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Office of Graduate Program

Faculty of computer and mathematical Sciences

Department of Mathematics

Project on

Maximum principles of second order elliptic equations

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Declaration

“I declare that this project has been composed by me and that no part of the project has formed the basis for the award of any degree, diploma, associate ship, fellowship or any other similar title to me.

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“This is to certify that this project is compiled /a recorded of the research work done / by haftu Adhena in the Department of mathematics, Addis Ababa University, under my supervision.

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Advisor’s Signature

List of figures

Abstract

In this project we discussed the maximum principle for elliptic equations state several theorems, corollaries, and give a number of examples. Although the maximum principle for Laplace's and some other equations has been known for about a hundred years, it is relatively recently that Hopf’s established strong maximum principles for general second-order elliptic equations

Hence the proofs of weak and strong maximum principles of second order linear elliptic operator and other related corollaries are discussed in detail.

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Table of Contents

INTRODUCTION

One of the most useful and best known tools employed in the study of partial differential equations is the maximum principle. This principle is a generalization of the elementary fact of calculus that any function $f(x)$ which satisfies the inequality $f" > 0$ on an interval $[a, b]$ achieves its maximum value at one of the endpoints of the interval. We say that solutions of the inequality $f" > 0$ satisfy a maximum principle.

More generally, functions which satisfy a differential inequality in a domain Ω and, because of it, achieve their maxima on the boundary of Ω are said to possess a maximum principle.

The study of partial differential equations frequently begins with a classification of equations into various types. The equations most frequently studied are those of elliptic, parabolic, and hyperbolic types. Because equations of these three types arise naturally in many physical problems, mathematicians interested in partial differential equations have tended to concentrate their efforts on those developments which are of both mathematical and physical interest.

Since many problems associated with equations of elliptic, parabolic, and hyperbolic types exhibit maximum principles, we feel that a study of the methods and techniques connected with these principles forms an excellent introduction or supplement to the study of partial differential equations.

There is usually a natural physical interpretation of the maximum principle in those problems in differential equations that arise in physics. In such situations the maximum principle helps us apply physical intuition to mathematical models. Consequently, anyone learning about the maximum principle becomes acquainted with the classically important partial differential equations and, at the same time, discovers the reasons for their importance.

This project paper has two chapters; classification of second order linear partial differential equations and the maximum principles.

The first chapter concerns about classification of second order linear partial differential equations for $n \geq 2$ independent variables, and elliptic equations.

The second chapter deals with maximum principle of elliptic differential equations. In this chapter we give the detail proofs for theorems on weak and strong maximum principles of second order linear elliptic partial differential equation and other related corollaries.

1 Classification of second order partial differential equations

Partial differential equations arise in physical science when the numbers of independent variable under the discussion are two or more than two. When such is the case, any dependant variable is likely to be a function of more than one variable so that it possesses not ordinary derivatives with respect to a single variable but partial derivatives with respect to several variables.

Let $u(x) = u(x_1, x_2, \ldots x_n)$ be a continuously differentiable function in a domain Ω. Then we denote the partial derivative of u with respect to the i^{th} $variable\ x_i\ by\ \frac{\partial u}{\partial x_i}$.

The second order partial derivative of twice continuously differentiable function with respect to $x_i\ and\ x_j$ will be denoted by $\frac{\partial^2 u}{\partial x_i \partial x_j}$.

1.1 Classification of second order linear PDEs in two independent variables

The general forms of second order linear partial differential equation in two independent variables $x\ and\ y$ is:

$$a(x,y)\frac{\partial^2 u}{\partial x^2} + b(x,y)\frac{\partial^2 u}{\partial x \partial y} + c(x,y)\frac{\partial^2 u}{\partial y^2} + d(x,y)\frac{\partial u}{\partial x} + e(x,y)\frac{\partial u}{\partial y} + f(x,y)u = g(x,y) \tag{1}$$

Where the variable $a, b, c, d, e, f\ and\ g\ are\ functions\ of\ x\ and\ y$, $\Omega \subseteq R^n.\ and\ (a, b, c) \neq (0,0,0)\ in\ \Omega$.

The expression $a(x,y)\frac{\partial^2}{\partial x^2} + b(x,y)\frac{\partial^2}{\partial x \partial y} + c(x,y)\frac{\partial^2}{\partial y^2}$ is called the principal parts of the equation. Since the principal part mainly determines the properties of solutions we shall classify the general second order linear partial differential equation based on the principal.

The function Δ defined by

$\Delta(x,y) = b^2(x,y) - 4a(x,y)c(x,y)$ is called the discriminate of the above equation(1).

Definition we say that the equation (1) at point $P(x,y) \in \Omega$ is:

1. Elliptic , if $\Delta(x,y) < 0$.
2. Parabolic , if $\Delta(x,y) = 0$
3. Hyperbolic, if $\Delta(x,y) > 0$.

Example

Show that linear operator $\mathcal{L}u = (1-x^2)\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + (1-y^2)\frac{\partial^2 u}{\partial y^2}$ is elliptic for

$x^2 + y^2 < 1$, parabolic for $x^2 + y^2 = 1$ and hyperbolic for $x^2 + y^2 > 1$ for all $x, y \in R$.

Solution

The principal matrix $a_{ij}(x,y) = \begin{bmatrix} 1-x^2 & xy \\ xy & 1-y^2 \end{bmatrix}$

To show that $\mathcal{L}$ is elliptic, we can use the elliptic formula $\Delta(x,y) = b^2 - 4ac < 0$. Where

$$a = 1 - x^2$$

$$b = 2xy$$

$$c = 1 - y^2$$

Then $b^2 - 4ac = (2xy)^2 - 4(1-x^2)(1-y^2) = (2xy)^2 - 4(1 - y^2 - x^2 + x^2y^2)$

$$= 4x^2y^2 - 4 + 4(x^2 + y^2) - 4((xy)^2)$$

$$= -4 + 4(x^2 + y^2)$$

From this we have

1 $b^2 - 4ac < 0 \; if \; x^2 + y^2 < 1$ and hence Elliptic

2 $b^2 - 4ac = 0 \; if \; x^2 + y^2 = 1$ and hence parabolic.

3 $b^2 - 4ac > 0 \; if \; x^2 + y^2 > 1$ and hence hyperbolic.

1.2 Classification of second order linear PDEs in n independent variables

Let Ω be a domain in the $n - dimensional$ Euclidean space R^n. And $x = (x_1, x_2, \dots, x_n)$ are point of R^n. A linear second order partial differential equation in R^n has the form

$$\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + F(x, u, \nabla u) = 0 \qquad (2)$$

Where the coefficient $a_{ij}(x)$ are assumed to be continuously differentiable function in **Ω**, $a_{ij} = a_{ji}, u(x)$ is unknown function and $\nabla u = \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_n}$ is gradient of u.

The linear operator

$$\mathcal{L} = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}$$

is called the principal part of equation (2).

A function $u(x) \in C^2(\Omega)$ is a solution of equation (2) in **Ω**,if the substitution of u and its derivative in (2) satisfies equation one for all $x \in \Omega$.

The classification of linear second order equation in n independent variables can be made with respect to the eigenvalues of the coefficient matrix ***A*,**

i.e. the roots of the equation

$$\begin{bmatrix} a_{11}(x_0)-\lambda_1 & \cdots & a_{1n}(x_0) \\ \vdots & \ddots & \vdots \\ a_{n1}(x_0) & \cdots & a_{nn}(x_0)-\lambda_n \end{bmatrix} = 0$$

From linear algebra it is known that since the matrix A is symmetric its eigenvalues of the matrix are real. Moreover the number of zero and negative eigenvalues of the matrix A remains invariant under nonsingular change independent variables.

Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalue of the matrix A of the principal part of equation (1)

Then the linear second order differential operator of equation (1) at point x_0 is said to be

1. Elliptic, if $\lambda_1, \lambda_2, \dots, \lambda_n$ is non zero and have the same sign.
2. Hyperbolic, if $\lambda_1, \lambda_2, \dots, \lambda_n$ are non zero and except one have the same sign
3. Parabolic, if any one of $\lambda_1, \lambda_2, \dots, \lambda_n$ is zero
4. Ultra hyperbolic, if $\lambda_1, \lambda_2, \dots, \lambda_n$ are non zero and at least two of them are positive and two of them are negative.

   For example

   1. The laplace equation $\Delta u = \frac{\partial^2 u}{\partial {x_1}^2} + \cdots + \frac{\partial^2 u}{\partial {x_n}^2} = 0$ is elliptic in R^n
   2. The wave equation $u_{tt} - C^2\left(\frac{\partial^2 u}{\partial {x_1}^2} + \cdots + \frac{\partial^2 u}{\partial {x_n}^2}\right) = 0$ where C is constant, is hyperbolic in R^{n+1}.
   3. The heat equation or $u_t - a^2\left(\frac{\partial^2 u}{\partial {x_1}^2} + \cdots + \frac{\partial^2 u}{\partial {x_n}^2}\right) = 0$,where a is a constant, is parabolic in R^{n+1}.
   4. $3\frac{\partial^2 u}{\partial {x_1}^2} + 2\frac{\partial^2 u}{\partial {x_2}^2} - 7\frac{\partial^2 u}{\partial {x_3}^2} - 4\frac{\partial^2 u}{\partial {x_4}^2} - \frac{\partial^2 u}{\partial {x_5}^2} = 0$ is ultrahyperbolic in R^5.

Proof: It is enough to show one of these examples, let us prove the first example

1. We can put the laplace equation in a matrix form as follows

$$\Delta u = \begin{bmatrix} 1 & 0 & 0 & & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial^2 u}{\partial {x_1}^2} \\ \vdots \\ \frac{\partial^2 u}{\partial {x_n}^2} \end{bmatrix}$$

Then the eigenvalue of this equation is given by

$$\begin{bmatrix} a_{11}(x_0) - \lambda_1 & \cdots & a_{1n}(x_0) \\ \vdots & \ddots & \vdots \\ a_{n1}(x_0) & \cdots & a_{nn}(x_0) - \lambda_n \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 1-\lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & 1-\lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & 1-\lambda_3 & & \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & & \cdots & 1-\lambda_n \end{bmatrix} = 0$$

From this we get $\lambda_i = 1$ which implies all eigenvalue are positive and hence have the some sign.

Therefore the laplace equation is elliptic.

Definition1.2.1 A domain Ω is an open connected subset of R^n. And the boundary of Ω is denoted by $\partial\Omega$.

$C(\Omega)$ is the space of all continuous real functions in Ω.

$C^2(\Omega)$ is the space of all twice continuously differentiable in Ω.

Definition1.2.2 A square symmetric matrix A with real entries is said to be

1) Positive definite if and only if $\mathcal{E}^t A \mathcal{E} > 0$ for all non-zero vectors $\mathcal{E}$ with real entries i.e. $\mathcal{E} \in R^n \setminus \{0\}$.

2) Positive semi definite if and only if $\mathcal{E}^t A\mathcal{E} \geq 0$ for all non-zero vectors Ɛ with real entries i.e. $\mathrm{Ɛ} \in R^n \setminus \{0\}$
3) Negative definite if and only if $\mathcal{E}^t A\mathcal{E} < 0$ for all non-zero vectors Ɛ with real entries i.e. $\mathrm{Ɛ} \in R^n \setminus \{0\}$
4) Negative semi definite if and only if $\mathcal{E}^t A\mathcal{E} \leq 0$ for all non-zero vectors Ɛ with real entries i.e. $\mathrm{Ɛ} \in R^n \setminus \{0\}$

Note that: $\mathcal{E}^t$ is the transpose of vector Ɛ.

Example 1 conceder matrix $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

For any non zero vector $\mathrm{Ɛ} = \begin{bmatrix} \mathrm{Ɛ}_1 \\ \mathrm{Ɛ}_2 \\ \mathrm{Ɛ}_3 \end{bmatrix}$

$$\mathrm{Ɛ} = \mathcal{E}^t A\mathcal{E} = [\mathrm{Ɛ}_1 \quad \mathrm{Ɛ}_2 \quad \mathrm{Ɛ}_3] \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} \mathrm{Ɛ}_1 \\ \mathrm{Ɛ}_2 \\ \mathrm{Ɛ}_3 \end{bmatrix}$$

$$= (\mathrm{Ɛ}_1 - \mathrm{Ɛ}_2)^2 + (\mathrm{Ɛ}_2 - \mathrm{Ɛ}_3)^2 + (\mathrm{Ɛ}_3 - \mathrm{Ɛ}_1)^2 \geq 0$$

This matrix is positive semi-definite for any non zero vector Ɛ.

And positive definite for any of the following condition:

$$\mathrm{Ɛ}_1 \neq \mathrm{Ɛ}_2 or\ \mathrm{Ɛ}_2 \neq \mathrm{Ɛ}_3 or\ \mathrm{Ɛ}_3 \neq \mathrm{Ɛ}_1$$

Example 2 $Matrix\ A = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} is\ negative\ semi\ definite.$

To show this for any none zero vector $\mathrm{Ɛ} = \begin{bmatrix} \mathrm{Ɛ}_1 \\ \mathrm{Ɛ}_2 \\ \mathrm{Ɛ}_3 \end{bmatrix}$ we have

$$\mathcal{E}^t A\mathcal{E} = [\mathrm{Ɛ}_1 \quad \mathrm{Ɛ}_2 \quad \mathrm{Ɛ}_3] \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} \mathrm{Ɛ}_1 \\ \mathrm{Ɛ}_2 \\ \mathrm{Ɛ}_3 \end{bmatrix}$$

$$= -2\mathcal{E}_1{}^2 - 2\mathcal{E}_2{}^2 - 2\mathcal{E}_3{}^2 + 2\mathcal{E}_1\mathcal{E}_2 + 2\mathcal{E}_1\mathcal{E}_3 + 2\mathcal{E}_2\mathcal{E}_3$$

$$= -(\mathcal{E}_1 - \mathcal{E}_2)^2 - (\mathcal{E}_2 - \mathcal{E}_3)^2 - (\mathcal{E}_1 - \mathcal{E}_3)^2 \le 0$$

And negative definite for any of the following condition:

$$\mathcal{E}_1 \ne \mathcal{E}_2 or\ \mathcal{E}_2 \ne \mathcal{E}_3 or\ \mathcal{E}_3 \ne \mathcal{E}_1$$

Definition 1.2.3 A square matrix $(a_{ij}(x))_{i,j=1\dots n} = A(x)\ for\ all\ x \in \Omega$ is symmetric if and only if $A^T(x) = A(x)$.

If $A(x) = \left(a_{ij}(x)\right)_{i,j=1\dots n}\ for\ all\ x \in \Omega\ is\ an\ n \times n$ symmetric positive definite

(Or respectively positive semi definite) matrix then:

$a_{ii}(x) > 0(or\ respectively\ a_{ii}(x) \ge 0)$.

Suppose $A(x)$ is $n \times n$ matrix and let λ be an eigenvalue of $A(x)$ corresponding to eigenvector $\mathcal{E}$.

Then we have that $\mathcal{E}^t A\mathcal{E} = \mathcal{E}^t(\lambda\mathcal{E}) = \lambda\,(\mathcal{E}^t\mathcal{E}) = \lambda|\mathcal{E}|^2$.

Therefore from the above we have

1. If $A(x)$ is positive definite $\lambda > 0$.
2. If $A(x)$ is positive semi definite $\lambda \ge 0$
3. If $A(x)$is negative definite $\lambda < 0$
4. If $A(x)$ is negative semi definite $\lambda \le 0$.

1.3 Elliptic equation

Consider the following second order linear partial differential equation

$$\mathcal{L}u(x) = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u(x)}{\partial x_i} + c(x)u(x)$$

$$for\ x \in \Omega \qquad (3)$$

Where

1) Ω is an open subset of R^n.

2) The matrix $(a_{ij}(x))_{i,j=1\dots n} = A(x)$ is $n \times n$ symmetric matrix and

$b_{i(x)}$= $(b_1(x), b_2(x), \dots b_n(x))$ for all $x \in \Omega$

3) The coefficient $a_{ij}(x), b_i(x)$ *and* $c(x)$ are continuous and bounded in $\overline{\Omega}$.

Let $\lambda_1(x)$ *and* $\lambda_2(x)$ be the smallest and largest eigenvalue of the matrix $a_{ij}(x)$

Definition1.3.1 The second order linear partial differential operator (3) is said to be

1) Elliptic in Ω if and only if $\lambda_1(x) > 0$ for all $x \in \Omega$
2) Uniformly elliptic in Ω if and only if $\mathcal{L}$ is elliptic in Ω and $\frac{\lambda_2(x)}{\lambda_1(x)}$ is bounded in Ω.
3) Strictly elliptic in Ω if and only if $\lambda_1(x) > \lambda_0(x)$ for all $x \in \Omega$ and for some positive constant $\lambda_0(x)$.

Example Show that the linear operator

$$\mathcal{L}u = (1-x^2)\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + (1-y^2)\frac{\partial^2 u}{\partial y^2}$$

is elliptic but not uniformly elliptic for $x^2 + y^2 < 1$, for all $x, y \in R$.

Solution

$\mathcal{L}$ is elliptic if the smallest eigenvalue of

$$a_{ij}(x,y) = \begin{pmatrix} 1-x^2 & xy \\ xy & 1-y^2 \end{pmatrix}$$

is positive.

Let the eigenvalue of $a_{ij}(x,y)\ be\ \lambda_i$ and it is given by

$$\begin{vmatrix} (1-x^2)-\lambda_i & xy \\ xy & (1-y^2)-\lambda_i \end{vmatrix} = 0 \ \ for\ i = 1,2$$

$$\Rightarrow \big((1-x^2)-\lambda_i\big)\big((1-y^2)-\lambda_i\big) - (xy)^2 = 0$$

$$\Rightarrow (1-x^2)(1-y^2) - (xy)^2 - \lambda_i(1-y^2) - \lambda_i(1-x^2) = 0$$

$$\lambda_i^{\ 2} - \lambda_i(1-x^2+1-y^2) + (1-x^2)(1-y^2) - (xy)^2 = 0$$

$$\lambda_i^{\ 2} - \lambda_i(2-(x^2+y^2)) + (1-x^2)(1-y^2) - (xy)^2 = 0$$

Then using quadratic formula $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ we have

$$\lambda_i = \frac{2-(x^2+y^2) \pm \sqrt{(2-(x^2+y^2))^2 - 4((1-x^2)(1-y^2)-(xy)^2)}}{2}$$

$$= \frac{2-(x^2+y^2) \pm \sqrt{4-4(x^2+y^2)+(x^2+y^2)^2 - 4((1-x^2-y^2+x^2y^2)-(xy)^2)}}{2}$$

$$= \frac{2-(x^2+y^2) \pm \sqrt{4-4(x^2+y^2)+(x^2+y^2)^2 - 4 + (x^2+y^2) - 4x^2y^2 + 4(xy)^2}}{2}$$

$$= \frac{2 - (x^2 + y^2) \pm \sqrt{(x^2 + y^2)^2}}{2}$$

$$= \frac{2 - (x^2 + y^2) \pm (x^2 + y^2)}{2}$$

$$\lambda_i = 1 \; or \; \lambda_i = 1 - (x^2 + y^2) > 0 \; since \; (x^2 + y^2) < 1$$

Therefore $\lambda_i > 0$ and hence $\mathcal{L}$ elliptic.

- To show that $\mathcal{L}$ in not uniformully elliptic.

We need to show that $\frac{\lambda_2(x)}{\lambda_1(x)}$ is unbounded. Where $\lambda_1(x)$ is the smallest eigenvalue and $\lambda_2(x)$ the largest eigenvalue.

We now that $(x^2 + y^2) < 1$.then

$$\text{As } (x^2 + y^2) \longrightarrow -\infty$$

$$1 - (x^2 + y^2) \longrightarrow \infty$$

$$\Rightarrow \lambda_2(x) \longrightarrow \infty$$

Therefore:

$$\frac{\lambda_2(x)}{\lambda_1(x)}) \longrightarrow \infty$$

Hence, $\mathcal{L}$ in not uniformully elliptic.

2 Maximum principle of elliptic equation

The maximum principle asserts that solution of certain elliptic equation of second order cannot have a maximum or minimum in the interior of the domain where they are defined. Consider the Laplace's equation $\Delta u = 0$. If u has maximum at a point x and the second derivative of u does not all vanish at x, then Δu is negatives at x, in contradiction to the equation.

The maximum principle is an important feature of second order elliptic equation that distinguishes them from higher order. This is because the second derivative of a function gives information on the function at the exterma.

The aim of this chapter is to prove maximum principle for a solution of

$$\mathcal{L}u(x) \ = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u(x)}{\partial x_i} \ + \ c(x)u(x) = 0 \ \ for \ x \in \Omega$$

To hold maximum principle we need to impose an additional condition on the sign of $c(x)$. Since otherwise no maximum principle can holds.

Example

$$\frac{\partial^2 u(x)}{\partial x^2} + u(x) = 0 \ on \ (0, \pi)$$

$$u(0) = u(\pi) = 0$$

Has a solution $u(x) = \beta sinx$.

For arbitrary u and depending on sign of β.

These solutions assume an interior maximum or minimum at $x = \frac{\pi}{2}$. Depending on β.

2.1 Weak maximum principles

2.1.1 The weak maximum principles of Laplace operator

In this section we shall prove some theorems related to weak maximum principles.

As introduction let us show the proof of the weak maximum principle for laplacian operator.

Let $u(x) = u(x_1, x_2, \dots x_n)$ be twice continuously differentiable function in a domain Ω. The Laplace operator Δ is defined as

$$\Delta = \frac{\partial^2}{\partial {x_1}^2} + \frac{\partial^2}{\partial {x_2}^2} + \cdots + \frac{\partial^2}{\partial {x_n}^2}$$

If the equation $\Delta u = 0$ is satisfied at each point of the domain Ω, we say that u is harmonic function in Ω.

Theorem 1 (weak maximum principle for the laplacian equation)

Let $u \in c^2(\Omega) \cap c^0(\bar{\Omega})$ with Ω a bounded domain in R^n.

1) If $\Delta u \geq 0 \ in \ \Omega$. Then $u \leq \max_{\partial\Omega} u$.

2) If $\Delta u \leq 0$ in Ω, then $u \geq \min_{\partial\Omega} u$

3) If $\Delta u = 0 \ in \ \Omega$, then $\min_{\partial\Omega} u \leq u \leq \max_{\partial\Omega} u$

Proof (1) First we consider the case $\Delta u > 0 \ in \ \Omega$.

Assume that u has an interior maximum at some point $x_0 \in \Omega$. Then at this point we have

$$\frac{\partial u}{\partial x_i} = 0 \; and \; \frac{\partial^2 u}{\partial {x_i}^2} \leq 0$$

Implies $\Delta u \leq 0$ contraduction to the assumption $\Delta u > 0$.

Therefore u has no an interior maximum.

$$u \leq \max_{\partial\Omega} u$$

Now let us see the general case $\Delta u \geq 0$.

Let $\epsilon > 0$.and define

$$v = u + \epsilon |x|^2$$

Then we get $\Delta v = \Delta u + \Delta \epsilon |x|^2 = \Delta u + 2n\epsilon > 0$ in Ω

Then by the above case v cannot have a maximum point within Ω.

If the maximum value of u on $\partial\Omega$ is M and the maximum value of $|x|^2$ on Ω is R^2.

Then the maximum value of v on the boundary of Ω is $M + R^2$.that is

$$u + \epsilon |x|^2 = v \leq M + R^2$$

Since R^2 is bounded by letting $\epsilon \to 0$ we get

$u \leq M = \max_{\partial\Omega} u$ (*)

To prove (2) let us take $u = -u \; in \; \Delta u \geq 0$. then we have

$$\Delta(-u) \geq 0$$

$$\Rightarrow -\Delta u \geq 0$$

$$\Rightarrow \Delta u \leq 0$$

And again using $-u\ in\ place\ of\ \ u\ at\ u \leq max_{\partial\Omega}\, u.$

$$-u \leq \max_{\partial\Omega}(-u)$$

$$\Rightarrow -u \leq -\min_{\partial\Omega} u$$

$$\Rightarrow u \geq \min_{\partial\Omega} u \qquad (**)$$

And hence from $(*)\&(**)$ we get (3)

$$\min_{\partial\Omega} u \leq u \leq \max_{\partial\Omega} u$$

2.1.2 Weak maximum principle of elliptic equations

Now let us extend the discussion to linear second order elliptic equation

Throughout this section we shall consider a second order linear partial differential equation of the form

$$\mathcal{L}u(x) = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u(x)}{\partial x_i} + c(x)u(x)\ for\ \ all\ x \in \Omega$$

The following assumptions are taken throughout, Ω is a domain in R^n .

The matrix $(\, a_{ij}(x))_{i,j=1\ldots n}$ is symmetric and strictly positive definite at every $x \in \overline{\Omega}$ i.e L is elliptic.

The coefficients $(\, a_{ij}(x))_{i,j=1\ldots n}, b_i(x), and\ c(x)$ are continuous and bounded on $\overline{\Omega}$. $u \in c^2(\Omega) \cap c^0(\,\overline{\Omega})\,, \frac{|b_i(x)|}{aii(x)} \leq C.$

Unless indicated otherwise this condition will be assumed throughout.

Definition

i. A function $u \in c^2(\Omega) \cap c^0(\overline{\Omega})$ is called sub solution if $\mathcal{L}u \leq 0$.

ii. A function $u \in c^2(\Omega) \cap c^0(\overline{\Omega})$ is called super solution $\mathcal{L}u \geq 0$.

The weak maximum principle of elliptic equation can be expressed by the following theorem.

Theorem 2 (**weak maximum principle of elliptic equation**)

1) Assume that $Lu \geq 0, c(x) = 0$ in a bounded domain Ω. Then the maximum of u(x) $on\ \overline{\Omega}\ is\ achieved\ on\ \partial\Omega$.
2) Assume that $Lu \leq 0, c(x) = 0$ in a bounded domain Ω.Then the minimum of $u(x) on\ \overline{\Omega}\ is\ achieved\ on\ \partial\Omega$.

To prove this theorem we need the following definition.

Definition The closure of Ω denoted by $\overline{\Omega}$ is the intersection of all closed sets that contain Ω.

The closure of Ω is the union of Ω and its boundary. $i.e. \overline{\Omega} = \Omega \cup \partial\Omega$.

Proof:

1) If $\mathcal{L}u(x) \geq 0$

Case 1 $If\ \mathcal{L}u(x) > 0\ for\ all\ x \in\ \Omega\ then\ u$ cannot achieve its maximum anywhere within Ω let us take the contrary i.e.

Suppose u has a maximum point at x_0 in Ω .Then at the interior maximum x_0 of u

We must have $\frac{\partial u(x_0)}{\partial x_i} = 0$ for $i = 1, 2 \dots . n$ and

$$\frac{\partial^2 u(x_0)}{\partial x_i \partial x_j} \leq 0$$

From this we have

$$\mathcal{L}u(x_0) = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u(x_0)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u(x_0)}{\partial x_i} + c(x)u(x_0)$$

$$= \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u(x_0)}{\partial x_i \partial x_j} \leq 0$$

This implies $\mathcal{L}u(x_0) \leq 0$

This contradicts the assumption $Lu(x) \geq 0 \quad in\ \Omega$.

This shows such interior maximum cannot occur within Ω.

Therefore $\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$

Case 2 For the general case $\mathcal{L}u(x) \geq 0\ for\ all\ x \in \Omega$.

Let us define an auxiliary function $\mathcal{W}(x) = e^{\gamma x_i}$ where γ is a positive constant to be chosen soon.

First let us prove the case $\mathcal{W}(x) = e^{\gamma x_1}$

$$\frac{\partial \mathcal{W}(x)}{\partial x_1} = \gamma\mathcal{W}(x), \qquad and \quad \frac{\partial^2 \mathcal{W}(x)}{\partial^2 x_1} = \gamma^2\mathcal{W}(x)$$

But $\frac{\partial \mathcal{W}(x)}{\partial x_i} = \frac{\partial^2 \mathcal{W}(x)}{\partial x_i \partial x_j}$=0 for $i, j > 1$

$$\mathcal{L}\mathcal{W}(x) = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 \mathcal{W}(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial \mathcal{W}(x)}{\partial x_i}$$

$$= a_{11}(x)\gamma^2\mathcal{W}(x) + b_1(x)\,\gamma\mathcal{W}(x)$$

$$= a_{11}(x))\,\gamma\mathcal{W}(x)\left(\gamma + \frac{b_1(x)}{a_{11}(x)}\right)$$

Since $a_{11}(x)$ and $b_1(x)$ are bounded then $\frac{b_1(x)}{a_{11}(x)}$≤C for some positive constant C

$$\geq a_{11}(x)\,\gamma\mathcal{W}(x)(\gamma - C)$$

Now choose $\gamma > C$ so that $\mathcal{L}\mathcal{W}(x) > 0$

Then $\mathcal{L}\big(u(x) + \epsilon\mathcal{W}(x)\big) = \mathcal{L}u(x) + \epsilon\mathcal{L}\mathcal{W}(x) > 0\ in\ \Omega$.

Hence by the above case (1) $u(x) + \epsilon\mathcal{W}(x)$ cannot have a maximum value within Ω.

Since $\epsilon w > 0$, $u \leq u + \epsilon w$

$$\max_{\Omega} u \leq \max_{\bar{\Omega}}(u + \epsilon w) = \max_{\partial\Omega}(u + \epsilon w) \leq \max_{\partial\Omega} u + \epsilon \max_{\partial\Omega} w$$

Since Ω is bounded we observe that $\mathcal{W}(x)$ is bounded in $\bar{\Omega}$ hence $\mathcal{W}(x)$ is finite

Then by letting $\epsilon \to 0$ we have

$$\max_{\Omega} u \leq \max_{\partial\Omega} u$$

$Since\ u\ is\ continuous,$ Ω is closed and bounded, hence the closure $\bar{\Omega}$ is compact.

Then the maximum of $u\ on\ \Omega$ coincides with the maximum of u on $\bar{\Omega}$.

$$\Rightarrow \max_{\bar{\Omega}} u = \max_{\Omega} u \leq \max_{\partial\Omega} u.$$

$$\max_{\bar{\Omega}} u \leq \max_{\partial\Omega} u$$

$Since\ \bar{\Omega} = \Omega \cup \partial\Omega\ , and\ \partial\Omega \subseteq \bar{\Omega}$ the $\max_{\bar{\Omega}} u\ cannot\ be\ less\ than\ the$ $\max_{\partial\Omega} u$

Hence $\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$

Remark In the above maximum principle, only the positive definiteness of $A(x)$ in Ω and the boundedness of $\frac{b_i(x)}{a_{ii}(x)}$ in Ω for some $i = 1, 2 \ldots n$ are needed in the proof.

Then one can take the auxiliary function $\mathcal{W}(x) = e^{\gamma x_i}$ in Ω in the above proof.

That is: If $LW(x) \geq 0$

Let $\mathcal{W}(x) = e^{\gamma x_i}$ in Ω where γ *is* positive constant to be determine soon

$$\frac{\partial \mathcal{W}(x)}{\partial x_i} = \gamma\, \mathcal{W}(x) \;\; i = 1,2 \dots n$$

$$\frac{\partial^2 \mathcal{W}(x)}{\partial x_i \partial x_j} = \frac{\partial^2 \mathcal{W}(x)}{\partial^2 x_i} = \gamma^2 \mathcal{W}(x) \; for \; i = j = 1,2,\dots n \;\; and \;\; \frac{\partial^2 \mathcal{W}(x)}{\partial x_i \partial x_j} = 0 \; for \; i \neq j.$$

Then

$$\mathcal{L}\mathcal{W}(x) = \sum_{i=1}^{n} a_{ii}(x) \frac{\partial^2 \mathcal{W}(x)}{\partial^2 x_i} + \sum_{i=1}^{n} b_i(x) \frac{\partial \mathcal{W}(x)}{\partial x_i} \;\; for \; i = 1,2,\dots n \; and \; x \in \Omega.$$

$$= \sum_{i=1}^{n} a_{ii}(x)\gamma^2 \mathcal{W}(x) + \sum_{i=1}^{n} b_i(x)\gamma\, \mathcal{W}(x)$$

$$= \sum_{i=1}^{n} a_{ii}(x)\gamma\, \mathcal{W}(x) \left(\gamma + \sum_{i=1}^{n} \frac{b_i(x)}{a_{ii}(x)} \right)$$

$$snice \; a_{ii}(x), b_i(x) \; and \; c(x) \; are \; bounded \; in \; \Omega \; then \; \frac{b_i(x)}{a_{ii}(x)} \; is \; bounded,$$

This implies $\sum_{i=1}^{n} \frac{b_i(x)}{a_{ii}(x)}$ is bounded. Say

$$\sum_{i=1}^{n} \frac{b_i(x)}{a_{ii}(x)} < C$$

For some positive constant C.

$$\geq \sum_{i=1}^{n} a_{ii}(x)\gamma\, \mathcal{W}(x)\, (\gamma - C)$$

$$choose \; \gamma \; large \; enaugh \; such \; that \;\; (\gamma - C) > 0 \; that \; is \; \gamma > C.$$

This implies

$$\mathcal{L}\mathcal{W}(x) \geq a_{ii}(x)\gamma\, \mathcal{W}(x)(\gamma - C) > 0$$

$$\Longrightarrow\ \mathcal{L}\mathcal{W}(x) > 0$$

$$\mathcal{L}\big(u(x) + \epsilon\mathcal{W}(x)\big) = \mathcal{L}u(x) + \epsilon\mathcal{L}\mathcal{W}(x) \geq \epsilon\mathcal{L}\mathcal{W}(x) > 0$$

Then by case 1 $u(x) + \epsilon\mathcal{W}(x)$ cannot have a maximum value within Ω.

It is clear that $u(x) \leq u(x) + \epsilon\mathcal{W}(x)\ for\ \epsilon > 0$

$$\max_{\Omega} u \leq \max_{\overline{\Omega}}(u + \epsilon w) = \epsilon \max_{\partial\Omega}(u + \epsilon w) \leq \max_{\partial\Omega} u + \epsilon \max_{\partial\Omega} w$$

$$\Rightarrow \max_{\Omega} u \leq \max_{\partial\Omega} u$$

$Since\ u\ is\ continuous,$ Ω is closed and bounded, hence the closure $\overline{\Omega}$ is compact.

Then the maximum $of\ u\ on\ \Omega$ coincides with the maximum of u on $\overline{\Omega}$.

$$\Rightarrow \max_{\overline{\Omega}} u = \max_{\Omega} u \leq \max_{\partial\Omega} u$$

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u$$

But $Since\ \overline{\Omega} = \Omega \cup \partial\Omega\,, and\ \partial\Omega \subseteq \overline{\Omega}$ the $\max_{\overline{\Omega}} u\ cannot\ be\ less\ than\ the$ $\max_{\partial\Omega} u$
Hence $\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$

2) If $\mathcal{L}u(x) \leq 0\ in\ \Omega$ then the minimum of $u\ in\ \overline{\Omega}$ achieves on $\partial\Omega$ i.e.

$$\min_{\Omega} u = \min_{\partial\Omega} u$$

Proof: Due to the linearity $of\ \mathcal{L}\ if\ u\ is\ super\ solution\ then\ -u$ is sub solution. Then we can substitute $-u(x)\ in\ in\ place\ of\ u(x)\ in\ (1)$ we have

$$\mathcal{L}u \geq 0$$

$$\Rightarrow \mathcal{L}(-u) \geq 0$$

$$\Rightarrow -\mathcal{L}u \geq 0$$

$$\Rightarrow \mathcal{L}u \leq 0$$

And again substitute $-u(x)$ *in place of* $u(x)$*in abave equation* we get

$$\max_{\overline{\Omega}}(-u) = \max_{\partial\Omega}(-u)$$

$$\Rightarrow -\min_{\overline{\Omega}} u = -\min_{\partial\Omega} u$$

$$\Rightarrow \min_{\overline{\Omega}} u = \min_{\partial\Omega} u$$

Corollary 3 *Suppose* $\mathcal{L}$ *is elliptic* , $c(x) \leq 0$, *in a* bounded domain Ω.

Let $u \in c^2(\Omega) \cap c^0(\overline{\Omega})$

$i)$ *if* $\mathcal{L}u \geq 0$ *in* Ω, *then* $\max_{\overline{\Omega}}(u) \leq \max_{\partial\Omega} u^+$

$ii)$ *if* $\mathcal{L}u \geq 0$ *in* Ω, *then* $\min_{\overline{\Omega}} u \geq -\max_{\partial\Omega} u^-$.

$iii)$ *if* $\mathcal{L}u = 0$ *in* Ω, *then* $\max_{\overline{\Omega}}|u| = \max_{\partial\Omega}|u|$

Where $u^+ = max\{u(x), 0\}$ and

$u^- = max\{-u(x), 0\}$

Proof: we shall first prove (i)

If $u(x) \leq 0$ throughout Ω *then* (i) trivially true

$$\max_{\overline{\Omega}}(u) \leq \max_{\partial\Omega} u^+$$

Suppose $u(x) \geq 0$ for some $x \in \Omega$ *then the set*

$\Omega^+ = \{x \in \Omega : u(x) > 0\}$ is nonempty

$$Let \ \mathcal{L}_0 u(x) = \mathcal{L}u(x) - c(x)u(x)$$

$$= \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i}$$

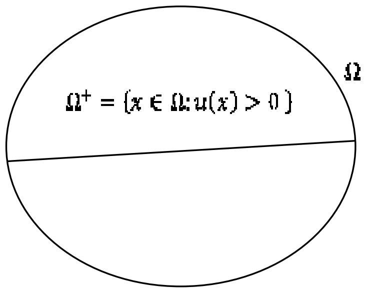


Figure 2. 1

On Ω^+ , we have

$\mathcal{L}_0 u(x) = \mathcal{L}u(x) - c(x)u(x)$ *Since* $u(x) > 0$ *for* $x \in \Omega^+$

$\geq -c(x)u(x)$ *since* $c(x) \leq 0$ *in* Ω *and hence in* Ω^+

≥ 0

By the weak maximum principle.

$$\max_{\bar{\Omega}^+} u(x) = \max_{\partial\Omega^+} u(x) \quad in \ \Omega^+ \qquad (1)$$

Since $u(x) \leq 0$ *on* $\partial\Omega^+ \cap \partial\Omega$ then the maximum u must achieve on $\partial\Omega$.

It is also clear that $\partial\Omega^+ \cap \partial\Omega \subseteq \partial\Omega$ and $\partial\Omega^+ \cap \partial\Omega \subseteq \partial\Omega^+$

$$\Rightarrow \max_{\partial\Omega^+\cap\partial\Omega} u(x) \le \max_{\partial\Omega^+} u(x) \le \max_{\partial\Omega} u(x) \le \max_{\partial\Omega} u^+(x)$$

.

$$\Rightarrow \max_{\partial\Omega^+} u(x) \le \max_{\partial\Omega} u^+(x) \qquad (2)$$

From $(1) and$ (2) we have

$$\max_{\bar{\Omega}^+} u(x) = \max_{\partial\Omega^+} u(x) \ \le \max_{\partial\Omega} u^+(x)$$

$$\max_{\bar{\Omega}^+} u(x) = \max_{\partial\Omega} u^+(x)$$

$$\max_{\Omega} u(x) \le \max_{\bar{\Omega}^+} u(x) \ \le \max_{\partial\Omega} u^+(x)$$

$$\max_{\Omega} u(x) \ \le \max_{\partial\Omega} u^+(x)$$

$Since\ u\ is\ continuous,$ Ω is closed and bounded, hence the closure $\bar{\Omega}$ is compact.

Then the maximum of $u\ on$ Ω coincides with the maximum of u on $\bar{\Omega}$.

$$\max_{\bar{\Omega}} u(x) = \max_{\Omega} u(x) \le \max_{\partial\Omega} u^+(x)$$

$$\max_{\bar{\Omega}} u(x) \le \max_{\partial\Omega} u^+(x)$$

$ii)$ $\boldsymbol{To\ proof}$ (ii)

Due to the linearity of $\mathcal{L}$ if $u(x)$ is super solution then $-u(x)$ is sub solution. Then we can substitute $-u(x)\ in\ in\ place\ of\ u(x)$ in the above result

Then $\mathcal{L}(-u) \ge 0 \Rightarrow \mathcal{L}u \le 0$ and $\max_{\bar{\Omega}}(-u) \le \max_{\partial\Omega}(-u)^+$ And since

$$\max_{\bar{\Omega}}(-u) = -\min_{\bar{\Omega}} u$$

$$\max_{\partial\Omega}(-u)^{+} = \max_{\partial\Omega}(-u^{-}) = -\min_{\partial\Omega}(u^{-})$$
$$as\ (-u)^{+} = max\{-u(x), 0\} = -min\{u(x), 0\} = u^{-}$$

$$\max_{\bar{\Omega}}(-u) \leq \max_{\partial\Omega}(-u)^{+}$$

$$\Rightarrow -\min_{\bar{\Omega}}(u) \leq \max_{\partial\Omega}(u)^{-}$$

Multiplying both sides by negative one we get $\min_{\bar{\Omega}} u \geq -\max_{\partial\Omega} u^{-}$ which proves (ii)

Finally we prove (iii) as follows

Since $\mathcal{L}u = 0 \Rightarrow \mathcal{L}u \geq 0\ and\ \mathcal{L}(-u) \geq 0\ on\ \Omega$ we see that for any $x \in \Omega$

$$\max_{\bar{\Omega}} u \leq \max_{\partial\Omega} u^{+} \leq \max_{\partial\Omega}|u|\ \ and$$

$$\max_{\bar{\Omega}}(-u) \leq \max_{\partial\Omega}(-u)^{+} \leq \max_{\partial\Omega}|u|$$

$$\Rightarrow \max_{\bar{\Omega}}|u| \leq \max_{\partial\Omega}|u|$$

But $Since\ \bar{\Omega} = \Omega \cup \partial\Omega\,, and\ \partial\Omega \subseteq \bar{\Omega}$ the
$\max_{\bar{\Omega}}|u|\ cannot\ be\ less\ than\ the\ \max_{\partial\Omega}|u|$

$$\therefore \max_{\bar{\Omega}}|u| = \max_{\partial\Omega}|u|$$

Corollary 4 let Ω be bounded domain, $\mathcal{L}\ is\ elliptic, u\ and\ v\ are\ in\ c^{2}(\Omega) \cap c^{0}(\bar{\Omega})$ and $c(x) \leq 0$, such that:

1) $\mathcal{L}u \geq \mathcal{L}v\ \ in\ \Omega\ and\ u \leq v\ on\ \partial\Omega\ \ then\ u \leq v\ in\ \Omega$.
2) $\mathcal{L}u = \mathcal{L}v\ \ in\ \ \Omega\ \ anb\ u = v\ on\ \partial\Omega\ then\ u = v\ in\ \Omega$.

Proof:

1) Let $w = u - v$ $then$ $\mathcal{L}w \geq 0$ in Ω and $w \leq 0$ on $\partial\Omega$.

By the weak maximum principle and the above corollary we have

$$\max_{\overline{\Omega}} w \leq \max_{\partial\Omega} w^{+} = 0$$

$$\Rightarrow \max_{\overline{\Omega}} w \leq 0$$

$$\Rightarrow w(x) \leq 0 \; in \; \overline{\Omega}.$$

$$\Rightarrow u(x) - v(x) \leq 0 \; in \; \overline{\Omega}.$$

$$\therefore u(x) \leq v(x) \; in \; \overline{\Omega}.$$

2) Let $w = u - v$ then $\mathcal{L}w = \mathcal{L}u - \mathcal{L}v = 0$ in Ω and $w = 0$ on $\partial\Omega$.

By $Corollary$ 3 we have

$$\max_{\overline{\Omega}} |w| = \max_{\partial\Omega} |w| = 0$$

$$\Rightarrow \max_{\overline{\Omega}} |w| = 0$$

$$\Rightarrow |w| = 0 \; in \; \overline{\Omega}$$

$$w = u - v = 0 \; in \; \overline{\Omega}$$

This implies $u = v$ in $\overline{\Omega}$.

A consequence of the maximum principle is the uniqueness for the dirichlet problem associated to the operator

$$\mathcal{L}u(x) = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u(x)}{\partial x_i} + c(x)u(x) \; for \; x \in \Omega$$

2.2 Uniqueness theorem of bounded value problem

We begin with a study of one of the simplest boundary value problems for second order partial differential equation on a bounded two dimensional domain Ω with a boundary $\partial\Omega$.

We pose the problem of determining a function $u(x, y)$ which is twice differentiable in Ω, continuous on $\Omega \cup \partial\Omega$. And satisfies the equation

$$\mathcal{L}u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x) \qquad (*)$$

$$u = g(s) \qquad (**)$$

$Equation$ $(*)$ is known as the poisons equation. The function f is prescribed throughout Ω, and the function g given in terms of the arc length S, is prescribed along $\partial\Omega$.

The problem of determining such a solution u is said to be dirchlet problem or first boundary value.

By a means of maximum principle alone, it is possible to show that if a solution of the first boundary value exists then it must be unique. That is we prove that there can be at most one solution of the problem.

Theorem 5 (Uniqueness theorem of bounded value problem)

Suppose u_1 and u_2 are two functions which satisfy $(*)$ & $(**)$ then $u_1 = u_2$.

Proof:

Define $v = u_1 - u_2$

We see that v satisfies

$$\mathcal{L}v = 0 \; in \; \Omega$$

$$v = 0 \; on \; \partial\Omega.$$

According to the maximum principle, v cannot have a maximum point in the interior of Ω. However the maximum of a continuous function on closed and bounded set must be attained. Since v is continuous on $\partial\Omega \cup \Omega$ and $v = 0$ on $\partial\Omega$,

We conclude that $v \leq 0 \; in \; \Omega$.

Similarly v cannot have a minimum point in Ω.

To show this substitute $v\ by - v$ in the above equation. We get

$$-v \leq 0\ in\ \Omega.$$

$$v \geq 0\ in\ \Omega$$

From these we obtain that

$$v = u_1 - u_2 \equiv 0\ in\ \Omega.$$

$$\Rightarrow u_1 = u_2\ in\ \Omega\,.$$

This completes the uniqueness of the solution.

Remark it is essential to assume that the domain Ω is bounded. Otherwise the above result is not holds true.

Example let Ω be the infinite string given by

$$\Omega = \begin{cases} -\infty < x < \infty \\ 0\ \leq y \leq \pi. \end{cases}$$

The function

$$u = \ e^x \sin y$$

Satisfies $\mathcal{L}u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

It vanishes on the boundary of Ω

i.e. $u(x.0) = u(x, \pi) = 0.$

Although the function u satisfies the maximum principle, it does not assume its maximum on the boundary of Ω.

We can see that $u\left(x, \frac{\pi}{2}\right) = e^x \longrightarrow +\infty\ as\ x \longrightarrow +\infty$

2.3 Strong maximum principle

The weak maximum principle states that $u(x)$ achieves its minimum or maximum for $Lu(x) \leq 0, or\ Lu(x) \geq 0$ on the boundary of Ω However $u(x)$ may attain its maximum or minimum at many points. And therefore it does not rule out the possibility that some of these points are interior.

In this section we will prove a strong version of the maximum principle when $\mathcal{L}\ is$ uniformly elliptic that rules out the possibility that u can take its maximum inside Ω.

Since strong maximum principle relies on the $hopf\ 's$ boundary lemma we need to prove the following lemma.

Definition A domain Ω is said to be satisfy an interior sphere condition at $x_0 \in \partial\Omega$ if there is exist a ball $B \subseteq \Omega$ such that $x_0 \in \partial B$. Type equation here.

Lemma 6 (***Hopf's boundary lemma***) let $\mathcal{L}\ be\ uniformly\ elliptic$.Assume that $c \equiv 0\ and\ u \in c^2(\Omega)$ and $\mathcal{L}u \geq 0.\ Let\ x_0 \in \partial\Omega$ such that

i. $u\ is\ continuous\ at\ x_0$.
ii. $u(x) \leq u(x_0)\ for\ all\ x \in \Omega$.
iii. $\Omega\ satisfies\ an\ interior\ sphere\ condition\ at\ x_0$.

Then the outer normal derivative $of\ u\ at\ x_0$, if it exists satisfies the strict

Inequality $\frac{\partial u}{\partial v}(x_0) > 0$.

Assume now that $\frac{|b(x)|}{\lambda_1(x)}$ and $\frac{|c(x)|}{\lambda_1(x)}$ is bounded in Ω.

If $c(x) \leq 0\ in\ \Omega$, the same conclusion holds if $u(x_0) \geq 0$, and if $u(x_0) = 0$ the same conclusion holds irrespective of the sign $of\ c(x)$.

Proof: since Ω satisfies the interior sphere condition at $x_0 \in \partial\Omega$ there exist a ball $B = B(z, R) \subseteq \Omega$. such that $x_0 \in \partial B$

Now let $0 < \rho \leq |x - z| \leq R$ and consider the annulus

$$A = B(z, R)\backslash\bar{B}(z, \rho) \; for \; all \; x \in A$$

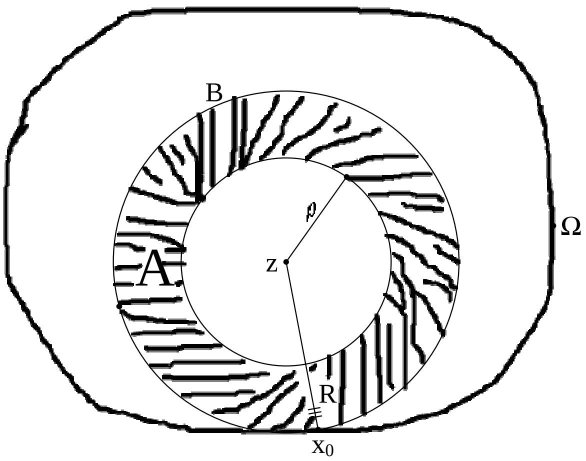


Figure2. 2

Define $w(x) = e^{-\gamma|x-z|^2} - e^{-\gamma R^2}$ in the annulus

$A = B(z, R)\backslash\bar{B}(z, \rho) \; for \; all \; x \in A$ where $\gamma > 0$ and will be determined soon.

Then $\frac{\partial w(x)}{\partial x_i} = -2\gamma(x_i - z_i)e^{-\gamma|x-z|^2}$ and

$$\frac{\partial^2 w(x)}{\partial x_i \partial x_j} = -2\gamma\delta_{ij}e^{-\gamma|x-z|^2} + 4\gamma^2(x_i - z_i)(x_j - z_j)e^{-\gamma|x-z|^2}$$

Where *the Kronecker delta,* δ_{ij} *is defined as*

$$\delta_{ij} = \begin{cases} 1 & if \; i = j \\ 0 & if \; i \neq j \end{cases}$$

$$\mathcal{L}w(x) = \sum_{i.j=1}^{n} a_{ij}(x)\frac{\partial^2 w(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial w(x)}{\partial x_i} + c(x)u(x)$$

$$= -2\gamma e^{-\gamma|x-z|^2}\sum_{i,j=1}^{n} a_{ij}\delta_{ij} + 4\gamma^2 e^{-\gamma|x-z|^2}\sum_{i,j=1}^{n} a_{ij}\,(x)(x_i - z_i)(x_j - z_j)$$

$$- 2\gamma e^{-\gamma|x-z|^2}\sum_{i=1}^{n} b_i(x)\,(x_i - z_i) + c(x)\left(e^{-\gamma|x-z|^2} - e^{-\gamma R^2}\right)$$

$$= e^{-\gamma|x-z|^2}\left[4\gamma^2\sum_{i,j=1}^{n} a_{ij}(x_i - z_i)(x_j - z_j) - 2\gamma\sum_{i=1}^{n} a_{ii}(x) - 2\gamma\sum_{i=1}^{n} b_i(x_i - z_i) + \frac{c(x)}{e^{-\gamma|x-z|^2}}(e^{-\gamma|x-z|^2} - e^{-\gamma R^2})\right]$$

$$= e^{-\gamma|x-z|^2}\left[4\gamma^2\sum_{i,j=1}^{n} a_{ij}(x_i - z_i)(x_j - z_j) - 2\gamma\sum_{i=1}^{n} a_{ii}(x) - 2\gamma\sum_{i=1}^{n} b_i(x_i - z_i) + c(x)(1 - e^{-\gamma R^2 + \gamma|x-z|^2})\right]$$

The ellipcity of L implies

$$\sum_{i,j=1}^{n} a_{ij}(x)\varepsilon_i\varepsilon_j > \lambda_1(x)|\varepsilon|^2$$

Thus

$$\mathcal{L}w(x) \geq e^{-\gamma|x-z|^2}\left[4\gamma^2\lambda_1(x)|x-z|^2 - 2\gamma n\lambda_2(x) - 2\gamma b(x)|x-z| + c(x)(1-e^{-\gamma R^2+\gamma|x-z|^2})\right]$$

$$= 2\gamma\lambda_1(x)e^{-\gamma|x-z|^2}\left[2\gamma|x-z|^2 - \frac{n\lambda_2(x)}{\lambda_1(x)} - \frac{|b(x)|}{\lambda_1(x)}|x-z| + \frac{c(x)}{2\gamma\lambda_1(x)}(1 - e^{-\gamma(R^2-|x-z|^2)})\right]$$

$$\geq 2\gamma\lambda_1(x)e^{-\gamma|x-z|^2}\left[2\gamma|x-z|^2 - \frac{n\lambda_2(x)}{\lambda_1(x)} - \frac{|b(x)|}{\lambda_1(x)}|x-z| - \frac{|c(x)|}{2\gamma\lambda_1(x)}(1 - e^{-\gamma(R^2-|x-z|^2)})\right]$$

$$\geq 2\gamma\lambda_1(x)e^{-\gamma|x-z|^2}\left[2\gamma\rho^2 - \frac{n\lambda_2(x)}{\lambda_1(x)} - \frac{R|b(x)|}{\lambda_1(x)} - \frac{|c(x)|}{2\gamma\lambda_1(x)}\right]$$

Since $\frac{\lambda_2(x)}{\lambda_1(x)}, \frac{|b(x)|}{\lambda_1(x)}$ and $\frac{|c(x)|}{\lambda_1(x)}$ are bounded on Ω we can choose $\gamma > 0$

$$i.e, 2\gamma\rho^2 > \left(\frac{n\lambda_2(x)}{\lambda_1(x)} + \frac{R|b(x)|}{\lambda_1(x)} + \frac{|c(x)|}{2\gamma\lambda_1(x)}\right) \textit{large enough such that } \mathcal{L}w(x) \geq 0$$

Now consider $\epsilon > o$ such that

$u(x) - u(x_0) + \epsilon w \leq 0$ For all $|x-z| = \rho$ *in the annulus* $A = B_R \backslash B_\rho$.

Then $u(x) - u(x_0) + \epsilon w \leq 0$ on the boundary ∂A *of the annuluse* A.

Since $u \in c^2(\Omega)$ and u is continuous at x_0 .

$$u - u(x_0) + \epsilon w \textit{ is in } c(\bar{A})$$

Moreover in A we have

$$\mathcal{L}(u - u(x_0) + \epsilon w) = \mathcal{L}(u + \epsilon w) - c(x)u(x_0) \geq -c(x)u(x_0)$$

$$\Rightarrow \ \mathcal{L}(u - u(x_0) + \epsilon w) \geq -c(x)u(x_0)$$

$if\ c(x) \leq 0$ Then $\mathcal{L}(u - u(x_0) + \epsilon w) \geq 0$ on $A\ iff\ \ u(x_0) \geq 0$.

$if\ u(x_0) = 0$ Then $\mathcal{L}(u - u(x_0) + \epsilon w) \geq 0$ on A without sign restriction of c.

Recall: $u - u(x_0) + \epsilon w \leq 0\ on\ \partial\Omega$

∴ By the weak maximum principle and $Corollary$ 3 we see that:

$$\max_{\bar{A}}(u(x) - u(x_0) + \epsilon w) \leq \max_{\partial A}(u(x) - u(x_0) + \epsilon w)^+ = 0$$

$$\Rightarrow \max_{\bar{A}}(u(x) - u(x_0) + \epsilon w) \leq 0$$

$$\Rightarrow \ u(x) - u(x_0) + \epsilon w \leq 0 \ \ on\ \bar{A}$$

$$\Rightarrow \ u(x) - u(x_0) \leq -\epsilon w\ on\ \bar{A}$$

Let υ be the outer unit normal to $\partial\Omega\ at\ x_0$.

Now assume that the normal derivative of $u\ at\ x_0$ exists.

Note that: $u - u(x_0) \leq -\epsilon w = -\epsilon\big(w(x) - w(x_0)\big)\ since\ w(x_0) = 0\ at\ x_0$.

$$\therefore \frac{\partial u}{\partial \upsilon}(x_0) = \lim_{t\to 0^-} \frac{u(x_0 + t\upsilon) - u(x_0)}{t}$$

$$\geq \ -\epsilon \lim_{t\to 0^-} \frac{w(x_0 + t\upsilon) - w(x_0)}{t}$$

$$= -\epsilon \frac{\partial w}{\partial \upsilon}(x_0).$$

In the last inequality we used the fact that

$u - u(x_0) \leq -\epsilon w = -\epsilon\big(w(x) - w(x_0)\big)$ And hence for $t < 0$ we have

$$\frac{u(x)-u(x_0)}{t} \le \frac{-\epsilon(w(x)-w(x_0))}{t}$$

$$\Rightarrow \lim_{t\to 0^-}\frac{u(x)-u(x_0)}{t} \ge \lim_{t\to 0^-}\frac{w(x)-w(x_0)}{t}$$

Substitute $x = x_0 + tv$ then from the above limit we get

$$\lim_{t\to 0^-}\frac{u(x_0+tv)-u(x_0)}{t} \ge -\epsilon \lim_{t\to 0^-}\frac{w(x_0+tv)-w(x_0)}{t}$$

$$\Rightarrow \frac{\partial u}{\partial v}(x_0) \ge -\epsilon\frac{\partial w}{\partial v}(x_0) = -\epsilon\frac{\partial w}{\partial v}(x_0).v(x_0)$$

Recall $\frac{\partial w}{\partial v}(x_0) = 2\gamma(z-x_0)e^{-\gamma R^2}$.

Since $v(x_0)$ points outward and $z - x_0$ points in to Ω it is clear that

$$(z-x_0).v(x_0) \le 0$$

$$\Rightarrow \frac{\partial u}{\partial v}(x_0) \ge -\epsilon\big(2\gamma(z-x_0)e^{-\gamma R^2}\big).\frac{x_0-z}{|x_0-z|} \quad Since\ v(x_0) = \frac{x_0-z}{|x_0-z|}$$

.

$$= -\epsilon(-\big(2\gamma(x_0-z)e^{-\gamma R^2}\big).\frac{x_0-z}{|x_0-z|}$$

$$= \epsilon\big(2\gamma(x_0-z)e^{-\gamma R^2}\big) > 0$$

$$\therefore \frac{\partial u}{\partial v}(x_0) > 0.$$

Theorem 7 (**strong maximum principle**).suppose Ω is a domain in R^n. Not necessary bounded, let $\mathcal{L}$ *be unformly elliptic* and $\mathcal{L}u \geq 0$ *in* Ω for $u \in c^2(\Omega)$.

1) If $c(x) \equiv 0$, and u achieves its maximum in the interior of Ω then u is constant.

2) If $c(x) \leq 0$ *and* $\frac{c(x)}{\lambda_{1(x)}}$ is bounded, then u cannot achieve a non-negative maximum in the interior of Ω. unless it is constant.

Proof:

1. Let $M = \max_\Omega u$

The weak maximum principle implies that this maximum u attains at the boundary of Ω. The weak maximum principle does not exclude the possibility of attaining M at the interior point of Ω.

We will now show that; unless u is constant this cannot exist.

So let $\Omega^- = \{x \in \Omega : u(x) < M\}$ be non empty unless if $\Omega^- = \emptyset$ then $u \equiv M$ in Ω. And $\Omega^+ = \{x \in \Omega : u(x) = M\}$

Suppose M attains inside Ω, and yet u is not constant.

Then $\Omega^- \neq \emptyset$ and $\Omega^- \neq \Omega$.

We first show that $\partial\Omega^- \cap \Omega \neq \emptyset$. Suppose $\partial\Omega^- \cap \Omega = \emptyset$.

Then $\partial\Omega^- \subseteq \partial\Omega$.now observe that.

$$\Omega \backslash \bar{\Omega}^- = \Omega \backslash (\Omega^- \cup \partial\Omega^-)$$

$$= (\Omega \backslash \Omega^-) \cap (\Omega \backslash \partial\Omega^-)$$

$$= (\Omega \backslash \Omega^-) \cap \Omega$$

$$= \Omega \backslash \Omega^-$$

$$\Rightarrow \Omega\backslash\bar{\Omega}^{-} = \Omega\backslash\Omega^{-}$$

But we know that $\Omega\backslash\bar{\Omega}^{-}$ is open. And $\Omega\backslash\Omega^{-}$ is closed.

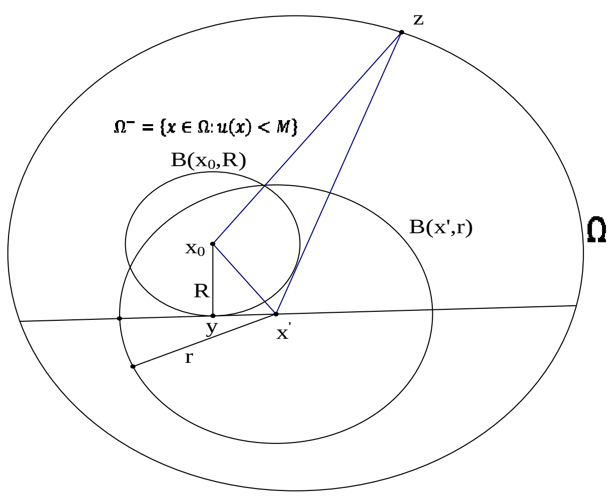
Since Ω is connected this implies $\Omega\backslash\Omega^{-} = \Omega$ or $\Omega\backslash\Omega^{-} = \emptyset$.

Hence $\Omega^{-} = \Omega$ or $\Omega^{-} = \emptyset$. which contradicts

Therefore $\partial\Omega^{-} \cap \Omega \neq \emptyset$.

So let $x' \in \partial\Omega^{-} \cap \Omega$.again let $x_0 \in B(x', r) \cap \Omega^{-}$

Where $r = \frac{1}{2} dist(x', \partial\Omega)$ then for any $z \in \partial\Omega$ we have



$\boldsymbol{Figure\ 2.3}$

By triangular inequality;

$$|x_0 - z| + |x' - x_0| \geq |z = x'|$$

$$\Rightarrow |x_0 - z| \geq |z - x'| - |x' - x_0|$$

$$\geq dist(x', \partial\Omega) - \frac{1}{2} dist(x', \partial\Omega)$$

$$= \frac{1}{2} dist(x', \partial\Omega)$$

$$> |x' - x_0|$$

$$\geq dist(x_0, \partial\Omega^-)$$

Therefore $dist\ (x_0, \partial\Omega^-) < dist\ (x_0, \partial\Omega)$. That is $x_0$is a point in Ω^-closer to the boundary of $(x_0, \partial\Omega^-)$ than to the boundary of Ω.

Let B (x_0, R) be the largest ball contained inΩ^-.

Then $u < M$ in $B\ (x_0, R)$ for some $y \in \partial B$.

Since Ω^- satsfies the interior sphere condition at y, by Hopf's lemma, the outer normal derivative of u at y is strictly positive.

But since $y \in \Omega$ we have $\frac{\partial u}{\partial x}(y) = 0$, which contradicts the assumption u is not constant.

Therefore u must be constant.

2. If u has none negative maximum at $y \in \Omega$, then by hopf's lemma this satisfies the strict inequality $\frac{\partial u}{\partial v}(y) > 0$ for $c(x) \leq 0$.

But since $y \in \Omega$ the first derivative of u at y vanishes, that is $\frac{\partial u}{\partial y} = 0$.

This contradicts

Therefore u must be constant.

Example

Let $u \in c^2(\Omega) \cap c(\overline{\Omega})$ satisfy the equation $\sum_{i=1}^{n} \frac{\partial^2 u}{\partial {x_i}^2} - u^3 = 0$ show that if

$$u = 0 \; on \; the \; boundary \; \partial\Omega,$$

Then $u \equiv 0 \; within \; \Omega$

Proof: Recall that if $u \in c^2(\Omega)$ has local maximum $x_0 \in \Omega$. Then

$$\sum_{i=1}^{n} \frac{\partial^2 u}{\partial {x_i}^2}(x_0) \leq 0.$$

. Likewise if u has local minimum at $x_0 \in \Omega$. Then

$$\sum_{i=1}^{n} \frac{\partial^2 u}{\partial {x_i}^2}(x_0) \geq 0.$$

Suppose $u(x) > 0$ at some point $x \in \Omega$,then since $u \in c(\overline{\Omega})$ and $u = 0 \; on \; \partial\Omega$, u will attain its maximum value at some point $y \in \Omega$.

$$\Rightarrow u(y) > 0$$

Now $\sum_{i=1}^{n} \frac{\partial^2 u}{\partial {x_i}^2}(y) = u^3(y) > 0$

But since u takes its maximum at $y \in \Omega$ we know that

$$\sum_{i=1}^{n} \frac{\partial^2 u}{\partial {x_i}^2}(y) \leq 0$$

This contradicts equation

Hence $u \leq 0$ (*)

Suppose now $u \leq 0$ at some point in Ω since $u = 0 \; on \; \partial\Omega$ and $u \in c(\overline{\Omega})$, u takes its minimum value at some point $z \in \Omega$.

$$\Rightarrow u(z) \leq 0$$

$$\sum_{i=1}^{n} \frac{\partial^2 u}{\partial {x_i}^2}(z) = u^3(z) \leq 0$$

However since u has minimum at $z \in \Omega$, we have $\sum_{i=1}^{n} \frac{\partial^2 u}{\partial {x_i}^2}(z) \geq 0$ which contradicts

Hence $u \leq 0$ (**)

From (*) & (**) we have

$$u \equiv 0.$$

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