A Graduate Seminar Report For Mathematics

on

Stability Theory for Differential Equations

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CONTENTS

PREFACE

The primary purpose of differential equation is to serve as a tool for the study of change in the physical world as nothing is permanent except change.

Both types of differential, i.e., Ordinary Differential Equations and Partial Differential Equations accordingly are applied in the investigation of problems for stability.

Systems of differential equations occur in various practical problems and their theory includes that of single equations. We must face the fact that it is usually difficult, if not even impossible, to find a solution of a given differential equation in a reasonably convenient and explicit form, especially if the differential equation is non-linear. Therefore, it is important to consider what is non-linear. Therefore, it is important to consider what qualitative information, rather than detailed quantitative information, can be obtained about the solutions of differential equations, particularly non-linear ones, with out actually solving the equations.

The representation of real world in mathematical terms so as to gain a more precise understanding of its significant properties, and which may also allow some form of prediction of future events is one branch of mathematics. This is the art of mathematics and we call it Mathematical Modelling. So, the general ideas of stability are fundamental for our perception of the universe and hence the fifth chapter deals with the structural stability of a model in a dynamical system.

The problems considered in this paper are mainly associated with the idea of "stability" of a solution. The second type of problem arises when a non-linear equation is approximated by a simpler linear one.

Examples in each of the subtopics for solving systems of differential equations, particularly those involving stability problems and finally some selected applications related to the topics are included.

I would like to express my deep appreciation to my advisor and instructor Dr. Tsegaye Gedif for the helpful suggestions and comments he has given me to have the seminar this form.

I am also grateful to W/t Tezerash Mare, the web master of HPR, for here patience and care in computer typesetting.

Finally, I would like to take this opportunity to thank my wife Abay Akemachew for her moral support throughout the courses.

1.Lyapunov Stability

1.1. Basic concepts and definitions

1. A critical point is *stable* if all paths that get sufficiently close to the point stays close to the point.
2. A critical point said to be *asymptotically stable* if it is stable and there exists a circle $x^2 + y^2 = r_0^2$ such that every path which is inside this circle for some $t = t_0$ approaches the origin as t tends to infinity.
3. A critical point that is not stable is *unstable.*
4. A system of the form $\begin{cases} x' = F(x, y) \\ y' = G(x, y) \end{cases} \quad \left({}' = \frac{d}{dt}\right)$ (*)

in which the independent variable doesn't explicitly appear in the functions F and G is said to be an *autonomous system.* The system can also be rewritten as $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where A is a 2x2 constant matrix. If the elements of the coefficient matrix **A** are a function of the independent variable t, then it is *non-autonomous.*

$$\mathbf{x}_i' = f_i(x_1, x_2, ..., x_n, t), i = 1,2,3,..., n \tag{1}$$

be given.

A solution $\varphi_i(t)$, i = 1,2,…, n of system (1) above satisfying the initial conditions $\varphi_i(t_0) = \varphi_{i0}$, i = 1,2,…,n, is said to be a *Lyapunov stable solution* as t→ ∞ if for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for each solution $x_i(t)$,i=1,2,3,…,n, of system (1) whose initial value satisfy the conditions

$$|x_i(t_0) - \varphi_{i0}| < \delta, i = 1,2,3,\dots,n \tag{2}$$

the inequalities

$$|x_i(t) - \varphi_i(t)| < \varepsilon, i = 1,2,3,\dots,n \tag{3}$$

hold for all $t \geq t_0$.

If for an arbitrary small $\delta > 0$ inequalities (3) fail to hold for at least one solution $x_i(t)$, i=1,2,…, n, then the solution $\varphi_i(t)$ is said to be *unstable.*

If under condition (2) besides inequalities (3) the conditions

$$\lim_{t\to\infty} |x_i(t) - \varphi_i(t)| = 0, i = 1,2,\dots, n \tag{4}$$

also holds, then the solution $\varphi_i(t)$, i = 1,2,…, n, is said to be asymptotically *stable.*

Investigating a solution $\varphi_i(t)$, i =1,2,…, n, of system (1) for stability can be reduced to investigating for stability the zero (trivial) solution $x_i \equiv 0$, i=1,2,…,n, of some system similar to system (1),

$$x_i' = F_i(x_1, x_2, \ldots, x_n, t), i = 1,2,3,\ldots,n \qquad (1')$$

where $F_i(0,0,\ldots,0,t) \equiv 0$, i =1,2,3,…,n.

5. A point $x_i = 0$, i=1,2,…, n, is said to be a *stationary point* of system (1').

The definitions of stability and instability as applied to the stationary point can be formulated as follows. A stationary point, $x_i = 0$, i = 1,2,…, n, is stable according to Lyapunov if whatever $\varepsilon > 0$ there exists $\delta > 0$ such that for any solution $x_i(t)$, i=1,2,…,n, whose initial data $x_{i0} = x_i(t_0)$,i=1,2,…, n, satisfy the condition

$$|x_{i0}| < \delta, i = 1,2,3,\ldots, n, \qquad (2')$$

the inequalities

$$|x_i(t)| < \varepsilon, i = 1,2,\ldots, n \qquad (3')$$

hold for all $t \geq t_0$

If besides inequalities (3') the condition

$\lim\limits_{t\to\infty}|x_i(t)| = 0$,i = 1,2,…, n, also holds, then the stability is asymptotic.

A stationary point $x_i = 0$, i = 1,2,3,…, n, is unstable if for an arbitrary small $\delta > 0$ condition (3') doesn't hold for at least one solution $x_i(t)$, i =1,2,…, n.

The concepts of stability, asymptotic stability and instability can be easily visualized in terms of an oscillating pendulum. The equation of an oscillating pendulum with a mass *m* attached to one end of a rigid, but weightless rod of length *l* and its position by the angle θ subjected in a damping force can be derived from the principle of angular momentum and the governing equation is $ml^2 \frac{d^2\theta}{dt^2} = -cl\frac{d\theta}{dt} - mgl\sin\theta$, where $c\left|\frac{d\theta}{dt}\right|$ is the damping force.

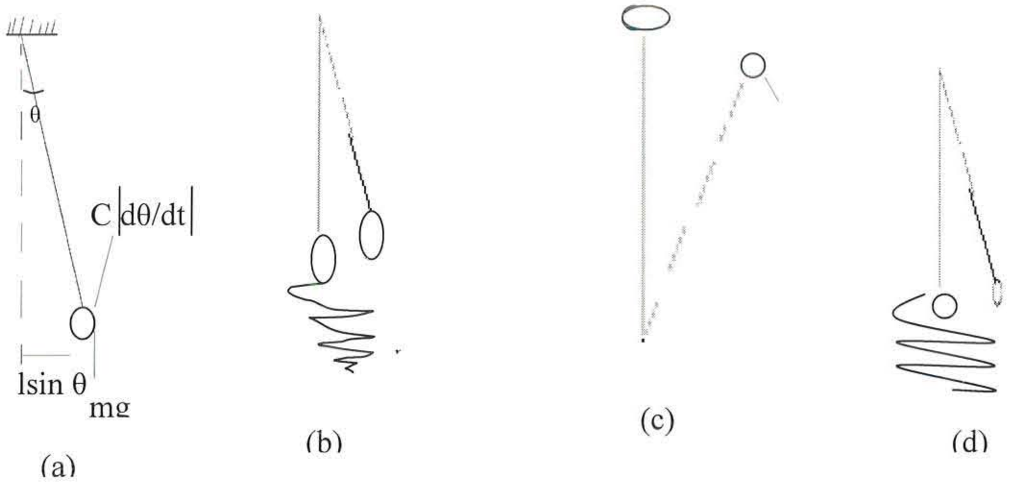


Diagram of an oscillating pendulum

The straightforward algebraic operations will give us in its stand form

$$\frac{d^2\theta}{dt^2}+\frac{c}{ml}\frac{d\theta}{dt}+\frac{g}{l}\sin\theta=0 \qquad (5)$$

To convert equation (5) to a system of two first order equations we let x= θ and $y=\frac{d\theta}{dt}$, then $\frac{dx}{dt}=y\,,\frac{dy}{dt}=-\frac{g}{l}\sin x-\frac{c}{ml}y$ (6)

Since *c, g, l* and *m* are all constants, the system (6) is an autonomous system of the form (*). Its critical points are found by solving the equations y = 0, $\frac{-g}{l}\sin x-\frac{c}{ml}y=0$. Consequently, y = 0 and x = ± nπ, where n is an integer. These correspond to two physical equilibrium positions, one with the mass directly below the point of support (θ = 0) and the other with the mass directly above the point of support (θ =π).

Our intuition suggests that the first is stable and the second is unstable.

More precisely, if the mass is slightly displaced from the lower equilibrium position, it will oscillate back and forth with gradually decreasing amplitude, eventually approaching the equilibrium position as the initial potential energy is dissipated by the damping force. This type of motion illustrates *asymptotic stability.*

On the other hand, if the mass is slightly displaced from the upper equilibrium position, it will rapidly fall, under the influence of gravity, and never again return to or approach the original equilibrium position. Indeed, the mass will ultimately approach the lower equilibrium position in this case also. This type of motion illustrates *instability*. In practice, it is impossible to maintain the pendulum in its upward equilibrium position for any length of time without an external constraint mechanism since the slightest perturbation will cause the mass to fall.

Finally, consider the ideal situation in which the damping coefficient c is zero. In this case, if the mass is displaced slightly from its lower equilibrium position, it will oscillate indefinitely with constant amplitude about the equilibrium position. Since there is no dissipation in the system, the mass will remain near the equilibrium position, but will not approach it asymptotically. This type of motion is *stable,* but not asymptotically stable. In general, this motion is impossible to achieve experimentally because the slightest degree of air resistance or friction at the point of support will eventually cause the pendulum to approach its rest position.

These three types of motion of a pendulum are illustrated schematically in the above figures (b) with air resistance, (c) with or without air resistance and (d) without air resistance.

Example 1: using the definition of Lyapunov stability, investigate for stability the solution of the equation $x(t) = 1+ t-x$ satisfying the initial condition $x(0) = 0$.

Solution: The given equation is a non-homogeneous linear equation and its general solution is $x(t) = ce^{-t} + t$. The initial condition $x(0) = 0$ is satisfied by the solution $\varphi(t) = t$. If $x(0) = x_0$, then $x(t) = x_0e^{-t} + t$. Now, consider the difference of the above solutions and we have $x(t) - \varphi(t) = x_0e^{-t} + t-t = (x_0-0)\, e^{-t}$. Hence it is seen that for any $\varepsilon > 0$, there exists $\delta > 0$ (say $\delta = \varepsilon$) such that for any solution $x(t)$ of the equation whose initial values satisfy the condition $|x_0-0| < \delta$ the inequality $|x(t)- \varphi(t)| = |x_0-0|\, e^{-t} < \varepsilon$, holds for all $t \geq 0$.

Therefore, the solution $\varphi(t) = t$ is stable. Moreover, since

$\lim_{t\to\infty}|x(t) - \varphi(t)| = \lim_{t\to\infty}|x_0 - 0|e^{-t} = 0$, the solution $\varphi(t) = t$ is asymptotically stable. But the solution $\varphi(t) = t$ is unbounded when $t \to +\infty$. This shows that the stability of the solution of a differential equation doesn't imply the boundness of the solution because of the characteristic of non-linear equations and systems, and the converse is not also true.

Example 2: Show that the solution of the system

$$\begin{cases} x' = -y \\ y' = x \end{cases}$$

satisfying the initial conditions $x(0) = 0$, $y(0) = 0$ is stable using the definition of Layapunov stability.

Solution: The solution of the system satisfying the given initial conditions is $x(t) \equiv 0$, $y(t) \equiv 0$. Any solution of the system satisfying the conditions $x(0) = x_0$, $y(0) = y_0$ is of the form $x(t) = x_o \cos t - y_0 \sin t$, $y(t) = x_0 \sin t + y_o \cos t$. Not we shall take an arbitrary $\varepsilon > 0$ and show that there exists $\delta(\varepsilon)>0$ such that for $|x_0-0| < \delta$, $|y_0-0| <\delta$ the inequalities

$$| x(t)- 0| = |x_0 \cos t-y_0 \sin t| < \varepsilon$$

$$| y(t)- 0| = |x_0 \sin t + y_0 \cos t| < \varepsilon, \text{ hold for all } t \geq 0.$$

This exactly means according to the definition that the zero solution $x(t) \equiv 0$, $y(t) \equiv 0$ of the system is a Lyapunov stable solution. Obviously, we have

$|x_0 \cos t - y_0 \sin t| \leq |x_0 \cos t| + |y_0 \sin t| \leq |x_0| + |y_0|$

$|x_0 \sin t + y_0 \cos t| \leq |x_0 \sin t | +| y_0 \cos t | \leq |x_0 | + |y_0|$, for all t.

Therefore, if $|x_0| + |y_0| < \varepsilon$, then so much the more

$|x_0 \cos t - y_0 \sin t| < \varepsilon$, $|x_0 \sin t + y_0 \cos t| < \varepsilon$ for all t.

Consequently, if we choose $\delta(\varepsilon) = \varepsilon/2$, the above inequalities will hold for all $t \geq 0$ when $|x_0| < \delta$ and $|y_0| < \delta$, i.e., the zero solution of the system is a Lyapunov stable solution, but its stability is not asymptotic.

Example 3: Consider the non-linear equation $x' = 1 - x^2(t)$. It has obvious solutions $\varphi(t) = -1$ and $\varphi(t) = 1$. The former solution is unstable and the latter is asymptotically stable as for $t \to +\infty$ all solutions of the equation

$$x(t) = \frac{(1+x_0)e^{2(t-t_0)}-(1-x_0)}{(1+x_0)e^{2(t-t_0)}+(1-x_0)}, (x_0 \neq -1), \text{ tends to } 1.$$

1.2 The simplest types of stationary points

Since many differential equations cannot be conveniently solved by analytical methods, we consider the idea of stability of a solution by geometrical analysis as follows.

1.2.1 Stability Properties of Linear Systems

Consider a system of two homogeneous linear differential equations with constant coefficients:

$$\begin{cases} x_1' = \alpha_{11}x_1 + \alpha_{12}x_2 \\ x_2' = \alpha_{21}x_1 + \alpha_{22}x_2 \end{cases} \quad \text{with} \quad \begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{vmatrix} \neq 0 \tag{1}$$

Such a system can also have the form

$$\mathbf{x}' = \mathbf{A}\mathbf{x} \tag{2}$$

where **A** is a 2x2 matrix and **x** is a 2x1 column vector.

If we seek solutions of the form $\mathbf{x} = \xi e^{\lambda t}$, then by substitution for **x** in equation (2) we find that $(\mathbf{A} - \lambda\mathbf{I})\,\xi = 0$, (3)

Thus λ must be an eigenvalue and ξ a corresponding eigenvector of the coefficient matrix **A**. The eigenvalues are the roots of the polynomial equation

$$\det(\mathbf{A} - \lambda\,\mathbf{I}) = 0 \tag{4}$$

and the eigenvectors are determined from equation (3).

The points for which the right side of equation (2) is zero are called *critical points.* These critical points are points where $\mathbf{Ax} = 0$. Since $\mathbf{x}' = \mathbf{0}$ also at such points, they correspond to constant solutions, or *equilibrium solutions*, of the differential equation (2). We will assume that **A** is non-singular, or that det $\mathbf{A} \neq 0$. It follows that $\mathbf{x} = 0$ is the only critical point of the system (2). Now, a vector function $x = \phi(t)$ is a solution for it and can be viewed as a parametric representation for a curve in the x_1x_2 – plane regarded as the path or trajectory. The x_1x_2- plane is called the *phase plane* and the set of trajectories is referred to as a *phase portrait.*

We consider different cases depending on the nature of the eigenvalues λ_1 and λ_2 of **A**. The general solution of equation (2) above is $\mathbf{x} = \mathbf{c}_1\xi_1\mathbf{e}^{\lambda_1 \mathbf{t}} + \mathbf{c}_2\xi_2\mathbf{e}^{\lambda_2 \mathbf{t}}$ where λ_1 and λ_2 are the eigenvalues and ξ_1 and ξ_2 are the eigenvectors.

Case i) Real unequal eigenvalues of the same sign: In this case λ_1 and λ_2 are either both positive or both negative. Suppose first that $\lambda_1 < \lambda_2 < 0$, and that the eigenvectors ξ_1 and ξ_2 are as shown in the figure below.

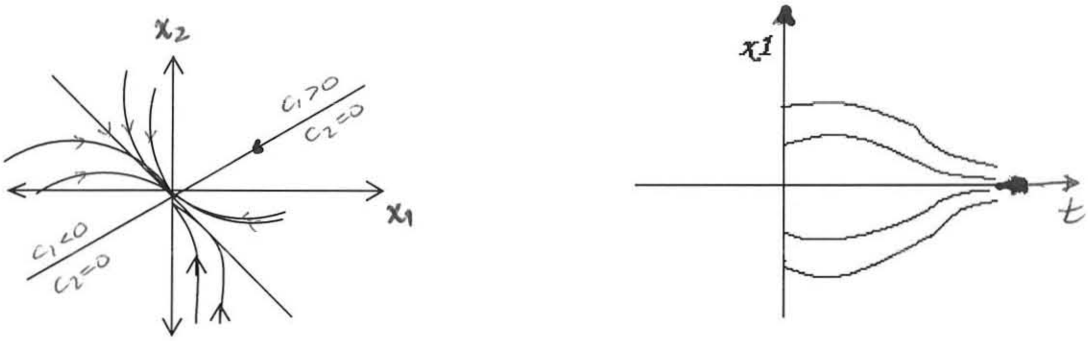


Figure 1 (a) the phase plane (b) x_1 V_s t

It follows that $\mathbf{x} \to \mathbf{0}$ as $t \to \infty$ regardless of the values of c_1 and c_2 , i.e., all solutions approach the critical point at the origin as $t \to \infty$. This type of critical point is called a node or sometimes an *improper node*.

If λ_1 and $\lambda_2$are both positive and $0 < \lambda_2 < \lambda_1$, then the trajectories have the same pattern as in figure (1), but the direction of motion is way from, rather than toward, the critical point at the origin. In example of an improper node occurs in the equation

$\mathbf{x}' = \begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix} \mathbf{x}$. Assuming that $\mathbf{x}=\xi e^{rt}$ is the solution, then

$\begin{pmatrix} -3-\lambda & \sqrt{2} \\ \sqrt{2} & -2-\lambda \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ gives the eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -4$ and the

corresponding eigen functions are $\xi_1 = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$ and $\xi_2 = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}$. Hence a fundamental set

of solutions of the system is $x_1(t) = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} c_1 e^{-t}$, $x_2(t) = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{-4t}$ and the general

solution is $x = c_1 x_1(t) + c_2 x_2(t) = c_1 \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{-4t}$. So, its trajectories are like

that of figure (1) above and the critical point is an unstable node.

Case ii) Real eigenvalues of opposite sign: suppose $\lambda_1 > 0$ and $\lambda_2 < 0$ and the eigenvectors ξ_1 and ξ_2 are as shown in the figure (2) below.

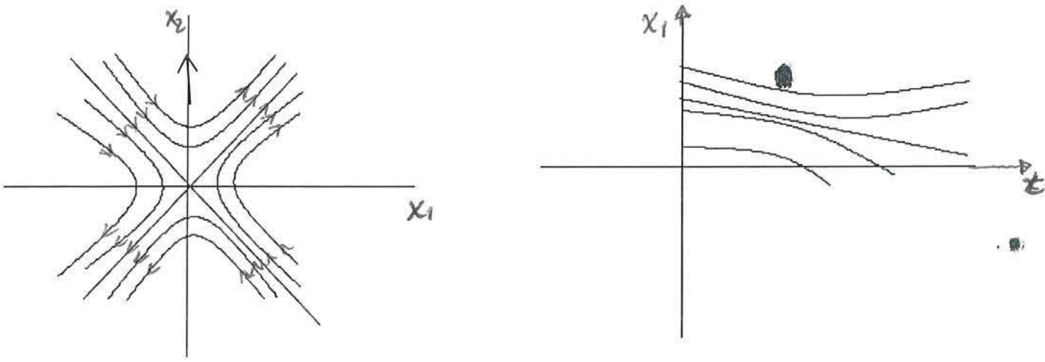


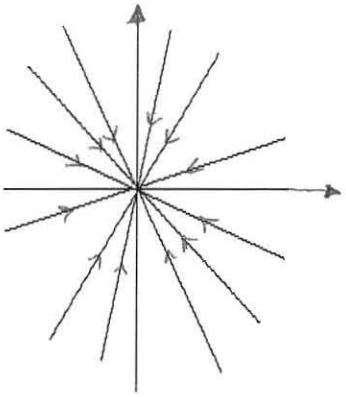
figure 2

If the solution starts at initial point on the line through ξ_1, then it follows that $c_2=0$. Consequently the solution remains on the line through ξ_1 for all t, and since $\lambda_1>0$, $\|\mathbf{x}\| \to \infty$ as $t \to \infty$. (Here $\|\cdot\|$ means magnitude). If the solution starts at an initial point on the line through ξ_2, then the situation is similar except $\|\mathbf{x}\| \to 0$ as $t \to \infty$ because $\lambda_2<0$. The origin in this case is called a *saddle point*. A specific example of a saddle point is for the system

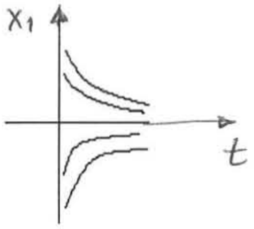
$$x' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} x. \text{ Here } \lambda_1 = 3, \lambda_2 = -1, \quad \xi_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \text{ and } x_1(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}, \quad x_2(t) = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$$

and hence the general solution $x = c_1x_1(t) + c_2x_2(t)$.

Case iii) Equal eigenvalues: we now suppose that $\lambda_1 = \lambda_2 = \lambda$ and the general solution is $x = c_1\xi_1 e^{\lambda t} + c_2 \xi_2 e^{\lambda t}$, where ξ_1 and ξ_2 are the independent vectors. The ratio x_2/x_1, is independent of t, but depends on the components of ξ_1 and ξ_2, and on the arbitrary constants c_1 and c_2. Thus every trajectory lies on a straight line through the origin as in the figure below for $\lambda < 0$.



fig(3): the phase plane



x_1 v_s t.

The critical point here is called a *proper node.* If $\lambda>0$, the trajectories are similar but the direction of motion is reversed, i.e., away from the critical point and hence unstable.

Case iv) Complex eigenvalues: Suppose that the eigenvalues are $\lambda_1 = p + iq$, $\lambda_2 = p - iq$, where p, q are real and $q > 0$. Then the general solution of equation (2) is

$$\begin{cases} x_1 = e^{pt}(c_{11}\cos qt + c_{12}\sin qt) \\ x_2 = e^{pt}(c_{21}\cos qt + c_{22}\sin qt) \end{cases}$$ when two of the constants are independent.

The critical point here is called a *spiral point* because the trajectories for $p \neq 0$ are always spirals and specifically if

a) $q \neq 0$, $p < 0$, the critical point is asymptotically stable. See figure (4)

b) $q \neq 0$, $p > 0$,the critical point is unstable. See figure (5)

c) $q \neq 0$, $p = 0$ (purely imaginary roots), the critical point is called a center. See figure(6)

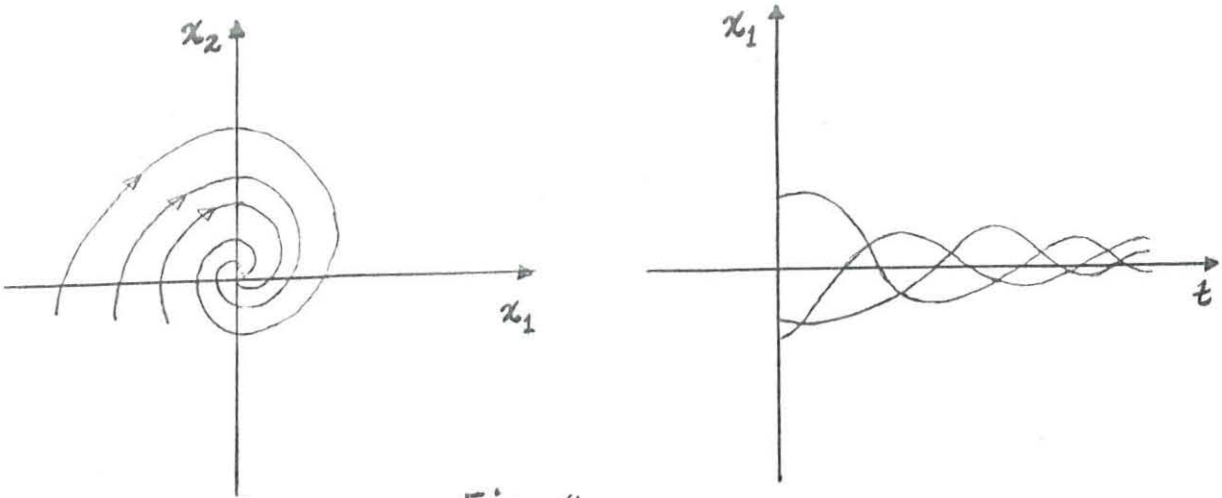


Fig. 4

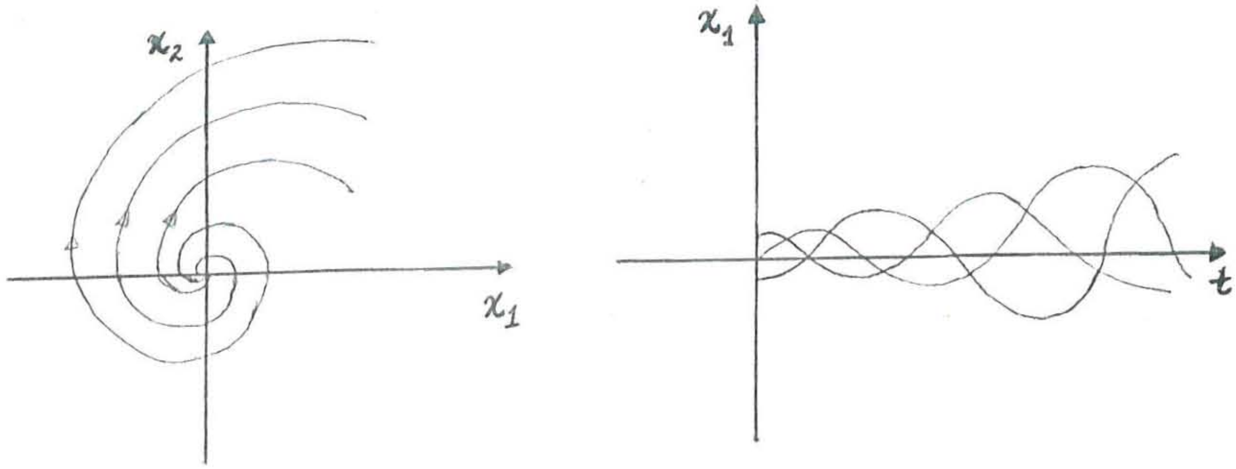


Fig. 5

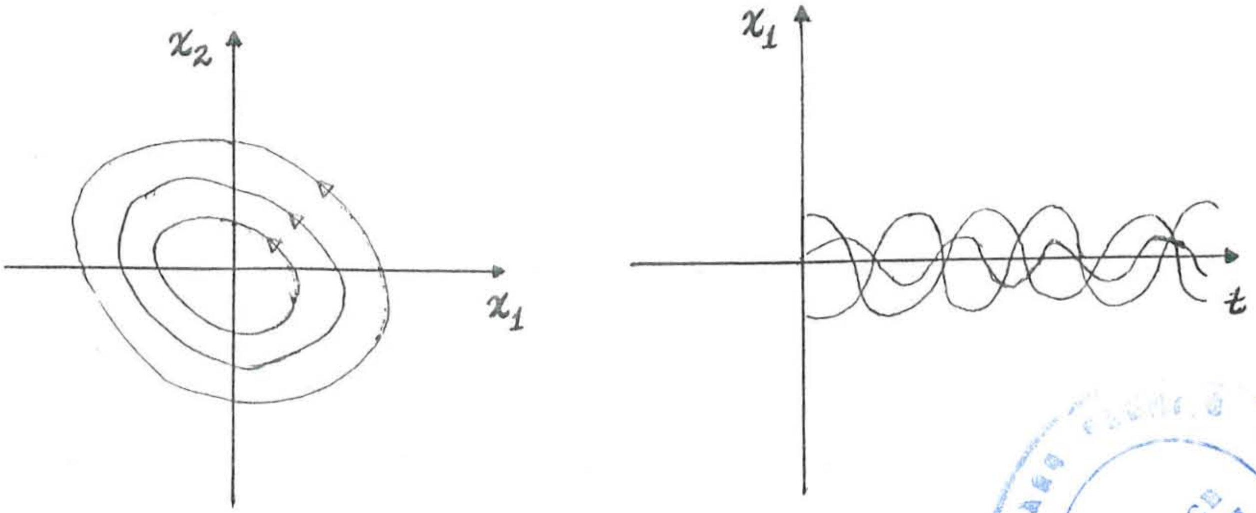


Fig. 6

A table for the summary of stability properties of linear systems

[det(**A**-**I**λ) = 0,det **A** = **0** for **x**$'$ = **Ax**:]

Eigenvalues	Types of Critical Points	Stability type
$\lambda_1 > \lambda_2 > 0$	Improper node	Unstable
$\lambda_1 < \lambda_2 < 0$	Improper node	Asymptotically stable
$\lambda_2 < 0 < \lambda_1$	Saddle point	Unstable
$\lambda_1 = \lambda_2 > 0$	Proper or improper node	Unstable
$\lambda_1 = \lambda_2 < 0$	Proper or improper node	Asymptotically stable
$\lambda_1, \lambda_2 = P \pm iq$	Spiral point	
$p > 0$		Unstable
$p < 0$		Asymptotically stable
$\lambda_1 = iq, \lambda_2 = -iq$	Center	Stable

Examples: Determine the character of the critical points **0**(0,0) of the systems:

1) $$\begin{cases} x_1' = -x_1 \\ x_2' = 2x_2 \end{cases}$$

The general solution of the system is

$$\begin{cases} x_1 = c_1 e^{-t} \\ x_2 = c_2 e^{2t} \end{cases} \Rightarrow \quad x_2 = \frac{c_2 c_1^2}{x_1^2}, \; x_1 \neq 0$$

Therefore, the trajectory will be a saddle point which is unstable.

2) $$\begin{cases} x_1' = -x_1 + 2x_2 \\ x_2' = -2x_1 - x_2 \end{cases}$$

The general solution of the given system is

$$\begin{cases} \mathbf{x_1 = e^{-t}(c_1 \cos 2t + c_2 \sin 2t)} \\ \mathbf{x_2 = e^{-t}(c_2 \cos 2t - c_1 \sin 2t)} \end{cases}$$

Hence, the critical point is asymptotically stable.

3) $$\begin{cases} \mathbf{x}_1' = 5\mathbf{x}_1 - \mathbf{x}_2 \\ \mathbf{x}_2' = 2\mathbf{x}_1 + \mathbf{x}_2 \end{cases}$$

Its eigenvalues are $\lambda_1 = 3 + \sqrt{2} > 0$ and $\lambda_2 = 3 - \sqrt{2} > 0$ so that both are real distinct and positive. Therefore, the critical point **0**(0,0) is an unstable node.

1.2.2 The almost linear systems

Consider a non-linear two-dimensional autonomous system $\mathbf{x}' = f(\mathbf{x})$ (1')

In order to examine the behavior of the trajectories of the system (1) in the neighbourhood of a critical point $\mathbf{x}^0$, it is convenient to choose the critical point to be the origin. If $\mathbf{x}^0 \neq \mathbf{0}$, it is always possible to make the substitution $\mathbf{u} = \mathbf{x} - \mathbf{x}^0$ in equation (1') so that **u** will satisfy an autonomous system with a critical point at the origin. We need to consider equation (1') that are close, in some appropriate sense to a linear system **x'=Ax**. Accordingly let us suppose that

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + g(\mathbf{x}) \tag{2'}$$

and that $\mathbf{x} = \mathbf{0}$ is an isolated critical point of the system (2').

This means that there is some circle about the origin within which there are no other critical points. We assume that the component of g have continuous partial derivatives and satisfy the limit condition

$$\frac{\|g(\mathbf{x})\|}{\|\mathbf{x}\|} \to 0 \textbf{ as } \mathbf{x} \to 0; \text{i.e., } \|\mathbf{g}\| \text{ is small in comparison to } \|\mathbf{x}\|$$

itself near the origin. Such a system is called an *almost linear system* in the neighbourhood of the critical point $\mathbf{x} = \mathbf{0}$.

If we let in scalar form $\mathbf{x} = (x,y)$ and $g(\mathbf{x}) = (g_1(x,y), g_2(x,y))$, then $\|\mathbf{x}\| = \sqrt{x^2 + y^2} = r$ and

$\|g(\mathbf{x})\| = \sqrt{g_1^2(x,y) + g_2^2(x,y)}$

Then it follows that $\frac{g_1(x,y)}{r} \to 0, \frac{g_2(x,y)}{r} \to 0$ as $r \to 0$

Example: Determine whether the system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & .5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -x^2 - xy \\ -0.75xy - 0.25y^2 \end{pmatrix}$$ is an almost linear in the neighbourhood

of the origin.

The system is of the form (2') with (0,0) a critical point and det $\mathbf{A} \neq 0$. It can be shown that (0,2), (1,0) and (0.5,0.5) are its other critical points; consequently, the origin is an isolated critical point. In checking the condition for $g_1$and g_2, let $x = r\cos\theta$, $y = r\sin\theta$ (polar coordinates). Then

$$\frac{g_1(x,y)}{r} = \frac{-x^2 - xy}{r} = \frac{-r^2\cos^2\theta - r^2\sin\theta\cos\theta}{r} = -r(\cos^2\theta + \sin\theta\cos\theta) \to 0 \text{ as } x \to 0.$$

In a similar way, one can show that $\frac{g_2(x,y)}{r} \to 0$ as $r \to 0$.

Hence, the given system is almost linear near the origin.

The undamped motion of a pendulum described earlier with c=0 is a practical example of an almost linear near the origin.

Let us now write the general non-linear system (1^1) in the scalar form

$$x' = F(x,y),\ y' = G(x,y) \tag{3'}$$

System (3^1) is almost linear in the neighbourhood of a critical point (x_0,y_0) whenever the function F and G have continuous partial derivatives up to order two. To show this, use Taylor's expansions about the point $(x_0,y_0)$to write F(x,y) and G(x,y) in the form

$$F(x,y) = F(x_0,y_0) + F_x(x_0,y_0)\,(x-x_0) + F_y(x_0,y_0)\,(y-y_0) + \eta_1(x,y)$$

$$G(x,y) = G(x_0,y_0) + G_x(x_0,y_0)\,(x-x_0) + G_y(x_0,y_0)\,(y-y_0) + \eta_2(x,y)$$

Where $\dfrac{\eta_1(x,y)}{[(x-x_0)^2]+(y-y_0)^2]^{\frac{1}{2}}} \to 0$ as $(x,y) \to (x_0,y_0)$ and similarly for η_2.

Note that $F(x_0,y_0) = G(x_0,y_0) = 0$ and $x' = (x-x_0)'$, $y' = (y-y_0)'$. Then $(3')$ reduces to

$$\frac{d}{dt}\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix} = \begin{pmatrix} F_x(x_0,y_0) & F_y(x_0,y_0) \\ G_x(x_0,y_0) & G_y(x_0,y_0) \end{pmatrix}\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix} + \begin{pmatrix} \eta_1(x,y) \\ \eta_2(x,y) \end{pmatrix}$$ or in vector notation;

$$\frac{\mathbf{du}}{\mathbf{dt}} = \frac{\mathbf{df}}{\mathbf{dx}}(\mathbf{x}^{\mathbf{0}}) + \boldsymbol{\eta}(\mathbf{x}),\ \text{where } \mathbf{u}^T = (x-x_0, y-y_0),\ \eta = (\eta_1,\eta_2)^T$$

This provides a general method for finding the linear system corresponding to an almost linear system near a given critical point. Its application on competing species and on predator-prey equations will be discussed in the last chapter.

1.3 Lyapunov's Direct Method

This method uses no knowledge of the solution of the system f differential equations; rather conclusions about the stability or instability of a critical point are obtained by constructing a suitable auxiliary function called *Lyapunov function* (**V**). The sign of **V** and its time derivative for the differential equation have to be considered and this technique is a very powerful one that provides a more global type of information.

Lyapunov's direct method is a generalization 0f two physical principles for conservative systems, namely

i. a rest position is stable if the potential energy is a local minimum, otherwise it is unstable, and

ii. the total energy is a constant during any motion.

Consider the autonomous system

$$\mathbf{x}' = F(x,y), \quad \mathbf{y}' = G(x,y) \tag{1}$$

And suppose that the point $x = 0$, $y = 0$ is an asymptotically stable critical point. Then there exists some domain D containing (0,0) such that every trajectory that starts in D must approach the origin at $t \to \infty$. Suppose that there exists an "energy" function **V** such that $\mathbf{V}(x,y) \geq 0$ for (x,y) in D with $\mathbf{V} = 0$ only at the origin. Since each trajectory in D approaches the origin as $t \to \infty$, then following any particular trajectory, $\mathbf{V} = 0 \to \infty$ as $t \to \infty$. But we need to prove the converse, i.e., if $\mathbf{V} \to 0$ as $t \to \infty$, then the trajectories must approach the origin as $t \to \infty$ and hence the origin as asymptotically stable.

Let **V** be defined on some domain D containing the origin. Then **V** is said to be *positive definite* on D if $\mathbf{V}(0,0) = 0$ and $\mathbf{V}(x,y) > 0$ for all other points in D. Similarly **V** is said to be *negative definite* on D if $\mathbf{V}(0,0) = 0$ and $\mathbf{V}(x,y) < 0$ for all other points in D. If the inequalities > and < are replaced by ≥ and ≤, then **V** is said to be *positive semi-definite and negative semi-definite*, respectively.

If $x = \phi(t)$ and $y = \psi(t)$ is a solution of the system (1), the

$$\frac{\mathbf{dV}}{\mathbf{dt}}[\phi(\mathbf{t}), \psi(\mathbf{t})]\frac{\mathbf{d}\phi(\mathbf{t})}{\mathbf{dt}} + \frac{\mathbf{dV}}{\mathbf{dt}}[\phi(\mathbf{t}), \psi(\mathbf{t})]\frac{\mathbf{d}\psi(\mathbf{t})}{\mathbf{dt}}$$

$$\Rightarrow \dot{\mathbf{V}}(x,y) = \mathbf{V}_x(x,y)\,F(x,y) + \mathbf{V}_y(x,y)\,G(x,y) \tag{2}$$

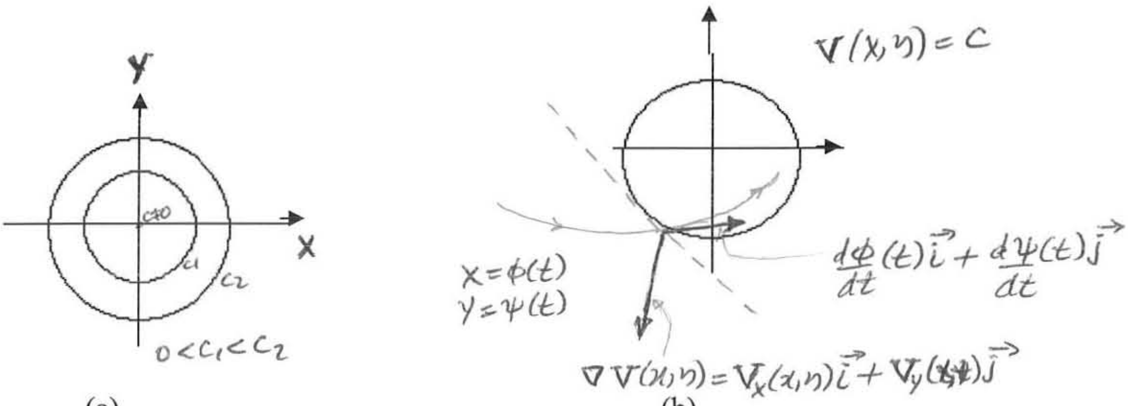
($\dot{}$ = total derivative)

We now state and prove Lyapunov Theorems

Theorem 1. (Lyapunov's stability theorem)

Suppose that the autonomous system (1) has an isolated critical point at the origin. If there exists a function **V** that is continuous and has a continuous first partial derivatives, is positive definite, and for which the function $\dot{\mathbf{V}}$ given by equation (2) is negative definite on some domain D in the xy-plane containing (0,0), then the origin is an asymptotically stable critical point. If $\dot{V}$ is negative semi-definite, then the origin is a stable critical point.

Proof: consider the second part of this theorem, i.e., the case $\dot{V} \le 0$. Let $c \ge 0$ be a constant and consider the curve in the xy-plan given by $\mathbf{V}(x,y) = c$. For $c = 0$, the curve reduces to the single point $x = 0$, $y = 0$. However, for $c > 0$ and sufficiently small, it can be shown by using the continuity of **V** that the curve is a closed curve containing the origin as illustrated in the figure below (a).



(a) (b)

Fig. (a) & (b): Geometrical interpretation of Lyapunov's method. We assume further that if $0 < c_1 < c_2$, then the curve $\mathbf{V}(x,y) = c_1$ lies within the curve $\mathbf{V}(x,y) = c_2$. We show that a trajectory starting inside a closed curve $\mathbf{V}(x,y) = c$ can not cross to the outside. Thus, given a circle of radius ε about the origin, by taking c sufficiently small we can ensure that every trajectory starting inside the closed curve $\mathbf{V}(x,y) = c$ stays within the circle of radius ε; indeed, it stays within the closed curve $\mathbf{V}(x,y) = c$ itself.

Thus the origin is a stable critical point. (Geometric proof). To show this, recall from calculus that the vector

$$\nabla \mathbf{V}(x,y) = \mathbf{V}_x(x,y)\,\vec{\mathbf{i}} + \mathbf{V}_y(x,y)\,\vec{\mathbf{j}}, \qquad (3)$$

known as the gradient of **V**, is normal to the curve $\mathbf{V}(x,y) = c$ and points in the direction of increasing **V**. In the present case **V** decrease outward from the origin, so $\nabla \mathbf{V}$ points away from the origin as in figure (b) above. Next consider a trajectory $x = \phi(t)$, $y = \psi(t)$ of the system (1), and recall that the vector $T(t) = \phi'(t)\,\vec{\mathbf{i}} + \psi'(t)\,\vec{\mathbf{j}}$ is tangent to the trajectory at each point. Let $x_1 = \phi'(t_1)$, $y_1 = \psi'(t_1)$ be a point of intersection of the trajectory and a closed curve $\mathbf{V}(x,y) = c$. At this point $\phi'(t_1) = F(x_1,y_1)$, $\psi'(t_1)=G(x_1,y_1)$. So, from equation (2) we obtain

$$
\begin{aligned}
(x_1,y_1) &= \mathbf{V}_x(x_1,y_1)\ \phi'(t_1) + \mathbf{V}_y(x_1,y_1)\ \phi'(t_1) \\
&= [\mathbf{V}_x(x_1,y_1)\ \vec{\mathbf{i}} + \mathbf{V}_y(x_1,y_1)\ \vec{\mathbf{j}}].[\phi'(t_1)\ \vec{\mathbf{i}} + \phi'(t_1)\ \vec{\mathbf{j}}] \\
&= \nabla \mathbf{V}(x_1,y_1).\ \mathbf{T}(t_1)
\end{aligned}
$$

Thus $\dot{\mathbf{V}}(x_1,y_1)$ is the scalar product of the vector $\nabla \mathbf{V}(x_1,y_1)$ and the vector $\mathbf{T}(t_1)$. Since $\dot{\mathbf{V}}(x_1,y_1) \leq 0$, it follows that the cosine of the angle between $\nabla \mathbf{V}(x,y)$ and $\mathbf{T}(t_1)$ is also less than or equal to zero; hence the angle itself is in the range $[\frac{\pi}{2}, \frac{3\pi}{2}]$. Thus the direction of motion one the trajectory is inward with respect to $\mathbf{V}(x,y) = c$ or, at worst, tangent to this curve. Trajectories starting inside a closed curve $\mathbf{V}(x,y) = c$ (no matter how small c is) can not escape, so the origin is a stable point. If $\dot{\mathbf{V}}(x,y) < o$, then the trajectories passing through points on the curve are actually pointed inward. As a consequence, it can be shown that trajectories starting sufficiently close to the origin must approach the origin; hence the origin is asymptotically stable.

Theorem 2: (Lyapunov's instability theorem)

Let the origin be an isolated critical point of the autonomous system (1). Let $\mathbf{V}$ be a function that is continuous and has continuous first partial derivatives. Suppose that $\mathbf{V}(0,0) = 0$ and that in every neighborhood of the origin there is at least one point at which $\mathbf{V}$ is positive (negative). Then if there exists a domain D containing the origin such that the function $\dot{\mathbf{V}}$ as given in equation (2) is positive definite (negative definite) on D, then the origin is an unstable critical point.

Proof: its geometrical proof follows by somewhat similar arguments. Briefly, suppose that $\dot{\mathbf{V}}$ is positive definite an suppose that given any circle about the origin there is an interior point (x_1,y_1) at which $\mathbf{V}(x_1,y_1) > 0$. Consider a trajectory that starts at (x_1,y_1). Along this trajectory it follows from equation (2) that $\mathbf{V}$ must increase, since $\dot{\mathbf{V}}(x,y) > 0$; furthermore, since $\mathbf{V}(x_1,y_1) > 0$ the trajectory cannot approach the origin because $\mathbf{V}(0,0) = 0$. This shows that the origin cannot be asymptotically stable. By further exploiting the fact that $\dot{\mathbf{V}}(x,y) > 0$, it is possible to show that the origin is an unstable point; however, we will not pursue this argument.

Note that the difficulty in using these theorems is that they tell us nothing about how to construct a Lyapunov function **V**, but we simply assume that one exists. However, these theorems are applicable in cases where the concept of physical energy is not pertinent. Even though there is not general method for constructing *V*, in the simplest cases it may be sought in the form $\mathbf{V}(x,y) = ax^2 + by^2$, $\mathbf{V}(x,y) = ax^4 + by^4$, $\mathbf{V}(x,y) = ax^4 + by^2$, $(a > 0, b > 0)$, etc. These theorems can also be extended for any n system of equations.

Illustrative examples:

1. Consider the system $\begin{cases} x' = y \\ y' = -x \end{cases}$

we choose $\mathbf{V}(x,y) = x^2+y^2$. It is positive definite. $\dot{\mathbf{V}} = 2x.x' + 2y.y' = 2xy - 2xy = 0$ Therefore it follows from theorem (1) that the critical point **0**(0,0) of the system is stable. However, its trajectories are circles and they don't tend to the point **0**(0,0) as $t\to+\infty$ and hence it is not asymptotically stable.

2. Consider the system $\begin{cases} x' = y - x^3 \\ y' = -x - 3y^3 \end{cases}$

Taking again $\mathbf{V}(x,y) = x^2 + y^2$, we find that $\dot{\mathbf{V}} = 2x(y - x^3) + 2y(-x - 3y^3) = -2(x^4 + 3y^4)$. Thus **V** is a negative definite function. Then by theorem 1, the critical point **0**(0,0) is asymptotically stable

3. Investigate the critical point $x = 0$, $y = 0$ of the system.

$$\begin{cases} x' = x \\ x' = -y \end{cases} \quad \text{for stability.}$$

Take $\mathbf{V}(x,y) = x^2 - y^2$. Then $\dot{\mathbf{V}} = \mathbf{V}_x.\ x' + \mathbf{V}_y.\ y' = 2x^2 + 2y^2$, which is a positive definite function. All the condition of theorem (2) hold since arbitrary close to the origin of coordination there are points in which $\mathbf{V} > 0$ (eg. $\mathbf{V}(x,y) = x^2 > 0$ along the line $y = 0$). Hence, the critical point **0**(0,0) is unstable (a saddle point).

2. Stability in the first approximation

Let the following system of differential equations be given

$$\dot{x}_i = f_i(x_1,x_2,\ldots,x_n), i=1,2,\ldots,n \qquad (1)$$

and let $x_i=0, i=1,2,\ldots,n$, be a stationary point of system (1), i.e., $f_i\,(0,0,\ldots,0)=0, i=1,2,\ldots,n$. We shall assume that functions $f_i(x_1,x_2,\ldots,x_n)$can be differentiated a sufficiently large number of times at the origin of coordinates.

We expand the function in the Taylor series of x in the neighbourhood of the origin of coordinates: $f_i(x_1,x_2,\ldots,x_n) = \sum\limits_{j=1}^{n} a_{ij}x_j + R_i(x_1,x_2,\ldots,x_n)$, here $a_{ij} = \dfrac{\partial f_i}{\partial x_j}(0,0,...,0)$ and R_i are terms of the second order of smallness with respect to $x_1,x_2,\ldots,x_n$.

The original system (1) will then be written as

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij}x_j + R_i(x_1,x_2,\ldots,x_n)\ a_{ij}x_j + R_i(x_1,x_2,\ldots,x_n),\ i = 1,2,\ldots,n \qquad (2)$$

Instead of system (2) we shall consider the system

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij}x_j\,,\ (i = 1,2,\ldots,n) \qquad (a_{ij} + \text{constant}) \qquad (3)$$

called the system of equations of *the first approximation* for system (1).

The following propositions hold.

1. If all roots of the characteristic equation

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22}-\lambda & \ldots\ldots & a_{2n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots \\ a_{n1} & a_{n2} & \ldots\ldots & a_{nn} \end{vmatrix} = 0$$

(4)

have negative real parts, then zero solutions $x_i \equiv 0$, $i=1,2,\ldots,n$ of system (3) and system (2) are asymptotically stable.

2. If at least one root of the characteristic equation (4) has a positive real part, then the zero solution of system (3) and system (2) is unstable.

It is said that investigation of stability in the first approximation is possible in cases (1) and (2). In critical cases when the real parts of all roots of the characteristic equation (4) are non-positive, with the real part of at least one root being zero, investigation for stability in the first approximation is in general impossible . (Because non-linear terms R_i start to exert influence.)

Examples:

Investigate the critical point x = 0, y = 0 of the system

$$1.\ \begin{cases} x' = 2x + y - 5y^2 \\ y' = 3x + y + \frac{1}{2}x^3 \end{cases} \tag{5}$$

for stability in the first approximation.

The system of the first approximation is $\begin{cases} x' = 2x + y \\ y' = 3x + y \end{cases}$ (6)

the nonlinear terms satisfy the necessary conditions, their order being greater than or equal to two. We setup the characteristic equation of system (6).

$$\begin{vmatrix} 2-\lambda & 1 \\ 3 & 1-\lambda \end{vmatrix} = 0 \quad \Rightarrow \lambda^2\text{-}3\,\lambda\text{-}1\text{=}0 \tag{7}$$

The roots of the characteristic equation (7) $\lambda_1 = \frac{3+\sqrt{13}}{3}$, $\lambda_2 = \frac{3-\sqrt{13}}{3}$ are real and $\lambda_1 > 0$.

Therefore the zero solution x=0, y=0 of system (5) is unstable.

$$2.\ \begin{cases} x' = -x + 3y + x^2 \sin y \\ y' = -x - 4y + 1 - \cos y^2 \end{cases};$$

Its system of the first approximation is

$$\begin{cases} x' = -x + 3y \\ y' = -x - 4y \end{cases} \quad \text{and} \quad \begin{vmatrix} -1-\lambda & 3 \\ -1 & -4-\lambda \end{vmatrix} = 0$$

Both λ_1 & λ_2 are negative and hence the zero solution x = 0, y = 0 of this system is stable.

3. Criterion approaches for stability

3.1 The Routh-Hurwitz Criterion

Consider a linear differential equation with constant real coefficients:

$$a_0y^{(n)} + a_1y^{(n-1)} + \ldots + a_ny = 0, \quad (a_0, a_1, \ldots, a_n \text{ constant}, a_0 > 0) \tag{1}$$

The zero solution $y \equiv 0$ of equation (1) is asymptotically stable, if all the roots of the characteristic equation $f(\lambda) \equiv a_0\lambda^n + a_1\lambda^{n-1} + \ldots + a_n = 0$, (2)

have negative real parts.

The Routh-Hurwitz Criterion: for all roots of equation (2) to have negative real parts, it is necessary and sufficient that all the principal diagonal minors of the Hurwitz matrix

$$\left|\begin{array}{ccccccc} a_1 & a_0 & 0 & 0 & 0 & 0 \ldots\ldots & 0 \\ a_3 & a_2 & a_1 & a_0 & 0 & 0 \ldots\ldots & 0 \\ a_5 & a_4 & a_3 & a_2 & 0 & 0 \ldots\ldots & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 \ldots\ldots & a_n \end{array}\right| \tag{3}$$

should be positive.

A Hurwitz matrix is composed as follows. The coefficients of polynomial (2), from a_1 to an, are written out in the main diagonal. The columns consist in turn of coefficients with only odd or only even subscripts, with the coefficient a_0 included in the latter. All the other entries of the matrix corresponding to coefficients with subscripts greater than n or less than 0 are set equal to zero. The principal diagonal minors of the Hurwitz matrix are of the form

$$\Delta_1 = a_1, \quad \Delta_2 = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix}, \quad \Delta_3 = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix}, \ldots, \quad \Delta_n = \begin{vmatrix} a_1 & a_0 & 0 & \ldots & 0 \\ a_3 & a_2 & a_1 & \ldots & 0 \\ a_5 & a_4 & a_3 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & a_n \end{vmatrix}$$

Thus the Hurwitz condition states: for the solution $y \equiv 0$ of equation (1) to be stable it is necessary and sufficient that the relations $\Delta_1 > 0$, $\Delta_2 > 0, \ldots, \Delta_n > 0$ should hold.

Since $\Delta_n = a_n \Delta_{n-1}$, the condition $\Delta_n > 0$ may be replaced by the requirement $a_n > 0$

Example: Investigate the zero solution of the equation

$Y^{iv} + 5y''' + 13y'' + 19y' + 10y = 0$ for stability.

Solution: We set up the characteristic equation

$$F(\lambda) = \lambda^4 + 5\lambda^3 + 13\lambda^2 + 19\lambda + 10 = 0$$

Here, $a_0 = 1$, $a_1 = 5$, $a_2 = 13$, $a_3 = 19$, $a_4 = 10$. We write out Hurwitz diagonal minors

$$\Delta_4 = \begin{vmatrix} 5 & 1 & 0 & 0 \\ 19 & 13 & 5 & 1 \\ 0 & 10 & 19 & 13 \\ 0 & 0 & 0 & 10 \end{vmatrix} = 4240 > 0, \Delta_3 = \begin{vmatrix} 5 & 1 & 0 \\ 19 & 13 & 5 \\ 0 & 10 & 19 \end{vmatrix} = 424 > 0,\ \Delta_2 = \begin{vmatrix} 5 & 1 \\ 19 & 13 \end{vmatrix} = 46 > 0,$$

$\Delta_1 = 5 > 0$, i.e., $\Delta_i > 0, i = 1,2,3,4$.

Hence the trivial solution $y \equiv 0$ of this equation is asymptotically stable.

Computation can be carried out as follows. We first compose the highest Hurwitz minor Δ_n. Using it, it is easy to write out all the lower minors $\Delta_{n-1},\ldots, \Delta_1$. We then begin to compute Δ_1, Δ_2, etc. in succession. If we come across a negative minor, the solution is unstable and further computation is unnecessary. For instance, one can easily show that the zero solution of $y^v + 3y^{iv} - 5y''' - 15y'' + 4y' + 10y = 0$ is unstable.

3.2. The geometrical criterion (the Mikhailov criterion)

Consider an nth order linear differential equation with constant real coefficients

$$a_0y^{(n)} + a_1y^{(n-1)} + \ldots + a_ny = 0 \tag{1}$$

Its characteristic equation is

$$f(\lambda) \equiv a_0\lambda^n + a_1\lambda^{n-1} + \ldots + a_n = 0 \tag{2}$$

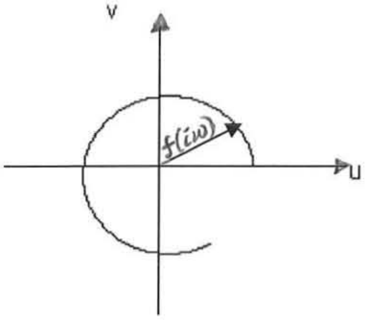
The Mikhailov criteria allows us to solve the equation about the location of the roots of the characteristic equation (2) in the complex plane and hence the question about the stability of the zero solution of equation (1). Setting $\lambda = iw$, we get

$f(iw) - u(w) + iv(w)$, where $u(w) = a_n - a_{n-2}w^2 + a_{n-4}w^4 + \ldots,$

$$v(w) = a_{n-1} - a_{n-3}w^3 + \ldots.$$

For a given value of the parameter w the quantity f(iw) can be represented as a vector in the u,v complex plane with the beginning at the origin of coordinates.

As w changes in the interval $(-\infty, \infty)$ the end of the vector describes some curve, the so-called *Mikhailov curve* as in the figure below.



Since the function u(w) is even, the Mikhailov curve is symmetrical with respect to the ou axis and so it is sufficient to construct that portion of the curve which corresponds to the change of the parameter w from o to ∞.

If the polynomial $f(\lambda)$ of degree n has m roots with a positive real part and n-m roots with a negative real part, then for the change of w from 0 to ∞ the angle of rotation φ of the vector f(iw) is equal to $\varphi = (n-2m)\ \frac{\Pi}{2}$.

It is clear that for a solution of equation (1) to be stable it is necessary and sufficient that m = 0.

The Mikhailov criterion: for the zero solution $y \equiv 0$ of equation (1) to be stable it is necessary and sufficient that

i. for the change of w from 0 to ∞ the vector f(iw) should rotate through an angle $\varphi = n\frac{\Pi}{2}$, i,e., it should make $\frac{\Pi}{4}$ rotations counterclockwise;
ii. the locus of f(iw) should not pass through the origin (0,0) as w changes from 0 to ∞.

If follows that for a solution of equation (1) to be stable it is necessary that all roots of the equation u(w) = 0, v(w) = 0 should be real and alternating, i.e. between any two roots of one equation there must be a root of the other equation.

Example: Investigate the zero solution $y \equiv 0$ of the equation

$y^{iv} + y''' + 4y'' + y' + y = 0$ for stability.

Solution: We set up the characteristic polynomial

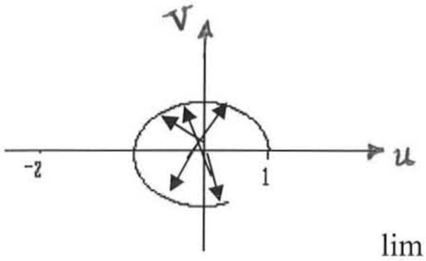
$$f(\lambda) = \lambda^4 + \lambda^3 + 4\lambda^2 + \lambda + 1$$

Further, $f(iw) = w^4 - iw^3 - 4w^2 + iw + 1$

Then $u(w) = w^4 - 4w^2 + 1$, $v(w) = -w^3 + w = w(1-w)(1+w)$

We construct the curve like this $u = u(w)$

$$V = v(w),\ 0 \leq w < \infty$$



$$\lim_{t \to \infty} \frac{v}{u} = 0$$

w	0	$\sqrt{2-\sqrt{3}}$	1	$\sqrt{2+\sqrt{3}}$
u	1	0	-2	0
v	0	$+$	0	$-$

The angle of rotation of the radius vector is $\varphi = 4\frac{\Pi}{2} = (n\text{-}2m)\ \frac{\Pi}{2}$.

Hence n-2m = 4 and, since n = 4, we have m = 0, i.e. all the roots of the characteristic equation are in the left-hand half-plane. So the trivial solution $y \equiv 0$ is asymptotically stable.

4. Equations with a small parameter of the derivative

Consider a differential equation $\frac{dx}{dt} = f(t, x(t), \varepsilon)$, (1)

where ε is a parameter.

If in some closed domain of t, x, ε the function $f(t,x,\varepsilon)$ is continuous with respect to the aggregate of variables and satisfies a Lipschitz condition in x:
$|f(t,x_2,\varepsilon) - f(t,x_1,\varepsilon)| \leq N|x_2-x_1|$, N being independent of t,x,ε ,then the solution of equation (1) is continuously dependent on ε.

In many physical problems one has to consider equations of the form

$$\varepsilon \frac{dx}{dt} = f(t, x) \quad (2)$$

ε being a small parameter.

On dividing both sides of equation (2) by ε we reduce it to the form $dx/dt = 1/\varepsilon f(t,x)$, (3) from which it is obvious that the right-hand side of (3) becomes discontinuous when $\varepsilon=0$, so that the theorem on the continuous dependence of solutions on the parameter ε can not be made use of in this case. We ask the question: under what conditions for small values of $|\varepsilon|$ is it possible to discard the term $\varepsilon dx/dt$ from (2) and consider the solution of the so-called "degenerate equation" $f(t,x)=0$ (4)
as an approximation to the solution of the differential equation (2). Let, for definiteness, $\varepsilon>0$ and let the degenerate solution (4) have only one solution $x = \varphi(t)$. Depending on the behaviour of $f(t,x)$ near the solution $x = \varphi(t)$ of (4) the solution $x(t,\varepsilon)$ of (2) tends to the solution $x = \varphi(t)$ of the degenerate equation or quickly recedes from it as $\varepsilon \to 0$. In the former case the solution $x = \varphi(t)$ is said to be stable in the latter case it is said to be unstable. That is, if in passing through the graph of the solution $x = \varphi(t)$ of the degenerate equation (4) the function $f(t,x)$ changes the sign from + to – as x increases with t fixed, then the solution of the degenerate equation, $x = \varphi(t)$, is stable and it can replace approximately the solution $x(t,\varepsilon)$ of (2)

If, however, the function $f(t, x)$ changes the sign from – to+, the solution $x = \varphi(t)$ of the degenerate equation (4) is unstable and so it is impossible to replace the solution $x(t,\varepsilon)$ of (2) by the solution of the degenerate equation (4).

Sufficient stability or instability conditions are expressed by the following propositions:

1. if $\frac{\partial f(t,x)}{\partial x} < 0$ for the solution x = $\varphi(t)$of (4), then the solution x = $\varphi(t)$ of the degenerate equation is stable.
2. if $\frac{\partial f(t,x)}{\partial x} > 0$ for the solution x = φ (t)of (4), then the solution x = φ (t)of the degenerate equation is unstable.

If the degenerate equation f(t,x) = 0 of (4) has several solutions x = $\varphi_i(t)$,i= 1,2,…,m, then each of them must be investigated for stability.

Example: Investigate the solution of the degenerate equation corresponding to the equation $\varepsilon dx/dt=x(e^x - 2)$ for stability

Solution: the degenerate equation x (e^x-2) has two solutions x = 0 and x = *ln*2. We have

$$\frac{\partial f(t,x)}{\partial x}\Big|_{x=0} = (e^x - 2 + xe^x)\Big|_{x=0} = -1 < 0$$

so that the solution x=o is stable and

$$\frac{\partial f(t,x)}{\partial x}\Big|_{x=\ln 2} = (e^x - 2 + xe^x)\Big|_{x=\ln 2} = 2\ln 2 > 0$$

so that the solution x=2*ln*2 of the degenerate equation is unstable.

5. Structural stability in Mathematical Models:

5.1 Introduction

We can approach a general definition of stability for a mathematical model by saying that a *model* is structurally stable if sufficiently small changes in the construction of the model itself will produce behaviour, which is in some sense qualitatively similar to the behaviour of the original model.

One of the simplest examples of stability, which we encounter in mathematics, is the notion of stable equilibrium for a dynamical system. It is important not only in mechanics, but also in the study of more general kinds of dynamical system – economic, biological, electrical and so on.

Suppose we have a system in equilibrium and the system itself is slowly changing by changing its various parameters. It is natural to expect that the equilibrium also changes slowly. As the parameters change, the equilibrium becomes less and less stable, until for a particular value or values of the parameters the equilibrium actually becomes unstable or even disappears altogether.

5.2 Systems governed by a smooth function

Given a dynamical system we can model its behaviour mathematically by picking out a number of observable and measurable quantities x_1, x_2,…,x_n, representing the state of the system at time t by the point $x(t)= (x_1(t),x_2(t),\ldots,x_n(t))$,in Euclidean n-space, and then looking at the kind of path that the point x(t) traces out as t changes.

If x (t) remains in one position for all time t, then the system is in equilibrium as far as the quantities x_i are concerned; if x (t) repeatedly traces out a closed loop, then the system is in some sense behaving in a periodic way, and so on. The information which we are given about the evolution of the system is in the form of a set of differential equations, telling us how the point x (t) varies, i.e. telling us what $\mathbf{x}$ (t)= dx(t)/dt is and perhaps also what the higher derivatives $\mathbf{x}(t)$ etc. are at each position and time. In this case, the equilibrium points are precisely those for which $\mathbf{x}(t)=0$ for all t.

Definition: A dynamical system is said to be governed by a *smooth function* **V** if **V** is a smooth (i.e. infinitely differentiable) function on the space of variables $(x_1,x_2,\ldots,x_n)$ such that

1. the equilibria of the system are precisely the critical points of **V** (the points where $\frac{\partial V}{\partial x_1} = \frac{\partial V}{\partial x_2} = \dots = \frac{\partial V}{\partial x_n} = 0$)

2. away from the equilibria the function **V**(x(t)) decreases as t increases.

Example: - Consider a biological cell whose behaviour is being influenced by the concentration x (t) of a certain chemical substance in the cell at each instant of time t. Suppose that the substance is entering the cell from outside at a constant rate S and is being destroyed inside the cell at a rate proportional to its concentration. This means that x (t) satisfies the differential equation $\dot{x} = -ax + S$

If there are two cells of exactly the same type, both influenced by the same substance but not interacting with each other in any way, then the concentrations $x_1(t), x_2(t)$ in each cell satisfy the equations

$$\dot{x}_1 = -ax_1 + S$$

$$\dot{x}_2 = -ax_2 + S$$

Now we complicate the model by assuming that the cells do interact at a rate, which is proportional to the difference in concentrations between the two. This converts the equations into the following system

$$\dot{x}_1 = -ax_1 + S + D(x_1 - x_2)$$

$$\dot{x}_2 = -ax_2 + S + D(x_2 - x_1)$$

Defining the function **V** by $V(x_1,x_2) = -S(x_1+x_2) + \frac{a}{2}(x_1^2+x_2^2) - \frac{D}{2}(x_1-x_2)^2$,

We see that

$$\dot{x}_1 = \frac{-\partial V}{\partial x_1} \quad , \quad \dot{x}_2 = \frac{-\partial V}{\partial x_2}$$

or simply, $\dot{x} = -\mathit{grad}\mathbf{V}$

This is an example of a *gradient system*, which is a particular case of a system governed by the potential function **V** for n = 2. In this case the only types of critical points, which are not removable by arbitrarily small perturbations of the function, are maxima, minima and saddle points. Similarly, for functions of n variables $x_1, x_2, \dots, x_n$, the important critical points are maxima, minima and (n-1) types of n-dimensional saddle point (with maximum in r dimensions, minimum in n-r dimensions, for each r =1,2,... (n-1))

Clearly the only such critical points which correspond to stable equilibria are the minima; in the other cases most (if not all) disturbances from equilibrium will cause the system to fall further a way from the equilibrium state.

5.3. Equilibria for systems controlled by k parameters

We consider the situation by supposing that the entire system depends on a number of parameters $c_1,c_2,\ldots,c_k$. For example, it may be that we have physical control over the system and can alter some of its properties (expressed by the c_i's as we wish). If we suppose that for each value of $c = (c_1,c_2,\ldots,c_k)$ the system is governed by a smooth function $\mathbf{V}_c$, then the equilibria of the system will vary with c. We make the following two assumptions about the way things depend on the parameters $(c_1,c_2,\ldots,c_k)$:

1. $\mathbf{V}_{(c_1,c_2,\ldots,c_k)}(x_1,x_2,\ldots,x_n)$ is a smooth function of $x_1,x_2,\ldots,x_n$, and of $c_1,c_2,\ldots,c_k$.
2. for each c the system governed by V_c attains equilibrium so rapidly that we can consider it as essentially instantaneous.

As the control point c is slowly moved around, the equilibrium state of the system usually varies slowly as well, but some times gives a sudden and possibly violent jump from one position to another. The explanation in terms of the potential function is that as c varies it is possible for a minimum and a maximum of $\mathbf{V}$ to coalesce and disappear. If the situation were in a state corresponding to the minimum which disappears it would have no option but to fly to the other minimum. Thom calls this a *catastrophe*.

Let $\mathbf{V}$ be any smooth function of n variables $(x_1,x_2,\ldots,x_n)$ with k parameters $(c_1,c_2,\ldots,c_k)$. Regarding $\mathbf{V}$ as governing a dynamical system, we call $(c_1,c_2,\ldots,c_k)$ – space the control space and we define the catastrophe set k to be the set of points $c=(c_1,c_2,\ldots,c_k)$ in the control space for which $\mathbf{V}_c$ (as a function of $x=(x_1,x_2,\ldots,x_n)$) has some coalescent critical points. When n =1,this means that k is the set of points c such that V'_c and V''_c both vanish simultaneously for some x. When n >1, the description is more complicated: here k is the set of points c such that all the partial derivatives

$$\frac{\partial V_c}{\partial x_1}=\frac{\partial V_c}{\partial x_2}=\ldots=\frac{\partial V_c}{\partial x_n}$$

and also the determinant of second derivatives

$$\begin{vmatrix} \frac{\partial^2 V_c}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 V_c}{\partial x_1 \partial x_2} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \frac{\partial^2 V_c}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 V_c}{\partial x_n \partial x_n} \end{vmatrix}$$

vanish simultaneously for some $(x_1, x_2, \ldots, x_n)$

Example: Let n =1, k = 2 and $\mathbf{V}_{(c_1,c_2)}(x) = x^3+c_1x^2+c_2x$.

The critical points of $\mathbf{V}_{(c_1,c_2)}$ occur where $V'_{(c_1,c_2)}(x) = 0$, i.e. $3x^2+2c_1x+c_2=0$.

Critical points coalesce where $V''_{(c_1,c_2)}(x) = 0$,i.e.$6x+2c_1= 0$ Eliminating x from these equation we get $c_1^2/3 -2c_1^2/3 + c_2 = 0$, i.e. $c_2 = c_1^2/3$

which is the equation of a parabola in the (c_1,c_2) – plane. This is the catastrophe set K for the given **V**. When $c_2 < 1/3c_1^2$, there is one minimum (and one maximum) for $\mathbf{V}_{(c1,c2)}$ when $c_1 > c_1^2/3$ there are no critical points. As the point (c_1,c_2) moves across **K** with c_2 decreasing the minimum coalesces with the maximum and disappears. A system governed by $\mathbf{V}_{(c_1,c_2)}$, and in equilibrium at the minimum of $\mathbf{V}_{(c_1,c_2)}$, would exhibit a catastrophic jump (towards $v = -\infty$, i.e. $x = -\infty$) as this happened.

5.4. Thom's theorem and his list of seven catastrophe models

5.4.1. Thom's Theorem

***Theorem*:** for systems governed by smooth functions with at most four parameters (but any number of variables) thee are essentially only seven possible types of local geometric structure for stable catastrophe sets.

To explain some of the words in the theorem, Thom provides a list of seven explicit functions (catastrophe models) with the property that any other function $\mathbf{V}_c(x)$ which has

- at most four parameter, and
- a catastrophe set K which is stable, can, in a neighbourhood of any point c in **K**, be converted essentially into one of these seven catastrophe models by means of a smooth change of coordinates which also converts **K** essentially into the catastrophe set of the model. The word 'essentially' in the theorem means that we must make all these allowances when trying to interpret a given catastrophe as one of Thom's seven.

5.4.2. Thom's list of seven catastrophe models

1. The *fold:* $V_{c_1}(x) \equiv x^3+c_1x$

For $c_1< 0$ there is one minimum and one maximum; for $c_1>0$ there are no critical points. The catastrophe set **K** consists of the one point $c = 0$. If we plot critical points of V_{c_1} against c_1 we obtain the parabola $3x^2 + c_1 = 0$ in the (c_1, x)-plane. The branch of the parabola with $x > 0$ corresponds to the minimum of V_{c_1}, and as c_1 passes through 0 from below the minimum disappears and the equilibrium "falls off" the curve towards $x = -\infty$. This indicates why this catastrophe is called the fold.

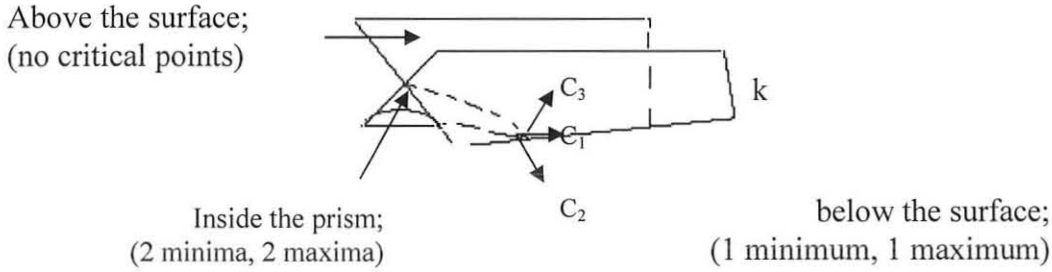
2. The *Cusp*: $V_{(c_1,c_2)}(x) \equiv x^4 + c_1x^2 + c_2x$

Critical points occur where $V'_{(c_1,c_2)}(x) = 0$,i.e. $4x^3 + 2c_1x + c_2 = 0$ and they coalesce where $V''_{(c_1,c_2)}(x) = 0$, i.e. $12x^2 + 2c_1 = 0$.

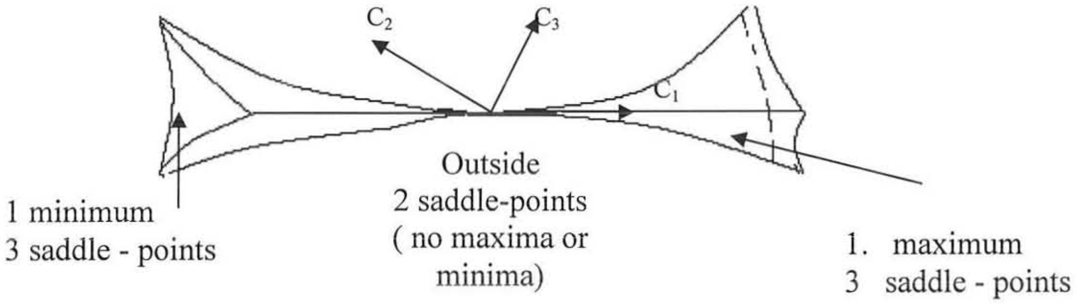
Eliminating x gives $8c_1^3 + 27c_2^2 = 0$ for the equation of the catastrophe set **K**, which is easily verified to be a cusp in the (c_1, c_2)-plane. For $8c_1^3+27c_2^2 > 0$ there is just one critical point (a minimum) for $V_{(c_1,c_2)}$; for $8c_1^3 + 27c_2^2 < 0$ there are two minima and one maximum.

For the next three models the calculations become very involved, so we will simply give the functions and sketch the catastrophe sets K in the control space, indicating the nature of the critical points of $\mathbf{V}_c$ for c in each region complementary to **K**. The control space has dimension three in all cases.

3. The *Swallow – tail*: $V_{(c_1,c_2,c_3)}(x) \equiv x^5 + c_1x^3 + c_2x^2 + c_3x$

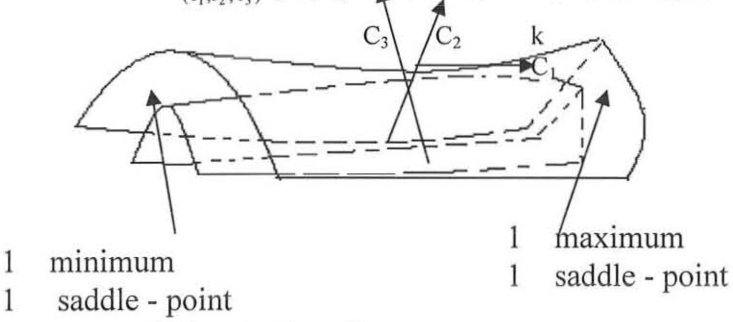


4. The *Elliptic umbilic, or hair:* $V_{(c_1,c_2,c_3)}(x_1,x_2) \equiv x_1^3-3x_1x_2^2+c_1(x_1^2+x_2^2)+c_2x_1+c_3x_2$



5. The *Hyperbolic umbilic,* or breaking wave:

$V_{(c_1,c_2,c_3)}(x_1,x_2) \equiv x_1^3+x_2^3+c_1x_1x_2+c_2x_1+c_3x_2$



Below both surfaces:

1 minimum, 1 maximum

2 saddle- points

The final two models each have control spaces of dimension four, so it is impossible to sketch the catastrophe sets. We simply give the functions $V_{(c_1,c_2,c_3,c_4)}$ in the two cases.

6. The *Butter fly:*

$$V_{(c_1,c_2,c_3,c_4)}(x) \equiv x^6+c_1x^4+c_2x^3+c_3x^2+c_4x$$

7. The *Parabolic Umbilic*, or the mushroom

$$V_{(c_1,c_2,c_3,c_4)}(x_1,x_2) \equiv x_2^4+x_1^2x_2+c_1x_1^2+c_2x_2^2+c_3x_1+c_4x_2$$

Example: (illustrating some points arising from the theorem)

Let $V_{(c_1,c_2)}(x_1,x_2) \equiv x_1^3 - 2x_1x_2 + c_1x_1^2 + x_2^2 + c_2x_1$

The critical points are given by

$$\frac{\partial V_{(c_1,c_2)}}{\partial x_1} = 3x_1^2 - 2x_2 + 2c_1x_1 + c_2 = 0$$

$$\frac{\partial V_{(c_1,c_2)}}{\partial x_2} = -2x_1 + 2x_2 = 0, \text{i.e.}, x_1 = x_2 \text{ and } 3x_1^2 + 2(c_1-1)x_1 + c_2 = 0$$

critical points coalesce where the determinant of second derivatives

$$\begin{vmatrix} 6x_1 + 2c_1 & -2 \\ -2 & 2 \end{vmatrix}$$

vanishes, i.e. $2(6x_1 + 2c_1) - 4 = 0$ or $x_1 = \frac{1}{3}(1 - c_1)$

Substituting this in the above quadratic equation for x_1 we get $c_2 = \frac{1}{3}(c_1-1)^2$

So the catastrophe set K is a parabolic in the (c_1,c_2)-plane and is a stable catastrophe set. Thom's theorem therefore implies that there is a smooth change of coordinates taking this model essentially to one of the seven.

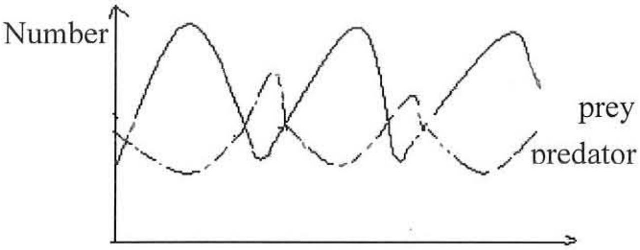
6. Some selected applications of stability theory

6.1 Population Models

An important problem in ecology is to investigate the question of coexistence of species and to decide what mankind should do to preserve this ecological balance of nature. In nature, species can interact in many different ways like one species of animal feeds on another species of animal, which in turn feeds on other things. We call the first species the *predator* and the second species the *prey*. We can also have an interaction between similar species competing for a limited food supply. We will also consider for this a competing species model.

6.1.1. **The predator-prey equations**

Theoretically, the predator can destroy all the prey so that the prey become extinct. However, if this happens, the predator will also become extinct, as it depends on the prey for its existence. What actually happens in nature is that a cycle develops where at some time the prey may be abundant and the predator few. Because of the abundance of prey, the predator population grows and reduces the population of prey, leading to points and this also results in a reduction of predators and consequent increase of prey and the cycle continues like the figure below.



Predator – prey cycle

In mathematical terms, let x and y be the prey and predator populations, respectively. The governing differential equations for two species interaction can be written as

$$dx/dt = f(x,y) \qquad (1)$$
$$dy/dt = g(x,y)$$

In the absence of predators, prey will grow unlimited according to $dx/dt = \alpha x$ and in the absence of prey, the predators will die out according to $dy/dt = -\delta y$. We model the interaction term by xy, positive for the predator and negative for the prey, resulting in the model.

$dx/dt = \alpha x - \beta xy$

$dy/dt = -\gamma y + \delta xy$

Where α, β, δ and γ are positive constants. The non-linear system (1) is known as the *Lotka-Volterra equations*. The equilibrium points are when dx/dt=0 and dy/dt=0.

Hence, we have

$dx/dt = 0$, for $x=0$ or $y = \alpha/\beta$

$dy/dt = 0$, for $y=0$ or $x = \gamma/\delta$

Thus, there are two possible equilibrium points $x=0, y=0$, $x = \gamma/\delta$, $y = \alpha/\beta$

Case I. Equilibrium point (0,0):

There are neither predator nor prey. Hence we can neglect the second terms βxy and δxy from equations (1)in comparison to the first term. So,

$$dx/dt = \alpha x, \quad dy/dt = -\gamma y \tag{2}$$

with the solutions

$$x = k_1 e^{\alpha t}, y = k_2 e^{-\gamma t} \tag{3}$$

As t increases, we see from equation (3) that x increases while y approaches zero. It is evident that if we should displace the particle slightly from the equilibrium point (0,0), it tends to move away from the point and we call the equilibrium point an unstable equilibrium point (or a point of instability). If, on the other hand, a slight displacement from the equilibrium point resulted in a tendency for the particle to turn or move towards the equilibrium point, as in the case of a mass on a stretched spring, we would call te point a stable equilibrium point.

The interpretation is that if at sometime there is a small number of predators and prey, there will be tendency for the number of preys to increase and the number of predators to decrease. However, in real life situation, x does not increase indefinitely since an increase in prey results in a subsequent increase in predator.

Case II. Equilibrium point $\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)$:

In this case, the predator and prey are in equilibrium state such that either the numbers do not change because $x = \gamma/\delta$, $y = \alpha/\beta$ is a solution of equations (1), which is independent of time. It is interesting to observe that what happens if there is a slight departure from this equilibrium state, which can occur if, hunters destroy either the prey or the predator (or both).

For this, we consider the transformation

$$x = (\gamma/\delta) + u, \quad y = (\alpha/\beta) + v \tag{4}$$

So that equations (1) become

$$du/dt = -(\beta\gamma/\delta)v - \beta uv, \quad dv/dt = (\alpha\delta/\beta)u + \delta uv \tag{5}$$

If the particle of our phase plane interpretation is close to the point $(\gamma/\delta, \alpha/\beta)$ of the xy-plane then u and v will be close to zero. In such a case, the second terms on the right of equations (5) can be neglected in comparison to the first so that we obtain the lineralised system.

$$du/dt = -(\beta\gamma/\delta)v, \quad dv/dt = (\alpha\delta/\beta)u \tag{6}$$

Eliminating v ,we obtain $d^2u/dt^2 + \alpha\gamma\, u = 0$ whose solution is

$$u = c_1\text{Cos}\left(\sqrt{\alpha\gamma}\right)t + c_2\text{Sin}\left(\sqrt{\alpha\gamma}\right)t$$

$$v = \frac{\delta}{\beta}\sqrt{\frac{\alpha}{\gamma}}\left[c_1\text{Sin}\left(\sqrt{\alpha\gamma}\right)t - c_2\text{Cos}\left(\sqrt{\alpha\gamma}\right)t\right] \tag{7}$$

Thus equations (4) become

$$x = \frac{\gamma}{\delta} + c_1\text{Cos}(\sqrt{\alpha\gamma})t + c_2\text{Sin}(\sqrt{\alpha\gamma})t \tag{8}$$

$$y = \frac{\alpha}{\beta} + \frac{\delta}{\beta}\sqrt{\frac{\alpha}{\gamma}}\left[c_1\text{Sin}\left(\sqrt{\alpha\gamma}\right)t + c_2\text{Cos}\left(\sqrt{\alpha\gamma}\right)t\right]$$

The parametric equation (8) represent concentric ellipses having a common center $(\gamma/\delta, \alpha/\beta)$. The fact that the phase curves are concentric ellipses can also be deduced directly from equations (6) by eliminating t, to obtain

$du/dv=-(\gamma\beta^2/\alpha\delta^2)v/u$ or $\alpha\delta^2 udu+\gamma\beta^2 vdv=0$ which on integration yields

$$u^2/\gamma\beta^2 + v^2/\alpha\delta^2 = c^2 \tag{9}$$

where c^2 is the constant of integration.

If we divide both sides of equation (9) by c^2, we see that it represents a family of concentric ellipses whose semi-major and semi-minor axes have lengths $c\beta\sqrt{\gamma}$ and $c\delta\sqrt{\alpha}$, respectively. The nature of the equilibrium point whether it is stable or unstable depends on the values of the parameters.

Example: discuss the solutions of the system

$dx/dt = x(1-0.5y) = x-0.5xy$ (10)

$dy/dt = y(-0.75+0.25x) = -0.75y+0.25xy$

for x and y positive. The critical points of this system are the solutions of the algebraic equations.

$x(1-0.5y) = 0$, $y(-0.75+0.25x) = 0$ namely, the points (0,0) and (3,2). Near the origin we can neglect the non-linear terms in (10) and thus

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 1 & -0.75 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \tag{11}$$

The eigenvalues and eigenvectors of (11) are

$$r_1=1,\ \xi_1=\begin{pmatrix} 1 \\ 0 \end{pmatrix};\ r_2=-0.75,\ \xi_2=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Therefore, the general solution is

$$\begin{pmatrix} x \\ y \end{pmatrix}=c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix}e^t+c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}e^{-0.75}$$

Thus, the origin is a saddle point of both (10) and (11) and therefore is an unstable critical point. To examine the critical point (3,2) we can make the substitution $x=3+u$, $y=2+v$ in (10) and neglecting the non-linear terms in u and v we have

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix}=\begin{pmatrix} 0 & -1.5 \\ 0.5 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \tag{12}$$

the eigenvalues and eigenvectors of this system are

$$r_1 = i\frac{\sqrt{3}}{2},\ \xi_1\begin{pmatrix}1\\ -i/\sqrt{3}\end{pmatrix} \quad ;r_2 = \frac{-i\sqrt{3}}{2},\ \xi_2 = \begin{pmatrix}1\\ i/\sqrt{3}\end{pmatrix}$$

Since the eigenvalues are imaginary, (3,2) is a center of (12) and thus a stable critical point of (10).

To find the trajectories of (12),

$dv/du = \frac{dv/dt}{du/dt} = 0.5u / -1.5v = -u/3v$ or $udu + 3vdv = 0$.

Thus, $u^2+3v^2 = k$, where k is an integration constant.

Turning to (10), $\frac{dy}{dx} = \frac{y(-0.75+0.25x)}{x(1-0.5y)}$ has a solution $0.75 \ln x + \ln y - 0.5y - 0.25x = c$, which is a center of (10) and the predator and prey population exhibit a cyclic vibration.

6.1.2. **Competing species and the almost linear system**

Suppose that in some closed environment there are two similar species competing for a limited food supply. Let x and y be the population of the two species at time t. Then the population of the two species, in the absence of the other, is governed by a logistic equation

$$dx/dt = x(Є_1 - б_1x) \qquad (1)$$
$$dy/dt = y(Є_2 - б_2y)$$

Where $Є_1$ and $Є_2$ are the growth rates of the two populations x and y respectively $Є_1/б_1$ and $Є_2/б_2$ are their saturation levels. The simplest expression for reducing the growth rate of species x due to the presence of species y is to replace the growth rate factor $Є_1-б_1x$ in (1) by $Є_1-б_1x - \alpha_1y$, where α_1 is a measure of the degree to which species y interferes with species x and similarly we replace $Є_2-б_2y$ by $Є_2-б_2y - \alpha_2x$. Thus we have the system of equations

$$dx/dt=x(Є_1- б_1x- \alpha_1y) \qquad (2)$$
$$dy/dt=y(Є_2- б_2y- \alpha_2x)$$

The values of the positive constants $Є_1$, $б_1$, α_1, $Є_2$, $б_2$ and α_2 depend on the particular species under consideration.

Example: Discuss the qualitative behavior of the solutions of the system

$$dx/dt = x(1-x-y) \qquad (3)$$

$$dy/dt = y(0.5-0.25y-0.75x)$$

where x and y are non negative.

Clearly, this system is a special case of the system (2) for two competing species. We find the critical points by solving the system of algebraic equations

$x(1-x-y) = 0$, $y(0.5-0.25y-0.75x) = 0$

So, there are four critical points, namely, (0,0), (1,0),(0,2) and (0.5,0.5) corresponding to equilibrium solutions of system (3). Here, we see that $dx/dt>0$ or $dx/dt < 0$ according to whether $1-x-y$ is positive or negative. Similarly, the sign of dy/dt is the same as the sign of $0.5-0.75x-0.25y$.By superposing the diagrams of the trajectories, the mixed equilibrium solution (0.5,0.5) is of particular interest since it corresponds to coexistence between the two species.

We can further analyze system (3) by looking at the linear approximations valid in the neighbourhood of each critical point.

Case 1.(0,0): Neglecting the non-linear terms, we get the linear system

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \qquad (4)$$

which is valid near the origin. The eigenvalues and eigenvectors of the system (4) are

$r_1=1$, $\xi_1=\begin{pmatrix} 1 \\ 0 \end{pmatrix}$; $r_2= 0.5$, $\xi_2=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, so the general solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix}e^{t} + c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}e^{0.5t}$$

Therefore, the origin is an unstable improper node of the linear system (4) and also of the non-linear system (3).

Case 2. (1,0): The corresponding linear system is

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 0 & -0.25 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \qquad (5)$$

Its eigenvalues and eigenvectors are

$r_1= -1$, $\xi_1=\begin{pmatrix} 1 \\ 0 \end{pmatrix}$; $r_2= -0.25$, $\xi_2=\begin{pmatrix} 4 \\ -3 \end{pmatrix}$

and its general solution is

$$\begin{pmatrix} u \\ v \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 4 \\ -3 \end{pmatrix} e^{-\frac{t}{4}}$$

Thus, the point (1,0) is an asymptotically stable improper node of the linear system (5) and of the non-linear system (3). If the initial value of x an y are sufficiently close to (1,0), then the interaction process will lead ultimately to that state, that is, to the survival of species x and the extinction of species y.

Case 3. (0,2): The analysis in this case is similar to that for the point (1,0). The appropriate linear system is

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -1.5 & -0.5 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix},$$

The eigenvalues and eigenvectors of this system are

$$r_1 = -1,\ \xi_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}\ ;\ r_2 = -0.5,\ \xi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

d its general solution is

$$\begin{pmatrix} u \\ v \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t/2}$$

Thus the critical point (0,2) is asymptotically stable improper node.

Case 4. (0.5,0.5): The corresponding linear system is

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -0.5 & -0.5 \\ -0.375 & -0.125 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix},$$

The eigenvalues and eigenvectors of this system are

$$r_1 = \frac{-5+\sqrt{57}}{16} \cong 0.1594,\ \xi_1 = \begin{pmatrix} 1 \\ (-3-\sqrt{57})/8 \end{pmatrix} \cong \begin{pmatrix} 1 \\ -1.3187 \end{pmatrix},$$

$$r_2 = \frac{-5-\sqrt{57}}{16} \cong -0.7844,\ \xi_2 = \begin{pmatrix} 1 \\ (-3+\sqrt{57})/8 \end{pmatrix} \cong \begin{pmatrix} 1 \\ 0.5687 \end{pmatrix},$$

so the general solution is

$$\begin{pmatrix} u \\ v \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1.3187 \end{pmatrix} e^{0.1594t} + c_2 \begin{pmatrix} 1 \\ 0.5687 \end{pmatrix} e^{-0.7844t}$$

Here, the critical point is a saddle point, which is unstable.

So, in the cases (1),(2) and (3), the interaction or competition is strong and the species can not exist – one must die out while in case (4), there is a weak competition and hence the species can coexist.

6.2. Applications to economics: A microeconomic market model

There has been an increasing interest in applications of Mathematics to Economics. Here, we see one of its application in a microeconomic market.

Let Q_s be the number of items of a commodity in supply per unit time, Q_d be the number of items of a commodity in demand per unit of time and P be the price in birrs of a single commodity. We now consider a relation in which Qs and Q_d depend linearly not only on P but also on dp/dt and d^2P/dt^2. This model incorporates the effect of trends in price on demand and supply. Here, dp/dt represents the effect of rising or fallingprices while d^2p/dt^2 denotes the effect of increasing or decreasing price rates of change.

Let $Q_d=a-Pb+AP'+BP''$, $Q_s=-c+Pd+CP'+DP''$ where a,b,c and d are positive constants and the constants A,B,C and D are positive, negative or zero.

For economic equilibrium to prevail, we have $Q_d = Q_s$, and this leads to the following differential equations for P:

$$(B-D)P'' + (A-c)P' - (b+d)P = -(a+c)$$

We consider here the case $B \neq D$ since this leads to the second order differential equation

$$P'' + \frac{A-C}{B-D}P' - \frac{b+d}{B-D}P = -\frac{a+c}{B-D} \tag{1}$$

We then have the positive constant particular solution

$P_p = \frac{a+c}{b+d}$ and this is called the equilibrium price. This equilibrium is said to be *dynamically stable* if and only if

$$\lim_{t \to \infty} P(t) = P_P$$

To determine whether a dynamically stable equilibrium prevails or not, we examine the roots of the auxiliary equation

$$\lambda^2 + \frac{A-C}{B-D}\lambda - \frac{b+d}{B-D} = 0 \text{ of equation (1).}$$

Since $b+d>0$, λ_1 and λ_2 are both non-zero. If λ_1 and λ_2 are real and unequal, the complete solution of (1) is

$$P = P(t) = c_1 e^{-\lambda_1 t} + c_2 e^{\lambda_2 t} + P_p .$$

The constants c_1 and c_2 can be found from the initial values $P(0)$ and $P'(0)$. Since c_1 (or c_2) will be zero only for very special conditions, for dynamical stability, we require that $\lim_{t\to\infty} P(t)=P_p$, where c_1 and c_2 are both different from zero. It can be easily seen that

$$\lim_{t\to\infty} P(t) = P_p \text{ if } \lambda_1<0 \text{ and } \lambda_2<0 \text{ (if } \lambda>0, \lim_{t\to\infty} e^{\lambda t} = \infty)$$

If $\lambda_1 = \lambda_2$, then $P = P(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t} + Pp$. In this case $\lim_{t\to\infty} p(t) = P_p$ iff $\lambda_1 <0$.

Finally, if $\lambda_1= \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$, then the solution of (1) is
$P=P(t)= e^{\alpha t} (c_1 \sin\beta t + c_2 \cos\beta t) + P_p$. Here, $\lim_{t\to\infty} P(t) =P_p$ iff $\alpha <0$.

(The absolute value of the harmonic factor $c_1\sin\beta t+c_2\cos\beta t$ cannot exceed $|c_1|+|c_2|$.) Thus it is possible to determine whether the solution of (1) does or does not converge to P_p without obtaining the complete solution.

Example: Let $Q_d=10-P-P'+2P''$, $Qs=-5+4P+P'+3P''$. Find the price P in terms of t if $P(0)=5$ and $P'(0)=1$.Determine whether or not the equilibrium is dynamically stable.

Solution: setting $Q_d = Qs$, we get $P''+2P'+5P=15$ (2)

The roots of the auxiliary equation are $m_1= -1+2i$ and $m_2=-1-2i$. A particular solution of (2) is $Pp=3$ and its complete solution is $P = P(t) = e^{-t} (c_1\sin 2t + c_2\cos 2t) +3$.

From $P(0) = 5$ and $P'(0) =1$, we find that $c_1=3/2$ and $c_2=2$

Hence, $P = P(t) = e^{-t}(3/2\sin 2t+2\cos 2t) + 3$.

Since $\alpha=-1<0$, the equilibrium is dynamically table and $\lim_{t\to\infty} P(t) = 3=Pp$.

6.3 Arms Race (Combat Model)

An interesting application that leads to a system of differential equation is the study of arms race. We will present here a model for arms race which is known as *Richardson model.* The following assumptions are made in this model.

i. The expenditure for armament of each country will increase at a rate which is proportional to the other country's expenditure.

ii. The expenditure for armament of each country will decrease at a rate which is proportional to its own expenditure.

iii. The rate of change of arms expenditure for a country has a constant component that measures the level of antagonism of that country towards the other

iv. The effects of all these assumptions are additive.

If x and y denote the expenditure incurred by two countries on armament, then under the assumptions (i)-(iv), the system of differential equation is

$dx/dt = ay-px+r$

$dy/dt= bx-qy+s$

The constants a,b,p,q are positive but the numbers r and s may have any value. Positive values arise if the countries have internal attitudes of distrust of each other.

In matrix notation, the problem may be written as

$$x' = \mathbf{Ax} + \mathbf{B},\ \mathbf{X(0)} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

where

$$x' = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix},\ \mathbf{x} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix},\ \mathbf{A} = \begin{pmatrix} -p & a \\ b & -q \end{pmatrix},\ B = \begin{pmatrix} r \\ s \end{pmatrix}$$

The nature of the solutionsof the system will depend upon the eigenvalues of the matrix A, i.e. on the roots of the characteristic equaiton $|\lambda I - A| = 0$.

$$\begin{vmatrix} -p-\lambda & a \\ b & -q-\lambda \end{vmatrix} = \lambda^2 + (p+q)\lambda + (pq-ab) = 0$$

The roots are

$$\frac{-(p+q)\pm\sqrt{(p+q)^2-4(pq-ab)}}{2}=\frac{-(p+q)\pm\sqrt{(p-q)^2+4ab}}{2}$$

Since a and b are positive, the eigenvalues are real and distinct. As $P > 0$, $q > 0$, it follows that if $pq - ab > 0$, the two eigenvalues are both negative, but if $pq-ab<0$,then the eigenvalues will have opposite signs. The presence of a positive eigenvalue is disturbing since it will lead to an exponential function that becomes unbounded as time increases, a situation that may result in a runaway arms race.

The possible consequences of this model has been illustrated by taking different parameters in the following examples.

1. If $a=4, b=2, p=3, q = 1, r=2, s=2, x_0=4$ and $y_0 = 1$, then we have

$$x'=\begin{pmatrix}-3 & 4\\ 2 & -1\end{pmatrix}\mathbf{x}+\begin{pmatrix}2\\2\end{pmatrix},\ \mathbf{x}(0)=\begin{pmatrix}4\\1\end{pmatrix}$$

The characteristic equation of the matrix equation is

$\begin{vmatrix}-3-\lambda & 4\\ 2 & -1-\lambda\end{vmatrix}=\lambda^2+4\lambda-5=0$. So that the eigenvalues are $\boldsymbol{\lambda}_1=1$ and $\boldsymbol{\lambda}_2=-5$.

Now, we solve the equation $\mathbf{A}x= \boldsymbol{\lambda}x$ to get the eigenvectors x corresponding to eigenvalues $\boldsymbol{\lambda}$. Thus, for $\lambda_1=1$,

$\mathbf{A}\begin{pmatrix}x_1\\x_2\end{pmatrix} = 1\begin{pmatrix}x_1\\x_2\end{pmatrix}$ gives $x_1= x_2$. If $x_1=1$, then for $\boldsymbol{\lambda}_1=1$, the eigenvector is $\begin{pmatrix}1\\1\end{pmatrix}$.

Similarly for $\boldsymbol{\lambda}_2=-5$, we have $x_1+2x_2=0$.

Taking $x_2=-1$,the eigenvector is $\begin{pmatrix}2\\-1\end{pmatrix}$

The general solution of the homogeneous system $x'=\mathbf{Ax}$ is $\mathbf{x}(t) = c_1\begin{pmatrix}1\\1\end{pmatrix} e^t + c_2\begin{pmatrix}2\\-1\end{pmatrix}e^{-5t}$

The homogeneous system $x'= \mathbf{Ax+B}$ has a solution of the form $\begin{pmatrix}e\\f\end{pmatrix}$. Substituting it into the system, we obtain

$$\begin{pmatrix}-3 & 4\\ 2 & -1\end{pmatrix}\begin{pmatrix}e\\ f\end{pmatrix}+\begin{pmatrix}2\\ 2\end{pmatrix}=0$$

a system with solution $\begin{pmatrix}-2\\ -2\end{pmatrix}$. Thus, the general solution of the non-homogeneous system is $x(t)=c_1\begin{pmatrix}1\\ 1\end{pmatrix}e^t+c_2\begin{pmatrix}2\\ -1\end{pmatrix}e^{-5t}+\begin{pmatrix}-2\\ -2\end{pmatrix}$

while the initial condition $\mathbf{x}(0)=\begin{pmatrix}4\\ 1\end{pmatrix}$ requires that

$$\begin{pmatrix}4\\ 1\end{pmatrix}=c_1\begin{pmatrix}1\\ 1\end{pmatrix}+c_2\begin{pmatrix}2\\ -1\end{pmatrix}+\begin{pmatrix}-2\\ -2\end{pmatrix}$$ so that $c_1=4, c_2=1$.

Therefore, the required solution is

$$x(t)=4\begin{pmatrix}1\\ 1\end{pmatrix}e^t+1\begin{pmatrix}2\\ -1\end{pmatrix}e^{-5t}+\begin{pmatrix}-2\\ -2\end{pmatrix}$$

or $x(t)=4e^t+2e^{-5t}-2$

$y(t)=4e^t-e^{-5t}-2$

Here, we have a runaway arms race.

2. If $a=4, b=2, p=3, r=-2, s=-2, x_0=2, y_0=1/2$, then the system has the same solution as above except for the sign of the particular solution. The general solution is

$$x(t)=c_1\begin{pmatrix}1\\ 1\end{pmatrix}e^t+c_2\begin{pmatrix}2\\ -1\end{pmatrix}e^{-5t}+\begin{pmatrix}2\\ 2\end{pmatrix}.$$

For the initial conditions, we have

$$\begin{pmatrix}2\\ 1/2\end{pmatrix}=c_1\begin{pmatrix}1\\ 1\end{pmatrix}+c_2\begin{pmatrix}2\\ -1\end{pmatrix}+\begin{pmatrix}2\\ 2\end{pmatrix}$$

so that $c_1=-1$ and $c_2=1/2$. The solution is, therefore,

$$x(t)=-e^t+e^{-5t}+2$$

$$y(t)=-e^t-\frac{1}{2}e^{-5t}+2$$

and each country will eventually decrease its expenditure for arms to zero –a condition of disarmament.

3. If $a=3, b=1, p=4, q=2, r=6, s=1$ and $x(0)=0, y(0)=0$, then

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -4 & 3 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 6 \\ 1 \end{pmatrix}$$

here, the eigenvalues are –1 and –5 and the corresponding eigenvalues are $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$.

Thus the general solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 3 \\ -1 \end{pmatrix} e^{-5t} + \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

From the initial conditions, we find that $c_1 = -9/4$, $c_2 = -1/4$ as that the general solution becomes

$$\begin{pmatrix} x \\ y \end{pmatrix} = -9/4 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} - 1/4 \begin{pmatrix} 3 \\ -1 \end{pmatrix} e^{-5t} + \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

which also yields

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{9}{4} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} - \frac{1}{4} \begin{pmatrix} 3 \\ -1 \end{pmatrix} e^{-5t}$$

Because of the negative exponents, the rate at which the expenditure is changing will tend towards zero and the arms expenditure will approach x=3 and y=2. There will be a stabilized arms race in this case.

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