

KURATOWSKI'S THEOREM

ADDIS ABABA UNIVERSITY



COLLEGE OF COMPUTATIONAL AND NATURAL SCIENCE
DEPARTMENT OF MATHEMATICS

A project submitted in partial fulfillment of the requirement of the degree of
master of science in mathematics

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Abstract

A graph is said to be planar if it can be drawn in the plane so that no two edges intersect except (possibly) at their ends vertices. It is of practical interest to be able to determine whether a graph is planar or not, our interest is how to check whether the graph is planar or not. In this project we present the proof of Kuratowski's theorem on planarity of graph and discuss examples on the application of the theorem.

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Introduction

Much of the work in graph theory is motivated and directed to the problem of planarity testing and construction of planar embeddings. Inversely, much of the development in graph theory is due to the study of planarity testing.

Auslander and Parter in 1961 and Goldstein in 1963 presented a first solution to the planarity-testing problem. Demoucron, Malgrance and Pertuiset also presented an algorithm in 1964. Lempel, Even and Ceberbaum, in 1967, gave a solution to the problem in time $o(n^3)$. In 1974, a significant breakthrough was achieved when Hopcroft and Tarjan, using path addition and depth first search, presented an algorithm to test planarity in linear time. Still, this algorithm only returns whether a graph is planar, but it does not actually find the planar embedding, although the authors showed that it could be constructed. Another disadvantage of this algorithm, like all other algorithms created later on to solve the problem in linear time, is that it is very complicated either to explain or to implement.

In 1976, two independent results showed together that the algorithm by Lempel, Even and Cederbaum mentioned above actually also can be implemented in $O(n)$ time. First, Even and Tarjan showed that an st-ordering of a bi-connected graph can be found in linear time. Then, Booth and Lueker introduced the PQ-tree, and showed that it can be used to test the consecutiveness of predecessors in overall linear time. Still, this algorithm does not compute the planar embedding of the graph, but in 1985, Chiba et al. showed how to use the algorithm to find the embedding in linear time.

In 1992, Hsu and Shih gave an entirely new approach to planarity testing, based on doing a depth-first search with appropriate bookkeeping. This approach was followed and clarified independently by Boyer and Myrvold, in 1999.

In 1996, Mehlhorn and Mutzel implemented Hopcroft and Tarjan's algorithm as part of the LEDA program package, and in the process, clarified the algorithm and explained how to find the planar embedding in linear time as well. Also in 1996, Di Battista and Tamassia developed an on-line planarity-testing algorithm. Here, the question is to decide whether the current graph is planar, under a sequence of changes to the graph that may or may not destroy or add planarity.

In graph theory, Kuratowski's theorem is a characterization of planar graphs, named after Kazimierz Kuratowski. It states that a finite graph is planar if and only if it does not contain a subgraph that is a subdivision of K_5 or of $K_{3,3}$.

Kazimierz Kuratowski published his theorem in 1930. Since then, multiple proofs of the theorem have been discovered. In the Soviet Union, Kuratowski's theorem was known as the ponyryagin

Kuratowski theorem, as its proof was allegedly first given in Lev Pontryagins unpublished notes. However, this usage has not spread to other places.

There are many practical situation in which it is important to decide whether a given graph is planar or not,for example in Electric circuit,Railway system etc. And,if so, then find a planar embedding of the graph .

This project work has two parts.

- In the first part about basic Graph theoretical concepts are discussed.
- In the second part,ideas about Bridges,and their properties; alongwith theorems used in the proof of Kuratowski's theorems, and of course, the proof of Kuratowski's theorem are discussed.Lastly an example is given to illustrate the application of the theorem.

Chapter 1

Preliminaries on Graphs

1.1 Graphs, The Incidence and Adjacency Matrices, Degree of a Vertex and Handshaking theorem

Definition 1.1.1. A graph G is an ordered triple $(V(G), E(G), \psi_G)$ consisting of a nonempty set $V(G)$ of vertices, a set $E(G)$, disjoint from $V(G)$, of edges, and an incidence function ψ_G that associates with each edge of G an unordered pair of (not necessarily distinct) vertices of G . If e is an edge and u and v are vertices such that $\psi_G(e) = uv$, then e is said to join u and v ; the vertices u and v are called the ends of e .

Example.

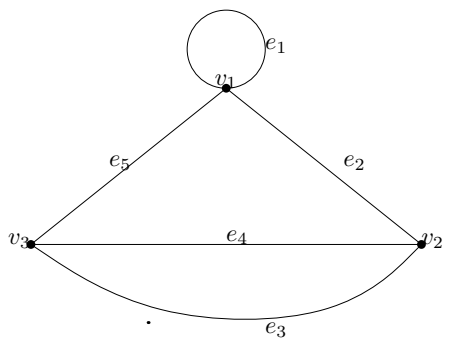


Figure 1.1: Graph

$$G = (V(G), E(G), \psi_G)$$

Where,

$$V(G) = \{v_1, v_2, v_3\}$$

$$E(G) = \{e_1, e_2, e_3, e_4, e_5\}$$

ψ_G is defined by

$$\psi_G(e_1) = v_1v_1, \psi_G(e_2) = v_1v_2, \psi_G(e_3) = v_2v_3, \psi_G(e_4) = v_2v_3, \psi_G(e_5) = v_1v_3$$

Definition 1.1.2. The number of vertices, is called the **order** of graph and denoted by $\nu(G)$ and the number of edges is called the **size** of graph and denoted by $\varepsilon(G)$.

Note We will use the symbol ν and ε instead of $\nu(G)$ and $\varepsilon(G)$ respectively for our next discussion.

Example.

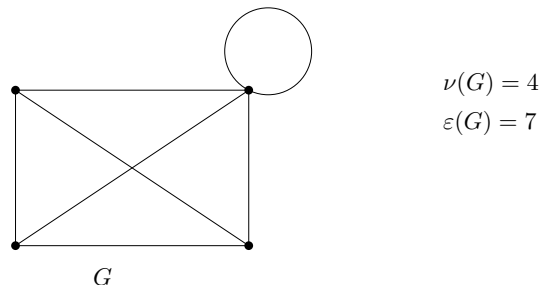


Figure 1.2:

- Definition 1.1.3.**
1. The ends of an edge are said to be incident with the edge, and vice versa.
 2. Two vertices which are incident with a common edge are adjacent, as are two edges which are incident with a common vertex.
 3. An edge with identical ends is called a **loop**, and an edge with distinct ends is a **link**.
 4. Parallel edges are two edges that are incident to the same two vertices.

Definition 1.1.4. A graph which does not contain loops or parallel edges is called a **simple graph**. A graph with only one vertex is trivial.

Example. Figure 1.3 (a) is simple graph and (b) is not simple graph .

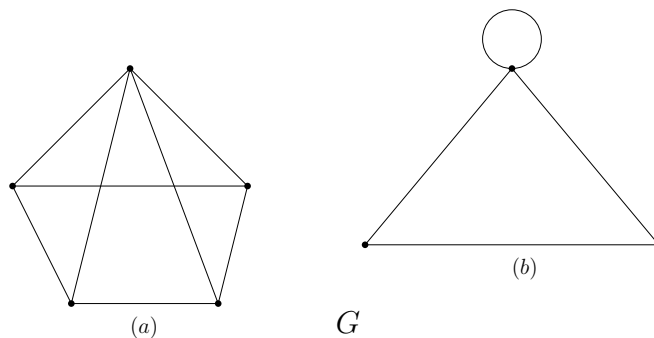
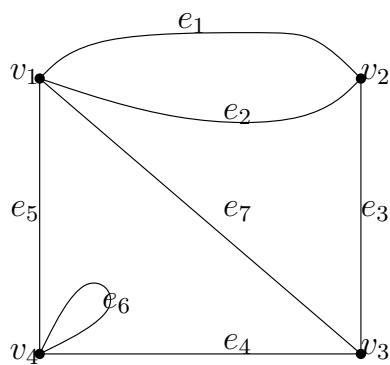


Figure 1.3: Graph

Definition 1.1.5. To any graph G there corresponds a $\nu \times \varepsilon$ matrix called the **incidence matrix** of G . Let us denote the vertices of G by v_1, v_2, \dots, v_ν and the edges by $e_1, e_2, \dots, e_\varepsilon$. Then the incidence matrix of G is the matrix $M(G) = [m_{ij}]$, where m_{ij} is the number of times that v_i and e_j are incident.

Another matrix associated with G is the **adjacency matrix**; this is the $\nu \times \nu$ matrix $A(G) = [a_{ij}]$, in which a_{ij} is the number of edges joining v_i and v_j .

Example.



G

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
v_1	1	1	0	0	1	0	1
v_2	1	1	1	0	0	0	0
v_3	0	0	1	1	0	0	1
v_4	0	0	0	1	1	2	0

$M(G)$

	v_1	v_2	v_3	v_4
v_1	0	2	1	1
v_2	2	0	1	0
v_3	1	1	0	1
v_4	1	0	1	1

$A(G)$

Figure 1.4:

Definition 1.1.6. The degree $d_G(v)$ of a vertex v in G is the number of edges of G incident with v , each loop counting as two edges. $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum degrees, respectively, of vertices of G .

A graph G is k -regular if each vertex in G has degree k

Example.

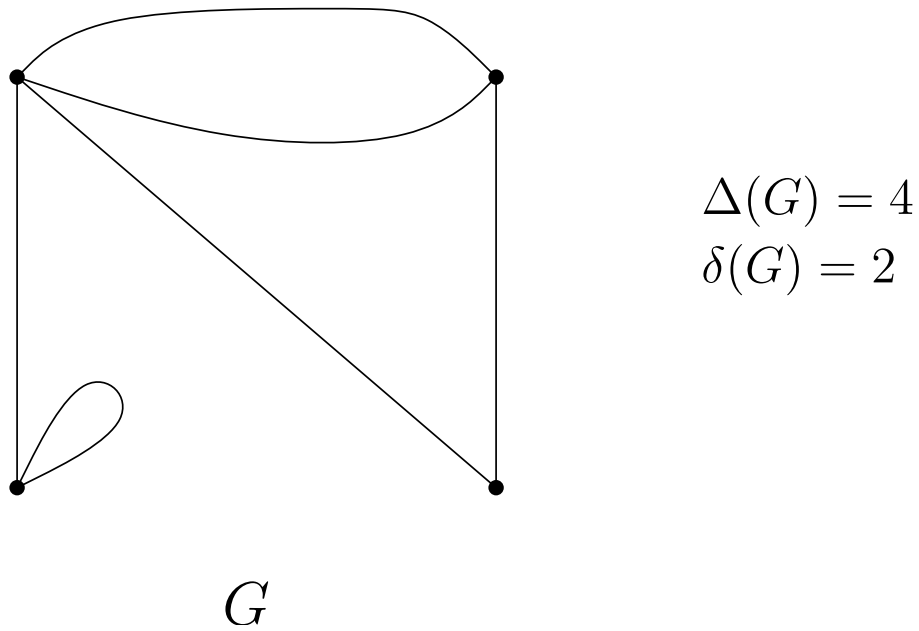


Figure 1.5:

Theorem 1.1. (Handshaking Theorem) $\sum_{v \in V} d(v) = 2\varepsilon$

Proof. Consider the incidence matrix M . The sum of the entries in the row corresponding to vertex v is precisely $d(v)$, and therefore $\sum_{v \in V} d(v)$ is just the sum of all entries in M . But this sum is also 2ε , since each of the e column sums of M is 2

1.2 Subgraphs, Path and Cycles, Connectedness

Definition 1.2.1. A graph H is a **subgraph** of G (written $H \subseteq G$) if $V(H) \neq \emptyset \subseteq V(G)$, $E(H) \subseteq E(G)$, and ψ_H is the restriction of ψ_G to $E(H)$. When $H \subseteq G$ but $H \neq G$, we write $H \subset G$ and call H a **proper subgraph** of G .

Example. H is a subgraph of G and since $H \neq G$ H is a proper subgraph of G

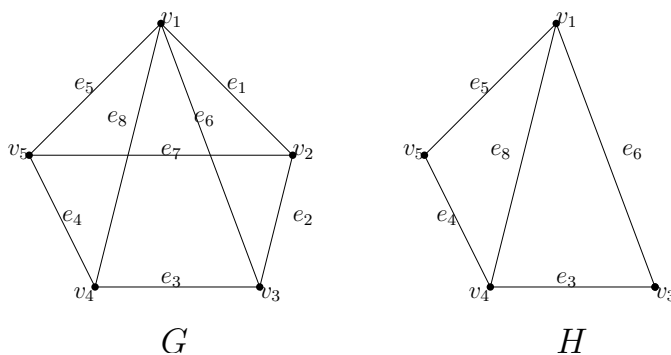


Figure 1.6: A graph G and a Subgraph H

Definition 1.2.2. Let H_1 and H_2 be subgraphs of G . We say that H_1 and H_2 are **disjoint** if they have no vertex in common, and **edge-disjoint** if they have no edge in common.

Definition 1.2.3. Suppose that V' is a nonempty subset of V . The subgraph of G whose vertex set is V' and whose edge set is the set of those edges of G that have both ends in V' is called the **subgraph of G induced by V'** and is denoted by $G[V']$; we say that $G[V']$ is an **induced** subgraph of G .

Example.

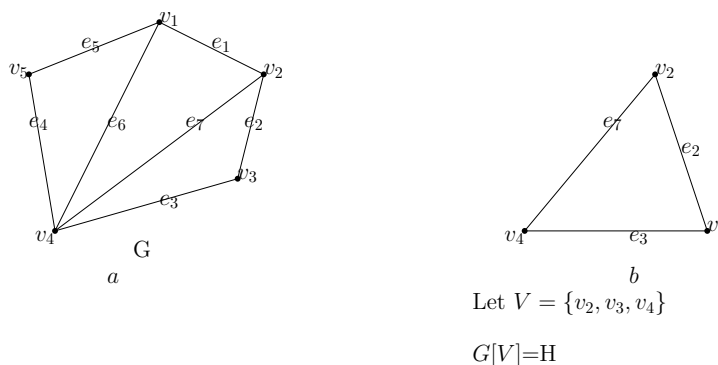


Figure 1.7: A graph G and Induced Subgraph H

Definition 1.2.4. Let $G = (V, E)$ be a graph and $H = (V', E')$ be a subgraph of G . The subgraph H is a **spanning** subgraph of G if $V' = V$.

Example.

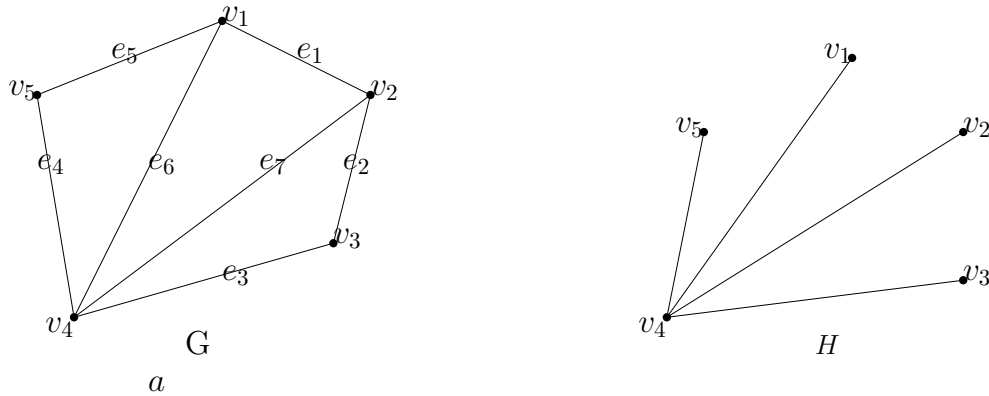


Figure 1.8: Spanning Subgraph H and a graph G

Definition 1.2.5. A **walk** in a graph G is a finite non-empty sequence $w = v_0e_1v_1e_2v_2\dots e_kv_k$, whose terms are alternately vertices and edges, such that, for $1 \leq i \leq k$, the ends of e_i are v_{i-1} and v_i . We say that w is a walk from v_0 to v_k , or a (v_0, v_k) -walk. The vertices v_0 and v_k are called the origin and terminus of w , respectively, and v_1, v_2, \dots, v_{k-1} its internal vertices. The integer k is the length of w . If $w = v_0e_1v_1e_2v_2\dots e_kv_k$ and $w' = v_ke_{k+1}v_{k+1}\dots e_1v_1$ are walks, the walk $v_ke_kv_{k-1}\dots e_1v_0$, obtained by reversing w , is denoted by w^{-1} and the walk $v_0e_1v_1\dots e_1v_1$, obtained by concatenating w and w at v_k , is denoted by ww' .

A section of a walk $w = v_0e_1v_1\dots e_kv_k$ is a walk that is a subsequence $v_ie_{i+1}v_{i+1}\dots e_jv_j$ of consecutive terms of w ; we refer to this subsequence as the (v_i, v_j) -section of w .

Definition 1.2.6. If the edges e_1, e_2, \dots, e_k of a walk w are distinct, w is called a **trail**; if the vertices v_0, v_1, \dots, v_k are distinct, w is called a **path**.

Example.

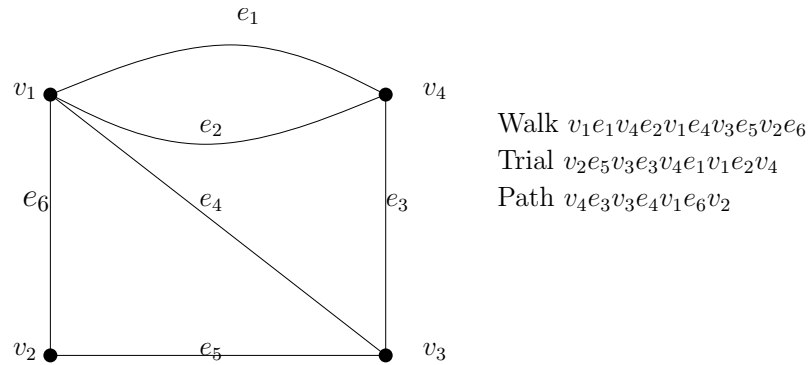


Figure 1.9:

Definition 1.2.7. The distance $d(u, v)$ between two vertices u and v of a finite graph is the number of edges in a shortest path connecting them

Well Ordering Axioms:-Every nonempty set of nonnegative integers has a least element.

Definition 1.2.8. A walk is closed if it has positive length and its origin and terminus are the same. A **Cycle** is a kind of walk which the starting and ending vertices are the same and edge and intermediate vertices are distinct. A cycle of length k is called a k -cycle; a k -cycle is odd or even depending on whether k is even or odd.

Definition 1.2.9. Two vertices u and v of G are said to be connected if there is a (u, v) -path in G . G is connected if there is a (u, v) -path for every $u, v \in V$.

Connection is an equivalence relation on the vertex set V . Thus There is a partition of V into nonempty subsets V_1, V_2, \dots, V_k such that two vertices u and v are connected if and only if u and v belong to the same set V_i where $i = 1, \dots, k$. The subgraphs $G[V_1], G[V_2], \dots, G[V_k]$ are called the **components** of G .

Note We denote the number of components of G by $\omega(G)$.

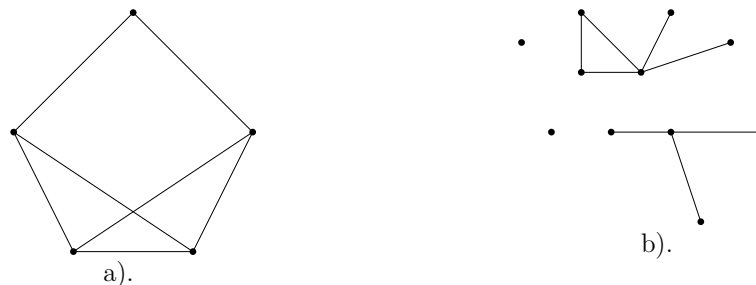


Figure 1.10: a).Connected Graph and b).Disconnected Graph with 4 Components

1.3 Some Classes of graphs

Definition 1.3.1. A **complete graph** is a simple graph in which every two distinct vertices are adjacent. A complete graph with n vertices is denoted by K_n .

Example.

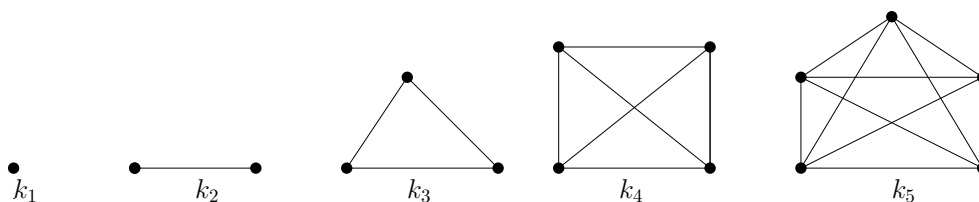


Figure 1.11: Complete Graph

Definition 1.3.2. A **bipartite graph** is a graph whose vertex set can be partitioned into two subsets X and Y such that each edge has one end in X and one end in Y . Such a partition (X, Y) is called a **bipartition of the graph**.

A **complete bipartite graph** is a bipartite graph with bipartition (X, Y) in which every vertex in X is adjacent to every vertex in Y . The complete bipartite graph with parts of size $|X| = n$ and $|Y| = m$ is denoted by $K_{n,m}$.

Example.

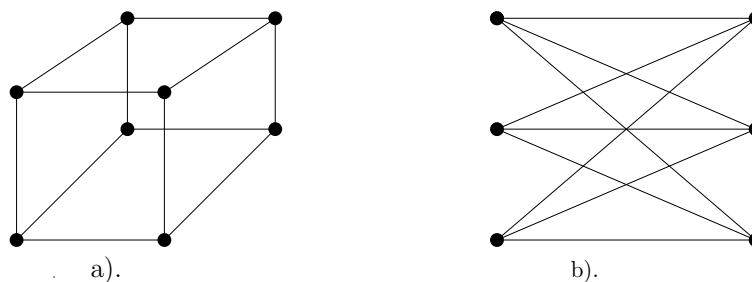


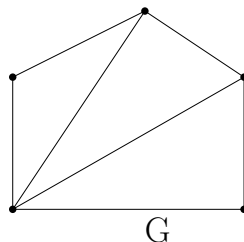
Figure 1.12: Bipartite and Complete Bipartite Graph

1.4 Cut vertex, Cut edge ,Connectivity and Blocks

Definition 1.4.1. A vertex v in G is a cut vertex if the number of components in $G - v$ is more than the number of components in G . An edge e in G is a cut edge if the number of components in $G - e$ is more than the number of components in G .

Definition 1.4.2. A vertex cut of a (connected) graph G is a subset $U \subset V$ such that $G - U$ is disconnected. A k -vertex cut is a vertex cut of k vertices. The connectivity, $\kappa(G)$, is the minimum k for which G has k -vertex cut. G is said to be k -connected if $\kappa(G) \geq k$.

Example.



$\kappa = 2$ (i.e. G is disconnected when we remove a minimum of 2 vertices).
 G is 1-connected, 2-connected

Figure 1.13:

Definition 1.4.3. A connected graph that has no cut vertices is called a **block**.

Example.

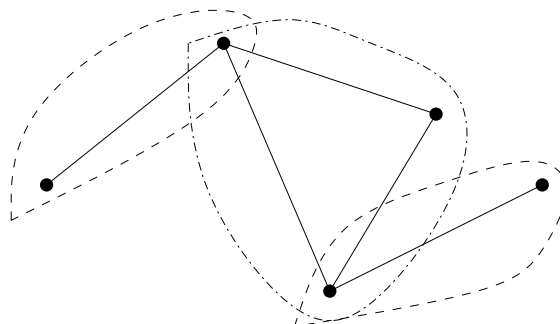


Figure 1.14: Block of Graph

Definition 1.4.4. A cut edge of G is an edge e such that $\omega(G - e) > \omega(G)$.

Definition 1.4.5. A vertex v of G is a cut vertex if E can be partitioned into two nonempty subsets E_1 and E_2 such that $G[E_1]$ and $G[E_2]$ have just the vertex v in common. If G is loopless and nontrivial, then v is a cut vertex of G if and only if $\omega(G - v) > \omega(G)$.

Theorem 1.2. An edge e of G is a cut edge of G if and only if e is Contained in no cycle of G .

Proof. Let e be a cut edge of G . Since $\omega(G - e) > \omega(G)$, there exist vertices u and v of G that are connected in G but not in $G - e$. Therefore some (u, v) -path P in G which traverses e . Suppose that x and y are the ends of e , and that x precedes y on P . In $G - e$, u is connected to x by a section of P and y is connected to v by a section of P . If e were in a cycle C , x and y would be connected in $G - e$ by the path $C - e$. Thus, u and v would be connected in $G - e$, a contradiction.

Conversely, suppose that $e = xy$ is not a cut edge of G ; thus, $\omega(G - e) = \omega(G)$. Since there is an (x, y) -path (namely xy) in G , x and y are in the same component of G . It follows that x and y are in the same component of $G - e$, and hence that there is an (x, y) -path P in $G - e$. But then e is in the cycle $P + e$ of G , a contradiction.

Definition 1.4.6. A family (collection) of paths in G is called **internally-disjoint (or independent)** if no vertex of G is an internal vertex of more than one path in the family.

A family (collection) of paths in G is called **edge-disjoint** if no edge of G belongs to more than one path in the family.

Example.

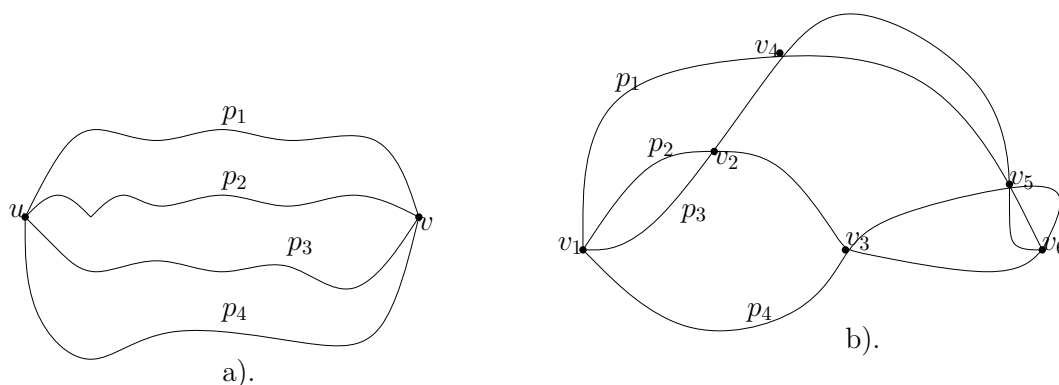


Figure 1.15: a).Internally Disjoint or Independent path and b).Edge Disjoint path

Theorem 1.3. A graph G with $\nu \geq 3$ is 2-connected if and only if any two vertices of G are connected by at least two internally-disjoint paths

Proof. If any two vertices of G are connected by at least two internally-disjoint paths then, clearly, G is connected and has no 1-vertex cut. Hence G is 2-connected.

Conversely, let G be a 2-connected graph. We shall prove, by induction on the distance $d(u, v)$ between u and v , that any two vertices u and v are connected by at least two internally-disjoint paths. Suppose, first, that $d(u, v) = 1$. Then, since G is 2-connected, the edge uv is not a cut edge and therefore, by theorem 1.2, it is contained in a cycle. It follows that u and v are connected by two internally-disjoint paths in G . Now assume that the theorem holds for any two vertices at distance less than k , and let $d(u, v) = k \geq 2$. Consider a (u, v) -path of length k , and let w be the vertex that precedes v on this path. Since $d(u, w) = k - 1$, it follows from the induction hypothesis that there are two internally-disjoint (u, w) -paths P and Q in G . Also, since G is 2-connected, $G - w$ is connected and so contains a (u, v) -path P' . Let x be the last vertex of P' that is also in $P \cup Q$ (see fig.1.16). Since u is in $P \cup Q$, there is such an x ; we do not exclude the possibility that $x = v$. We may assume, without loss of generality, that x is in P . Then G has two internally-disjoint (u, v) -paths, one composed of the section of P from u to x together with the section of P' from x to v , and the other composed of Q together with the path wv .

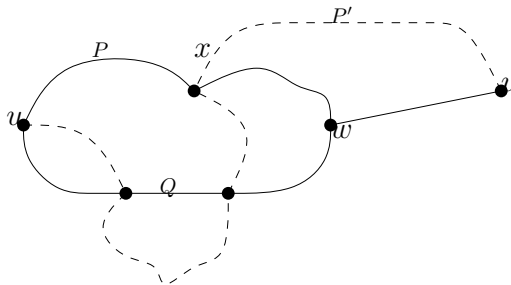


Figure 1.16:

Corollary 1.1. 1 If G is 2-connected, then any two vertices of G lie on a common cycle.

Proof. This follows immediately from the above theorem since two vertices lie on a common cycle if and only if they are connected by two internally-disjoint paths

1.5 Trees

Definition 1.5.1. An acyclic graph is one that contains no cycles. A tree is a connected acyclic graph.

Theorem 1.4. If G is a tree, then $\varepsilon = v - 1$.

Proof. By induction on v . When $v = 1$, and $\varepsilon = 0 = v - 1$.

Suppose the theorem true for all trees on fewer than v vertices, and let G be a tree on $v \geq 2$ vertices. Let $uv \in E$. Then $G - uv$ contains no (u, v) -path, since uv is the unique (u, v) -path in G . Thus $G - uv$ is disconnected and so $\omega(G - uv) = 2$. The components G_1 and G_2 of $G - uv$, being acyclic, are trees. Moreover, each has fewer than v vertices. Therefore, by the induction hypothesis

$$\varepsilon(G_i) = \nu(G_i) - 1 \text{ for } i = 1, 2$$

$$\text{Thus } \varepsilon(G) = \varepsilon(G_1) + \varepsilon(G_2) + 1 = \nu(G_1) + \nu(G_2) - 1 = \nu(G) - 1$$

1.6 Subdivision of a Graph

Definition 1.6.1. An edge e is said to be **subdivided** when it is replaced by a path of length two, the internal vertex of this path being a new vertex.

Example.

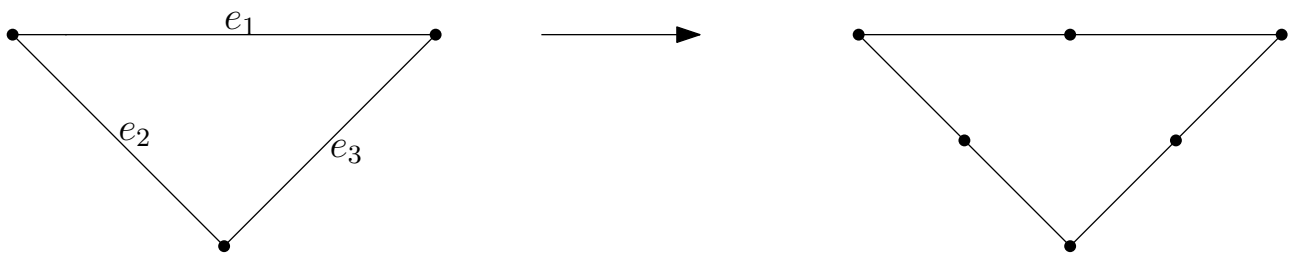


Figure 1.17: Subdivision of Graphs

Definition 1.6.2. a subdivision of a graph G is a graph resulting from the subdivision of edges in G .

Note:- Let us expand a graph $G = (V, E)$ by a sub division one of its edge $e = (u, v) \in E$ we put a new node w one and replace it by two new edge $e_1 = (u, w)$ and $e_2 = (w, v)$, the new graph is thus given by $G' = \{V \cup \{w\}, E \cup \{e_1, e_2\} \setminus \{e\}\}$

Chapter 2

Kuratowski's Theorem

2.1 Planar Graph

Definition 2.1.1. A graph G is said to be planar or embeddable in the plane if it can be drawn in the plane so that no two edges intersect except (possibly) at their end vertices; otherwise it is said to be a **nonplanar graph**. A planar graph embedded in the plane is called a **plane graph**.

Definition 2.1.2. A Minimal nonplanar graph is a nonplanar graph such that every proper subgraph is planar.

Example. Figure 2.1 shows a planar graph G_1 , two plane graphs G_2, G_3

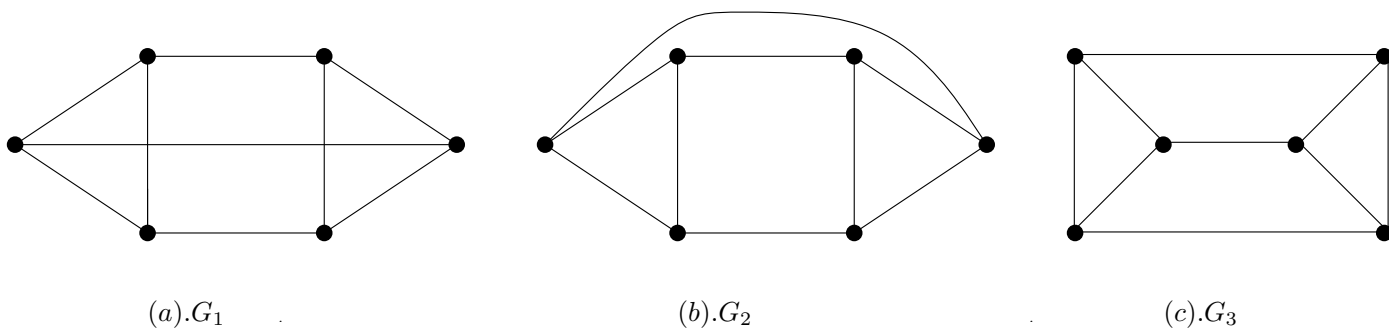


Figure 2.1: Planar Graph, Plane Graphs

2.1.1 Jordan curve and a nonplanar graph K_5 and $K_{3,3}$

Definition 2.1.3. Given any two points a and b in the plane, any non-self intersecting continuous curve from a to b is called **Jordan curve** and it denoted by $J[a, b]$ if $a = b$, then J is called a **closed Jordan curve**

Let J be a closed Jordan curve in the plane. Then the rest of the plane is partitioned into two disjoint open sets called the interior and exterior of J . We denote the interior and exterior of J , respectively, by $intJ$ and $extJ$. Clearly $IntJ \cap ExtJ = J$. The Jordan curve theorem states that any line joining a point in $intJ$ to a point in $extJ$ must meet J at some point.

Theorem 2.1. K_5 is nonplanar.

Proof. This proof uses the Jordan curve theorem. Assume the contrary, namely, K_5 is planar. Let v_1, v_2, v_3, v_4 , and v_5 be the vertices of K_5 in a plane representation of K_5 . The cycle $C = v_1v_2v_3v_4v_1$ (as a closed Jordan curve) divides the plane into two faces, namely, the interior and the exterior of C . The vertex v_5 must belong either to $intC$ or to $extC$. without loss of generality, we can assume that v_5 belongs to $intC$. Suppose that v_5 belongs to $intC$. Draw the edges v_5v_1, v_5v_2, v_5v_3 , and v_5v_4 in $intC$ because they are not conflicting (don't cross each other). Now there remain two more edges v_1v_3 and v_2v_4 to be drawn. None of these can be drawn in $intC$, since it is assumed that K_5 is planar. Thus, v_1v_3 lies in $extC$. Then one of v_2 and v_4 belongs to the interior of the closed Jordan curve $C_1 = v_1v_5v_3v_1$ and the other to its exterior (see Fig. 2.2). Hence, v_2v_4 cannot be drawn without violating planarity.

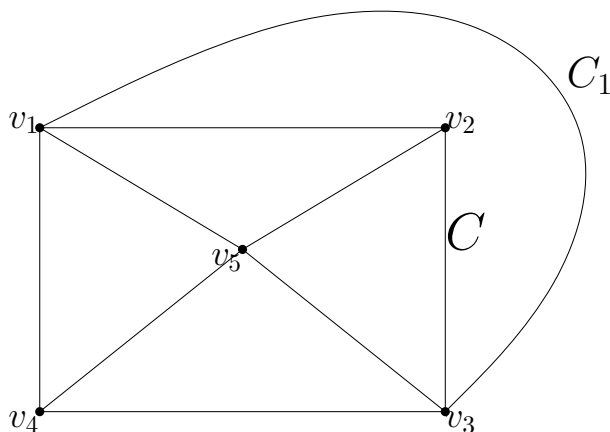


Figure 2.2: Graph for the proof

Theorem 2.2. $K_{3,3}$ is nonplanar.

Proof. The proof is by the use of the Jordan curve theorem. Suppose that $K_{3,3}$ is planar. Let $U = \{u_1, u_2, u_3\}$ and $V = \{v_1, v_2, v_3\}$ be the bipartition of $K_{3,3}$ in a plane representation of the graph. Consider the cycle $C = u_1v_1u_2v_2u_3v_3u_1$. Since the graph is assumed to be planar, the edge u_1v_2 must lie either in the interior of C or in its exterior. For the sake of definiteness, assume that it lies in $\text{int}C$ (a similar proof holds if one assumes that the edge u_1v_2 lies in $\text{ext}C$). Two more edges remain to be drawn, namely, u_2v_3 and u_3v_1 . None of these can be drawn in $\text{int}C$ without crossing the edge u_1v_2 . Hence, both of them are to be drawn in $\text{ext}C$. Now draw u_2v_3 in $\text{ext}C$. Then one of v_1 and u_3 belongs to the interior of the closed Jordan curve $C_1 = u_1v_2u_2v_3u_1$ and the other to the exterior of C_1 (see Fig. 2.3). Hence, the edge v_1u_3 cannot be drawn without violating planarity. This shows that $K_{3,3}$ is nonplanar.

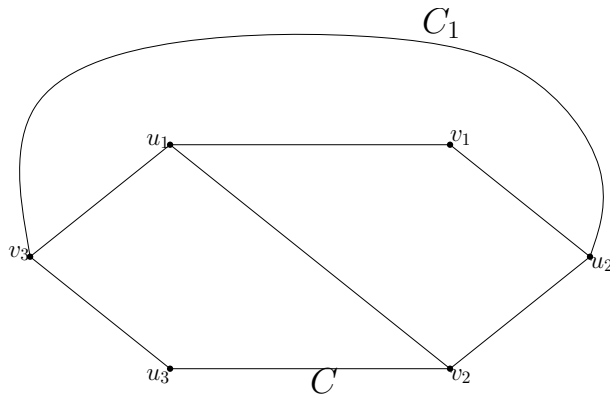


Figure 2.3: Graph for the proof

We observe that the two graphs K_5 and $K_{3,3}$ have the following common properties.

1. Both are regular.
2. Both are nonplanar.
3. Removal of one edge or a vertex makes each a planar graph.
4. K_5 is a nonplanar graph with the smallest number of vertices, and $K_{3,3}$ is the nonplanar graph with smallest number of edges (Hence, any nonplanar graph must have at least five vertices and nine edges.)

2.1.2 Plane Graphs and its Dual Graphs

Definition 2.1.4. A plane graph partitions the plane into a number of regions. These regions are called **faces**.

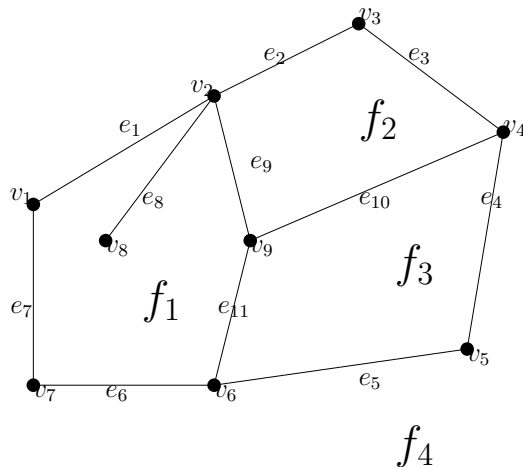
Note

- there is one unbounded face . This is called the **exterior face**.
- We denote $F(G)$ and $\phi(G)$, respectively, the set of faces and the number of faces of a plane graph G .

Definition 2.1.5. The boundary of a face f is denoted by $b(f)$. The boundary can be regarded as a closed walk with cut-edges traversed twice.

Definition 2.1.6. A face f is said to be incident with the vertices and edges in its boundary. If e is a cut-edge, there is only one face incident to it. Otherwise, there are two faces incident with e . We say that an edge separates the faces incident with it. The degree, $deg(f)$, of a face f is the number of edges with which it is incident (cut edges are counted twice).

Example.



from the this figure f_4 is the exterior region or infinite face
and f_1, f_2 and f_3 are interior region

$$b(f_1) = v_1e_1v_2e_8v_8e_8v_2e_9v_9e_9e_{11}v_6e_6v_7e_7v_1 \quad b(f_2) = v_2e_2v_3e_3v_4e_{10}v_9e_9v_2$$

$$deg(f_2) = 4 \text{ and } deg(f_4) = 7$$

Figure 2.4:

Definition 2.1.7. Given a plane graph G , let's define another graph G^* as follows: corresponding to each face f of G there is a vertex f^* of G^* , and corresponding to each edge e of G there is an edge e^* of G^* ; two vertices f^* and g^* are joined by the edge e^* in G^* if and only if their corresponding faces f and g are separated by the edge e in G .

Definition 2.1.8. G^* is called the dual of G .

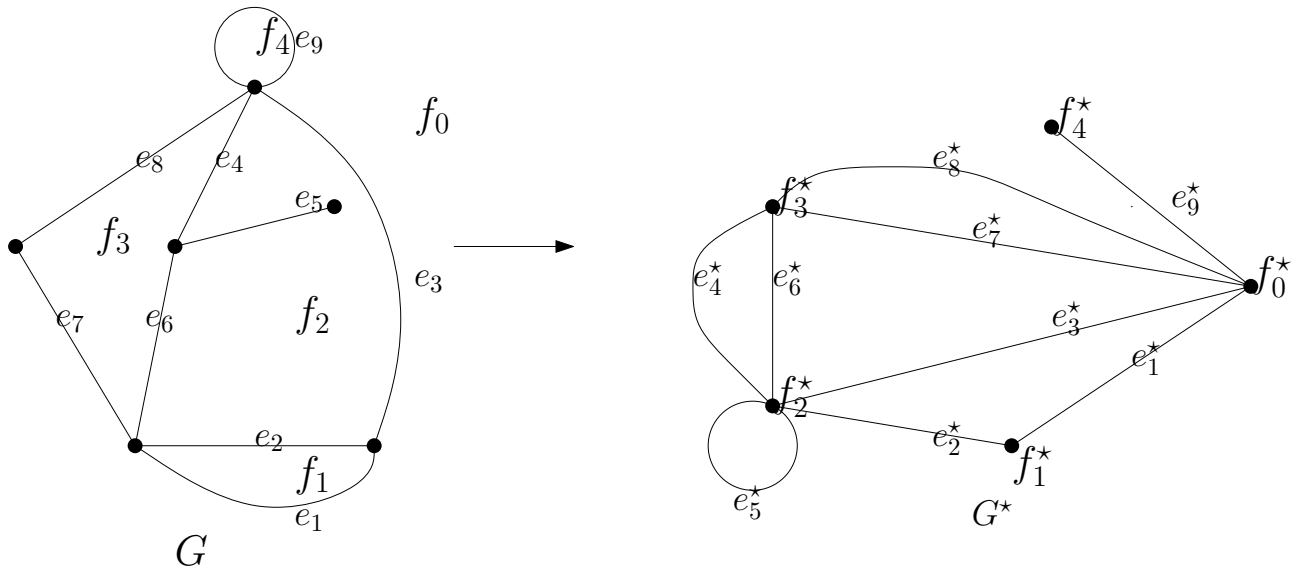


Figure 2.5: Plane Graph and its Dual Graph

Note:- if e is a loop of G , then e^* is a cut edge of G^* , and vice versa. The following relations are direct consequences of the definition of G^* :

- $\nu(G^*) = \phi(G)$
- $\varepsilon(G^*) = \varepsilon(G)$
- $d_{G^*}(f^*) = d_G(f)$ for all $f \in F(G)$ and $f^* \in V(G^*)$

Theorem 2.3. If G is plane graph, Then

$$\sum_{f \in F} d(f) = 2\varepsilon$$

Proof. Let G^* be the Dual of G , Then

$$\sum_{f \in F(G)} d(f) = \sum_{f^* \in V(G^*)} d(f^*) \text{ by above relation}$$

$$2\varepsilon(G^*) \text{ by theorem 1.1}$$

$$2\varepsilon(G) = 2\varepsilon \text{ by the above relation}$$

2.1.3 Euler's Formula

There is a simple formula relating the numbers of vertices, edges and faces in a connected plane graph. It is known as Euler's formula

Theorem 2.4. *If G is a connected plane graph, then $\nu - \varepsilon + \phi = 2$*

Proof. *By induction on ϕ , the number of faces of G . If $\phi = 1$, then each edge of G is a cut edge and so G , being connected, is a tree. In this case $\varepsilon = \nu - 1$, by theorem 1.4, and the theorem clearly holds. Suppose that it is true for all connected plane graphs with fewer than n faces, and let G be a connected plane graph with $n \geq 2$ faces. Choose an edge e of G that is not a cut edge. Then $G - e$ is a connected plane graph and has $n - 1$ faces, since the two faces of G separated by e combine to form one face of $G - e$. By the induction hypothesis*

$$\nu(G - e) - \varepsilon(G - e) + \phi(G - e) = 2$$

and, using the relations

$$\nu(G - e) = \nu(G), \varepsilon(G - e) = \varepsilon(G) - 1, \phi(G - e) = \phi(G) - 1$$

We obtain

$$\nu(G) - \varepsilon(G) + \phi(G) = 2$$

The theorem follows by principle of Induction.

Corollary 2.1. *If G is a simple planar graph with $\nu \geq 3$, then $\varepsilon \leq 3\nu - 6$.*

Proof. *It clearly suffices to prove this for connected graphs. Let G be a simple connected graph with $\nu \geq 3$. Then $d(f) \geq 3$ for all $f \in F$, and*

$$\sum_{f \in F} d(f) \geq 3\phi$$

by theorem 2.3

$$2\varepsilon \geq 3\phi$$

Thus, from theorem 2.4

$$\nu - \varepsilon + \frac{2\varepsilon}{3} \geq 2 \text{ or } \varepsilon \leq 3\nu - 6$$

Corollary 2.2. *K_5 is nonplanar.*

Proof. *If K_5 were planar, Then by corollary 2.1, we would have*

$$10 = \varepsilon(K_5) \leq 3\nu(K_5) - 6 = 9$$

Thus K_5 must be nonplanar.

Corollary 2.3. *$K_{3,3}$ is nonplanar.*

Proof. *Suppose that $K_{3,3}$ is planar and let G be a planar embedding of $K_{3,3}$. Since $K_{3,3}$ has no cycles of length less than four, every face of G must have degree at least four. Therefore, by theorem 2.3, we have*

$$4\phi \leq \sum_{f \in F} d(f) = 2\varepsilon = 18$$

That is $\phi \leq 4$

Theorem 2.4 now implies that

$$2 = \nu - \varepsilon + \phi \leq 6 - 9 + 4 = 1$$

Which is contradict our supposition .

Therefore $k_{3,3}$ is nonplanar.

2.2 Bridges

In the study of planar graphs, certain subgraphs, called **bridges**, play important roles. Properties of these subgraphs will be discussed in this section. Let H be a given subgraph of a graph G . We define a relation " \sim " on $E(G) \setminus E(H)$ by the condition that $e_i \sim e_j$ if there exists a walk w such that

- i). The first and last edges of w are e_i and e_j , respectively.
- ii). w is internally-disjoint from H (that is, no internal vertex of w is a vertex of H).

" \sim " is an equivalence relation on $E(G) \setminus E(H)$.

Proof.

1. Reflexive:-

Let e_i be in $E(G) \setminus E(H)$ then there exist a walk $w' = e_i$ such that

- i The first and last edges of w' is e_i
- ii Since e_i does not have an internal vertex, w' is internally-disjoint from H (that is, no internal vertex of w' is a vertex of H). Hence $e_i \sim e_i$.

2 .Symmetric:-

Suppose that $e_i \sim e_j$ then by definition there exists a walk w such that

- i . The first and last edges of w are e_i and e_j respectively, from this by reversing w we get another walk w^{-1} (let us denote by $w^{-1} = w'$) such that the first and last edges of w' are e_j and e_i respectively.
- ii . w is internally-disjoint from H (that means, no internal vertex of w is a vertex of H). w' also have a vertices of w and edges of w , we change only the direction of the walk. Therefore, no internal vertex of w' is a vertex of H (i.e. w' is internally-disjoint from H). Hence $e_j \sim e_i$.

3 .Transitivity:- Suppose that $e_i \sim e_j$ and $e_j \sim e_k$ then by definition there exist a walk w_1 and w_2 such that

- i .The first and last edges of w_1 are e_i and e_j respectively, and that of w_2 are e_j and e_k respectively. By concatenating w_1 and w_2 , we get another walk $w' = w_1w_2$ such that the first and last edges of w' are e_i and e_k respectively.
- ii .No internal vertex of w_1 is a vertex of H and no internal vertex of w_2 is a vertex of H . This implies that no internal vertex of w' is a vertex of H . Therefore, w' is internally-disjoint from H . Hence $e_i \sim e_k$.

Therefore, " \sim " is an equivalence relation on $E(G) \setminus E(H)$.

Definition 2.2.1. A subgraph of $G - E(H)$ induced by an equivalence class under the relation " \sim " is called a **bridge** of H in G .

Example.

In figure 2.6 B_1 and B_2 are bridges of H in G .

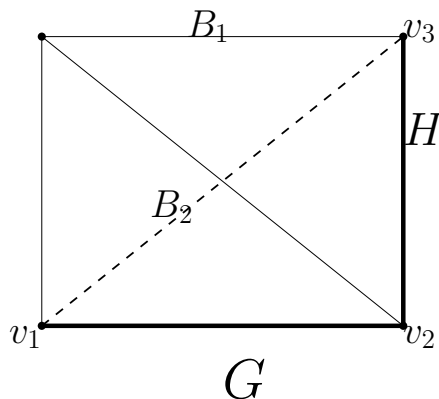


Figure 2.6: Bridges in Graph

Note

- 1 .If B is a bridge of H , then B is a connected graph and, moreover, that any two vertices of B are connected by a path that is internally-disjoint with H .
- 2 .Two bridges of H have no vertices in common except, possibly, for vertices of H .

Definition 2.2.2. For a bridge B of H , we write $V(B) \cap V(H) = V(B, H)$, and call the vertices in this set the vertices of attachment of B to H .

Example. In Figure 2.6 the vertices v_1 , v_2 and v_3 are the vertices of attachment of B_1 to H .

Now let us consider with bridges of a cycle C . let as abbreviated 'bridge of C ' to 'bridge'

Definition 2.2.3. A bridge with k vertices of attachment is called a **k -bridge**. Two k -bridges with the same vertices of attachment are equivalent **k -bridges**.

Definition 2.2.4. The vertices of attachment of a k bridge B with $k \geq 2$ effect a partition of C into edge-disjoint paths, called the **segments** of B .

Definition 2.2.5. Two bridges **avoid one another** if all the vertices of attachment of one bridge lie in a single segment of the other bridge; otherwise they **overlap**.

Example. In figure 2.7 B_1 and B_2 share all their 3 vertices of attachment or are equivalent 3-bridges.

B_2 and B_4 avoid one another, but B_3 and B_4 overlap.

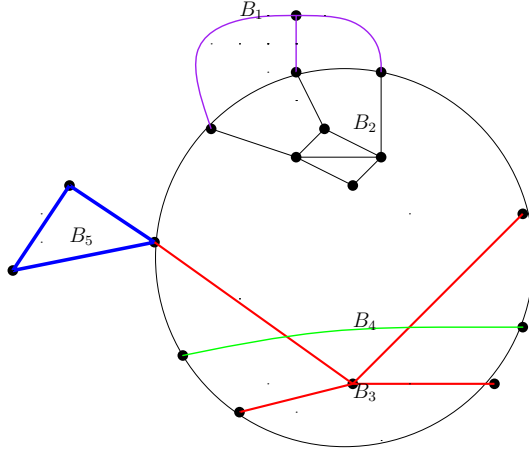


Figure 2.7: Bridges in Graph

Definition 2.2.6. Two bridges B and B' are **skew** if there are four distinct vertices u, v, u' and v' of C such that u and v are vertices of attachment of B , u' and v' are vertices of attachment of B' , and the four vertices appear in the cyclic order u, u', v, v' on C .

Example. In figure 2,7 , B_3 and B_4 are skew, but B_2 and B_4 are not skew.

Theorem 2.5. If two bridges overlap, then either they are **skew** or else they are equivalent **3-bridges**.

Proof. Suppose that the bridges B and B' overlap. Each must have at least two vertices of attachment, since the bridges B and B' are k -bridges with $k \geq 2$. Now if either B or B' is a 2-bridge, it is easily verified that they must be skew. We may therefore assume that both B and B' have at least three vertices of attachment. There are two cases.

Case. 1. B and B' are not equivalent bridges. Then B' has a vertex of attachment u' between two consecutive vertices of attachment u and v of B . Since B and B' overlap, some vertex of attachment v' of B' does not lie in the segment of B connecting u and v . It now follows that B and B' are skew.

Case. 2. B and B' are equivalent k -bridges, $k \geq 3$. If $k \geq 4$, then B and B' have at least four vertices of attachment, let say u, v, w, z in B and B' . Take vertices of attachment u, w in B and vertices of attachment v, z in B' then the four vertices appear in the cyclic order u, v, w, z on C . Hence B and B' are skew. If $k = 3$, they are equivalent 3-bridges.

Theorem 2.6. If a bridge B has three vertices of attachment v_1, v_2 and v_3 , then there exists a vertex v_0 in $V(B) \setminus V(C)$ and three paths P_1, P_2 and P_3 in B joining v_0 to v_1, v_2 and v_3 respectively, such that, for $i \neq j$, P_i and P_j have only the vertex v_0 in common.

Proof. Let P be a (v_1, v_2) -path in B , internally-disjoint from C . P must have an internal vertex v , since otherwise the bridge B would be just P , and would not contain a third vertex v_3 . Let Q be

a (v_3, v) -path in B , internally-disjoint from C , and let v_0 be the first vertex of Q on P . Denote by P_1 the (v_0, v_1) -section of P^{-1} , by P_2 the (v_0, v_2) -section of P , and by P_3 the (v_0, v_3) -section of Q^{-1} . $V(P_1) \cap V(P_2) = v_0$, $V(P_1) \cap V(P_3) = v_0$ and $V(P_2) \cap V(P_3) = v_0$ (See below fig.2.8)

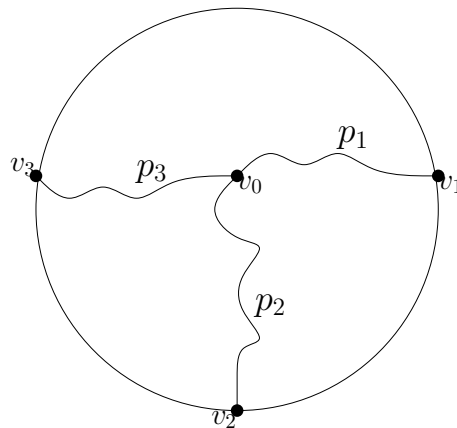


Figure 2.8:

Now consider bridges in plane graphs. Suppose that G is a plane graph and that C is a cycle in G . Then C is a Jordan curve in the plane, and each edge of $E(G) \setminus E(C)$ is contained in one of the two regions $IntC$ or $ExtC$. It follows that a bridge of C is contained entirely in $IntC$ or $ExtC$.

Definition 2.2.7. A bridge contained in $IntC$ is called an **inner bridge**, and a bridge contained in $ExtC$, an **outer bridge**.

Example. In figure 2.9 B_1 and B_2 are inner bridges, and B_3 and B_4 are outer bridges.

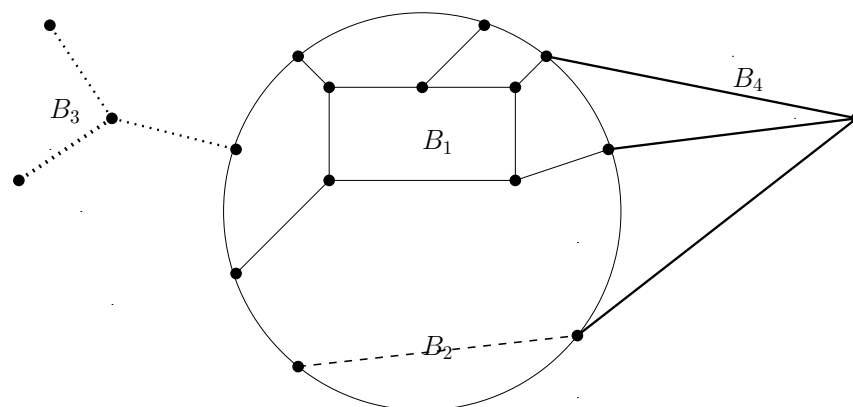


Figure 2.9: Bridges in Plane Graph

Theorem 2.7. *Inner (outer) bridges avoid one another.*

Proof. *By contradiction. Let B and B' be two inner bridges that overlap. Then, by theorem 2.5, they must be either skew or equivalent 3-bridges.*

Case. 1. *B and B' are skew. By definition, there exist distinct vertices u and v in B and u' and v' in B' , appearing in the cyclic order u, u', v, v' on C . Let P be a (u, v) -path in B and P' a (u', v') -path in B' , both internally disjoint from C . The two paths P and P' cannot have an internal vertex in common because they belong to different bridges. At the same time, both P and P' must be contained in $\text{Int}C$ because B and B' are inner bridges. Lets take the Jordan curve $C' = uu'vu$ and to join u' from v' we must meet C' in some point. By the Jordan curve theorem, G cannot be a plane graph, contrary to hypothesis(see fig. 2.10).*

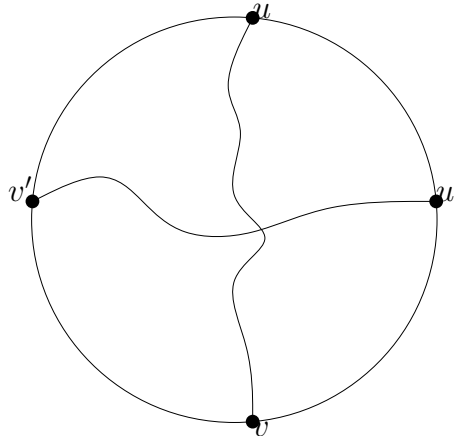


Figure 2.10:

Case. 2. *B and B' are equivalent 3-bridges. Let the common set of vertices of attachment be v_1, v_2, v_3 . By theorem 2.6, there exist in B a vertex v_0 and three paths P_1, P_2 and P_3 joining v_0 to v_1, v_2 and v_3 , respectively, such that, for $i \neq j, P_i$ and P_j have only the vertex v_0 in common. Similarly, B' has a vertex v'_0 and three paths P'_1, P'_2 and P'_3 joining v'_0 to v_1, v_2 and v_3 , respectively, such that, for $i \neq j, P'_i$ and P'_j have only the vertex v'_0 in common (see figure 2.11). Now the paths P_1, P_2 and P_3 divide $\text{Int}C$ into three regions, and v'_0 must be in the interior of one of these regions. Since only two of the vertices v_1, v_2 and v_3 can lie on the boundary of the region containing v'_0 , we may assume, by symmetry, that v_3 is not on the boundary of this region. By the Jordan curve theorem, the path P'_3 must cross either P_1, P_2 or C . But since B and B' are distinct inner bridges, this is clearly impossible. We conclude that inner bridges avoid one another. Similarly, outer Bridges avoid one another*

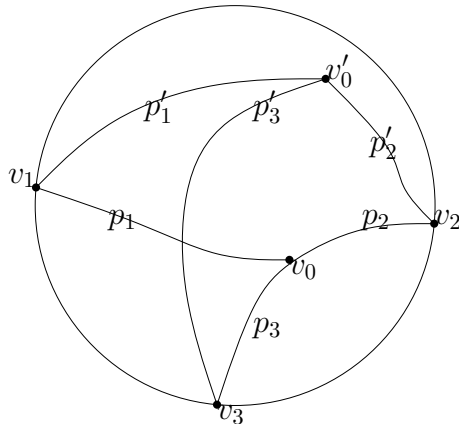


Figure 2.11:

Definition 2.2.8. Let G be a plane graph. An inner bridge B of a cycle C in G is **transferable** if there exists a planar embedding G' of G which is identical to G itself, except that B is an outer bridge of C in G' . The plane graph G' is said to be obtained from G by transferring B .

Example.

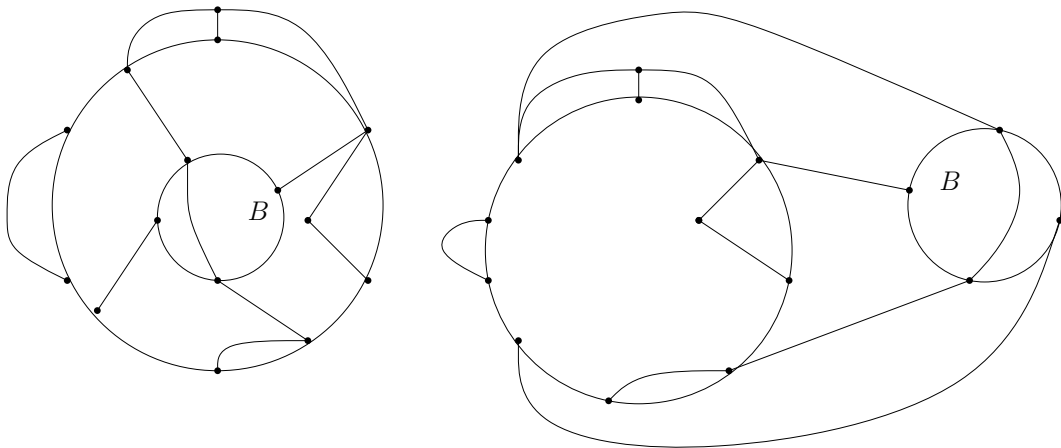


Figure 2.12: The Transfer of Bridges

Theorem 2.8. An inner bridge that avoids every outer bridge is transferable.

Proof. Let B be an inner bridge that avoids every outer bridge. Then the vertices of attachment of B to C all lie on the boundary of some face of G contained in $ExtC$. B can now be drawn in this face.

2.3 Kuratowski's Theorem

We have already noted that K_5 and $K_{3,3}$ are minimally nonplanar and that any proper subgraph of either of these graphs is planar. A remarkably simple characterisation of planar graphs was given by Kuratowski (1930). This section is devoted to a proof of Kuratowski's theorem.

Lemma 2.1. *If G is nonplanar, then every subdivision of G is nonplanar.*

Proof. *Let H be a subdivision of G , then H is obtained from G by adding vertices at least in to one edge of G . Since G is nonplanar, by adding vertices in to edges of G does not affect planarity of G . Therefore H is nonplanar, but H is chosen arbitrarily so every subdivision of G is nonplanar*

Lemma 2.2. *If G is planar, then every subgraph of G is planar.*

Proof. *Let H be a subgraph of a graph G , since G is planar it can be drawn in the plane without none of its edges crossing. We have two cases to show H is a planar graph.*

Case. 1. *if $H = G$, we are done .*

Case. 2. *: if $H \neq G$, now H is obtained from G by removing vertices and edges or either vertices or edges from G . But removing vertices and edges or either vertices or edges does not affect planarity. Therefore H can be drawn in the plane without edge crossing (i.e. H is planar), but H is chosen arbitrarily so every subgraph of G is planar*

Let G be a graph with a 2- vertex cut $\{u, v\}$. Then there exist edge-disjoint subgraphs G_1 and G_2 such that $V(G_1) \cap V(G_2) = \{u, v\}$ and $G_1 \cup G_2 = G$. Consider such a separation of G into subgraphs. In both G_1 and G_2 join u and v by a new edge e to obtain graphs H_1 and H_2 , as in figure 2.13. Clearly $G = (H_1 \cup H_2) - e$. It is also easily seen that $\varepsilon(H_i) < \varepsilon(G)$ for $i = 1, 2$.

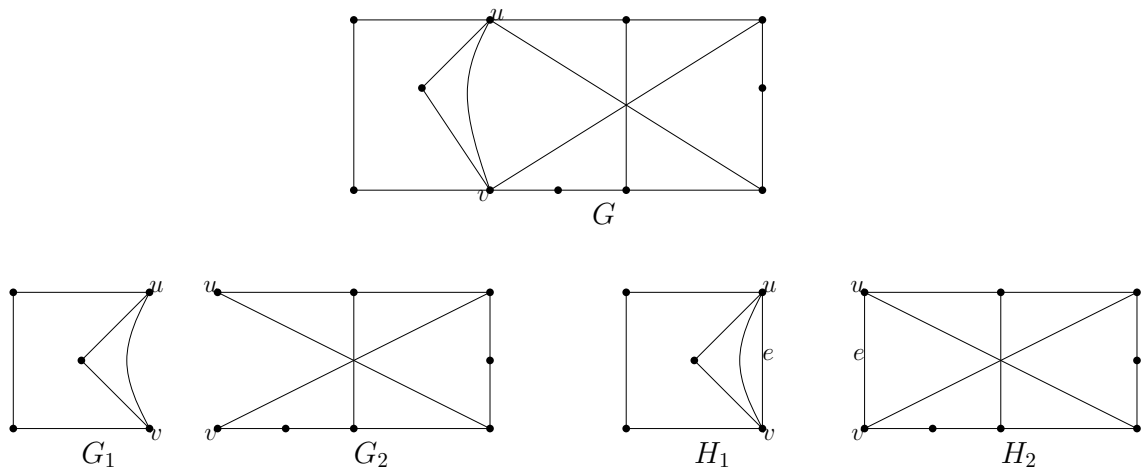


Figure 2.13:

Lemma 2.3. *If G is nonplanar, then at least one of H_1 or H_2 is also nonplanar.*

Proof. *By contradiction, suppose that both H_1 and H_2 are planar. Let H_1' be a planar embedding of H_1 , and let f be a face of H_1' incident with e . If H_2' is an embedding of H_2 in f such that H_1' and H_2' have only the vertices u and v and the edge e in common, then $(H_1' \cup H_2') - e$ is a planar embedding of G . This contradicts the hypothesis that G is nonplanar. Hence at least one of H_1 and H_2 is nonplanar*

Lemma 2.4. *Let G be a nonplanar connected graph that contains no subdivision of K_5 or $K_{3,3}$ and has as few edges as possible. Then G is simple and 3-connected.*

Proof. *By contradiction. Let G satisfy the hypotheses of the lemma. Then G is a minimal nonplanar graph, because it has as few edge as possible, and therefore must be a simple block. If G is not 3-connected, let u, v be a 2-vertex cut of G and let H_1 and H_2 be the graphs obtained from this cut as described above. By lemma 2.3, at least one of H_1 and H_2 , say H_1 , is nonplanar. Since $\varepsilon(H_1) < \varepsilon(G)$, H_1 must contain a subgraph K which is a subdivision of K_5 or $K_{3,3}$; moreover $K \not\subseteq G$, and so the edge e is in K . Let P be a (u, v) -path in $H_2 - e$. Then G contains the subgraph $(K \cup P) - e$, which is a subdivision of K and hence a subdivision of K_5 or $K_{3,3}$. This contradiction establishes the lemma.*

We shall find it convenient to adopt the following notation in the proof of Kuratowski's theorem. Suppose that C is a cycle in a plane graph. Then we can regard the two possible orientations of C as 'clockwise' and 'anticlockwise'. For any two vertices, u and v of C , we shall denote by $C[u, v]$ the (u, v) -path which follows the clockwise orientation of C ; similarly we shall use the symbols $C(u, v)$, $C[u, v)$ and $C(u, v)$ to denote the paths $C[u, v] - u$, $C[u, v] - v$ and $C[u, v] - \{u, v\}$. We are now ready to prove Kuratowski's theorem.

Theorem 2.9. (Kuratowsk's) *A graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$.*

Proof. *By contradiction. Suppose a graph G is planar and assume that G contains a subdivision of K_5 or $K_{3,3}$. We know that K_5 and $K_{3,3}$ are nonplanar. Hence G contains a nonplanar graph. Let say the nonplanar graph which is contained in G is H , which is a subgraph of G (i.e. there exist a nonplanar graph H which is a subgraph of G), by lemma 2.2 G is nonplanar. This contradicts the hypothesis that G is planar. Therefore G contains no subdivision of K_5 or $K_{3,3}$.*

We shall prove the Converse by contradiction. If possible, choose a nonplanar graph G that contains no subdivision of K_5 or $K_{3,3}$ and has as few edges as possible. From lemma 2.4 it follows that G is simple and 3-connected. Since G has as few edges as possible, hence it must also be a minimal nonplanar graph. Let uv be an edge of G , and let H be a planar embedding of the planar graph $G - uv$. Since G is 3-connected, H is 2-connected and, by corollary 1.1, u and v are contained together in a cycle of H . Choose a cycle C of H that contains u and v and is such that the number of edges in $\text{Int}C$ is as large as possible. Since H is simple and 2-connected, each bridge of C in H must have at least two vertices of attachment. Now all outer bridges of C must be 2-bridges that overlap uv because, if some outer bridge were a k -bridge for $k \geq 3$ or a 2-bridge that avoided uv , then there would be a cycle C' containing u and v with more edges in its interior than C ,

contradicting the choice of C . These two cases are illustrated in figure 2.14 (with C' indicated by heavy lines)

In fact, all outer bridges of C in H must be single edges. For if a 2-bridge with vertices of attachment x and y had a third vertex, the set $\{x, y\}$ would be a 2-vertex cut of G , contradicting the fact that G is 3-connected. By theorem 2.7, no two inner bridges overlap. Therefore some inner

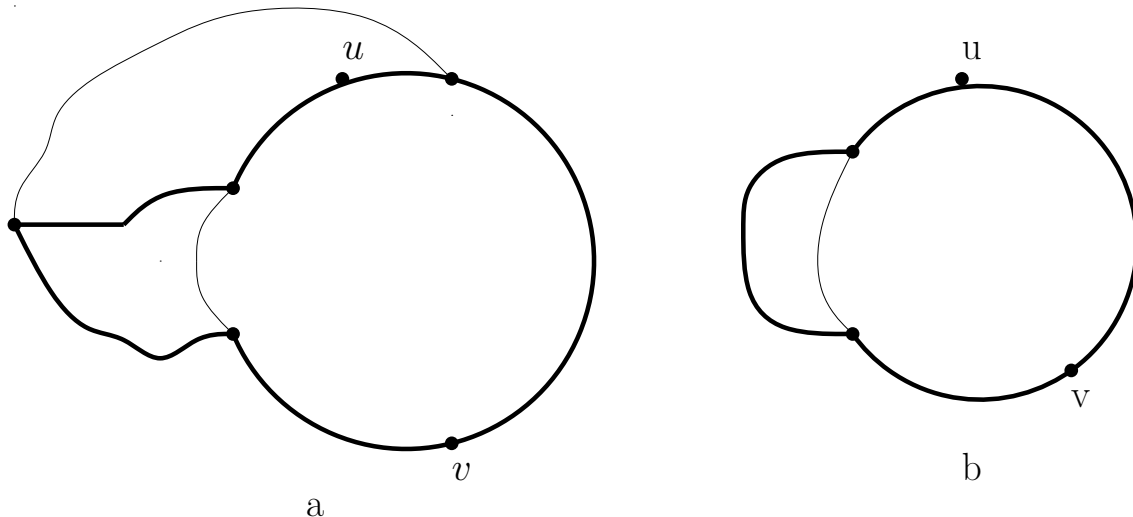


Figure 2.14:

bridge skew to uv must overlap some outer bridge. For otherwise, by theorem 2.8, all such bridges could be transferred (one by one), and then the edge uv could be drawn in $\text{Int}C$ to obtain a planar embedding of G ; since G is nonplanar, this is not possible. Therefore, there is an inner bridge B that is both skew to uv and skew to some outer bridge xy .

Two cases now arise, depending on whether B has a vertex of attachment different from u, v, x and y or not.

Case. 1. B has a vertex of attachment different from u, v, x and y . We can choose the notation so that B has a vertex of attachment v_1 in $C(x, u)$ (see figure 2.15). We consider two subcases, depending on whether B has a vertex of attachment in $C(y, v)$ or not.

Case. 1.1 B has a vertex of attachment v_2 in $C(y, v)$. In this case there is a (v_1, v_2) -path P in B that is internally-disjoint from C . But then $(C \cup P) + \{uv, xy\}$ is a subdivision of $K_{3,3}$, in G , a contradiction (see fig. 2.15).

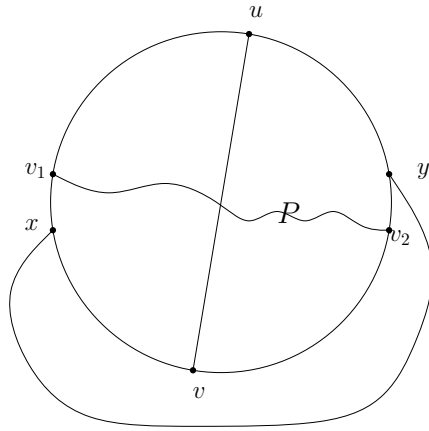


Figure 2.15:

Case. 1.2 B has no vertex of attachment in $C(y, v)$. Since B is skew to uv and to xy , B must have vertices of attachment v_2 in $C(u, y)$ and v_3 in $C[v, x]$. Thus B has three vertices of attachment v_1, v_2 and v_3 . By theorem 2.5, there exists a vertex v_0 in $V(B) \setminus V(C)$ and three paths P_1, P_2 and P_3 in B joining v_0 to v_1, v_2 and v_3 , respectively, such that, for $i \neq j$, P_i and P_j have only the vertex v_0 in common. But now $(C \cup P_1 \cup P_2 \cup P_3) + \{uv, xy\}$ contains a subdivision of $K_{3,3}$, a contradiction. This case is illustrated in figure 2.16. The subdivision of $K_{3,3}$ is indicated by heavy lines.

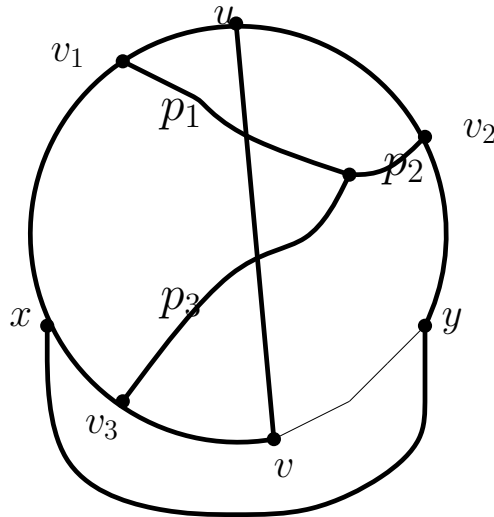


Figure 2.16:

Case. 2. B has no vertex of attachment other than u, v, x and y . Since B is skew to both uv and xy , it follows that u, v, x and y must all be vertices of attachment of B . Therefore there exists a (u, v) -path P and an (x, y) -path Q in B such that (i) P and Q are internally-disjoint from C , and (ii) $|V(P) \cap V(Q)| \geq 1$. We consider two sub cases, depending on whether P and Q have one or more vertices in common.

Case. 2.1 $|V(P) \cap V(Q)| = 1$. In this case $(C \cup P \cup Q) + \{uv, xy\}$ is a subdivision of K_5 , in G , again a contradiction (see figure 2.17).

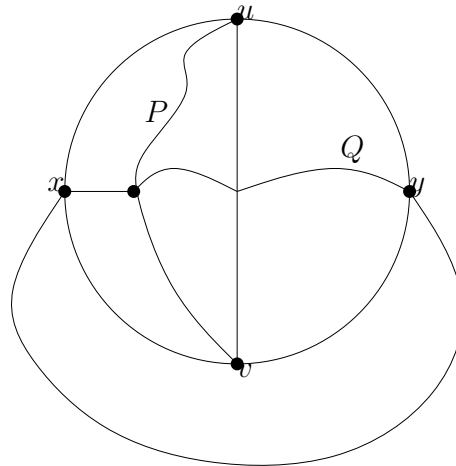


Figure 2.17:

Case. 2.2 $|V(P) \cap V(Q)| \geq 2$. Let u' and v' be the first and last vertices of P on Q , and let P_1 and P_2 denote the (u, u') - and (v', v) -sections of P . Then $(C \cup P_1 \cup P_2 \cup Q) + \{uv, xy\}$ contains a subdivision of $K_{3,3}$ in G , once more a contradiction (see figure 2.18)

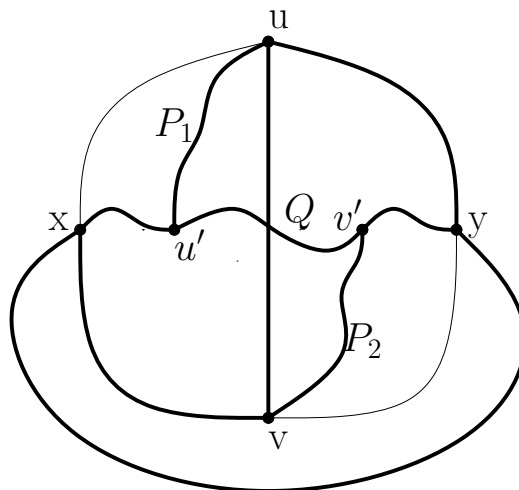


Figure 2.18:

Thus all the possible cases lead to contradictions, therefore the graph G must be planar and the proof is complete .

Example. show that that the peterson graph is non-planar using kuratowsk's theorem.

Solution. We say a given graph G is nonplanar if it contains a subdivision of either K_5 or $K_{3,3}$

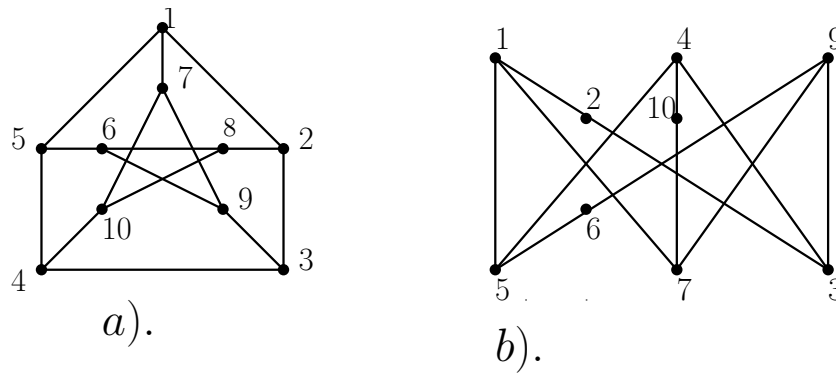


Figure 2.19:

Thus peterson graph contain a subdivision of $k_{3,3}$
Therefore peterson graph is nonplanar graph.

Conclusion

In this report, the aim is to identify the planarity of a given graph. There are some ways to know which graphs are planar and which are not. In this report we use Kuratowski's theorem whether the given graph is planar or not.

In Kuratowski's theorem we only check whether a given graph is planar or not by checking if the graph contains a subdivision of K_5 or $K_{3,3}$. If it contains a subdivision of K_5 or $K_{3,3}$ it is nonplanar, otherwise it is planar.

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