COLLEGE OF NATURAL SCIENCE
DEPARTMENT OF MATHEMATICS

GRADUATE PROJECT REPORT ON SCALAR CONSERVATION LAW AND NON-LINEAR EQUATIONS OF FIRST ORDER

Submited in partial fulfilment of requirements for the Degree of Master of Science in Mathematics

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Contents

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The undersigned hereby certify that they have read and recommend to the school of graduate studies for acceptance of a project entitled **Scalar Conservation Law and Non-linear Equation of First Order** by Temesgen Alemu in partial fulfillment of the requirements for the degree of master of Science.

Dated: August, 2014

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Abstract

This project shows a simple method of solving first order non-linear equation and scalar conservation law of the form $u_t + A(u)u_x = 0$ with initial condition $u(x, 0) = f(x)$ in one dimensional wave equation,shock wave and traffic flow model with illustrative example and figures.

Introduction

First order, nonlinear partial differential equations arise in various areas of physical sciences which include geometrical optics, fluid dynamics, and analytical dynamics. An important example of such equations is the Hamilton-Jacobi equation used to describe dynamical systems. Another famous example of the first-order nonlinear equations is the Eikonal equation which arises in nonlinear optics and also describes the propagation of wave fronts and discontinuities for acoustic wave equations Maxwell's equation and equations of elastic wave propagation. Evidently ,first-order, nonlinear equations play an important role in the development of these diverse areas. First order nonlinear equation are also of great importance in modeling of traffic flow, waves breaking the sound barrier and many other branches of applied mathematics.

A conservation law states that the rate of change of the total amount of material contained in a fixed domain of volume V is equal to the flux of that material across the closed bounding surface S of the domain. If we denote the density of the material by $\rho(x,t)$ and the flux vector by $q(x,t)$, then, the conservation law is given by

$$\frac{d}{dt}\int_V \rho dV = -\int_S q.ndS. \tag{1}$$

where dV is the volume element and dS is the surface element of the boundary surface S, n denotes the outward unit normal vector to S , and the right-hand side measures the outward flux ,hence the minus sign is used. Applying the Gauss divergence theorem and taking $\frac{d}{dt}$ inside the integral sign, we obtain

$$\int_V (\frac{\partial \rho}{\partial t} + divq)dV = 0 \tag{2}$$

This result is true for any arbitrary volume V, and, if the integrand is continuous, it must vanish everywhere in the domain. Thus, we obtain the differential form of the conservation law

$$\rho_t + divq = 0 \tag{3}$$

Chapter 1

Preliminaries

1.1 Directional Derivative

Definition 1.1.1. *For a function f, the directional derivative of f at (x_0, y_0) in the direction of a unit vector $u = \langle a, b \rangle$ (i.e.,$a^2 + b^2 = 1$)is denoted $D_u f(x_0, y_o)$ and is defined as:*

$$D_u f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

If the limit exists. The special case of $u = i = \langle 1, 0 \rangle$ gives the partial derivative $f_x(x, y)$ and the case $u = j = \langle 0, 1 \rangle$ gives the partial derivative $f_y(x, y)$.

1.2 The Gradient Vector

Definition 1.2.1. *The gradient vector of a differentiable function $f(x, y)$ at the point (a, b) is $f(a, b) = f_x(a, b)i + f_y(a, b)j$.*

1.3 First order partial differential equation

Definition 1.3.1. *A first order partial differential equation for $u = u(x)$ is given by $F(Du, u, x) = 0$, Where: $F : R^n \times R \times \Omega \to R$ is a given function and Du is the vector of partial derivatives of u.*

1.4 Existence and Computation of Directional Derivative

It turns out that there is a notion of "differentiable" for a function of two variables and if a function is differentiable at a point in that sense, then it has well-defined directional derivatives along all unit vector, the directional derivative in the direction of $u = \langle a, b \rangle$ is given by:

$$D_u f(x_0, y_o) = a f_x(x_0, y_0) + b f_y(x_0, y_0)$$

A sufficient condition for a function to be differentiable is that both the partial derivative f_x *and* f_y exist and are continuous around the point.

1.5 Method of Characteristics

Consider a partial differential equation of the form

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u) \tag{1.1}$$

Suppose that the values of u are known on some initial curve

$$x(0) = x_0 \quad y(0) = y_0 \quad u(0) = u_0 \tag{1.2}$$

Where y_0 and u_0 are given interms of x_0 by solving the characteristic equations

$$\frac{dx}{ds} = a, \quad \frac{dy}{ds} = b \;\; and \;\; \frac{du}{ds} = c \tag{1.3}$$

Subject to the initial conditions (1.2) one can obtain a unique local solution for the original system (1.1)-(1.3) provided that the Jacobin

$$\jmath = det \begin{pmatrix} \frac{dx}{ds} & \frac{dy}{ds} \\ \frac{dx}{dx_0} & \frac{dy}{dy_0} \end{pmatrix} \text{ is non- zero along the initial curve (1.2)}$$

In fact solvability condition $\jmath \neq 0$ is equivalent to the condition that the initial curve (1.2) is not characteristic curve.

1.6 General Technique Used to Solve First Order Linear PDEs.

The Method of Characteristics is a general technique used to solve first order linear PDEs However, one could always try this method on nonlinear equations. The typical form to be consider

$$a(x, t)\frac{\partial u(x, t)}{\partial t} + b(x, t)\frac{\partial u(x, t)}{\partial x} + f[u(x, t), x, t] = 0 \tag{1.4}$$

Initial condition $u(x,t=0) = u_0(x)$,this form can be readily generalized to with higher dimensions. Note that $u(x,t)$ is a function of two independent variables x and t.The solution $u(x,t)$defines a surface above the xt plane. In the Method of Characteristics, one tries to find a relationship between x and t such that along the curve $x(t)$, the equation for u simplifies and can be solved. Often, $u(x,t)$ is a constant along $x(t)$. However, $x(t)$ is only one trajectory that winds through xt space. However, if we can find $u(x,t)$ for each of the infinite number of non-crossing trajectories $x(t)$, then we have reconstructed $u(x,t)$ on the xt plane.To go about this, we try to find general trajectories of both $x(s)$ and $t(s)$ as functions of a new coordinate s
Along the coordinates s

$$\frac{du(x(s),t(s))}{ds} = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial t}\frac{dt}{ds} \tag{1.5}$$

Therefore, along s, the rate of change $\frac{du(x(s),t(s))}{ds}$ is identical to the first two terms of equation (1.4) provided

$$\frac{dt}{ds} = a(x(s),t(s)) \; and \; \frac{dx}{ds} = b(x(s),t(s)) \tag{1.6}$$

Often,$a(x,t) = 1$, and we can simply take $s = t$. Therefore, the only non-trivial trajectory is that of $x(t)$, and the ordinary differential equation that it solves is

$$\frac{dx(t)}{dt} = b(x(t),t) \tag{1.7}$$

Therefore along $x(t)$,

$$\tfrac{du(x(t),t)}{dt} + f[u(x(t),t),x(t),t] = 0$$

If this ODE is integrable, then we have some hope for an analytic solution. To give a specific example of the procedure, lets also assume $f = 0$ and $b(x,t) = x$, so that

$$\frac{du(x(t),t)}{dt} = 0 \tag{1.8}$$

and $x(t)$ determined by equation(1.7): $\frac{dx}{dt} = x$. Therefore, along the characteristic curves

$$x(t) = x(t=0)e^t = x_0 e^t \tag{1.9}$$

$u(x(t),t) = constant$. Since we define the initial condition as $u(x,t=0) = u_0(x)$, different values x_0 of the position give the relationship

$$u(x_0 e^t, t) = u_0(x_0) \tag{1.10}$$

Upon redefining variables, our final solution that traces every value of x to a value of x_0 is

$$u(x,t) = u_0(xe^{-t}) \tag{1.11}$$

Here first order non-linear equation can be solved using ODE and PDE however in this paper we will see non-linear first order PDE using

$$u_t + A(u)u_x = 0 \text{ with initial condition } u(x,0) = f(x)$$

Where:$A(u)$ is a function of u.

Chapter 2

Scalar Conservation Law and Nonlinear Equation of First Order

2.1 First Order Nonlinear Equation

First order nonlinear equation of the form

$$u_t + A(u)u_x = 0,$$

With initial condition $u(x, 0) = f(x)$
Where:$A(u)$ is a function of u.

Interpret as directional derivative and conclude that $u(x(t), t)$ is constant on the curve $x = x(t)$,with direction field given by

$$\frac{dx}{dt} = A(u(x(t), t)) \tag{2.1}$$

Solve the partial differential equation

$$u_t + A(u)u_x = 0, with \ \ initial \ \ condition \ \ u(x, 0) = f(x) \tag{2.2}$$

Solution: The partial differential equation given can be rewritten as follows:

$$\nabla u(x, t).\langle A(u), 1\rangle = 0 \tag{2.3}$$

where $\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial t}\rangle$ and $\langle A(u), 1\rangle$ is a vector in the direction $A(u)i + j$ at the point (x, t) .I have chosen my coordinate system to be the right handed Cartesian x t z-coordinates, where vector i is a unit vector in the x direction, j is a unit vector in the t direction, k is a unit vector in the z direction. Geometrically, the solution to equation (2.3) $z = u(x,t)$ will be a surface in the x t z-coordinate system. Equation (2.3) a directional derivative,$D_v u(x,t) = \nabla u(x,t).V$, and so equation (2.3) tells us that the rate of change of the function $u(x,t)$ in the direction $\langle A(u), 1\rangle$ at the point (x,t) is zero.This means the function must be constant. $u(x,t) = c$, in this direction. This is just a level curve of the function $u(x,t)$ .All solutions must have this form to be constant in this direction. Functions without this form will not satisfy the partial differential equation. The vector $\langle A(u), 1\rangle$ is a direction field. It is the direction field associated with the ordinary differential equation

$$\frac{dx}{dt} = A(u) \tag{2.4}$$

Before we solve equation (2.4), we need to make some simplifications. Since the solution to the pde we seek is a constant on curves which satisfy the ode, we have

$$u(x,t) = c$$

$$u(x,t) = u(x_0, 0)$$

$$u(x,t) = f(x_0) \tag{2.5}$$

Where: the point $(x_0, 0)$ is a constant We chose to work with the point $(x_0, 0)$ since that allowed us to relate the solution $u(x,t)$ to the initial condition f. We now must solve the ordinary differential equation given in equation (2.4). Equation (2.5) allows us to integrate the differential equation without knowing the form of u(x,t). We will do the integration as a definite integral between the points $(x_0, 0)$ and (x,t). This is like integrating from the initial point to the final point.

$$\tfrac{dx}{dt} = A(u(x,t))$$

$$\tfrac{dx}{dt} = A(f(x_0))$$

$$dx = A(f(x_0))dt$$

$$\textstyle\int_{x_0}^{x} dx = \int_0^t (A(f(x_0)dt$$

$$x - x_0 = A(f(x_0))t$$

$$t = \frac{x - x_0}{A(f(x_0))} \tag{2.6}$$

These are the characteristic curves of the pde. They are straight lines with slope given by $\frac{1}{A(f(x_0))}$ .This is as far as we can go without knowing the specific form of A, so we will start to look at some specific examples. The solution to the boundary value problem is given by $u(x,t) = f(x_0)$ where x_0 is found by solving equation (2.6).

Example 2.1.1. *Solve the partial differential equation*

$$u_t + (\ln u)u_x = 0$$

Subject to the conduction

$$u(x, 0) = e^x$$

Solution: *First we should identify the functions as the functions as they relate to our previous analysis.*

$$f(x) = e^x, A(u) = \ln u$$

Solving equation (2.5) for x_0 we find

$$t = \frac{x-x_0}{A(f(x_0))}$$

$$t = \frac{x-x_0}{\ln e^{x_0}}$$

$$t = \frac{x-x_0}{x_0}$$

$$x_0 t = x - x_0$$

$$x_0 = \frac{x}{t+1}$$

The solution to the boundary value problem we were given is

$$u(x,t) = f(x_0)$$

$$u(x,t) = e^{(x_0)}$$

$$u(x,t) = e^{\frac{x}{t+1}}$$

2.2 First Order Nonlinear Equation Where $A(u) = u$

First order nonlinear equation of the form

$$u_t + A(u)u_x = 0,$$

With initial condition $u(x, 0) = f(x)$
Where: $A(u) = u$

Step 1

Rewrite the following outlined solution of $u_t + A(u)u_x = 0$, based on the notation of characteristic curve. That is

$$\frac{du}{dt} = 0 \;\; and \;\; \frac{dx}{dt} = u \tag{2.7}$$

These are the characteristic curves of $u_t + A(u)u_x = 0$,

Step 2

Let $x(t)$ denote a characteristic curve.since u is constant along $(x(t), t)$,it follows that $u(x(t), t) = u(x(0), 0) = f(x(0))$. since $u(x(t), t)$ is constant and equals $f(x(0))$. we conclude from (2.7) that the characteristic curves are straight lines with slopes $f(x(0))$. We write these lines in the form $x = tf(x(0)) + x(0)$

Step 3

We complete the solution of $u_t + A(u)u_x = 0$, by solving for $x(0)$. In preceding equation to get an implicit relation for the characteristic lines of the form $L(x, t) = x(0)$ .The final solution of $u_t + A(u)u_x = 0$, will be the form $u(x, t) = g(L(x, t))$,where g is a function chosen so as to satisfy the initial condition $u(x, 0) = f(x)$. That is $g(L(x, 0)) = f(x)$.As an illustration of this method let us solve (step2) with the initial condition $u(x, 0) = x$, Where $A(u) = u$ and $f(x) = x$ from step 2,the characteristic lines are of the form

$$x = t\; x(0) + x(0)$$

Solving for $x(0)$ we get $x(0) = \frac{x}{t+1}$ which yields the implicit relation $L(x, t) = \frac{x}{(t+1)}$, thus the solution is

$$u(x,t) = g(\frac{x}{t+1})$$

Setting $t = 0$ and using the initial condition we obtain $g(x) = x$ which yields the solution;

$$u(x,t) = \frac{x}{t+1}.$$

Example 2.2.1. *If $u_t + uu_x = o$ and $u(x,0) = 3x$, then find the solution of $u(x,t)$. The equivalent system of ODEs is*

$$\frac{du}{dt} = 0 \tag{2.8}$$

$$\frac{dx}{dt} = u \tag{2.9}$$

$$\frac{dx}{dt} = 3x(0), dx = 3x(0)dt$$

Integrating gives:

$$x = 3x(0)t + x(0)$$

And solve for $x(0)$

$$x(0) = \frac{x}{3t+1}$$

$$u(x,t) = u(x(0),0) = 3x(0) = \frac{3x}{3t+1}$$

2.3 One Dimensional Wave Equation and Method of Characteristics

The simplest first order non-linear wave equation is given by

$$u_t + c(u)u_x = 0, \qquad -\infty < x < \infty,\ t > 0 \tag{2.10}$$

Where $c(u)$ is a given function of u.
We solve this nonlinear equation subject to the initial condition

$$U(x,0) = f(x), \qquad -\infty < x < \infty \tag{2.11}$$

First, unlike linear differential equation, the principal of super position cannot be applied to find the general solution of non-linear partial differential equation
Second, the effect of nonlinearity can change the entire nature of the solution

Third,a study of the above initial-value problem reveals most of the important ideas for non-linear hyperbolic wave. Finally a large number of physical and engineering problems are governed by the above non-linear system or an extension system of it.
Although the non-linear system governed by 1 and 2 look simple it poses non trivial problems in applied mathematics and it leads surprisingly to new phenomena.
We solve the system by the method of characteristics. In order to continuous solution we consider the total differential du given by

$$du = \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}dx \tag{2.12}$$

so that the points (x, t) are assumed to lie on a curve Γ.Then, $\frac{\partial x}{\partial t}$ represent the slope of the curve Γ at any point p on Γ.
Thus,equation 2.12 become

$$\frac{du}{dt} = u_t + (\frac{dx}{dt})u_x \tag{2.13}$$

It follows from this result that (2.8) can be regarded as ODE

$$\frac{du}{dt} = 0 \tag{2.14}$$

Along any member of the family Γ of curves Γ which are the solution curves

$$\frac{dx}{dt} = c(u) \tag{2.15}$$

These curve are called the characteristics curve of the main equation (2.10) thus, the solution (2.10) has been reduced to the solution of a pair of simultaneous ODE (2.14) and (2.15).Clearly, both the characteristic speed and characteristics depended on the solution u.
Equation (2.14) implies that $u = constant$ along each characteristics curve Γ and each $c(u)$ remains constant on Γ.Therefore (2.15) shows that the characteristic curve of (2.10) from a family of straight line in the (x, t) plane with slope $c(u)$.This indicate that the general solution of (2.10) depends on finding the family of line also each line with slope $c(u)$ corresponds to the value of u on it. If the initial point on the characteristics curve Γ is denoted by ξ and it one of the curve Γ intersect $t = 0$ at $x = \xi$, then $u(\xi, 0) = f(\xi)$ on the whole at that curve Γ as shown in the figure (1)

Thus, we have the following characteristic from on Γ

$$\frac{dx}{dt} = c(u), x(0) = \xi \tag{2.16}$$

$$\frac{du}{dt} = 0, u(\xi, 0) = f(\xi) \tag{2.17}$$

These constitute a pair of coupled ODE on Γ equation (2.16) cannot be solved independently because c is a function of u.However (2.17) can readily be solved to obtain $u = constant$ and

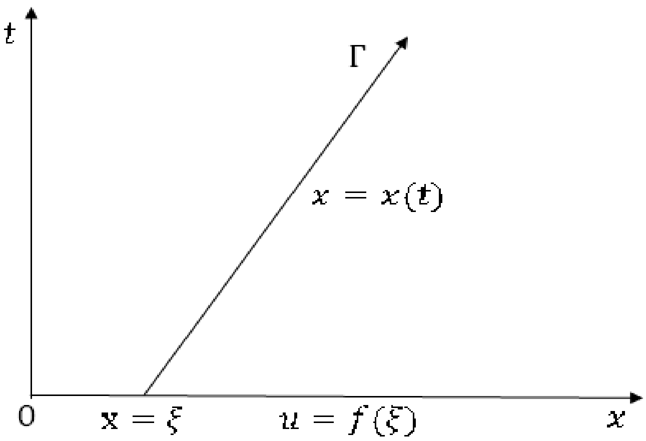


Figure - 1

Hence $u = f(\xi)$ on the whole of Γ, thus (2.16) leads to

$$\frac{dx}{dt} = F(\xi), x(0) = \xi \tag{2.18}$$

$$Where : F(\xi) = c(f(\xi)) \tag{2.19}$$

Integrating equation (2.18) gives

$$x = tF(\xi) + \xi \tag{2.20}$$

This represents the characteristic curve which is a straight line whose slope is not a constant but depends on ξ
Combining these results we obtain the solution of the initial-value problem in parametric form

$$u(x, t) = f(\xi), \quad x = \xi + tF(\xi) \tag{2.21}$$

Where $F(\xi) = c(f(\xi))$

Physical significance of (2.21) We assume $c(u) > 0$the graph of u at $t = 0$ is the graph of f in view of

$$u(x,t) = u(\xi + tF(\xi), t) = f(\xi) \tag{2.22}$$

$$u(\xi, 0) = f(\xi)$$

Point $(\xi, f(\xi))$ move parallel to the $x - axis$ in the positive direction through a distance $tf(\xi) = ct$ and the distance moved $(x = \xi + ct)$ depends on ξ. This is a typical non- linear phenomenon.in the linear case,the curve moves parallel to the $x - axis$ with constant velocity c,and the solution represents waves travelling, without change of shape.Thus,there is a striking difference between the linear and the non-linear solution.
The solution of the non-linear initial-value problem exists provided.

$$1 + tF^{'}(\xi) \neq 0, x = \xi + tF(\xi) \tag{2.23}$$

However the former condition is always satisfied for sufficiently small time t. By solution of the problem, we mean differentiable function $u(x,t)$. It follows from results(2.22) that both u_x and u_t tends to infinity as $1 + tF'(\xi)) \to 0$. This means that the solution develops a singularity (discontinuity) when $1 + tF'(\xi) = 0$, we consider a point $(x,t) = (\xi, 0)$ so that this condition is satisfied on the characteristics through the point $(\xi, 0)$ at time t such that

$$t = \frac{-1}{F^{'}(\xi)}$$

Which is positive provided $F^{'}(\xi) = c^{'}(f)f^{'}(\xi) < 0$.if you assume $c^{'}(f) > 0$,the above inequality implies that $f^{'}(\xi) < 0$.Hence, the solution(2.21) cease to exist for all time if the initial data is such that $f^{'}(\xi) < 0$ for some value of ξ.Suppose $t = \tau$ is the time when the solution first develops a singularity (discontinuity for some value of ξ then

$$\tau = \frac{-1}{min{-\infty<\xi<\infty}(c^{'}(f)f^{'}(\xi))} > 0$$

When draw the graphs of the nonlinear equation $u(x,t) = f(\xi)$ below for different values of $t = 0, \tau, 2\tau, \ldots$.The shape of the initial curve for $u(x,t)$ changes with increasing values of the solution becomes multiple valued for $t \geq \tau$.Therefore the solution breaks down when $F^{'}(\xi) < 0$ for some ξ and such breaking is atypical nonlinear phenomenon. In linear theory such breaking will never occur.More precisely the development of a singularity in the solution for $t \geq \tau$ can be seen by the following consideration.If $f^{'}(\xi) < 0$, then we can find two values of $\xi = \xi_1, \xi_2 (\xi_1 < \xi_2)$ on the initial line such that the characteristics through them have different slopes $\frac{1}{c(u_1)}$ and $\frac{1}{c(u_2)}$ where

$$u_1 = f(\xi_1) \text{ and } u_2 = f(\xi_2) \text{ and } c(u_2) < c(u_1).$$

Thus,these two characteristics will intersect at a point in the (x, t) plane for some $t > 0$. Since the characteristics carry constant values of u the solution cease to be single-valued at their point of intersection. Figure 2 shows that the wave profile progressively distorts itself and at any instant of time there exists an interval on the x-axis where u assumes three values for a given x. The end result is the development of a non-unique solution and this leads to breaking. Therefore when condition (2.23) is violated the solution develops a discontinuity known as a shock.

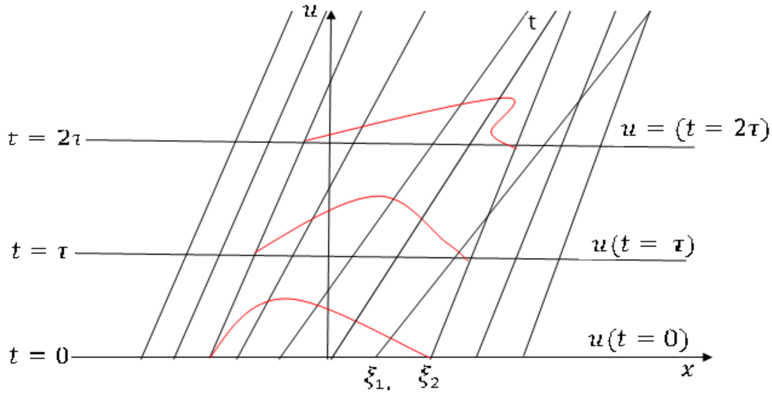


Figure-2

Shock waves result when solving the nonlinear equation $u_t + uu_x = 0$
Initial conditions: $x(s = 0) = x_0$, $t(s = 0) = 0$

$$U(x, t = 0) = f(x) \text{ or } U(s = 0) = f(x_0)$$

Now the wave speed is not constant but depends on the amplitude $u(x, t)$. The characteristic equations are:

$$\frac{dt}{ds} = 1, \qquad \Rightarrow t = s \text{ ,(using} \quad t(0) = 0)$$

$$\begin{aligned} \frac{du}{ds} &= \frac{\partial u}{\partial t}\frac{dt}{ds} + \frac{\partial u}{\partial x}\frac{dx}{ds} \\ &= u_t + uu_x = 0 \end{aligned}$$

$$u = f(x_0) = f(x - ut)$$

So $u = f(x - ut)$ is given implicitly since u is a function of itself.The characteristic curves given by

$$t = \frac{1}{u}(x - x_0)$$

$$t = \frac{1}{(f(x_0))}(x - x_0)$$

The characteristic curves no longer have constant slope, they may cross (meaning U is multiply defined shock waves) or be discontinuous (regions with no solution for $U \to$ expansion waves) as we will see in the next example.

Example 2.3.1. *Solving $u_t + uu_x = 0$ with the following initial conditions:*

$$U(x, t = 0) = f(x) = \begin{cases} u_1, x > 0 \\ u_2, x < 0 \end{cases}$$

$$t = \begin{cases} \frac{1}{u_1}(x - x_0), x > 0 \quad or \quad x = u_1 t + x_0 \\ \frac{1}{u_2}(x - x_0), x < 0 \quad or \quad x = u_2 t + x_0 \end{cases}$$

We have 2 cases:
$u_1 < u_2$ - compression wave $\to$ shock
$u_1 > u_2$ - expansion wave $\to$ rarefaction

Case 1: Shock wave $u_1 < u_2$

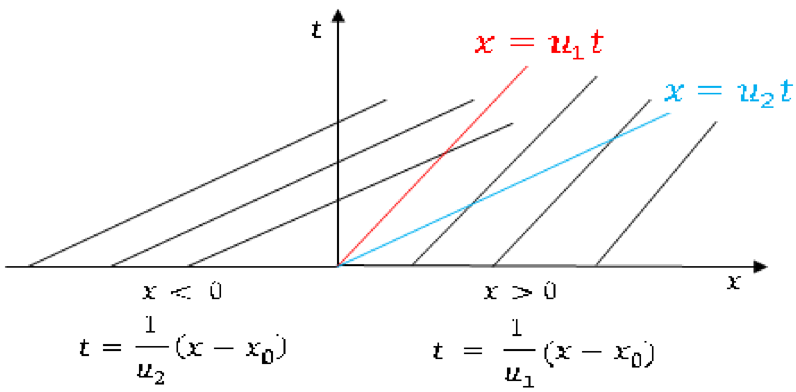


Figure- 3

In the fan bounded by $x = u_1 t$ and $x = u_2 t$ the characteristic curves are multi-valued leading to shocks (breaking waves).
We illustrate this below

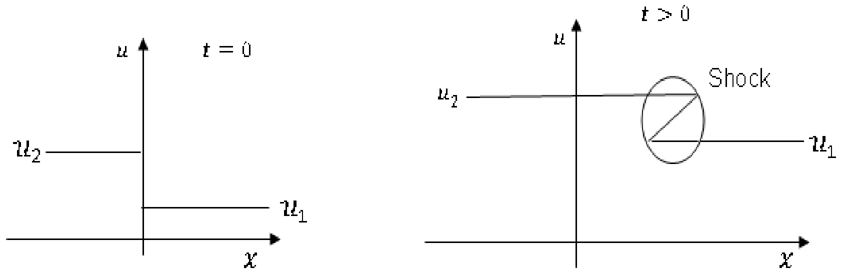


Figure-4

Case 2: Rarefaction or expansion wave $u_1 > u_2$

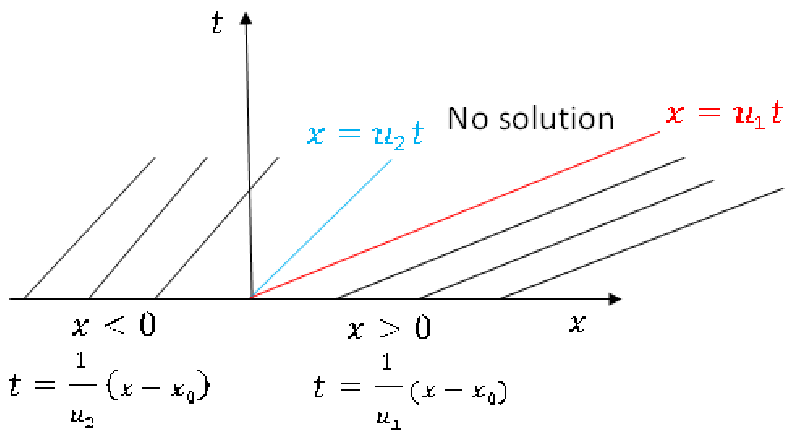


Figure-5

The solution is single-valued for $t > 0$ unlike the shock wave case. However in wedge $x = u_2 t$ and $x = u_1 t$ there is no information. We assume $x = ut$ in wedge since $u_2 t \leq x \leq u_1 t$ and speeds vary $u_2 \leq u \leq u_1$ and add solution to the wedge

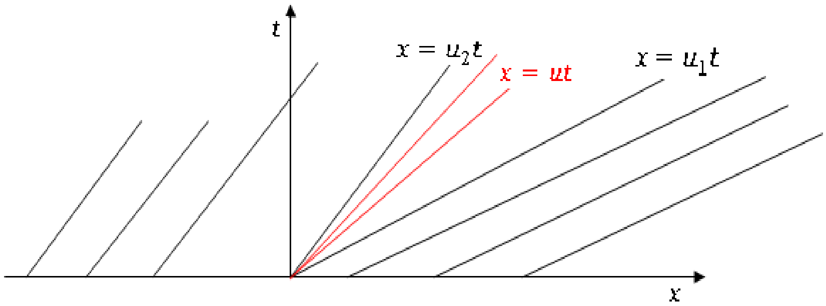


Figure-6

$$Thus \quad u = \begin{cases} u_2, \frac{x}{t} < u_2 \\ \frac{x}{t}, u_2 < \frac{x}{t} < u_1 \\ u_1, \frac{x}{t} > u_1 \end{cases}$$

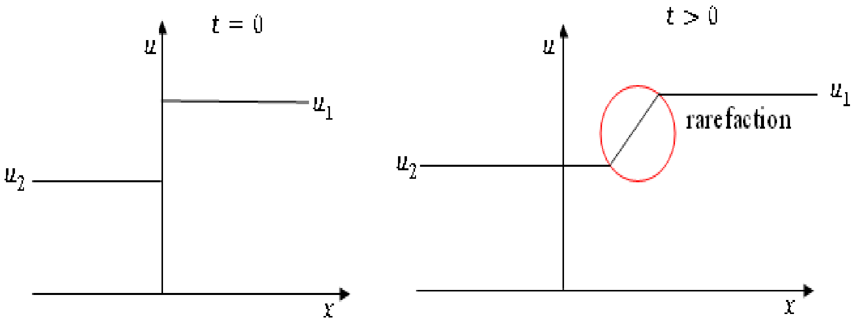


Figure-7

2.4 The scalar conservation law

A scalar conservation law in one space dimension is a first order partial differential equation of the form

$$u_t + f(u)_x = 0 \tag{2.24}$$

Here $u = u(t, x)$ is called the conserved quantity, while f is the flux. The variable t denotes time, while x is the one-dimensional space variable. Equations of this type often describe transport phenomena. Integrating (2.24) over a given interval $[a, b]$ one obtains

$$\frac{d}{dt}\int_a^b u(t,x)dx = \int_a^b u_t(t,x)dx = -\int_a^b f(u(t,x))_x dx \tag{2.25}$$

$$= f(u(t,a)) - f(u(t,b)) = [inflowat\ a] - [outflowat\ b]\ .$$

In other words, the quantity u is neither created nor destroyed: the total amount of u contained inside any given interval [a, b] can change only due to the flow of u across boundary points.

Example 2.4.1. *(**traffic flow**).*
Let $\rho(t, x)$ be the density of cars on a highway, at the point x at time t. For example, u may be the number of cars per kilometer(figure 8) . In first approximation, we shall assume that ρ is continuous and that the velocity v of the cars depends only on their density, say

$$v = v(\rho)\ ,\ \tfrac{dv}{d\rho} < 0$$

Given any two points a and b on the highway, the number of cars between a and b therefore varies according to the law

$$\int_a^b \rho_t(t,x)dx = \frac{d}{dt}\int_a^b \rho(t,x)dx = [inflowat\ a] - [outflowat\ b] \tag{2.26}$$

$$= v(\rho(t,a)) * \rho(t,a) - v(\rho(t,b)) * \rho(t,b)\ = -\textstyle\int_a^b [v(\rho)\rho]_x dx$$

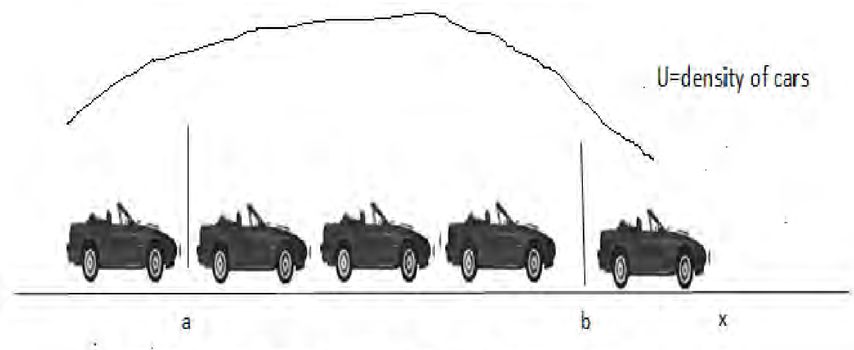


Figure-8 The density of cars can be described by a conservation law.

Since (2.26) holds for all a, b, this leads to the conservation law

$$\rho_t + [v(\rho)\rho]_x = 0,$$

where ρ *is the conserved quantity and* $f(\rho) = v(\rho)\rho$ *is the flux function.*

Chapter 3

The Traffic Flow Model

People like to drive, especially in the United States. In fact, we can often tell where people come from by how they refer to highways: people on Americas east coast talk about taking the turnpike (or the pike) or the interstate, while on the west coast we get on the freeway or we take the 5 or the 101, referring to a particular highway by its number. In order to design the roads and the cars that enable and facilitate such personal transportation, we model both the behavior of individual cars with their drivers in a single line of autos, and that of groups of cars in one or more lanes of traffic. However, our concern is not with modeling the ergonomics of operating a car. Rather, we focus on the interactions of autos on single highway lanes, both individually and in dense lines.

3.1 Can We Really Make Sense of Freeway Traffic?

No matter how we refer to traffic arteries, the flow of traffic on them is modeled, analyzed, and predicted with traffic flow theory, which we now detail at two levels. The macroscopic modeling of traffic assumes a sufficiently large number of cars in a lane or on a road such that each stream of autos can be treated as we would treat fluid flowing in a tube or stream. Thus, to maintain the biological metaphor, traffic flow is treated as a flow of a fluid field in an artery. Macroscopic models are expressed in terms of three gross or average variables for a whole line of traffic: the number of cars passing a fixed point per unit of time, called the rate of flow; the distance covered per unit time, the speed of the traffic flow; and the number of cars in a traffic line or column of given length, which we identify as the traffic density. The relationship between the speed and the density is embodied by macroscopic modelers in a plot of these two variables called the fundamental diagram. The second level of traffic modeling, microscopic modeling, addresses the interaction of individual cars in a line of traffic. Microscopic models describe how an individual follower car responds to an individual leader car by modeling its acceleration as a function of various perceived stimuli, which might be the distance between the leader and follower cars, the relative speeds of the two cars, or the reaction time of the operator of the follower car. Car-following models come in several varieties, and they can be used to construct the speed-density curves that are the underpinning of macroscopic modeling. Such speed-density plots, supported by data taken from real traffic arteries, enable traffic experts to model and understand road or freeway capacity as a function of traffic speed and density-even if everyday drivers feel they do not fully understand what is happening around them. (The microscopic models are also used to support the modeling of vehicular control, that is, to implement control strategies that enable lines of traffic to maintain high flow rates at high speeds. However, we will not delve into control theory and its applications here.) We will start our brief overview of traffic modeling at the macroscopic level, applying conservation principles for cars aggregated into a field (or sufficiently large collection of cars) to define the fundamental diagram for the flow of traffic on a highway populated with multiple vehicles. Then we will examine how the continuum hypothesis influences our view of individual cars (and drivers), as a guide to developing car-follower models that model the interaction between a single car as its driver reacts to another auto immediately ahead. These car-follower models are then used to derive the speed-density relationships that allow us to put specific models and numbers into the more general macroscopic traffic flow theory.

3.2 Macroscopic Traffic Flow Models

We start by asserting the validity of an analogy, namely, that the flow of a stream of cars can be modeled as a field, much as we would model the flow of a fluid. Thus, the collection of cars taking the 10 east out of Los Angeles on any given evening is mathematically similar to the flow of blood in an artery or water in a home piping system. We want to relate the speed of a line of traffic to the amount of traffic in that line (or lane). We use three variables to describe such traffic flows:

1. the rate of flow, $q(x,t)$, measured in the number of cars per unit time;

2. the density of the flow, $\rho(x,t)$, which is the number of vehicles per

unit length of road; and

3. the speed of the flow, $v(x, t)$.

How are these three variables related?

3.2.1 Conservation of Cars

We can provide one answer to the foregoing question by applying the conservation principle to traffic moving (in one direction) along an arbitrary stretch of a road. The conservation principle states that the change in the number of cars within that stretch of road results from the flow of traffic into and out of that road interval, and from the generation or consumption of cars within the interval. Notwithstanding the occasional pictures we have all seen of horrific mega- accidents that occur during severe fogs or major storms, we will (safely) assume that cars are neither generated nor consumed within that road interval. Thus, imagine a coordinate, x, along a particular stretch or interval of road under consideration that has endpoints defined by $x = x$ and $x = x + \Delta x$. The number of cars within this road interval of length Δx is given by $\Delta N(x, t)$. Given our assumption that we will neither generate or consume cars, the conservation principle of states that the change in the number of cars within the interval $\Delta N(x, t)$ during a time interval Δt is, in the limit, equal to the rate of traffic flow, $q(x, t)$:

$$q(x, t) \equiv \lim_{\Delta t \to 0} \frac{\Delta N(x, t)}{\Delta t} \tag{3.1}$$

The change in the number of cars within the road interval, $\Delta N(x, t)$, is simply the difference between the number of cars going in and out of that stretch of road at each end, $N(x, t)$ and $N(x + \Delta x, t)$, respectively:

$$\Delta N(x, t) = N(x, t) - N(x + \Delta x, t), \tag{3.2}$$

If Δx denotes the length of road interval that is traveled during the time, Δt , the statement of conservation of cars (3.1) can also be written as

$$q(x, t) \equiv \lim_{\Delta t \to 0} \frac{\Delta N(x, t)}{\Delta x} (\frac{\Delta x}{\Delta t}) \tag{3.3}$$

where the fraction introduced in equation. (3.3) is the speed of the traffic, $v(x, t)$, in the interval:

$$v(x, t) = (\frac{\Delta x}{\Delta t}) \tag{3.4}$$

Equations (3.2) and (3.4) are now substituted into the conservation of cars (3.3) to yield

$$q(x,t) = (\lim_{\Delta t \to 0} \frac{N(x,t) - N(x+\Delta x, t)}{\Delta x} v(x,t) \tag{3.5}$$

Note that the limit in equation. (3.5) is now taken as $\Delta x \to 0$, and that its dimensions correspond to the number of vehicles per unit length of road, which we define as the density of the traffic flow:

$$\rho(x,t) = \lim_{\Delta t \to 0} \frac{N(x,t) - N(x+\Delta x, t)}{\Delta x} \tag{3.6}$$

Thus, equation. (3.5) can be rewritten for the last time to cast the principle of conservation of cars in the form

$$q(x,t) = \rho(x,t)v(x,t). \tag{3.7}$$

Beyond preserving the notion that what goes in must go out, what does equation. (3.7) mean? First, we note that the equation is dimensionally consistent and correct . Second, we note that equation. (3.7) can be shown to make physical sense by a rather simple argument derived by looking at two different ways of counting the number of cars passing a (specified) point on the road during a very small time interval. One measure of the traffic count is that the number of cars, ΔN, passing a point during a time interval, Δt , is simply the product of the flow rate, q, and the time interval: $\Delta N = q\Delta t$. The second measure count assumes that during the same small interval of time a car moving with a speed, v, will cover a distance, $\Delta x = v\Delta t$. The number of vehicles passing through that distance is found from another simple product: of density, , times distance: $\Delta N = \rho\Delta x$. Hence, equating the two measures of the number of cars passing a point yields the result

$$q\Delta t = \rho\Delta x, \tag{3.8}$$

which is clearly an averaged version of equation. (3.7) that accords well with this elementary physical reasoning . We also observe that the single equation (3.7) is expressed in three variables: q, ρ , and v. Therefore, it is of very limited use in this form without substantial further information. However, it is clear that traffic density, ρ, and speed,v, are the two fundamental traffic variables because we can determine the rate,q, at which traffic flows by inserting them into equation. (3.7). Further, if we could relate speed directly to density, i.e., $v = v(\rho)$, then we could write a direct relationship between the traffic flow rate, q, and the density, ρ:

$$q(\rho) = \rho \; v(\rho) \tag{3.9}$$

As we will see in Section 3.2.3, plots of traffic flow rate, q, against density, ρ, are so widely used in modeling traffic flow that they are identified under the rubric of the fundamental diagram of road traffic. Speed-density relationships (e.g., $v = v(\rho)$) are clearly central to our understanding of traffic flow, so we turn to them next.

3.2.2 Relating Traffic Speed to Traffic Density

Even inexperienced drivers would agree that traffic speed and traffic density are related. Drivers speed up when traffic is sparse, and they slow down to clog up arteries when traffic is thick. Thus, we are tempted to postulate that there is a direct relationship between traffic speed and traffic density:

$$v = v(\rho). \tag{3.10}$$

Let us now reason a bit further about this relationship to determine any conditions that need to be applied to any particular functional form, $v(\rho)$, that might be proposed. Building on the intuition just mentioned, we expect that a driver will drive fastest, v_{max}, when the density is at its smallest value, $\rho \to 0$. The speed decreases as the density increases, which is a statement about the slope of the v versus ρ curve. Finally, traffic grinds to a halt, $v = 0$, at some maximum or jam density,ρ_{jam} , presumably when the traffic is bumper-to-bumper. We can summarize these experience-born intuitions in mathematical requirements on the function, $v(\rho)$:

$$\begin{gathered} v(\rho = 0) = v_{max} \\ \tfrac{dv}{d\rho} \leq 0, \\ v(\rho = \rho_{jam}) = 0 \end{gathered} \tag{3.11}$$

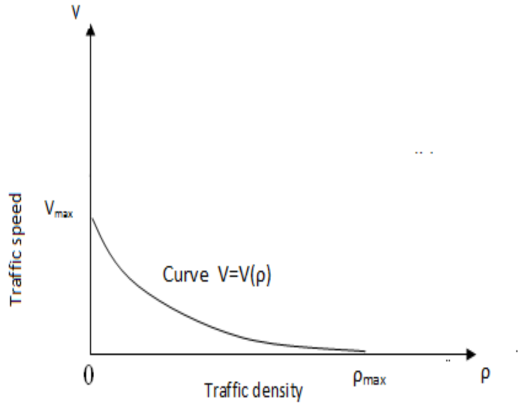


Figure 3.1 A generic schematic of the variation of traffic velocity with density. It displays the endpoints, $[(0, v_{max})$ and $(\rho_{max}, 0)$, respectively] , and shows that the slope is always non-positive, $\frac{dv}{d\rho} < 0$, which results from our experience that traffic speed drops off as traffic density increases.

We can also display these results graphically, in the generic curve shown in Figure 3.1. Note that the precise shape of the curve is unknown; only the endpoint values and the sign of the slope are specified at this point. The elementary modeling assumptions just outlined do not exhaust all of the possibilities, although experience suggests that equations. (3.10) and (3.11) adequately reflect the behavior of traffic that is accelerating or decelerating. Models behind traffic speed-density relations will reflect human behavior-rather than mechanical laws-because they reflect how drivers respond to stimuli. That is, drivers can respond to perceived distances between cars, to relative speeds, to the perceived density further down the road, and so on. In fact, speed-density relations such as equation. (3.10) are found both from empirical data and from the very stuff of the modeling of car-following interactions that we address in Section 3.3.

3.2.3 Relating Traffic Flow to Traffic Density: The Fundamental Diagram

From the viewpoint of the traffic engineer who is designing a road and all of its facilities (including entrance and exit ramps, traffic signs and signals, toll booths, etc.), the most relevant variable is the capacity (or maximum flow rate) that the road system must accommodate, as reflected in its traffic flow rate, $q(x,t)$. For macroscopic models we can take the speed to be homogeneous, which means that it does not explicitly depend on the road coordinate, x, or on time, t . Then, we can write $v = v(\rho)$, anticipating as in equation. (3.9), that traffic flow ultimately depends only on the density, ρ We can now extend our qualitative analysis of the speed-density relationship (of Section 3.2.2) to the relationship between the traffic flow rate and the density. Thus, because a drivers fastest speed, v_{max}, occurs when the density is at its smallest, $\rho = 0$, equation. (3.9) tells us that $q(\rho = 0) = 0$, that is, that the flow rate is zero. Similarly, when traffic slows to a halt at its maximum density, $v(\rho_{jam}) = 0$, equation. (3.9) tells us once again that the traffic flow rate is zero: $q(\rho_{jam}) = \rho_{jam} v(\rho_{jam}) = 0$. The traffic flow rate must be positive for all values of the density $(0 < \rho < \rho_{jam})$, and must attain its maximum value q_{max} somewhere in that interval.Further, the slope of the traffic flow rate is given by

$$\frac{dq}{d\rho} = v(\rho) + \rho \frac{dv}{d\rho} \tag{3.12}$$

The qualitative results just found are embodied in the generic curve shown in Figure 3.2, which is called the fundamental diagram of traffic flow. As with Figure 3.1, the precise shape of the curve is unknown: the endpoint values are specified and the variation of the slope can be inferred.

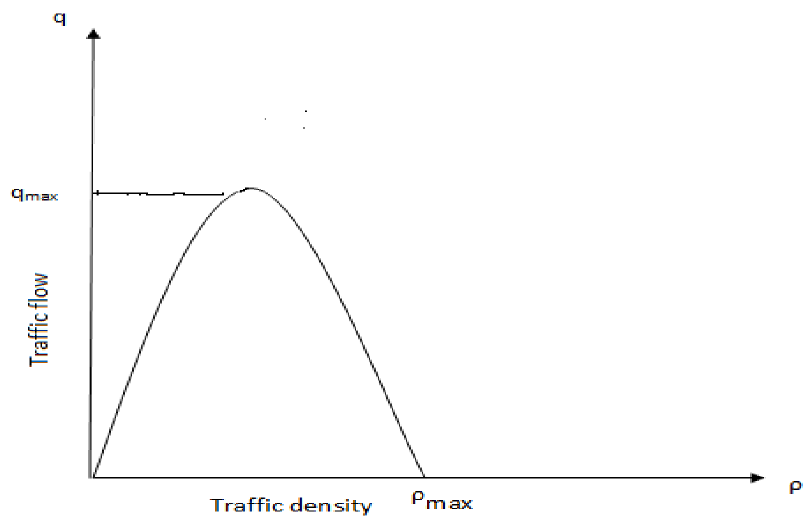


Figure 3.2 A generic schematic of the variation of the traffic flow rate with density. It displays the endpoints, $[(0,0)$ and $(\rho_{max}, 0)$, respectively], and

shows that the slope is positive until the maximum flow rate or capacity, q_{max}, is reached, and negative thereafter.

To make some of these qualitative ideas more specific, consider the following linear speed-density relationship:

$$v(\rho) = v_{max}(1 - \frac{\rho}{\rho_{jam}}) \tag{3.13}$$

This relationship clearly satisfies all of the conditions required by equation. (3.11).Moreover, as the simplest (linear) mathematical expression that satisfies these conditions, it is particularly attractive as a building block for further modeling, provided that it adequately models reality. When substituted into equation. (3.9), it produces a relationship for the traffic flow rate as a function of density that is parabolic:

$$q(\rho) = v_{max}(\rho - \frac{\rho^2}{\rho_{jam}}) \tag{3.14}$$

The maximum flow rate occurs when its slope vanishes:

$$\frac{dq(\rho)}{d\rho} = v_{max}(1 - \frac{2\rho}{\rho_{jam}}) \tag{3.15}$$

Equation (3.15) shows that the maximum traffic flow rate under these assumptions occurs at the mid-point of the fundamental diagram, when $\rho = \rho_{jam}$, and that its value is

$$q_{max} = \frac{1}{4}\rho_{jam}v_{max} \tag{3.16}$$

We consider the flow of cars on a long highway under the assumptions that cars do not enter or leave the highway at any one of its points. We take the x-axis along the highway and assume that the traffic flows in the positive direction. Suppose $\rho(x,t)$ is the density representing the number of cars per unit length at the point x of the highway at time t, and $q(x,t)$ is the flow of cars per unit time. We assume a conservation law which states that the change in the total amount of a physical quantity contained in any region of space must be equal to the flux of that quantity across the boundary of that region. In this case, the time rate of change of the total number of cars in any segment $x_1 \leq x \leq x_2$ of the highway is given by

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho(x,t)dx = \int_{x_1}^{x_2} \frac{\partial \rho}{\partial t}dx \tag{3.17}$$

This rate of change must be equal to the net flux a cross x_1 andx_2 given by

$$q(x_1,t)-q(x_2,t)$$

which measures the flow of cars entering the segment at x_1 minus the flow of cars leaving the segment at x_2. Thus, we have the conservation equation

$$\frac{d}{dt}\int_{x_1}^{x_2}\rho(x,t)dx = q(x_1,t)-q(x_2,t) \tag{3.18}$$

$$\int_{x_1}^{x_2}\frac{\partial\rho}{\partial t}dx = -\int_{x_1}^{x_2}\frac{\partial q}{\partial x}dx$$

$$\int_{x_1}^{x_2}(\frac{\partial\rho}{\partial t}+\frac{\partial q}{\partial x})dx = 0 \tag{3.19}$$

Since the integrand in (3.19) is continuous and (3.19) holds for every Segment $[x_1,x_2]$, it follows that the integrand must vanish so that we have the partial differential equation

$$\frac{\partial\rho}{\partial t}+\frac{\partial q}{\partial x}=0 \tag{3.20}$$

We now introduce an additional assumption which is supported by both theoretical and experimental findings. According to this assumption, the flow rate q depends on x and t only through the density, that is, $q=Q(\rho)$ for some function Q . This assumption seems to be reasonable in the sense that the density of cars surrounding a given car indeed controls the speed of that car. The functional relation between q and ρ depends on many factors, including speed limits, weather conditions, and road characteristics.

We consider here a particular relation $q=\rho v$ where v is the average local velocity of cars. We assume that v is a function of ρ to a first approximation. In view of this relation, (2.20) reduces to the nonlinear hyperbolic equation

$$\frac{\partial\rho}{\partial t}+c(\rho)\frac{\partial q}{\partial x}=0 \tag{3.21}$$

where

$$c(\rho)=q'(\rho)=v(\rho)+\rho v'(\rho) \tag{3.22}$$

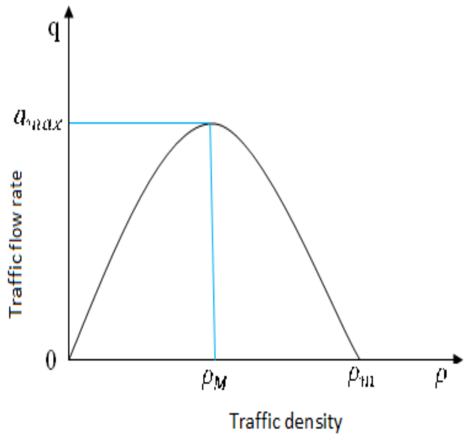
In general, the local velocity $v(\rho)$ is a decreasing function of ρ so that $v(\rho)$ has a finite maximum value v_{max} at $\rho=0$ and decreases to zero at $\rho=\rho_{max}=\rho_m$. For the value of $\rho=\rho_m$ the cars are bumper to bumper. Since $q=\rho v$, $q(\rho)=0$ when $\rho=0$ and $\rho=\rho_m$. This means that q is an increasing function of ρ until it attains a maximum value $q_{max}=q_M$ for some $\rho=\rho_M$ and then decreases to zero at $\rho=\rho_m$. Equation (3.21) is similar to (2.10) with the wave propagation velocity $c(\rho)=v(\rho)+\rho v'(\rho)$. Since $v'(\rho)<0, c(\rho)<v(\rho)$, that is, the propagation velocity is less than the car velocity. In other words, waves propagate backwards through the stream of cars, and drivers

are warned of disturbances ahead. It follows in the Figure below that $q(\rho)$ is an increasing function in $[0, \rho_M]$, a decreasing function in $[\rho_M, \rho_m]$, and attains a maximum at ρ_M . Hence, $c(\rho) = q'(\rho)$ is positive in $[0, \rho_M]$, zero at ρ_M and negative in $[\rho_M, \rho_m]$. All these mean that waves propagate forward relative to the highway in $[0, \rho_M]$, are stationary atρ_M , and then travel backwards in $[\rho_M, \rho_m]$. to solve the initial-value problem for the nonlinear equation (3.21) with the initial condition $\rho(x, 0) = f(x)$. The solution is

$$\rho(x,t) = f(\xi), x = \xi + tF(\xi) \tag{3.23}$$

where

$$F(\xi) = c(f(\xi))$$



3.3 Microscopic Traffic Models

We now turn from macroscopic models that use averaged variables to microscopic models that look at individual cars. Our interest is in using the microscopic models to develop the traffic speed-density relations that we need to do macroscopic evaluations of capacity, which we require if were going to design highway systems. As we noted in Section 3.2.2, we are looking for models that describe how drivers respond to the stimuli of their traffic situations. The driver will perceive a variety of stimuli, including the distance between vehicles, their relative speed, and their perceived relative acceleration. We thus seek psychological, not mechanical, models in order to model human behavior. The drivers response will depend on the responders sensitivity to the given stimuli, as well as on the speed with which the response is undertaken. Thus, some time delay should also be incorporated into such models.

3.3.1 Linear Car-following Model

Imagine a line of cars traversing a given road, as shown in Figure 3.3. Each car is identified by a discrete coordinate that varies in time, so that the location of the n^{th} car is given by $x_n t$. We also assume that the line has a reasonable value of local density and does not permit passing or overtaking. Then the basic equation of car-following for such a single lane of traffic is the psychological one:

$$response = sensitivity \cdot stimulus.$$

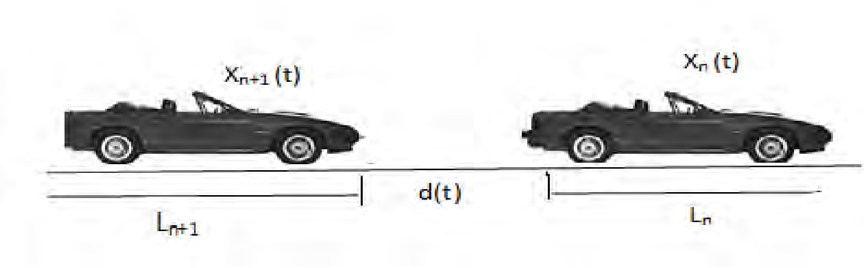


Figure 3.3 The nomenclature for a line (or lane) of cars on a highway of total length, L_R . Each car has the same length, L, and is separated from its neighbors by a common distance, $d(t)$. The discrete functions, $x_{n+1}(t)$ and $x_n t$, represent, respectively, the coordinates of the follower and leader cars. The response will generally be modeled as the acceleration of the $(n + 1)st$ follower car, $\ddot{x}_{n+1}(t)$, as it moves behind the n^{th} leader car. The stimulus will be modeled in terms of the coordinate of the follower car relative to the leader car, which can in turn be written in terms of the traffic density, ρ. The acceleration is then integrated to determine the speed of that car as

Predict? a function of the traffic density, which is the input we require for our macroscopic modeling. Consider a simple linear car-following model in which the driver of the follower car responds to the speed of the leader car relative to the follower car:

$$\frac{d^2x_{n+1}(t)}{dt^2} = -K_p(\frac{dx_{n+1}(t)}{dt} - \frac{dx_n(t)}{dt}) \tag{3.24}$$

The coefficient, K_p, introduced here is a sensitivity parameter that has dimensions of per unit time. Note, that with $K_p > 0$, the follower car will decelerate to avoid hitting the car in front if it is slowing down, relatively speaking.We will discuss this in further detail later. We can model the time it takes the following driver to respond to events by building in a reaction time that slows the followers acceleration by the delay time T:

$$\frac{d^2x_{n+1}(t+T)}{dt^2} = -K_p(\frac{dx_{n+1}(t)}{dt} - \frac{dx_n(t)}{dt}) \tag{3.25}$$

Assuming that the sensitivity parameter, K_P, is a constant, equation. (3.25) is a linear ordinary differential equation with constant coefficients that can be integrated once to yield

$$\frac{dx_{n+1}(t+T)}{dt} = -K_p(x_{n+1}(t) - x_n(t)) + C_{n+1}, \tag{3.26}$$

where C_{n+1} is the arbitrary constant, with dimensions of speed, that results from the integration just performed. Note that equation. (3.26) clearly relates the speed of the follower car to the distance maintained between the follower and leader cars. Thus, it is a natural precursor of the speed-density relationship that we seek. Let us further assume that all of the cars have the same length, L, and that the spacing between common points on any pair of cars (see Figure 3.3) is given by $d(t)$:

$$d(t) = x_n(t) - L - x_{n+1}(t). \tag{3.27}$$

It then follows that the number of cars, N_R, found in a stretch of road of length,L_R, is

$$N_R = \frac{L_R}{L + d(t)} \tag{3.28}$$

which means that the density of cars on that road is

$$\rho = \frac{L_R}{N_R} = \frac{1}{L + d(t)} = \frac{1}{x_n(t) - x_{n+1}(t)} \tag{3.29}$$

where we have used the spacing defined in equation. (3.27) to obtain the final form of equation. (3.29). Thus, we have in equation. (3.29) a relationship between the (macroscopic) traffic density,ρ, and the (microscopic) coordinates of the leader and follower cars.

Summary

Directional derivative and characteristic method are used to solve
$u_t + A(u)u_x = 0$ with initial condition $u(x, 0) = f(x)$.
Directional derivative solve the partial differential equation using gradient vector $\nabla u(x,t)\langle A(u), 1\rangle = 0$ with ordinary differential equation the solution $u(x,t) = f(x_0)$,where $x_0 = x - A(f(x_0))t$ and method of characteristics solve $u_t + A(u)u_x = 0$ using $\frac{du}{dt} = 0, \frac{dx}{dt} = u$ rewrite ordinary differential equation in to characteristic curve such that $x = tf(x_0) + x_0$, when $A(u) = u$ then the result will be

$$u(x,t) = f(\tfrac{x}{t+1}) \ .$$

At $t > 0$ and if non-unique solution exist,then one dimensional wave equation leads to braking and therefore the result known as a shock.the equation are great importance for traffic flow model.
some of the most fundamental ideas of traffic modeling as they are applied in the engineering of traffic systems. We described macroscopic models that predict the average variables of traffic density and traffic flow rates because they are very important for calculating the capacity of roads and highways.We then pointed out the role of scaling and of the continuum hypothesis in moving from macroscopic models to microscopic and in beneficially integrating the two.We introduced microscopic models that predict how speed varies with driver sensitivities and responses to various traffic stimuli because they provide a basis for obtaining the gross traffic density and flow rates needed in macroscopic models. Finally, we also noted in passing that the microscopic models are increasingly used to investigate the control of individual vehicles, as well as lines (or lanes) of vehicles.

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