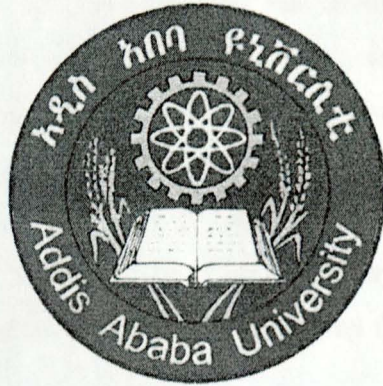


**ADDIS ABABA UNIVERSITY  
SCHOOL OF GRADUATE STUDIES  
DEPARTMENT OF MATHEMATICS**



**GRADUTE SEMINAR REPORT**

**ON**

**DIFFERENCE EQUATIONS**

**FOR**

**TIME-INVARIANT SYSTEMS**

**(Submitted in partial fulfillment of M.Sc.Degree in Mathematics)**

**Compiled by : Nigussie Chanie**

**Advisor : Dr. Taddesse Abdi**

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**Addis Ababa**

PREFACE

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**PREFACE**

This paper attempts to present the most important methods to evaluate  $A^n$  for any square matrix  $A$ ; some basic theories and applications of Time-Invariant systems. The report divided into four sections. The first section introduces an algorithm to compute  $A^n$ . The second section deals with systems of linear homogeneous and inhomogeneous difference equations. The third section presents the method of Jordan form. Finally, in the last section two applications of Time-Invariant systems are discussed.

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# INTRODUCTION

## 1.1 Definitions

### LIST OF SYMBOLS

$\Delta$  = The difference operator

$\mathfrak{R}^{k \times k}$  =  $k \times k$  real matrix

$Z^+$  = The set of non-negative integers

$Z$  = The Set of Integers

$\prod_{i=1}^k$  = Product

$\det A$  = The determinant of matrix A

$A^T$  = Transpose of matrix A

$C(n)$  = The Casorati

$diag$  = Diagonal matrix

$\rho(A)$  = Spectral Radius of matrix A



# 1 INTRODUCTION

## 1.1 Definitions

Difference equations are the discrete version of differential equations. Any functional relation of the form;

$$F[n, x(n), x(n+1), \dots, x(n+k)] = 0, \text{ or}$$

$$x(n+k) = f[n, x(n), x(n+1), \dots, x(n+k-1)], n \in \mathbb{Z}^+ \text{ is a difference equation.}$$

This paper is concerned with systems of linear first order difference equations. Particularly, it focuses on systems of  $k$ -linear first order difference equations with constant coefficients.

Consider the following system of equations:

$$\begin{cases} x_1(n+1) = a_{11}x_1(n) + a_{12}x_2(n) + \dots + a_{1k}x_k(n) \\ x_2(n+1) = a_{21}x_1(n) + a_{22}x_2(n) + \dots + a_{2k}x_k(n) \\ \vdots \\ x_k(n+1) = a_{k1}x_1(n) + a_{k2}x_2(n) + \dots + a_{kk}x_k(n) \end{cases} \dots (1.1)$$

Then equation (1.1) can be written in vector form as;

$$x(n+1) = Ax(n) \dots \dots \dots (1.2)$$

where,  $A(n) = (a_{ij})_{k \times k}$  is real nonsingular matrix and  $f(n) = (x_1(n), x_2(n), \dots, x_k(n))^T \in \mathbb{R}^k$

Since the values of  $A$  are constants, equation (1.2) is called Time-Invariant (autonomous). If for some  $n_0 \geq 0$ ,  $x(n_0) = x_0$  is specified, then equation (1.2) is called initial value problem

Now, suppose  $x(n_0) = x_0$

$$x(n_0 + 1) = Ax_0$$

$$x(n_0 + 2) = Ax(n_0 + 1) = A^2 x_0$$

Inductively, we can conclude that:  $x(n) = A^{n-n_0} x_0$

Hence, the unique solutions of the initial value problem is given by

$$x(n) = A^{n-n_0} x_0, \quad n \geq n_0 \geq 0$$

If  $n_0 = 0$ , then  $x(n) = A^n x_0$

But, the question is how to compute  $A^n$ . In differential equation, we use Putzer Algorithm to calculate  $e^{At}$ . Similarly, we can develop Putzer Algorithm to compute  $A^n$ .

## 1.2 Discrete Putzer Algorithm

Let  $A$  is a real  $k \times k$  matrix. Then an eigenvalues of  $A$  is a real or complex number  $\lambda$  such that  $Au = \lambda u, u \neq 0$

$$\Rightarrow (A - \lambda)u = 0, u \neq 0 \dots\dots\dots (1.3)$$

Equation (1.3) has a non-zero solution if and only if  $\det(A - \lambda I) = 0$

Or:  $\lambda^k + a_1\lambda^{k-1} + a_2\lambda^{k-2} + \dots + a_{k-1}\lambda + a_k = 0$

Equation (1.3) is called characteristic equation of matrix  $A$ , whose roots  $\lambda$  are called eigenvalues of  $A$  and  $p_A(\lambda) = \det(A - \lambda I) = 0$  is called characteristic polynomial.

If  $\lambda_1, \lambda_2, \dots, \lambda_k$  be eigenvalues of  $A$  (some of them may be repeated) then  $p_A(\lambda)$  can be expressed as in the form:

$$P_A(\lambda) = \prod_{j=1}^k (\lambda - \lambda_j)$$

**Theorem (Cayley Hamilton’s Theorem):**

Every matrix satisfies its characteristic equation.

That is  $p_A(A) = \prod_{j=1}^k (A - \lambda_j I) = 0$  or  $A^k + a_{k-1}A^{k-1} + \dots + a_1A + a_0I = 0$

**Proof:** Let  $A$  be a  $k \times k$  matrix with characteristic polynomial

$$p_A(\lambda) = \lambda^k + a_{k-1}\lambda^{k-1} + \dots + a_1\lambda + a_0, \text{ for some real numbers } a_0, a_1, \dots, a_{k-1}$$

Consider the adjoint  $B(\lambda)$  of  $(A - \lambda I)$

$$(A - \lambda I) \text{adjoint}(A - \lambda I) = \det(A - \lambda I)I$$

$$\Rightarrow (A - \lambda I)B(\lambda) = p_A(\lambda)I$$

Now the cofactors of  $A - \lambda I$  are of degree at most  $k-1$  in  $\lambda$ , so the same is true of the elements of  $B$ .

Hence we may represent  $B$  as a matrix polynomial

$$B(\lambda) = B_{k-1}\lambda^{k-1} + B_{k-2}\lambda^{k-2} + \dots + B_1\lambda + B_0, \text{ where } B_m \text{ is a } k \times k \text{ matrix whose elements are the coefficients of } \lambda^m \text{ in the corresponding elements of } B.$$

$$\begin{aligned} \therefore (A - \lambda I)B(\lambda) &= (A - \lambda I)(B_{k-1}\lambda^{k-1} + B_{k-2}\lambda^{k-2} + \dots + B_1\lambda + B_0) \\ &= (\lambda^{k-1}AB_{k-1} - \lambda^k B_{k-1}) + (\lambda^{k-2}AB_{k-2}\lambda^{k-1} - \lambda^{k-1}B_{k-2}) + \dots + (\lambda AB_1 - \lambda^2 B_1) + AB_0 \\ &= -B_{k-1}\lambda^k + \lambda^{k-1}(AB_{k-1} - B_{k-2}) + \dots + \lambda(AB_1 - B_0) + AB_0 \end{aligned}$$

Setting the coefficients of  $\lambda^m$  in the expression equal to the coefficients of  $\lambda^m$  in  $p_A(A)I$

$$\begin{cases} -B_{k-1} = I \\ AB_m - B_{m-1} = a_m I, 1 \leq m \leq k-1 \\ AB_0 = a_0 I \end{cases}$$

Hence 
$$\begin{aligned} \rho_A(A) &= A^k + a_{k-1}A^{k-1} + a_{k-2}A^{k-2} + \dots + a_1A + a_0I \\ &= A^k I + A^{k-1}(a_{k-1}I) + A^{k-2}(a_{k-2}I) + \dots + A(a_1I) + a_0I \\ &= A^k(-B_{k-1}) + A^{k-1}(AB_{k-1} - B_{k-2}) + \dots + A(AB_1 - B_0) + AB_0 \\ &= A^k(B_{k-1} - B_{k-1}) + A^{k-1}(B_{k-2} - B_{k-2}) + A^{k-2}(B_{k-3} - B_{k-3}) + \dots + A(B_0 - B_0) \\ &= 0 \end{aligned}$$

Now let us develop the discrete Putzer's Algorithm for  $A^n$

Let A be a  $k \times k$  real matrix . We seek for a representation of  $A^n$  in the form

$$A^n = \sum_{j=1}^k u_j(n)M_{j-1} \dots \dots \dots (1.4.)$$

Where, the  $u_j(n)$ 's are scalar functions to be determined later and

$$M_j = (A - \lambda_j I)M_{j-1}, M_0 = I \dots \dots \dots (1.5)$$

$$\text{Or : } M_{j+1} = (A - \lambda_{j+1} I)M_j, M_0 = I) \dots \dots \dots (1.6)$$

By iteration, we may show that:

$$\begin{aligned} M_n &= (A - \lambda_n I)(A - \lambda_{n-1} I) \dots (A - \lambda_1 I)M_{j-1} \\ \Rightarrow M_n &= \prod_{j=1}^n (A - \lambda_j I) \dots \dots \dots (1.7) \end{aligned}$$

By Cayley Hamilton Theorem, 
$$M_k = \prod_{j=1}^k (A - \lambda_j I) = 0$$

Consequently,  $M(n) = 0, \forall n \geq k$  .

Hence, we may rewrite (1.4) as

$$A^n = \sum_{j=1}^k u_j(n) M_{j-1} \dots\dots\dots (1.8)$$

If we let  $n=0$ , in (1.8), we obtain:

$$A^0 = \sum_{j=1}^k u_j(0) M_{j-1}$$

$$\Rightarrow I = u_1(0)I + u_2(0)M_1 + \dots + u_k(0)M_{k-1}$$

$$\Rightarrow u_1(0) = 1, u_2(0) = u_3(0) = \dots u_k(0) = 0 \dots\dots\dots (1.9)$$

From (1.8), we have:

$$\sum_{j=1}^k u_j(n+1) M_{j-1} = A^{n+1} = \sum_{j=1}^k u_j(n) A M_{j-1} \dots\dots\dots (1.10)$$

From (1.5), we have,

$$A M_{j-1} = M_j + \lambda_j M_{j-1} \dots\dots\dots (1.11)$$

Solving equations (1.10) and (1.11) simultaneously, we get

$$\sum_{j=1}^k u_j(n+1) M_{j-1} = \sum_{j=1}^k u_j(n) [M_j + \lambda_j M_{j-1}] \dots\dots (1.12)$$

Comparing the coefficients of  $M(j), 1 \leq j \leq k$ , in (1.12) and applying conditions (1.9), we obtain:

$$\begin{aligned} u_1(n+1) &= \lambda_1 u_1(n), u_1(0) = 1 \\ u_j(n+1) &= \lambda_j u_j(n) + u_{j-1}(n), u_j(0) = 0, j = 2, 3, \dots, k \end{aligned} \dots\dots\dots (1.13)$$

Thus, the solutions of (1.13) are given by:

$$\begin{aligned} u_1(n) &= \lambda_1^n \\ u_j(n) &= \sum_{i=0}^{n-1} \lambda_j^{n-1-i} u_{j-1}(i), j = 2, 3, \dots, k \end{aligned} \dots\dots\dots (1.14)$$

### Discrete Putzer Algorithm:

Let A be a  $k \times k$  matrix with eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_k$ .

$$\text{Then: } A^n = \sum_{j=1}^k u_j(n) M(j-1)$$

where  $M_0 = I, M_j = (A - \lambda_j I) M_{j-1}$ , and

$$u_1(n) = \lambda_1^n$$

$$u_j(n) = \sum_{i=0}^{n-1} \lambda_j^{n-1-i} u_{j-1}(i), j=2, 3, \dots, k$$

Table 2.2 Summation formula

	Summation
1	$\sum_{i=1}^n i = \frac{n(n+1)}{2}$
2	$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$
3	$\sum_{i=1}^n i^3 = \left[ \frac{n(n+1)}{2} \right]^2$
4	$\sum_{i=1}^n i^4 = \frac{n(6n^4 + 15n^3 + 10n^2 - 1)}{30}$
5	$\sum_{i=1}^{n-1} a^i = \begin{cases} \frac{a^n - 1}{a - 1}, & \text{if } a \neq 1 \\ n - 1, & \text{if } a = 1 \end{cases}$
6	$\sum_{i=1}^{n-1} ia^i = \frac{(a^n - 1)(n+1)a^{n+1} - a^{n+2} + a}{(a-1)^2}, \text{ if } a \neq 1$

**Example**

Find the solution of the system

$$x(n+1) = Ax(n), \text{ where } A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

**Solution**

$$\rho_A(\lambda) = \begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 3-\lambda \end{vmatrix} = -(\lambda-2)^2(\lambda-3)$$

$$\rho_A(\lambda) = 0 \Rightarrow \lambda = 2, \lambda = 3$$

Hence  $\lambda_1 = 2 = \lambda_2, \lambda_3 = 3$  are eigenvalues of A

$$\text{Thus } M(0) = I, M(1) = A - \lambda_1 I = A - 2I = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \text{ and}$$

$$M(2) = (A - \lambda_2 I)M(1) = (A - 2I)^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$u_1(n) = \lambda_1^n = 2^n$$

$$u_2(n) = \sum_{i=0}^{n-1} \lambda_2^{n-1-i} u_1(i) = \sum_{i=0}^{n-1} 2^{n-1-i} 2^i = n2^{n-1}$$

$$u_3(n) = \sum_{i=0}^{n-1} \lambda_3^{n-1-i} u_2(i) = \sum_{i=0}^{n-1} 3^{n-1-i} i2^i = 3^n - 2^n - n2^{n-1}$$

Therefore  $A^n = u_1(n)M(0) + u_2(n)M(1) + u_3(n)M(2)$

$$= 2^n \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + n2^{n-1} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} + [3^n - 2^n - n2^{n-1}] \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2^n & n2^{n-1} & 3^n - 2^n - n2^{n-1} \\ 0 & 2^n & 3^n - 2^n \\ 0 & 0 & 3^n \end{pmatrix}$$

So the solution of the given system is given by

$$x(n) = A^n x(0) = \begin{pmatrix} 2^n & n2^{n-1} & 3^n - 2^n - n2^{n-1} \\ 0 & 2^n & 3^n - 2^n \\ 0 & 0 & 3^n \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix}$$

$$= \begin{pmatrix} x_1(0)2^n + x_2(0)n2^{n-1} + (3^n - 2^n - n2^{n-1})x_3(0) \\ x_2(0)2^n + (3^n - 2^n)x_3(0) \\ x_3(0)3^n \end{pmatrix}$$

## 2. THE BASIC THEORY

In this section we see different theorems, definitions, lemmas, formulas and facts.

Consider the autonomous system

$$x(n+1) = Ax(n) + g(n) \dots\dots\dots (2.1)$$

where  $A$  is a  $k \times k$  matrix,  $\det A \neq 0$  and  $g(n) \in \mathbb{R}^k$ . Equation (2.1) is called inhomogeneous linear difference Time-Invariant (autonomous) system. The corresponding homogeneous system is given by

$$x(n+1) = Ax(n) \dots\dots\dots (2.2)$$

### 2.1. Homogeneous Linear Difference Time-Invariant Systems

We now establish the existence and uniqueness of solution of the homogeneous linear difference autonomous system (2.2)

#### Theorem - 1 (unique- existence theorem)

For each  $x_0 \in \mathbb{R}^k$  and  $n_0 \in \mathbb{Z}^+$  there exists a unique solution  $x(n)$  of the system

$$x(n+1) = Ax(n) \quad , \quad x(n_0) = x_0$$

**Proof:**

$$x(n_0) = x_0 \quad (\text{Given})$$

$$x(n_0 + 1) = Ax(n_0) = Ax_0$$

$$x(n_0 + 2) = Ax(n_0 + 1) = A^2 x_0$$

Inductively we obtain:  $x(n) = A^{n-n_0} x_0$

Hence there is a solution  $x(n) = A^{n-n_0} x_0$ . To show the uniqueness part, let  $y(n)$  be a solution of the system.

$$\text{Then } y(n+1) = Ay(n), \text{ and } y(n_0) = x_0$$

$$\Rightarrow y(n) = A^{n-n_0} y(n_0) = A^{n-n_0} x_0 = x(n)$$

Therefore the solution is unique

Next let us develop the notation of a fundamental matrix.

## Definition - 2

The solutions  $x_1(n), x_2(n), \dots, x_k(n)$ , of equation (2.2) are said to be linearly independent for  $n \geq n_0 \geq 0$  if whenever  $x(n) = \sum_{i=1}^k c_i x_i(n) = 0$ , for all  $n \geq n_0$ , then  $c_i = 0, 1 \leq i \leq k$

But it is not practical to check the linearly independent of a set of solutions using this definition. Fortunately there is a simple method to check the linearly independent of the solutions. This method is called Casoratian of the solution  $C(n)$

$$C(n) = \begin{vmatrix} x_1(n) & x_2(n) & \cdots & x_k(n) \\ x_1(n+1) & x_2(n+1) & \cdots & x_k(n+1) \\ \vdots & \vdots & \vdots & \vdots \\ x_1(n+k-1) & x_2(n+k-1) & \cdots & x_k(n+k-1) \end{vmatrix}$$

Let  $\phi(n)$  be a  $k \times k$  matrix whose columns are solutions of the system

$$x(n+1) = Ax(n)$$

We write  $\phi(n) = [x_1(n), x_2(n), \dots, x_k(n)]$ .

Then  $\phi(n+1) = [x_1(n+1), x_2(n+1), \dots, x_k(n+1)]$

$$= [Ax_1(n), Ax_2(n), \dots, Ax_k(n)]$$

$$= A [x_1(n), x_2(n), \dots, x_k(n)]$$

$$= A \phi(n)$$

Thus  $\phi$  satisfies the difference equation:

$$\phi(n+1) = A\phi(n)$$

Moreover the solutions  $x_1(n), x_2(n), \dots, x_k(n)$ , are linearly independent for  $n \geq n_0$  if and only if  $\det \phi(n) \neq 0$  for all  $n \geq n_0$ . This leads us the following definition.

### Definition-3:

If  $\phi(n)$  is a non-singular matrix for all  $n \geq n_0$  and  $\phi(n+1) = A\phi(n)$ , then it is said to be fundamental matrix for the system  $x(n+1) = Ax(n)$

Note that: if  $\phi(n)$  is a fundamental matrix and  $C$  is any matrix with  $\det C \neq 0$ , then  $\phi(n)C$  is also a fundamental matrix. Thus there are infinitely many fundamental matrices for a given system.

To compute a fundamental system for autonomous system we use Putzer algorithm.

**Lemma-4: (Abel's Formula)**

For any  $n \geq n_0 \geq 0$  and  $A$  a  $k \times k$  constant real matrix, if  $\phi(n) = A^{n-n_0} \phi(n_0)$  is a solution of  $\phi(n+1) = A\phi(n)$  then  $\det \phi(n) = (\det A)^{n-n_0} \det \phi(n_0)$

**Proof:** For any two  $k \times k$  real matrix  $A$  and  $B$ ,  
 $\det(AB) = \det A \cdot \det B$

$$\begin{aligned} \text{Hence } \det \phi(n) &= \det(A^{n-n_0} \phi(n_0)) = (\det A)^{n-n_0} \det \phi(n_0) \\ &= \det(A \times A \times \cdots \times A) \det \phi(n_0) \\ &= \det A \times \det A \times \cdots \times \det A \times \det \phi(n_0) \\ &= (\det A)^{n-n_0} \det \phi(n_0) \end{aligned}$$

Corollary-5:

- i). Let  $\phi(n)$  be a fundamental matrix. Then  $\det \phi(n) \neq 0$  for  $n \geq n_0 \Leftrightarrow \det \phi(n_0) \neq 0$
- ii). The solutions  $x_1(n), x_2(n), \dots, x_k(n)$  of  $\phi(n+1) = A\phi(n)$  are linearly independent for  $n \geq n_0$  if and only if  $\det \phi(n_0) \neq 0$ .

**Proof:** (i).  $\Rightarrow$  Suppose  $\det \phi(n) \neq 0, \forall n \geq n_0$

$$\begin{aligned} \phi(n) &= A^{n-n_0} \phi(n_0). \text{ By Abel's lemma } \det \phi(n) = (\det A)^{n-n_0} \det \phi(n_0) \neq 0 \\ &\Rightarrow \phi(n_0) \neq 0 \end{aligned}$$

$$\Leftarrow \text{Suppose } \det \phi(n_0) \neq 0.$$

Since  $\det \phi(n) = (\det A)^{n-n_0} \det \phi(n_0)$  and  $\det A \neq 0$ ,

$$\det \phi(n) = (\det A)^{n-n_0} \det \phi(n_0) \neq 0$$

(ii).  $\Rightarrow$  Suppose  $x_1(n), x_2(n), \dots, x_k(n)$  are linearly independent.

$$\text{Let } \phi(n) = [x_1(n), x_2(n), \dots, x_k(n)] \text{ and } \phi(n+1) = A\phi(n)$$

$$\Rightarrow \det \phi(n) \neq 0, \forall n \geq n_0$$

$$\text{Since } \det \phi(n) = (\det A)^{n-n_0} \det \phi(n_0), \det \phi(n_0) \neq 0.$$

⇐ Suppose  $\det \phi(n_0) \neq 0$ .

Since  $\det \phi(n) = (\det A)^{n-n_0} \det \phi(n_0)$ , and  $\det A \neq 0$ ,

$$\det \phi(n) = (\det A)^{n-n_0} \det \phi(n_0) \neq 0$$

∴  $\{x_1(n), x_2(n), \dots, x_k(n)\}$  is a fundamental set of solution of  $\phi(n+1) = A\phi(n)$

Hence the solutions  $x_1(n), x_2(n), \dots, x_k(n)$  are linearly independent  $\forall n \geq n_0$

**Theorem-6:** There are  $k$ -linearly independent solutions of system

$$x(n+1) = Ax(n), \quad n \geq n_0$$

**Proof:**  $e_i = (0, \dots, 0, 1, 0, \dots, 0)^T$  be the standard unit vectors for  $1 \leq i \leq k$  in  $\mathbb{R}^k$

By unique-existence theorem, for each  $i = 1, 2, \dots, k$  there is a solution

$$x(n, n_0, e_i) \text{ with initial condition } (n, n_0, e_i) = e_i$$

Since  $\phi(n_0) = I, \det \phi(n_0) = 1 \neq 0$

⇒  $\{x(n, n_0, e_i) / 1 \leq i \leq k\}$  is linearly independent

⇒ there are  $k$  - linearly independent solutions of system (3.1.2) for  $n \geq n_0$

**Linearity principle (Superposition):**

An important feature of the solution of system (2.2) is that, they are closed under addition and scalar multiplication. That is, if  $x_1(n)$  and  $x_2(n)$  are solutions of (2.2) and  $c \in \mathbb{R}$ , then

i)  $x_1(n) + x_2(n)$  is a solution of (2.2)

ii)  $cx_1(n)$  is a solution of (2.2)

**Proof:** i) Let  $x(n) = x_1(n) + x_2(n)$

$$\text{Then } x(n) = x_1(n+1) + x_2(n+1) = Ax_1(n) + Ax_2(n) = A[x_1(n) + x_2(n)] = Ax(n)$$

ii) Let  $x(n) = cx_1(n)$

$$\text{Then } x(n+1) = cx_1(n+1) = cAx_1(n) = A[cx_1(n)] = Ax(n)$$

An immediate consequence of the linearity principle is that if  $x_1(n), x_2(n), \dots, x_k(n)$  are

solutions of system (2.2), then  $x(n) = \sum_{i=1}^k c_i x_i(n)$  is also a solution of (2.2),

where,  $c_i \in \mathbb{R}, 1 \leq i \leq k$ . This leads the following definition

**Definition-7:**

Assuming that  $\{x(n)/1 \leq i \leq k\}$  is any linearly independent set of solution of system (2.2). Then the general solution of (2.2) is defined to be:

$$x(n) = \sum_{i=1}^k c_i x_i(n), \text{ where } c_i \in \mathbb{R} \text{ and at least one } c_i \neq 0$$

The above equation may be written as  $x(n) = \phi(n)C$ , where  $\phi(n) = [x_1(n), x_2(n), \dots, x_k(n)]$  is a fundamental matrix of

$$\phi(n + 1) = A\phi(n) \text{ and } C = (c_1, c_2, \dots, c_k) \in \mathbb{R}^k$$

**2.2. Inhomogeneous Autonomous Systems**

Consider the inhomogeneous autonomous system (2.1). We define a particular solution  $y_p(n)$  of (2.1) as any k-vector function that satisfy the inhomogeneous difference system

**Theorem-8:**

Any solution  $y(n)$  of (2.1) can be written as:

$$y(n) = \phi(n)C + y_p(n) \dots\dots\dots (2.3)$$

for any appropriate choice of the constant vector  $C$ , and a particular solution  $y_p(n)$

**Proof:**

Let  $y(n)$  be a solution of (2.1) and  $y_p(n)$  be a particular solution of (2.1)

$$\text{Let } x(n) = y(n) - y_p(n)$$

$$\text{Then } x(n+1) = y(n+1) - y_p(n+1)$$

$$= Ay(n) - Ay_p(n)$$

$$= A[y(n) - y_p(n)]$$

$$= Ax(n)$$

Hence  $x(n)$  is a solution of the homogeneous equation (2.2)

Thus  $x(n) = \phi(n)C$ , for some constant  $C$

$$\Rightarrow y(n) - y_p(n) = \phi(n)C$$

$$\Rightarrow y(n) = \phi(n)C + y_p(n)$$

Next, let's derive a formula to evaluate  $y_p(n)$ .

**Lemma-9:**

A particular solution of (2.1) may be given by:

$$y_p(n) = \sum_{r=n_0}^{n-1} A^{n-r-1} g(r), \text{ with } y_p(n_0) = 0$$

**Proof:** 
$$\begin{aligned} y_p(n+1) &= \sum_{r=n_0}^n A^{n-r} g(r) \\ &= \sum_{r=n_0}^{n-1} A^{n-r} g(r) + g(n) \\ &= \sum_{r=n_0}^{n-1} A A^{n-r-1} g(r) + g(n) \\ &= A \sum_{r=n_0}^{n-1} A^{n-r-1} g(r) + g(n) \\ &= A y_p(n) + g(n) \end{aligned}$$

**Theorem-10: (Variation of Constant Formula)**

The unique solution of the initial value problem

$$y(n+1) = Ay(n) + g(n), y(n_0) = y_0, \forall n \geq n_0 \geq 0 \text{ is given by:}$$

$$y(n, n_0, y_0) = A^{n-n_0} y_0 + \sum_{r=n_0}^{n-1} A^{n-r-1} g(r)$$

**Proof:** By theorem8 and lemma9, we have

$$\begin{aligned} y(n) &= \phi(n)y_0 + y_p(n) \\ &= A^{n-n_0} y_0 + \sum_{r=n_0}^n A^{n-r} g(r) \end{aligned}$$

**Example:** Solve  $\begin{cases} y_1(n+1) = 2y_1(n) + 3y_2(n) + 1 \\ y_2(n+1) = y_1(n) + y_2(n) \end{cases}, y_1(0) = 0, y_2(0) = -1$

**Solution:** Let  $y(n+1) = \begin{pmatrix} y_1(n+1) \\ y_2(n+1) \end{pmatrix}, A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}, g(n) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, y(0) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

Then the given system may be written as:

$$y(n+1) = Ay(n) + g(n)$$

$$\rho_A(\lambda) = \begin{vmatrix} 2-\lambda & 3 \\ 1 & 4-\lambda \end{vmatrix} = (\lambda-1)(\lambda-5)$$

Thus  $p_A(\lambda) = 0 \Rightarrow \lambda_1 = 1, \lambda_2 = 5$

$$M(0) = I, M(1) = A - \lambda_1 I = \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix}$$

$$u_1(n) = \lambda_1^n = 1^n = 1$$

$$u_2(n) = \sum_{i=0}^{n-1} \lambda_2^{n-i-1} u_1(i) = \sum_{i=0}^{n-1} 5^{n-i-1} \cdot 1 = 5^{n-1} \sum_{i=0}^{n-1} 5^{-i} = \frac{5^n - 1}{4}$$

$$A^n = u_1(n)I + u_2(n)M_1$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \left( \frac{5^n - 1}{4} \right) \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} \frac{5^n + 3}{4} & \frac{3(5^n - 1)}{4} \\ \frac{5^n - 1}{4} & \frac{3 \cdot 5^n + 1}{4} \end{pmatrix}$$

$$A^n y(0) = \begin{pmatrix} \frac{5^n + 3}{4} & \frac{3(5^n - 1)}{4} \\ \frac{5^n - 1}{4} & \frac{3 \cdot 5^n + 1}{4} \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{-3(5^n - 1)}{4} \\ \frac{-3 \cdot 5^n - 1}{4} \end{pmatrix}$$

$$y_p(n) = \sum_{r=n_0}^{n-1} A^{n-r-1} g(r)$$

$$= \sum_{r=0}^{n-1} \begin{pmatrix} \frac{5^{n-r-1} - 3}{4} & \frac{3(5^{n-r-1} - 1)}{4} \\ \frac{5^{n-r-1} - 1}{4} & \frac{3 \cdot 5^{n-r-1} + 1}{4} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \sum_{r=0}^{n-1} \begin{pmatrix} \frac{5^{n-r-1} - 3}{4} \\ \frac{5^{n-r-1} - 1}{4} \end{pmatrix} = \begin{pmatrix} \frac{3 \cdot 5^n - 12n - 3}{16} \\ \frac{5^n - 4n - 1}{16} \end{pmatrix}$$

$$\text{Hence } y(n) = \left( \frac{-3(5^n - 1)}{4} \right) + \left( \frac{3.5^n - 12n - 63}{16} \right)$$

$$= \left( \frac{-9.5^n - 12n + 9}{16} \right)$$

$$= \left( \frac{-(11.5^n + 4n + 5)}{16} \right)$$

Thus,  $y_1(n) = \frac{-9.5^n - 12n + 9}{16}$ ,  $y_2(n) = \frac{-(11.5^n + 4n + 5)}{16}$  is the solution of the system.

Next, let's see how to transform the k-order linear autonomous difference equation in to a k-dimensional system of first order equations.

Consider the equation  $y(n+k) + p_1y(n+k-1) + \dots + p_ky(n) = g(n)$  to represent equation as a system of first order equations of dimension-k.

We let

$$x_1(n) = y(n)$$

$$x_2(n) = y(n+1) = x_1(n+1)$$

⋮

$$x_k(n) = y(n+k-1) = x_{k-1}(n+1)$$

$$\text{Hence } \begin{cases} x_1(n+1) = x_2(n) \\ \vdots \\ x_{k-1}(n+1) = x_k(n) \\ x_k(n) = -p_kx_1(n) - p_{k-1}x_2(n) - \dots - p_1x_k(n) + g(n) \end{cases}$$

In vector notation, we write this system as:

$$x(n+1) = Ax(n) + f(n)$$

where,  $x(n) = [x_1(n), x_2(n), \dots, x_k(n)]^T$ ,

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -p_k(n) & -p_{k-1}(n) & -p_{k-2}(n) & \dots & -p_1(n) \end{pmatrix},$$

$$f(n) = \begin{pmatrix} 0 \\ \vdots \\ g(n) \end{pmatrix}$$

### 3 . THE JORDAN FORM

The Jordan form of a matrix is vital for both theoretical and computational purpose in autonomous systems. In this section, we will describe the Jordan form and derive a method for computing fundamental matrices.

#### 2.1 Diagonalizable Matrices

The matrix  $A = [a_{ij}]_{k \times k}$  is called diagonal if  $a_{ij} = 0$  whenever  $i \neq j$ . Conventionally, we denote such a matrix as  $D = \text{diag}[a_{11}, a_{22}, \dots, a_{kk}]$ . The diagonal matrix is nonsingular if and only if all its diagonal entries are non-zero.

#### Definition 3.1:

Let A and B be two  $k \times k$  matrices. A is said to be similar to B if there exists a nonsingular matrix P such that  $P^{-1}AP = B$

#### Theorem 3.2:

Similar matrices have the same eigenvalues.

#### Proof:

Let A is similar to B. Then  $B = P^{-1}AP$ , for some nonsingular matrix P. Thus,

$$\begin{aligned} \det(B - \lambda I) &= \det(P^{-1}AP - \lambda I) \\ &= \det(P^{-1}AP - \lambda P^{-1}P) \\ &= \det(P^{-1}AP - P^{-1}\lambda P) \\ &= \det[P^{-1}(A - \lambda I)P] \\ &= \det(P^{-1})\det(A - \lambda I)\det P \\ &= \det(A - \lambda I)\det(P^{-1})\det P \\ &= \det(A - \lambda I) \end{aligned}$$

Thus, A and B have the same characteristic polynomial. Since eigenvalues are roots of the characteristic polynomial A and B have the same eigenvalues

Therefore from theorem 3.2 we may conclude that similar transformation preserves eigenvalues.

**Definition 3.3:**

The  $k \times k$  matrix  $A$  said to be diagonalizable if it is similar to a diagonal matrix.

Not every square matrix is diagonalizable. A necessary and sufficient condition for  $A$  to be diagonalizable is that its eigenvectors form linearly independent set.

Let  $A$  have eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_k$  with associated eigenvectors  $\xi_1, \xi_2, \dots, \xi_k$ .

Then, for  $i=1, 2 \dots k$ , we have  $A\xi_i = \lambda_i \xi_i$ .

$$\begin{aligned} \text{Consider } AP &= A[\xi_1, \xi_2, \dots, \xi_k] = [A\xi_1, A\xi_2, \dots, A\xi_k] = [\lambda_1 \xi_1, \lambda_2 \xi_2, \dots, \lambda_k \xi_k] \\ &= [\xi_1, \xi_2, \dots, \xi_k] \text{diag}[\lambda_1, \lambda_2, \dots, \lambda_k] = PD \end{aligned}$$

$$\Rightarrow A = PDP^{-1}, \text{ where } P = [\xi_1, \xi_2, \dots, \xi_k], \text{ and } D = \text{diag}[\lambda_1, \lambda_2, \dots, \lambda_k]$$

$D$  is unique up to ordering of the eigenvalues.

Let  $A$  be diagonalizable. Then there exists a nonsingular matrix  $P$  such that

$$\begin{aligned} P^{-1}AP &= D = \text{diag}[\lambda_1, \lambda_2, \dots, \lambda_k] \\ \Rightarrow A &= PDP^{-1} \end{aligned}$$

$$\Rightarrow A^n = PD^n P^{-1} = P \begin{pmatrix} \lambda_1^n & & 0 \\ & \ddots & \\ 0 & & \lambda_k^n \end{pmatrix} P^{-1}$$

We are interested in finding another (but simpler) fundamental matrix of the equation

$$x(n+1) = Ax(n)$$

Then we let

$$\phi(n) = A^n P = P \begin{pmatrix} \lambda_1^n & & 0 \\ & \ddots & \\ 0 & & \lambda_k^n \end{pmatrix} \dots \dots \dots (3.5)$$

From formula (3.5) we have  $\phi(0) = P$  and consequently,

$$A^n = \phi(n) \cdot \phi^{-1}(0) \dots \dots \dots (3.6)$$

Let  $P = [\xi_1, \xi_2, \dots, \xi_k]$  where  $\xi_i$  is the  $i^{\text{th}}$  column of  $P$

$$P^{-1}AP = D$$

Then  $\Rightarrow AP = PD$

$$\Rightarrow A\xi_i = \lambda_i \xi_i, i = 1, 2, \dots, k$$

Thus,  $\xi_i$ ,  $1 \leq i \leq k$  is the eigenvector of  $A$  corresponding to  $\lambda_i$ , and hence the  $i^{\text{th}}$  column of  $P$  is the eigenvector of  $A$  corresponding to the  $i^{\text{th}}$  eigenvalue of  $A$ . Since  $\det P \neq 0$ , the eigenvectors  $\xi_1, \xi_2, \dots, \xi_k$  of  $A$  are linearly independent.

Now let  $\lambda_1, \lambda_2, \dots, \lambda_k$  be eigenvalues of  $A$  and let  $\xi_1, \xi_2, \dots, \xi_k$  be the corresponding linearly independent eigenvectors of  $A$ .

Then from formula (3.3) we have

$$\phi(n) = [\xi_1, \xi_2, \dots, \xi_k] \begin{pmatrix} \lambda_1^n & & 0 \\ & \ddots & \\ 0 & & \lambda_k^n \end{pmatrix} = [\lambda_1^n \xi_1, \lambda_2^n \xi_2, \dots, \lambda_k^n \xi_k] \dots \dots \dots (3.7)$$

Hence the general solution of (2.2) may be given by;

$$x(n) = \sum_{i=1}^k c_i \lambda_i^n \xi_i \dots \dots \dots (3.8)$$

**Example;** Find the general solution of

$$x(n+1) = Ax(n), \quad A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$$

**Solution:**

$$p_A(\lambda) = \det \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix} = -(\lambda - 1)^2 (\lambda - 5)$$

$$p_A(\lambda) = 0 \Rightarrow \lambda = 1, \lambda = 5$$

Hence  $\lambda_1 = 5, \lambda_2 = 1 = \lambda_3$  are eigenvalues of  $A$ .

To find the corresponding eigenvectors of  $A$ , we solve the equation  $(A - \lambda I)\xi = 0$ .

Hence for  $\lambda_1 = 5$ ,  $\begin{pmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Solving the system gives us the first eigenvector  $\xi_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

For  $\lambda_2 = 1 = \lambda_3$ , we have 
$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Consequently,  $d_1 + 2d_2 + d_3 = 0$  is the only equation obtained from this algebraic system.

Let  $d_1 = s, d_2 = t$  then  $s + 2t + d_3 = 0$

$$\Rightarrow d_3 = -s - 2t$$

Then 
$$\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = \begin{pmatrix} s \\ t \\ -s - 2t \end{pmatrix} = \begin{pmatrix} s \\ 0 \\ -s \end{pmatrix} + \begin{pmatrix} 0 \\ t \\ -2t \end{pmatrix}$$

Obviously, there are infinitely many choices for  $\xi_2$  and  $\xi_3$ .

Thus let  $s=1, t=0$ , then  $\xi_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$  and let  $s=0, t=1$  then  $\xi_3 = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$

Therefore using formula (3.8) we see that the general solution is

$$\begin{aligned} x(n) &= c_1 5^n \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} c_1 5^n + c_2 \\ c_1 5^n + c_3 \\ c_1 5^n - c_2 - 2c_3 \end{pmatrix} \end{aligned}$$

Next we will examine the case where the matrix  $A$  has a complex eigenvalues

Let  $A$  be a real  $k \times k$  matrix with eigenvalue  $\lambda = \alpha + i\beta$ ,  $\alpha, \beta$  are real.

Then  $\bar{\lambda} = \alpha - i\beta$  is also an eigenvalue of  $A$ . Moreover, if  $\xi$  is the eigenvector of  $A$  corresponding to the eigenvalue  $\lambda = \alpha + i\beta$ , then  $\bar{\xi}$  is the eigenvector of  $A$  corresponding to the eigenvalue  $\bar{\lambda} = \alpha - i\beta$ .

Suppose that  $\xi = \xi_1 + i\xi_2$ . A solution of (2.2) may be given by

$$x(n) = (\alpha + i\beta)^n (\xi_1 + i\xi_2)$$

Let  $r = \sqrt{\alpha^2 + \beta^2}$ ,  $\theta = \tan^{-1}(\beta/\alpha)$ . Thus the solution may be written as

$$x(n) = [r(\cos \theta + i \sin \theta)]^n (\xi_1 + i \xi_2)$$

$$= r^n [(\cos n\theta)\xi_1 - (\sin n\theta)\xi_2] + ir^n [(\cos n\theta)\xi_2 + (\sin n\theta)\xi_1]$$

$$= u(n) + iv(n), \text{ where } u(n) = r^n [(\cos n\theta)\xi_1 - (\sin n\theta)\xi_2], \text{ and}$$

$v(n) = r^n [(\cos n\theta)\xi_2 + (\sin n\theta)\xi_1]$  are linearly independent solutions of (2.2)

Hence we do not need to consider the solution generated by  $\bar{\lambda}$  and  $\bar{\xi}$

**Example:**

Find a general solution of the system

$$x(n+1) = Ax(n), \quad A = \begin{pmatrix} 1 & -5 \\ 1 & -1 \end{pmatrix}$$

**Solution:**  $\rho_A(\lambda) = \begin{vmatrix} 1-\lambda & -5 \\ 1 & -1-\lambda \end{vmatrix} = \lambda^2 + 4$

$$p_A(\lambda) = 0 \quad \lambda = \pm 2i$$

Thus  $\lambda_1 = 2i, \lambda_2 = -2i$  are eigenvalues of  $A$  and the corresponding eigenvectors are

$$\xi_1 = \begin{pmatrix} 1-2i \\ 5 \\ 1 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} 1+2i \\ 5 \\ 1 \end{pmatrix}$$

Hence  $x(n) = (2i)^n \begin{pmatrix} 1-2i \\ 5 \\ 1 \end{pmatrix}$  is a solution.

Since  $r = 2, \theta = \pi/2$ ,  $(2i)^n = 2^n [\cos \frac{n\pi}{2} + i(\sin \frac{n\pi}{2})]$

Hence  $x(n) = 2^n [\cos \frac{n\pi}{2} + i(\sin \frac{n\pi}{2})] \begin{pmatrix} 1-2i \\ 5 \\ 1 \end{pmatrix}$

$$= 2^n \begin{pmatrix} \frac{1}{5} \cos \frac{n\pi}{2} + \frac{2}{5} \sin \frac{n\pi}{2} \\ \cos \frac{n\pi}{2} \\ \sin \frac{n\pi}{2} \end{pmatrix} + i2^n \begin{pmatrix} -\frac{2}{5} \cos \frac{n\pi}{2} + \frac{1}{5} \sin \frac{n\pi}{2} \\ \sin \frac{n\pi}{2} \\ \cos \frac{n\pi}{2} \end{pmatrix}$$

Thus  $u(n) = 2^n \begin{pmatrix} \frac{1}{5} \cos \frac{n\pi}{2} + \frac{2}{5} \sin \frac{n\pi}{2} \\ \cos \frac{n\pi}{2} \end{pmatrix}$  and  $v(n) = 2^n \begin{pmatrix} -\frac{2}{5} \cos \frac{n\pi}{2} + \frac{1}{5} \sin \frac{n\pi}{2} \\ \sin \frac{n\pi}{2} \end{pmatrix}$  are linearly

independent solutions of the given system.

Hence a general solution of the system may be given as

$$x(n) = c_1 2^n \begin{pmatrix} \frac{1}{5} \cos \frac{n\pi}{2} + \frac{2}{5} \sin \frac{n\pi}{2} \\ \cos \frac{n\pi}{2} \end{pmatrix} + c_2 2^n \begin{pmatrix} -\frac{2}{5} \cos \frac{n\pi}{2} + \frac{1}{5} \sin \frac{n\pi}{2} \\ \sin \frac{n\pi}{2} \end{pmatrix}$$

$$= 2^n \begin{pmatrix} (\frac{1}{5} c_1 - \frac{2}{5} c_2) \cos \frac{n\pi}{2} + (\frac{2}{5} c_1 + \frac{1}{5} c_2) \sin \frac{n\pi}{2} \\ c_1 \cos \frac{n\pi}{2} + c_2 \sin \frac{n\pi}{2} \end{pmatrix}$$

So far we have discuss about the solution of system  $x(n+1)=Ax(n)$ , where  $A$  is diagonalizable.

We now turn our attention to the general case where the matrix  $A$  is not diagonalizable. This happens when  $A$  has repeated eigenvalues and one is not able to generate  $k$  linearly independent eigenvectors. For example  $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$  is not diagonalizable

## 2.2 Jordan Canonical Form

The Jordan canonical form is a set of “almost diagonal matrices”, called Jordan matrices, which includes the diagonal matrices. A Jordan matrix that is similar to a given matrix is called the Jordan canonical form (or sometimes the Jordan normal form) of the matrix. Although every square matrix is not diagonalizable, it is possible to show that every matrix is similar to a Jordan form matrix  $J$ , i.e  $A = P^{-1}JP$ , where  $J$  is a block diagonal matrix

$$J = \text{diag}[J_1, J_2, \dots, J_r] = \begin{pmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_r \end{pmatrix}, 1 \leq r \leq k \dots\dots\dots (3.9)$$

with each block  $J_i$  being of size  $s_i \times s_i$  with  $\sum s_i = k$  and of form

$$J_i = \begin{pmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{pmatrix} \dots\dots\dots (3.10)$$

where  $\lambda_i$  belongs to the spectrum of A. The matrix  $J_i$  is called a **Jordan block**.

The Jordan form of A is unique up to ordering of the blocks. If A is diagonalizable, then its Jordan form coincides with its diagonalized form.

The number of Jordan blocks corresponding to one eigenvalue  $\lambda$  is called the **geometric multiplicity of  $\lambda$** , and this number in turn, equals the **number of linearly independent eigenvectors** corresponding to A. The algebraic multiplicity of an eigenvalue  $\lambda$  is the number of times it is repeated.

**Note that** a matrix of the form  $\begin{pmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{pmatrix}$  has only one eigenvector, namely, the unit

vector  $e_1 = (1 \ 0 \ \dots \ 0)^T$ . This shows us that the linearly independent eigenvectors of the Jordan form J given by formula (3.9) are:  $e_1, e_{s_1+1}, e_{s_1+s_2+1}, \dots, e_{s_1+s_2+\dots+s_{r-1}+1}$ .

Now, since  $P^{-1}AP = J$ , then  $AP = PJ \dots\dots\dots (3.11)$

Let  $P = (\xi_1, \xi_2, \dots, \xi_k)$ . Equating the first  $s_1$  columns of both sides in formula 3.11), we obtain:

$$A\xi_1 = \lambda_1\xi_1, \dots, A\xi_i = \lambda_1\xi_i + \xi_{i-1}, i = 2, 3, \dots, s_1 \dots\dots\dots (3.12)$$

Clearly,  $\xi_1$  is the only eigenvector of A in the Jordan chain  $\xi_1, \xi_2, \dots, \xi_{s_1}$ . The other vectors  $\xi_2, \xi_3, \dots, \xi_{s_1}$  are called **generalized eigenvectors of A**, and they may be obtained by using the equation

$$(A - \lambda_1 I)\xi_i = \xi_{i-1}, i = 2, 3, \dots, s_1 \dots\dots\dots (3.13)$$

Repeating this process for the remainder of the Jordan blocks, one may find the generalized eigenvectors corresponding to the  $m^{th}$  Jordan block using the equation:

$$(A - \lambda_1 I)\xi_{m_i} = \xi_{m_i-1}, i = 2, 3, \dots, s_m \dots\dots\dots (3.14)$$

**Definition**

A non-zero vector  $u \in \mathfrak{R}^k$  is said to be a generalized eigenvector of order  $m$  of matrix  $A \in \mathfrak{R}^{k \times k}$  corresponding to the eigenvalue  $\lambda \in \mathfrak{R}$  if  $(A - \lambda I)^m u = 0$ , but  $(A - \lambda I)^{m-j} u \neq 0$ , for  $j=1, 2, \dots, m$ .

In this case the set of vectors  $u_j = (A - \lambda I)^{m-j} u$  for  $j=1, 2, \dots, m$  is said to be form a Jordan chain of length  $m$ ; they satisfy the following chain of equalities:

$$(A - \lambda I)u_1 = 0,$$

$$(A - \lambda I)u_2 = u_1$$

$$(A - \lambda I)u_3 = u_2$$

⋮

$$(A - \lambda I)u_m = u_{m-1}$$

If  $A = PJP^{-1}$  is the Jordan canonical form of  $A$ , then

$$A^n = PJ^n P^{-1}, \text{ where, } J^n = \begin{pmatrix} J_1^n & 0 & \dots & 0 \\ 0 & J_2^n & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & J_r^n \end{pmatrix}$$

Notice that for any  $J_i, i=1,2,\dots,r$  we have  $J_i = \lambda_i I + N_i$ , where  $N_i = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & \ddots & \vdots & \\ \vdots & \vdots & \ddots & 1 & \\ 0 & 0 & \dots & 0 & \end{pmatrix}$  is a

$s_i \times s_i$  nilpotent matrix. i.e.  $N_i^m = 0, \forall m \geq s_i$

Hence  $J_i^n = (\lambda_i I + N_i)^n = \lambda_i^n I + \binom{n}{1} \lambda_i^{n-1} N_i + \binom{n}{2} \lambda_i^{n-2} N_i^2 + \dots + \binom{n}{s_i-1} \lambda_i^{n-s_i+1} N_i^{s_i-1}$

$$= \begin{pmatrix} \lambda_i^n & \binom{n}{1} \lambda_i^{n-1} & \dots & \binom{n}{s_i-1} \lambda_i^{n-s_i+1} \\ & \lambda_i^n & \ddots & \vdots \\ & & \ddots & \binom{n}{1} \lambda_i^{n-1} \\ & & & \lambda_i^n \end{pmatrix} \dots \dots \dots (3.15)$$

We know that  $x(n) = A^n C = PJ^n P^{-1} C$

$$\Rightarrow x(n) = PJ^n \hat{C} \quad \text{where, } \bar{C} = P^{-1} C \dots\dots\dots (3.16)$$

Hence, a fundamental matrix of system  $x(n+1) = Ax(n)$  may be given by  $\phi(n) = PJ^n$ . Also, the state transition matrix may be given by  $\phi(n, n_0) = PJ^{n-n_0} P^{-1}$  and thus  $\phi(n, n_0, x_0) = PJ^{n-n_0} P^{-1} x_0$

**Theorem:**

For any  $k \times k$  matrix  $A$ ,  $\lim_{n \rightarrow \infty} A^n = 0$  if and only if  $|\lambda| < 1$  for all eigenvalues  $\lambda$  of  $A$

**Proof:**

Since  $J$  is the direct sum of the Jordan blocks, it suffices to consider the powers of the Jordan blocks,  $J_i, i=1, 2, \dots, r$

$$[J_i]^n = [\lambda_i I + N_i]^n = \sum_{j=0}^n \binom{n}{j} \lambda_i^j N_i^{n-j} = \sum_{j=n-s_i+1}^n \binom{n}{j} \lambda_i^j N_i^{n-j} \quad \text{for all } n \geq s_i$$

Since the diagonal elements are all  $\lambda_i^n$  if  $J^n \rightarrow 0$ , it is necessarily that  $\lambda_i^n \rightarrow 0$ , which means that  $|\lambda_i| < 1$ .

Conversely if  $|\lambda_i| < 1$  we would like to prove that

$$\binom{n}{n-j} \lambda_i^{n-j} \rightarrow 0, \text{ as } n \rightarrow \infty \text{ for each } j=0, 1, 2, \dots, s_i - 1$$

But  $\left| \binom{n}{n-j} \lambda_i^{n-j} \right| = \left| \frac{n(n-1)\dots(n-j+1)}{j! \lambda_i^j} \lambda_i^n \right| \leq \left| \frac{n^j \lambda_i^n}{j! \lambda_i^j} \right|$

So that it will suffice to show that  $n^j |\lambda_i^n| \rightarrow 0$  as  $n \rightarrow \infty$ . An easy way to see this is take

logarithms and observes that  $j \log n + n \log |\lambda_i| \rightarrow -\infty$ . Therefore  $n^j |\lambda_i^n| \rightarrow 0$

In general, the generalized eigenvectors corresponding to an eigenvalue  $\lambda$  of algebraic multiplicity of  $m$  are solutions of the equation.

$$(A - \lambda I)^m \xi = 0 \dots\dots\dots (3.17)$$

The first eigenvector  $\xi_1$  corresponding to  $\lambda$  is obtained by solving the equation

$$(A - \lambda I) \xi = 0$$

The second eigenvector or generalized eigenvector  $\xi_2$  is obtained by the equation

$$(A - \lambda I)^2 \xi = 0 \quad \text{and so on.}$$

Now, if  $J$  is the Jordan form of  $A$ ; that is,  $P^{-1}AP = J$  or  $A = PJP^{-1}$  then,  $\lambda$  is an eigenvalue of  $A$  if and only if it is an eigenvalue of  $J$ . Moreover, if  $\xi$  is an eigenvector of  $A$  then,  $\overline{\xi} = P^{-1}\xi$  is an eigenvector of  $J$ .

We need like to know the structure of the eigenvectors  $\overline{\xi}$  of  $J$ . for this we appeal to the following simple lemma from Linear Algebra.

**Lemma:**

Let  $C = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$  be a  $k \times k$  block-diagonal matrix such that  $A$  is an  $r \times r$  and  $B$  is an  $s \times s$  matrix, with  $r + s = k$ . Then the following statements hold true:

i). If  $\lambda$  is an eigenvalue of  $A$ , then it is an eigenvalue of  $C$ . Moreover, the eigenvector and the generalized eigenvectors corresponding to  $\lambda$  are of the form  $\xi = (a_1, a_2, \dots, a_r, 0, \dots, 0)^T$

for some  $a_i \in \mathfrak{R}$

ii). If  $\lambda$  is an eigenvalue of  $B$ , then it is an eigenvalue of  $C$ . Moreover, the eigenvector and the generalized eigenvectors corresponding to  $\lambda$  are of the form  $\xi = (0, 0, \dots, 0, a_{r+1}, a_{r+2}, \dots, a_s)^T$  for

some  $a_i \in \mathfrak{R}$

**Proof:**

i). suppose that  $\lambda$  is an eigenvalue of  $A$ , and  $v = (a_1, a_2, \dots, a_r)^T$  is the corresponding eigenvector

Define  $\xi = (a_1, a_2, \dots, a_r, 0, \dots, 0)^T$   $a_i \in \mathfrak{R}$

Then clearly  $C\xi = \lambda\xi$  and thus  $\lambda$  is an eigenvalue of  $C$ .

Let the  $k \times k$  identity matrix  $I$  be written in the form  $I = \begin{pmatrix} I_r & 0 \\ 0 & I_s \end{pmatrix}$ , where  $I_r$  and  $I_s$  are,

respectively, the  $r \times r$  and  $s \times s$  identity matrices.

Let  $\lambda$  is an eigenvalue of A with algebraic multiplicity m.

$$\text{Then } (C - \lambda I)\xi = \begin{pmatrix} A - \lambda I_r & 0 \\ 0 & B - \lambda I_s \end{pmatrix} \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_r \\ \vdots \\ \xi_s \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\text{Hence } (A - \lambda I_r) \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_r \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \text{ has a non trivial solution } \bar{\xi} = \begin{pmatrix} a_1 \\ \vdots \\ a_r \end{pmatrix}$$

$$\text{However } (B - \lambda I_s) \begin{pmatrix} \xi_{r+1} \\ \vdots \\ \xi_s \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \text{ has only the trivial solution } \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

Then  $\xi = (a_1, a_2, \dots, a_r, 0, 0, \dots, 0)^T$  is an eigenvector of C corresponding to  $\lambda$ . The same analysis can be done for generalized eigenvectors by solving  $(C - \lambda I)^i \xi = 0, 1 \leq i \leq m$

ii) The proof of the second part is analogous.

**Example:** Solve the system  $x(n+1) = Ax(n)$ , where  $A = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{5}{2} & -\frac{1}{2} \\ 0 & 1 & 2 \end{pmatrix}, x(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

**Solution:**  $P_A(\lambda) = \det \begin{pmatrix} \frac{3}{2} - \lambda & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{5}{2} - \lambda & -\frac{1}{2} \\ 0 & 1 & 2 - \lambda \end{pmatrix} = -(2 - \lambda)^3$

Hence  $P_A(\lambda) = 0 \Rightarrow \lambda = 2$

Thus the eigenvalues of A are  $\lambda_1 = \lambda_2 = \lambda_3 = 2$ .

To find the eigenvectors for  $\lambda = 2$ , we solve  $(A - 2I)\xi = 0$

Let  $(A - \lambda I)\xi_1 = 0$ .

Then 
$$\begin{pmatrix} -1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & -1 \\ 2 & 2 & 2 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} -a_1 + a_2 + a_3 = 0 \\ a_1 + a_2 - a_3 = 0 \\ a_2 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} a_1 = 1 \\ a_2 = 0 \\ a_3 = 1 \end{cases}$$

Hence  $\xi_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

We must now find two generalized eigenvectors  $\xi_2$  and  $\xi_3$  using formula (4.2.6).

Thus  $(A - \lambda I)\xi_2 = \xi_1$

$$\Rightarrow \begin{pmatrix} -1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & -1 \\ 2 & 2 & 2 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} -a_1 + a_2 + a_3 = 2 \\ a_1 + a_2 - a_3 = 0 \\ a_2 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} a_1 = 1 \\ a_2 = 1 \\ a_3 = 2 \end{cases}$$

Hence  $\xi_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

Next  $(A - \lambda I)\xi_3 = \xi_2$

$$\Rightarrow \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

$$\Rightarrow \begin{cases} -a_1 + a_2 + a_3 = 2 \\ a_1 + a_2 - a_3 = 2 \\ a_2 = 2 \end{cases}$$

$$\Rightarrow \begin{cases} a_1 = 1 \\ a_2 = 2 \\ a_3 = 1 \end{cases}$$

Therefore,  $\xi_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

So,  $P = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix}$  and

$$P^{-1} = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -1 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

Thus  $J = P^{-1}AP = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

## 4. APPLICATIONS

This section of the paper deals with the application of concepts of first order difference equations.

These concepts are applied to models such as physics, economics and the explanation of the transition in trade model in economics and

### 4.1. A Trade Model

Example 7: (a Trade Model)

Now,  $\phi(n, x_0) = PJ^n P^{-1} x_0$

Consider a model of two countries with the following assumptions:

- (1) National income in period  $n$  is given by  $y_p(n)$  and  $y_f(n)$  for country 1 and 2 respectively.
- (2) Total consumption in period  $n$  is given by  $c_p(n)$  and  $c_f(n)$  for country 1 and 2 respectively.
- (3) Time is divided into periods of equal length, denoted by  $n = 0, 1, 2, \dots$

Let, for country  $p = 1, 2,$

$y_p(n)$  = National income in period  $n$

$c_p(n)$  = Total consumption in period  $n$

$x_p(n)$  = Exports in period  $n$

$i_p(n)$  = Net investment in period  $n$

$w_p(n)$  = Imports in period  $n$

$d_p(n)$  = Consumption of domestic products in period  $n$

Now from assumption (1) and (2) we have

$$y_p(n) = c_p(n) + x_p(n) - i_p(n) + i_p(n)$$

$$d_p(n) = c_p(n) - w_p(n) \tag{4.1}$$

$$\Rightarrow y_p(n) = d_p(n) + x_p(n) + i_p(n) \tag{4.2}$$

#### 4. APPLICATIONS

This section of the paper deals with the application of systems of first order difference equations. These equations can be applied in various fields of scientific inquiry such as physics, economics and biology. However, we only focus on the application of the equations in trade model in economics and heat equation in physics.

##### 4.1. A Trade Model

###### Example 1: (a Trade Model)

Consider a model of the trade between two countries restricted by the following assumptions

- (1) 
$$\text{National income} = \text{Consumption outlays} + \text{Net investment} + \text{Exports} - \text{Imports}$$
- (2) 
$$\text{Domestic consumption outlays} = \text{Total consumption} - \text{Imports}$$
- (3) Time is divided into periods of equal length, denoted by  $n=0, 1, 2, 3; \dots$

Let, for country  $j=1, 2,$

- $y_j(n)$  = National income in period  $n$
- $c_j(n)$  = Total consumption in period  $n$
- $x_j(n)$  = Exports in period  $n$
- $i_j(n)$  = Net investment in period  $n$
- $m_j(n)$  = Imports in period  $n$
- $d_j(n)$  = Consumption of domestic products in period  $n$

Now from assumptions (1) and (2) we have

$$y_j(n) = c_j(n) + x_j(n) - m_j(n) + i_j(n)$$

$$d_j(n) = c_j(n) - m_j(n) \dots\dots\dots (4.1)$$

$$\Rightarrow y_j(n) = d_j(n) + x_j(n) + i_j(n) \dots\dots\dots (4.2)$$

Now, we make the following reasonable assumption

The domestic consumption  $d_j(n)$  and the imports  $m_j(n)$  of each country at a period  $(n+1)$  are proportional to the country's national income  $y_j(n)$  one time earlier.

Thus there are positive constants  $a_{ij}$ ,  $i, j=1, 2$  such that

$$d_1(n+1) = a_{11}y_1(n) \quad m_1(n+1) = a_{21}y_1(n) \dots \dots \dots (4.3)$$

$$d_2(n+1) = a_{22}y_2(n) \quad m_2(n+1) = a_{12}y_2(n) \dots \dots \dots (4.4)$$

The constants  $a_{ij}$  are called marginal propensities

Since the trade is between country 1 and country 2, the exports of one must be equal to the imports of the other. That is;

$$m_1(n) = x_2(n) \quad , \quad m_2(n) = x_1(n) \dots \dots \dots (5.1.5)$$

From (4.2) we get

$$\begin{cases} y_1(n+1) = d_1(n+1) + x_1(n+1) + i_1(n+1) \\ y_2(n+1) = d_2(n+1) + x_2(n+1) + i_2(n+1) \end{cases} \dots \dots (4.6)$$

Using equations (4.3)-(4.6), we obtain

$$\begin{cases} y_1(n+1) = a_{11}y_1(n) + a_{12}y_2(n+1) + i_1(n+1) \\ y_2(n+1) = a_{21}y_1(n+1) + a_{22}y_2(n) + i_2(n+1) \end{cases} \dots \dots \dots (4.7)$$

$$\Rightarrow \begin{pmatrix} y_1(n+1) \\ y_2(n+1) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} i_1(n+1) \\ i_2(n+1) \end{pmatrix} \dots \dots \dots (4.8)$$

Let us assume the net investments  $i_1(n) = i_1$  and  $i_2(n) = i_2$  are constants

Then (4.8) becomes:

$$\begin{pmatrix} y_1(n+1) \\ y_2(n+1) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} i_1 \\ i_2 \end{pmatrix} \dots \dots \dots (4.9)$$

Hence (4.9) can be written as:

$$y(n+1) = Ay(n) + i \quad \dots \dots \dots (4.10)$$

Where  $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$  and  $i = \begin{pmatrix} i_1 \\ i_2 \end{pmatrix}$

By variation constant formula, we obtain:

$$y(n) = A^n y(0) + \sum_{r=0}^{n-1} A^{n-r-1} i$$

$$y(n) = A^n y(0) + \sum_{r=0}^{n-1} A^r i \dots\dots\dots (4.11)$$

The two countries have stable economy if the sum of the domestic consumption  $d_j(n+1)$  and the imports  $m_j(n+1)$  in period  $(n+1)$  is less than the national income  $y_j(n)$  in period  $n$ . that is:

$$d_j(n+1) + m_j(n+1) < y_j(n), j = 1,2$$

$$\Rightarrow a_{11} + a_{21} < 1, \quad a_{12} + a_{22} < 1, \quad \dots\dots\dots (4.12)$$

Under condition (4.12),  $|\lambda| < 1$  where,  $\lambda$  is an eigenvalue of  $A$

Hence the spectral radius of  $A$ , denoted by  $\rho(A) := \max \{|\lambda|\} < 1$

Thus Neumann series  $\sum_{r=0}^{\infty} A^r$  converges uniformly and  $A^n \rightarrow 0$ , as  $n \rightarrow \infty$

This implies that  $\sum_{r=0}^{\infty} A^r i = (I - A)^{-1} i$

$$\Rightarrow \lim_{n \rightarrow \infty} y(n) = (I - A)^{-1} i$$

Therefore, the national incomes of the two countries approach equilibrium values independent of the initial values of the national incomes  $y_1(0), y_2(0)$

For example, let  $a_{11} = 0.4, a_{21} = 0.5, a_{12} = 0.3, a_{22} = 0.6$ ,  $i_1 = 25$  billion dollars, and  $i_2 = 20$  billion dollars. If  $y_1(n)$  and  $y_2(n)$  denote the national incomes of country 1 and country 2, respectively, and  $y_1(0) = 500$  billion dollars and  $y_2(0) = 650$  billion dollars, then:

- (a) Find  $y_1(3)$ , and  $y_2(3)$
- (b) What are the equilibrium national incomes for the nations 1 and 2?

**Solution:** (a)  $A = \begin{pmatrix} 0.4 & 0.3 \\ 0.5 & 0.6 \end{pmatrix}$   $i = \begin{pmatrix} 25 \\ 20 \end{pmatrix}$ , and  $y(0) = \begin{pmatrix} 500 \\ 650 \end{pmatrix}$

$$y(n) = A^n y(0) + \sum_{r=0}^{n-1} A^r i$$

$$y(3) = A^3 y(0) + \sum_{r=0}^2 A^r i$$

$$= \begin{pmatrix} 314.45 \\ 523.9 \end{pmatrix} + \begin{pmatrix} 54.75 \\ 67.22 \end{pmatrix} = \begin{pmatrix} 369.2 \\ 591.1 \end{pmatrix}$$

Hence  $y_1(3) = 369.2$  billion dollars,  $y_2(3) = 591.1$  billion dollars

$$(b) \quad I - A = \begin{pmatrix} 0.6 & -0.3 \\ -0.5 & 0.4 \end{pmatrix}, \text{ then } (I - A)^{-1} = \begin{pmatrix} \frac{40}{9} & \frac{30}{9} \\ \frac{50}{9} & \frac{60}{9} \end{pmatrix}$$

$$\Rightarrow (I - A)^{-1} \begin{pmatrix} 25 \\ 20 \end{pmatrix} = \begin{pmatrix} \frac{1600}{9} \\ \frac{2450}{9} \end{pmatrix} = \begin{pmatrix} 177.78 \\ 272.22 \end{pmatrix}$$

Hence the equilibrium national incomes of country 1 and 2 are 177.78 billion dollars and 272.22 billion dollars respectively.

#### 4.2. The Heat Equation

##### Example:

Consider the distribution of heat through a thin bar composed of a homogeneous material. Let  $x_1, x_2, \dots, x_k$  be  $k$  equidistant points on the bar. Let  $T_i(n)$  be the temperature at time  $t_n = (\Delta t)n$  at the point  $x_i$ ,  $1 \leq i \leq k$ . Denote the temperatures at the left and the right ends of the bar at time  $t_n$  by  $T_0(n)$ ,  $T_{k+1}(n)$ , respectively.

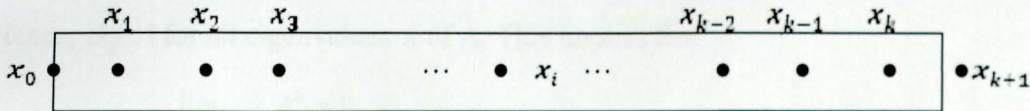


Figure 5.1 Heat transfer.

Assume that the sides of the bar are sufficiently well insulated that no heat energy is lost through them. The only thing, then, that affects the temperature at the point  $x_i$  is the temperature of the points next to it, which are  $x_{i-1}$ ,  $x_{i+1}$

Assume that the left end of the bar is kept at  $b$  degrees Celsius and the right end of the bar at  $c$  degrees Celsius. These conditions imply that  $T_0(n) = b$ , and  $T_{k+1}(n) = c$ , for  $n \geq 0$ . We assume that the temperature at a given point  $x_i$  is determined only by the temperature at the nearby points  $x_{i-1}$  and  $x_{i+1}$ . Then according to Newton's law of cooling, the change in temperature  $T_i(n+1) - T_i(n)$  at a point  $x_i$  from time  $n$  to  $n+1$  is directly proportional to the temperature difference between the point  $x_i$  and the nearby points  $x_{i-1}$  and  $x_{i+1}$ .

In other words, 
$$T_i(n+1) - T_i(n) = \alpha([T_{i-1}(n) - T_i(n)] + [T_{i+1}(n) - T_i(n)])$$

$$= \alpha([T_{i-1}(n) - 2T_i(n) + T_{i+1}(n)]) \dots \dots (4.13)$$

Or:

$$T_i(n+1) = \alpha T_{i-1}(n) + (1 - 2\alpha)T_i(n) + \alpha T_{i+1}(n), i = 2, 3, \dots, k-1$$

Similarly, one may also derive the following two equations:

$$T_1(n+1) = (1 - 2\alpha)T_1(n) + \alpha T_2(n) + \alpha b$$

$$T_k(n+1) = \alpha T_{k-1}(n) + (1 - 2\alpha)T_k(n) + \alpha c$$

This correlation may be written in the compact form

$$T(n+1) = AT(n) + g,$$

Where 
$$A = \begin{pmatrix} (1-2\alpha) & \alpha & & & \\ \alpha & (1-2\alpha) & \ddots & & \\ & \ddots & \ddots & \alpha & \\ & & & \alpha & (1-2\alpha) \end{pmatrix}, g = \begin{pmatrix} \alpha b \\ 0 \\ \vdots \\ \alpha c \end{pmatrix}$$

This is a tri-diagonal Toeplitz matrix. Its eigenvalues may be found by the formula

$$\lambda_n = (1 - 2\alpha) + \alpha \cos\left(\frac{n\pi}{k+1}\right), n = 1, 2, \dots, k.$$

Hence,  $|\lambda| < 1$  for all eigenvalues  $\lambda$  of A. This implies that

$$\lim_{n \rightarrow \infty} A^n = 0$$

From the variation of constant formula, it follows that

$$T(n) = A^n T(0) + \sum_{r=0}^{n-1} A^r g,$$

Thus,  $\lim_{n \rightarrow \infty} T(n) = (I - A)^{-1} g$ .

Finally, this equation points out that the temperature at the point  $x_i, 1 \leq i \leq k$ , approaches the  $i^{\text{th}}$  component of the vector  $(I - A)^{-1}g$ , regardless of the initial temperature at the point  $x_i$ .

Consider the above problem with  $k=3, \alpha = 0.4, T_0(n) = 10^{\circ}C, T_4(n) = 20^{\circ}C$

Then, 
$$A = \begin{pmatrix} 0.2 & 0.4 & 0 \\ 0.4 & 0.2 & 0.4 \\ 0 & 0.4 & 0.2 \end{pmatrix}, g = \begin{pmatrix} 4 \\ 0 \\ 8 \end{pmatrix}$$

$$(I - A)^{-1} = \begin{pmatrix} 0.8 & -0.4 & 0 \\ -0.4 & 0.8 & -0.4 \\ 0 & -0.4 & 0.8 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{15}{8} & \frac{5}{4} & \frac{5}{8} \\ \frac{5}{4} & \frac{5}{4} & \frac{5}{8} \\ \frac{4}{8} & \frac{2}{4} & \frac{4}{8} \\ \frac{13}{8} & \frac{5}{4} & \frac{15}{8} \\ \frac{8}{8} & \frac{4}{4} & \frac{8}{8} \end{pmatrix}$$

$$\text{Hence } \lim_{n \rightarrow \infty} T(n) = \begin{pmatrix} \frac{15}{8} & \frac{5}{4} & \frac{5}{8} \\ \frac{5}{4} & \frac{5}{4} & \frac{5}{8} \\ \frac{4}{8} & \frac{2}{4} & \frac{4}{8} \\ \frac{13}{8} & \frac{5}{4} & \frac{15}{8} \\ \frac{8}{8} & \frac{4}{4} & \frac{8}{8} \end{pmatrix} \begin{pmatrix} 4 \\ 0 \\ 8 \end{pmatrix} = \begin{pmatrix} \frac{25}{2} \\ 15 \\ \frac{43}{2} \end{pmatrix}$$

**Remark:** Let  $\Delta x = x_i - x_{i-1}$  and  $\Delta t = t_i - t_{i-1}$ . If we assume that the constant of proportionality  $\alpha$  depends on  $\Delta t$  and  $\Delta x$  then we may write

$$\alpha = \left[ \frac{\Delta t}{(\Delta x)^2} \right] \beta, \dots \dots \dots (4.14)$$

where  $\beta$  is a constant that depends the material of the bar. Formula (4.14) simply states that the smaller the value of  $\Delta t$ , the smaller should be the change in the temperature at a given point. Moreover, the smaller the separation of points, the larger should be their influence on the temperature changes in nearby points.

Using formula (4.14) in (4.13) yields

$$\frac{T_i(n+1) - T_i(n)}{\Delta t} = \beta \left[ \frac{T_{i+1}(n) - 2T_i(n) + T_{i-1}(n)}{(\Delta x)^2} \right] \dots \dots \dots (4.15)$$

If we let  $\Delta t \rightarrow 0, \Delta x \rightarrow 0$  as  $n \rightarrow \infty$ ,  $x_i = (\Delta x)i = x$ , and  $t_i = (\Delta t)i = t$ , then (4.15) gives the partial differential equation

$$\frac{\partial T(x,t)}{\partial t} = \beta \frac{\partial^2 T(x,t)}{\partial x^2} \dots\dots\dots (4.16)$$

Equation (4.16) is known as the Heat Equation

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