



**DEGENERATE THREE-LEVEL ATOM DRIVEN BY  
COHERENT LIGHT AND IN A CAVITY COUPLED  
TO SQUEEZED VACUUM RESERVOIR**

By

Abebaw Gizachew Biyazen

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE  
REQUIREMENTS FOR THE DEGREE OF  
MASTERS IN PHYSICS (QUANTUM OPTICS)

AT

ADISS ABABA UNIVERSITY  
COLLEGE OF NATURAL AND COMPUTATIONAL SCIENCES  
ADISS ABABA, ETHIOPIA

APRIL 18, 2022

© Copyright by Abebaw Gizachew Biyazen, 2022

ADISS ABABA UNIVERSITY  
COLLEGE OF NATURAL AND COMPUTATIONAL SCIENCES  
DEPARTMENT OF PHYSICS

The undersigned hereby certify that they have read and recommend to the College of Natural and Computational Sciences for acceptance a project entitled **“DEGENERATE THREE-LEVEL ATOM DRIVEN BY COHERENT LIGHT AND IN A CAVITY COUPLED TO SQUEEZED VACUUM RESERVOIR ”** by **Abebaw Gizachew Biyazen** in partial fulfillment of the requirements for the degree of **Masters in Physics (Quantum Optics)**.

Dated: April 18, 2022

Advisor:

\_\_\_\_\_  
Dr. Deribe Hirpo

Examiners:

\_\_\_\_\_  
Dr. Ayana Yihunie

\_\_\_\_\_  
Dr. Merid Tufa

Chairman:

\_\_\_\_\_  
Dr. Yitagsu Elfagd

ADISS ABABA UNIVERSITY

Date: **April 18, 2022**

Author: **Abebaw Gizachew Biyazen**

Title: **DEGENERATE THREE-LEVEL ATOM DRIVEN BY  
COHERENT LIGHT AND IN A CAVITY COUPLED  
TO SQUEEZED VACUUM RESERVOIR**

Department: **College of Natural and Computational Sciences  
Department of Physics**

Degree: **MSc**

Convocation: **April**

Year: **2022**

Permission is here with granted to Adiss Ababa University to circulate and to have copied for non-commercial purposes, at its discretion, the above title upon the request of individuals or institutions.

---

Signature of Author

THE AUTHOR RESERVES OTHER PUBLICATION RIGHTS, AND NEITHER THE THESIS NOR EXTENSIVE EXTRACTS FROM IT MAY BE PRINTED OR OTHERWISE REPRODUCED WITHOUT THE AUTHOR'S WRITTEN PERMISSION.

THE AUTHOR ATTESTS THAT PERMISSION HAS BEEN OBTAINED FOR THE USE OF ANY COPYRIGHTED MATERIAL APPEARING IN THIS THESIS (OTHER THAN BRIEF EXCERPTS REQUIRING ONLY PROPER ACKNOWLEDGEMENT IN SCHOLARLY WRITING) AND THAT ALL SUCH USE IS CLEARLY ACKNOWLEDGED.

# Table of Contents

List of Figures	v
Abstract	vi
Acknowledgements	vii
<b>1 Introduction</b>	<b>1</b>
<b>2 The Time Evolution of Operators</b>	<b>2</b>
2.1 The Correlation properties of the cavity mode noise operators . . . . .	13
<b>3 Photon Statistics</b>	<b>15</b>
3.1 The mean photon number . . . . .	15
3.2 Power spectrum . . . . .	17
<b>4 Quadrature Squeezing</b>	<b>20</b>
<b>5 Conclusion</b>	<b>24</b>
<b>Bibliography</b>	<b>25</b>

# List of Figures

2.1	Schematic diagram of coherently driven degenerate three-level atom in a cavity coupled to squeezed vacuum reservoir . . . . .	3
3.1	Plot of steady-state mean photon number[Eq. (3.1)] versus $\epsilon$ , for $\kappa = 0.8$ , $r = 0.5$ , and $\gamma_c = 0.5$ . . . . .	16
3.2	Plot of steady-state mean photon number versus $\gamma_c$ for $\kappa = 0.8$ , and $r = 0.5$ , $\epsilon = 0.5$ . . . . .	16
3.3	Plot of steady-state mean photon number versus $r$ for $\kappa = 0.8$ , $\epsilon = 0.5$ , and $\gamma_c = 0.5$ . . . . .	17
3.4	$\bar{n}_{\pm\lambda}$ versus $\lambda$ , for $\kappa = 0.8$ , $\epsilon = 0.9$ , $r = 0$ (blue) and $r = 0.2$ (red) . . . . .	19
4.1	Plus quadrature variance(Red), minus quadrature variance in vacuum level(black) and minus quadrature variance(blue) versus $\gamma_c$ , for $\epsilon = 0.5$ , $\kappa = 0.8$ , $r = 0.5$ . .	21
4.2	Plot of squeezing(S) versus $\epsilon$ , for $\kappa = 0.8$ , $r = 0.5$ (blue), $r = 0$ (red) and $\gamma_c = 0.5$	22
4.3	Plot of squeezing(S) versus $r$ , for $\kappa = 0.8$ , $\epsilon = 0.5$ and $\gamma_c = 0.5$ . . . . .	22
4.4	Plot of squeezing(S) versus $\gamma_c$ , for $\kappa = 0.8$ , $\epsilon = 0.5$ , and $r=0.5$ . . . . .	23

# Abstract

In this thesis, we analyze the squeezing and statical properties of light produced by degenerate three-level atom in a cavity coupled to squeezed vacuum reservoir via a port-mirror. The three-level atom available in the cavity is driven by coherent light from the bottom to the top level. Employing the master equation we obtain the differential equation of the atomic and cavity mode operators. Applying the large time approximation to the time evolution of cavity mode operator and using steady-state solutions of the expectation values of cavity mode and atomic operators, we obtain the mean of the cavity photon number, power spectrum, quadrature variance and quadrature squeezing of the cavity light. We have seen that the amplitude of the coupling coherent light, the squeezing parameter, and the stimulated emission decay constant have the effect of increasing both the global and local mean photon number. Moreover, we find that the quadrature squeezing of the cavity light increases with the increase of the amplitude of the driving coherent light, the rate of the stimulation decay constant, and squeeze parameter. The squeeze parameter and the amplitude of the driving coherent light have the effect of enhancing the quadrature squeezing, with maximum squeezing of 58% for  $r = 0.5$  and 32% for  $r=0$ .

# Acknowledgements

First of all I would like to thank the almighty God, for keeping me alive to accomplish this task. I would like also to express my gratitude to my advisor and instructor Dr. Deribe Hirpo for his marvelous and remarkable advise, suggestions and constant support in academic and social affairs. He is always on the bottom of my heart in gravitating me to study and love quantum optics. I wish to him forever success, peace and grace throughout his life.

Next I express my sincere thanks to physics department of Addis Ababa university facilitating conditions for Specialization master in physics.

I would also extend my sincere thanks to my all family, for their moral support.

# Chapter 1

## Introduction

Interaction of three-level atom with light has attracted a great deal of interest in recent year[1,2,3]. In addition, some authors have studied the quadrature squeezing and statical properties of a light produced by three-level atoms in which the crucial role is played by the coupling of the top and bottom levels[1-7]. The coupling of the top and bottom levels are responsible for the interesting non classical feature of the generated light. In general the atomic coherence can be induced in three-level atom by coupling the level between which direct transition is dipole forbidden by coherent light or by preparing the atom initially in coherent supper position of these two level[1,7,8,9]. In three-level atom the top, intermediate, and bottom levels are denoted by  $|a\rangle$ ,  $|b\rangle$ , and  $|c\rangle$  respectively, in which direct transition between  $|a\rangle$  and  $|c\rangle$  is dipole forbidden[1,9,10,11,12]. When the three-level atom decays from  $|a\rangle$  to  $|c\rangle$  via level  $|b\rangle$  two photons are emitted. If the two photons have the same frequency, then three-level atom is referred as a degenerate three-level atom, other wise it is referred as a non degenerate three-level atom[3,13]. It has been estimated that a three-level atom under certain condition generates squeezed light[13-16]. In this thesis we seek to analyze the squeezing and statistical properties of light emitted by a degenerate three-level atom whose bottom and top levels are coupled by coherent light and available in a cavity coupled to squeezed vacuum reservoir via a port mirror. In order to determine the squeezing and statistical properties of the light produced by the quantum optical system, we first drive equation of evolution of atomic and cavity mode operators, applying the master equation. Using the steady-state solution of the resulting equation, we calculate mean photon number, power spectrum, quadrature variance, and quadrature squeezing.

# Chapter 2

## The Time Evolution of Operators

We here consider degenerate three-level atom driven by coherent light and in a cavity coupled to a squeezed vacuum reservoir. The Hamiltonian describing the coupling of the top and bottom levels of the atom by the coherent light is expressible as[1]

$$\hat{H}_1 = i\lambda(\hat{\sigma}_c^\dagger \hat{c} - \hat{c}^\dagger \hat{\sigma}_c), \quad (2.1)$$

where  $\hat{c}$  is annihilation operator for driving coherent light,  $\hat{\sigma}_c = |c\rangle\langle a|$  is atomic operator, and  $\lambda$  is coupling constant between driving coherent light and the a degenerate three-level atom. With the driving coherent light treated classically, the Hamiltonian can be written as

$$\hat{H}_1 = i\epsilon(\hat{\sigma}_c^\dagger - \hat{\sigma}_c), \quad (2.2)$$

where  $\epsilon$  is constant proportional to the amplitude of the deriving coherent light. Moreover, the interaction of a three-level atom with the cavity mode is described by the Hamiltonian[3]

$$\hat{H}_2 = ig[\hat{\sigma}_a^\dagger \hat{a} - \hat{a}^\dagger \hat{\sigma}_a + \hat{\sigma}_b^\dagger \hat{a} - \hat{a}^\dagger \hat{\sigma}_b], \quad (2.3)$$

where  $\hat{\sigma}_a = |b\rangle\langle a|$ ,  $\hat{\sigma}_b = |c\rangle\langle b|$ , are atomic operator,  $g$  is the atom-cavity mode coupling constant, and  $\hat{a}$  is the annihilation operator for the cavity mode.

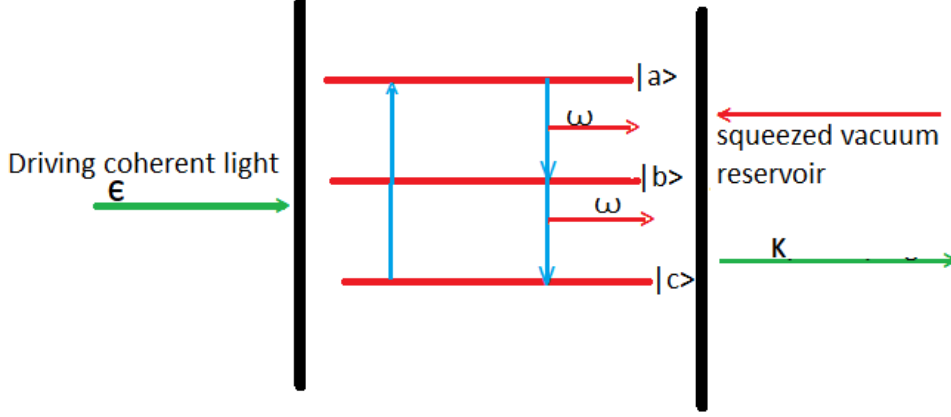


Figure 2.1: Schematic diagram of coherently driven degenerate three-level atom in a cavity coupled to squeezed vacuum reservoir

We denote the upper level by  $|a\rangle$ , the middle level by  $|b\rangle$ , and the lower level by  $|c\rangle$ , as shown in Figure 2.1. The dipole allowed transition between level  $|a\rangle$  and  $|b\rangle$  and between level  $|b\rangle$  and  $|c\rangle$  are resonant with the cavity mode[3,14]. The direct transition between level  $|a\rangle$  and  $|c\rangle$  is dipole forbidden[1,3]. The interaction of three-level atom with the deriving coherent light and the cavity mode can be described by the Hamiltonian

$$\hat{H} = i\epsilon(\hat{\sigma}_c^\dagger - \hat{\sigma}_c) + ig(\hat{\sigma}_a^\dagger \hat{a} - \hat{a}^\dagger \hat{\sigma}_a + \hat{\sigma}_b^\dagger \hat{a} - \hat{a}^\dagger \hat{\sigma}_b). \quad (2.4)$$

The master equation for the light produced by our quantum optical system expressible as[3]

$$\begin{aligned} \frac{d\hat{\rho}}{dt} = & -i[\hat{H}, \hat{\rho}] + \frac{\kappa(N+1)}{2}(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}) \\ & + \frac{\kappa N}{2}(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger) - \frac{\kappa M}{2}(2\hat{a}^\dagger\hat{\rho}\hat{a}^\dagger - \hat{a}^{\dagger 2}\hat{\rho} - \hat{\rho}\hat{a}^{\dagger 2}) \\ & - \frac{\kappa M}{2}(2\hat{a}\hat{\rho}\hat{a} - \hat{a}^2\hat{\rho} - \hat{\rho}\hat{a}^2), \end{aligned} \quad (2.5)$$

where  $k$  is the cavity dumping constant,  $\hat{\rho}$  is density operators, and  $\hat{H}$  is Hamiltonian. The effects of the reservoir are incorporated via the parameters  $N = \sinh^2 r$  and  $M = \sinh r \cosh r$ ,

with  $r$  being the squeeze parameter. The time evolution of expectation value of an operator  $\hat{A}$  can be written in the Schrodinger picture, as[3,12]

$$\frac{d\langle\hat{A}\rangle}{dt} = Tr\left(\frac{d\hat{\rho}}{dt}\hat{A}\right). \quad (2.6)$$

According to Eq. (2.5) and Eq. (2.6), the time evolution of the expectation value of the cavity mode operator  $\hat{a}$  can be written as,

$$\begin{aligned} \frac{d\langle\hat{a}\rangle}{dt} &= Tr\left(\frac{d\hat{\rho}}{dt}\hat{a}\right) \\ &= iT_r([\hat{H}, \hat{\rho}]\hat{a}) + \frac{\kappa(N+1)}{2}Tr(2\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}) \\ &\quad + \frac{\kappa N}{2}Tr(2\hat{a}^\dagger\hat{\rho}\hat{a}^2 - \hat{a}\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a}) - \frac{\kappa M}{2}Tr(2\hat{a}^\dagger\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^{\dagger 2}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^{\dagger 2}\hat{a}) \\ &\quad - \frac{\kappa M}{2}Tr(2\hat{a}\hat{\rho}\hat{a}^2 - \hat{a}^2\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^2\hat{a}) \\ &= -iT_r(\hat{H}\hat{\rho}\hat{a} - \hat{\rho}\hat{H}\hat{a}) + \frac{\kappa(N+1)}{2}Tr(2\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}) \\ &\quad + \frac{\kappa N}{2}Tr(2\hat{a}^\dagger\hat{\rho}\hat{a}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a}) - \frac{\kappa M}{2}Tr(2\hat{a}^\dagger\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^{\dagger 2}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^{\dagger 2}\hat{a}) \\ &\quad - \frac{\kappa M}{2}Tr(2\hat{a}\hat{\rho}\hat{a}\hat{a} - \hat{a}^2\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^2\hat{a}), \end{aligned} \quad (2.7)$$

or

$$\frac{d\langle\hat{a}\rangle}{dt} = -iZ_1 + \frac{\kappa(N+1)}{2}Z_2 + \frac{\kappa N}{2}Z_3 - \frac{\kappa M}{2}Z_4 - \frac{\kappa M}{2}Z_5, \quad (2.8)$$

where

$$\begin{aligned} Z_1 &= Tr(\hat{H}\hat{\rho}\hat{a} - \hat{\rho}\hat{H}\hat{a}), \\ Z_2 &= Tr(2\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}), \\ Z_3 &= Tr(2\hat{a}^\dagger\hat{\rho}\hat{a}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a}), \\ Z_4 &= Tr(2\hat{a}^\dagger\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^{\dagger 2}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^{\dagger 2}\hat{a}), \\ Z_5 &= Tr(2\hat{a}\hat{\rho}\hat{a}\hat{a} - \hat{a}^2\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^2\hat{a}). \end{aligned} \quad (2.9)$$

Next we evaluate these traces applying the cyclic property of trace and commutation relation for cavity mode operator. We thus have

$$\begin{aligned}
Z_1 &= Tr(\hat{H}\hat{\rho}\hat{a} - \hat{\rho}\hat{H}\hat{a}) \\
&= Tr(\hat{\rho}\hat{a}\hat{H} - \hat{\rho}\hat{H}\hat{a}) \\
&= Tr(\hat{\rho}(\hat{a}\hat{H} - \hat{H}\hat{a})) \\
&= Tr(\hat{\rho}([\hat{a}, \hat{H}])) \\
&= \langle [\hat{a}, \hat{H}] \rangle.
\end{aligned} \tag{2.10}$$

$$\begin{aligned}
Z_2 &= Tr(2\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}^2) \\
&= Tr(2\hat{\rho}\hat{a}^\dagger\hat{a}^2 - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}^2) \\
&= Tr(\hat{\rho}(2\hat{a}^\dagger\hat{a}^2 - \hat{a}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}^2)) \\
&= Tr(\hat{\rho}(\hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger)\hat{a}) \\
&= Tr(\rho[\hat{a}^\dagger, \hat{a}]\hat{a}) \\
&= -Tr(\hat{\rho}\hat{a}) \\
&= -\langle \hat{a} \rangle.
\end{aligned} \tag{2.11}$$

Similarly

$$\begin{aligned}
Z_3 &= Tr(2\hat{a}^\dagger\hat{\rho}\hat{a}^2 - \hat{a}\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a}) \\
&= Tr(2\hat{\rho}\hat{a}^2\hat{a}^\dagger - \hat{\rho}\hat{a}^2\hat{a}^\dagger - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a}) \\
&= Tr(\hat{\rho}(\hat{a}^2\hat{a}^\dagger - \hat{a}\hat{a}^\dagger\hat{a})) \\
&= Tr(\hat{\rho}\hat{a}(\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a})) \\
&= Tr(\hat{\rho}\hat{a}) \\
&= \langle \hat{a} \rangle.
\end{aligned} \tag{2.12}$$

Moreover

$$\begin{aligned}
Z_4 &= Tr(2\hat{a}^\dagger\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^{\dagger 2}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^{\dagger 2}\hat{a}) \\
&= Tr(2\hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}^\dagger - \hat{\rho}\hat{a}\hat{a}^{\dagger 2} - \hat{\rho}\hat{a}^{\dagger 2}\hat{a}) \\
&= Tr(\hat{\rho}(\hat{a}^\dagger\hat{a}\hat{a}^\dagger - \hat{a}\hat{a}^{\dagger 2} + \hat{a}^\dagger\hat{a}\hat{a}^\dagger - \hat{a}^{\dagger 2}\hat{a})) \\
&= Tr(\hat{\rho}(\hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger)\hat{a}^\dagger + \hat{a}^\dagger(\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a})) \\
&= 0.
\end{aligned} \tag{2.13}$$

And

$$\begin{aligned}
Z_5 &= Tr(2\hat{a}\hat{\rho}\hat{a}^2 - \hat{a}^2\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^2\hat{a}) \\
&= Tr(2\hat{\rho}\hat{a}^3 - \hat{\rho}\hat{a}^3 - \hat{\rho}\hat{a}^3) \\
&= 0.
\end{aligned} \tag{2.14}$$

Substituting the results given in Eqs. (2.10)-(2.14) in to Eq. (2.8), we get

$$\frac{d\langle\hat{a}\rangle}{dt} = -i\langle[\hat{a}, \hat{H}]\rangle - \frac{\kappa}{2}\langle\hat{a}\rangle. \tag{2.15}$$

Following a similar procedure, it can be readily verified that

$$\frac{d\langle\hat{a}^2\rangle}{dt} = -i\langle[\hat{a}^2, \hat{H}]\rangle - k\langle\hat{a}^2\rangle + \kappa M, \tag{2.16}$$

$$\frac{d\langle\hat{a}^\dagger\hat{a}\rangle}{dt} = -i\langle[\hat{a}^\dagger\hat{a}, \hat{H}]\rangle - k\langle\hat{a}^\dagger\hat{a}\rangle + \kappa N, \tag{2.17}$$

$$\frac{d\langle\hat{a}\hat{a}^\dagger\rangle}{dt} = -i\langle[\hat{a}\hat{a}^\dagger, \hat{H}]\rangle - k\langle\hat{a}\hat{a}^\dagger\rangle + \kappa N + \kappa. \tag{2.18}$$

We seek to obtain the equation of evolution for the expectation value of atomic operators with aid of Eqs. (2.5) and Eqs. (2.6). We thus see that

$$\begin{aligned}
\frac{d\langle\hat{\sigma}_a\rangle}{dt} &= Tr\left(\frac{d\hat{\rho}}{dt}\hat{\sigma}_a\right) \\
&= -iTr([\hat{H}, \hat{\rho}]\hat{\sigma}_a) \\
&+ \frac{k(N+1)}{2}Tr(2\hat{a}\hat{\rho}\hat{a}^\dagger\hat{\sigma}_a - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{\sigma}_a - \hat{\rho}\hat{a}^\dagger\hat{a}\hat{\sigma}_a) \\
&+ \frac{kN}{2}Tr(2\hat{a}^\dagger\hat{\rho}\hat{a}\hat{\sigma}_a - \hat{a}\hat{a}^\dagger\hat{\rho}\hat{\sigma}_a - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{\sigma}_a) - \frac{kM}{2}Tr(2\hat{a}^\dagger\hat{\rho}\hat{a}^\dagger\hat{\sigma}_a - \hat{a}^{\dagger 2}\hat{\rho}\hat{\sigma}_a - \hat{\rho}\hat{a}^{\dagger 2}\hat{\sigma}_a) \\
&- \frac{kM}{2}Tr(2\hat{a}\hat{\rho}\hat{\sigma}_a - \hat{a}^2\hat{\rho}\hat{\sigma}_a - \hat{\rho}\hat{a}^2\hat{\sigma}_a),
\end{aligned} \tag{2.19}$$

or

$$\frac{d\langle\hat{\sigma}_a\rangle}{dt} = -iL_1 + \frac{k(N+1)}{2}L_2 + \frac{kN}{2}L_3 - \frac{kM}{2}L_4 - \frac{kM}{2}L_5, \tag{2.20}$$

where

$$\begin{aligned}
L_1 &= Tr([\hat{H}, \hat{\rho}] \hat{\sigma}_a) \\
L_2 &= Tr(2\hat{a}\hat{\rho}\hat{a}^\dagger \hat{\sigma}_a - \hat{a}^\dagger \hat{a} \hat{\rho} \hat{\sigma}_a - \hat{\rho} \hat{a}^\dagger \hat{a} \hat{\sigma}_a), \\
L_3 &= Tr(2\hat{a}^\dagger \hat{\rho} \hat{a} \hat{\sigma}_a - \hat{a} \hat{a}^\dagger \hat{\rho} \hat{\sigma}_a - \hat{\rho} \hat{a} \hat{a}^\dagger \hat{\sigma}_a), \\
L_4 &= Tr(2\hat{a}^\dagger \hat{\rho} \hat{a}^\dagger \hat{\sigma}_a - \hat{a}^{\dagger 2} \hat{\rho} \hat{\sigma}_a - \hat{\rho} \hat{a}^{\dagger 2} \hat{\sigma}_a), \\
L_5 &= Tr(2\hat{a} \hat{\rho} \hat{a} \hat{\sigma}_a - \hat{a}^2 \hat{\rho} \hat{\sigma}_a - \hat{\rho} \hat{a}^2 \hat{\sigma}_a).
\end{aligned} \tag{2.21}$$

Evaluation of the above traces results in

$$L_1 = \langle [\hat{\sigma}_a, \hat{H}] \rangle, \tag{2.22}$$

$$L_2 = 0, \tag{2.23}$$

$$L_3 = 0, \tag{2.24}$$

Moreover

$$L_4 = 0, \tag{2.25}$$

$$L_5 = 0. \tag{2.26}$$

Substituting trace from  $L_1$  up to  $L_5$  into Eq. (2.20) we obtain

$$\frac{d\langle \hat{\sigma}_a \rangle}{dt} = -i\langle [\hat{\sigma}_a, \hat{H}] \rangle. \tag{2.27}$$

In similar manner, one can establish that

$$\frac{d\langle \hat{\sigma}_b \rangle}{dt} = -i\langle [\hat{\sigma}_b, \hat{H}] \rangle, \tag{2.28}$$

$$\frac{d\langle \hat{\sigma}_c \rangle}{dt} = -i\langle [\hat{\sigma}_c, \hat{H}] \rangle, \tag{2.29}$$

$$\frac{d\langle \hat{\eta}_a \rangle}{dt} = -i\langle [\hat{\eta}_a, \hat{H}] \rangle, \tag{2.30}$$

$$\frac{d\langle\hat{\eta}_b\rangle}{dt} = -i\langle[\hat{\eta}_b, \hat{H}]\rangle, \quad (2.31)$$

$$\frac{d\langle\hat{\eta}_c\rangle}{dt} = -i\langle[\hat{\eta}_c, \hat{H}]\rangle, \quad (2.32)$$

where  $\hat{\eta}_a = |a\rangle\langle a|$ ,  $\hat{\eta}_b = |b\rangle\langle b|$ , and  $\hat{\eta}_c = |c\rangle\langle c|$  are atomic operators representing the probability of finding the atom in the upper, intermediate, and bottom level respectively. Employing the Hamiltonian specified by Eq. (2.4) in Eqs. (2.15)-(2.18) as well as in Eqs. (2.27)-(2.32), the time evolution of the expectation values of cavity mode and atomic operators are expressible as

$$\frac{d\langle\hat{a}\rangle}{dt} = -\frac{\kappa}{2}\langle\hat{a}\rangle - g(\langle\hat{\sigma}_a\rangle + \langle\hat{\sigma}_b\rangle), \quad (2.33)$$

$$\frac{d\langle\hat{a}^2\rangle}{dt} = -\kappa\langle\hat{a}^2\rangle - g(\langle\hat{a}\hat{\sigma}_a\rangle + \langle\hat{\sigma}_a\hat{a}\rangle + \langle\hat{a}\hat{\sigma}_b\rangle + \langle\hat{\sigma}_b\hat{a}\rangle) + kM, \quad (2.34)$$

$$\frac{d\langle\hat{a}^\dagger\hat{a}\rangle}{dt} = -\kappa\langle\hat{a}^\dagger\hat{a}\rangle - g(\langle\hat{\sigma}_a^\dagger\hat{a}\rangle + \langle\hat{a}^\dagger\hat{\sigma}_a\rangle + \langle\hat{a}\hat{\sigma}_b^\dagger\rangle + \langle\hat{a}^\dagger\hat{\sigma}_b\rangle) + kN, \quad (2.35)$$

$$\frac{d\langle\hat{a}\hat{a}^\dagger\rangle}{dt} = -\kappa\langle\hat{a}\hat{a}^\dagger\rangle - g(\langle\hat{a}\hat{\sigma}_a^\dagger\rangle + \langle\hat{\sigma}_a\hat{a}^\dagger\rangle + \langle\hat{a}\hat{\sigma}_b^\dagger\rangle + \langle\hat{\sigma}_b\hat{a}^\dagger\rangle) + \kappa N + \kappa, \quad (2.36)$$

$$\frac{d\langle\hat{\sigma}_a\rangle}{dt} = g(\langle\hat{\eta}_b\hat{a}\rangle - \langle\hat{\eta}_a\hat{a}\rangle + \langle\hat{a}^\dagger\hat{\sigma}_c\rangle) + \epsilon\langle\hat{\sigma}_b^\dagger\rangle, \quad (2.37)$$

$$\frac{d\langle\hat{\sigma}_b\rangle}{dt} = g(\langle\hat{\eta}_c\hat{a}\rangle - \langle\hat{\eta}_b\hat{a}\rangle - \langle\hat{a}^\dagger\hat{\sigma}_c\rangle) - \epsilon\langle\hat{\sigma}_a^\dagger\rangle, \quad (2.38)$$

$$\frac{d\langle\hat{\sigma}_c\rangle}{dt} = g(\langle\hat{\sigma}_b\hat{a}\rangle - \langle\hat{\sigma}_a\hat{a}\rangle) + \epsilon(\langle\hat{\eta}_c\rangle - \langle\hat{\eta}_a\rangle), \quad (2.39)$$

$$\frac{d\langle\hat{\eta}_a\rangle}{dt} = \epsilon(\langle\hat{\sigma}_c^\dagger\rangle + \langle\hat{\sigma}_c\rangle) + g(\langle\hat{\sigma}_a^\dagger\hat{a}\rangle + \langle\hat{a}^\dagger\hat{\sigma}_a\rangle), \quad (2.40)$$

$$\frac{d\langle\hat{\eta}_b\rangle}{dt} = -g(\langle\hat{\sigma}_a^\dagger\hat{a}\rangle + \langle\hat{a}^\dagger\hat{\sigma}_a\rangle - \langle\hat{a}^\dagger\hat{\sigma}_b\rangle), \quad (2.41)$$

$$\frac{d\langle\hat{\eta}_c\rangle}{dt} = -\epsilon(\langle\hat{\sigma}_c\rangle + \langle\hat{\sigma}_c^\dagger\rangle) - g(\langle\hat{a}^\dagger\hat{\sigma}_b\rangle + \langle\hat{\sigma}_b^\dagger\hat{a}\rangle). \quad (2.42)$$

The quantum Langevin equation for the cavity mode operator  $\hat{a}$  can be written, based on Eq. (2.33), as

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} - g(\hat{\sigma}_a + \hat{\sigma}_b) + \hat{f}(t), \quad (2.43)$$

where  $\hat{f}(t)$  is a cavity mode noise operator whose correlation properties remain to be determined. It is clear that the expectation value of Eq. (2.43) is equal to Eq. (2.33) if  $\langle \hat{f}(t) \rangle = 0$ . We observe that Eqs. (2.34)-(2.36) and Eqs. (2.37)-(2.42) are nonlinear coupled differential equations and hence it is difficult to obtain their time-dependent solution[3,4]. We thus apply the large time approximation scheme to Eq. (2.43) and write its solution as

$$\hat{a}(t) = -\frac{2g}{\kappa}(\hat{\sigma}_a + \hat{\sigma}_b) + \frac{2}{\kappa}\hat{f}(t). \quad (2.44)$$

It then follows that

$$\langle \hat{a}(t) \rangle = -\frac{2g}{\kappa}(\langle \hat{\sigma}_a \rangle + \langle \hat{\sigma}_b \rangle), \quad (2.45)$$

and

$$\hat{a}^\dagger(t) = -\frac{2g}{\kappa}(\hat{\sigma}_a^\dagger + \hat{\sigma}_b^\dagger) + \frac{2}{\kappa}\hat{f}^\dagger(t). \quad (2.46)$$

Substituting Eq. (2.44) into Eq. (2.34), we obtain

$$\begin{aligned} \frac{d\langle \hat{a}^2 \rangle}{dt} &= -\kappa\langle \hat{a}^2 \rangle + \frac{2g^2}{\kappa}(\langle (\hat{\sigma}_a + \hat{\sigma}_b)\hat{\sigma}_a \rangle + \langle \hat{\sigma}_a(\hat{\sigma}_a + \hat{\sigma}_b) \rangle + \langle \hat{\sigma}_b(\hat{\sigma}_a + \hat{\sigma}_b) \rangle + \kappa M) \\ &= -\kappa\langle \hat{a}^2 \rangle + \frac{2g^2}{\kappa}(|b\rangle\langle a|b\rangle\langle a| + |c\rangle\langle b|b\rangle\langle a| + |b\rangle\langle a|b\rangle\langle a| + |b\rangle\langle a|c\rangle\langle b| + |c\rangle\langle b|b\rangle\langle a| \\ &\quad + |c\rangle\langle b|c\rangle\langle b|) + \kappa M \\ &= -\kappa\langle \hat{a}^2 \rangle + \frac{2g^2}{\kappa}(|c\rangle\langle a| + |c\rangle\langle a|) + \kappa M \\ &= -\kappa\langle \hat{a}^2 \rangle + \frac{4g^2}{\kappa}\langle \hat{\sigma}_c \rangle + \kappa M, \end{aligned} \quad (2.47)$$

where  $\langle \hat{\sigma}_a \rangle \langle \hat{\sigma}_a \rangle = |b\rangle\langle a|b\rangle\langle a| = 0$ ,

$\langle \hat{\sigma}_a \rangle \langle \hat{\sigma}_b \rangle = |b\rangle\langle a|c\rangle\langle b| = 0$ ,

$\langle \hat{\sigma}_b \rangle \langle \hat{\sigma}_a \rangle = |c\rangle\langle b|b\rangle\langle a| = \langle \hat{\sigma}_c \rangle$ ,

$\langle \hat{\sigma}_b \rangle \langle \hat{\sigma}_b \rangle = |c\rangle\langle b|c\rangle\langle b| = 0$ .

Following the same procedure, we find that

$$\frac{d\langle \hat{a}^\dagger \hat{a} \rangle}{dt} = -\kappa\langle \hat{a}^\dagger \hat{a} \rangle + \frac{4g^2}{\kappa}(\langle \hat{\eta}_a \rangle + \langle \hat{\eta}_b \rangle) + \kappa N, \quad (2.48)$$

$$\frac{d\langle\hat{a}\hat{a}^\dagger\rangle}{dt} = -\kappa\langle\hat{a}\hat{a}^\dagger\rangle + \frac{4g^2}{\kappa}(\langle\hat{\eta}_b\rangle + \langle\hat{\eta}_c\rangle) + \kappa N + \kappa. \quad (2.49)$$

Substituting Eq. (2.44) and its conjugate into Eq. (2.37), we note that

$$\begin{aligned} \frac{d\langle\hat{\sigma}_a\rangle}{dt} &= g(\langle\hat{\eta}_b(-\frac{2g}{\kappa}(\hat{\sigma}_a + \hat{\sigma}_b) - \langle\hat{\eta}_a(-\frac{2g}{k}(\hat{\sigma}_a + \hat{\sigma}_b)) \\ &\quad + \langle(-\frac{2g}{\kappa}(\hat{\sigma}_a^\dagger + \hat{\sigma}_b^\dagger)\hat{\sigma}_c)\rangle + \epsilon\langle\hat{\sigma}_b^\dagger\rangle \\ &= -\frac{2g^2}{\kappa}(\langle\hat{\eta}_b\hat{\sigma}_a\rangle + \langle\hat{\eta}_b\hat{\sigma}_b\rangle) - \frac{2g^2}{k}(\langle\hat{\sigma}_a^\dagger\hat{\sigma}_c\rangle + \langle\hat{\sigma}_b^\dagger\hat{\sigma}_c\rangle) \\ &\quad + \langle\hat{\eta}_a\hat{\sigma}_a\rangle + \langle\hat{\eta}_a\hat{\sigma}_b\rangle + \epsilon\langle\hat{\sigma}_b^\dagger\rangle. \end{aligned} \quad (2.50)$$

Utilizing the results  $\hat{\eta}_b\hat{\sigma}_a = \hat{\sigma}_a$ ,  $\hat{\eta}_b\hat{\sigma}_b = 0$ ,  $\hat{\eta}_a\hat{\sigma}_a = 0$ ,  $\hat{\eta}_a\hat{\sigma}_b = 0$ , we can express Eq. (2.50) as

$$\frac{d\langle\hat{\sigma}_a\rangle}{dt} = -\frac{2g^2}{k}(\langle\hat{\sigma}_a\rangle + \langle\hat{\sigma}_a\rangle) + \epsilon\langle\hat{\sigma}_b^\dagger\rangle, \quad (2.51)$$

or

$$\frac{d\langle\hat{\sigma}_a\rangle}{dt} = -\gamma_c\langle\hat{\sigma}_a\rangle + \epsilon\langle\hat{\sigma}_b^\dagger\rangle, \quad (2.52)$$

where the parameter defined by  $\gamma_c = \frac{4g^2}{\kappa}$  is called stimulated emission decay constant.

Following a similar procedure, we readily obtain

$$\frac{d\langle\hat{\sigma}_b\rangle}{dt} = -\frac{\gamma_c}{2}(\langle\hat{\sigma}_b\rangle - 2\langle\hat{\sigma}_a\rangle) - \epsilon\langle\hat{\sigma}_a^\dagger\rangle, \quad (2.53)$$

$$\frac{d\langle\hat{\sigma}_c\rangle}{dt} = -\frac{\gamma_c}{2}\langle\hat{\sigma}_c\rangle + \epsilon(\langle\hat{\eta}_c\rangle - \langle\hat{\eta}_a\rangle), \quad (2.54)$$

$$\frac{d\langle\hat{\eta}_a\rangle}{dt} = \epsilon(\langle\hat{\sigma}_c^\dagger\rangle + \langle\hat{\sigma}_c\rangle) - \gamma_c\langle\hat{\eta}_a\rangle, \quad (2.55)$$

$$\frac{d\langle\hat{\eta}_b\rangle}{dt} = \gamma_c(\langle\hat{\eta}_a\rangle - \langle\hat{\eta}_b\rangle), \quad (2.56)$$

$$\frac{d\langle\hat{\eta}_c\rangle}{dt} = -\epsilon(\langle\hat{\sigma}_c\rangle + \langle\hat{\sigma}_c^\dagger\rangle) + \gamma_c\langle\hat{\eta}_b\rangle. \quad (2.57)$$

The steady-state solution of Eq. (2.47), Eq. (2.48), Eq. (2.49), Eq. (2.52), Eq. (2.53), Eq. (2.54), Eq. (2.55), and Eq. (2.56), can be written, respectively, as

$$\langle \hat{a}^2 \rangle = \frac{4g^2}{\kappa^2} \langle \hat{\sigma}_c \rangle + M, \quad (2.58)$$

$$\langle \hat{a}^\dagger \hat{a} \rangle = \frac{4g^2}{\kappa^2} (\langle \hat{\eta}_a \rangle + \langle \hat{\eta}_b \rangle) + N, \quad (2.59)$$

$$\langle \hat{a} \hat{a}^\dagger \rangle = \frac{4g^2}{\kappa^2} (\langle \eta_b \rangle + \langle \eta_c \rangle) + N + 1, \quad (2.60)$$

$$\langle \hat{\sigma}_a \rangle = \frac{\epsilon}{\gamma_c} \langle \hat{\sigma}_b^\dagger \rangle, \quad (2.61)$$

$$\langle \hat{\sigma}_b \rangle = 2\langle \hat{\sigma}_a \rangle - \frac{2\epsilon}{\gamma_c} \langle \hat{\sigma}_a^\dagger \rangle, \quad (2.62)$$

$$\langle \hat{\sigma}_c \rangle = \frac{2\epsilon}{\gamma_c} (\langle \hat{\eta}_c \rangle - \langle \hat{\eta}_a \rangle), \quad (2.63)$$

$$\langle \hat{\eta}_a \rangle = \frac{\epsilon}{\gamma_c} (\langle \hat{\sigma}_c^\dagger \rangle + \langle \hat{\sigma}_c \rangle), \quad (2.64)$$

$$\langle \hat{\eta}_b \rangle = \langle \hat{\eta}_a \rangle. \quad (2.65)$$

Complex conjugates of Eq. (2.61) and Eq. (2.62) becomes

$$\langle \hat{\sigma}_a^\dagger \rangle = \frac{\epsilon}{\gamma_c} \langle \hat{\sigma}_b \rangle, \quad (2.66)$$

$$\langle \hat{\sigma}_b^\dagger \rangle = 2\langle \hat{\sigma}_a^\dagger \rangle - \frac{2\epsilon}{\gamma_c} \langle \hat{\sigma}_a \rangle. \quad (2.67)$$

Substituting Eqs. (2.66) and (2.61) into Eq. (2.62), we obtain

$$\langle \hat{\sigma}_b \rangle = \frac{2\epsilon}{\gamma_c} \langle \hat{\sigma}_b^\dagger \rangle - \frac{2\epsilon^2}{\gamma_c^2} \langle \hat{\sigma}_b \rangle, \quad (2.68)$$

and its complex conjugate is

$$\langle \hat{\sigma}_b^\dagger \rangle = \frac{2\epsilon}{\gamma_c} \langle \hat{\sigma}_b \rangle - \frac{2\epsilon^2}{\gamma_c^2} \langle \hat{\sigma}_b^\dagger \rangle, \quad (2.69)$$

Subtracting Eq. (2.68) from Eq. (2.69), we obtain  $\langle \hat{\sigma}_b^\dagger \rangle = \langle \hat{\sigma}_b \rangle$ , so that Eq. (2.61) becomes

$$\langle \hat{\sigma}_a \rangle = \frac{\epsilon}{\gamma_c} \langle \hat{\sigma}_b \rangle, \quad (2.70)$$

and

$$\langle \hat{\sigma}_a^\dagger \rangle = \frac{\epsilon}{\gamma_c} \langle \hat{\sigma}_b^\dagger \rangle. \quad (2.71)$$

Moreover, substituting Eqs. (2.70) and (2.71), in to Eq. (2.62), we get

$$\langle \hat{\sigma}_b \rangle = 0. \quad (2.72)$$

Putting Eq. (2.72) in Eq. (2.61), we have

$$\langle \hat{\sigma}_a \rangle = 0. \quad (2.73)$$

Employing Eq. (2.65) along with the relation

$$\langle \hat{\eta}_a \rangle + \langle \hat{\eta}_b \rangle + \langle \hat{\eta}_c \rangle = 1, \quad (2.74)$$

we see that

$$2\langle \hat{\eta}_a \rangle + \langle \hat{\eta}_c \rangle = 1. \quad (2.75)$$

On the basis of Eq. (2.63), we notice that  $\hat{\sigma}_c = \hat{\sigma}_c^\dagger$ . Taking this in to account and combining Eqs. (2.63), (2.64), and (2.75), we obtain

$$\langle \hat{\eta}_a \rangle = \frac{4\epsilon^2}{\gamma_c^2 + 12\epsilon^2}, \quad (2.76)$$

which with the aid of Eq. (2.65) leads to

$$\langle \hat{\eta}_b \rangle = \frac{4\epsilon^2}{\gamma_c^2 + 12\epsilon^2}. \quad (2.77)$$

Next substituting Eq. (2.76) into Eq. (2.75), we get

$$\langle \hat{\eta}_c \rangle = \frac{\gamma_c^2 + 4\epsilon^2}{\gamma_c^2 + 12\epsilon^2}. \quad (2.78)$$

In addition, using Eq. (2.76) and Eq. (2.78) in Eq. (2.63), we have

$$\langle \hat{\sigma}_c \rangle = \frac{2\epsilon\gamma_c}{\gamma_c^2 + 12\epsilon^2}. \quad (2.79)$$

Now substituting Eq. (2.72), and (2.73) into Eq. (2.45), we obtain

$$\langle \hat{a} \rangle = 0. \quad (2.80)$$

Putting Eq. (2.79) into Eq. (2.58), we find

$$\langle \hat{a}^2 \rangle = \frac{2\epsilon\gamma_c^2}{\kappa(\gamma_c^2 + 12\epsilon^2)} + \sinh r \cosh r, \quad (2.81)$$

$$\langle \hat{a}^{\dagger 2} \rangle = \frac{2\epsilon\gamma_c^2}{\kappa(\gamma_c^2 + 12\epsilon^2)} + \sinh r \cosh r. \quad (2.82)$$

## 2.1 The Correlation properties of the cavity mode noise operators

Upon substituting Eq. (2.43) and its conjugate in to the relation

$$\frac{\langle d\hat{a}^\dagger \hat{a} \rangle}{dt} = \frac{\langle d\hat{a}^\dagger \rangle}{dt} \hat{a} + \frac{\langle \hat{a}^\dagger d\hat{a} \rangle}{dt}, \quad (2.83)$$

we obtain

$$\frac{\langle d\hat{a}^\dagger \hat{a} \rangle}{dt} = -\kappa \langle \hat{a}^\dagger \hat{a} \rangle - g(\langle \hat{\sigma}_a^\dagger \hat{a} \rangle + \langle \hat{\sigma}_b^\dagger \hat{a} \rangle + \langle \hat{a}^\dagger \hat{\sigma}_a \rangle + \langle \hat{a}^\dagger \hat{\sigma}_b \rangle) + \langle \hat{a}^\dagger \hat{f}(t) \rangle + \langle \hat{a}^\dagger \hat{f}(t) \rangle. \quad (2.84)$$

Comparing Eq. (2.48) with Eq. (2.84), we obtain

$$\langle \hat{a}^\dagger \hat{f}(t) \rangle + \langle \hat{f}^\dagger(t) \hat{a} \rangle = \kappa N. \quad (2.85)$$

The solution of Eq. (2.44) can be written as

$$\hat{a}(t) = \hat{a}(0)e^{-\frac{\kappa t}{2}} + \int_0^t e^{-\frac{\kappa(t-t')}{2}} (-g(\hat{\sigma}_b(t') + \hat{\sigma}_a(t')) + \hat{f}(t')) dt'. \quad (2.86)$$

Multiplying Eq. (2.86) from the left by  $\hat{f}^\dagger(t)$  and taking the expectation value of the product, we have

$$\langle \hat{f}^\dagger(t) \hat{a}(t) \rangle = \langle \hat{f}^\dagger(t) \hat{a}(0) \rangle e^{-\frac{\kappa t}{2}} + \int_0^t e^{-\frac{\kappa(t-t')}{2}} (-g(\langle \hat{f}^\dagger(t) \hat{\sigma}_b(t') \rangle + \langle \hat{f}^\dagger(t) \hat{\sigma}_a(t') \rangle) + \langle \hat{f}^\dagger(t) \hat{f}(t') \rangle) dt'. \quad (2.87)$$

Since the noise operators should not affect cavity mode and atomic operators at earlier times, we can write

$\langle \hat{f}^\dagger(t) \hat{\sigma}_b \rangle + \langle \hat{f}^\dagger(t) \langle \hat{\sigma}_a \rangle = \langle \hat{f}^\dagger(t) \rangle \langle \hat{\sigma}_b \rangle + \langle \hat{f}^\dagger(t) \rangle \langle \hat{\sigma}_a \rangle = 0$ , so that Eq. (2.87) can be put as

$$\langle \hat{f}^\dagger(t) \hat{a}(t) \rangle = \int_0^t e^{-\frac{\kappa(t-t')}{2}} \langle \hat{f}^\dagger(t) \hat{f}(t') \rangle dt'. \quad (2.88)$$

Following a similar procedure, we readily obtain

$$\langle \hat{a}^\dagger(t) \hat{f}(t) \rangle = \int_0^t e^{-\frac{\kappa(t-t')}{2}} \langle \hat{f}^\dagger(t') \hat{f}(t) \rangle dt'. \quad (2.89)$$

Combination Eqs. (2.88) and (2.89) with Eq. (2.85) leads to

$$\int_0^t e^{-\frac{\kappa(t-t')}{2}} \langle \hat{f}^\dagger(t) \hat{f}(t') \rangle dt' + \int_0^t e^{-\frac{\kappa(t-t')}{2}} \langle \hat{f}^\dagger(t') \hat{f}(t) \rangle dt' = \kappa N, \quad (2.90)$$

and assuming  $\langle \hat{f}^\dagger(t) \hat{f}(t') \rangle = \langle \hat{f}^\dagger(t') \hat{f}(t) \rangle$ , we obtain

$$\int_0^t e^{-\frac{\kappa(t-t')}{2}} \langle \hat{f}^\dagger(t) \hat{f}(t') \rangle dt' = \frac{\kappa N}{2}, \quad (2.91)$$

$$\langle \hat{f}^\dagger(t) \hat{f}(t') \rangle = \kappa N \delta(t - t'). \quad (2.92)$$

In a similar manner, it can be verified that

$$\langle \hat{f}(t') \hat{f}^\dagger(t) \rangle = \kappa(N + 1) \delta(t - t'), \quad (2.93)$$

$$\langle \hat{f}(t') \hat{f}(t) \rangle = \kappa M \delta(t - t'). \quad (2.94)$$

# Chapter 3

## Photon Statistics

Here we investigate the statistical properties of the light generated by coherently driven degenerate three-level atom in a cavity coupled to squeezed vacuum reservoir via a single port mirror. In light of this, we evaluate the mean of the photon number and the power spectrum of the cavity light.

### 3.1 The mean photon number

Employing Eqs. (2.76) and (2.77) in (2.59), the steady-state cavity mean photon number is expressible as

$$\bar{n} = \frac{\gamma_c}{k} \left( \frac{8\epsilon^2}{\gamma_c^2 + 12\epsilon^2} \right) + \sinh^2 r. \quad (3.1)$$

Moreover, substituting Eq. (2.77) and Eq. (2.78) into Eq. (2.60), we obtain

$$\langle \hat{a}\hat{a}^\dagger \rangle = \frac{\gamma_c(8\epsilon^2 + \gamma_c^2)}{\kappa(\gamma_c^2 + 12\epsilon^2)} + \sinh^2 r + 1. \quad (3.2)$$

We see from Eq. (3.1) that the steady-state cavity mean photon number depends on the amplitude of the coupling coherent light, stimulated emission decay constant ( $\gamma_c$ ), cavity dumping constant, and the squeeze parameter. If there is no driving coherent light ( $\epsilon = 0$ ) the steady-state mean photon number becomes,

$$\bar{n} = \sinh^2 r, \quad (3.3)$$

which is the mean photon number of the cavity mode coupled to squeezed vacuum reservoir.

For a cavity coupled to vacuum reservoir ( $r = 0$ ), the mean photon number reduces to

$$\bar{n} = \frac{\gamma_c}{k} \left( \frac{8\epsilon^2}{\gamma_c^2 + 12\epsilon^2} \right), \quad (3.4)$$

or

$$\bar{n} = \frac{\gamma_c}{k} \left( \frac{8}{12 + (\frac{\gamma_c}{\epsilon})^2} \right). \quad (3.5)$$

We observe from this result that coherently driven three-level atom in a cavity coupled to vacuum reservoir emits light as the result of stimulated emission due to its interaction with the driving coherent light.

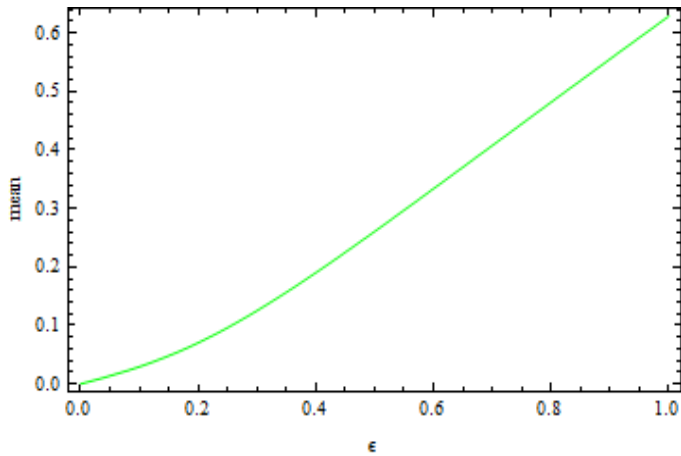


Figure 3.1: Plot of steady-state mean photon number [Eq. (3.1)] versus  $\epsilon$ , for  $\kappa = 0.8$ ,  $r = 0.5$ , and  $\gamma_c = 0.5$

From Figure 3.1 we see that the mean photon number increases with the increase of the values of amplitude of driving coherent light ( $\epsilon$ ).

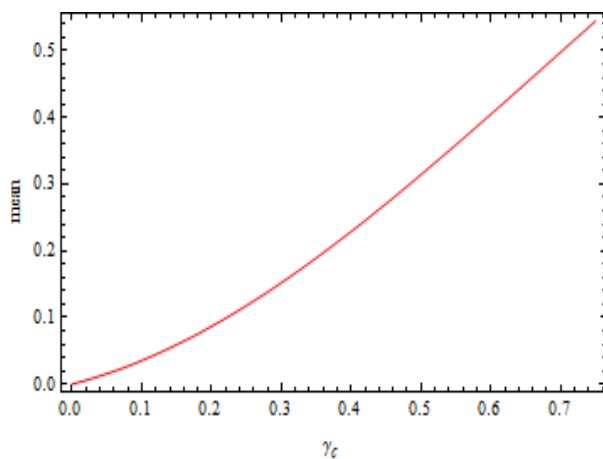


Figure 3.2: Plot of steady-state mean photon number versus  $\gamma_c$  for  $\kappa = 0.8$ , and  $r = 0.5$ ,  $\epsilon = 0.5$

From Figure 3.2 we see that the mean photon number enhances with stimulated emission

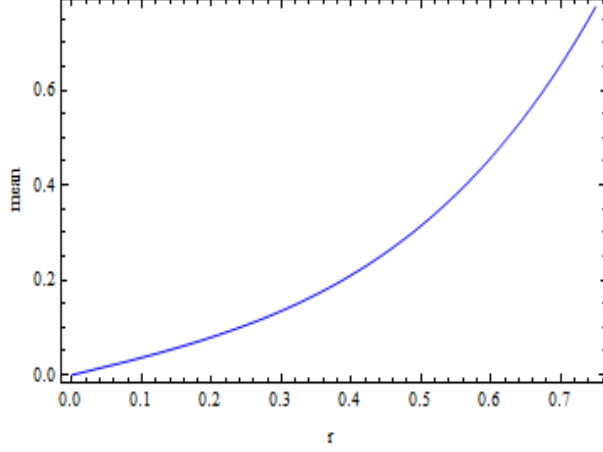


Figure 3.3: Plot of steady-state mean photon number versus  $r$  for  $\kappa = 0.8$ ,  $\epsilon = 0.5$ , and  $\gamma_c = 0.5$

decay( $\gamma_c$ ).

From Figure (3.3) we see that the mean photon number enhances with squeezed parameter( $r$ ). Generally, the plots Figures 3.1, 3.2, and 3.3 show that the values of amplitude of driving coherent light ( $\epsilon$ ), stimulated emission decay( $\gamma_c$ ), and squeeze parameter( $r$ ) increase mean photon number of the cavity light produced by the system.

## 3.2 Power spectrum

In nearly all cases the frequency of a single-mode light is not sharply defined[3]. In general there is some variation about the central frequency [1,3]. In this section we want to evaluate the spectrum of the mean photon number usually known as the power spectrum of light emitted by coherently driven degenerated three-level atom represented by cavity mode operators  $\hat{a}$  and  $\hat{a}^\dagger$ . The power spectrum of a single mode light with central frequency  $\omega_o$  is expressible as [3,12]

$$P(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \langle \hat{a}^\dagger(t) \hat{a}(t + \tau) \rangle_{ss} e^{i(\omega - \omega_o)\tau} d\tau. \quad (3.6)$$

Upon integrating both sides of this equation over  $\omega$ , we see that

$$\int_{-\infty}^{\infty} P(\omega) d\omega = \int_{-\infty}^{\infty} \langle \hat{a}^\dagger(t) \hat{a}(t + \tau) \rangle_{ss} e^{-i\omega_o\tau} d\tau \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega\tau} d\omega \quad (3.7)$$

and in view of the relation

$$\delta(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega\tau} d\omega, \quad (3.8)$$

we find that

$$\int_{-\infty}^{\infty} P(\omega) d\omega = \int_{-\infty}^{\infty} \langle \hat{a}^\dagger(t) \hat{a}(t + \tau) \rangle_{ss} e^{-i\omega_o \tau} \delta(\tau) d\tau, \quad (3.9)$$

which with the aid of the relation

$$\int_{-\infty}^{\infty} f(x) \delta(x) dx = f(x)|_{x=0}, \quad (3.10)$$

becomes

$$\int_{-\infty}^{\infty} P(\omega) d\omega = \langle \hat{a}^\dagger(t) \hat{a}(t) \rangle_{ss}, \quad (3.11)$$

or

$$\int_{-\infty}^{\infty} P(\omega) d\omega = \bar{n}. \quad (3.12)$$

Based on the result described by Eq. (3.12), we observe that  $P(\omega) d\omega$  is the steady-state mean photon number the frequency interval  $d\omega$  [3]. Rewriting Eq. (3.6) as

$$P(\omega) = \frac{1}{2\pi} \int_{-\infty}^0 \langle \hat{a}^\dagger(t) \hat{a}(t + \tau) \rangle_{ss} e^{i(\omega - \omega_o)\tau} d\tau + \frac{1}{2\pi} \int_0^{\infty} \langle \hat{a}^\dagger(t) \hat{a}(t + \tau) \rangle_{ss} e^{i(\omega - \omega_o)\tau} d\tau \quad (3.13)$$

by replacing  $\tau$  by  $-\tau$  and then  $t$  by  $t + \tau$  in the first integral, we obtain

$$P(\omega) = \frac{1}{2\pi} \int_0^{\infty} \langle \hat{a}^\dagger(t) \hat{a}(t - \tau) \rangle_{ss} e^{-i(\omega - \omega_o)\tau} d\tau + \frac{1}{2\pi} \int_0^{\infty} \langle \hat{a}^\dagger(t) \hat{a}(t + \tau) \rangle_{ss} e^{i(\omega - \omega_o)\tau} d\tau, \quad (3.14)$$

The first term is the complex conjugate of the second term .

Hence the power spectrum can be express as,

$$P(\omega) = \frac{1}{\pi} \text{Re} \int_0^{\infty} \langle \hat{a}^\dagger(t) \hat{a}(t + \tau) \rangle_{ss} e^{i(\omega - \omega_o)\tau} d\tau, \quad (3.15)$$

where Re stands for the real part of the integration. We proceed to determine the two time correlation function that appear in Eq. (3.15) for the cavity light. Taking Eqs. (2.72) and (2.73) into account, we can write the solution of Eq. (2.33) as

$$\langle \hat{a}(t + \tau) \rangle = \langle \hat{a}(t) \rangle e^{-\frac{\kappa}{2}\tau}. \quad (3.16)$$

Apply the quantum regression theorem to this equation, we obtain

$$\langle \hat{a}^\dagger(t) \hat{a}(t + \tau) \rangle = \langle \hat{a}^\dagger(t) \hat{a}(t) \rangle e^{-\frac{\kappa}{2}\tau}. \quad (3.17)$$

Substituting this result in Eq. (3.15) and performing the integration, the power spectrum for the cavity light is expressible as

$$P(\omega) = \bar{n} \left[ \frac{\kappa/2\pi}{(\omega - \omega_o)^2 + (\kappa/2)^2} \right]. \quad (3.18)$$

In view of Eq. (3.12) the local mean photon number in the interval between  $\omega' = -\lambda$  and  $\omega' = \lambda$  is expressible as[4]

$$\bar{n}_{\pm\lambda} = \int_{-\lambda}^{\lambda} P(\omega') d\omega', \quad (3.19)$$

in which  $\omega - \omega_o = \omega'$ .

Substituting Eq. (3.18) in to Eq. (3.19) and carrying out the integration applying the relation

$$\int_{-\lambda}^{\lambda} \frac{dx}{x^2 + d^2} = \frac{2}{d} \arctan\left(\frac{\lambda}{d}\right), \quad (3.20)$$

we arrive

$$\bar{n}_{\pm\lambda} = (2\bar{n}/\pi) \arctan(2\lambda/\kappa). \quad (3.21)$$

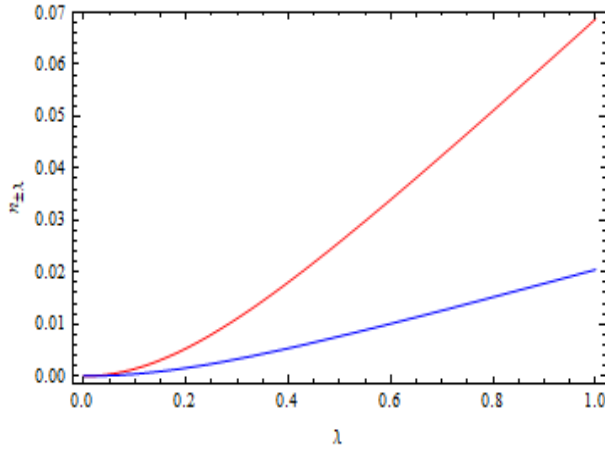


Figure 3.4:  $\bar{n}_{\pm\lambda}$  versus  $\lambda$ , for  $\kappa = 0.8$ ,  $\epsilon = 0.9$ ,  $r = 0$ (blue) and  $r = 0.2$ (red)

Figure 3.4 show that the local mean photon number increases with the squeeze parameter  $r$  and frequency interval  $\lambda$ .

# Chapter 4

## Quadrature Squeezing

In a squeezed state the quantum noise in one quadrature is below the coherent state level at the expense of enhanced fluctuations in the conjugate quadrature, with the product of the uncertainties in the two quadratures satisfying the uncertainty relation[1,3]. The plus and minus quadrature operators are respectively defined as

$$\hat{a}_+ = \hat{a}^\dagger + \hat{a}, \quad (4.1)$$

$$\hat{a}_- = i(\hat{a}^\dagger - \hat{a}). \quad (4.2)$$

The variance of the quadrature operators are expressible as,

$$(\Delta a_\pm)^2 = \langle \hat{a}_\pm^2 \rangle - \langle \hat{a}_\pm \rangle^2. \quad (4.3)$$

Substituting Eq. (4.1) and Eq. (4.2) in to Eq. (4.3), we get

$$(\Delta a_+)^2 = \langle \hat{a}^\dagger \hat{a} \rangle + \langle \hat{a} \hat{a}^\dagger \rangle + \langle \hat{a}^{\dagger 2} \rangle + \langle \hat{a}^2 \rangle, \quad (4.4)$$

$$(\Delta a_-)^2 = \langle \hat{a}^\dagger \hat{a} \rangle + \langle \hat{a} \hat{a}^\dagger \rangle - \langle \hat{a}^{\dagger 2} \rangle - \langle \hat{a}^2 \rangle. \quad (4.5)$$

Substituting Eqs. (2.80)-(2.82), and(3.1)-(3.2) in to Eqs. (4.4) and (4.5), we easily get

$$\begin{aligned} (\Delta a_+)^2 &= \frac{\gamma_c}{\kappa} \left[ \frac{16\epsilon^2 + \gamma_c^2 + 2\epsilon\gamma_c}{12\epsilon^2 + \gamma_c^2} \right] + 2(\sinh^2 r + \sinh r \cosh r) + 1 \\ &= \frac{\gamma_c}{\kappa} \left[ \frac{16\epsilon^2 + \gamma_c^2 + 4\epsilon\gamma_c}{12\epsilon^2 + \gamma_c^2} \right] + e^{2r} \end{aligned} \quad (4.6)$$

$$\begin{aligned}
(\Delta a_-)^2 &= \frac{\gamma_c}{\kappa} \left[ \frac{16\epsilon^2 + \gamma_c^2 - 4\epsilon\gamma_c}{12\epsilon^2 + \gamma_c^2} \right] + 2(\sinh^2 r - \sinh r \cosh r) + 1 \\
&= \frac{\gamma_c}{\kappa} \left[ \frac{16\epsilon^2 + \gamma_c^2 - 4\epsilon\gamma_c}{12\epsilon^2 + \gamma_c^2} \right] + e^{-2r}.
\end{aligned} \tag{4.7}$$

The quadrature variance of the cavity mode in a vacuum state is obtained by setting  $r = 0$  and  $\epsilon = 0$  in Eq. (4.7). It then follows that

$$(\Delta a_-)_v^2 = \frac{\gamma_c}{\kappa} + 1. \tag{4.8}$$

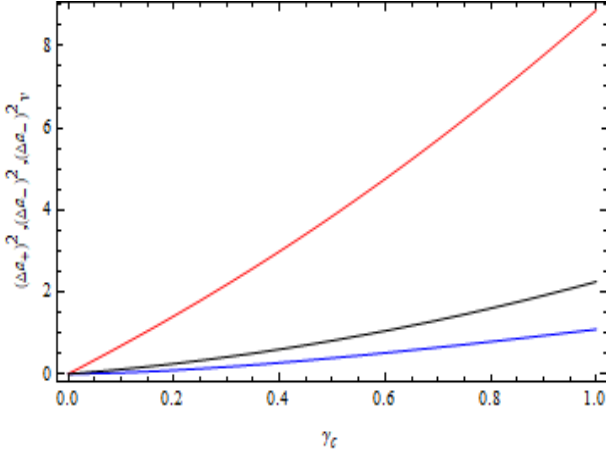


Figure 4.1: Plus quadrature variance(Red), minus quadrature variance in vacuum level(black) and minus quadrature variance(blue) versus  $\gamma_c$ , for  $\epsilon = 0.5$ ,  $\kappa = 0.8$ ,  $r = 0.5$

From the plots in Figure 4.1, we see that the cavity light is in squeezed state and the squeezing occurs in the minus quadrature. The quadrature squeezing(S)of the cavity light relative to the quadrature variance of the cavity vacuum is defined as

$$S = \frac{(\Delta a_-)_v^2 - (\Delta a_-)^2}{(\Delta a_-)_v^2}, \tag{4.9}$$

where  $(\Delta a_-)_v^2$  is the quadrature variance of vacuum light.

Substituting Eq. (4.7) and Eq.(4.8) in to Eq. (4.9), we obtain

$$\begin{aligned}
S &= \frac{4\gamma_c^2\epsilon - 4\epsilon^2\gamma_c - 2\kappa(12\epsilon^2 + \gamma_c^2)(\sinh r \cosh r - \sinh^2 r)}{(12\epsilon^2 + \gamma_c^2)(\gamma_c + \kappa)} \\
&= \frac{4\gamma_c^2\epsilon - 4\epsilon^2\gamma_c + \kappa(12\epsilon^2 + \gamma_c^2)(1 - e^{-2r})}{(12\epsilon^2 + \gamma_c^2)(\gamma_c + \kappa)}.
\end{aligned} \tag{4.10}$$

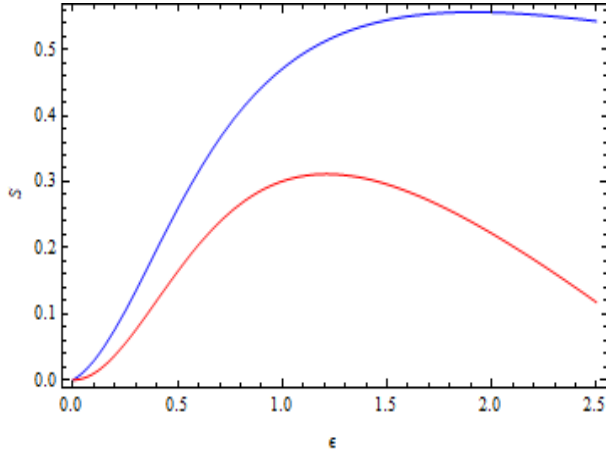


Figure 4.2: Plot of squeezing( $S$ ) versus  $\epsilon$ , for  $\kappa = 0.8$ ,  $r = 0.5$ (*blue*),  $r = 0$ (*red*) and  $\gamma_c = 0.5$

Figure 4.2 shows that the degree of squeezing increases with the amplitude of the coherent light coupling the levels of the atom and the maximum squeezing is 58% for  $r=0.5$  and 32% for  $r=0$  below the vacuum level.

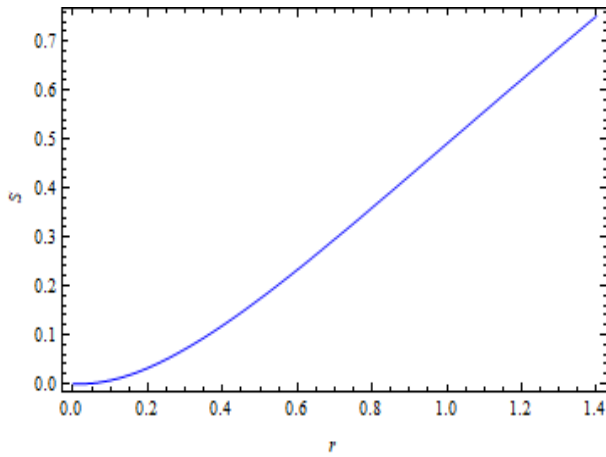


Figure 4.3: Plot of squeezing( $S$ ) versus  $r$ , for  $\kappa = 0.8$ ,  $\epsilon = 0.5$  and  $\gamma_c = 0.5$

The plot in Figure 4.3 indicates that the increase of the squeeze parameter enhances squeezing.

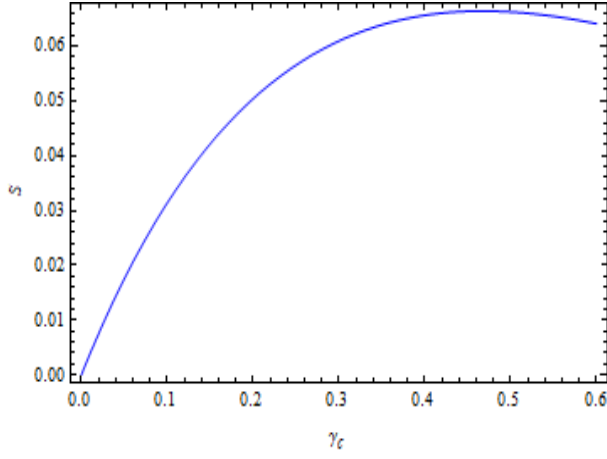


Figure 4.4: Plot of squeezing(S) versus  $\gamma_c$ , for  $\kappa = 0.8$ ,  $\epsilon = 0.5$ , and  $r=0.5$

Figure 4.4 indicates that the increase of emission decay( $\gamma_c$ ) enhances squeezing. We notice from the plots in Figures 4.2, 4.3 and 4.4 the degree of squeezing of the radiation increases with the increasing of the amplitude of the coupling coherent light, squeezed parameter ( $r$ ), and the stimulated emission decay constant.

# Chapter 5

## Conclusion

In this thesis, we have studied the squeezing and statistical properties of cavity light generated by degenerate three-level atom whose top and bottom levels are coupled by a driving coherent light; and available in a cavity coupled to squeezed vacuum reservoir. Employing the master equation for the system, we obtained the cavity mode and atomic dynamic equations. With the use of the large time approximation and steady-state solutions of the dynamic equations, we have calculated the global and local mean photon number, and the quadrature squeezing of the cavity light. Our results have shown that the mean photon number increases with increase of the amplitude of the driving coherent light, the squeeze parameter, and stimulated emission decay constant. Moreover, the local mean photon number increases as the frequency interval increases. We have also found that the cavity light is in a squeezed state and the squeezing occurs in the minus quadrature. The plot in Figure 4.2 demonstrated that the squeeze parameter and the amplitude of the driving coherent light have the effect of enhancing the quadrature squeezing, with maximum squeezing of 58% for  $r = 0.5$  and 32% for  $r=0$ .

# Bibliography

- [1] Fesseha Kassahun, Fundamentals of Quantum Optics (Lulu Press Inc.,North Carolina 2010).
- [2] E.Alebachew, k. Fesseha, Opt. Commun,265,314(2006).
- [3] Yosef Terefe, MSc Thesis, (Addis Ababa University, 2013).
- [4] Mulugeta Melaku, MSc Thesis, (Addis Ababa University, 2014).
- [5] Fesseha Kassahun, Opt. Commun. 284, 1357 (2011).
- [6] Fesseha Kassahun, arXiv1105. 1438v3[quant-ph]25Sep2012.
- [7] Gilbert Grynberg, Alian Aspect and Claude, Introduction to Quantum Optics(Cambridge University, 2010).
- [8] N.A. Ansari, Phys Rev A, 4686 (1993).
- [9] S.Tesfa, Phys Rev A, 77, 013815(2008).
- [10] V.M Akulin, Coherent Daynamics of Complex Quantum system(Paris Spiringer,2004).
- [11] B. Daniel and K. Fesseha, Opt. Commun. 151, 384 (1998).
- [12] Lijalem Minewagaw, MSc Thesis, (Addis Ababa University, 2019).
- [13] M.O. Scully, and M.S. Zubairy, Quantum Optics(Cambridge University Press, 1997).
- [14] N. Lu and S.Y. Zhu, phys Rev A, 41, 2865 (1990).
- [15] K.Fesseha, Phys. Rev. A, 63, 033811 (2001).
- [16] Ayana Yihunie, arXiv:1909. 01004v1[quant-ph]3sep2019.

## DECLARATION

*▷I, hereby declare that this thesis is my original work and has not been presented for a degree in any other un*

Name: Abebaw Gizachew

Signature: \_\_\_\_\_.

e-mail:-abebawgi89@gmail.com

Place and time of submission:

Addis Ababa University

Department of physics

April 18 2022

This thesis has been submitted for examination with my approval as University advisor.

Name: Dr. Deribe Hirpo

Signature: \_\_\_\_\_.

e-mail:-deribeh@gmail.com