u(x) \quad \text{and}$$

$$\min_{x \in \bar{\Omega}} u(x) = \min_{x \in \partial\Omega_h} u(x)$$

Corollary 2.4 Let $h. > 0$. If $u : \bar{\Omega}_h \to \Re$ satisfies

$$\Delta_h u(x) \geq 0 \qquad \forall x \in \Omega_h,$$

$$u(x) \leq a \qquad \forall x \in \partial\Omega_h,$$

For some $a \in \Re$, then $\quad u(x) \leq a \qquad \forall x \in \bar{\Omega}_h$.

<u>Proof</u> By discrete maximum principles, $\max_{x \in \bar{\Omega}h} u = \max_{x \in \partial\Omega h} u \leq a$.

The discrete maximum principle can be used in comparison function arguments. These work as follows. Let $u : \bar{\Omega}_h \to \Re$ be a solution of (1.6), and suppose that, for some wisely chosen

$$v : \bar{\Omega}_h \to \Re, \ x \in \Omega_h \ \Delta h(u(x) - v(x)) = -\Delta hv(x) \geq 0, i.e \Delta hv(x) \leq 0, \quad \text{and}$$

$$x \in \partial\Omega_h \Rightarrow u(x) - v(x) = f_h(x) - v(x) \leq 0, i.e \ \ v(x) \geq f_h(x).$$

Then, $\forall x \in \bar{\Omega}_h$, $u(x) - v(x) \leq 0$

So $u(x) \leq v(x)$.

Many variations of this argument can be used to get bounds for solution, u of (1.6), without knowing its form, or even its exists . This is why these spots of estimates are a priori bound. We will now apply the discrete maximum principle to show that the discrete problem (1.6) has unique solutions. First note that (1.6) is no more than a system of linear equations. In fact, the set of grid functions with domain $\bar{\Omega}_h$, with the usual(point wise) sum and scalar product, is a vector space of finite dimension , and its dimension equals the number of elements of $\bar{\Omega}_h$. So, (1.6) consists exactly of $\left|\bar{\Omega}_h\right|$ equation in $\left|\bar{\Omega}_h\right|$ unknowns.

Example 2.5 Let $\Omega = (0,1)^2$, and $h = \frac{1}{n}$ (see Fig.1 on page 7).

Then $\left|\bar{\Omega}_h\right| = (n+1)^2$, $\left|\partial\Omega_h\right| = 4n$, and $\left|\Omega_h\right| = (n-1)^2$ .The dimension of $\Re^{\bar{\Omega}_h}$ is $(n+1)^2$.

Lemma 2.6 Let $h > 0$. Let $f : \partial\Omega \to \Re$. Then the problem (1.6) has a unique solution

$$u : \bar{\Omega}_h \to \Re .$$

<u>Proof</u> Write $\bar{\Omega}_h = \{x_{i,j} : (i,j) \in I \}$ where $I = \{(i,j) \in z^2 : x_{i,j} \in \bar{\Omega}_h \}$. Well order I, and set $v_k = u(x_{i,j})$, where (i,j) is the k'th element of I. We know $1 \le k \le \left|\bar{\Omega}\right|$.Then, equation (1.6)
Setup a system of linear equation:

$$AV = b.................................(2.1)$$

The vector b comes from right hand side of (1.3), so its entries are ether 0 or $f_h(x)$, for some $x \in \partial\Omega_h$. A is a $\left|\bar{\Omega}_h\right| \times \left|\Omega_h\right|$ matrix . To show that (2.1) has a unique solution it is enough to show that the linear map,

$$\Re^{\bar{\Omega}_h} \ni V \mapsto AV \in \Re^{\bar{\Omega}_h} \quad \text{is injective .}$$

Suppose $AV = 0$. From (1.6), this means that $f_h(x) = 0$, for all $x \in \partial\Omega_h$. By discrete maximum principle, $V = 0$ .We now work in a superstructure $(V(\Re), {}^*V(\Re), *)$. We will omit the stars on all standard functions of one or several variables and usual binary relations. Each finite $x \in {}^*\Re$ can be uniquely decomposed as $x = r + £$,where $r \in \Re$ and $£$ is an infinitesimal; r is called standard part of x ,and denoted by $st \quad x$ .If $x - y \in {}^*\Re$ such that $x - y$ is infinitesimal ,then we say that x is infinitesimally close to y ,and write $x \approx y$ .Similarly , if $x, y \in {}^*\Re^n$:

$x \approx y$ iff $|x-y|=0$ iff $x_i = y_j$ for each $i=1,2,3,...,n$

If $x \in {}^*\Re^n$ is finite then let:

$^0x = {}^0(x_1, x_2, x_3, ..x_n) = (stx_1, ..., stx_n)^4$. For other function

$F: A \subset {}^0\Re^n \to {}^0\Re^m$, denoted by ^0F by:

$$^0F({}^0x) = {}^0F(x) \quad \forall x \in A$$

For sets $A \in \Re^n$, $^0A = \{{}^0x : x$ is finite and $x \in A\}$

Each circle map as introduced above is sometimes called a standard map. Whenever $h \approx 0$, the sets $\bar{\Omega}_h$, $\partial\Omega_h$, and Ω_h will be internal sets; also, f_h will be an internal function. In this chapter, we stick to the convention that objects subscripted by $h \approx 0$ will be internal subsets of $\Re_h^2$ or grid functions .The capital letters u, v, ,... will preferably be used to designate internal grid functions . Since we will sometimes need to designate standard elements of some other sets, we will not omit the stars on sets. By transfer, the size of $\bar{\Omega}_h$ will be some hyper integer $N \in {}^*N$ .Since Ω is open, whenever h is infinitesimal, N will be infinitely large. Recall the construction of f_h. By construction $\left|\left(\bar{a}, \bar{b}\right) - (a,b)\right| \leq h$, so if $h \approx 0$ and f is continuous, then for all $x \in \partial\Omega_h$, $^0f_h(x) = f\ ({}^0x)$. By transfer of Lemma (2.6) and internal definition principle, we obtain:

Lemma 2.7 Let $h \approx 0$. Let $f : \partial\Omega \to {}^*\Re$. Then the problem (1.6) has a unique solution.
We now turn to the approximate version of maximum principle.

Definition 2.8 The relations $\leq$ and $\geq$ are defined in $^*\Re^2$ by:

$$a \leq b \Leftrightarrow a < b \vee a \approx b$$
$$a \geq b \Leftrightarrow b \leq a$$

Lemma 2.9 (Approximate Maximum Principle)

Let $h. > 0$, $h \approx 0$. Let $u : \bar{\Omega}_h \to {}^*\Re$ be Internal and $a \in \Re$. Suppose : $\begin{matrix} \Delta_h u(x) \geq 0, \forall x \in \Omega_h \\ u(x) \leq a, \forall x \in \partial\Omega_h \end{matrix}$ then , $u(x) \leq a, \forall x \in \bar{\Omega}_h$.

Proof We use a comparison function argument. Fix some $\bar{x} = (x_1, x_2) \in \Omega_h$. For each $c \in \Re^+$

$v(x) = u(x) + w(x), \forall x \in \Omega_h$, where $w(x) = c\left|x - \bar{x}\right|^2$. Let $\Re \in \Re^+$ be such that

$B_R\left(\bar{x}\right) \supset \bar{\Omega}$. Computing the Laplacian of the (standard) function w yields:

$$\Delta w(x_1, x_2) = \left(\frac{\partial^2}{\partial x_1^{\ 2}} + \frac{\partial^2}{\partial x_2^{\ 2}}\right)\left(c\left(x_1 - \bar{x}_1\right)^2 + c\left(x_2 - \bar{x}_2\right)^2\right) = 4c$$

From our hypothesis on u and the differentiability of w:

$$\Delta_h v(x) = \Delta_h u(x) + \Delta_h w(x) \geq 0 + 4c > 0$$

Hence $\Delta_h v(x) > 0$.

On the other hand ,if $x \in \partial\Omega_h$ then,

$$v(x) = u(x) + c\left|x - \bar{x}\right|^2 \leq u(x) + c\Re^2 \leq a + c\Re^2 \leq a + 2c\Re^2$$

By transfer of discrete maximum principle,

$$v(x) \leq a + 2c\Re^2, \forall x \in \bar{\Omega}_h .$$

Hence, $$u(x) \leq v(x) \leq a + 2c\Re^2, \forall x \in \bar{\Omega}_h .$$

Since c was arbitrary chosen in $\Re^+$,we get that

$$u(x) \leq a, \forall x \in \bar{\Omega}_h .$$

Theorem 2.10 (Approximate Minimum Principles)

Let $h \approx 0$. If $u : \bar{\Omega}_h \rightarrow^* \Re$ in an interval and, $\Delta_h u(x) \leq 0, \forall x \in \Omega_h$:

$u(x) \geq 0, \forall x \in \partial\Omega_h$, then $u(x) \geq 0, \forall x \in \bar{\Omega}_h$

Proof Apply the approximate maximum principle to $-u$.

Theorem 2.11 (Analytical Maximum Principle) Let $\Omega \in \Re^2$ be bounded and open, and consider $u \in c^2(\Omega, \Re) \cap c\left(\bar{\Omega}, \Re\right)$. If $a \in \Re$, and $\begin{matrix} \Delta_h u(x) \geq 0, \forall x \in \Omega_h \\ u(x) \leq a, \forall x \in \partial\Omega_h \end{matrix}$ then $u(x) \leq a, \forall x \in \bar{\Omega}_h$.

Proof Let $h \approx 0$,and condsider $\bar{\Omega}, \Omega_h, \partial\Omega_h$ and Δ_h as introduced before. Since u is c^2 in $\Omega, \Delta_h u(x) \approx \Delta u(x) \geq 0$ at every x such that $dist(x, \partial\Omega) \neq 0$. Consider the set:

$$E = \{¢ \in^* \Re^+ : \forall x \in \Omega_h, \forall y \in^* \partial\Omega |x - y| > ¢ \Rightarrow \Delta_h U(x) > -¢ \}.$$

E is internal and includes all $¢ \neq 0$. Hence, it must contain some positive $¢ \approx 0$. Consider

$$\bar{\Omega}^{¢} = \left\{ x \in \bar{\Omega}_h : \forall y \in *\partial\Omega |x - y| \geq ¢ \right\},$$

and let

$$\Omega_h^{¢} = \left\{ x \in \bar{\Omega}_h^{¢} : x \text{ has four neighbors } \right\},\ \partial\bar{\Omega}_h^{¢} = \bar{\Omega}_h^{¢} - \Omega_h^{¢}.$$

Then $\Delta_h U(x) > -¢ \approx 0, \forall x \in \Omega_h^{¢}, U(x) \approx U({}^{\circ}x) \leq a, \forall x \in \partial\Omega_h^{¢}$

Note that $*dist(\partial\Omega_h^{¢}, {}^*\partial\Omega) \approx 0$, so ${}^{\circ}x \in \partial\Omega$, $\forall x \in \partial\Omega_h^{¢}$. By the approximate maximum principles, we conclude that $U(x) \geq a, \forall x \in \bar{\Omega}_h^{¢}$. Since ${}^{\circ}\left(\bar{\Omega}_h^{¢}\right) = \bar{\Omega}$, this implies our result.

2.2 Uniqueness of solution to classical Dirichlet problem

We revisit our Dirichlet problem (1.1)

$$\begin{aligned} \Delta u(x) &= 0, x \in \Omega \\ u(x) &= f(x), x \in \partial\Omega \end{aligned}$$

Corollary 1.12 Let $\Omega \in \Re^2$ be bounded and open and $f \in c(\partial\Omega, \Re)$,

Then, $\begin{aligned} \Delta u(x) &= 0, x \in \Omega \\ u(x) &= f(x), x \in \partial\Omega \end{aligned}$ has no more than one solution in $c^2(\Omega, \Re) \cap c\left(\bar{\Omega}, \Re\right)$

Proof **Let** $w \in C^2(\Omega, \Re) \cap C\left(\bar{\Omega}, \Re\right)$ be another solution. Then $\Delta(w - u) = 0$ in Ω and $(u - w)(x) = f(x) - f(x) = 0$ on $\partial\Omega$. By the analytic maximum principle (applied to $w - u$ and $u - w$) we conclude that $w = u$ in $\bar{\Omega}$.

Chapter -3

One Dimensional Heat Transfer

In this section we will discuss about maximum principle and uniqueness of the solution of the heat equation.

3.1 Maximum principle

Theorem 1 If $u\ (x,t)$ satisfies the heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

on the rectangle $(x,t)\ \epsilon\ \ [0,l] \times [0,T]$

Then the maximum value of $u(x,t)$ is equal to either the maximum value at a time t=0 or the maximum value on either of the lateral sides $x = 0$ or $x = l$.

Proof

To prove the principle we will look at new function $\mathrm{v}(x,t)$ that is a slight perturbation of the original function $u(x,t)$.

We define $\mathrm{v}(x,t) = u\ (x,t) + \epsilon x^2$ for $\epsilon > 0$.

Since $u(x,t)$ is a continuous function it must obtain it's maximum on the closed in the rectangle $[0,l]\times[0,T]$. We donate the maximum value in the rectangle by M. It follows

$$\mathrm{v}(x,t) \leq M + \epsilon x^2 \text{ when t=0 and } x = 0 \text{ or } x = l \qquad (1)$$

Next we plug V in to the differential heat equation and from the definition of u and v

$$\frac{\partial v}{\partial t} - c^2\frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial t}\left(u + £x^2\right) - c^2\frac{\partial^2}{\partial x^2}\left(u + £x^2\right)$$

$$\frac{\partial u}{\partial t} - c^2\frac{\partial^2 u}{\partial x^2} - 2£c = -2£c < 0 \qquad (2)$$

The inequality $\frac{\partial v}{\partial t} - c^2 \frac{\partial^2 v}{\partial x^2} = -2£c < 0$ is essential to our proof and will be referred to as the heat inequality. Next we use calculus combined with this inequality to prove that the maximum V can neither occur on the interior of the rectangle nor on the top edge of the rectangle t =T.

Suppose maximum of V occurs on the interior of the rectangle at (a, b). Then

$$\frac{\partial v}{\partial x}(a,b) = 0 \quad , \quad \frac{\partial v}{\partial t} \text{ (a, b)} = 0, \ \frac{\partial^2 v}{\partial x^2} \text{ (a, b)} \leq 0 \tag{3}$$

But this contradicts the heat inequality (2) so the maximum cannot occur at interior point. Next we consider the top of the rectangle when t=T. In this case by the same reasoning as before

$$\frac{\partial v}{\partial t} \ (a,T) = 0 \text{ and } \frac{\partial^2 v}{\partial x^2} \text{ (a, T)} \leq 0$$

In order for v to have a maximum on the boundary of the rectangle, at the point (a, T) we must have

$$\text{v(a, T)} > \text{v (a, T -h)}, \ \forall \text{ h} > 0 \tag{4}$$

Then by the definition of derivative

$$\frac{\partial v}{\partial t} = \lim_{h\to 0^+} \frac{\text{V (a,T)} - \text{V (a,T -h)}}{h} \geq 0 \tag{5}$$

But this implies $\frac{\partial v}{\partial t} - c^2 \frac{\partial^2 v}{\partial x^2} \geq 0$ which contradicts the heat inequality. Since V is continuous function on a compact rectangle it can only attain its maximum on the edge when t=0 or when x=0 or $x = l$ by definition v

$$\text{v } (x,t) = \text{u } (x,t) + \epsilon x^2 \leq M + \epsilon l^2 \tag{6}$$

$$\text{u } (x,t) \leq M + \epsilon(l^2 - x^2), \ \forall \epsilon > 0 \tag{7}$$

Then by taking epsilon arbitrarily small we prove

$$\text{u } (x,t) \leq M \tag{8}$$

The maximum of u (x,t) is obtained on either the bottom side of the rectangle when t=0 or on the edge of the rectangle when x= 0 or x= l.

Theorem 2 If $u(x,t)$ satisfies the heat equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ on the rectangle $(x, t) \in [0, l] \times [0, T]$ then the minimum value of $u(x,t)$ is equal to either the minimum value at the t= 0 or the minimum value on either the lateral side x= 0 or x= l.

Proof from (8), we have $u(x,t) \le M$, when t= 0 or x= 0 or x= l

Then $-M \le -u(x,t)$, put $-M$=m (9)

We have $m \le -u(x,t) \le u(x,t) \le M$ (10)

$$m \le u(x,t) \tag{11}$$

The minimum of $u(x,t)$ is obtained on either the bottom side of the rectangle when t= 0 or on the edge of the rectangle when x= 0 or x= l.

3.2 Uniqueness of the solution to heat equation

To show that there is at most one solution to the heat equation. Consider heat equation

$$\frac{\partial u}{\partial t} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \qquad \text{For } x \in (0, l) \text{ and } t > 0 \tag{12}$$

With boundary conditions and initial condition

$$u(0, t) = g(t),\ u(l, t) = h(t) \tag{13}$$

$$u(x, 0) = \emptyset(x) \tag{14}$$

Assume that $y(x, t)$ and u(x, t) are two solutions to the problem and define $w = y$ -u. By linearity we know w must also satisfy the homogenous heat equation, in particular it must satisfy the maximum principle. Plugging w in to the differential heat equation

$$\frac{\partial w}{\partial t} - c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial y}{\partial t} - c^2 \frac{\partial^2 y}{\partial x^2} - \frac{\partial u}{\partial t} + c^2 \frac{\partial^2 u}{\partial x^2} = 0 \tag{15}$$

Furthermore we know

$$w(x, 0) = y(x, 0) - u(x, 0) = \emptyset(x) - \emptyset(x) = 0$$

$$w(l, t) = y(l, t) - u(l, t) = h(t) - h(t) = 0 \text{ and}$$

$$w(0, t) = y(0, t) - u(0, t) = g(t) - g(t) = 0$$

We have

$$w(x, 0) = 0,\ w(0, t) = 0,\ w(h, t) = 0 \tag{16}$$

By the maximum principle, the maximum of w must be equal to the maximum of w on the bottom of the rectangle on the edges which implies $(x, t) \leq 0$. By the same reasoning the minimum principle implies that $u\ (x, t) \geq 0$ so w must be identically zero.

Therefore $y\ (x, t) = u\ (x, t)$ if $t \geq 0$.

3.3 Stability of the solution to heat equation

Definition: The solution to a PDE is uniformly stable if u_1 and u_2 are to solutions to the PDE in the interval $[0, l]$ satisfying the initial conditions $u_1\ (x, 0) = \emptyset_1\ (x)$ and $u_2\ (x, 0) = \emptyset_2\ (x)$, then

$$\text{Max } |u_1(x,t) - u_2(x,t)| \leq \text{max } |¤_1(x) - ¤_2(x)|$$

$$x \in [0,l] \qquad\qquad x£[o,l], \forall t > 0$$

Now, to show that the solution to the heat equation is uniformly stable

Assume y and u are two solutions to the problem with boundary conditions

$$u\ (0,t) = \text{u}(l, t) = 0 \text{ but } y\ (x, 0) = \emptyset_1\ (x) \text{ and u } (x, 0) = \emptyset_2\ (x).$$

On the rectangle we define $w = y - \text{u}$ with $w = 0$ on the lateral sides. On the bottom of the rectangle when t=0, $w = \emptyset_1 - \emptyset_2$ so by maximum principle we have

$$y\ (x, t) - \text{u } (x, t) \leq \text{max } |¤_1 - ¤_2|$$

And by the minimum principle

$$y\ (x, t) - \text{u } (x, t) \geq -\text{max}|¤_1 - ¤_2|.$$

This means

$$\text{Max } |y(x,t) - u(x,t)| \leq \text{max } |¤_1(x) - \phi_2(x)|$$

$$x \in [0,l] \qquad\qquad x \in [0,l], \forall t > 0$$

Chapter 4

Some of its Application to Microeconomics

4.1 Basic Concepts

4.1.1 Budget Constraint

According to economic theory of consumer ,economists assume that the consumers choose of the best bundle is some of sets of goods they can afford. Suppose that there is some of sets of goods from which the consumer can choose. In real life there are many goods to consume, but for our purpose it is convenient to consider only the case of two goods, since we can then depict the consumers choose behavior graphically. Let the two goods be x_1 and x_2 with respective price p_1 and p_2 , then the budget constraint will take the form $p_1x_1 + p_2x_2 \leq m$.

In the above expression, the money spent on good 1, p_1x_1 plus the amount of money spent on good 2, p_2x_2 must be less than or equal to the total amount of money the consumer has to spend, m.

4.1.2 Properties of the budget set

The equation of budget line is the set of bundles that cost exactly m

$$P_1x_1+p_2x_2 = m\text{………………………}(4.2)$$

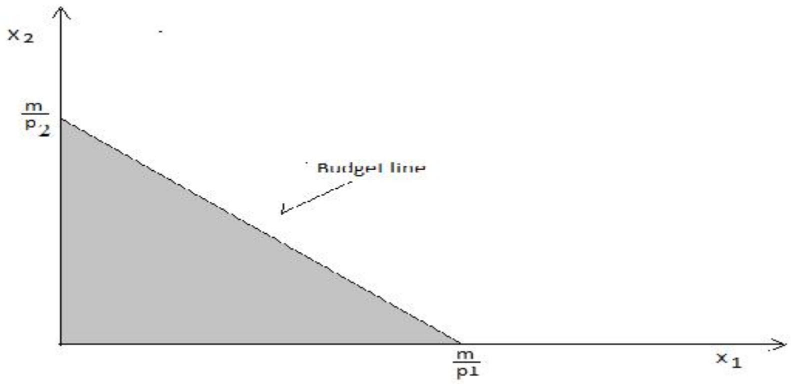


Figure 4.1 The budget set. The budget consistes of all bundeles that are affordable at a given prices and Income

We can arrange the budget line equation (4.2) it gives as the formula

$$x_2 = \frac{m}{p_2} - \frac{p_1}{p_2} x_1 \text{..................................(4.3)}$$

This is the formula for a straight line with a vertical intercept of $\frac{m}{p_2}$ and a slope of - $\frac{p_1}{p_2}$.

The formula tells us how money units of good 2 the consumer needs to consume in order to just satisfy the budget constraint if she/he is consuming x_1 unit of good 1.The slope of the budget line has economic interpretation. It measures the rate at which the market is to substitute good 1 for good 2. Suppose for example that the consumer is going to increase her/his consumption of good 1 by Δx_1. How much will her consumption of good 2 have to change in order to satisfy his/ her budget constraint? Let us use Δx_2 to indicate her change in consumption of good 2. Now note that if she satisfies her budget constraint before and after making a change must satisfy

$$p_1 x_1 + p_2 x_2 = m \quad \text{and}$$

$$p_1(x_1 + \Delta x_1) + p_2(x_2 + \Delta x_2) = m$$

Subtracting the first equation from the second it gives;

$$p_1 \Delta x_1 + p_2 \Delta x_2 = 0$$

This implies that the total volume of the change in her consumption must be zero. Solving for $\frac{\Delta x_2}{\Delta x_1}$, the rate at which good 2 an be substituted for good 1 while still satisfying the budget constraint. Gives $\frac{\Delta x_2}{\Delta x_1} = -\frac{p_1}{p_2}$

This is just the slope of the budget line. The negative sign implies, if you consume more of good 1, you have to consume less of good 2 and vice verse if you continue to satisfy the budget constraint.

Economists sometimes say that the slope of the budget line measures the opportunity cost of consuming good 1. In order to consume more of good 1 you have to give up some consumption of good 2. Giving up the opportunity to consume good 2 is the true economic cost of more good 1consumption; and that cost is measured by the slope of the budget line.

4.1.3 How the budget line changes

When prices and incomes change, the set of goods that a consumer can afford changes as well. How do these changes affect the budget set?

Change in income

Using equation (4.3) an increase in income will increase the vertical intercept and does not affect the slope of the line. Thus an increase in income will result in a parallel shift toward the budget line as in figure (4.2). Similarly, a decrease in income will cause a parallel shift inward.

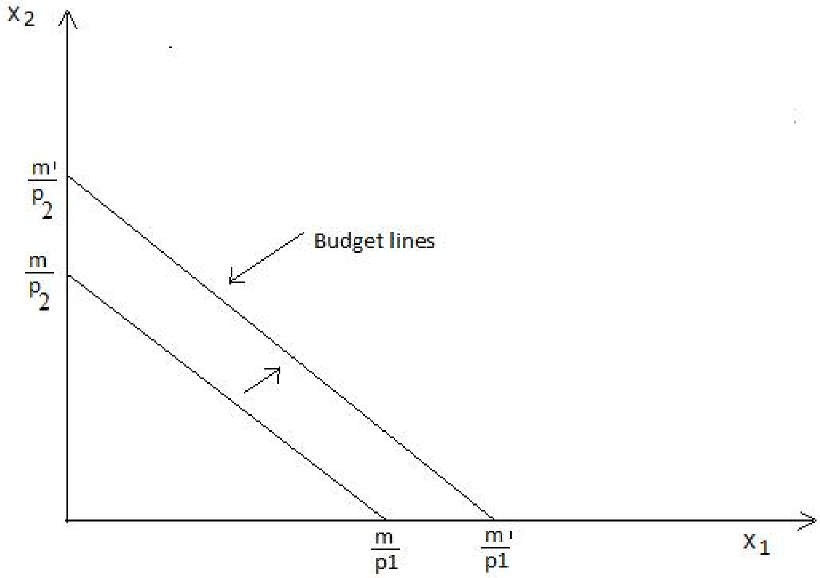


Figure 4.2 An increase in income causes a parallel shift out ward of the budet line.

Change in price

Let us first consider increasing price 1 while holding price 2 and income fixed. According to equation (4.3), increasing p_1 will not change the vertical intercept, but it will make the budget line steeper since $\frac{p_1}{p_2}$ will become larger.

Another way to see how the budget line changes is to use the trick described earlier for drawing the budget line. If you are spending all your money on good 2, then increasing the price of good 1 does not change the maximum amount of good 2 you could buy thus the vertical intercept of the budget line does not change. But if you are spending all of your money on good 1, and good 1 become more expensive, then your consumption of good 1 must decrease. Thus the horizontal intercept of the budget line must shift in ward, resulting in the flit depicted in the figure 4.3

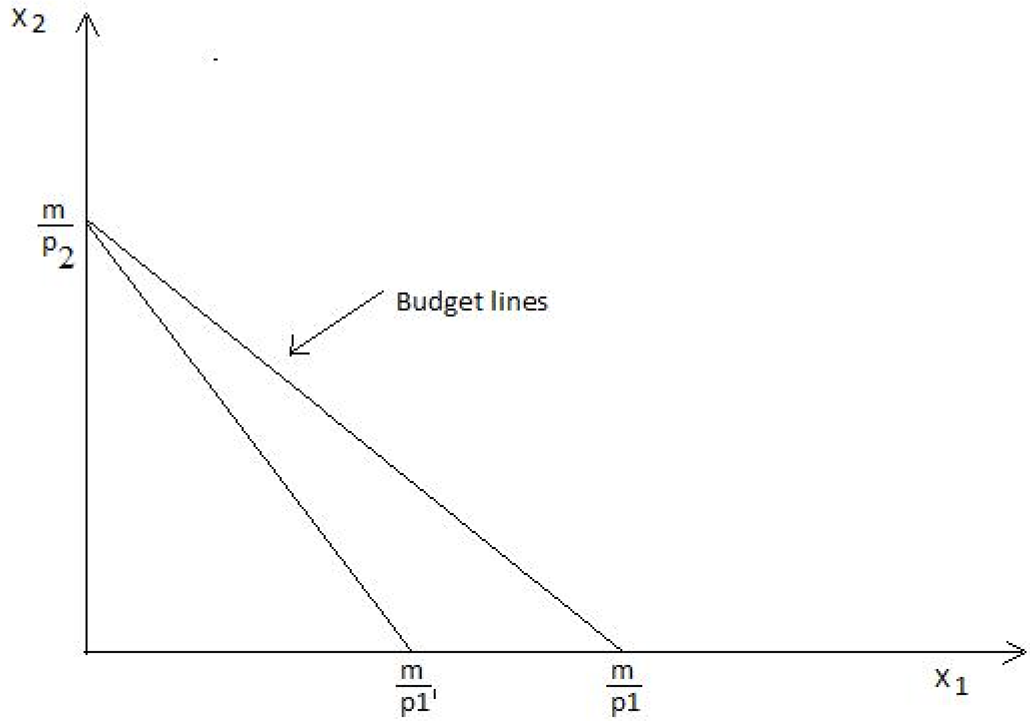


Figure 4.3 If good 1 become more expensive, the budget line becomes steeper.

Suppose for example, if we double the prices of both goods 1 and 2. In this case both the horizontal and vertical intercepts shift inward by a factor of one-half, and therefore the budget line shifts in ward by one-half as well. Multiplying both prices by two is just like dividing by 2. We can also consider price and income changes together. If M decreases and p_1 *and* p_2 both increase, then the intercepts $\frac{m}{p_1}$ *and* $\frac{m}{p_2}$ must both decrease. This means that the budget line will shift in ward. If price 2 increases more than price 1, so that $\frac{-p_1}{p_2}$ decreases (in absolute value), then the budget line will be flatter, if price 2 increases less than price 1, the budget line will be steeper.

4.2 Preferences and Indifference curves

4.2.1 Preferences

Economists usually make some assumptions about the consistency of consumers' preferences. So we usually make some assumptions about how the preference relation works. Some of the assumptions about preferences are so fundamental that we can refer to them as axioms of consumers' preference.

Complete

We assume that any two bundles can be compared. That is given any x-bundles and any y-bundles we assume that $(x_1, x_2) \geq (y_1, y_2)$, or $(y_1, y_2) \geq (x_1, x_2)$ or both, in which case the consumer is indifference between the two bundles.

Reflexive

We assume that any bundle is at least as good as itself; $(x_1, x_2) \geq (x_1, x_2)$.

Transitive

If $(x_{1,} x_2) \geq (y_1, y_2)$ and $(y_1, y_2) \geq (z_1, z_2)$, then we assume that $(x_1, x_2) \geq (z_1, z_2)$. In other words, if the consumer thinks that x is at least as good as y and y is at least as good as z, then the consumer thinks that x is at least as good as z.

4.2.2 Indifference curves

The indifference curve through any consumption bundle consists of all bundles of goods that leave the consumer indifferent to the given bundle.

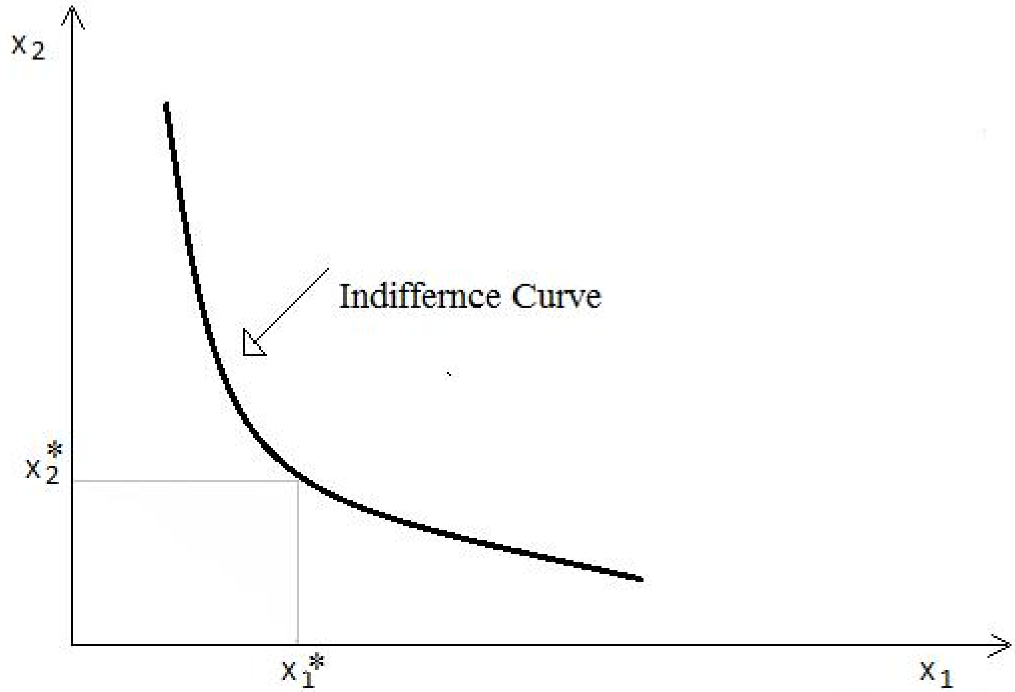


Figure 4.4 Indiffernce curve : bunded in differente to (x_1^* , x_2^*)

Properties of indifference curve

Indifference curves can represent distinct levels of preference cannot cross each. That is, the situation shown in figure 4.5 cannot occur. In order to prove this, let us choose three bundles of goods, x, y, and z, such that x lies only on indifference curve, z lies at intersection of the indifference curves. By assumption the indifference curves represent distinct levels of preference, so one of the bundles, say x, is strictly preferred to the other bundle, y. We know

x~z and z~y, and the axiom of transitivity therefore implies that x~y. But this contradicts the assumption that $x > y$. This contradiction establishes the result indifference curves represent distinct level of preference cannot cross.

Perfect substitutes

Two goods are perfect substitutes if the consumer is willing to substitute one good for the other at a constant rate. The simplest cases of perfect substitute the goods on one to one basis. Suppose, for example, that we are considering a choice between red pencils and blue pencils, and the consumer involved likes pencils, but do not care about a color at all. Pick a consumption bundle; say $(10,10)$.Then for this consumer, any other consumption bundle that has 20 pencils in it is just as good as $(10,10)$.Let us draw the indifferent curves for perfect substitutes.

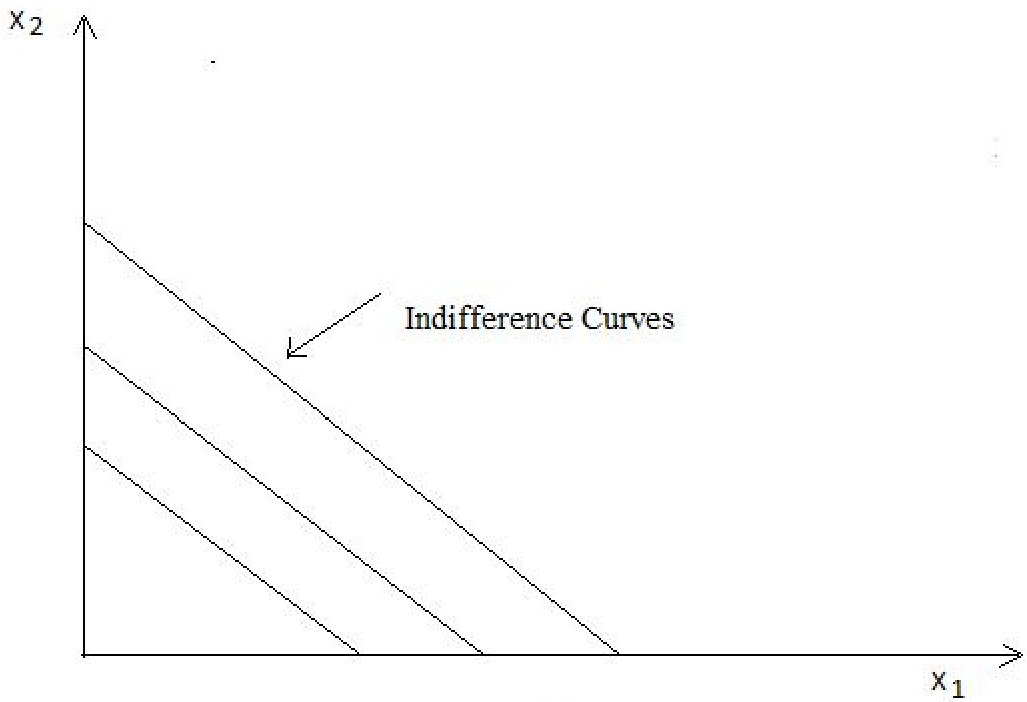


Figure 4.5 Perfect substitutes. The consumer only cares about the total number of penciles, not about their colors. The Indifference curves are stright lines with a slop of -1.

Perfect complements

Perfect complements are goods that are always consumed together in fixed proportion. In some sense the goods “complement” each other. A nice example is that of right shoes and left shoes. The consumer likes shoes, but always wears right and left shoes together. Having only one of the pair of shoes does not do the consumer a bit of good. Let us draw the indifference curves for perfect complements.

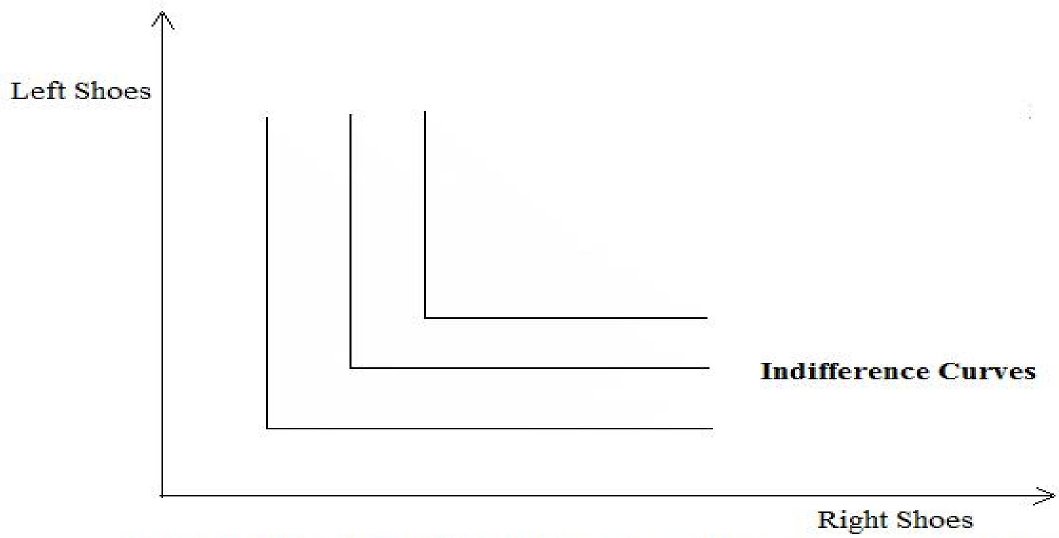


Figure 4.6 Perfect complements. The consumer always wants to consume the goods in fixed proportion to each others. Thus the indifference curves are L-Shaped.

4.3 Utility

The value of a consumer places on a unit of a good or service depends on the pleasure or satisfaction he or she expects to derive from having or consuming it at a point of making a consumption choice. In economics the satisfaction or pleasure consumer's drive from the consumption of consumer goods is called utility. As regards the measurement of utility, there are two basic approaches;

4.3.1 The Cardinal Utility Theory

The cardinals postulated the utility could be measured by the amount of money the consumer is willing to pay for another unit of commodity and its measuring unit is called util. According to this approach, the utility or satisfaction of each commodity is measurable. Money is the most convenient measurement of utility. In the other words, the monetary unit that consumer is prepared to pay for another unit of commodity measures utility or satisfaction.

4.3.2 Ordinal Utility Approach

In the ordinal utility approach, utility cannot be measured absolutely but different consumption bundles are ranked according to preference. The concept is based on the fact that it may not be possible for consumers to express the utility of various commodities they consume in absolute terms, like, 1util, 2 util, or, 3 util, but it is always possible for the consumers to express the utility in relative terms. It is practically possible for the consumers to rank commodities in order of their preference as 1^{st},2^{nd} , 3^{rd} and so on.

Assumption of ordinal utility theory

1. The consumers are rational –they aim at maximizing their satisfaction or utility given their income and market prices.

2. Utility is ordinal; i.e. utility is not absolutely (cardinally) measurable. Consumers are required only to order or rank their preference for various bundles of commodities.

3. The total utility of the consumer depends on the quantities of the commodities consumed, i.e. $u = f(x_1, x_2, ..., x_n)$

4. Preferences are transitive and consistent.

It is transitive in the senses that if the consumer prefers market baskets X to market basket Y, and prefers Y to Z. When we say consistent it means that if market basket X is greater than market basket Y (X>Y) then Y is not greater than X (Y not >X).

The ordinal utility approach is expressed or explained with the help of indifference curves. An indifference curve is a concept used to represent an ordinal measure of the tastes and preferences of the consumer and to show how he/she maximizes utility in spending income. Since it uses indifference curves to study the consumer's behavior, the ordinal utility theory is known as the indifference curve analysis.

4.3.3 Cobb-Douglas utility function

Another commonly used utility function is the Cobb-Douglas utility function

$$U(x_1, x_2) = x_1^{\ c} x_2^{\ d}$$

where c and d are positive numbers that describe the preference of the consumer. Paul Douglas was a twentieth-century economist at the University of Chicago who later become US senator. Charles Cobb was a mathematician at Amherst College. The Cobb-Douglas functional form was originally used to study production behavior.

4.3.4 Lagrange Function

Given utility function and budget constraint: $u(x_1, x_2)$ and income constraint $p_1x_1 + p_2x_2 = m$.

The Lagrange multiplier will be

$$L = u(x_1, x_2) - \text{ª}(p_1x_1 + p_2x_2 - m) = 0$$

$$\frac{\partial L}{\partial x_1} = \frac{\partial u(x_1, x_2)}{\partial x_1} - \text{ª}p_1 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$\Rightarrow \frac{\partial u(x_{1,}\ x_2)}{\partial x_1} = \text{ª}p_1$$

$$\frac{\partial L}{\partial x_2} = \frac{\partial u(x_1, x_2)}{\partial x_2} - \text{ª}p_2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Therefore

$$\frac{\frac{\partial u(x_{1,} x_2)}{\partial x_1}}{\frac{\partial u(x_1, x_2)}{\partial x_2}} = \frac{p_1}{p_2}$$

The Lagrange function is used as transforming the exponential function in to logarithm function. For example given a Cobb-Douglas utility function

$$U(x_1, x_2) = x_1^{\ c} x_2^{\ d}$$

Can be transformed in to

$$L = c \ln x_1. + d \ln x_2. - \text{ª}(p_1x_1 + p_2x_2 - m) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

The marginal utility function for x_1 is

$$\frac{\partial c \ln U(x_1, x_2)}{\partial x_1} = \frac{c}{\partial x_1}$$

And marginal utility for x_2 is $$\frac{\partial d \ln U(x_1, x_2)}{\partial x_2} = \frac{d}{\partial x_2}$$

4.4 Maximization of Utility

The consumer problem can be formulated mathematically as:

$$\max_{x_1, x_2} U(x_1, x_2)$$
$$s.t\, p_1 x_1 + p_2 x_2 = m \quad \text{..}(4)$$

We can solve using Lagrange method

$$L = U(x_1, x_2) + \lambda (m - p_1 x_1 - p_2 x_2) \quad \text{..}(5)$$

Assuming the utility function is differentiable, and there exists interior solution. Then, the first conditions are:

$$\frac{\partial L}{\partial x_1} = \frac{\partial U(x_1, x_2)}{\partial x_1} - \lambda p_1 = 0 \quad \text{..}(6)$$

$$\frac{\partial L}{\partial x_2} = \frac{\partial U(x_1, x_2)}{\partial x_2} - \lambda p_2 = 0 \quad \text{..}(7)$$

$$\frac{\partial L}{\partial \lambda} = m - p_1 x_1 - p_2 x_2 = 0 \quad \text{..}(8)$$

From the first two first order conditions, we have

$$\frac{\frac{\partial U(x_1, x_2)}{\partial x_1}}{\frac{\partial U(x_1, x_2)}{\partial x_2}} = \frac{p_1}{p_2} \quad \text{..}(9)$$

Which is the same as $MRS = \frac{p_1}{p_2}$

Suppose that the consumer has Cobb-Douglas preferences.

$$U(x_1, x_2) = x_1^{c} x_2^{d} \quad \text{..}(10)$$

Then

$$MU_1 = \frac{\partial U}{\partial x_1} = c x_1^{c-1} x_2^{d} \quad \text{..}(11)$$

$$and MU_2 = \frac{\partial U}{\partial x_2} = d x_1^{c} x_2^{d-1} \quad \text{..}(12)$$

So the $MRS = \frac{\frac{\partial U}{\partial x_1}}{\frac{\partial U}{\partial x_2}} = \frac{c x_1^{c-1} x_2^{d}}{d x_1^{c} x_2^{d-1}} = \frac{c x_2}{b x_1}$ ....(13)

at $(x_1^*, x_2^*), MRS = \frac{p_1}{p_2}$

$$\Rightarrow \frac{c x_2^*}{b x_1^*} = \frac{p_1}{p_2}$$

$$\Rightarrow x_2{*} = \frac{d p_1}{c p_2} x_1{*} \text{ ....}(14)$$

And also we have budget constraint

$$p_1 x_1^* + p_2 x_2^* = m \text{ ....}(15)$$

From equation (14) and (15), we have

$x_1{*} = \frac{cm}{(c+d) p_1}$ and substituting for x_1^* in equation (15), we have

$x_2^* = \frac{dm}{(c+d) p_2}$ so we get the most preferred affordable bundle for a consumer with Cobb-Douglas preferences

$U(x_1, x_2) = x_1^c x_2^d$ is

$$(x_1^*, x_2^*) = \left(\frac{cm}{(c+d) p_1}, \frac{dm}{(c+d) p_2} \right) \text{ ....}(16)$$

The expenditures on each good:

$(p_1 x_1^*, p_2 x_2^*) = \left(\frac{c}{c+d} m, \frac{d}{c+d} m \right)$, It is a property of Cobb-Douglas utility function , that expenditure share on a particular good is constant. i.e. $\frac{c}{c+d} m + \frac{d}{c+d} m = m$

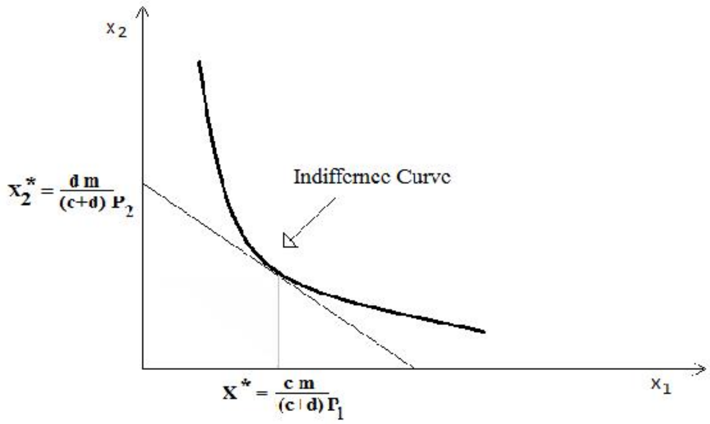


Figure 4.7 Optimum Solution.

4.4.1 Corner solution of utility function

If either $x_1^* = 0$ or $x_2* = 0$, then the ordinary demand (x_1^*, x_2^*) is at a corner solution to the problem of maximizing utility subject to a budget constraint. It happens when indifference curves are too flat or steep relative to the budget line.

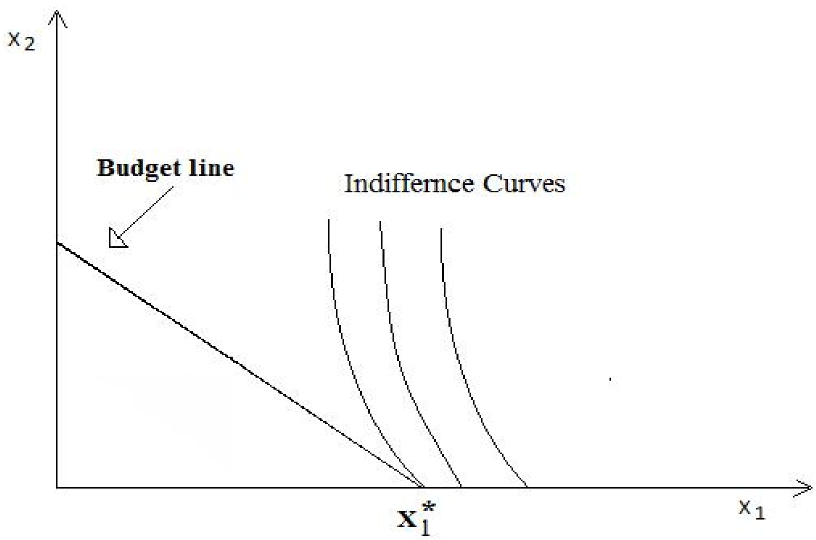


Figure 4.8 Boundery Optimum

The indifference curves are too steep in this case to touch the budget line. The willingness to pay for good 2 is still lower than the price of good 2 even when x_2 is 0.

In perfect substitutes case consider the utility function $U(x_1, x_2) = x_1 + x_2$

Case1. If $p_1 > p_2$

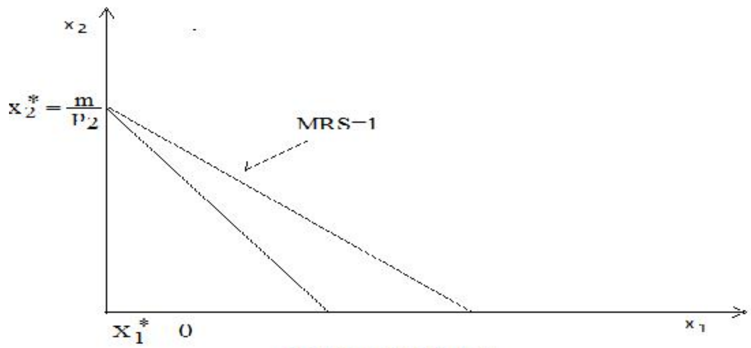


Figure 4.9 If $P_1 > P_2$

Case2. If $p_1 < p_2$

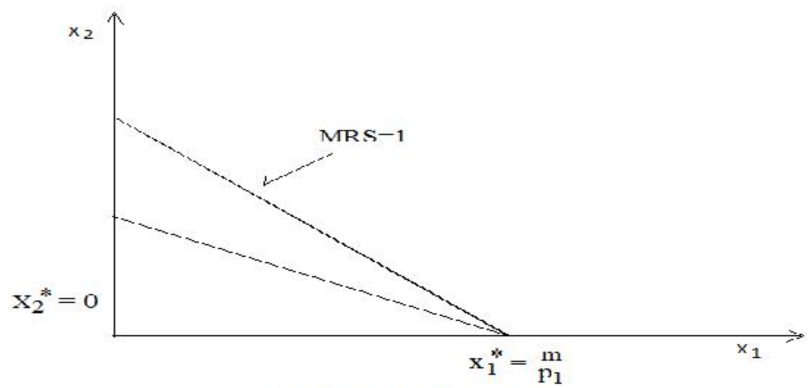
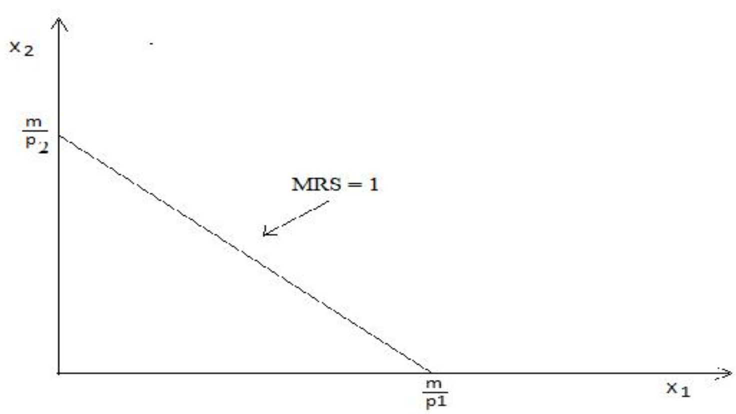


Figure 4.10 If $P_1 < P_2$

Therefore, the most preferred affordable bundle is (x_1^*, x_2^*). Where

$$(x_1^*, x_2^*) = \left(\frac{m}{p_1}, 0\right) \quad \text{if } p_1 < p_2 \text{ and } (x_1^*, x_2^*) = \left(0, \frac{m}{p_2}\right) \text{ if } p_1 > p_2 .$$

Case3. If $p_1 = p_2$



Figuer 4.11 All the bundels in the constraints are equally the most preferred affordable when $P_1 = P_2$.

4.4.2 Kinky solutions of utility function

In perfect complements consider the utility function $U(x_1, x_2) = \min(ax_1, x_2)$.

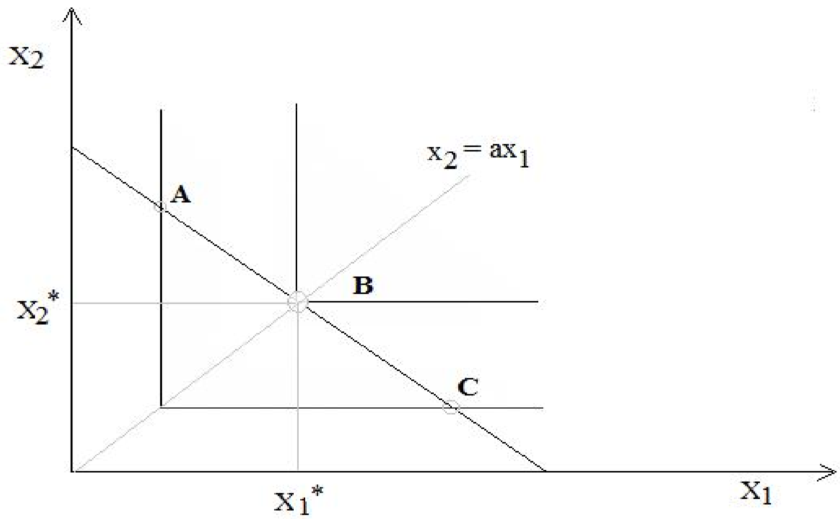


Figure 4.12 Optimum solution is at point B.

The most preferred affordable bundle is at B. To find kinky solutions, consider the budget constraint $p_1x_1^* + p_2x_2^* = m$...(17) and

$x_2^* = ax_1^*$...(18)

Substituting equation (18) in to equation (17) we get

$$x_1^* = \frac{m}{p_1 + ap_2} \text{ and } x_2^* = \frac{am}{p_1 + ap_2}$$

Therefore, the optimal consumption bundle of perfect complement is $\left(x_1^*, x_2^*\right) = \left(\frac{m}{p_1 + ap_2}, \frac{am}{p_1 + ap_2}\right)$.

Summary

In this particular paper, we discussed uniqueness of Laplace equation and uniqueness and stability of heat equation. Also we have seen how utility of the consumer is maximixed under fixed budget constraint .

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