GRADUATE SEMINAR REPORT ON MATHEMATICAL MODELLING FOR OPTIMIZATION OF AN ASSIGNMENT PROBLEM

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PREFACE

Mathematics, which is one part of a science, has so many applications in different fields. The main target of this seminar paper is to show how we can apply mathematics in solving problems that occur in our daily activities by presenting various aspects of mathematical modeling for optimization of an assignment problem.

This seminar paper is divided in to three topics.

Chapter 1. ***Mathematical modeling***

Under this topic the meaning of mathematical modeling, the basic steps involved in developing a mathematical model for a certain real world problems, the advantages of using it and classification of mathematical models are discussed.

Chapter 2. ***Assignment problem***

In our daily activities one of the most important questions that should be answered is "How to assign a certain number of facilities to the same number of jobs one and only once such that to be the effectiveness optimum?" This and other related problems should be solved in such a way that to perform a best decision. In order to solve assignment problem many algorithms have been developed. But in this paper I consider only one of these methods namely the **ASSIGNMENT ALGORITHM or HUNGARIAN METHOD.**

The assignment problem in general form can be stated as follows.

Given n facilities, n jobs and the effectiveness of each facility for each job, the problem is to assign each facility to one and only one job in such a way that the measure of effectiveness is optimized (maximum or minimum).

Chapter 3. ***The traveling sales man:***

Several problems of management have a structure identical with the assignment problems. Because of this nature the algorithm that developed for assignment problem can be applied in solving other related problems. The traveling salesman problem is one of these related problems that can be solved using assignment algorithm, and in this chapter the procedure of solving the problem is discussed using some examples.

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Table of Contents

CHAPTER-ONE

1. Mathematical Modeling for Optimization

1.1. Introduction

The use of mathematics in solving real- world problems has become widespread in the recent times. This is partly due to the use of the systems approach to problem solving, and to the increasing computational power of digital computers, and computing methodology, both of which have made many more problems amenable to mathematical solutions.

It is important to realize at the outset that learning to apply mathematics is a very different activity from learning mathematics. The skills needed to be successful in applying mathematics are quite different from those needed to understand concepts, to prove theorems or to solve equations. The difficulty is not in learning and understanding the mathematics involved but in seeing where and how to apply it.

The steps involved in using mathematics to solve real-world problems are: -

Step-1 problem formulation

Step-2 mathematical description

Step-3 mathematical analysis

Step-4 interpretation of the analysis to obtain solutions.

Step-5 implementing and maintaining the solution.

The most crucial and important step is the satisfactory translation of the problem from the real physical world into a mathematical description. Once this is done, standard techniques of mathematical analysis can be used to obtain a solution to the problem. The validity of the solution depends on how well the mathematical description models the real world. The mathematical description is called a mathematical model, and the process of obtaining it is called mathematical modeling.

A model is defined as a representation of an actual object or situation. Since a model is an abstraction of reality, it thus appears to be less complete than reality itself. The main objective of a model is to provide means for analyzing the behavior of the system for the purpose of improving its performance. Mathematical model is one that employs a set of mathematical symbols to represent the decision variables of the system.

These variables are related together by means of mathematical equations to describe the behavior of the system.

A real world problem, in all its generality can seldom be translated in to a mathematical problem and even if it can be so translated, it may not be possible to solve the resulting mathematical problem. Thus it is quit often necessary to "idealize" or" simplify" the problem or appropriate it by another which is quit close to the original problem and yet it can be translated and solved mathematically.

Since the optimum value of the solution improve the problem system's performance only if the model is a good representation of system, the correspondence of the model of reality must be tested and the solution must be evaluated.

1.2. Examples of Model Formulation

Now it becomes necessary to present a few interesting examples to explain the real life situations where mathematical modeling may arise. The outlines of formulation of real world problems are explained with the help of these examples.

Example 1. (Production allocation problem)

A company produces two types of hats. Each hat of the first type requires twice as much labor time as the second type. If all hats are of the second type only, the company can produce a total of 500 hats a day. The market limits daily sales of the first and second type to 150 and 250 hats. Assuming that the profits per hat are 8 Dollars in type A and 5 Dollars in type B; formulate a mathematical model in order to determine the number of hats to be produced of each type so as to maximize the profit.

Formulation: - Let the company produces x_1 hats of type A and x_2 hats of type B each day. So the profit P after selling these two products is given by the linear function:

$$P= 8x_1 + 5x_2 \qquad \text{(objective function)}$$

Since the company can produce at most 500 hats in a day and A type of hats required twice as much time as that of type B, production restriction is given by

$2tx_1 + tx_2 \leq 500t$, where t is the labor time per unit of second

i.e., $2x_1 + x_2 \leq 500$.

But, there are limitations on the sales of hats, therefore further restrictions are: -

$x_1 \leq 150, x_2 \leq 250$.

Also, since the company cannot produce negative quantities?

$x_1 \geq 0$ and $x_2 \geq 0$

Hence the problem can be finally put in the form:

Find x_1 and x_2 such that the profit $P = 8x_1 + 5x_2$,

is minimum subject to the restrictions:

$2x_1 + x_2 \leq 500$, $x_1 \leq 150$, $x_2 \leq 250$, $x_1 \geq 0, x_2 \geq 0$.

Example 2

A firm can produce three types of cloths, say A, B and C. Three kinds of wools are required for it, say: red, green and blue wools. One unit of length type A cloth needs 2 meters of red wool and 3 meters of blue wool; one unit length of type B cloth needs 3 meters of red wool, 2 meters of green wool and 2 meters of blue wool; and one unit of type C cloth needs 5 meters of green wool and 4 meters of blue wool. The firm has only a stock of 5 meters of red wool; 10 meters of green wool and 15 meters of blue wool. It is assumed that the income obtained from one unit length of type A, B and C are 3, 5 and 4 dollars respectively.

Determine how the firm should use the available material so as to maximize the income from the finished cloth.

Formulation: - It is often convenient to construct the following table after understanding the problem carefully.

Quality of Wool	**Type of cloth** A (x_1)	B (x_2)	C (x_3)	**Total quantity of wools available**
Red	2	3	0	8
Green	0	2	5	10
Blue	3	2	4	15
Income per unit length of cloth	$ 3.00	$ 5.00	$ 4.00	

Let x_1, x_2 and x_3 be the quantities (in meters) produced of cloth type A, B and C respectively. Since 2 meters of red wool are required for each meters of cloth A and x_1 meters of this type of cloth are produced, so $2x_1$ meters of red wool will be produced for cloth A.

Similarly, cloth B requires $3x_2$ meters of red wool and cloth C doesn't require red wool. Thus, total quantity of red wool becomes:

$$2x_1 + 3x_2 + 0x_3 \qquad \text{(red wool)}$$

Following similar arguments for green and blue wool:

$$0x_1 + 2x_2 + 5x_3 \qquad \text{(green wool)}$$

$$3x_1 + 2x_2 + 4x_3 \qquad \text{(blue wool)}$$

Since not more than 8 meters of red, 10 meters of green and 15 meters of blue wool are available, the variables x_1, x_2, x_3 must satisfy the following restrictions:

$$2x_1 + 3x_2 + 0x_3 \leq 8 \qquad (1)$$

$$0x_1 + 2x_2 + 5x_3 \leq 10 \qquad (2)$$

$$3x_1 + 2x_2 + 4x_2 \leq 15 \qquad (3)$$

Also, negative quantities cannot be produced. Hence x_1, x_2 and x_3 must satisfy the non-negativity restrictions:

i.e. $x_1 \geq 0, x2 \geq 0, x_3 \geq 0$ (4)

The total income from the finished cloth is given by:

$$P = 3x_1 + 5x_2 + 4x_3 \qquad (5)$$

Thus the problem now becomes to find x1, x2 and x3 satisfying the restrictions 1-4 and maximizing the profit function P. The above three examples can be solved using different methods.

1.3. Simplification of Mathematical Modeling

While constructing a model, two conflicting objectives usually strike in our mind:

(i) The model should be as accurate as possible.

(ii) It should be as easy as possible.

Besides, the management must be able to understand the solution of the model and must be capable of using it. So the reality of the problem understudy should be simplified to the extent where there is no loss of accuracy.

The model can be simplified by:

(a) Omitting certain variables

(b) Changing the nature of variables

(c) Aggregating the variables

(d) Changing the relationship between variables and

(e) Modifying the constraints, etc.

Not that the process of mathematical modeling starts in the real world and it ends in the real world.

1.4. Applications of Mathematical Modeling

The following are some important areas of applications of mathematical modeling in our life: -

1. **Personnel assignment problem:** Suppose we are given m persons, n- jobs and the expected productivity C_{ij} of i^{th} person on the j^{th} job. We want to find an assignment of persons $x_{ij} \geq 0$ for all i and j, to n jobs so that the average productivity of person assigned is maximum, subject to the conditions:

$$\Sigma x_{ij} \leq a_i \text{ and } \Sigma x_{ij} \leq b_j ,$$

where a_i is the number of persons in personnel category i and b_j is the number of jobs in personnel category j.

2. **Transportation problem:** We suppose that m factories (called sources) supply n warehouses (called destinations) with a certain product. Factory F_i (i=1,2,3,----,m) produces a_i units and warehouse W_j (j= 1,2,3,----, n) requires b_j units. Suppose that the cost shipping from factory F_i to warehouse W_j is directly proportional to the amount shipped; and that the unit cost is C_{ij}. Let the decision variables, x_{ij} , be the amount shipped from factory F_i to warehouse W_j. The objective is to determine the number of units transported from factory F_i to warehouse W_j so that the total transportation cost $\Sigma\, \Sigma\, C_{ij}x_{ij}$ is minimized. In the mean time the supply and demand must satisfied exactly.

3. **Efficiency on operations of systems of dams:** In this problem, we determine variation in water storage of dams, which generate power so as to maximize the energy obtained from the entire system. The physical limitations of storage appear as inequalities.

4. **Agricultural applications:** Mathematical modeling can be applied in agricultural planning for allocating the limited resources such as acreage, labor, water, supply and working capital, etc. so as to maximize the net revenue.

5. **Marketing management:** Mathematical modeling helps in analyzing the effectiveness of advertising campaign and time based on the available advertising media. It also helps traveling salesman in finding the shortest route for his tour.

Besides the above, mathematical modeling is applicable in the area of administration, production management, manpower management, education, inventory control and capital budgeting etc.

1.5. Advantage of Mathematical Modeling

Mathematical modeling is often used to help us make decision through optimization. We use it to select the best alternative out of a large number of possibilities. Mathematical modeling has been successfully used by almost all scientists, engineers, managers and researchers throughout the ages in all fields of life, but its importance has been realized only during the last three decades. Though some mathematical modeling problems may require more than one technique for their solution, yet in practice most mathematical problems use one determinant technique. The following indicates somewhat the importance of using mathematical modeling:

1. A model provides logical and systematic approach for solving or understanding a problem.
2. Models help in finding scope of new method for solving a problem and improvement in the system.
3. Models help in incorporating useful changes even without disturbing the system or problem under consideration
4. Models indicate the nature of measurable quantities in a problem.
5. Models help in understanding and economic explanation of the operations of the system.
6. A model permits to examine the behavior of a system without interfering with on going operations.

All these and many other problems can and have been solved through mathematical modeling. In general to solve a given biological, physical, chemical, financial or social problems, we first develop a mathematical model, then solve it mathematically and finally interpret the solution in terms of the original problem. For example, the power industry provides many applications of mathematical modeling in order to solve their Problems such as concerning flow of water, electricity, gas and oil. Another activity in which mathematical modeling plays an important role is planning. Many national and local government departments depend on mathematics to predict, for example, changes in transportation, education and leisure requirements as population shift or change in structure.

1.6. Limitation of Mathematical Modeling

In spite of wide area of applications, some limitations are associated with Mathematical modeling. These are stated as follows: -

There are so many mathematical models, which have been successfully developed and applied to get insight into tens of thousands of situations. In fact mathematical physics, mathematical economics, operations research, biomathematics etc, are almost synonymous with mathematical modeling.

However there is still equality large or even a large number of situations that have not yet been mathematically modeled either because the situations are sufficiently complex or because mathematical models formed are mathematically intractable.

The development of powerful computers has enabled a much large number of situations to be mathematically modeled. Moreover it has been possible to make more realistic models and to obtain better agreement with observations.

However, successful guidelines are not available for choosing the number of parameters and of estimating the values for these parameters. In fact reasonably accurate models can be developed to fit any data by choosing number of parameters to be even five or six. We want a minimum number of parameters and we want to be able to estimate them accurately.

In general, an optimal solution is one that minimizes or maximizes the performance measure in a model, subject to the conditions and constraints represented in that model. Optimization is therefore yields the best solution to the problem that is modeled. But since a model is never perfect representation of the problem, the optimal solution is never the best solution.

CHAPTER TWO

2. Assignment Problems

2.1. Introduction.

One of the early applications of linear programming techniques has been the formulation and the solution of the transportation problem as a linear programming problem. The transportation problem involves a number of manufacturing points (factories or warehouses) and a number of destinations (dealers or customers). Each factory has a certain capacity constraint and each dealer has a certain requirement with a known unit cost of transportation of the commodity from the factory to the dealer. The objective is to satisfy the dealer requirements with in the factory capacities at a minimum transportation cost. The transportation model is a special case of a linear programming problem. Yet another class of special type of linear programming problems is known as the assignment problem.

The assignment model is very closely related to the transportation model. In fact assignment problem is considered as a special case of transportation problem in which the objective is to assign a number of "origins "to the same number of "destinations "at a minimum total cost. The assignment is made on a one-to-one basis, i.e; each origin can be associated with one and only one destination such as allocation of "n "jobs to "n "workers. Consequently the quantity assigned must be either zero or one.

There are several managerial problems studied with the help of assignment model such as assigning of tasks to persons, salesmen to sale territories, pilots to air flights, projects to contractors, machines to factories, etc.

2.2. Mathematical Statement of the Problem.

Associated to each assignment problem there is a matrix called cost or effectiveness matrix (C_{ij}), where C_{ij} is the cost associated with assigning i^{th} resource (workers) to the j^{th} activity (project). The cost matrix for such a problem is shown below:

Effectiveness or Cost Matrix

Resources (Workers)	Activities (projects) P_1	P_2	...	P_n	Capacity (a_i)
W_1	C_{11}	C_{12}	...	C_{1n}	1
W_2	C_{21}	C_{22}	...	C_{2n}	1
.	.	.	...	.	.
.	.	.	...	.	.
.	.	.	...	.	.
W_n	C_{n1}	C_{n2}	...	C_{nn}	1
Requirement (b_j)	1	1	...	1	

Let x_{ij} denotes the assignment of i^{th} worker to j^{th} project such that

$$x_{ij} = \begin{cases} 1, & \text{if worker } i \text{ is assigned to job } j \\ 0, & \text{otherwise} \end{cases}$$

Then the mathematical model of the assignment problem can be stated as:

$$\text{Minimize } Z = \sum_{j=1}^{n} \sum_{i=1}^{n} c_{ij} x_{ij}$$

Subject to $\sum_{j=1}^{n} x_{ij} = 1$, for all i, (workers available), which means that only one person should be assigned to the j^{th} job, $j = 1, 2, 3, \ldots, n$.

$\sum_{i=1}^{n} x_{ij} = 1$, for all j, (job requirements), which means that only one job is done by the i^{th} person, $i = 1, 2, 3, \ldots, n$, and $x_{ij} = 0$ or 1, for all i and j

Example 1: Consider problem of assigning five operators to five machines. The assignment costs are given below:

Operators→		I	II	III	IV	V
Machines	A	10	5	13	15	16
	B	3	9	18	3	6
	C	10	7	2	2	2
	D	5	11	9	7	12
	E	7	9	10	4	12

Formulate a linear programming model to determine an optimal assignment. Write the objective function and the constraints in detail.

Solution: -

Step 1. Key decision is to determine which operate should be assigned to which machine.

Step 2. Feasible alternatives are set of values of x_{ij}, where each x_{ij} =0 or 1.

Step 3. Objective is to minimize the total cost involved,

i.e. to minimize $Z = (10x_{11}+5x_{12}+13x_{13}+15x_{14}+16x_{15})+(3x_{21}+9x_{22}+18x_{23}+3x_{24}+6x_{25})$

$+ (10x_{31}+7x_{32}+2x_{33}+2x_{34}+2x_{35}) + (5x_{41}+11x_{42}+9x_{43}+7x_{44}+12x_{45})$

$+ (7x_{51}+9x_{52}+10x_{53}+4x_{54}+12x_{55})$

subject to the following constraints:

(i) Each operator must be assigned to one and only one machine

$$\sum_{j=1}^{5} x_{1j} = 1, \text{ for machine A}$$

$$\sum_{j=1}^{5} x_{2j} = 1, \text{ for machine B}$$

$$\sum_{j=1}^{5} x_{3j} = 1, \text{ for machine C}$$

$$\sum_{j=1}^{5} x_{4j} = 1, \text{ for machine D}$$

$$\sum_{j=1}^{5} x_{5j} = 1, \text{ for machine E}$$

(II) Each machine must be assigned to one and only one Operator i.e.

$$\sum_{i=1}^{5} x_{i1} = 1 \text{ , for operator I}$$

$$\sum_{i=1}^{5} x_{i2} = 1 \text{, for operator II}$$

$$\sum_{i=1}^{5} x_{i3} = 1 \text{, for operator III}$$

$$\sum_{i=1}^{5} x_{i4} = 1 \text{, for operator IV}$$

$$\sum_{i=1}^{5} x_{i5} = 1 \text{, for operator V}$$

2.3. Methods for Solving an Assignment Problem

The cost matrix is the same as that of transportation cost matrix except that the capacity at each of the resources and the requirement at each of the destinations are taken to be unity due to the fact that assignment is made on a one to one basis. Thus all the methods that are applied to solve a transportation problem can be also applied in solving an assignment problem.

However, because of its special structure there is a very interesting class of "Allocation Method" which is applied to a lot of very practical problems called "Assignment problem" which depends on the following fundamental theorems.

Theorem* 1. *(Reduction theorem)

If, in an assignment problem, a constant is added or subtracted to every element of a row (or column) of the cost matrix, then an assignment that minimizes the total cost for the new matrix also minimizes the total cost for the original matrix.

Mathematically the above theorem may be restated as follows.

If $x_{ij} = X_{ij}$, minimizes $Z = \sum_{j=1}^{n} \sum_{i=1}^{n} c_{ij} x_{ij}$

Over all x_{ij} such that $\sum_{i=1}^{n} x_{ij} = 1 = \sum_{j=1}^{n} x_{ij}$

and $x_{ij} \geq 0$, then $x_{ij} = X_{ij}$ also minimizes $Z^* = \sum_{j=1}^{n}\sum_{i=1}^{n} c^*_{ij}x_{ij}$

where $C^*_{ij} = C_{ij} + a_i + b_j$

with a_i and b_j any constants, i= 1, 2,…, n, j = 1, 2, …, n.

Proof: -We have $Z^* = \sum_{j=1}^{n}\sum_{i=1}^{n} c^*_{ij}x_{ij} = \sum_{j=1}^{n}\sum_{i=1}^{n}(c_{ij} + a_i + b_j)x_{ij}$

$$= \sum_{j=1}^{n}\sum_{i=1}^{n} c_{ij}x_{ij} + \sum_{j=1}^{n}\sum_{i=1}^{n} a_i x_{ij} + \sum_{j=1}^{n}\sum_{i=1}^{n} b_j x_{ij}$$

$$= Z + \sum_{i=1}^{n} a_i \sum_{i=1}^{n} x_{ij} + \sum_{j=1}^{n} b_j \sum_{j=1}^{n} x_{ij} = Z + .\sum_{i=}^{n} a_i + \sum_{j=1}^{n} b_j$$

Since $\sum_{i=}^{n} a_i$ & $\sum_{j=1}^{n} b_j$ are independent of x_{ij}, it follows that Z^* is minimized whenever Z is minimized. Hence $x_{ij} = X_{ij}$ which minimizes Z also minimizes Z^* //

The result of the above theorem can be used in two different ways to solve the assignment problem. If in an assignment problem some cost elements are negative, we may convert them into an equivalent assignment problem where all the cost elements are non-negative by adding a suitably large constant to the elements of the relevant row or column. Next we look for a feasible solution which has zero assignment cost after adding suitable constants to the elements of various rows and columns. Since it has been assumed that all the cost elements are non-negative, this assignment must be optimum.

Corollary 1. If x_{ij}, i=1, 2, ---, n; j=1, 2, ---, n is an optimal solution for an assignment problem with cost (C_{ij}), then it is also optimal for the problem with cost (C^*_{ij})
when $C^*_{ij} = C_{ij}$ for i,j = 1,2,---,n; $j \neq k$, $C^*_{ik} = C_{ik} - A$, where A is a constant

Proof: For $j \neq k$ we have: $Z^* = \sum_{j=1}^{n}\sum_{i=1}^{n} C^*_{ij}\, x_{ij} = \sum_{j=1}^{n}(\sum_{i=1}^{n} C^*_{ij} + C^*_{ik})x_{ij}$

$$=\sum_{j=1}^{n}(\sum_{i=1}^{n}C^*_{ij}+C^*_{ik}-A)x_{ij}=\sum_{j=i}^{n}\sum_{i=1}^{n}C_{ij}x_{ij}-A\sum_{j=1}^{n}x_{ij}=Z-A$$

Thus if x_{ij} minimizes Z so will it Z^* //

***Theorem** 2.*In an assignment problem with cost C_{ij}, if all $C_{ij} \geq 0$ then, a feasible solution x_{ij}, which satisfies $\sum_{j=i}^{n}\sum_{i=1}^{n}C_{ij}x_{ij}=0$, is an optimalsolution for the problem.

Proof: - Since all $C_{ij} \geq 0$ and all $x_{ij} \geq 0$, the objective function $Z=\sum_{j=1}^{n}\sum_{i=1}^{n}C_{ij}x_{ij}$ cannot be negative. The minimum possible value that Z can attain is 0.Thus, any feasible solution x_{ij} that satisfies $\sum_{j=1}^{n}\sum_{i=1}^{n}C_{ij}x_{ij}=0$ will be optimal //

2.3.1 Assignment Algorithm (Hungarian Method)

The method is listed below in the form of a series of computational steps, when the objective function is that of minimization type.

Step 1. Subtract the minimum element of each row of the effectiveness matrix; from all the elements of the respective rows.

Step 2. Further modify the resulting matrix by subtracting the minimum elements of each column from all the elements of the respective columns. Thus obtained the first modified matrix.

Step 3. Examine the rows successively until a row-wise exactly single zero is found, mark this zero by “As” to make the assignment. Then, mark a cross (x) at the right bottom of all zeros if lying in the column of the marked “As" zeros, showing that they cannot be considered for future assignment. Continue in this manner until all the rows have been examined. Repeat the same procedure for columns also.

Step 4. Repeat step 3 successively until one of the following situations arises:

i) If no unmarked zero is left, then proceed to step 5; or

ii) If there lay more than one of the unmarked zeros in any column or row, then mark "As" one of the unmarked zeros arbitrarily and mark a cross (x) in the cells of remaining zeros in the corresponding rows and columns. Repeat the process until no unmarked zero is left in the matrix.

Step 5. Then, draw the minimum number of horizontal and vertical lines to cover all the zeros in the resulting matrix at least once. Let the minimum numbers of lines be N. Now there may be two possibilities:

i) If N = n, the number of rows (or columns) of the given matrix, then proceed to step 10

ii) If N < n, then proceed to step 6.

Step 6. Determine the smallest element in the matrix, not covered by the N lines. Subtract this minimum element from all uncovered elements and add this element to the values at the intersection of the horizontal and vertical lines. Thus, the second modified matrix is obtained.

Step 7. Again repeat steps 5 and 6 until minimum number of lines become equal to the number of rows (or columns) of the given matrix i.e., N = n.

Step 8. (To make a zero-assignment). Examine the rows successively until a row-wise exactly single zero is found, mark this zero by "As" to make the assignment. Then, mark a cross (x) at the right bottom of all zeros if lying in the column of the marked "As" zeros, showing that they cannot be considered for future assignment. Continue in this manner until all the rows have been examined. Repeat the same procedure for columns also.

Step 9. Repeat step 8 successively until one of the following situations arises:

i) If no unmarked zero is left, then the process ends; or

ii) If there lay more than one of the unmarked zeros in any column or row, then mark "As" one of the unmarked zeros arbitrarily and mark a cross (x) in the cells of remaining zeros in the corresponding rows and columns. Repeat the process until no unmarked zero is left in the matrix.

Step 10. Thus exactly one marked "As" zero in each row and column of the matrix is obtained. The assignment corresponding to these marked zeros will give the optimal assignment.

2.3.2 A Rule to Draw Minimum Number of Lines

A very convenient rule of drawing minimum number of lines to cover all the Zeros of the reduced matrix is given in the following steps:

Step 1. Tick (√) all rows that do not have any marked ("As") zero.

Step 2.Tick (√) columns, which have zeros in the ticked rows.

Step 3.Tick (√) all rows having assignments in the ticked columns.

Step 4. Repeat steps 2 and 3 until the chain of ticking is complete.

Step 5. Draw lines through all unticked rows and ticked columns.

This will give us the minimal system of lines.

Example 2. A department head has four subordinates and four tasks to be performed. The subordinates differ in efficiency and the tasks differ in their intrinsic difficulty. His estimates of the times each man would take to perform each task are given in the matrix below:

Tasks→		I	II	III	IV
Subordinates	A	8	26	17	11
	B	13	28	4	26
	C	38	19	18	15
	D	19	26	24	10

How the tasks should be allocated to subordinates so as to minimize the total Man-hour?

Solution:

Step 1: Subtract the smallest number in each row and in each column to get the following reduced matrix.

Table-1

Tasks→		I	II	III	IV
Subordinates	A	0	14	9	3
	B	19	20	0	22
	C	23	0	3	0
	D	9	12	14	0

Step 2. Thus, using Hungarian method we obtain the assignment in the matrix below:

Table-2

Tasks→		I	II	III	IV
Subordinates	A	0_{As}	14	9	3
	B	19	20	0_{As}	22
	C	23	0_{As}	3	0_{x}
	D	9	12	14	0_{As}

Now, the complete set of assignments of subordinates to perform different tasks along with their respective man-hours is as follows:

Subordinate	Tasks	Man-hour
A	I	8
B	III	4
C	II	19
D	IV	10
	Total	41

Hence, the department head should assign the task to his subordinates as given above and the total minimum man-hour required is 41.

2.4. Special cases in assignment problems

2.4.1 Maximization case in assignment problem.

In some cases, the pay off elements of the assignment problem may represent revenues or profits instead of costs so that the objective will be to maximize the total revenue or profit. The Hungarian method explained earlier can also be used for maximization case. The problem of maximization can be converted in to a minimization case by selecting the largest element among all elements of the profit matrix and then subtracting all other elements in the matrix from this largest element and assigning to the corresponding cell. We can then proceed as usual and obtain the optimal solution.

Example 3. A company has four territories open, and four salesmen available for assignment. The territories are not equally rich in their sales potential, it is estimated that a typical salesman operating in each territory would bring in the following annual sales:

Territories	I	II	III	IV
Annual sales ($)	60,000	50,000	40,000	30,000

The four territories are also considered to be different in ability, it is estimated that, working under the same conditions their yearly sales would be proportionally as follows:

Salesman;	A	B	C	D
Proportion:	7	5	5	4

If the criterion is maximum expected total sales, the intuitive answer is to assign the best salesman to the richest territory the next best salesman to the second richest territory, and so on. Verify this answer by the assignment technique.

Solution: -

Step 1. To construct the effectiveness matrix

The sum of proportions of sales of the four salesmen is: 7 + 5 + 5 + 4 = 21

Taking the salesmen in the four territories sales are as follows:
For A, we have 7/21*6, 7/21*5, 7/21*4 and 7/21*3
For B, we have 5/21*6, 5/21*5, 5/21*4 and 5/21*3

For C, we have 5/21*6, 5/21*5, 5/21*4 and 5/21*3

For D, we have 3/21*6, 4/21*5, 4/21*4 and 4/21*3

To avoid fractions we consider the sales in 21 years, which are as follows:

For A:	42,	35,	28,	and 21
For B:	30,	25,	20,	and 15
For C:	30,	25,	20,	and 15
For D:	18,	20,	16,	and 12

Thus the problem is to determine the assignment of (salesmen to territories) which makes the total sales maximum. The effectiveness matrix of the problem is given by the following table.

Territories

Salesmen

	I	II	III	IV
A	42	35	28	21
B	30	25	20	15
C	30	25	20	15
D	18	20	16	12

Step 2. Since the matrix represents the sales, which can be generated by each salesman in each territory, the objective function of the assignment problem is, therefore; to maximize the total sales generated. But the algorithm for assignment problem is for minimization of the objective function. We, therefore, convert the given problem to minimization problem by subtracting all the elements of the given matrix from the maximum element 42 to obtain the following matrix.

Table-2

	I	II	III	IV
A	0	7	14	21
B	12	17	22	27
C	12	17	22	27
D	24	22	26	30

Step-3: After row and column subtractions, simply by applying the procedures of Hungarian Algorithm, we get the following two different optimal solutions:

Solution-1: A→I, B→II, C→III and D→IV

Solution-2: A→I, B→III, C→II and D→IV

Thus it is obvious that the best salesman A is assigned to the richest territory I and the worst salesman D assigned to the poorest territory IV. The salesmen B and C are equal in efficiency, so either of them may be assigned to territories II and III.

2.4.2 Multiple optimal Solutions.

Sometimes it is possible to have two or more ways to assign all zero elements in the final reduced matrix for a given problem. This implies that there are more than the required numbers of independent zero elements. In such cases, there will be optimal multiple solutions with the same total cost of assignment.

In such type of situations, managers may exercise their judgment or preference and select that set of optimal assignments, which is more suited to their requirement.

Example 4. A solicitors firm employs typists on hourly pieces-rate basis for their daily work. There are five typists for service and their charges and speed are different. According to an earlier understanding only one job is given to one typist and the typist is paid for full hour even if he/she works for a fraction of an hour. Find the least cost allocation for the following data:

Typist	Rate per hour	No. of pages typed/hour	Job	No. of pages
A	5	12	P	199
B	6	14	Q	175
C	3	8	R	145
D	4	10	S	298
E	4	11	T	178

Solution;

Step 1. First transfer the following real world problem into a mathematical form, which is represented in the following matrix.

Table- 1

Job→		P	Q	R	S	T
Typist	A	85	75	65	125	75
	B	90	78	66	132	78
	C	75	66	57	114	69
	D	80	72	60	120	72
	E	76	64	56	112	68

Table-2. After row subtraction

	Job→	P	Q	R	S	T
Typist	A	20	10	0	60	10
	B	24	12	0	66	12
	C	18	9	0	57	12
	D	20	12	0	60	12
	E	20	8	0	56	12

Step 2. Subtract the minimum element of each column from all its elements; in turn the above matrix reduces to.

Table-3

Typist

Job→	P	Q	R	S	T
A	2	2	0	4	0
B	6	4	0	10	2
C	0	1	0	1	2
D	2	4	0	4	2
E	2	0	0	0	2

Step 3. Since there are only 4 lines, which cover all zeros, minimal assignment cannot be made. we, therefore subtract the minimal uncovered element (2) from all uncovered elements, add this value to all junction values and leave the other elements undisturbed as shown in the following matrix.

Table-4

Typist

Job→	P	Q	R	S	T
A	2	2	2	4	0
B	4	2	0	8	0
C	0	1	2	1	2
D	0	2	0	2	0
E	2	0	2	0	2

Step 4. Still the minimum number of lines, which cover all the zeros, is less than the dimensions of the matrix. Thus, optimal assignment cannot be made at this stage also. Repeat step 3 to get the final matrix given below.

Table-5

Typist

Job→	P	Q	R	S	T
A	2	1	2	3	0_{As}
B	4	1	0_{As}	7	0_x
C	0_x	0_{As}	2	0_x	2
D	0_{As}	1	0_x	1	0_x
E	3	0_x	3	0_{As}	3

Thus the optimum assignment is given below

Typist	Job	Cost
A	T	75
B	R	66
C	Q	66
D	P	80
E	S	112

Remark. In this case the above solution is not unique. The alternative solution is the following.

Typist	Job	Cost
A	T	75
B	R	66
C	S	114
D	P	80
E	Q	64

In both cases the total cost is 399.

2.4.3 Unbalanced assignment problem.

Whenever, the payoff matrix of an assignment problem is not a square matrix (i.e. number of rows are not equal to the number of columns), the assignment problem is called unbalanced assignment problem. In such cases, dummy rows and/or columns are added in the matrix to make it a square matrix. Then we can apply the Hungarian method to this resulting balanced assignment problem. For example, if four workers are to be assigned to five machines, a dummy row is simply added to transform the assignment problem in to a square (5X5) matrix. Creating a dummy rows or columns will give us a matrix of equal dimensions and allow us to solve the problem as discussed earlier. The cost associated to this dummy row or column is assigned zero elements in the matrix.

Example 5. A method Engineer wants to assign four new methods to three work centers. The assignment of the new methods will increase production and they are given below. If only one method can be assigned to a work center.

Increase in production

Work Center→		A	B	C
Methods	I	10	7	8
	II	8	9	7
	III	7	12	6
	IV	10	10	8

i) Determine the optimum assignment.

ii) Which method should be ignored?

Solution. The given problem is of maximization type since the elements of the given matrix related to increase in production of units due to introduction of new methods. First of all, convert it into minimization problem by subtracting each element of the given matrix from maximum element 12. Since the problem is an unbalanced one, introduce a dummy work center. We get the following new matrix:

Table-1

Work Center→		A	B	C	Dummy
Methods	I	2	5	4	0
	II	4	3	5	0
	III	5	0	6	0
	IV	2	2	4	0

Step 2. Subtracting the smallest element of each column from all the elements of that column. we get the reduced matrix, which gives us the optimal solution as follow:

Table-2

Work Center→		A	B	C	Dummy
Methods	I	0_{As}	5	0_x	0_x
	II	2	3	1	0_{As}
	III	3	0_{As}	2	0_x
	IV	0_x	2	0_{As}	0_x

The allocations as obtained from the above process are:

I→ A, II→ Dummy, III → B and IV→ C. The total production under the above assignment is:

10 units + 12 units + 8 units = 30 units.

An alternate solution also exists which is given by:

I→ C, II → Dummy, III→ B and IV→ A,

with production = 8 units + 12 units + 10 units =30 units, thus method II should be ignored.

Not that this problem can be an example of the special cases; multiple, maximization and unbalanced assignment problems.

2.4.4 Prohibited assignments

Sometimes due to certain reason an assignment cannot be made in a particular cell. For example a particular machine cannot be installed at particular place or a worker cannot be

given a particular job to perform. To resolve this, we put a very large cross (X) to avoid assignment in those cells where there is restriction of assignment.

Example 6: An airline has drawn up a new flight schedule involving five flights. To assist in allocating five pilots to the flights, it has asked them to state their preference scores by giving each flight a number out of 10. The higher the number, the greater the preference is. Certain of these flights are unsuitable to some pilots owing to domestic reasons represented by a large cross (X). What should be the allocation of the pilots to flights in order to meet as many preferences as possible?

Pilot	Flight number				
↓	I	II	III	IV	V
A	8	2	X	5	4
B	10	9	2	8	4
C	5	4	9	6	X
D	3	6	2	8	7
E	5	6	10	4	3

Solution:

Step 1. Since we have to make the assignment, which maximizes the total preference score, and the method for assignment is applicable only for minimization of the objective function, we therefore, convert the given problem to minimization problem by subtracting all the elements of the given matrix from the maximum element 10 to obtain the following matrix. Subtract the smallest number in each column and in each row to obtain the following reduced matrix.

Table-1

Pilot	Flight number				
↓	I	II	III	IV	V
A	2	8	X	5	6
B	0	1	8	2	6
C	5	6	1	4	X
D	7	4	8	2	3
E	5	4	0	6	7

Step 2. After row and column subtractions we obtain the table below:

Table-2

Pilot	Flight number				
↓	I	II	III	IV	V
A	0	5	X	3	3
B	0	0	8	2	5
C	4	4	0	3	X
D	5	1	6	0	0
E	5	3	0	6	6

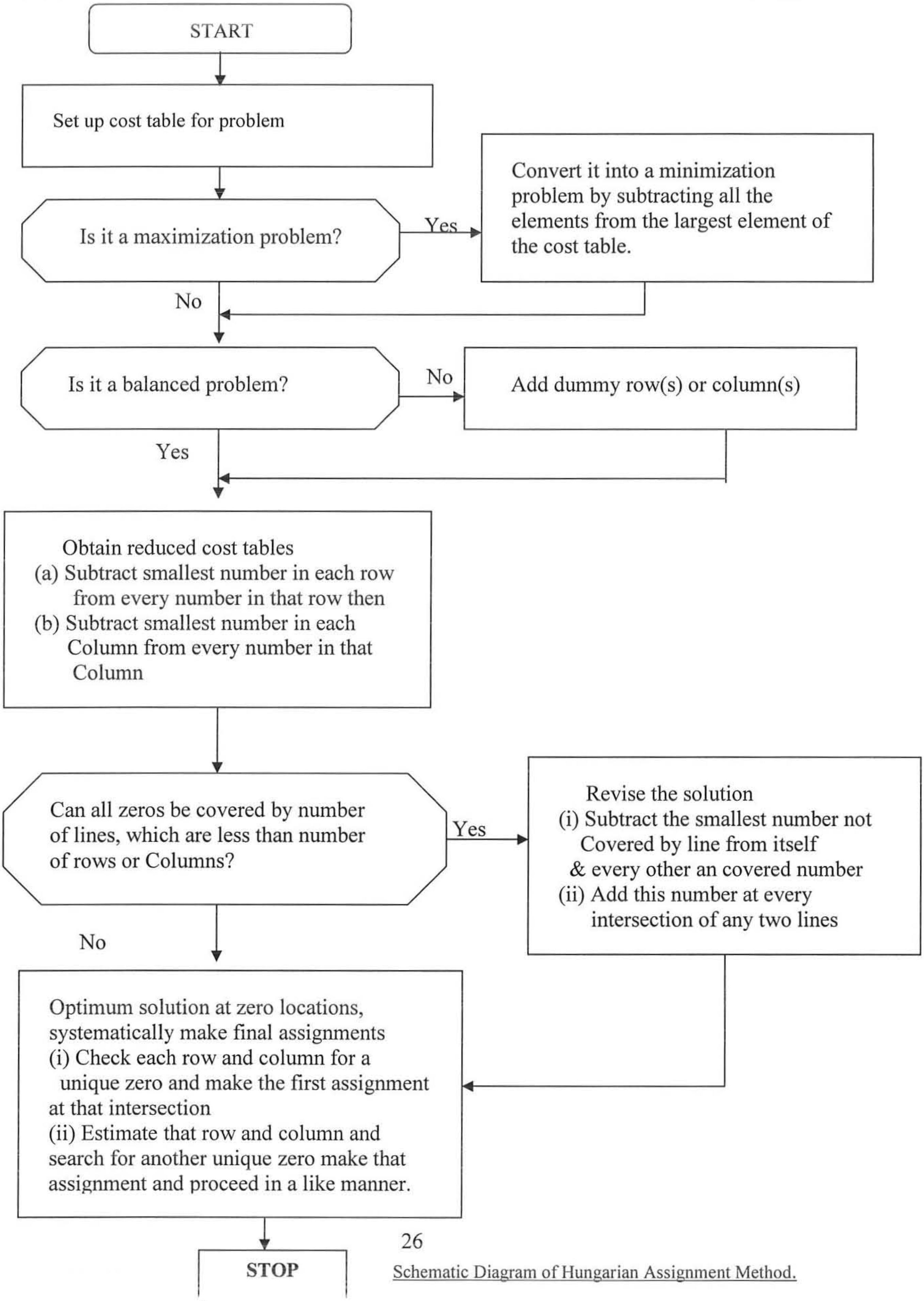
Step 3. As shown in the above matrix the number of lines to cover all the zeros is only 4 which is less than the dimension of the matrix, so subtract the smallest uncovered number 3 from all uncovered numbers and add it to the values at the intersection of the lines. The final table will be the following:

Table-3

Pilot	Flight number				
↓	I	II	III	IV	V
A	0_{As}	2	X	0_x	0_x
B	3	0_{As}	11	2	5
C	4	1	0_x	0_{As}	X
D	8	1	9	0_x	0_{As}
E	5	0_x	0_{As}	3	3

Hence the optimal assignment is:

A-to-I, B-to-II, C-to- IV, D-to-V & E-to-III.



Schematic Diagram of Hungarian Assignment Method.

2.5. Sensitivity in an assignment problem

In many practical linear programming problems we want to find not only an optimal solution but also to determine what happens to this optimal solution when certain changes are made in the system. We would like to determine the effects of these changes without having to solve a new problem or a series of problems from the very beginning. For example, after solving a problem, one may find that certain values of the variables changed and therefore he would like to know whether the current optimal solution is still optimal under the changed circumstance or else, he wants to know how far the input parameter values can vary without causing changes in the optimal solution of the system. Such an investigation is known as sensitivity analysis.

The structure of assignment problem is of such a type that there is very little scope for sensitivity analysis. Modest alternations in the conditions (such as one being able to do two jobs) c an be considered by r epeating the row and adding a dummy column to square up the matrix.

Addition of a constant through any row or column also makes no difference to this position of optimal assignment. In an assignment problem multiplying each element of element of the effectiveness (cost) matrix by some fixed non-zero number did not change the optimal solution. However, sometimes equiproportionate change throughout a row or column can make a difference. So in reference to assignment problems there is no scope for altering the level of an assignment.

Example 7. A department head has four subordinates and four tasks to be performed. The subordinates differ in efficiency and the tasks differ in their intrinsic difficulty. His estimates of the times each man would take to perform each task are given in the matrix below:

Tasks→		I	II	III	IV
Subordinates	A	8	26	17	11
	B	13	28	4	26
	C	38	19	18	15
	D	19	26	24	10

A) A dd a constant number 3 to each entry and allow subordinate B to do two tasks. How the tasks should be allocated to subordinates so as to minimize the total Man-hour?

B) Multiply the effectiveness matrix by 4 and allow subordinate D to do two tasks. How the tasks should be allocated to subordinates so as to minimize the total Man-hour?

Solution of: A)

Step 1: After adding 3 and using dummy column to make it a square matrix we get the following table.

Table-1

Tasks→		I	II	III	IV	V
Subordinates	A	11	29	20	14	0
	B	16	31	7	29	0
	B	16	31	7	29	0
	C	41	23	21	18	0
	D	23	29	27	13	0

Step 2: After row and column subtraction the reduced effectiveness matrix will be the following:

Table-2

Tasks→		I	II	III	IV	V
Subordinates	A	0_{As}	6	13	1	0_x
	B	5	8	0_{As}	16	0_x
	B	5	8	0_x	16	0_{As}
	C	30	0_{As}	14	5	0_x
	D	12	6	20	0_{As}	0_x

Thus, the optimal solution is obtained by the following allocation:

Subordinate A → I,

B → III,

C → II and

D → IV

Solution of: B)

Step 1. Multiply the effectiveness matrix by 4 and use a fifth dummy column to balance the repetition of the fourth row. So the new reduced matrix will be:

Table-1

Tasks→		I	II	III	IV	V
Subordinates	A	32	104	68	44	0
	B	52	112	16	108	0
	C	152	76	72	60	0
	D	76	104	96	40	0
	D	76	104	96	40	0

Step 2.subtracting the smallest number of each column from each numbers in that column will give us the next reduced table:

Table-2

Tasks→		I	II	III	IV	V
Subordinates	A	0_{As}	28	52	4	0_x
	B	20	36	0_{As}	68	0_x
	C	120	0_{As}	56	20	0_x
	D	44	28	80	0_{As}	0_x
	D	44	28	80	0_x	0_{As}

As we can observe from table 2 the optimal solution is obtained by the following assignment:

Subordinate A → I,

B → III

C → II

D → IV

Now compare the solutions of the above two questions with the original question on pages 20 & 21. We can observe that no change on the solutions

2.6. Illustrative Examples

Example 1: Five jobs are available to work with the machines and the respective costs associated with each job – machine assignment is given below.

Machines

		M_1	M_2	M_3	M_4	M_5
Jobs	J_1	12	3	6	X	5
	J_2	4	11	X	5	X
	J_3	8	2	10	9	7
	J_4	X	7	8	6	12
	J_5	5	8	9	4	6

i) Determine the optimal solution.

Solution:

Step 1: After subtracting the smallest number from each column and from each row we will get the following reduced matrix;

Machine Table-1.

		M_1	M_2	M_3	M_4	M_5
Jobs	J_1	9	0_x	1	X	0_{As}
	J_2	0_{As}	7	X	1	X
	J_3	6	0_{As}	6	7	3
	J_4	X	1	0_{As}	0_x	4
	J_5	1	4	3	0_{As}	0_x

Step 2: From the above table we can obtain the assignment as follows:

$J_1 \to M_5$, $J_2 \to M_1$, $J_3 \to M_2$, $J_4 \to M_3$ and $J_5 \to M_4$.

Thus, the minimum cost =5 + 4 + 2 + 8 + 4 =23.

ii) If a sixth machine is available to replace one of the existing machines and the associated costs are also given below in the square matrix obtained by adding the dummy row.

Table-2 Machines

		M_1	M_2	M_3	M_4	M_5	M_6
Jobs	J_1	12	3	6	X	5	9
	J_2	4	11	X	5	X	8
	J_3	8	2	10	9	7	5
	J_4	X	7	8	6	12	10
	J_5	5	8	9	4	6	1
	J_6	0	0	0	0	0	0

Then, Should the new machine be accepted? If so determine the associated saving in cost.
Solution:

Step 1 . S ubtract the smallest element from each row; the reduced matrix will be the following;

Table-3 Machines

		M_1	M_2	M_3	M_4	M_5	M_6
Jobs	J_1	9	0	3	X	2	7
	J_2	0	7	X	1	X	4
	J_3	6	0	8	7	5	3
	J_4	X	1	2	0	6	4
	J_5	4	7	8	3	5	0
	J_6	0	0	0	0	0	0

From the above-reduced matrix we can observe that the minimum number of lines that covers all the zeros is less than the number of columns (or rows). So subtracting the smallest uncovered number from each uncovered numbers and adding it to the values, which are at the intersection of the lines, then applying the Hungarian algorithm will give us the next assignment.

Table-4 Machines

		M_1	M_2	M_3	M_4	M_5	M_6
Jobs	J_1	7	0_x	1	X	0_{As}	5
	J_2	0_{As}	9	X	1	X	4
	J_3	4	0_{As}	6	5	3	1
	J_4	X	3	2	0_{As}	6	4
	J_5	4	9	8	3	5	0_{As}
	J_6	0_x	2	0_{As}	0_x	0_x	0_x

Then the assignment becomes:

$J_1 \rightarrow M_5$, $J_2 \rightarrow M_1$, $J_3 \rightarrow M_2$, $J_4 \rightarrow M_4$ and $J_5 \rightarrow M_6$,

i.e. the minimum cost is: 5+ 4 + 2 + 6 + 1 + 0 = 18, this implies that the machine is accepted and saving in cost is: 23 - 18 = 5 units.

Example 2. To stimulate interest and provide an atmosphere for intellectual discussion a mathematics department decides to hold special seminars on four fields; Numerical analysis, O ptimization t heory, A lgebra a nd D ifferential. S uch s eminars s hould b e h old once per week in the afternoons. However, scheduling these seminars (one for each field and not more that one seminar per afternoon) has to be done carefully so that the number of students unable to attend is kept to a minimum. A careful study indicates that the number of students who cannot attend a particular seminar on a special day is as follows:

	Optimization	Numerical	Differential	Algebra
Monday	50	40	60	20
Tuesday	40	30	40	30
Wednesday	60	20	30	20
Thursday	30	30	20	30
Friday	10	20	10	30

Find an optimal schedule of the seminars. Also find out the total number of students who will be missing at least one seminar and determine the day on which no need of presenting seminar.

Solution:

Step 1. Since it is unbalanced assignment problem first change in to a square matrix using a dummy column as follow:

Table-1

	Optimization	Numerical	Differential	Algebra	Dummy
Monday	50	40	60	20	0
Tuesday	40	30	40	30	0
Wednesday	60	20	30	20	0
Thursday	30	30	20	30	0
Friday	10	20	10	30	0

Step 2. Subtract the smallest element in each column from every element in that column to get the following reduced matrix.

Table-2

	Optimization	Numerical	Differential	Algebra	Dummy
Monday	40	20	50	0_{As}	0_x
Tuesday	30	10	30	10	0_{As}
Wednesday	50	0_{As}	20	0_x	0_x
Thursday	20	10	10	10	0_x
Friday	0_{As}	0_x	0_x	10	0_x

Step 3. The minimum number of lines, which cover all the zeros at least once, is less than the number of rows (columns) as seen in table-3 below. Which day should be excluded in the scheduling?

Table-3

	Optimization	Numerical	Differential	Algebra	Dummy
Monday	~~40~~	~~20~~	~~50~~	~~0_{As}~~	~~0_x~~
Tuesday	30	10	30	10	0_{As}
Wednesday	~~50~~	~~0_{As}~~	~~20~~	~~0_x~~	~~0_x~~
Thursday	~~20~~	~~10~~	~~10~~	~~10~~	~~0_x~~
Friday	0_{As}	0_x	0_x	10	0_x

Step 4. The smallest element among all uncovered elements of table - 3 is 10. Subtract this number from all uncovered values and add to all those values that lie at the intersection of the lines. Thus identify the smallest uncovered element and subtract it from each uncovered numbers and add to the values, which are covered twice to get the next final table. Make an assignment as follow:

Table-4

	Optimization	Numerical	Differential	Algebra	Dummy
Monday	40	20	50	0_{As}	10
Tuesday	20	0_x	20	0_x	0_{As}
Wednesday	50	0_{As}	20	0_x	10
Thursday	10	0_x	0_{As}	0_x	0_x
Friday	0_{As}	0_x	0_x	10	10

Thus it is obvious that the optimal schedule will be:

Monday → Algebra

Tuesday → No seminar

Wednesday → Numerical

Thursday → Differential

Friday → Optimization

And the total number of students who will miss at least one seminar is:

20+ 0 +20 +10 +20 = 70 Students

Example 3. An air - line operating seven days a week has timetable shown below. Crews must have a minimum layover (rest) time of 5 hrs between flights. Obtain the pair of flights that minimizes layover time away from home. For any given pair, the crew will be based at the city that results in the smaller layover. For each pair, mention the town where the crews should be based.

Addis Ababa - Mekelle			Mekelle - Addis Ababa		
Flight No.	Depart	Arrive	Flight No.	Depart	Arrive
1	7:00 A.M.	8:00 A.M.	101	8:00A.M.	9:15 AM
2	8:00 A.M.	9:00 A. M.	102	8:30 A.M.	9:45 A.M.
3	1:30 P.M.	2:30 P.M.	103	12:00 Noon	1:15 P.M.
4	6:30 P.M	7:30 P.M.	104	5:30 P.M.	6:45 P.M.

Solution:

In solving the problem it is assumed that a plane flying from Addis Ababa to Mekelle must come back to Addis Ababa at the immediate next opportunity. It is further assumed that each plane will make only one forward and one return trip and thus there must be four planes for four forward and return back flights.

Step 1: construct the table for layover times between flights when crew is based in Addis Ababa. For simplicity, consider 15 minutes = 1 unit.

Table-1

Flights	101	102	103	104
1	96	98	112	38
2	92	94	108	34
3	70	72	86	108
4	50	52	66	88

Since the crews have a minimum layover of 5hrs between flights, the layover time between flights 1 and 101 will be 24 hrs (96 units) from 8.00 A.M. to 8.00A.M. next day.

Likewise, calculation as follows: Table-2

Flight No	Layover Times	No. of units (1hr.= 4 units
1→ 102	8:00A.M.- 8:30 A.M.= 24hrs & 30 min	98
2→103	9:00 A.M. – 12:00 Noon =27 hrs	108
3→104	2:30 P.M. – 5:30 P.M. =27hrs	108
4→101	7:30 P.M. – 8:00 A.M. =12:30hrs	50

Step-2: Similarly, layover times for other pair of flights can also be calculated as shown in the following table –3

Table -3 of layover times when crew based at Mekelle

Flights	101	102	103	104
1	87	85	71	49
2	91	89	75	53
3	113	111	97	75
4	37	35	21	95

Since the plane arrives Addis Ababa at 9:15 A.M. by flight number 101 and again depart to Mekelle at 7:00 A.M. by flight number 1,the layover time is obviously 21hrs and 45 min (i.e. 87 units). Similarly, layover times between other pairs of flights can also be computed as shown in table -3.

Step -3: Construct the table for the smaller layover times between flights with the help of Tables 1 & 3. Layover times marked " * " denoted that the crew is based at Mekelle. Thus the following table is obtained.

Table-4. Smaller layover times

Flight No.	101	102	103	104
1	87*	85*	71*	38
2	91*	89*	75*	34
3	70	72	86	75*
4	37*	35*	21*	88

Step 4. Finally applying the assignment technique in the usual manner, we get the following reduced Table.

Table-5

Flight No.	101	102	103	104
1	4	0_x	0_{As}	0_x
2	12	8	8	0_{As}
3	0_{As}	0_x	28	50
4	4	0_{As}	0_x	100

CHAPTER - THREE

3. The Traveling - Salesman (Routing) Problem

3.1. Introduction:

The traveling salesman problem is one of the problems considered as puzzles by the mathematician.

Suppose a salesman wants to visit a certain number of cities allocated to him. He knows the distance (cost or time) of journey between every pair of cities, usually denoted by C_{ij} , i.e. from city i to city j. His problem is to select such a route that starts from his home city, passes through each city once and only once, and returns to his home city in the shortest possible distance (or at the least cost or in the leas time).

The problem may be classified in two terms:

1.Symmetrical. The problem is said to be symmetrical, if the distance (cost or time) between every pairs of cities is independent of the directions of his journey.

2.Asymmetrical. The problem is said to be asymmetrical if for one or more pair of cities, the distance (or cost or time) changes with the direction. For example, Going up a hill from city A to B instead of coming down the hill from city B to city A needs more cost, Furthermore, if number of cities is only two, obviously there is no choice. If number of cities become three, say A, B &C, one of them (say A) is the home base, then there are two possible routes:

A ->B - > C and A - > C - > B

For four cities A, B, C and D, there are 3! = 6 possible routes, these are:

A → B → C →D, A → B → D → C, A → C → B → D,

A → C → D → B, A → D → C → B and A →D → B → C.

In general if there are n cities, there is (n-1)! Possible routes.

3.2. Mathematical Formulation of Traveling Salesman Problem

The formulation of traveling salesman problem is as follow:

$$\text{Minimize } Z = \sum_{i}\sum_{j}\sum_{k} d_{ij}x_{ijk}\ ,\ i\neq j$$

where d_{ij} denotes the distance from town i to town j, and i, j, k are integers varying from town 1 to n.

$$X_{ijk} = \begin{cases} 1, \text{if the } k^{th} \text{directed arc is from i to town j} \\ 0, \text{ othewise.} \end{cases}$$

The constraints are of the following type:

$$\text{(i)} \sum_{i}\sum_{j} x_{ijk} = 1, k = 1,2,3,---,n \qquad i \neq j$$

This implies that only one directed arc might be assigned to a specific value of k.

$$\text{(ii)} \sum_{j}\sum_{k} x_{ijk} = 1, i = 1,2,3,---,n$$

This implies that only one town may be reached from a specific town i.

$$\text{(iii)} \sum_{i}\sum_{k} x_{ijk} = 1, j=1,2,3,---,n$$

This implies that only one other town can initiate directed arc to a specific tow j.

$$\text{(iv)} \sum_{\substack{i \\ i\neq j}} x_{ijk} = \sum_{\substack{r \\ r\neq j}} x_{jr(k+1)}, \text{for all j and k.}$$

This constraint will ensure that the round trip will consists of connected arcs. It is given that the k^{th} directed arc ends at some specific town j, the $(k+1)^{th}$ directed arc must start at the same town j. This problem has several practical applications.

At present, the best procedure is to solve the problem as if it were an assignment problem. It becomes necessary to formulate this type of sequencing problem in the form of an assignment problem with the additional restriction on his choice of route.

3.3. Formulation of Traveling Salesman Problem as an Assignment Problem

Suppose C_{ij} is the distance (or cost or time) from city i to city j and $x_{ij} = 1$, if the salesman goes directly from city i to city j, and $x_{ij} = 0$ otherwise. Thus minimize $\Sigma\Sigma x_{ij}c_{ij}$ with the additional restriction that the x_{ij} must be so chosen that no city be visited twice before the tour of all cities is completed. In particular, he cannot go directly from city i to city i itself. This possibility may be avoided in the minimization process by adapting the convention $C_{ii} = \infty$ which ensures that x_{ii} can never be unity.

It is also important to note that only single $x_{ij} = 1$ for each value of i and j. The distance (or cost or time) matrix for this problem can be represented in the following table.

To

From

	A_1	A_2	---	A_n
A_1	∞	C_{12}	---	C_{1n}
A_2	C_{21}	∞	---	C_{2n}
⋮	⋮	⋮	⋮	⋮
A_n	C_{n1}	C_{n2}	---	∞

3.4. Solution Procedure

For solving such problems, detailed procedure is explained with reference to an equivalent example involving the order in which five products A_1, A_2, A_3, A_4 and A_5 are processed over a production facility. To find an order, in which products are produced so that the set- up costs is minimized, is an asymmetrical traveling salesman problem. The change -over/ set -up costs between products must be produced once and only once and production must return to the first product. The following examples will make the procedure clear.

Example-1. Given the matrix of set-up cost. Show how to sequence the production so as to minimize the set- up cost per cycle.

The Cost Matrix

To

From

	A_1	A_2	A_3	A_4	A_5
A_1	∞	2	5	7	1
A_2	6	∞	3	8	2
A_3	48	7	∞	4	7
A_4	12	4	6	∞	5
A_5	1	3	2	8	∞

Solution: Consider the problem as an assignment problem.

Step-1: Subtract the smallest number in each row and column from the corresponding rows and columns to get the following reduced matrix:

To :Table-1

From

	A_1	A_2	A_3	A_4	A_5
A_1	∞	1	3	6	0_{As}
A_2	4	∞	0_{As}	6	0_x
A_3	4	3	∞	0_{As}	3
A_4	8	0_{As}	1	∞	1
A_5	0_{As}	2	0_x	7	∞

Although, zeros of this matrix gives a solution to the assignment problem, but this is not a solution of the traveling salesman problem. This solution which assigns A_1 to A_5, A_5 to A_1, A_2 to A_3, A_3 to A_4 and A_4 to A_2; indicates to produce the products A_1, then A_5 and again A_1, without producing the products A_2, A_3 & A_4; thereby violating the additional restrictions of producing each product once and only once before returning to the first product.

Step-2: Again examine the matrix for some of the "next best" solutions to the assignment problem, and try to find out one solution which satisfies the additional restriction. The smallest element other than zero is 1, so try the effect of putting such an element in the solution.

To: Table-2

From

	A_1	A_2	A_3	A_4	A_5
A_1	∞	1_{As}	3	6	0_x
A_2	4	∞	0_{As}	6	0_x
A_3	4	3	∞	0_{As}	3
A_4	8	0_x	1	∞	1_{As}
A_5	0_{As}	2	0_x	7	∞

Start by making unity assignment in the cell (1,2) instead of zero assignment in the cell (1,5) as shown in the above table-2. Delete row 1 and column 2 and get the remaining (4x4) matrix. Although, there is no solution to the assignment problem among zeros, it is easy to see the best solution lies in the marked "As" elements. Thus the required solution of the problem is:

A_1 - >A_2 ->A_3 ->A_4 ->A_5 ->A_1.

The cost in the reduced matrix (as given in table-2) is 2 for this solution.

Step-3: It is seen that any solution where the cost exceeds 2 is not optimal. Now, only examine solutions containing the element 1 in the cell (4,3) no so far used to see if a better solution exists. After deleting row 4 and column 3 from table-3. The remaining (4x4) matrix does not have a solution among the zeros, and it is conclude that the cost for the reduced matrix is 2. Hence, the most suitable sequence is A_1 -> A_2 ->A_3 -> A_4 -> A_5 ->A_1 and the minimum set-up cost is: 2 + 3 + 4 + 5 + 1 = 15

3.5. Selected Solved Problems:

Problem 1: A salesman estimates that the following would be the cost in his route; visiting the six cities as shown below:

To city

From city

	I	II	III	IV	V	VI
I	∞	20	23	27	29	34
II	21	∞	19	26	31	24
III	26	28	∞	15	36	26
IV	25	16	25	∞	23	18
V	23	40	23	31	∞	10
VI	27	18	12	35	16	∞

The salesman can visit each of the cities once and only once. Determine the optimum sequence he should follow to minimize the total distance traveled. What is the total distance traveled?

Solution:

Step-1: Apply the usual procedure of assignment algorithm to obtain the table showing an optimum assignment solution indicated by marked zeros in the following matrix:

To city: Table-1

From city

	I	II	III	IV	V	VI
I	∞	0_x	0_{As}	7	2	4
II	0_{As}	∞	0_x	10	8	8
III	6	13	∞	0_{As}	14	14
IV	4	0_{As}	6	∞	0_x	2
V	8	30	10	21	∞	0_{As}
VI	13	6	0_x	26	0_{As}	∞

This table gives the optimum assignment solution I ->III ->IV ->II->I and V->VI ->V with minimum distance 1 01 units. But, this table does not provide the solution to the traveling salesman problem, as it is not allowed to go from city II to I without visiting the cities V& VI.

Step-2: Once and only once we try to find the "next best" solution which satisfies the additional restriction. The smallest element other than zero is 2. So, we try to bring 2 into the solution. Since the element 2 occurs at two places, we shall consider both cases separately until the acceptable solution is obtained. We start making assignment with the cell (1, 5) h aving t he n ext m inimum e lement 2 i nstead o f z ero i n t he c ell (1,3). A fter making this assignment we observe that no other assignment can be made in the first row and fifth column, and thus the resulting feasible solution will be I ->V ->VI ->III ->IV ->II ->I. The assigned elements for this solution are marked in the following table. (The cost corresponding to this feasible solution is 2)

To city: T

From city

	I	II	III	IV	V	VI
I	∞	0_x	0_x	7	2_{As}	4
II	0_{As}	∞	0_x	10	8	8
III	6	13	∞	0_{As}	14	11
IV	4	0_{As}	6	∞	0_x	2
V	8	30	10	21	∞	0_{As}
VI	13	6	0_{As}	26	0_x	∞

Step-3: again, if we make an assignment problem in the cell (4,6) having the next best element 2 instead of zero marked in cell (4,2), then no feasible solution is obtained in terms of zeros. Hence the best solution is:

I ->V ->VI ->III ->IV ->II ->I,

and the total set-up cost according to this result is 103.

Problem 2: A medical representative has to visit five stations A, B, C, D and E. He does not want to visit any station twice before completing his tour of all the stations, and he wishes to return to the starting station. Costs of going from one station to another are given below. Determine the optimal route:

	A	B	C	D	E
A	∞	2	4	7	1
B	5	∞	2	8	2
C	7	6	∞	4	6
D	10	3	5	∞	4
E	1	2	2	8	∞

Solution:

Step 1 after row and column subtraction the new reduced matrix will be the following:

Table 1

	A	B	C	D	E
A	∞	1	3	6	0_{As}
B	3	∞	0_{As}	6	0_X
C	3	2	∞	0_{As}	2
D	7	0_{As}	2	∞	1
E	0_{As}	1	1	7	∞

When the problem solve as an assignment problem the optimal solution, which is clearly shown in the above table, will be:

A→ E → A and B → C → D → B.

But this cannot be a solution for the problem as he returns back to station A without visiting the stations B, C and D.

Step 2: We try to find the "next best" solution which satisfies the additional restriction. The smallest element other than zero is 1.So we try to bring 1 into the solution. Since 1 occurs at four places, we shall consider all cases separately until the acceptable solution is attained. We start making assignment with the cell (1, 2) having the next minimum element 1 instead of zero in the cell (1,5). After making this assignment we no other assignment can make in the first row and second column.

The assigned element for this solution is marked in the following table.

Table 2

	A	B	C	D	E
A	∞	1_{As}	3	6	0_x
B	3	∞	0_{As}	6	0_X
C	3	2	∞	0_{As}	2
D	7	0_x	2	∞	1_{As}
E	0_{Aa}	1	1	7	∞

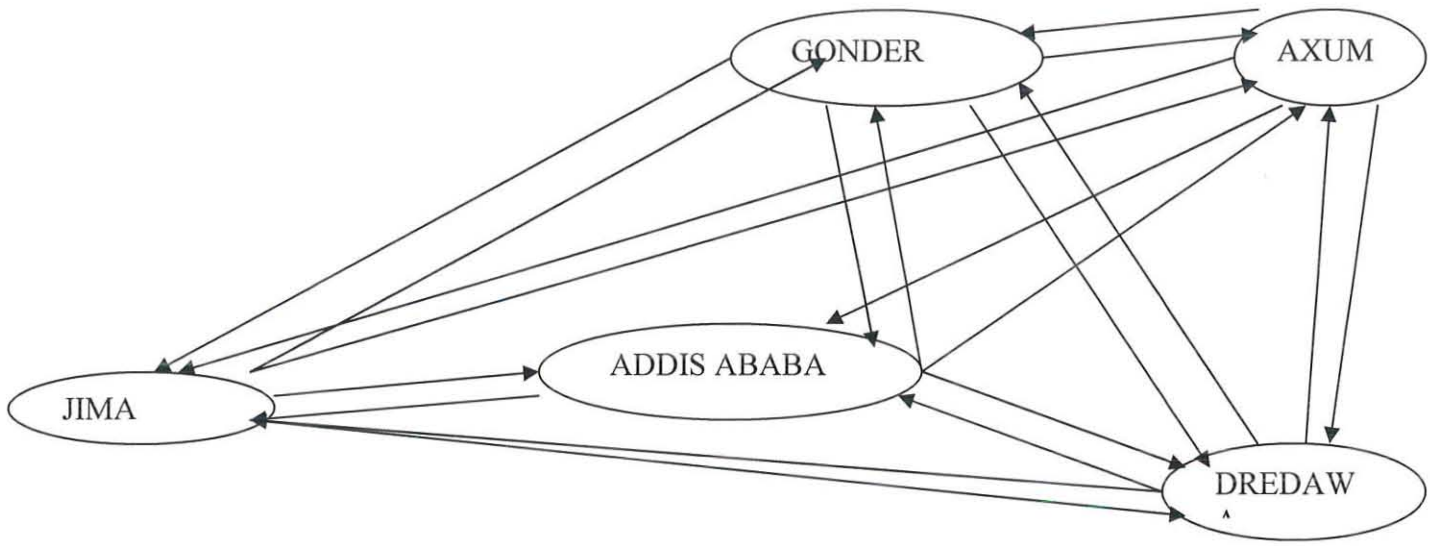
Thus the resulting solution will be: A → B → C → D → E →A

Step 3: Again, if we make an assignment for the remain three cases, no solution is obtained for the problem. Hence the best solution is:

A → B → C → D → E →A, and the total cost is: 2 + 2 + 4 + 4 + 1 =13

Problem 3. A person who lives in Addis Ababa wants to visit four historical places, which are found in Gonder, Axum, Dredawa, and Jima. He decides to pass through each city one and only once and return back to his home city Addis Ababa. If the distance (in Nautical mile) between any pair of the cities is given as a cost matrix below.

	Addis Ababa	Gonder	Axum	Dredawa	Jima
Addis Ababa	∞	227	308	195	152
Gonder	227	∞	121	344	293
Axum	308	121	∞	327	402
Dredawa	195	344	327	∞	320
Jima	152	293	402	320	∞



What should be the sequence of his trip in order to minimize the total distance traveled?

Solution:

Step 1:Subtract the smallest element 121 from all the entries to get the next reduced matrix.

Table 1

	Addis Ababa	Gonder	Axum	Dredawa	Jima
Addis Ababa	∞	106	187	74	31
Gonder	106	∞	0	223	172
Axum	187	0	∞	206	281
Dredawa	74	223	206	∞	199
Jima	31	172	281	199	∞

Step 2: Subtract the smallest element of each row and column from the corresponding rows and columns to get the next reduced matrix.

Table 2

	Addis Ababa	Dredawa	Axum	Dredawa	Jima
Addis Ababa	∞	75	156	0	0
Gonder	106	∞	0	180	172
Axum	187	0	∞	163	281
Dredawa	0	149	132	∞	125
Jima	0	141	250	125	∞

As we observe in the above table the number of lines, which cover all the zeros, is 4, which is less than 5. Thus, subtract the smallest uncovered number 125 from all uncovered elements and add it to the numbers at the intersection of the lines. The new matrix will be:

Table 3

	Addis Ababa	Gonder	Axum	Dredawa	Jima
Addis Ababa	∞	75	156	0_{As}	0_x
Gonder	231	∞	0_{As}	180	172
Axum	312	0_{As}	∞	163	281
Dredawa	0_x	24	7	∞	0_{As}
Jima	0_{As}	16	125	0_x	∞

From the above table the optimal solution is:
Addis Ababa → Dredawa → Jima → Addis Ababa and Gonder → Axum → Gonder, which is impossible. Thus, we try to find the "next best" Solution which satisfies the additional restriction. The smallest element other than zero is 7. So, consider the assignment in the fourth row and in the third column instead of the assignment in (Dredawa, Jima) and (Gonder, Axum). Then the new assignment will be given in the lower matrix.

Table-4

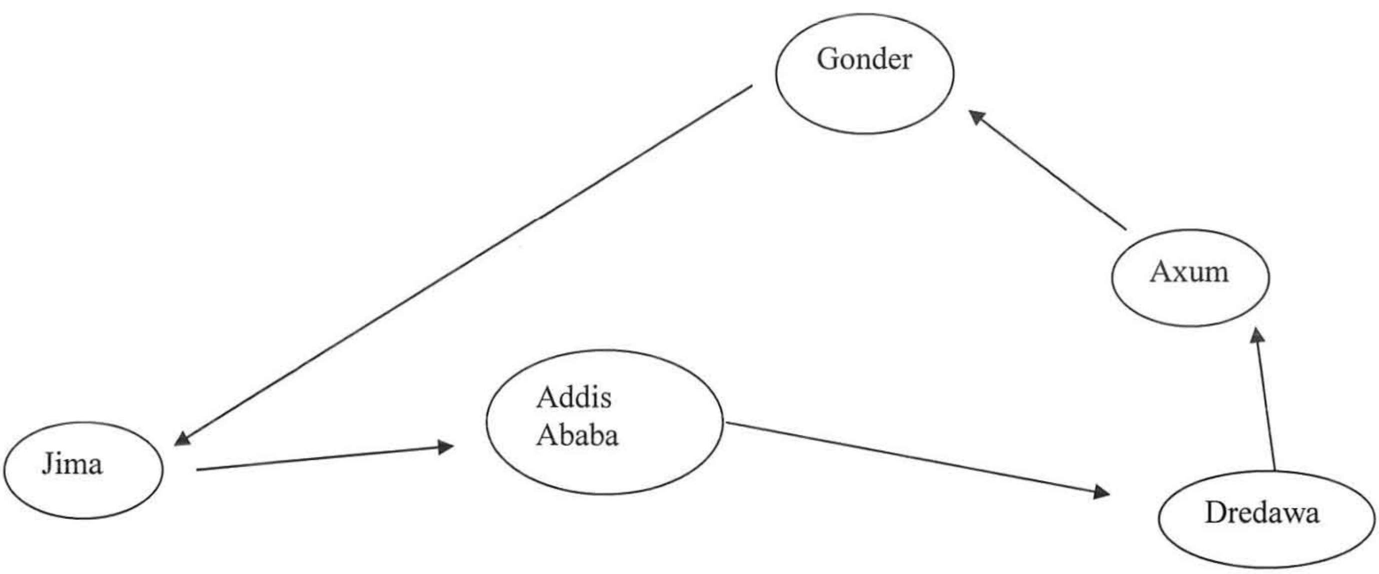
	Addis Ababa	Gonder	Axum	Dredawa	Jima
Addis Ababa	∞	75	156	0_{As}	0_x
Gonder	231	∞	0_x	180	172_{As}
Axum	312	0_{As}	∞	163	281
Dredawa	0_x	24	7_{As}	∞	0_x
Jima	0_{As}	16	125	0_x	∞

Thus, the visitor should have to travel as follow:

Addis Ababa → Dredawa → Axum → Gonder → Jima → Addis Ababa, and the total distance what he has covered will be:

195Nm + 327Nm + 121Nm + 293Nm + 152Nm = 1088Nm

He has to follow the next pattern of a trip



4. Conclusion:

From this seminar paper we can conclude that it is possible to solve any assignment problem using Hungarian method after translating into mathematical expression. This assignment algorithm also helps to solve different managerial decision problems. Even though there are many applications of this algorithm; I present only one of these, that is, the traveling salesman problem.

As we can observe from the examples what already discussed, it is too difficult solving such type of problems using the algorithm manually especially if the number of entries is large, as a result of this difficulty developing a C^{++} language for the algorithm is very important.

As a limitation of this seminar paper the following remains open for those who want to continue on this topic.

i) Presenting a theorem and its proof to ensure the application of assignment algorithm in solving a traveling salesman problem.

ii) Presenting a method of solving a traveling salesman problem using methods other than assignment algorithm.

iii) Generalizing the traveling salesman problem, that is, solving a traveling salesman problem for m cities and n men.

5. References

[1] Gv .Shenoy: Linear Programming methods and applications

2. Ed, Willey Eastern, 1992

[2] K.V Mital: Optimization Methods in Operational Research and Systems Analysis.

2. Ed, Willey Eastern, 1990

[3] V.K Kapoor: Problems and Solutions in Operational Research.

2. Ed, Sultan Chad and Sons, New Delhi, 1994

[4] S.D Sharma, Ph.D.: Operational Research for Management Decision.

1. Ed, Kedar Nath Ram. 1999.

[5] D.N.P Murthy and N.W.Page, Australia and E.Y.Rodin, Washington Mathematical Modeling.

1. Ed, Pergamon Press, 1990

[6] Harvey M. Wagner: Principle of Operations Research.

2. Ed, Prentice- Hall of India, 2001

[7] JN Kapur: Mathematical Modeling.

3. Ed, Willey Eastern, New Delhi, 1994.

[8] R.K. Gupta: Linear Programming.