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GRADUATE SEMINAR REPORT

ON

INTERPOLATION, CURVE FITTING AND THEORY OF APPROXIMATION

(Submitted in Partial Fulfillment of Master of Science in Mathematics)

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PREFACE

We have said that numerical analysis is concerned with the solution of mathematical problems by arithmetic processes. Clearly, then, the need to approximate non-arithmetic quantities by arithmetic-and to ascertain the errors associated with such approximation-lies at the heart of much of numerical Analysis. Hence most numerical methods are based on replacing complicated objects, equations, etc. by simpler ones.

Today, the problem of approximating a function is a central problem in numerical analysis due to its importance in the development of software for digital computers. Moreover, in experimental work, the problem of fitting a curve to data that are subject to error is also encountered. It is clear that numerical method can give approximate solutions, in an efficient way, when ordinary analytical methods fail to these problems.

In this report, I consider the problem of approximating a general function by class of simpler functions. There are two uses for approximating functions:

i) The first is to replace complicated functions by some simpler functions so that many common operations such as differentiation and integration or even evaluation cab be more easily performed.
ii) The second major use is for recovery of a function from partial information about it, eg., from a table of (possibly only approximate) values.

The most commonly used classes of approximating functions are algebraic polynomials, trigonometric polynomials, and, lately, piece-wise polynomial functions. We consider best, and good, approximation by each of these classes.

An algebraic polynomial is the most convenient function to be handled in practice. To define a polynomial, it is only necessary to specify a finite number of its coefficients. Algebraic polynomials are readily evaluated, differentiated, integrated, and so forth. Therefore they are widely used for approximating various functions. The whole text deals with the practical methods of approximating functions of one variable on an interval or on a set of tabulated points.

Discussions about *interpolation*, *curve fitting*, and *theory of approximations* are made in three distinct chapters.

The first chapter deals with methods of obtaining polynomial approximations by means of interpolation, which is the simplest and widely used technique of approximation.

The second chapter deals with least-squares methods of fitting a curve to given data points, orthogonalization process and approximating continuous functions on a closed interval by orthogonal polynomials.

The last chapter deals with Chebyshev polynomials, which have got important applications in the approximation of functions in digital computers, and in economization of power series, also approximation of functions with periodic behavior by trigonometric functions, as well as piecewise polynomial approximation by the method of B-splines. In each chapter, the methods discussed have been well illustrated by appropriate examples and detailed solutions.

I take this opportunity to thank my advisors: Dr. G.L. Reddy, Dr. Adinew Alemayehu and Dr. Devendra Kumar for their help, encouragement and continuous follow up and made suggestions for the improvement of the report.

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CHAPTER ONE

INTERPOLATION AND POLYNOMIAL APPROXIMATION

1.1. Introduction

Polynomials are used as the basic means of approximation in nearly all areas of numerical analysis. They are used in the solution of equations and in the approximations of functions, of integrals, and derivatives, of solutions of integral and differential equations, etc. Polynomials owe this popularity to their simple structure, which makes it easy to construct effective approximations and then make use of them. For this reason, the representation and evaluation of polynomials is a basic topic in numerical analysis. We discuss this topic in the present chapter in the context of polynomial interpolation, the simplest and certainly the most widely used technique for obtaining polynomial approximations. More advanced methods for getting good approximations by polynomials and other approximating functions are given in chapter 2 and 3.

Given the set of tabular values (x_0, y_0), (x_1, y_1), ...(x_n, y_n), satisfying the relation y = f(x) where the explicit nature of f (x) is not known, it is required to find a simpler function, say $\phi(x)$, such that f (x) and $\phi(x)$ agree at the set of tabulated points. Such a process is called interpolation. If $\phi(x)$ is a polynomial, then the process is called polynomial interpolation and $\phi(x)$ is called the interpolating polynomial.

The existence of a polynomial function p(x) which approximates any continuous function f(x) on a finite interval [a, b] is guaranteed by the *weierstrass approximation theorem.*

Theorem (Weierstrass):

If the function f(x) is continuous on a finite interval [a, b], then given any $\varepsilon > 0$, there exists an N = N(ε) and a polynomial $p_n(x)$ of degree n such that $|f(x) - p_n(x)| < \varepsilon$ for all x in [a, b] and n > N(ε).

Proof:

Before going to see the proof of weierstrass approximation theorem, we have to define the Bernstein polynomial for which the proof depends up on it.

Definition: The Bernstein polynomial of degree n associated with the function f on [a, b] is defined by

$$B_n(f;x)=\sum_{k=0}^{n}\binom{n}{k}(x-a)^k(b-x)^{n-k}f(x_k) \qquad (1.1)$$

where the points $x_k = a+kh = a+(\frac{k}{n})(b-a)$ for k = 0, 1, 2, ..., n, n = 0, 1, 2, ...

In the special case where the interval is [0, 1], equation (1.1) reduces to

$$B_n(f;x)=\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}f(\tfrac{k}{n})=\sum_{k=0}^{n}f(\tfrac{k}{n})B_{n,k}(x) \qquad (1.2)$$

where the Bernstein basis polynomials are defined by

$$B_{n,k}(x)=\binom{n}{k}x^k(1-x)^{n-k}, \text{ for } k=0,1,2,\ldots,n \text{ and } n=0,1,2, \qquad (1.3)$$

Now in view of the binomial expansion, it is clear that we have

$$\sum_{k=0}^{n}B_{n,k}(x)=\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}=(x+(1-x))^n=1 \qquad (1.4)$$

From which it follows that the Bernstein polynomial for the function f(x) = 1 is itself 1.

Now, differentiating (1.4) we obtain

$$0=\sum_{k=0}^{n}\binom{n}{k}x^{k-1}(1-x)^{n-k-1}[k(1-x)-(n-k)x]$$

$$=\sum_{k=0}^{n}\binom{n}{k}x^{k-1}(1-x)^{n-k-1}(k-nx)$$

and multiplying by x(1-x) yields

$$0=\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}(k-nx) \qquad (1.5)$$

Hence for f(x) =x, $f(k/n)=k/n$, we have:

$$B_n(f;x)=\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}(\tfrac{k}{n})=\sum_{k=0}^{n}\binom{n}{k}x^{k+1}(1-x)^{n-k}=x. \text{ (by (1.4) and (1.5))} \qquad (1.6)$$

That is, Bernstein polynomial for the function f(x) = x reproduces this function exactly. Again differentiating (1.6) we have

$$1=\sum_{k=0}^{n}\binom{n}{k}(\tfrac{k}{n})x^{k-1}(1-x)^{n-k-1}.(k-nx)$$, and on multiplying by $\frac{x(1-x)}{n}$ we have

$$\frac{x(1-x)}{n}=\sum_{k=0}^{n}\binom{n}{k}(\tfrac{k}{n})x^{k}(1-x)^{n-k}.(\frac{k}{n}-x)$$

$$=\sum_{k=0}^{n}\binom{n}{k}(\tfrac{k}{n})^{2}x^{k}(1-x)^{n-k}-x^{2})$$

Now for f(x) = x^2, we have by using this relation,

$$B_n(x^2;x)=\sum_{k=0}^{n}\binom{n}{k}(\tfrac{k}{n})^{2}x^{k}(1-x)^{n-k}=\frac{x(1-x)}{n}+x^{2} \qquad (1.7)$$

Since $\max\limits_{0\le x\le 1} x(1-x)=\frac{1}{4}$, it follows that

$$|f(x)-B_n(f;x)|=|x^2-B_n(x^2;x)|\le\tfrac{1}{4n}\to 0,\ as\ n\to\infty.$$

On this occasion, we can guess that these Bernstein polynomials $B_n(f;x)$ give us good approximations to f(x) over [0, 1].

Now let us go to the proof of the theorem stated above:

This constructive proof is for functions, which are continuous on [0, 1], the modification to other intervals is achieved by a straight forward transformation of the interval [a, b] in to [0, 1]. Or by letting x = (b-a) t +a, then as x varies from a to b, t varies from 0 to1. Now putting f (x) = f ((b-a) t +a) = g (t), which is continuous on [0, 1], we can construct the required polynomial for it.

Hence by using the Bernstein polynomial and with out loss of generality assuming [a, b] = [0, 1], we must show that for any $\varepsilon>0$, $|f(x)-B_n(f;x)|<\varepsilon$, $\forall x\in[0,1]$ and n sufficiently large.

Let x $\in$ [0, 1], then using (1.4) which implies that

$$f(x)=\sum_{k=0}^{n}\binom{n}{k}x^{k}(1-x)^{n-k}f(x)$$, we get

$$|f(x)-B_n(f;x)| \le \sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}\left|f(x)-f(\tfrac{k}{n})\right| \tag{1.8}$$

Now, since f is continuous on [0, 1], it follows that it is uniformly continuous also.

(Continuity on a compact set $\Leftrightarrow$ uniform continuity)

Therefore, there exists $\delta > 0$ such that $|\mathrm{f(s)} - \mathrm{f(t)}| < \frac{\varepsilon}{2}$ whenever $|s-t| < \delta$.

Denote the set {k: |x-k/n| < δ} by A. Clearly, the sum on the right side of (1.8) can be separated in to two sums for those k in A and those that are not.

For the first of these, we have

$$\sum_{k\in A}\binom{n}{k}x^k(1-x)^{n-k}\left|f(x)-f(\tfrac{k}{n})\right| \le \sum_{k\in A}\binom{n}{k}x^k(1-x)^{n-k}\frac{\varepsilon}{2} \le \frac{\varepsilon}{2}$$

It remains to prove that we can choose n to make the second sum less than $\frac{\varepsilon}{2}$ also.

Now, f is continuous and so bounded by say, M. Therefore,

$$\sum_{k\notin A}\binom{n}{k}x^k(1-x)^{n-k}\left|f(x)-f(\tfrac{k}{n})\right| \le 2M\sum_{k\notin A}\binom{n}{k}x^k(1-x)^{n-k} \tag{1.9}$$

(as $|f(x)-f(\frac{k}{n})| \le |f(x)|+|f(\frac{k}{n})| \le M+M=2M, \forall x\in[0,1]$)

Multiplying (1.5) by x/n, we obtain

$0=\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}(\frac{k}{n}-x)$, while from the derivation of (1.7), we have

$$\frac{x(1-x)}{n}=\sum_{k=0}^{n}(\tfrac{k}{n})\binom{n}{k}x^k(1-x)^{n-k}(\tfrac{k}{n}-x).$$

Subtracting these two equations, we get $\frac{x(1-x)}{n}=\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}(\frac{k}{n}-x)^2$.

Since, for $k\notin A$, $\delta \le \left|\frac{k}{n}-x\right|$, from the denotation of A, it follows that

$$\begin{aligned}\sum_{k\notin A}\binom{n}{k}x^k(1-x)^{n-k}\delta^2 &\le \sum_{x\notin A}\binom{n}{k}x^k(1-x)^{n-k}(\tfrac{k}{n}-x)^2\\ &\le \sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}(\tfrac{k}{n}-x)^2\\ &=\frac{x(1-x)}{n}\le\frac{1}{4n}.\end{aligned}$$

Hence from (1.9),

$$\sum_{k \notin A} \binom{n}{k} x^k (1-x)^{n-k} \left| f(x) - f(\tfrac{k}{n}) \right| \le 2M \sum_{k \notin A} \binom{n}{k} x^k (1-x)^{n-k}$$

$$\le \frac{2M}{4n\delta^2}.$$

It is therefore sufficient to choose $n > \frac{M}{\varepsilon\delta^2}$ to achieve the desired bound as required.

$\therefore \ | f(x) - B_n(f; x) | \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ and so $B_n(f; x)$ converges uniformly to f.

i.e. given any function, defined and continuous on a closed interval, there exists a polynomial that is as close to the given function as desired.

1.2. **Errors In Polynomial Interpolation**

Let the function f(x) defined by the (n+1) points (x_i, y_i), i = 0, 1, 2, …,n be continuous and differentiable (n+1) times, and let f(x) be approximated by a polynomial $\phi_n(x)$ of degree not exceeding n such that $\phi_n(x_i) = y_i$, i = 0, 1, 2, …,n. (1.10)

If we now use $\phi_n(x)$ to obtain approximate values of f(x) at some points other than those defined by (1.1), what would be the accuracy of this approximation?

Since the expression $f(x) - \phi_n(x)$ vanishes for x = x_0, x_1, …, x_n, we put

$$f(x) - \phi_n(x) = L\pi_{n+1}(x) \text{ where} \tag{1.11}$$

$$\pi_{n+1}(x) = (x - x_0)(x - x_1)(x - x_2)\ldots(x - x_n) \tag{1.12}$$

and L is to be determined such that equation (1.11) holds for any intermediate value of x, say x = x′, $x_0 < x < x_n$. Clearly, $L = \frac{f(x') - \phi_n(x')}{\pi_{n+1}(x')}$. (1.13)

We construct a function F(x) such that

$$F(x) = f(x) - \phi_n(x) - L\pi_{n+1}(x), \text{ where L is given by (1.13).} \tag{1.14}$$

Therefore F (x_0) = F (x_1) = … = F (x_n) = F (x′) = 0, i.e. F (x) vanishes (n+2) times in the interval $x_0 < x < x_n$; consequently, by the repeated application of Rolle's theorem, $F'(x)$ must vanish (n+1) times, $F''(x)$ must vanish n times, etc., in the interval $x_0 < x < x_n$

In particular, $F^{(n+1)}(x)$ must vanish once in the interval. Let this point be given by $x = \varepsilon$ $x_0 < \varepsilon < x_n$. On differentiating (1.14) (n+1) times with respect to x and putting $x = \varepsilon$, we obtain $0 = f^{(n+1)}(\varepsilon) - L(n+1)!$ so that $L = \frac{f^{(n+1)}(\varepsilon)}{(n+1)!}$. (1.15)

Comparison of (1.13) and (1.15) yields the results $f(x') - \phi_n(x') = \frac{f^{(n+1)}(\varepsilon)}{(n+1)!}\pi_{n+1}(x')$.

Dropping the prime on x′, we obtain

$$f(x) - \phi_n(x) = \frac{\pi_{n+1}(x)}{(n+1)!} f^{(n+1)}(\varepsilon), \quad x_0 < \varepsilon < x_n. \tag{1.16}$$

which is the required expression for the error. Here f (x) is, generally, unknown and hence we do not have any information concerning $f^{(n+1)}(x)$.

Therefore formula (1.16) is almost useless in practical computations, while it is extremely useful in theoretical work in different branches of numerical analysis.

1.3. <u>The Taylor's Polynomial Approximation</u>

This section is concerned with how to find an approximating polynomial that agree with the given function and as many of its derivatives as possible at a single point x_0. Hence the shape of the graph of the polynomial be as close as possible to that of the given function near x_0.

Let $p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ be the required polynomial which approximates the given function f(x) near a point x_0.

Then $p(x_0) = f(x_0)$, $p'(x_0) = f'(x_0)$, $p''(x_0) = f''(x_0)$, and so on. Then the polynomial gets the following form:

Theorem : (*Taylor's Theorem*):

Suppose $f \in C^n[a,b]$ (a space of n-times continuously differentiable functions) and $f^{(n+1)}$ exists on [a, b]. Let $x_0 \in [a, b]$. For every $x \in [a, b]$, there exists $\xi(x)$ between x_0 and x with $f(x) = P_n(x) + R_n(x)$, where

$$P_n(x) = \sum_{k=0}^{n} \frac{f^k(x_0)}{k!}(x - x_0)^k \quad \text{and}$$

$$R_n(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)^{n+1}, \quad x_0 < \xi(x) < x.$$

$P_n(x)$ is the nth degree Taylor polynomial for f about x_0 and $R_n(x)$ is called the remainder term or truncation error associated with $P_n(x)$.

Note: 1. Truncation error refers to the error involved on using a truncated or finite summation to approximate the sum of an infinite series.

2. The infinite series obtained by taking the limit $P_n(x)$ as $n \to \infty$ is called the Taylor series for f about x_0.

Example:

1. a) Obtain the third-degree Taylor polynomial for $f(x) = (1+x)^{-2}$ about $x_0 = 0$.

b) Use this polynomial to approximate f (0.05) and $\int_0^{0.05} (1+x)^{-2} dx$.

Find an error bound for these approximations and compare your result to the exact (actual) values.

c) Use the same polynomial to approximate f (0.5), f (1), and f (10) and obtain the error committed.

Solution:

a) Performing the necessary differentiation on f(x) yields:

$f(x) = (1+x)^{-2}, \qquad f(0) = 1$

$f'(x) = -2(1+x)^{-3} \quad , f'(0) = -2$

$f''(x) = 6(1+x)^{-4} \quad , f''(0) = 6$

$f'''(x) = -24(1+x)^{-5}, f'''(0) = -24$

$f^{iv}(x) = 120(1+x)^{-6}, \quad f^{iv}(\xi) = 120(1+\xi)^{-6}$, where $0 < \xi < x$.

Hence from Taylor's theorem, we have

$p_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3.$

$= 1 - 2x + 3x^2 - 4x^3$ is the required polynomial.

b) i) $f(0.05) \approx P_3(0.05) = 1 - 2(0.05) + 3(0.05)^2 - 4(0.05)^3 = 0.907$. And the exact value is $f(0.05) = (1+0.05)^{-2} = 0.907029478$.

The error involved is given by $R_3(0.05) = \frac{f^{(IV)}(\xi)}{4!}(0.05)^4$.

$$|R_3(0.05)| = \frac{|120(1+\xi)^{-6}|}{4!}(0.05)^4 \le \frac{120}{24}(0.05)^4.\max(1+\xi)^{-6}, \xi \in [0, 0.05]$$

$$= 3.125\text{x}10^{-5},$$

and the actual error is about $|0.907 - 0.907029478| = 2.95\text{x}10^{-5}$.

ii) $\int_0^{0.05} (1+x)^{-2}dx \approx \int_0^{0.05} (1 - 2x + 3x^2 - 4x^3)dx = 0.04761875$, with an error given by

$\int_0^{0.05} R_3(x)dx$.

Hence $\left|\int_0^{0.05} R_3(x)dx\right| = \frac{120}{24}\int_0^{0.05} (1+\xi)^{-6}x^4dx \le 5\int_0^{0.05} x^4dx = 3.125\text{x}10^{-7}$.

Since the true value of $\int_0^{0.05} (1+x)^{-2}dx = 0.047619047$, the actual error is about $2.97\text{x}10^{-7}$ which is inside the error bound.

c) $f(0.5) \approx P_3(0.5) = 1 - 2(0.5) + 3(0.5)^2 - 4(0.5)^3 = 0.25$

$f(1) \approx P_3(1) = 1 - 2(1) + 3(1)^2 - 4(1)^3 = -2$

$f(10) \approx P_3(10) = 1 - 2(10) + 3(10)^2 - 4(10)^3 = -3719$.

And the actual values are $f(0.5) = (1+0.5)^{-2} = 0.444444$.

$f(1) = (1+1)^{-2} = 0.25$

$f(10) = (1+10)^{-2} = 8.26446281\text{x}10^{-3}$.

x	0.05	0.5	1	10
$P_3(x)$	0.97	0.25	-2	-3719
f(x)	0.907029478	0.444444	0.25	$8.26446281\text{x}10^{-3}$
$\lvert P_3(x) - f(x)\rvert$	$2.95\text{x}10^{-5}$	0.194	2.25	$3.719\text{x}10^{3}$

This table suggests that the Taylor polynomial of degree three for the function $f(x) = (1+x)^{-2}$, expanded about $x_0 = 0$ gives less accurate results as x moves away from x_0.

Since the Taylor polynomials have the property that all the information used in the approximation is concentrated at one point, x_0, the type of difficulty that occurs in the example part (c) is quite common. This limits the use of approximation by Taylor's polynomial to the situation where approximations are needed at points very close to x_0. Consequently, the Taylor polynomial is often of little use, and alternative methods of approximations must be sought.

Note: The primary use of Taylor polynomials in numerical analysis is not for approximations purposes, but for use in the derivation of numerical technique.

1.4. Interpolation with Equally Spaced Points

Assume that we have a table of values (x_i, y_i), $i = 0, 1, 2, \ldots, n$ of any function $y = f(x)$, the values of x being equally spaced, i.e. $x_i = x_0 + ih$, $i = 0, 1, 2, \ldots, n$. Suppose we are required to find a polynomial of the nth degree $P_n(x)$ such that $f(x)$ and $P_n(x)$ agree at the tabulated points so that we can manipulate any operation on it as a substitute of the given function with tolerable error of operation.

There are different methods of obtaining such a polynomial called *interpolation formulae* with equally spaced points of the arguments. We discuss in this section the following such formulae:

i) Newton's Interpolation Formulae
   a) Newton's Forward Difference Interpolation Formulae
   b) Newton's Backward Difference Interpolation Formulae

ii) Central Difference Interpolation Formulae:
   a) Gauss's Forward and Backward Interpolation Formula
   b) Stirling's Interpolation Formula
   c) Bessel's and Everett's Interpolation Formula.

1.4.1. **Finite Differences**

i) Forward Differences:

Finite differences deals with the changes that take place in the value of the function due to change in the independent variable. i.e. it the study of the relations that exist between the values assumed by the function, whenever the independent variable changes by finite jumps.

Consider y = f(x) and consecutive values of x differing by h. If the first such number is a, then the numbers are a, a + h, a + 2h, …, a + nh, ….

The corresponding value of f (x) is f (a), f (a+h), f (a+2h), …, f (a+nh), ….

$$\left.\begin{array}{l} f(a+h)-f(a) \\ f(a+2h)-f(a+h) \\ f(a+3h)-f(a+2h) \\ \vdots \end{array}\right\} \text{first difference; denoted by } \Delta f(a),\ \Delta f(a+h),$$

Δ f(a+2h), … where Δ is called the forward difference operator.

∴ The first forward difference is given by $\Delta y_i = y_{i+1} - y_i$ or

$$\Delta f(a) = f(a+h) - f(a)$$

The differences of these first differences are called second differences, denoted by

$\Delta^2 f(a) = \Delta f(a+h) - \Delta f(a) = f(a+2h) - 2f(a+h) + f(a)$

$\Delta^2 f(a+h) = \Delta f(a+2h) - \Delta f(a+h) = f(a+3h) - 2f(a+2h) + f(a+h)$ and so on.

The difference of these second differences are called third differences

$\Delta^3 f(a) = \Delta^2 f(a+h) - \Delta^2 f(a) = f(a+3h) - 3f(a+2h) + 3f(a+h) - f(a)$ and so on.

Hence it is clear that any higher order difference can easily be expressed in the above form.

The following table shows how the forward differences of all orders can be formed:

x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
a	f(a)					
		$\Delta f(a)$				
a+h	f(a+h)		$\Delta^2 f(a)$			
		$\Delta f(a+h)$		$\Delta^3 f(a)$		
a+2h	f(a+2h)		$\Delta^2 f(a+h)$		$\Delta^4 f(a)$	
		$\Delta f(a+2h)$		$\Delta^3 f(a+h)$		$\Delta^5 f(a)$
a+3h	f(a+3h)		$\Delta^2 f(a+2h)$		$\Delta^4 f(a+h)$	
		$\Delta f(a+3h)$		$\Delta^3 f(a+2h)$		
a+4h	f(a+4h)		$\Delta^2 f(a+3h)$			
		$\Delta f(a+4h)$				
a+5h	f(a+5h)					

Table 1. Forward Difference table

ii) Backward Difference

The differences f(a+h) – f(a), f(a+2h) – f(a+h), …, f(a+nh) – f(a+(n-1)h), … are called first backward differences if they are denoted by $\nabla f(a+h)$, $\nabla f(a+2h)$, …, $\nabla f(a+nh)$, respectively, i.e. $\nabla Y_i = Y_i - Y_{i-1}$ where ∇ is called the backward difference operator. In a similar way as that of forward difference we can have higher order backward differences:

$\nabla^2 f(a+2h) = \nabla f(a+2h) - \nabla f(a+h) = f(a+2h) - 2f(a+h) + f(a)$

$\nabla^3 f(a+3h) = \nabla^2 f(a+3h) - \nabla^2 f(a+2h) = f(a+3h) - 3f(a+2h) + 3f(a+h) - f(a)$ and so on.

The following table shows how the backward differences of all orders can be formed:

x	f(x)	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$	$\nabla^5 f(x)$
a	f(a)					
		∇ f(a+h)				
a+h	f(a+h)		∇^2f(a+2h)			
		∇ f(a+2h)		∇^3f(a+3h)		
a+2h	f(a+2h)		∇^2f(a+3h)		∇^4f(a+4h)	
		∇ f(a+3h)		∇^3f(a+4h)		∇^5f(a+5h)
a+3h	f(a+3h)		∇^2f(a+4h)		∇^4f(a+5h)	
		∇ f(a+4h)		∇^3f(a+5h)		
a+4h	f(a+4h)		∇^2f(a+5h)			
		∇ f(a+5h)				
a+5h	f(a+5h)					

Table 2. Backward Difference table.

iii) Central Differences

Definitions:

1. The central difference operator δ is defined by the relations:

$\delta f(x) = f(x + h/2) - f(x-h/2)$

i.e. $\delta Y_i = Y_{i+1/2} - Y_{i-1/2}$

Hence $\delta Y_{1/2} = Y_1 - Y_0 = \nabla Y_1 = \Delta Y_0$

$\delta Y_{3/2} = Y_2 - Y_1 = \nabla Y_2 = \Delta Y_1$

$\vdots$

$\delta Y_{n-1/2} = Y_n - Y_{n-1} = \nabla Y_n = \Delta Y_{n-1}$

2. The shift operator E is defined by the relation

$Ef(x) = f(x+h)$; i.e. $E Y_i = Y_{i+1}$.

$E^2 f(x) = f(x+2h)$; i.e. $E^2 Y_i = Y_{i+2}$.

$\vdots$

$E^n f(x) = f(x+nh)$; i.e. $E^n Y_i = Y_{i+n}$. And also

$E^{-n} f(x) = f(x-nh)$; i.e. $E^{-n} Y_i = Y_{i-n}$.

3. The averaging (mean) operator μ is defined by the relation:

$$\mu f(x) = \frac{f(x+\frac{h}{2})+f(x-\frac{h}{2})}{2}$$

$$= \frac{E^{\frac{1}{2}}+E^{\frac{-1}{2}}}{2}$$

From the definitions, the following relations can easily be established.

a) $\Delta f(a) = f(a+h) - f(a) = Ef(a)-f(a) = (E-1)f(a).$

$\Rightarrow \Delta = E-1$, as f(a) is arbitrary.

b) $\nabla f(a+h) = f(a+h) - f(a) = f(a+h) - E^{-1} f(a+h) = (1 - E^{-1})f(a+h)$

$\Rightarrow \nabla = 1 - E^{-1}$, as f(a+h) is arbitrary.

c) $\delta Y_i = Y_{i+1/2} - Y_{i-1/2} = E^{\frac{1}{2}} Y_i - E^{\frac{-1}{2}} Y_i = (E^{\frac{1}{2}} - E^{\frac{-1}{2}})Y_i$

$\Rightarrow \delta = (E^{\frac{1}{2}} - E^{\frac{-1}{2}})$

d) $\mu Y_i = \frac{Y_{i+\frac{1}{2}} + Y_{i-\frac{1}{2}}}{2} = (\frac{E^{\frac{1}{2}} + E^{\frac{-1}{2}}}{2})Y_i$

$\Rightarrow \mu = \frac{E^{\frac{1}{2}} + E^{\frac{-1}{2}}}{2}$

e) $\mu^2 = 1 + \frac{1}{4}\delta^2 \quad and \ \Delta = \nabla E = \delta E^{\frac{1}{2}}$, etc.

1.4.2. **Newton's Interpolation Formulae**

Let y = f(x) denote a function which takes the values f(a), f(a + h), f(a + 2h), …, f(a + nh) for (n+1) equidistant values a, a + h, a + 2h, …, a + nh of the independent variable x.

Let $P_n(x)$ be a polynomial of degree n such that f(x) and $P_n(x)$ agree at the tabulated points. Since $P_n(x)$ is a polynomial of the nth degree, it may be written as:

$$P_n(x) = A_0 + A_1(x-a) + A_2(x-a)(x-a-h) + A_3(x-a)(x-a-h)(x-a-2h) + \dots$$
$$+ A_n(x-a)(x-a-h)(x-a-2h). \dots . (x-a-(n-1)h) \qquad (1.7)$$

Where $A_0, A_1, A_2, \dots, A_n$ are coefficients to be determined such that

$P_n(a) = f(a),\ P_n(a+h) = f(a+h), \dots, P_n(a+nh) = f(a+nh).$

Putting, x = a, a + h, a + 2h, …, a + nh successively in (1.7) and writing the values of $P_n(a)$, $P_n(a+h)$,…, $P_n(a+nh)$, we obtain:

$f(a) = A_0 \Rightarrow \boxed{A_0 = f(a)}$

$f(a+h) = A_0 + hA_1 \Rightarrow A_1 = \dfrac{f(a+h)-f(a)}{h} \Rightarrow \boxed{A_1 = \dfrac{\Delta f(a)}{h}}$

$f(a+2h) = A_0 + 2h\,A_1 + 2h^2\,A_2 \Rightarrow \boxed{A_2 = \dfrac{\Delta^2 f(a)}{2h^2}}$

⋮

Similarly $\boxed{A_n = \dfrac{\Delta^n f(a)}{n!h^n}}$

Substituting these in (1.7), we have:

$$P_n(x) = f(a) + \frac{\Delta f(a)}{h}(x-a) + \frac{\Delta^2 f(a)}{2!h^2}(x-a)(x-a-h) + \frac{\Delta^3 f(a)}{3!h^3}(x-a)(x-a-h)(x-a-2h) + \ldots + \frac{\Delta^n f(a)}{n!h^n}(x-a)(x-a-h)(x-a-2h).\ldots.(x-a-(n-1)h)$$

This is called as *Newton Gregory Forward Interpolation Formula.*

Letting $\frac{(x-a)}{h} = p$ or $x = a + ph$, the above formula takes the following form:

$$Y_p = P_n(x) = f(a) + p\,\Delta f(a) + \frac{p(p-1)}{2!}\Delta^2 f(a) + \ldots + \frac{p(p-1)(p-2)\cdots(p-n+1)}{n!}\Delta^n f(a)$$

Note: This formula is used to interpolate a value of f(x) near the beginning of the tabular values. While to interpolate near the end of the tabular values, we use Newton Gregory backward difference formula, which will be discussed below.

Assume a polynomial of nth degree,

$$P_n(x) = A_0 + A_1(x-a-nh) + A_2(x-a-nh)(x-a-nh+h) + A_3(x-a-nh)(x-a-nh+h)(x-a-nh+2h) + \ldots + A_n(x-a-nh)(x-a-nh+h)(x-a-nh+2h)\ldots(x-a-h). \quad (1.8)$$

We choose $A_0, A_1, A_2, \ldots, A_n$ so as to make, $P_n(a+nh) = f(a+nh), \ldots, P_n(a) = f(a)$.

Putting $x = a + nh, a + nh - h, \ldots, a+h$ in (1.8), we obtain:

$$A_0 = f(a+nh)$$

$$A_1 = \frac{f(a+nh)-f(a+nh-h)}{h} = \frac{\nabla f(a+nh)}{h}$$

$$A_2 = \frac{\nabla^2 f(a+nh)}{2h^2}$$

$$\vdots$$

$$A_n = \frac{\Delta^n f(a+nh)}{n!h^n}$$

$$P_n(x) = f(a+nh) + \frac{\nabla f(a+nh)}{h}(x-a-nh) + \frac{\nabla^2 f(a+nh)}{2h^2}(x-a-nh)(x-a-nh+h) + \ldots + \frac{\Delta^n f(a+nh)}{n!h^n}(x-a-nh)(x-a-nh+h)(x-a-nh+2h)\ldots(x-a-h)$$

Which is Newton's *backward* difference interpolation formula.

Letting $\frac{x-(a+nh)}{h} = p$ or x = a +nh+ ph, the above formula takes the following form:

$$Y_p = P_n(x) = Y_n + p\,\nabla Y_n + \frac{p(p+1)}{2!}\nabla^2 Y_n + \ldots + \frac{p(p+1)(p+2)\cdots(p+n-1)}{n!}\nabla^n Y_n,$$

where $Y_n = f(a+nh)$.

To find the error committed in replacing the function y = f(x) by means of the polynomial $P_n(x)$, we use the formula derived in section 1.2.

$$f(x)-\phi_n(x) = \frac{\pi_{n+1}(x)}{(n+1)!} f^{(n+1)}(\varepsilon),\ x_0 < \varepsilon < x_n$$

$$= \frac{(x-x_0)(x-x_1)\cdots(x-x_n)}{(n+1)!} f^{(n+1)}(\varepsilon)$$

But as remarked in that section, we do not have any information concerning $f^{(n+1)}(x)$, however a useful estimate of the derivative can be obtained in the following way: Expanding f(x+h) by Taylor's series about the point x, we obtain $f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \cdots$. Neglecting the terms containing h^2 and higher powers of h, this gives $f'(x) \approx \frac{1}{h}[f(x+h) - f(x)] = \frac{1}{h}\Delta f(x)$.

Writing $f'(x)$ as $f'(x)=\frac{d}{dx}f(x)=Df(x)$, the differentiation operator, the above equation gives the operator relation $D=\frac{1}{h}\Delta$ and hence $D^{n+1}=\frac{1}{h^{n+1}}\Delta^{n+1}$.

We thus obtain, $f^{(n+1)}(x)=\frac{d^{n+1}}{dx^{n+1}}f(x)\approx D^{n+1}f(x)=\frac{1}{h^{n+1}}\Delta^{n+1}f(x)$.

Hence $$f(x)-P_n(x)=\frac{(x-x_0)(x-x_1)\cdots(x-x_n)}{(n+1)!}\frac{\Delta^{n+1}}{h^{n+1}}f(\varepsilon)=\frac{p(p-1)(p-2)\cdots(p-n)}{(n+1)!}\Delta^{n+1}f(\varepsilon)$$

in which it is suitable for computations.

Similarly the error committed in Newton's backward formula is written as:

$f(x)-P_n(x)=\frac{p(p+1)(p+2)\cdots(p+n)}{(n+1)!}h^{n+1}f^{(n+1)}(\varepsilon)$, where $x_0 < \varepsilon < x_n$ and $x = x_n + ph$.

$$=\frac{p(p+1)(p+2)\cdots(p+n)}{(n+1)!}h^{n+1}\frac{\nabla^{n+1}}{h^{n+1}}f(\varepsilon)$$

$$=\frac{p(p+1)(p+2)\cdots(p+n)}{(n+1)!}\nabla^{n+1}f(\varepsilon).$$

This relation is true because, expanding f(x-h) by Taylor's series about x, we have $f(x-h)=f(x)-hf'(x)+\frac{h^2}{2}f''(x)-\cdots$. Neglecting the terms containing h^2 and higher powers of h, gives $f'(x)\approx\frac{1}{h}[f(x)-f(x-h)]=\frac{1}{h}\nabla f(x)$.

Writing $f'(x)$ as $f'(x)=\frac{d}{dx}f(x)=Df(x)$, the differentiation operator, we have a relation $D=\frac{1}{h}\nabla$, and hence $D^{n+1}=\frac{1}{h^{n+1}}\Delta^{n+1}$.

We thus obtain, $f^{(n+1)}(x)=\frac{d^{n+1}}{dx^{n+1}}f(x)\approx D^{n+1}f(x)=\frac{1}{h^{n+1}}\nabla^{n+1}f(x)$.

So that $f(x)-P_n(x)=\frac{p(p+1)(p+2)\cdots(p+n)}{(n+1)!}\nabla^{n+1}f(\varepsilon)$ is obtained.

Example:2. The population of a town in decennial census was as under.

Estimate the population for the year

a) 1925 b) 1955 c) 1905 d) 1965

Year	1921	1931	1941	1951	1961
Population (in thousands)	46	66	81	93	101

Solution:

a) Here interpolation is desired at the beginning of the table and so we use forward interpolation formula.

h = spacing b/n arguments = 10, x_0 = first term = 1921, x = 1925 and using the relation $\frac{(x-x_0)}{h}=p$, we have $\frac{(1925-1921)}{10}=0.4$

The difference table is as follows:

x	f(x)	Δ f(x)	Δ^2f(x)	Δ^3f(x)	Δ^4f(x)
1921	46				
		20			
1931	66		-5		
		15		2	
1941	81		-3		-3
		12		-1	
1951	93		-4		
		8			
1961	101				

Hence substituting all the information above in the formula derived we have

$$P_n(1925) = 46+(0.4)20+\frac{0.4(0.4-1)}{2!}(-5)+\frac{0.4(0.4-1)(0.4-2)}{6}(2)+\frac{0.4(0.4-1)(0.4-2)(0.4-3)}{24}(-3)$$

= 54.85 thousands.

b) Interpolation is desired at the end of the table and so we use backward interpolation formula.

Given that: h = 10, x_n = last term = 1961, x = 1955 and using the relation $\frac{(x-x_n)}{h}=p$,

we have $\frac{(1955-1961)}{10}=-0.6$.

Hence using the formula $P_n(x) = Y_n + p\nabla Y_n + \frac{p(p+1)}{2!}\nabla^2 Y_n + \ldots +$

$\frac{p(p+1)(p+2)\cdots(p+n-1)}{n!}\nabla^n Y_n$,

$$P_n(1955) = 101 + (-0.6)(8) + \frac{(-0.6)(-0.6+1)}{2!}(-4) + \frac{(-0.6)(-0.6+1)(-0.6+2)}{6}(-1) + \frac{(-0.6)(-0.6+1)(-0.6+2)(-0.6+3)}{24}(-3)$$

= 96.84 thousands.

c) Proceeding as in the case (b) above, we obtain:

$$P_n(1915) = 101 + (-4.6)(8) + \frac{(-4.6)(-4.6+1)}{2!}(-4) + \frac{(-4.6)(-4.6+1)(-4.6+2)}{6}(-1) + \frac{(-4.6)(-4.6+1)(-4.6+2)(-4.6+3)}{24}(-3), \text{ where } p = \frac{(1915-1961)}{10} = -4.6.$$

= 29.64 thousands.

d) And proceeding as in the case (a) above, we obtain:

$$P_n(1965) = 46 + (4.4)20 + \frac{4.4(4.4-1)}{2!}(-5) + \frac{4.4(4.4-1)(4.4-2)}{6}(2) + \frac{4.4(4.4-1)(4.4-2)(4.4-3)}{24}(-3), \text{ where } p = \frac{(1965-1921)}{10} = 4.4.$$

= 102.28 thousands.

Note: The process of finding the values of y for some x outside the given range is called extrapolation and if the tabulated function is a polynomial, then interpolation and extrapolation would give exact values, while if a tabulated function is other than a polynomial, then in case of extrapolation the error would be quite considerable, i.e. the values obtained are far from the table limits.

1.4.3. Central Difference Interpolation Formula.

In the preceding section, we derived and discussed Newton's forward and backward interpolation formulae that are applicable near the beginning and end of the tabulated values, respectively.

We shall, in this section discuss the central difference formulae which are most suited for interpolation near the middle of a tabulated set.

i) Gauss's Central Difference Formulae.

a) Gauss's Forward Formula.

Consider the following difference table in which the central ordinate is taken for convenience as y_0 corresponding to $x = x_0$.

x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$	$\Delta^6 f(x)$
x_{-3}	y_{-3}						
		Δy_{-3}					
x_{-2}	y_{-2}		$\Delta^2 y_{-3}$				
		Δy_{-2}		$\Delta^3 y_{-3}$			
x_{-1}	y_{-1}		$\Delta^2 y_{-2}$		$\Delta^4 y_{-3}$		
		Δy_{-1}		$\Delta^3 y_{-2}$		$\Delta^5 y_{-3}$	
x_0	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$		$\Delta^6 y_{-3}$
		Δy_0		$\Delta^3 y_{-1}$		$\Delta^5 y_{-2}$	
x_1	y_1		$\Delta^2 y_0$		$\Delta^4 y_{-1}$		
		Δy_1		$\Delta^3 y_0$			
x_2	y_2		$\Delta^2 y_1$				
		Δy_2					
x_3	y_3						

Table 3

The differences used in this formula lie on the broken line arrow shown above and the formula is $y_p = y_0 + G_1 \Delta y_0 + G_2 \Delta^2 y_{-1} + G_3 \Delta^3 y_{-1} + G_4 \Delta^4 y_{-2} + \ldots$,

where $G_1, G_2, G_3, \ldots$ have to be determined.

Now $y_p = E^p y_0$, where E- shift operator, defined as $Ey_i = y_{i+1}$

$= (1+\Delta)^p y_0$, using the relation $E = 1+\Delta$.

$= y_0 + p\,\Delta y_0 + \frac{p(p-1)}{2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \ldots$,

using binomial expansion theorem.

And to express the right hand side of the formula in terms of Δy_0 and higher order differences:

$\Delta^2 y_{-1} = \Delta^2 E^{-1} y_0$

$= \Delta^2 (1+\Delta)^{-1} y_0$

$= \Delta^2 (1-\Delta+\Delta^2-\Delta^3+\Delta^4- \ldots) y_0$, by using Taylor's series expansion of $(1+\Delta)^{-1}$ about $\Delta = 0$.

$= \Delta^2 y_0 - \Delta^3 y_0 + \Delta^4 y_0 - \Delta^5 y_0 + \ldots$.

Similarly

$\Delta^3 y_{-1} = \Delta^3 E^{-1} y_0 = \Delta^3 (1+\Delta)^{-1} y_0$

$= \Delta^3 y_0 - \Delta^4 y_0 + \Delta^5 y_0 - \Delta^6 y_0 + \ldots$.

$\Delta^4 y_{-2} = \Delta^4 E^{-2} y_0$

$= \Delta^4 (1+\Delta)^{-2} y_0$

$= \Delta^4 (1-2\Delta+3\Delta^2-4\Delta^3+5\Delta^4- \ldots) y_0$, using Taylor's series expansion of $(1+\Delta)^{-1}$ about $\Delta = 0$.

$= \Delta^4 y_0 - 2\Delta^5 y_0 + 3\Delta^6 y_0 - 4\Delta^7 y_0 + \ldots$.

Substituting all these in the right side of the formula and equating with y_p, we have

$$y_0 + p\Delta y_0 + \frac{p(p-1)}{2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \ldots =$$

$$y_0 + G_1 \Delta y_0 + G_2(\Delta^2 y_0 - \Delta^3 y_0 + \Delta^4 y_0 - \Delta^5 y_0 + \ldots) + G_3(\Delta^3 y_0 - \Delta^4 y_0 + \Delta^5 y_0 - \Delta^6 y_0 + \ldots)$$
$$+ G_4(\Delta^4 y_0 - 2\Delta^5 y_0 + 3\Delta^6 y_0 - 4\Delta^7 y_0 + \ldots) + \ldots$$

Equating the coefficients of Δy_0, $\Delta^2 y_0$, $\Delta^3 y_0$, $\Delta^4 y_0$, etc. on both sides, we obtain

$G_1 = p$,

$G_2 = \frac{p(p-1)}{2}$

$-G_2 + G_3 = \frac{p(p-1)(p-2)}{3!} \Rightarrow G_3 = \frac{(p+1)p(p-1)}{3!}$

$G_4 = \frac{(p+1)p(p-1)(p-2)}{4!}$, etc.

Gausses forward formula is given by:

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \ldots$$

b) Gauss's Backward Formula

This formula is derived using the differences, which lie on the solid line shown above: Hence the formula is then assumed to be of the form

$y_p = y_0 + G_1'\Delta y_{-1} + G_2'\Delta^2 y_{-1} + G_3'\Delta^3 y_{-2} + G_4'\Delta^4 y_{-2} + \ldots.$

Where G_1', G_2', G_3', … have to be determined and following the same procedure as in case (a), we obtain: $G_1' = p$ $\quad G_3' = \frac{(p+1)p(p-1)}{3!}$

$G_2' = \frac{p(p+1)}{2} \quad G_4' = \frac{(p+2)(p+1)p(p-1)}{4!}$, etc.

Hence Gauss's backward formula is expressed as:

$$y_p = y_0 + p\Delta y_{-1} + \frac{p(p+1)}{2}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2} + \frac{(p+2)(p+1)p(p-1)}{4!}\Delta^4 y_{-2} + \ldots.$$

ii) Stirling's Formula:

Taking the mean of Gauss's forward and backward formula, we obtain:

$$y_p = \frac{(y_0 + y_0)}{2} + p\frac{(\Delta y_0 + \Delta y_{-1})}{2} + \frac{1}{2}\left(\frac{p(p-1)}{2} + \frac{p(p+1)}{2}\right)\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\,\frac{1}{2}(\Delta^3 y_{-1} + \Delta^3 y_{-2})$$

$$+\frac{1}{2}\left(\frac{(p+1)p(p-1)(p-2)}{4!} + \frac{(p+2)(p+1)p(p-1)}{4!}\right)\Delta^4 y_{-2} + \ldots.$$

$$= y_0 + p\frac{(\Delta y_0 + \Delta y_{-1})}{2} + \frac{p^2}{2}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\,\frac{1}{2}(\Delta^3 y_{-1} + \Delta^3 y_{-2}) + \frac{p^2(p^2-1)}{4!})\Delta^4 y_{-2} + \ldots$$

which is called Stirling's Formula.

iii) Bessel's Formula:

This formula uses the differences as shown in the following table.

$\vdots$	$\vdots$						
x_{-2}	y_{-2}						
x_{-1}	y_{-1}						
x_0	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$		$\Delta^6 y_{-3}$
		Δy_0		$\Delta^3 y_{-1}$		$\Delta^5 y_{-2}$	
x_1	y_1		$\Delta^2 y_0$		$\Delta^4 y_{-1}$		$\Delta^6 y_{-2}$
x_2	y_2						
$\vdots$	$\vdots$						

Hence, Bessel's formula can be assumed in the form

$$y_p = \frac{(y_0 + y_1)}{2} + B_1 \Delta y_0 + B_2 \frac{(\Delta^2 y_0 + \Delta^2 y_{-1})}{2} + B_3 \Delta^3 y_{-1} + B_4 \frac{(\Delta^4 y_{-1} + \Delta^4 y_{-2})}{2} + \ldots$$

$$= y_0 + (B_1 + \tfrac{1}{2}) \Delta y_0 + B_2 \frac{(\Delta^2 y_0 + \Delta^2 y_{-1})}{2} + B_3 \Delta^3 y_{-1} + B_4 \frac{(\Delta^4 y_{-1} + \Delta^4 y_{-2})}{2} + \ldots$$

Following the same procedure as in case (i) above, we obtain:

$$B_1 + \frac{1}{2} = p. \qquad B_2 = \frac{p(p-1)}{2!}$$

$$B_3 = \frac{p(p-1)(p-\frac{1}{2})}{3!} \qquad B_4 = \frac{(p+1)p(p-1)(p-2)}{4!}, \text{ etc.}$$

Hence Bessel's interpolation formula is written as:

$$y_p = y_0 + p \Delta y_0 + \frac{p(p-1)}{2!} \frac{(\Delta^2 y_0 + \Delta^2 y_{-1})}{2} + \frac{p(p-1)(p-\frac{1}{2})}{3!} \Delta^3 y_{-1}$$

$$+ \frac{(p+1)p(p-1)(p-2)}{4!} \frac{(\Delta^4 y_{-1} + \Delta^4 y_{-2})}{2} + \ldots$$

iv) Everett's Formula

This formula uses only even order differences as shown in the following table:

x_0	y_0	$\Delta^2 y_{-1}$	$\Delta^4 y_{-2}$	$\Delta^6 y_{-3}$
	--	--	--	--
x_1	y_1	$\Delta^2 y_0$	$\Delta^4 y_{-1}$	$\Delta^6 y_{-2}$

Hence the formula has the form

$y_p = E_0\, y_0 + E_2\, \Delta^2 y_{-1} + E_4\, \Delta^4 y_{-2} + \ldots + F_0\, y_1 + F_2\, \Delta^2 y_0 + E_4\, \Delta^4 y_{-1} + \ldots .$

The coefficients E_0, F_0, E_2, F_2, E_4, E_4, ... can be determined by the same method as in the preceding cases, and we obtain:

$E_0 = 1\text{-}p = q;$ $\qquad F_0 = p$

$E_2 = \dfrac{q(q^2-1^2)}{3!};$ $\qquad F_2 = \dfrac{p(p^2-1^2)}{3!}$

$E_4 = \dfrac{q(q^2-1^2)(q^2-2^2)}{5!};$ $\qquad E_4 = \dfrac{p(p^2-1^2)(p^2-2^2)}{5!}, \ldots .$

Hence Everett's formula is given by:

$$y_p = q\, y_0 + \frac{q(q^2-1^2)}{3!}\,\Delta^2 y_{-1} + \frac{q(q^2-1^2)(q^2-2^2)}{5!}\,\Delta^4 y_{-2} + \ldots + p\, y_1 + \frac{p(p^2-1^2)}{3!}\,\Delta^2 y_0$$

$$+ \frac{p(p^2-1^2)(p^2-2^2)}{5!}\,\Delta^4 y_{-1} + \ldots, \text{ where } q = 1 - p.$$

Remark: i) If the interpolation is desired near the beginning or end of a table, there is no alternative to Newton's forward and backward difference formulae, simply because higher order central differences do not exist at the beginning or end of the table of values.

ii)For interpolation near the middle of a table, Stirling's formula gives the most accurate result for $-1/4 \le p \le 1/4$, and Bessel's formula is most efficient near $p = ½$, say $1/4 \le p \le 3/4$.

Example: 3. The population of a certain town was as under.

a) Estimate the population for the year 1936.

Year	1901	1911	1921	1931	1941	1951
Population (In thousands)	12	15	20	27	33	52

Solution:

X(year)	Y(population)	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
$x_{-3} = 1901$	$y_{-3} = 12$					
		3				
$x_{-2} = 1911$	$y_{-2} = 15$		2			
		5		0		
$x_{-1} = 1921$	$y_{-1} = 20$		2		-3	
		7		-3		20
$x_0 = 1931 \rightarrow$	$y_0 = 27$		-1		17	
		6		14		?
$x_1 = 1941$	$y_1 = 33$		13			
		19				
$x_2 = 1951$	$y_2 = 52$					

Here as we observe from the table above (going along the line drawn), Gauss's forward and Stirlings formula cannot be applied, hence

i) using Gauss's backward formula, we have the following:

$x_0 = 1931$ is taken as the origin, and h = 10 as the unit value (spacing).

The value of y required will be for $p = \frac{x - x_0}{h} = \frac{1936 - 1931}{10} = 0.5.$

$$y_p = 27 + 0.5(7) + \frac{0.5(0.5+1)}{2}(-1) + \frac{(0.5+1)0.5(0.5-1)}{3!}(-3) + \frac{(0.5+2)(0.5+1)0.5(0.5-1)}{4!}(17) + \frac{(0.5+2)(0.5+1)0.5(0.5-1)(0.5-2)}{5!}(20).$$

$= 29.883.$

ii) Higher order differences does not exist for eg. $\Delta^4 y_{-1}$, $\Delta^5 y_{-2}$, hence it is not possible to use Bessel's or Everett's formula.

1.5. Interpolation with Unequally Spaced Points

The goal of this section is to derive a general interpolation formulae which fit the given data points on which the arguments are not necessarily equally spaced. We discuss, in this section, four such formulae:

i) Lagrange's Interpolation formula which uses only the function values;

ii) Hermite's Interpolation formula;

iii) Newton's General Interpolation formula which uses what are called divided differences;

iv) Aitken's method of interpolation by iteration.

1.5.1. Lagrange's Interpolation formula

Let $f(x_0)$, $f(x_1)$, $f(x_2)$, …, $f(x_n)$ be the values of the function $y = f(x)$ corresponding to the arguments $x_0, x_1, x_2, \ldots, x_n$ not necessarily equally spaced.

Consider the construction of a polynomial $P_n(x)$ of degree at most n that passes through the (n+1) points, such that $f(x_k) = P_n(x_k)$, for all $k = 0, 1, 2, \ldots, n$.

To start with the simplest one, let a polynomial of degree one, which passes through $(x_0, f(x_0))$ and $(x_1, f(x_1))$ is given by:

$$P_1(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1).$$

Here the quotients $\frac{x - x_1}{x_0 - x_1}$ and $\frac{x - x_0}{x_1 - x_0}$ are involved.

Denoting $L_0(x) = \frac{x - x_1}{x_0 - x_1}$ and $L_1(x) = \frac{x - x_0}{x_1 - x_0}$, we have :

When $x = x_0$, $L_0(x_0) = 1$ and $L_1(x_0) = 0$.

When $x = x_1$, $L_0(x_1) = 0$ and $L_1(x_1) = 1$.

In a similar way, the Lagrange polynomial of degree two passing through three points $(x_0, f(x_0))$, $(x_1, f(x_1))$ and $(x_2, f(x_2))$ is written as:

$$P_2(x)=\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}f(x_0)+\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}f(x_1)+\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}f(x_2).$$

$P_2(x)=\sum_{k=0}^{2} L_k(x)f(x_k)$, where the $L_k(x)$ satisfy the conditions $L_k(x_i)=\begin{cases}1, if\ i=k\\ 0, if\ i\neq k\end{cases}$.

For the general case we need to construct, for each k = 0, 1, 2, …, n, a quotient $L_k(x)$ with the property that $L_k(x_i) = 0$, when $i \neq k$ and $L_k(x_k) = 1$.

In order to satisfy $L_k(x_i) = 0$, for each $i \neq k$ requires that the numerator of L_k contains the product $(x - x_0)(x - x_1)(x - x_2) \dots (x - x_{k-1})(x - x_{k+1}) \dots (x - x_n)$. …………………. (*)

To satisfy $L_k(x_k) = 1$, the denominator of L_k must be equal to (*), when $x = x_k$.

Thus $L_k(x)=\dfrac{(x-x_0)(x-x_1)(x-x_2)\dots(x-x_{k-1})(x-x_{k+1})\dots(x-x_n)}{(x_k-x_0)(x_k-x_1)(x_k-x_2)\dots(x_k-x_{k-1})(x_k-x_{k+1})\dots(x_k-x_n)}$

$=\prod_{\substack{i=0\\ i\neq k}}^{n}\dfrac{(x-x_i)}{(x_k-x_i)}$, such that $L_k(x_i) = 0$, when $i \neq k$ and $L_k(x_k) =1$, k = 0, 1, …, n.

Hence the polynomial is given by

$$P_n(x)=a_0L_0(x)+a_1L_1(x)+\cdots+a_nL_n(x)=\sum_{k=0}^{n}a_kL_k(x).$$

$P_n(x_i)=\sum_{k=0}^{n}a_kL_k(x_i)=a_i$, i = 0, 1, 2, …, n.

i.e. the coefficients in the Lagrange form $a_0, a_1, a_2, \dots, a_n$ are simply the values of the polynomial $P_n(x)$ at the points $x_0, x_1, x_2, \dots, x_n$.

Consequently $P_n(x)=\sum_{k=0}^{n}L_k(x)f(x_k)$ is the required polynomial.

If we now set $\Pi_{n+1}(x)=(x-x_0)(x-x_1)\dots(x-x_{k-1})(x-x_k)(x-x_{k+1})\dots(x-x_n)$, then

$$\Pi'_{n+1}(x_k)=\frac{d}{dx}[\Pi_{n+1}(x)]_{x=x_k}$$

$=(x_k-x_0)(x_k-x_1)(x_k-x_2)\dots(x_k-x_{k-1})(x_k-x_{k+1})\dots(x_k-x_n)$ so that

$$L_k(x)=\frac{\Pi_{n+1}(x)}{(x-x_k)\Pi'_{n+1}(x_k)}.$$

Therefore the polynomial $P_n(x) = \sum_{k=0}^{n} \frac{\Pi_{n+1}(x)}{(x - x_k)\Pi'_{n+1}(x_k)} f(x_k)$ is now called *Lagrange's Interpolation formula.* The coefficients $L_k(x)$, defined above are called Lagrange's Interpolation *Coefficients.* Interchanging x and y in this formula, we obtain $P_n(y) = \sum_{k=0}^{n} \frac{\Pi_{n+1}(y)}{(y - y_k)\Pi'_{n+1}(y_k)} x_k$ which is useful for inverse interpolation.

Example: 4. a) Find the Lagrange interpolating polynomial of degree two approximating the function y = lnx defined by the following table of values.

a) Hence determine the value of ln2.7 and $e^{0.72}$.

b) Estimate the error in the values of y obtained in b)

x	2	2.5	3.0
y = lnx	0.69315	0.91629	1.09861

Solution:

a) we have:

$$P_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

$= (2x^2 - 11x + 15)(0.69315) - (4x^2 - 20x + 24)(0.91629) + (2x^2 - 9x + 10)(1.09861)$

$= - 0.08164x^2 + 0.81366x - 0.60761$, which is the required quadratic polynomial.

b) Putting x = 2.7 in the above polynomial $P_2(x)$, we obtain:

$\ln 2.7 \approx P_2(2.7) = - 0.08164(2.7)^2 + 0.81366(2.7) - 0.60761 = 0.9941164$.

Actual value of ln2.7 = 0.9932518, so that |error| = 0.0008646.

To find $e^{0.7}$: $y = \ln x \Rightarrow x = e^y$ so that assuming y = 0.7, we have:

$$x = P_2(y) = \frac{(y-y_1)(y-y_2)}{(y_0-y_1)(y_0-y_2)}(x_0) + \frac{(y-y_0)(y-y_2)}{(y_1-y_0)(y_1-y_2)}(x_1) + \frac{(y-y_0)(y-y_1)}{(y_2-y_0)(y_2-y_1)}(x_2)$$

$\therefore$ x = $P_2(2.7)$ = 2.013515906.

Actual value of $e^{0.7}$ = 2.013752707.

c) Before going to do this particular problem, consider error in Lagrange's Interpolation Formula.

From section 1.2, we have a formula to estimate the error of the Lagrange interpolation formula for the class of functions which have continuous derivatives of order up to (n+1) on [a,b].

We have therefore $f(x) - P_n(x) = R_n(x) = \frac{\Pi_{n+1}(x)}{(n+1)!} f^{(n+1)}(\varepsilon),\ a < \varepsilon < b$.

In our case: $f(x) - P_2(x) = R_2(x) = \frac{\Pi_3(x)}{(3)!} f^{(3)}(\varepsilon),\ 2 < \varepsilon < 3$.

Since y = lnx, we obtain $y' = \frac{1}{x},\ y''(x) = \frac{-1}{x^2},\ and\ y'''(\varepsilon) = \frac{2}{\varepsilon^3}$.

For example the continuity conditions on f(x) and its derivatives are satisfied in [2, 3].

Hence $R_2(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{6} \frac{2}{\varepsilon^3},\ 2 < \varepsilon < 3$.

When x = 2.7, $|R_2(2.7)| = \left|\frac{(2.7-2)(2.7-2.5)(2.7-3)}{6}\right| \left|\frac{2}{\varepsilon^3}\right|$, but $\left|\frac{2}{\varepsilon^3}\right| \leq \frac{1}{2^3} = \frac{1}{8}$.

$$\leq \frac{0.7x0.2x0.3}{3.8} = 0.00175,$$ which agrees with the actual error 0.0008646.

1.5.2. Hermite's Interpolation Formula

The interpolation formulae so far considered make use of only a certain number of function values. We now derive an interpolation formula in which both the function and its first derivative values are to be assigned at each point of interpolation.

Before we go to this specific formula, consider the following general case:

The set of osculating polynomials is a generalization of the Taylor polynomials, the Lagrange polynomials and others at the (n+1) points. These polynomials have the property that, given (n+1) distinct points $x_0, x_1, \ldots, x_n$ and non-negative integers $m_0, m_1, \ldots, m_n$, the osculating polynomial approximating a function $f \in C^m[a,b]$, where $m = \max\{m_0, m_1, \ldots, m_n\}$ and $x_i \in [a,b]$ for each i = 0, 1, 2, …, n, is the polynomial of least degree with the property that it agrees with the function f and all of its derivatives of order less than or equal to m_i at x_i for each i = 0, 1, 2, …, n.

The degree of this osculating polynomial will be at most $M = \sum_{i=0}^{n} m_i + n$. This is because the number of conditions to be satisfied is $M = \sum_{i=0}^{n} m_i + (n+1)$, and a polynomial of degree M has M+1 coefficients that can be used to satisfy these conditions.

Definition:

Let $x_0, x_1, \ldots, x_n$ be (n+1) distinct points in [a, b] and m_i be a non-negative integer associated with x_i for i = 0, 1, 2, …, n. Let $m = \max_{0 \le i \le n} m_i$, and $f \in C^m[a,b]$. The osculating polynomial approximating f is the polynomial P of least degree such that $\frac{d^k P(x_i)}{dx^k} = \frac{d^k f(x_i)}{dx^k}$, for each i = 0, 1, 2, …, n and k = 0, 1, 2, …, m_i.

Note that:

- When n = 0, the osculating polynomial approximating f is simply the Taylor polynomial of degree m_0 for f at x_0.
- When $m_i = 0$ for i = 0, 1, 2, …, n, the osculating polynomial is the polynomial interpolating f on $x_0, x_1, \ldots, x_n$, that is , the Lagrange polynomial.

The situation which occurs when $m_i = 1$ for $i = 0, 1, 2, \ldots, n$ gives a class of polynomials called Hermite polynomials.

For a given function f, these polynomials not only agree with f at $x_0, x_1, \ldots, x_n$, but since their first derivatives agree with those of f, they have the same 'shape' as the function at $(x_i, f(x_i))$ in the sense that the tangent lines to the polynomial and the function agree.

The procedure is as follows:

The problem is to find a polynomial of least degree, say $H_{2n+1}(x)$, such that:

$$\mathbf{f}(\mathbf{x}_j) = \mathbf{H}_{2n+1}(\mathbf{x}_j)$$

$$\mathbf{f}'(\mathbf{x}_j) = \mathbf{H}'_{2n+1}(\mathbf{x}_j),\ j = 0, 1, 2, \ldots, n. \tag{1.9}$$

By counting the data (i.e. 2n+2 conditions), we find that a polynomial of degree 2n+1 has the required number of undetermined coefficients. Thus in analogy with the Lagrange interpolation formula, we seek a representation in the form:

$$\mathbf{H}_{2n+1}(\mathbf{x}) = \sum_{j=0}^{n} \mathbf{f}(\mathbf{x}_j) u_j(x) + \sum_{j=0}^{n} \mathbf{f}'(\mathbf{x}_j) v_j(x) \ . \tag{1.10}$$

Here the polynomials $u_j(x)$ and $v_j(x)$ are required to be of degree at most 2n+1 and to satisfy:

$$u_j(x_i) = \begin{cases} 1, & if\ \ j = i \\ 0, & if\ \ j \neq i \end{cases}; \qquad v_j(x) = 0 \text{ for all j.}$$

$$u'_j(x) = 0 \text{ for all j} \quad ; \qquad v'_j(x_i) = \begin{cases} 1, & if\ \ j = i \\ 0, & if\ \ j \neq i \end{cases} \tag{1.11}$$

Since $u_j(x)$ and $v_j(x)$ are polynomials in x of degree (2n+1), and $L_j(x)$ is the jth Lagrange coefficient polynomial of degree n defined before, we write:

$$u_j(x) = A_j(x)[L_j(x)]^2$$

$$v_j(x) = B_j(x)[L_j(x)]^2, \tag{1.12}$$

it is easy to see that $A_j(x)$ and $B_j(x)$ are both linear functions in x.

Therefore $\quad u_j(x) = (a_j x + b_j)[L_j(x)]^2$

$$v_j(x) = (c_j x + d_j)\,[L_j(x)]^2 \tag{1.13}$$

Now using conditions (1.11) in (1.13), we obtain

$$a_j x_j + b_j = 1 \qquad a_j + 2L'_j(x_j) = 0$$

$$c_j x_j + d_j = 0 \qquad c_j = 1 \tag{1.14}$$

which follows that

$$a_j = -2L'_j(x_j)\,, \qquad b_j = 1 + 2x_j\, L'_j(x_j)$$

$$c_j = 1 \quad , \qquad d_j = -\,x_j\,.$$

Hence equation (1.13) become:

$$u_j(x) = [-2x\,L'_j(x_j) + 1 + 2x_j\, L'_j(x_j)\,]\,[L_j(x)]^2$$

$$= [1 - 2(x - x_j)\,L'_j(x_j)\,]\,[L_j(x)]^2$$

$$v_j(x) = (x - x_j\,)\,[L_j(x)]^2 \qquad (1.15)$$

Using these expressions for $u_j(x)$ and $v_j(x)$ in (1.10), we obtain the required Hermite's Interpolation Formula:

$$\mathbf{H}_{2n+1}(x) = \sum_{j=0}^{n} f(x_j)[1 - 2(x - x_j)L'_j(x_j)]\,[L_j(x)]^2 + \sum_{j=0}^{n} f'(x_j)(x - x_j\,)\,[L_j(x)]^2 \qquad (1.16)$$

Example: 5.

Determine the Hermite polynomial of degree 5, which fits the following data, and hence find an appropriate value of ln2.7.

x	2	2.5	3.0
$y = \ln x$	0.69315	0.91629	1.09861
$y' = \dfrac{1}{x}$	0.5	0.4000	0.33333

Solution:

To apply the above formula, equation (1.16), we must first determine $L_j(x)$ and $L'_j(x)$, j = 0, 1, 2.

$$L_0(x) = \frac{(x-2.5)(x-3.0)}{(2-2.5)(2-3.0)} = 2x^2 - 11x + 15$$

$$L_1(x) = \frac{(x-2)(x-3.0)}{(2.5-2)(2.5-3.0)} = -(4x^2 - 20x + 24)$$

$$L_2(x) = \frac{(x-2)(x-2.5)}{(3.0-2)(3.0-2.5)} = 2x^2 - 9x + 10.$$

Hence $L'_0(x) = 4x-11$, $L'_1(x) = -8x+20$, and $L'_2(x) = 4x-9$.

$\therefore\ L'_0(x_0) = L'_0(2) = -3$, $L'_1(x_1) = L'_1(2.5) = 0$ and $L'_2(x_2) = L'_2(3.0) = 3$.

So that $H_{2n+1}(x)=H_5=\sum_{j=0}^{n} f(x_j)[1-2(x-x_j)L_j'(x_j)][L_j(x)]^2 + \sum_{j=0}^{n} f'(x_j)(x-x_j)[L_j(x)]^2$

$$
\begin{aligned}
H_{2n+1}(x)=H_5 = {} & f(x_0)[1-2(x-x_0)L_0'(x_0)][L_0(x)]^2 \\
& + f(x_1)[1-2(x-x_1)L_1'(x_1)][L_1(x)]^2 \\
& + f(x_2)[1-2(x-x_2)L_2'(x_2)][L_2(x)]^2 \\
& + f'(x_0)(x-x_0)[L_0(x)]^2 \\
& + f'(x_1)(x-x_1)[L_1(x)]^2 \\
& + f'(x_2)(x-x_2)[L_2(x)]^2.
\end{aligned}
$$

$$
\begin{aligned}
&= (0.69315)(6x-11)(2x^2-11x+15)^2 + (0.91629)(4x^2-20x+24)^2 \\
&\quad + (1.09861)(19-6x)(2x^2-9x+10)^2 + (x-2)(2x^2-11x+15)^2(0.5) \\
&\quad + (0.4)(x-2.5)(4x^2-20x+24)^2 + (0.33333)(x-3)(2x^2-9x+10)^2.
\end{aligned}
$$

Putting x = 2.7 and simplifying, we obtain $\ln(2.7) \approx H_5(2.7) = 0.993252$, this is a more accurate result than that obtained by using the Lagrange interpolation formula.

1.5.3. **Divided Differences and Newton's General Interpolation Formula**

The Lagrange interpolation formula has the disadvantage that the work done in calculating the approximation by the second- degree polynomial does not lessen the work needed to calculate the third-degree approximation; nor is the fourth-degree approximation easier to obtain once the third-degree approximation is known. It is the purpose of this subsection to derive the approximating polynomials in a manner that utilizes the previous calculations to greatest advantage. Newton's General Interpolation Formula is one such formula and it employs what are called divided differences.

Let $f(x_0), f(x_1), \ldots, f(x_n)$ be the entries corresponding to the (n+1) arguments $x_0, x_1, \ldots x_n$, where the intervals, $x_1 - x_0, x_2 - x_1, \ldots, x_n - x_{n-1}$ may not be equally spaced. Then the first order divided difference is defined as:

$$f[x_0,x_1]=\frac{f[x_1]-f[x_0]}{x_1-x_0}=\frac{f(x_1)-f(x_0)}{x_1-x_0},$$ as $f[x_0] = f(x_0)$, the zero order.

Higher order differences can be computed in a similar way as:

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

$$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0}$$

$$\vdots$$

$$f[x_0, x_1, x_2, \ldots, x_n] = \frac{f[x_1, x_2, \ldots, x_n] - f[x_0, x_1, \ldots, x_{n-1}]}{x_n - x_0}$$

Property: The nth divided difference of a polynomial of the nth degree is constant.

Proof:

Consider $f(x) = x^n$.

The first divided difference is given by

$$f[x_r, x_{r+1}] = \frac{f[x_{r+1}] - f[x_r]}{x_{r+1} - x_r} = \frac{x^n{}_{r+1} - x^n{}_r}{x_{r+1} - x_r} = x^{n-1}_{r+1} + x_r x^{n-2}_{r+1} + \ldots + x_r^{n-2} x_{r+1} + x_r^{n-1}$$

which is a polynomial of degree (n-1) in x_r and x_{r+1} .

The second divided difference is given by

$$f[x_r, x_{r+1}, x_{r+2}] = \frac{f[x_{r+1}, x_{r+2}] - f[x_r, x_{r+1}]}{x_{r+2} - x_r}$$

$$= \frac{x^{n-1}_{r+2} + x_{r+1} x^{n-2}_{r+2} + \ldots + x^{n-2}_{r+1} x_{r+2} + x^{n-1}_{r+1} - (x^{n-1}_{r+1} + x_r x^{n-2}_{r+1} + \ldots + x_r^{n-2} x_{r+1} + x_r^{n-1})}{x_{r+2} - x_r}$$

after some simplification,

$$= (x^{n-2}_{r+2} + x_r x^{n-3}_{r+2} + \ldots + x_r^{n-3} x_{r+2} + x_r^{n-2}) + x_{r+1} (x^{n-3}_{r+2} + x_r x^{n-4}_{r+2} + \ldots + x_r^{n-4} x_{r+2} + x_r^{n-3}) + \ldots + x^{n-2}_{r+1}$$

which is a polynomial of degree (n-2).

By induction it can be shown that $f[x_r, x_{r+1}, \ldots, x_{r+m}]$ is a polynomial of degree (n-m).

Hence the nth divided difference, $f[x_r, x_{r+1}, \ldots, x_{r+n}]$ is a polynomial of degree (n-n) = 0, i.e. a constant. Hence the result follows.

Newton's General Interpolation Formula

Let $f(x_0)$, $f(x_1)$, …, $f(x_n)$ be the values of f(x) corresponding to the (n+1) arguments x_0, x_1, …x_n, not necessarily equally spaced. Then from the definition of divided difference,

$$f[x,x_0]=\frac{f[x]-f[x_0]}{x-x_0}$$

$$\Rightarrow f(x) = f(x_0) + (x-x_0)f[x,x_0] \qquad (1.17)$$

Again, $f[x,x_0,x_1] = \dfrac{f[x,x_0]-f[x_0,x_1]}{x-x_1}$

$$\Rightarrow f[x,x_0] = f[x_0,x_1] + (x-x_1)f[x,x_0,x_1].$$

Substituting this value of $f[x,x_0]$ in (1.17), we obtain:

$$f(x) = f(x_0) + (x-x_0)\, f[x_0,x_1] + (x-x_0)\,(x-x_1)\, f[x,x_0,x_1] \qquad (1.18)$$

Again, $f[x,x_0,x_1,x_2] = \dfrac{f[x,x_0,x_1]-f[x_0,x_1,x_2]}{x-x_2}$,

and so $f[x,x_0,x_1] = f[x_0,x_1,x_2] + (x-x_2)f[x,x_0,x_1,x_2]$.

Substituting this result of $f[x,x_0,x_1]$ in (1.18), gives:

$$f(x) = f(x_0) + (x-x_0)\, f[x_0,x_1] + (x-x_0)\,(x-x_1)\, f[x_0,x_1,x_2] + (x-x_0)\,(x-x_1)\,(x-x_2)f[x,x_0,x_1,x_2] \qquad (1.19)$$

Proceeding in this way we obtain:

$$f(x) = f(x_0) + (x-x_0)\, f[x_0,x_1] + (x-x_0)\,(x-x_1)\, f[x_0,x_1,x_2] + (x-x_0)\,(x-x_1)\,(x-x_2)f[x_0,x_1,x_2,x_3] + \ldots + (x-x_0)(x-x_1)\ldots(x-x_n)f[x,x_0,x_1,\ldots,x_n] \qquad (1.20)$$

The last term is the remainder after (n+1) terms. Hence

$$P_n(x) = f(x_0) + (x-x_0)\, f[x_0,x_1] + (x-x_0)(x-x_1)\, f[x_0,x_1,x_2] + (x-x_0)(x-x_1)(x-x_2)f[x_0,x_1,x_2,x_3] + \ldots + (x-x_0)(x-x_1)\ldots(x-x_{n-1})f[x_0,x_1,\ldots,x_n]. \qquad (1.21)$$

Which agrees with f(x) at x_0, x_1, …, x_n. This formula is called *Newton's General Interpolation Formula* with divided differences. From which it follows that the error is given by:

$$f(x) - P_n(x) = (x-x_0)(x-x_1)\ldots(x-x_n)\, f[x,x_0,x_1,\ldots,x_n] = \Pi_{n+1}(x)\, f[x,x_0,x_1,\ldots,x_n]$$

From section 1.2, we have an error expression

$$f(x) - P_n(x) = \frac{\Pi_{n+1}(x)}{(n+1)!} f^{(n+1)}(\varepsilon),\ x_0 < \varepsilon < x.$$

These two must agree, hence $f[x, x_0, x_1, \ldots, x_n] = \dfrac{f^{(n+1)}(\varepsilon)}{(n+1)!}$.

Example: 6

Find the Newton's divided difference-interpolating polynomial of degree 3 approximating the function y = lnx defined by the following table values and hence determine the value of ln2.7.

x	2	2.5	3.0	3.5
y = lnx	0.69315	0.91629	1.09861	1.25276

Solution:

To apply the Newton's General Interpolation Formula, we first compute the divided differences as shown below.

x	y = lnx	f[,]	f[, ,]	f[, , ,]
2.0	0.69315			
		0.44628		
2.5	0.91629		- 0.08164	
		0.36464		0.01687
3.0	1.09861		- 0.05634	
		0.3083		
3.5	1.25276			

Hence equation (5) gives

$P_3(x) = (0.69315) + (x-2)(0.44628) + (x-2)(x-2.5)(-0.08164) + (x-2)(x-2.5)(x-3)(0.01687)$

and so $f(2.7) = \ln 2.7 \approx P_3(2.7) = 0.993408$.

Actual value: $\ln 2.7 = 0.9932518$, so that $|\text{error}| = 1.56 \times 10^{-4}$, which shows that Newton's General Interpolation Formula is easier to compute and more accurate than that of Lagrange's Interpolation Formula.

Note: If $x_0, x_1, \ldots, x_n$ are equally spaced, then $(x_1-x_0) = (x_2-x_1) = \ldots = (x_n-x_{n-1}) = h$, so that $f[x_0, x_1] = \dfrac{f(x_1)-f(x_0)}{x_1 - x_0} = \dfrac{\Delta f(x_0)}{h}$

$f[x_0, x_1, x_2] = \dfrac{f[x_1, x_2]-f[x_0, x_1]}{x_2 - x_0} = \dfrac{1}{2h}\left[\dfrac{\Delta f(x_1)}{h} - \dfrac{\Delta f(x_0)}{h}\right] = \dfrac{\Delta^2 f(x_0)}{2!h^2}$ and in general,

$f[x_0, x_1, \cdots, x_n] = \dfrac{\Delta^n f(x_0)}{n!h^n}$.

Hence Newton's General divided difference formula coincides with the Newton's Forward Interpolation Formula.

1.5.4. Interpolation by Iteration

Given the (n+1) points (x_0, y_0), (x_1, y_1),..., (x_n, y_n), where the values of x need not necessarily be equally spaced, then to find the values of y corresponding to any given value of x, we proceed iteratively as follows:

Obtain a first approximation to y by considering the first two points only; then obtain its second approximation by considering the first three points, and so on. We denote the different interpolation polynomials by $\Delta(x)$ (with suitable subscripts)

for eg. $\Delta_{01}(x)$, first degree interpolating polynomial,

$\Delta_{012}(x)$, second degree interpolating polynomial and so on, so that at the first stage of approximation, we have

$$\Delta_{01}(x) = y_0 + (x-x_0)f[x_0, x_1] = \frac{1}{(x_1 - x_0)}\begin{vmatrix} y_0 & x_0 - x \\ y_1 & x_1 - x \end{vmatrix}$$

Similarly, we can form $\Delta_{02}(x)$, $\Delta_{03}(x)$, ...

Next we form $\Delta_{012}(x)$ by considering the first three points,

$$\Delta_{012}(x) = \frac{1}{(x_2 - x_1)}\begin{vmatrix} \Delta_{01}(x) & x_1 - x \\ \Delta_{02}(x) & x_2 - x \end{vmatrix}$$

Similarly, we obtain $\Delta_{013}(x)$, $\Delta_{014}(x)$, etc. At the nth stage of approximation we obtain:

$$\Delta_{012\ldots n}(x) = \frac{1}{(x_n - x_{n-1})}\begin{vmatrix} \Delta_{012\ldots\overline{n-1}}(x) & x_{n-1} - x \\ \Delta_{012\ldots\overline{n-2}n}(x) & x_n - x \end{vmatrix}$$

The computations may conveniently be arranged as in the following form:

x	y				
x_0	y_0				
		$\Delta_{01}(x)$			
x_1	y_1		$\Delta_{012}(x)$		
		$\Delta_{02}(x)$		$\Delta_{0123}(x)$	
x_2	y_2		$\Delta_{013}(x)$		$\Delta_{01234}(x)$
		$\Delta_{03}(x)$		$\Delta_{0124}(x)$	
x_3	y_3		$\Delta_{014}(x)$		
		$\Delta_{04}(x)$			
x_4	y_4				

Table 4. Aitken's Scheme

Note: An obvious advantage of Aitken's method is that it gives good idea of the accuracy of the result at any stage.

Example: 7.

Certain corresponding values of x and $\log_{10}^{x}$ are (300, 2.4771), (304, 2.4829), (305, 2.4843) and (307, 2.4871).

Find $\log_{10}^{301}$ by applying Aitken's method of iteration.

Solution: Aitken's Scheme is:

x	$\log_{10}^{x}$			
300	2.4771			
		2.47855		
304	2.4829		2.47858	
		2.47854		2.47860
305	2.4843		2.47857	
		2.47853		
307	2.4871			

Hence the successive interpolation polynomials evaluated at x = 301 are given above and so $\log_{10}^{301}$ = 2.47860, which is the latest approximation.

1.6. Inverse Interpolation

The problem of direct interpolation is to find the value of y = f(x) for an x lying between two tabulated values of the arguments, say x_0, x_1.

The problem in inverse interpolation is to find the value of x corresponding to a y lying between two tabulated values of y, say y_0, y_1. Since $x = f^{-1}(y)$, inverse interpolation is ordinary interpolation of the inverse function $f^{-1}(y)$; therefore of necessity, a direct interpolation of the inverse function demands the use of Lagrange's interpolation formula for unequal intervals of the arguments. The use of Lagrange's formula was illustrated in the previous example.

When the values of x are equally spaced, the method of successive approximations described below, should be used.

We start with Newton's forward difference formula written as:

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{6}\Delta^3 y_0 + \ldots,\ where\ p = \frac{x - x_0}{h}.$$

From this we obtain:

$$p = \frac{1}{\Delta y_0}[y_p - y_0 - \frac{p(p-1)}{2}\Delta^2 y_0 - \frac{p(p-1)(p-2)}{6}\Delta^3 y_0 - ...].$$

Neglecting the second and higher differences, we obtain the first approximation to p and this we write as follows:

$$p_1 = \frac{1}{\Delta y_0}[y_p - y_0]$$

Next, we obtain the second approximation to p by including the term containing the second differences. Thus

$p_2 = \frac{1}{\Delta y_0}[y_p - y_0 - \frac{p_1(p_1-1)}{2}\Delta^2 y_0]$, where we have used the value of p_1 for p in the coefficient of $\Delta^2 y_0$, similarly, we obtain:

$p_3 = \frac{1}{\Delta y_0}[y_p - y_0 - \frac{p_2(p_2-1)}{2}\Delta^2 y_0 - \frac{p_2(p_2-1)(p_2-2)}{6}\Delta^3 y_0]$ and so on.

This process should be continued till two successive approximations to p agree with each other to the required accuracy.

Hence as $p = \frac{x - x_0}{h}$, the required result is therefore $\boldsymbol{x = ph + x_0}$.

Conclusion of the chapter:

Polynomial interpolation is a method of approximating the value of a function at a point by means of a polynomial passing through known functional values. A major virtue of the method of approximation is its ease of implementation. Another virtue is that it leads to an expression for the truncation error in the approximation, which can often be estimated or bounded.

Implicit in our definition of polynomial interpolation was the assumption that truncation and not round off error is the major source of error.

In the next chapter, we will consider the problem of approximating a function whose values at a sequence of points are generally known only empirically and thus are subject to inherent errors, which may be large.

Thus round off will be a serious source of error and often the controlling source.

CHAPTER TWO

APPROXIMATION THEORY

The study of approximation theory involves two general types of problems. One problem arises when a function is given *explicitly* but it is desirable to find a "simpler" type of functions, such as a polynomial, that can be used to determine approximate values of the given function. The other problem in approximation theory is concerned with fitting functions to given data and finding the "best" function in a certain class that can be used to represent the data.

Both problems have been touched up on in the previous chapter. The Taylor polynomial of degree n about the point x_0 was discussed as an excellent approximation to an (n+1)-times differentiable function f in a small neighborhood of a point x_0. The Lagrange interpolating polynomials were discussed both as approximating polynomials and as polynomials to fit certain data. Other interpolating polynomials were also discussed in that chapter.

In this chapter, limitations to these techniques will be pointed out and other avenues of approach will be discussed.

2.1. Curve Fitting to Data

We have so far discussed the approximation of a function by means of *interpolation* at certain points. Such a procedure presupposes that the values of f (x) at these points are known free of errors. Hence interpolation is of little use in the following common situation:

The function f (x) describes the relation-ship between two physical quantities x and y = f(x), and, through *measurement* or other *experiment*, one has obtained numbers y_i, which merely approximate the values of f(x) at x_i, i.e. which are subject to error. The problem of data fitting is to recover f (x) from the given (approximate) data y_i, $i = 1, 2, \ldots, n$.

The task is to approximate or fit the data by some function F(x) in such a way that F(x) contains or represents most (if not all) the information about f(x) contained in the data and little (if any) of the error.

This is accomplished in practice by picking a function

$$F(x)=a_0\phi_0(x)+ a_1\phi_1(x)+..., a_n\phi_n(x),$$

where { ϕ_i } are selected set of functions and {a_i} are parameters which must be determined, so that the deviations $y_i - F(x_i)$, i = 0, 1, 2, ..., n are simultaneously made as small as possible.

2.1.1. Least Squares Curve Fitting Procedures

The method of least squares is probably the most systematic procedure to fit a unique curve through given data points and is widely used in practical computations. It can also be easily implemented on digital computers.

Let the set of data points be (x_i, y_i), i = 1, 2, .., n, and let the curve given be Y = f(x) be fitted to this data. At $x = x_i$, the experimental (or observed) value of the ordinate is y_i and the corresponding value on the fitting curve is $f(x_i)$. If e_i is the error of approximation at $x = x_i$, then we have $e_i = y_i - f(x_i)$, i = 1, 2, ..., n.

If we write $E = [y_1 - f(x_1)]^2 + [y_2 - f(x_2)]^2 + \ldots + [y_n - f(x_n)]^2$

$$= e_1^2 + e_2^2 + \ldots + e_n^2$$

$$= \sum_{i=1}^{n} e_i^2 , \qquad (2.1)$$

then the method of least squares consists in minimizing this sum.

In the following subsections, we shall study the linear and non-linear least square fitting to given data , (x_i, y_i), i = 1, 2, .., n.

a) Fitting a straight Line

Let $Y = a_0 + a_1x$ be the straight line to be fitted to the given data. Letting $a_0 + a_1x_i$ denote the ith value on the approximating line and y_i the ith given value, then corresponding to equation (2.1), the constants a_0 and a_1 be found that minimize

$$E = \sum_{i=1}^{n}[y_i - (a_0 + a_1x_i)]^2.$$

As E is a function of two variables a_0 and a_1, for a minimum to occur at (a_0, a_1), it is necessary for

$$0 = \frac{\partial}{\partial a_0}\sum_{i=1}^{n}[y_i - (a_0 + a_1x_i)]^2$$

and

$$0 = \frac{\partial}{\partial a_1}\sum_{i=1}^{n}[y_i - (a_0 + a_1x_i)]^2$$

These equations simplify to:

$$0 = -2\sum_{i=1}^{n}[y_i - (a_0 + a_1x_i)] \quad \text{and}$$

$$0 = -2\sum_{i=1}^{n}[y_i - (a_0 + a_1x_i)]x_i.$$

Or $$\sum_{i=1}^{n} y_i = na_0 + a_1\sum_{i=1}^{n} x_i \quad \text{and} \quad \sum_{i=1}^{n} y_ix_i = a_0\sum_{i=0}^{n} x_i + a_1\sum_{i=1}^{n} x_i^2 \qquad (2.2)$$

Since the x_i and y_i are known quantities, equation (2.2) called the *normal equations,* can be solved for the two unknowns a_0 and a_1.

Example: 1.

Certain experimental values of x and y are given below

x	2	4	6	8
y	2	11	28	40.

Find the least-squares line approximating this data?

Solution:

Extend the table and sum the columns, as shown below:

x_i	y_i	x_i^2	$x_i y_i$
2	2	4	4
4	11	16	44
6	28	36	168
8	40	64	320
20	81	120	536

Now equation (2.2) implies that

$$4a_0 + 20a_1 = 81$$
$$20a_0 + 120a_1 = 536$$

which simplifies to

$$4a_0 + 20a_1 = 81$$
$$5a_0 + 30a_1 = 134.$$

The solution to this system of equations is $a_0 = -12.5$ and $a_1 = 6.55$.
Hence the best linear equation in the list-squares sense is $y = 6.55x - 12.5$.

b) Non-linear Curve Fitting

In this part, we consider a polynomial of the nth degree, a power function and an exponential function to fit the given data points, (x_i, y_i), $i = 1, 2, 3, \ldots, n$.

i) Polynomial of the nth degree:

Let the polynomial of the nth degree $Y = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$, be fitted to the data points (x_i, y_i), $i = 1, 2, 3, \ldots, n$. We then have:

$$E = \sum_{i=1}^{n} [y_i - (a_0 + a_1x_i + a_2x_i^2 + \ldots + a_nx_i^n)]^2.$$

Equating, as before, the first partial derivatives to zero (i.e. $\frac{\partial E}{\partial a_i} = 0, \forall i = 0,1,\ldots,n$) and simplifying, we get the following normal equations:

$$\sum_{i=1}^{n} y_i = na_0 + a_1\sum_{i=1}^{n} x_i + a_2\sum_{i=1}^{n} x_i^2 + \ldots + a_n\sum_{i=1}^{n} x_i^n$$

$$\sum_{i=1}^{n} x_iy_i = a_0\sum_{i=1}^{n} x_i + a_1\sum_{i=1}^{n} x_i^2 + \ldots + a_n\sum_{i=1}^{n} x_i^{n+1} \qquad (2.3)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\sum_{i=1}^{n} x_i^n y_i = a_0\sum_{i=1}^{n} x_i^n + a_1\sum_{i=1}^{n} x_i^{n+1} + \ldots + a_n\sum_{i=1}^{n} x_i^{2n}$$

These are (n+1) equations with (n+1) unknowns and hence can be solved for $a_0, a_1, a_2, \ldots, a_n$. Hence $Y = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ is the required polynomial of nth degree.

Example: 2.

Fit a polynomial of the second degree to the data points given in the following table.

x	0	1.0	2.0
y	1.0	6.0	17.0

Solution:

In equation (2.3), we require the quantities

$\sum_{i=1}^{n} x_i$, $\sum_{i=1}^{n} x_i^2$, $\sum_{i=1}^{n} x_i^3$, $\sum_{i=1}^{n} x_i^4$, $\sum_{i=1}^{n} x_i y_i$, $\sum_{i=1}^{n} y_i$, and $\sum_{i=1}^{n} x_i^2 y_i$. These are computed in the following table.

x_i	y_i	x_i^2	x_i^3	x_i^4	$x_i y_i$	$x_i^2 y_i$
0	1	0	0	0	0	0
1	6	1	1	1	6	6
2	17	4	8	16	34	68
3	24	5	9	17	40	74

We now obtain the normal equations: (using equation (2.3))

$$3a_0 + 3a_1 + 5a_2 = 24$$

$$3a_0 + 5a_1 + 9a_2 = 40$$

$$5a_0 + 9a_1 + 17a_2 = 74$$

To solve this system, we can use Gauss-elimination method, and the solution of which is $a_0 = 1$, $a_1 = 2$, $a_2 = 3$. The required polynomial is then given by $Y = 1 + 2x + 3x^2$.

ii) *Power function and exponential function:*

Although least squares using polynomials is the most extensively used procedure, occasionally it is appropriate to assume that the data is exponentially related. This requires the approximating function to be of the form:

$y = be^{ax}$ *or* $y = bx^a$ for some constants a and b.

The method that is usually followed when the data is suspected to be exponentially related is to consider the logarithm of the approximating equation:

i.e. $\ln y = \ln b + ax$

or $\ln y = \ln b + \ln x$.

In either case a linear problem now appears and solutions for lnb and a can be determined.

Example .3:

Consider the collection of data in the following table:

x	1.00	1.25	1.50	1.75	2.00
y	5.10	5.79	6.53	7.45	8.46

Determine the constants a and b such that $y = be^{ax}$ fits the data.

Solution:

Taking logarithms of both sides of $y = be^{ax}$, we obtain $lny = lna + bx$.

Setting lny = Y, x = X, lna = a_0 and b = a_1, the above relation takes the form Y = $a_0 + a_1X$, which is a straight line.

Extending the table and summing each column gives the following.

X = x	y	Y = lny	X^2	XY
1.00	5.10	1.629	1.0000	1.629
1.25	5.79	1.756	1.5625	2.195
1.50	6.53	1.876	2.2500	2.814
1.75	7.45	2.008	3.0625	3.514
2.00	8.46	2.135	4.0000	4.270
7.50		**9.404**	**11.875**	**14.422**

Formula (2.2) give $5a_0 + 7.50a_1 = 9.404$

$7.5a_0 + 11.875a_1 = 14.422.$

which yield the solution; a_0 =1.122, and a_1 = 0.5056.

Hence the approximation assumes the form $y = 3.071e^{0.5.56x}$

2.2. Approximation of functions

The commonly used classes of approximating functions include polynomials, exponential function, trigonometric functions, and rational functions. However, from the application viewpoint, the polynomial functions are mostly used, although in special cases the other functions are also used.

The existence of a polynomial function p (x) that approximates any continuous function f(x) on a finite interval [a, b] is guaranteed by the *Weierstrass approximation theorem* stated before.

2.2.1. Least Squares Approximation Procedure

In the previous section, we have minimized the sum of squares of the errors. A more general approach is to minimize the weighted sum of squares of the errors taken over all data points.

Definition: An integrable function W is called a weight function on [a, b] if $w(x) \geq 0 \, for \forall x \in [a,b]$, but $w(x) \neq 0$ on any subinterval of [a, b]. The purpose of a weight function is to assign varying degrees of importance to approximation on certain portions of the interval.

For functions whose values are given at (n+1) points, $x_0, x_1, x_2, \ldots, x_n$, we have

$$E(a_0, a_1, \ldots, a_n) = \sum_{i=0}^{n} w(x_i)[f(x_i) - \sum_{j=0}^{n} a_j \phi_j(x_i)]^2 = \text{minimum} \qquad (2.4)$$

where $w(x_i)$ are the weight functions,

$f(x_i)$ are values of the function corresponding to the points x_i, i= 0, 1, 2, …, n.

$\phi_i(x_i)$ are coordinate functions and a_i, i = 0, 1, 2, …, n are parameters to be determined so that (2.4) is as small as possible.

For functions that are continuous on [a, b] and are given explicitly, we have

$$\mathbf{E}(\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_n) = \int_a^b w(x)[f(x) - \sum_{j=0}^{n} a_j \phi_j(x)]^2 \, dx = \text{minimum} \qquad (2.5)$$

Note: For polynomial approximation, the coordinate functions $\phi_i(x)$ are usually chosen as $\phi_i(x) = x^i$, i = 0, 1, …, n. and w(x)=1.

The necessary condition for (2.4) and (2.5) above to have a minimum value is that

$$\frac{\partial E}{\partial a_i} = 0, \forall i = 0,1,2,\dots,n$$

This gives a system of (n+1) linear equations in (n+1) unknowns $a_0, a_1, \dots, a_n$. These equations are called normal equations and are given for (2.4) and (2.5) respectively as follows:

$$\sum_{i=0}^{n} w(x_i)[f(x_i) - \sum_{j=0}^{n} a_j \phi_j(x_i)]\phi_j(x_i) = 0, \text{and}$$

$$\int_a^b w(x)[f(x) - \sum_{j=0}^{n} a_j \phi_j(x)]\phi_j(x_i) dx = 0, \; j = 0,1,2,\dots, n. \qquad (2.6)$$

Example: 4.

Obtain the least-square polynomial approximation of degree one and two for $f(x) = \sqrt{x}$, on [0, 1], take w(x)=1.

Solution:

i) For n =1, we have

$$E(a_0, a_1) = \int_0^1 [f(x) - (a_0 + a_1 x)]^2 dx = \text{minimum}.$$

$$\frac{\partial E}{\partial a_0} = 0 = \int_0^1 (x^{\frac{1}{2}} - a_0 - a_1 x) dx$$

$$\frac{\partial E}{\partial a_1} = 0 = \int_0^1 (x^{\frac{1}{2}} - a_0 - a_1 x) x dx$$

Hence $a_0 + \frac{1}{2} a_1 = \frac{2}{3}$ and $\frac{1}{2} a_0 + \frac{1}{3} a_1 = \frac{2}{5}$.

Now solving these equations, we obtain $a_0 = \frac{4}{15}$ and $a_1 = \frac{4}{5}$.

Therefore the first degree least square approximation to $f(x) = \sqrt{x}$, on [0, 1] is

$$p_1(x) = \frac{4}{5}x + \frac{4}{15} = \frac{4}{15}(3x+1)$$

For example: $p_1(\frac{3}{4}) = 0.86, and, f(\frac{3}{4}) = 0.866025$.

ii) For n =2, we have the normal equations:

$$\int_0^1 (x^{\frac{1}{2}} - a_0 - a_1 x - a_2 x^2)dx = 0$$

$$\int_0^1 (x^{\frac{1}{2}} - a_0 - a_1 x - a_2 x^2)xdx = 0$$

$$\int_0^1 (x^{\frac{1}{2}} - a_0 - a_1 x - a_2 x^2)x^2 dx = 0$$

which gives:

$$a_0 + \frac{1}{2}a_1 + \frac{1}{3}a_2 = \frac{2}{3}$$

$$\frac{1}{2}a_0 + \frac{1}{3}a_1 + \frac{1}{4}a_2 = \frac{2}{5}$$

$$\frac{1}{3}a_0 + \frac{1}{4}a_1 + \frac{1}{5}a_2 = \frac{2}{7}.$$

Solving the system of linear equations, we obtain

$$a_0 = \frac{6}{35}, a_1 = \frac{48}{35}, a_2 = \frac{-20}{35}$$

Hence the required quadratic least squares approximation to $f(x) = \sqrt{x}$, on [0, 1]

is: $p_2(x) = \frac{-20}{35}x^2 + \frac{48}{35}x + \frac{6}{35}$

Again to compare the result P_2 (0.5)=0.714286 and f(0.5) = 0.70711.

2.2.2. **Orthogonal Functions and Least Squares Approximation**

In the previous section, we have seen that the method of determining a least-squares approximation to a continuous function gives satisfactory results. However, this method possesses the disadvantage of solving a large linear system of equations. Besides a small change in any of the parameters of the system may introduce large errors in the solution. The degree of these ill-conditioning increases with order of the system. This difficulty can be avoided if the functions $f_i(x)$ are so chosen that they are orthogonal with respect to the weight function w(x) on the interval [a, b].

Definition: 1.

A set of functions { $f_i(x)$ } is said to be orthogonal over a set of points { x_i } with respect to the weight function w(x) if:

$$\sum_{i=1}^{n} w(x_i) f_j(x_i) f_k(x_i) = \begin{cases} 0, if, j \neq k \\ \sum_{i=1}^{n} w(x_i) f_j^2(x_i), if, j = k \end{cases}$$

Definition: 2.

A set of functions { $f_i(x)$ } is said to be orthogonal on an interval [a, b], with respect to the weight function w(x) if :

$$\int_a^b w(x) f_j(x) f_k(x) dx = \begin{cases} 0, if, j \neq k \\ \int_a^b w(x) f_j^2(x) dx, if, j = k \end{cases}.$$

This method of using orthogonal polynomials possesses the great advantage that it does not require a linear system to be solved and is described below. Choosing the approximation in the form $Y(x) = a_0f_0(x) + a_1f_1(x) + \ldots a_nf_n(x)$ where $f_j(x)$ is a polynomial in x of degree j. Then, we write:

$$E(a_0, a_1, a_2, \ldots, a_n) = \sum_{i=0}^{n} w(x_i)[y(x_i) - \sum_{j=0}^{n} a_j f_j(x_i)]^2$$

and for continuous functions on [a, b],

$$E(a_0, a_1, a_2, \ldots, a_n) = \int_a^b w(x)[y(x) - \sum_{j=0}^{n} a_j f_j(x)]^2 dx \cdot$$

Now if the functions $f_j(x)$, j = 0, 1, 2, …, n are, orthogonal, then from the normal equations: $\sum_{i=0}^{n} w(x_i)[y(x_i) - \sum_{j=0}^{n} a_j f_j(x_i)] f_j(x_i) = 0$ and

$\int_a^b w(x)[y(x) - \sum_{j=0}^{n} a_j f_j(x)] f_j(x) dx = 0$, we obtain

$\sum_{i=0}^{n} w(x_i) y(x_i) f_j(x_i) = \sum_{j=0}^{n} a_j w(x_i)] f_j^{\,2}(x_i)$ and

$$\int_a^b w(x) y(x) f_j(x) dx = a_j \int_a^b w(x) f_j^2(x) dx, j = 0,1,2,\ldots, n. \qquad (2.7)$$

From the above, we obtain

$$a_j = \frac{\int_0^1 w(x) y(x) f_j(x) dx}{\int_0^1 w(x) f_j^2(x)}, j = 0,1,2,\ldots, n. \qquad (2.8)$$

Now substitution of $a_0, a_1, a_2, \ldots, a_n$ in $Y(x) = a_0 f_0(x) + a_1 f_1(x) + \ldots a_n f_n(x)$, yields the required least-squares approximation, but the functions $f_j(x)$ are yet to be determined. These are obtained by using the *Gram-Schmidt Orthogonalization Process,* which has important applications in numerical analysis.

Some examples of orthogonal polynomials, which have been extensively studied, are:

a) the Legendre polynomial,

$$p_n(x) = \frac{(-1)^n}{2^n n!} \cdot \frac{d^n}{dx^n}[(1 - x^2)^n], n \geq 1, with, p_0(x) = 1, p_n(1) = 1, \forall n.$$

$$\int_{-1}^{1} p_m(x) p_n(x) dx = \begin{cases} 0, m \neq n \\ \frac{2}{2n+1}, m = n \end{cases}, w(x) = 1.$$

b) the Laguerre polynomial,

$$L_n(x) = \frac{1}{n! e^{-x}} \cdot \frac{d^n}{dx^n}\{x^n e^{-x}\}, n \geq 0,$$

$$\int_0^{\infty} e^{-x} L_m(x) L_n(x) dx = \begin{cases} 0, m \neq n \\ \frac{2}{2n+1}, m = n \end{cases}, w(x) = e^{-x}.$$

c) the Chebysheve polynomial,

$$T_n(x) = \cos(n \cos^{-1} x), n \geq 0, with, T_0(x) = 1, T_n(1) = 1, \forall n.$$

$$\int_{-1}^{1} \frac{T_m(x)T_n(x)}{\sqrt{1-x^2}}dx = \begin{cases} 0, m \neq n \\ \pi, m=n=0 \\ \frac{\pi}{2}, m=n>0. \end{cases}, w(x) = \frac{1}{\sqrt{1-x^2}}$$

d) the Hermite polynomial,

$$H_n(x) = (-1)^n e^{x^2}.\frac{d^n}{dx^n}\{e^{-x^2}\}, n \geq 0,$$

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x)H_n(x)dx = \begin{cases} 0, m \neq n \\ 2^n n!\sqrt{\pi}, m=n \end{cases}, w(x) = e^{-x^2} \text{ and so on.}$$

2.2.3. **Gram-Schmidt Orthogonalization Process**

Suppose that the orthogonal polynomial $f_i(x)$, valid on the interval [a, b], has the leading term x^i. Then, starting with $f_0(x) = 1$, we find that the linear polynomial $f_1(x)$, with leading term x can be written as:

$$f_1(x) = x + k_{1,0}f_0(x). \tag{2.9}$$

Where $k_{1,0}$ is a constant to be determined. Since $f_0(x)$ and $f_1(x)$ are orthogonal, we have

$$\int_a^b w(x)f_0(x)f_1(x)\,dx = 0 = \int_a^b xw(x)f_0(x)dx + k_{1,0}.\int_a^b w(x)f_0^2(x)dx. \text{ (using (2.9))}$$

From this we obtain $$k_{1,0} = -\frac{\int_a^b xw(x)f_0(x)dx}{\int_a^b w(x)f_0^2(x)dx} \tag{2.10}$$

And hence (2.9) gives $f_1(x) = x - \dfrac{\int_a^b xw(x)f_0(x)dx}{\int_a^b w(x)f_0^2(x)dx}$, as $f_0(x) = 1$.

Again, the polynomial $f_2(x)$, of degree two in x and leading term x^2, may be written as:

$$f_2(x) = x^2 + k_{2,0}f_0(x) + k_{2,1}f_1(x) \tag{2.11}$$

where the constants $k_{2,0}$ and $k_{2,1}$ are to be determined by using the orthogonality condition. Since $f_2(x)$ is orthogonal to $f_0(x)$, we have:

$$\int_a^b w(x) f_0(x)[\, x^2 + k_{2,0} f_0(x) + k_{2,1} f_1(x)] dx = 0$$

$$\Rightarrow \int_a^b w(x) f_0(x)\, x^2 dx + k_{2,0} \int_a^b w(x) f_0(x) dx = 0 \text{ ,since}$$

$$\int_a^b w(x) \mathrm{f}_0\,(\mathrm{x})\, \mathrm{f}_1(\mathrm{x})\, dx = 0.$$

Hence $k_{2,0} = -\frac{\int_a^b x^2 w(x) f_0(x) dx}{\int_a^b w(x) f_0^2(x) dx} = -\frac{\int_a^b x^2 w(x) dx}{\int_a^b w(x) f_0^2(x) dx}$ (2.12)

Again, since $f_2(x)$ is orthogonal to $f_1(x)$, we have

$$\int_a^b w(x) f_1(x)[\, x^2 + k_{2,0} f_0(x) + k_{2,1} f_1(x)] dx = 0$$

$$\Rightarrow \int_a^b w(x) f_1(x)\, x^2 dx + k_{2,1} \int_a^b w(x) f_1^2\, (x) dx = 0,$$

since $\int_a^b w(x) \mathrm{f}_0\,(\mathrm{x})\, \mathrm{f}_1(\mathrm{x})\, dx = 0.$

Hence we have $k_{2,1} = -\frac{\int_a^b x^2 w(x) f_1(x) dx}{\int_a^b w(x) f_1^2(x) dx}.$ (2.13)

Now since $k_{2,0}$ and $k_{2,1}$ are known, $f_2(x)$ becomes determined. Proceeding in this way, the method can be generalized and we write

$$f_i(x) = x^i + k_{i,0} f_0(x) + k_{i,1} f_1(x) + \ldots + k_{i,i-1} f_{i-1}(x). \quad (2.14)$$

where the constants $k_{i,j}$ are so chosen that $\mathrm{f}_i(\mathrm{x})$ is orthogonal to $f_0(x), f_1(x), \ldots, f_{i-1}(x)$. These conditions yield:

$$k_{i,j} = -\frac{\int_a^b x^i w(x) f_j(x)dx}{\int_a^b w(x) f_j^2(x)dx}. \qquad (2.15)$$

Since the a_i and $f_i(x)$ in the equation $Y(x) = a_0f_0(x) + a_1f_1(x) + \ldots a_nf_n(x)$ are known, the approximation can now be determined.

Example: 5.

Obtain the first four orthogonal polynomials $f_n(x)$ on [0, 1] which are orthogonal with respect to the weight function $w(x) = 1$.

Using the polynomials obtained, determine the least square approximation of the third degree for the function $g(x) = x^{\frac{1}{2}}$ on [0, 1].

Solution:

i) Let $f_0(x) = 1$, then by (2.10), $k_{1,0} = -\frac{\int_0^1 xdx}{\int_0^1 dx} = -\frac{1}{2}$,

we then obtain from (2.9), $f_1(x) = x - \frac{1}{2}$.

Now from (2.12) and (2.13), we have $k_{2,0} = -\frac{\int_0^1 x^2 dx}{\int_0^1 dx} = -\frac{1}{3}$

and $k_{2,1} = -\frac{\int_0^1 x^2(x-\frac{1}{2})dx}{\int_0^1 (x-\frac{1}{2})^2 dx} = -1$

Hence from (2.11), we get $f_2(x) = x^2 - (x - \frac{1}{2}) - \frac{1}{3} = x^2 - x + \frac{1}{6}$.

In a similar manner, we obtain $k_{3,0} = -\frac{\int_0^1 x^3 dx}{\int_0^1 dx} = -\frac{1}{4}$

$$k_{3,1} = -\frac{\int_0^1 x^3(x-\frac{1}{2})dx}{\int_0^1 (x-\frac{1}{2})^2 dx} = -\frac{9}{10}$$

$$k_{3,2} = -\frac{\int_0^1 x^3(x^2-x+\frac{1}{6})dx}{\int_0^1 (x^2-x+\frac{1}{6})^2\,dx} = -\frac{3}{2}$$

Therefore $f_3(x) = x^3 - \frac{3}{2}(x^2 - x + \frac{1}{6}) - \frac{9}{10}(x - \frac{1}{2}) - \frac{1}{4}$.

$$= x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20},$$

And hence the required first four orthogonal polynomials are:

$f_0(x) = 1,$ $\quad f_1(x) = x - \frac{1}{2},$ $\quad f_2(x) = x^2 - x + \frac{1}{6},$

$f_3(x) = x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20}.$

ii) Now to determine the least-square approximation of the function $g(x) = x^{\frac{1}{2}}$ on [0, 1] with these polynomials, it remains to find the constants a_i, i = 0, 1, 2, Such that the required polynomial will be $Y(x) = a_0f_0(x) + a_1f_1(x) + a_2f_2(x)+a_3f_3(x)$. For this we use equation (2.8) and orthogonality conditions:

i.e. $E(a_0, a_1, a_2, a_3) = \int_0^1 w(x)[g(x) - \sum_{j=0}^{3} a_j f_j(x)]^2 dx = \text{minimum}.$

For a minimum to occur, $\frac{\partial E}{\partial a_j} = 0$, j = 0, 1, 2, 3. Hence we have

$$a_j = \frac{\int_0^1 w(x)x^{\frac{1}{2}} f_j(x)dx}{\int_0^1 w(x) f_j^2(x)dx}, \; j = 0,1,2,3.$$ Since w(x) = 1, we have

$$a_0 = \frac{\int_0^1 w(x)g(x)f_0(x)dx}{\int_0^1 f_0^2(x)dx} = \frac{\int_0^1 (x^{\frac{1}{2}})dx}{\int_0^1 dx} = \frac{2}{3}$$

$$a_1 = \frac{\int_0^1 g(x)f_1(x)dx}{\int_0^1 f_1^2(x)dx} = \frac{\int_0^1 (x^{\frac{1}{2}})(x-\frac{1}{2})dx}{\int_0^1 (x-\frac{1}{2})^2\,dx} = \frac{4}{5}$$

$$a_2 = \frac{\int_0^1 g(x)f_2(x)dx}{\int_0^1 f_2^2(x)} = \frac{\int_0^1 (x^{\frac{1}{2}})(x^2-x+\frac{1}{6})dx}{\int_0^1 (x^2-x+\frac{1}{6})^2\,dx} = -\frac{4}{7}$$

$$a_3 = \frac{\int_0^1 g(x) f_3(x) dx}{\int_0^1 f_3^2(x)} = \frac{\int_0^1 (x^{\frac{1}{2}})(x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20}) dx}{\int_0^1 (x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20})^2 dx} = \frac{8}{9}$$

Therefore, the required least-square approximation for $g(x) = x^{\frac{1}{2}}$ on [0, 1] is:

$$p(x) = \tfrac{2}{3} f_0(x) + \tfrac{4}{5} f_1(x) - \tfrac{4}{7} f_2(x) + \tfrac{8}{9} f_3(x).$$

Or $p(x) = \frac{8}{9}x^3 - \frac{40}{21}x^2 + \frac{40}{21}x + \frac{8}{63}$ is the required polynomial.

Hence on [0, 1], $g(x) = x^{\frac{1}{2}} \approx \frac{8}{9}x^3 - \frac{40}{21}x^2 + \frac{40}{21}x + \frac{8}{63}$.

CHAPTER-3

CHEBYSHEV POLYNOMIALS AND OTHER APPROXIMATING FUNCTIONS

3.1. Chebyshev Polynomials

This section will be concerned with the theory of a class of orthogonal polynomials that are the basis for fitting non-algebraic functions with polynomials of maximum efficiency. i.e. a means of reducing the degree of an approximating polynomial with minimal loss of accuracy.

3.1.1. Chebyshev Polynomials and Approximations

We turn now to the problem of representing a function with a minimum error. This is a central problem in the software development of digital computers because it is more economical to compute the values of the common functions using an efficient approximation than to store a table of values and employ interpolation techniques.

One way to approximate a function by a polynomial is to use a truncated Taylor series. This is not the best way in most cases. To study better ways, we first need to introduce the Chebyshev polynomials.

The chebyshev polynomial of degree n over the interval [-1, 1] is defined by the relation:

$$T_n(x) = \cos [n\cos^{-1}x] \quad , \text{for } x \in [-1, 1] \text{ and } n = 0, 1, 2, \ldots \qquad (3.1)$$

From which follows immediately the relation $T_n(x) = T_{-n}(x)$. Let $\cos^{-1}x = \theta$ so that $x = \cos\theta$ and (3.1) gives:

$$T_n(x) = \cos (n\theta). \text{ for } x \in [-1, 1] \text{ and } n = 0, 1, 2, \ldots .$$

Hence $T_0(x) = 1$ and $T_1(x) = x$.

Using the trigonometric identity, $\cos(n-1)\theta + \cos(n+1)\theta = 2\cos n\theta.\cos\theta$, we obtain:

$$T_{n-1}(x) + T_{n+1}(x) = 2xT_n(x), \text{ which is the same as:}$$

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \text{ , for } x \in [-1, 1] \text{ and } n = 0, 1, 2, \ldots \qquad (3.2)$$

This is the recurrence relation which can be used to compute successively all $T_n(x)$, since we know $T_0(x)$ and $T_1(x)$.

The first six-chebyshev polynomials are:

$T_0(x) = 1,$

$T_1(x) = x,$

$T_2(x) = 2x^2 – 1,$

$T_3(x) = 4x^3 - 3x,$

$T_4(x) = 8x^4 – 8x^2 +1,$

$T_5(x) = 16x^5 – 20x^3 +5x,$

$T_6(x) = 32x^6 – 48x^4 +18x^2 – 1$, and so on (3.3)

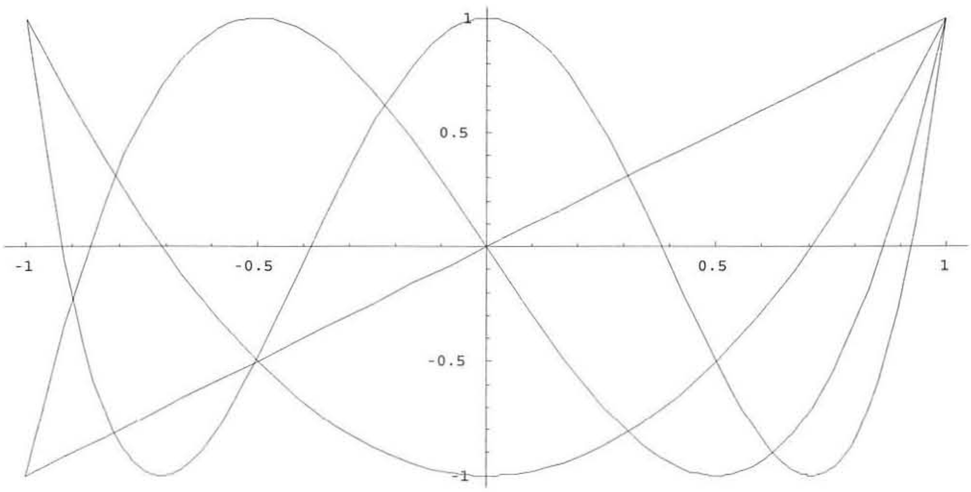

Figure 3.1. Chebyshev polynomials, $T_n(x)$, n = 1, 2, 3, 4.

Note that the coefficient of x^n in $T_n(x)$ is always 2^{n-1}.

Some properties of chebyshev polynomials

- If we set $y = T_n(x) = \cos n\theta$, then we get

$$\frac{dy}{dx} = \frac{n \sin n\theta}{\sin\theta}, \text{ and}$$

$$\frac{d^2y}{dx^2} = \frac{-n^2\cos n\theta + n\sin n\theta \cot\theta}{\sin^2\theta} = \frac{-n^2 y + x\frac{dy}{dx}}{1-x^2},$$

So that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + n^2 y = 0$

Which is the differential equation satisfied by $T_n(x)$.

- An important property of $T_n(x)$ is given by:

$$\int_{-1}^{1}\frac{T_m(x)T_n(x)dx}{\sqrt{1-x^2}}=\begin{cases}0, & m\neq n\\ \frac{\pi}{2}, & m=n\neq 0\\ \pi, & m=n=0\end{cases} \tag{3.4}$$

i.e. the polynomials form an orthogonal set w.r.t. the weight function $\frac{1}{\sqrt{1-x^2}}$.

This property is easily proved since by putting x $=\cos\theta$, the above integral becomes,

$$\int_{0}^{\pi}T_m(\cos\theta)T_n(\cos\theta)d\theta=\int_{0}^{\pi}(\cos m\theta)(\cos n\theta)d\theta$$

➢ If $n\neq m$:

$$\int_{-1}^{1}\frac{T_m(x)T_n(x)dx}{\sqrt{1-x^2}}=\frac{1}{2}\int_{0}^{\pi}(\cos((n+m)\theta)d\theta+\frac{1}{2}\int_{0}^{\pi}(\cos((n-m)\theta)d\theta=$$

$$\left[\frac{\sin(m+n)\theta}{2(m+n)}+\frac{\sin(m-n)\theta}{2(m-n)}\right]_0^{\pi}=0$$

➢ If $n=m\neq 0$:

$$\int_{-1}^{1}\frac{T_m(x)T_n(x)dx}{\sqrt{1-x^2}}=\int_{-1}^{1}\frac{T_n^2(x)dx}{\sqrt{1-x^2}}=\frac{\pi}{2}, \text{ for each } n\geq 0$$

➢ If $n=m=0$:

$$\int_{-1}^{1}\frac{T_m(x)T_n(x)dx}{\sqrt{1-x^2}}=\frac{1}{2}\int_{0}^{\pi}(\cos(0)d\theta+\frac{1}{2}\int_{0}^{\pi}(\cos(0)d\theta=\frac{\pi}{2}+\frac{\pi}{2}=\pi.$$

- The Chebyshev polynomial T_n of degree $n\geq 1$ has n simple roots in [-1, 1] at:

$$\bar{x}_k=\cos\left(\frac{2k-1}{2n}\pi\right) \text{ for each } k = 1, 2, \ldots, n.$$

Moreover, T_n assumes its absolute extrema at the points

$$\bar{x}_k'=\cos\left(\frac{k\pi}{n}\right) \text{ for each } k = 1, 2, \ldots, n, \text{ with}$$

$T_n(\bar{x}_k') = (-1)^k$ for each k = 1, 2, …, n.

To prove the first part:

If $\bar{x}_k = \cos(\frac{2k-1}{2n}\pi)$ for k = 1, 2, …, n, then

$$T_n(\bar{x}_k) = \cos(n \cos^{-1} \bar{x}_k) = \cos(n \cos^{-1}(\cos(\frac{2k-1}{2n}\pi))) = \cos(\frac{2k-1}{2}\pi) = 0;$$

So $\bar{x}_k$ is a zero of T_n for each k = 1, 2, …, n. Since T_n is a polynomial of degree n, all zeros of T_n must be of this form.

TO show the second, note that $T_n'(x) = \frac{d}{dx}[\cos(n\cos^{-1}(x))] = \frac{n\sin(n\cos^{-1} x)}{\sqrt{1-x^2}}$, and that, when $1 \le k \le n-1$,

$$T_n'(\bar{x}_k') = \frac{n\sin(n\cos^{-1}(\cos(\frac{k\pi}{n})))}{\sqrt{1-\cos^2(\frac{k\pi}{n})}} = \frac{n\sin(k\pi)}{\sin(\frac{k\pi}{n})} = 0.$$

Moreover, since T_n is a polynomial of degree n, T_n' is a polynomial of degree (n-1) and all zeros of T_n' occur at these points. The only other possibilities for extrema of the function T_n occur at the end points of the interval [-1, 1]; i.e., at $\bar{x}_0' = 1$ and $\bar{x}_n' = -1$.

Since $T_n(\bar{x}_k') = \cos(n \cos^{-1}(\cos(\frac{k\pi}{n}))) = \cos(k\pi) = (-1)^k$, a maximum occurs at each even value of k and a minimum at each odd value.

- Because of the relation $T_n(x) = \cos n\theta$, it is apparent that the Chebyshev polynomial have a succession of maximums and minimums of alternating signs, each of magnitude one. Further, because $|\cos n\theta| = 1$ for $n\theta = 0, \pi, 2\pi, \ldots$, and because θ varies from 0 to π, as x varies from 1 to -1, $T_n(x)$ assumes its maximum magnitude of unity n+1 times on the interval [-1, 1].

We have seen above that $T_n(x)$ is a polynomial of degree n in x and that the coefficient of x^n in $T_n(x)$ is 2^{n-1}.

Of all polynomials of degree n where the coefficient of x^n is unity, the polynomial $\widehat{T}_n(x) = 2^{1-n}.\ T_n(x)$, ($n \geq 1$) has a smaller upper bound to its magnitude in the interval [-1, 1] than any other.

Theorem

The polynomials of the form $\widehat{T}_n$, when $n \geq 1$, have the property that

$$\frac{1}{2^{n-1}} = \max_{x\in[-1,1]}\left|\widehat{T}_n(x)\right| \leq \max_{x\in[-1,1]}\left|P_n(x)\right| \quad \text{for all monic polynomials } P_n .$$

Equality occurs only if $P_n = \widehat{T}_n$.

Proof:

Suppose not, i.e. $\frac{1}{2^{n-1}} = \max_{x\in[-1,1]}\left|\widehat{T}_n(x)\right| \geq \max_{x\in[-1,1]}\left|P_n(x)\right|$ for all monic polynomials P_n.

Let Q = $\widehat{T}_n$ - P_n. Since P_n and $\widehat{T}_n$ are both monic polynomials of degree n, Q is a polynomial of degree at most (n-1). Moreover, at the extreme points of $\widehat{T}_n$,

$$Q(\bar{x}_k') = \widehat{T}_n(\bar{x}_k') - P_n(\bar{x}_k') = \frac{(-1)^k}{2^{n-1}} - P_n(\bar{x}_k').$$

The fact that $\frac{1}{2^{n-1}} \geq \left|P_n(x)\right|$ implies that, for k = 1, 2, …, n, $Q(\bar{x}_k') \leq 0$, when k is odd, and $Q(\bar{x}_k') \geq 0$. when k is even.

Since Q is continuous, the Intermediate Value Theorem can be used to show that the (n-1)st-degree polynomial Q must have at least n zeros in the interval [-1, 1], which is clearly impossible unless Q = 0. This implies $\widehat{T}_n = P_n$, which establishes the result. //

Thus, in chebyshev approximation, the maximum error is kept down to a minimum. This is often referred to as *minimax principle*. By this process, we can obtain the best lower order (degree) approximation, called the *minimax approximation*, to a given polynomial.

And this is important because we will be able to write power series representations of functions whose maximum errors are given in terms of this upper bound.

- By rearranging the chebyshev polynomials, we can express powers of x in terms of them:

$$1 = T_0$$

$$x = T_1$$

$$x^2 = \frac{1}{2}(T_0 + T_2)$$

$$x^3 = \frac{1}{4}(3T_1 + T_3),$$

$$x^4 = \frac{1}{8}(3T_0 + 4T_2 + T_4), \tag{3.5}$$

$$x^5 = \frac{1}{16}(10T_1 + 5T_3 + T_5),$$

$$x^6 = \frac{1}{32}(10T_0 + 15T_2 + 6T_4 + T_6),$$

$$x^7 = \frac{1}{64}(35T_1 + 21T_3 + 7T_5 + T_7),$$

$$x^8 = \frac{1}{128}(35T_0 + 56T_2 + 28T_4 + 8T_6 + T_8), \text{ and so on.}$$

Example: 1.

Obtain the minimax approximation to the cubic $1/3x^3 + 2x^2$ on [-1, 1].

Solution

Using the relations given in (3.5), we write

$$1/3x^3 + 2x^2 = \frac{1}{12}(3T_1 + T_3) + 2x^2$$

$$= 2x^2 + 1/4T_1(x) + 1/12T_3(x)$$

$$= 2x^2 + 1/4\,(x) + 1/12T_3(x), \text{ as } T_1(x) = x.$$

Since $T_3(x)$ is a polynomial of degree 3, the required lower-order approximation to the given cubic is $2x^2 + 1/4\,(x)$. This implies $1/3x^3 + 2x^2 - (2x^2 + 1/4\,(x)) = 1/12T_3(x)$, and hence the maximum error of this approximation on [-1, 1] is: $\max\limits_{-1\le x\le 1} \frac{1}{12}|T_3(x)| = 0.08333$.

3.1.2. Economization of Power Series

To describe this process, we consider the power series expansion of f(x) in the form:

$$f(x) = A_0 + A_1x + A_2x^2 + \ldots + A_nx^n, \ (-1 \le x \le 1) \qquad (3.6)$$

Using the relations given in (3.5), we convert the above series into an expansion in Chebyshev polynomials. We obtain

$$f(x) = B_0 + B_1T_1(x) + B_2T_2(x) + \ldots + B_nT_n(x), \qquad (3.7)$$

For a large number of functions, an expansion as in (3.7) above converges more rapidly than the power series given by (3.6). This is known as economization of the power series. We are now ready to use Chebyshev polynomials to "economize" a power series.

Example: 2.

Consider the Maclaurin series for e^x:

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \ldots.$$

If we would like to use a truncated series to approximate e^x on the interval [0, 1] with a precision of 0.001, we will have to retain terms through that in x^6, because the error after the term in x^5 will be more than 1/720. Suppose we subtract $(\frac{1}{720})(\frac{T_6}{32})$ from the truncated series. We note from eq. (3.3) that this will exactly cancel the x^6 term and at the same time make adjustments in other coefficients of the Maclaurin series. Because the maximum value of T_6 on the interval [0, 1] is unity, this will change the sum of the truncated series by only, $(\frac{1}{720}).(\frac{1}{32}) < 0.00005$, which is small with respect to our required precision of 0.001. Performing the calculations, we have:

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} - \frac{1}{720}.(\frac{1}{32})(32x^6 - 48x^4 + 18x^2 - 1)$$

$$e^x = 1.000043 + x + 0.499219x^2 + \frac{x^3}{6} + 0.043750x^4 + \frac{x^5}{120}. \qquad (3.8)$$

This gives a fifth-degree polynomial that approximates e^x on [0, 1] almost as well as the sixth-degree one derived from Maclaurin series. (The actual maximum error of the fifth-degree expression is 0.000270; for the sixth-degree expression it is 0.000226.) We hence have "economized" the power series in that we get nearly the same precision with fewer terms.

By subtracting $(\frac{1}{120})(\frac{T_5}{16})$ we can economize further, getting a fourth-degree polynomial that is almost as good as the economized fifth-degree one.

Because of the relative ease with which they can be developed, such economized power series are frequently used for approximations to functions and are much more efficient than power series of the same degree obtained by merely truncating a Taylor or Maclaurin series.

x	e^x	Maclaurin, sixth-degree	Economized, fifth-degree	Economized, fourth-degree	Maclaurin, fourth-degree
0	1.00000	1.00000	1.00004	1.00004	1.00000
0.2	1.22140	1.22140	1.22142	1.22098	1.22140
0.4	1.49182	1.49182	1.49179	1.49133	1.49173
0.6	1.82212	1.82211	1.82208	1.82212	1.82140
0.8	2.22554	2.22549	2.22553	2.22605	2.22240
1.0	2.71828	2.71806	2.71801	2.71749	2.70833
Maximum error		0.00023	0.00027	0.00078	0.00995

Table 3.1 Comparison of errors of economized power series and a Maclaurin series for e^x.

The maximum error in the economized fifth-degree polynomial is only slightly greater than in the sixth-degree Maclaurin series. The economized fourth-degree polynomial incurs a maximum error about three and one-half times as much, but still within the 0.001 limits that was initially imposed. In contrast, a fourth-degree Maclaurin series has an error nearly ten times greater than the 0.001 tolerance, and its error is over twelve times that of the fourth-degree economized form.

Chebyshev Series:

By substituting the identities given by (3.5) into an infinite Taylor series and collecting terms in $T_i(x)$, we create a Chebyshev series. For example, we can get the first four terms of a Chebyshev series by starting with the Maclaurin expansion for e^x. Such a series converges more rapidly than does a Taylor series on [-1, 1]:

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots .$$

Replacing terms by eq. (3.5), but omitting polynomials beyond $T_3(x)$ because we want only four terms, we have:

$$e^x = T_0 + T_1 + \frac{1}{4}(T_0 + T_2) + \frac{1}{24}(3T_1 + T_3) + \frac{1}{192}(3T_0 + T_2 + \dots) +$$

$$\frac{1}{1920}(10T_1 + 5T_3 + \dots) + \frac{1}{23{,}040}(10T_0 + 15T_2 + \dots) + \dots .$$

$$= 1.2661T_0 + 1.1302T_1 + 0.2715T_2 + 0.0443T_3 + \dots .$$

To compare the Chebyshev expansion with the Maclaurin series, we convert back to powers of x, using eq. (3.3):

$$e^x = 1.2661 + 1.1302(x) + 0.2715(2x^2-1) + 0.0443(4x^3-3x) + \dots$$

$$= 0.9946 + 0.9973x + 0.5430x^2 + 0.1772x^3 + \dots . \quad (3.9)$$

x	e^x	Chebyshev	Error	Maclaurin	Error
-1.0	0.3679	0.3631	0.0048	0.3333	0.0346
-0.8	0.4493	0.4536	-0.0042	0.4346	0.0147
-0.6	0.5488	0.5534	-0.0046	0.5440	0.0048
-0.4	0.6703	0.6712	-0.0009	0.6693	0.0010
-0.2	0.8187	0.8154	0.0033	0.8187	0.0001
0	1.0000	0.9946	0.0054	1.0000	0.0000
0.2	1.2214	1.2172	0.0042	1.2213	0.0001
0.4	1.4918	1.4917	0.0001	1.4907	0.0012
0.6	1.8221	1.8267	-0.0046	1.8160	0.0061
0.8	2.2255	2.2307	-0.0051	2.2054	0.0202
1.0	2.7183	2.7121	0.0062	2.6667	0.0516

Table 3.2. Comparison of Chebyshev series for e^x with Maclaurin series:

Table 3.2 compare the error of Chebyshev expansion, eq. (3.9), with the Maclaurin series, using terms through x^3 in each case. The table shows how Chebyshev expansion attains a smaller maximum error by permitting the error at the origin to increase. The error distributes more or less uniformly through out the interval. In contrast to this, the Maclaurin expansion, which gives very small errors near the origin, allows the error to bunch up at the ends of the interval.

<u>*Example*: 3.</u>

Economize the series

$\text{Sinhx} = x + \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^7}{5040}$, on the interval [-1, 1], allowing for a tolerance of 0.0006.

<u>*Solution*</u>

Since 1/5040 = 0.000198, the truncated series

$\text{Sinhx} = x + \frac{x^3}{6} + \frac{x^5}{120}$ will produce a change in the fourth decimal place only which is below 0.0006. We know convert the powers of x in the above expression in to chebyshev polynomials by using the relations given in (3.5). This gives

$$\text{Sinhx} = T_1(x) + \frac{1}{24}(3T_1(x) + T_3(x)) + \frac{1}{120(16)}(10T_1(x) + 5T_3(x) + T_5(x))$$

$$= \frac{216}{192}T_1(x) + \frac{17}{384}T_3(x)) + \frac{1}{1920}T_5(x)$$

Since $\frac{1}{1920}$ = 0.00052 which is again less than our tolerance, the required economized series is therefore given by $\sin hx = \frac{216}{192}T_1(x) + \frac{17}{384}T_3(x))$

$$= \frac{216}{192}x + \frac{17}{384}(4x^3 - 3x)$$

$$= \frac{381}{384}x + \frac{17}{96}x^3$$

3.2. Trigonometric Polynomial Approximations

In practical interpolation and approximation problems, we encounter special properties of the function to be approximated. One common situation is that the function is known to be periodic; that is, there is some constant, τ - say the period of the function – such that $f(x + \tau) = f(x)$ for all x.

Many physical phenomena, such as most wave theories – electromagnetic, sound, and water waves, for example – exhibit such periodic behaviour.

Since the only periodic polynomials are the constant functions, one has to use other function classes for the effective approximation of periodic functions. Hence trigonometric functions are often be employed.

To simplify the theory, we fix the period as being 2π and the interval over which we seek our approximation to be $[-\pi, \pi]$. Other periods or intervals can easily be handled with a simple linear change of the independent variable.

We say that $S_n(x)$ is a trigonometric sum (polynomial) of order at most n, if

$$S_n(x) = \frac{1}{2}a_0 + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx), \qquad (3.10)$$

the coefficients a_k and b_k are real numbers. It is a trigonometric polynomial obtained by truncating a trigonometric series (sum) of the form

$$S_n(x) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$$

We consider the representation of a function f(x) defined in the interval $[-\pi, \pi]$ in terms of trigonometric polynomials.

If f(x) is defined in some other interval [a, b], a simple linear change of variable can reduce the problem to the case of the interval $[-\pi, \pi]$.

A basic result on approximation by such trigonometric sums is again due to *Weierstrass* and can be stated as:

Let $f(\theta)$ be continuous on $[-\pi, \pi]$ and periodic with period 2π. Then for any $\varepsilon>0$, there exists an $n = n(\varepsilon)$ and a trigonometric sum, $S_n(\theta)$, such that $|f(\theta)-S_n(\theta)| < \varepsilon$ for all θ.

If f(x) is periodic of period 2π and square integrable on $[-\pi, \pi]$, we can seek a trigonometric sum of the form (3.10) for which

$$\|f-S_n\|_2 = \left(\left(\int_{-\pi}^{\pi}[f(x)-S_n(x)]^2\right)^{\frac{1}{2}}\right. \qquad (3.11)$$

is a minimum with respect to all such sums.

The trigonometric functions satisfy the orthogonality relations

$$\int_{-\pi}^{\pi}\cos jx\cos kx dx = \begin{cases} 0 & j \neq k \\ \pi & j = k \neq 0 \end{cases},$$

$$\int_{-\pi}^{\pi}\sin jx\sin kx dx = \begin{cases} 0 & j \neq k \\ \pi & j = k \neq 0 \end{cases}$$

$$\int_{-\pi}^{\pi}\sin jx\cos kx dx = 0$$

By using these results in the normal system obtained by minimizing (3.10), we find

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos kx\, f(x)\, dx, \quad (k = 0, 1, 2, \dots, n)$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi}\sin kx\, f(x)\, dx, \quad (k = 1, 2, \dots, n) \qquad (3.12)$$

The trigonometric sum (3.10) with coefficients given in (3.12) determines the best least squares approximation of order n to f(x) by such sums.

<u>Remark:</u> Given a function f(x) which is integrable over the interval $[-\pi, \pi]$, we can define the *Fourier Series* for f(x) by $F(x) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$

Where a_k and b_k are given by allowing $n \to \infty$ in (3.12).

Hence the truncated Fourier series $F_n(x)$ for f(x) is the best trigonometric polynomial of degree n or less in the sense of minimizing (3.11)

The Fourier Series converges uniformly to f(x) if f(x) is periodic with period 2π and is continuous with a piecewise-continuous first derivative.

Note: If the period of f(x) is some number p, then the change of variable $\varepsilon = \frac{2\pi x}{p}$ results in a function $g(\varepsilon) = f(\frac{p\varepsilon}{2\pi})$ which has period 2π.

Example4.

Find the coefficients of the Fourier Series expansion of the saw tooth wave defined by $f[x + 2k\pi] = x, x \in (-\pi, \pi]$

Solution:

The coefficients a_k of the approximation will all be zero since x.coskx is an odd function and the integration in (3.12) is over a symmetric interval. For the coefficients of the sine terms, we get $b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin kx \, dx = (2/k)(-1)^{k+1}$. So that the Fourier series for the function x over $[-\pi, \pi]$ is $2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}\sin kx}{k} = 2\sin x - \sin 2x + 2/3 \sin 3x - \ldots$.

This expansion converges to $f[x + 2k\pi] = x$, and hence the function is approximated by this trigonometric series.

3.3. **Spline Approximation**

The previous chapters were concerned with the approximation of arbitrary functions on closed intervals by the use of polynomials. While the oscillatory nature of high degree polynomials and the property that a fluctuation over a small portion of the interval can induce large fluctuations over the entire range, restricts their use when approximating many of the functions that arise in actual physical situations.

An alternative approach that can be used to obtain interpolatory functions is to divide the interval in to a collection of subintervals and construct a (generally) different approximating polynomial on each subinterval. Approximating by functions of this type is called *piece-wise polynomial approximation.*

3.3.1. **Cubic Spline Interpolation**

The simplest types of piece wise polynomial functions on an interval $[x_0, x_1]$ is the function obtained by fitting a linear and quadratic polynomials between each successive pair of nodes. i.e. constructing a linear or a quadratic on each subinterval agreeing with the function at the end points of the subintervals.
However there is a difficulty with these procedures because the curves has to be chosen in such a way that not only the curve itself, but also its slope and curvature are continuous functions. Linear and quadratic functions fail to satisfy these conditions.

One of the most popular techniques presently in use is the type of piece wise polynomial approximation using cubic polynomials between each successive pair of nodes, which is called cubic spline interpolation. A general cubic polynomial involves four constants; so there is sufficient flexibility in the cubic spline procedure to ensure not only that the interpolant is continuously differentiable on the interval, but that it has a continuous second derivative on the interval as well.

Definition:

Given a function f defined on [a, b] and a set of numbers, called, ***nodes*** or *knots*, $a = x_0 < x_1 < x_2 < \ldots < x_n = b$, a cubic spline interpolant, S, for f is a function that satisfies the following conditions:

a) S(x) is a cubic polynomial, say $S_i(x)$, in each subinterval $[x_i, x_{i+1}]$ $i = 0, 1, 2, \ldots, n-1$; from this condition, we obtain:

$$S(x) = \begin{cases} S_0(x), & x_0 \le x \le x_1 \\ S_1(x), & x_1 \le x \le x_2 \\ \cdots\cdots & \cdots\cdots \quad \cdots\cdots . \\ S_{n-1}(x), & x_{n-1} \le x \le x_n \end{cases}$$

b) $S(x_i) = f(x_i) = y_i$ for each $i = 0, 1, 2, \ldots, n$;

This interpolation condition gives:

$S(x_0) = S_0(x_0) = y_0;$

$S(x_1) = S_1(x_1) = y_1;$

....

$S(x_n) = S_{n-1}(x_n) = y_n.$

c) S(x), $S'(x)$ and $S''(x)$ are continuous on [a, b]

This continuity condition yield relations of the type:

i) $S_i(x_{i+1}) = S_{i+1}(x_{i+1})$

ii) $S_i'(x_{i+1}) = S_{i+1}'(x_{i+1})$

iii) $S_i''(x_{i+1}) = S_{i+1}''(x_{i+1})$, for each $i = 0, 1, 2, \ldots, n-2$.

d) One of the following set of boundary conditions is satisfied:

i) $S''(x_0) = S''(x_n) = 0$, (Free boundary condition)

ii) $S'(x_0) = f'(x_0)$ and $S'(x_n) = f'(x_n)$,(Clamped boundary conditions)

When the free boundary conditions occur, the spline is called a *natural spline*.

To construct the cubic-spline interpolant for a given function f, the conditions in the definition can be applied to the cubic polynomials.

$S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3$ for each $i = 0, 1, 2, \ldots, n-1$.

Clearly, $f(x_i) = S(x_i) = S_i(x_i) = a_i$, And if condition c(i) is applied,

$a_{i+1} = S_{i+1}(x_{i+1}) = S_i(x_{i+1})$

$= a_i + b_i(x_{i+1} - x_i) + c_i(x_{i+1} - x_i)^2 + d_i(x_{i+1} - x_i)^3$ for each i = 0,1, 2, …, n-2.

If we denote $x_{i+1} - x_i = h_i$ and in addition, if we define $a_n = f(x_n)$, it can be seen that the equation

$a_{i+1} = a_i + b_i h_i + c_i(h_i)^2 + d_i(h_i)^3$ holds for each i = 0,1, 2, …, n-1. (3.13)

In a similar manner, define $b_n = S'(x_n)$ and observe that

$S_i'(x) = b_i + 2c_i(x - x_i) + 3d_i(x - x_i)^2$ it follows that:

$S_i'(x_i) = b_i$ for each i = 0,1, 2, …, n-1.

Applying condition c(ii),

$b_{i+1} = b_i + 2c_i h_i + 3d_i(h_i)^2$ for each i = 0,1, 2, …, n-1. (3.14)

Another relation between the coefficients of S_i can be obtained by defining $c_n = \dfrac{S''(x_n)}{2}$

and again applying condition c(iii),

$c_{i+1} = c_i + 3d_i h_i$, for each i = 0,1, 2, …, n-1. (3.15)

Solving for d_i in eq.(3.15) and substituting this value into eq.(3.13) and (3.14) gives the new equations:

$a_{i+1} = a_i + b_i h_i + (h_i)^2 1/3\ (2c_i + c_{i+1})$ (3.16)

and $b_{i+1} = b_i + h_i (c_i + c_{i+1})$ for each i = 0,1, 2, …, n-1. (3.17)

The final relationship involving the coefficients can be obtained by solving the appropriate equation in the form of eq. (3.16), first for b_i;

$$b_i = \frac{1}{h_i}(a_{i+1} - a_i) - \frac{h_i}{3}(2c_i + c_{i+1}) \quad (3.18)$$

and then, with a reduction of the index, for b_{i-1},

$$b_{i-1} = \frac{1}{h_{i-1}}(a_i - a_{i-1}) - \frac{h_{i-1}}{3}(2c_{i-1} + c_i).$$

Substituting these values in to the equation derived from eq. (3.17), when the index is reduced by one, gives the linear system of equations:

$$h_{i-1}c_{i-1} + 2(h_{i-1} + h_i)c_i + h_i c_{i+1} = \frac{3}{h_i}(a_{i+1} - a_i) - \frac{3}{h_{i-1}}(a_i - a_{i-1}) \quad (3.19)$$

for each i = 0,1, 2, …, n-1.

This system involves, as unknowns, only $\{c_i\}_{i=0}^{n}$, since the values of $\{h_i\}_{i=0}^{n-1}$ and $\{a_i\}_{i=0}^{n}$ are given by the spacing of the nodes $\{x_i\}_{i=0}^{n}$ and the values of f at the nodes.

Note that once the values of $\{c_i\}_{i=0}^{n}$ are known it is a simple matter to find the remainder of the constants $\{b_i\}_{i=0}^{n-1}$ from eq.(3.18), and $\{d_i\}_{i=0}^{n-1}$ from eq. (3.15) and to construct the cubic polynomials $\{S_i\}_{i=0}^{n-1}$.

Equation (3.19) constitute (n-1) equations in (n+1) unknowns, $c_0, c_1, \ldots, c_n$.

Clearly two further relations are required in order that a unique interpolating spline may be found. These conditions are end conditions (boundary conditions) stated in (d).

Example.4.

Fit a natural cubic spline to the following data

x	1	2	3
y	-8	-1	18

and compute i) y(1.5)

ii) y′(1)

Solution:

Here n =2 and by (d), $S''(x_0) = S''(1) = S''(x_2) = S''(3) = 0$ (Free boundary condition).

$$S_i''(x) = 2c_i + 6d_i(x-x_i),$$

$$\therefore\ S_i''(x_i) = 2c_i\,.$$

Hence $S''(x_0) = S_0''(x_0) = 2c_0 = 0$, implies $c_0 = 0$. And

$S''(x_2) = S_2''(x_2) = 2c_2 = 0$, implies $c_2 = 0$.

So it remains to find c_1. To do so, first:

$$f(x_i) = S(x_i) = S_i(x_i) = a_i,\ i = 0, 1, 2.$$

$\therefore$ Hence $a_0 = f(x_0) = f(1) = -8$,

$a_1 = f(x_1) = f(2) = -1$,

$a_2 = f(x_2) = f(3) = 18$.

Using eq.(3.19), and using $\{h_i\}_{i=0}^{n-1}$ = spacing of the nodes, we have:

$4c_1 = 3(19)-3(7)$. This implies $c_1= 9$.

Now using eq. (3.18) $b_i = \frac{1}{h_i}(a_{i+1} - a_i) - \frac{h_i}{3}(2c_i + c_{i+1})$, we have

$b_0 = 1(7)-1/3(9) = 4$

$b_1 = 1(18+1)-1/3(18) = 13$

And from eq.(3.15), we have: $d_i = \frac{c_{i+1} - c_i}{3h_i}$

$d_0 = (c_1-c_0)/3 =3$

$d_1 = (c_2-c_1)/3 = -3$

For the interval $1 \le x \le 2$, the cubic spline obtained is given by:

$S_0(x) = a_0 + b_0(x-x_0) + c_0(x-x_0)^2 + d_0(x-x_0)^3$.

$= -8 + 4(x-1) +3(x-1)^3$

$= 3(x-1)^3 + 4x-12$.

And therefore; $y(1.5) \approx S_0(1.5) = S(1.5) = (-45)/8$.

$S'_0(1) = S'(1) = b_0 + 2c_0(1-x_0) + 3d_0(1-x_0)^2$

$= b_0 = 4$.

Or $S'_0(x) = 9(x-1)^2 +4$

Hence $S'_0(1) = 4$.

Note: The cubic spline has produced a better approximation when the interval is halved.

3.3.2. **B- Splines**

B- splines can be of any degree, but this subsection will concentrate on the discussion of cubic B-splines only, because in computer graphics and other applications, the B-splines of degree 2 or 3 are generally found to be sufficient.

The cubic B-spline resembles the ordinary cubic splines of the previous subsection in that a separate cubic is derived for each pair of points in the set (interval). However, the B-spline need not pass through any or all of the data points that are used in its definition. Further, the B-spline curves are non global.

Specifically, a cubic B-spline (or a B-spline of order 4), denoted by $B_{4i}(x)$, is a cubic spline with knots k_{i-4}, k_{i-3}, k_{i-2}, k_{i-1}, and k_i, which is zero everywhere except in the range $k_{i-4} < x < k_i$. In such a case, the B-spline $B_{4i}(x)$ is said to have support $[k_{i-4}, k_i]$.

The B-splines may be defined in several ways. A useful representation is that based on divided differences and this will be given in the next section.

Let the set of data points be (x_i, y_i), $i = 0, 1, 2, \ldots, n$ and $a \le x \le b$. Let $S(x)$ be the cubic spline with knots $k_1, k_2, \ldots, k_p$ where $a < k_1 < k_2 < \ldots < k_p < b$. Then the cubic spline $B_{4,5}(x)$ with knots k_1, k_2, k_3, k_4, and k_5 must satisfy the following properties

i) On each interval, the B-spline must be a polynomial of degree 3 or less,

ii) The B-spline and its first two derivatives must be continuous over the entire curves,

iii) $B_{4,5}(x) > 0$ inside $[k_1, k_5]$, i.e., the B-spline is nonzero only over four successive intervals,

iv) $B_{4,5}(x)$ is identically zero outside $[k_1, k_5]$, and

v) For each knot value, the sum of all the B-splines in the given range is equal to 1.

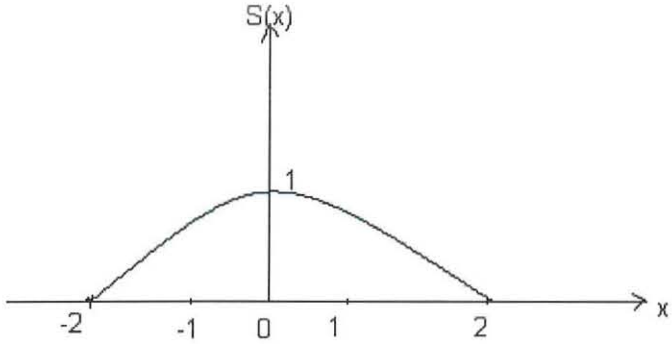

Fig. 3.2. The graph of a B-spline of degree 3 with knots –2, -1, 0, 1, 2.

In this figure, S(x) has the following properties

i) $S(-2) = S(2) = 0$ and $S(0) = 1$

ii) $S'(-2) = S'(2) = 0$ and

iii) $S''(-2) = S''(2) = 0$ (3.20)

Suppose now we have p knots, $k_1, k_2, \ldots, k_p$. To compute the cubic B-splines at k_1 and k_p, we require eight additional knots. To obtain the full set of B-splines, we then introduce eight additional knots, k_{-3}, k_{-2}, k_{-1}, k_0, k_{p+1}, k_{p+2}, k_{p+3} and k_{p+4}.

These are chosen such that:

$$k_{-3} < k_{-2} < k_{-1} < k_0 = a$$

and $$b = k_{p+1} < k_{p+2} < k_{p+3} < k_{p+4} \tag{3.21}$$

where [a, b] is the given range such that $a < k_1$ and $k_p < b$. We have now p+4 cubic B-splines in the range $a \le x \le b$ and then the cubic spline S(x) can be represented as a linear combination of the (p+4) cubic B-splines in the unique form.

$$S(x) = \sum_{i=1}^{p+4} \alpha_i B_{4,i}(x) \tag{3.22}$$

a) <u>Representations of B-splines</u>

To define the cubic B-spline at $x = k_i$, we first consider the five knots k_{i-4}, k_{i-3}, k_{i-2}, k_{i-1} and k_i, where $a < k_{i-4}$ and $k_i < b$. We also define the function

$$P_+^3 = \begin{cases} P^3, & when\ P \ge 0 \\ 0, & when\ P \le 0 \end{cases} \tag{3.23}$$

Then a unique representation of the cubic B-spline with knots $k_{i-4}, \ldots, k_i$ is given by

$$S(x) = B_{4,i}(x) = \sum_{j=0}^{3} \alpha_j x^j + \sum_{m=i-4}^{i} \beta_m (x - k_m)_+^3 \tag{3.24}$$

Another representation of the B-spline, a traditional one, is through divided differences. The divided difference of fourth order of the function $(k_p - x)_+^3$ with respect to the knots k_{i-4}, k_{i-3}, k_{i-2}, k_{i-1} and k_i as arguments is denoted by $[k_{i-4}, k_{i-3}, k_{i-2}, k_{i-1}, k_i]$. We then have:

$$B_{4,i}(x) = [k_{i-4}, k_{i-3}, k_{i-2}, k_{i-1}, k_i]$$

$$= \frac{(k_{i-4} - x)_+^3}{(k_{i-4} - k_{i-3})(k_{i-4} - k_{i-2})(k_{i-4} - k_{i-1})(k_{i-4} - k_i)} +$$

$$= \frac{(k_{i-3} - x)_+^3}{(k_{i-3} - k_{i-4})(k_{i-3} - k_{i-2})(k_{i-3} - k_{i-1})(k_{i-3} - k_i)} +$$

....

$$= \frac{(k_i - x)_+^3}{(k_i - k_{i-4})(k_{i-4} - k_{i-3})(k_{i-4} - k_{i-2})(k_{i-4} - k_{i-1})} \tag{3.25}$$

Setting $\pi_{4,i}(x) = (x - k_{i-4})(x - k_{i-3})(x - k_{i-2})(x - k_{i-1})(x - k_i)$, (3.26)

equation (3.25) can be expressed in the more compact form

$$B_{4,i}(x) = \sum_{m=i-4}^{i} \frac{(k_m - x)_+^3}{\pi'_{4,i}(k_m)} \tag{3.27}$$

More generally, a B-spline of order n (degree n-1) is defined by

$$B_{n,i}(x) = [k_{i-n}, k_{i-n+1}, \ldots, k_i]$$

$$= \sum_{m=i-n}^{i} \frac{(k_m - x)_+^{n-1}}{\pi'_{n,i}(k_m)} \tag{3.28}$$

where $\pi_{n,i}(x) = (x - k_{i-n})(x - k_{i-n+1})\ldots(x - k_i)$. (3.29)

Recalling that

$$[k_{i-4}, k_{i-3}, k_{i-2}, k_{i-1}, k_i] = \frac{[k_{i-3}, k_{i-2}, k_{i-1}, k_i] - [k_{i-4}, k_{i-3}, k_{i-2}, k_{i-1}]}{k_i - k_{i-4}} \tag{3.30}$$

We obtain the relation

$$B_{4,i}(x) = \frac{B_{3,i}(x) - B_{3,i-1}(x)}{k_i - k_{i-4}} \tag{3.31}$$

Which is a recurrence relation. Similarly, for B-splines of order n, we obtain the relation

$$B_{n,i}(x) = \frac{B_{n-1,i}(x) - B_{n-1,i-1}(x)}{k_i - k_{i-n}} \tag{3.32}$$

for a recursive computation of the B-splines $B_{n,i}(x)$. Unfortunately, computational algorithms based on formula (3.32) have been found to be numerically unstable even for simple examples.

Example.5.

Using the relation (3.24), determine the cubic B-spline S(x) with support [0, 4] on the knots 0, 1, 2, 3, 4. Show further that such a representation will be unique if S(1) is specified.

Solution

Since S(x) is a cubic B-spline over [0, 4], we have

i) $S(0) = S'(0) = S''(0) = S(4) = S'(4) = S''(4) = 0$

Also, ii) $S'(2) = 0$,

iii) $S(1) = S(3)$ by symmetry.

On [0, 1], let S(x) be given by

iv) $S(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3$

Since $S(0) = 0$, we obtain $\alpha_0 = 0$.

Also $S'(0) = S''(0) = 0$ give $\alpha_1 = \alpha_2 = 0$

Accordingly, (iv) becomes

v) $S(x) = \alpha_3 x^3$ which is the cubic B-spline on [0, 1] satisfying the conditions $S'(0) = 0 = S''(0)$.

For definiteness, let

vi) $S(1) = S(3) = \beta_0$. Then (v) gives

vii) $\beta_0 = S(1) = \alpha_3$ so that $\alpha_3 = \beta_0$ and (v) is written as

viii) $S(x) = \beta_0 x^3$

Let the cubic B-spline on [0, 2] be written as

ix) $S(x) = \beta_0 x^3 + \beta_1 (x-1)_+^3$ where β_1 is to be determined.

Using the condition $S'(2) = 0$, we obtain

$$0 = 12\beta_0 + 3\beta_1 \text{ so that}$$

x) $\beta_1 = -4\beta_0$. Hence (ix) becomes

xi) $S(x) = \beta_0 x^3 - 4\beta_0 (x-1)_+^3$ which represents the cubic B-spline valid in the interval [0, 2].

On the interval [0, 3] let the cubic B-spline be written as

xii) $S(x) = \beta_0 x^3 - 4\beta_0 (x-1)_+^3 + \beta_1 (x-2)_+^3$

Since $S(3) = \beta_0$, we obtain

$$\beta_0 = \beta_0 (27) - 4\beta_0 (8) + \beta_1 \text{ so that } \beta_1 = 6\beta_0\text{. Then (xii) becomes}$$

xiii) $S(x) = \beta_0 [x^3 - 4(x-1)_+^3 + 6(x-2)_+^3]$

Finally, on the interval [0, 4], let the cubic B-spline be represented by

xiv) $S(x) = \beta_0 [x^3 - 4(x-1)_+^3 + 6(x-2)_+^3] + \beta_2 (x-3)_+^3$

Since $S(4) = 0$, we obtain

$$0 = S(4) = \beta_0 [64 - 108 + 48] + \beta_2 \text{ so that } \beta_2 = -4\beta_0.$$

Hence, the cubic B-spline with support [0, 4] may be written as

xv) $S(x) = \beta_0 [x^3 - 4(x-1)_+^3 + 6(x-2)_+^3 + -4(x-3)_+^3$

If the value $\beta_0 = S(1)$ is specified, then the representation given by (xv) is unique.

Further, it is easily verified that $S'(4) = S''(4) = 0$.

b) The Cox-de Boor Recurrence Formula

Unlike to formula (3.32), algorithms based on the Cox-de Boor Recurrence relation have been found to be numerically stable and efficient.

The Cox-de Boor Recurrence formula for calculating B-splines of order n is given by

$$B_{n,i}(x) = \frac{(x - k_{i-n})B_{n-1,i-1}(x) + (k_i - x)B_{n-1,i}(x)}{k_i - k_{i-n}} \tag{3.33}$$

and holds for all values of x.

Proof

The method of proof given below is that essentially due to Cox [1972]. For the sake of clarity, we prove the formula for n = 4 and the general result can be deduced in an analogous manner. We have

$$B_{3,i}(x) = \sum_{m=i-3}^{i} \frac{(k_m - x)_+^2}{\pi'_{3,i}(k_m)} \tag{3.34}$$

and $$B_{3,i-1}(x) = \sum_{m=i-4}^{i-1} \frac{(k_m - x)_+^2}{\pi'_{3,i-1}(k_m)} \tag{3.35}$$

Substituting (3.34) and (3.35) in (3.33) with n = 4, we obtain

$$B_{4,i}(x) = \left(\frac{x - k_{i-4}}{\mathbf{k_i} - k_{i-4}}\right) \sum_{m=i-4}^{i-1} \frac{(k_m - x)_+^2}{\pi'_{3,i-1}(k_m)} + \left(\frac{k_i - x}{\mathbf{k_i} - k_{i-4}}\right) \sum_{m=i-3}^{i} \frac{(k_m - x)_+^2}{\pi'_{3,i}(k_m)} \tag{3.36}$$

But $$\sum_{m=i-4}^{i-1} \frac{(k_m - x)_+^2}{\pi'_{3,i-1}(k_m)} = \frac{(k_{i-4} - x)_+^2}{\pi'_{3,i-1}(k_{i-4})} + \sum_{m=i-3}^{i-1} \frac{(k_m - x)_+^2}{\pi'_{3,i-1}(k_m)}$$

and $$\sum_{m=i-3}^{i} \frac{(k_m - x)_+^2}{\pi'_{3,i}(k_m)} = \sum_{m=i-3}^{i-1} \frac{(k_m - x)_+^2}{\pi'_{3,i}(k_m)} + \frac{(k_i - x)_+^2}{\pi'_{3,i}(k_i)}$$

Hence (3.36) simplifies to

$$B_{4,i}(x) = \left(\frac{x - k_{i-4}}{\mathbf{k_i} - k_{i-4}}\right)\left[\frac{(k_{i-4} - x)_+^2}{\pi'_{3,i-1}(k_{i-4})} + \sum_{m=i-3}^{i-1} \frac{(k_m - x)_+^2}{\pi'_{3,i-1}(k_m)}\right]$$

$$+ \left(\frac{k_i - x}{\mathbf{k_i} - k_{i-4}}\right)\left[\frac{(k_i - x)_+^2}{\pi'_{3,i}(k_i)} + \sum_{m=i-3}^{i-1} \frac{(k_m - x)_+^2}{\pi'_{3,i}(k_m)}\right]$$

$$= \left(\frac{x - k_{i-4}}{\mathbf{k_i} - k_{i-4}}\right) \frac{(k_{i-4} - x)_+^2}{\pi'_{3,i-1}(k_{i-4})} + \left(\frac{k_i - x}{\mathbf{k_i} - k_{i-4}}\right) \frac{(k_i - x)_+^2}{\pi'_{3,i}(k_i)}$$

$$= \sum_{m=i-3}^{i-1} [\left(\frac{x - k_{i-4}}{\mathbf{k_i} - k_{i-4}}\right) \frac{(k_m - x)_+^2}{\pi'_{3,i-1}(k_m)} + \left(\frac{k_i - x}{\mathbf{k_i} - k_{i-4}}\right) \frac{(k_m - x)_+^2}{\pi'_{3,i}(k_m)}] \tag{3.37}$$

From (3.29), we have

$$\pi_{3,i}(x) = (x - k_{i-3})(x - k_{i-2})(x - k_{i-1})(x - k_i)$$

and $$\pi_{3,i}(x) = (x - k_{i-3})(x - k_{i-2})(x - k_{i-1})(x - k_i) \tag{3.38}$$

Hence $$\pi'_{3,i-1}(k_{i-4}) = (k_{i-4} - k_{i-3})(k_{i-4} - k_{i-2})(k_{i-4} - k_{i-1}) \tag{3.39}$$

Also from (3.26), we obtain

$$\pi'_{4,i}(k_{i-4}) = (k_{i-4} - k_{i-3})(k_{i-4} - k_{i-2})(k_{i-4} - k_{i-1})(k_{i-4} - k_i) \quad (3.40)$$

From (3.39) and (3.40), it follows that

$$(k_{i-4} - k_i)\pi'_{3,i-1}(k_{i-4}) = \pi'_{4,i}(k_{i-4}) \quad (3.41)$$

Again, from (3.38), we obtain

$$\pi'_{3,i}(k_i) = (k_i - k_{i-3})(k_i - k_{i-2})(k_i - k_{i-1}) \quad (3.42)$$

and from (3.26)

$$\pi'_{4,i}(k_i) = (k_i - k_{i-4})(k_i - k_{i-3})(k_i - k_{i-2})(k_i - k_{i-1}) \quad (3.43)$$

From (3.42) and (3.43), it follows that

$$(k_i - k_{i-4})\pi'_{3,i-1}(k_i) = \pi'_{4,i}(k_i) \quad (3.44)$$

Equation (3.37) can now be simplified as

$$B_{4,i}(x) = \frac{(x - k_{i-4})(k_{i-4} - x)_+^2}{-\pi'_{4,i}(k_{i-4})} + \frac{(k_i - x)(k_i - x)_+^2}{\pi'_{4,i}(k_i)}$$

$$= + \sum_{m=i-3}^{i-1} \frac{(k_m - x)_+^2}{\mathbf{k_i} - k_{i-4}}\left[\frac{x - k_{i-4}}{\pi'_{3,i-1}(k_m)} + \frac{k_i - x}{\pi'_{3,i}(k_m)}\right]$$

$$= \frac{(k_{i-4} - x)_+^3}{\pi'_{4,i}(k_{i-4})} + \frac{(k_i - x)_+^3}{\pi'_{4,i}(k_i)}$$

$$+ \sum_{m=i-3}^{i-1} \frac{(k_m - x)_+^2}{\mathbf{k_i} - k_{i-4}}\left[\frac{x - k_{i-4}}{\pi'_{3,i-1}(k_m)} + \frac{k_i - x}{\pi'_{3,i}(k_m)}\right]$$

$$= \frac{(k_{i-4} - x)_+^3}{\pi'_{4,i}(k_{i-4})} + \frac{(k_i - x)_+^3}{\pi'_{4,i}(k_i)}$$

$$+ \sum_{m=i-3}^{i-1} \frac{(k_m - x)_+^2}{\mathbf{k_i} - k_{i-4}}\left[\frac{(x - k_{i-4})(k_m - k_i)}{\pi'_{4,i}(k_m)} + \frac{(k_i - x)(k_m - k_{i-4})}{\pi'_{4,i}(k_m)}\right],$$

on substituting equations of the type (3.41) and (3.44).

$$= \sum_{m=i-4}^{i} \frac{(k_m - x)_+^3}{\pi'_{4,i}(k_m)}, \text{ on simplification.}$$

This completes the proof of (3.33) for n = 4.

c) Computation of B-Splines

To compute the cubic B-spline, say B_{4i} based on the knots k_{i-4}, k_{i-3}, k_{i-2}, k_{i-1} and k_i we need to compute the elements in the following array

$B_{1,i-4}$			
	$B_{2,i-3}$		
$B_{1,i-3}$		$B_{3,i-2}$	
	$B_{2,i-2}$		$B_{4,i-1}$
$B_{1,i-2}$		$B_{3,i-1}$	
	$B_{2,i-1}$		$B_{4,i}$
$B_{1,i-1}$		$B_{3,i}$	
	$B_{2,i}$		
$B_{1,i}$			

In computing the above array, advantage can be taken of the fact that some of the elements are zero. Thus, for example, if $k_{i-4} \leq x < k_{i-3}$, then using the relation

$$B_{1,i} = \begin{cases} \dfrac{1}{x_i - x_{i-1}}, & \text{if } x_{i-1} \leq x < x_i \\ 0, & \text{otherwise} \end{cases}$$

The above array takes the form

0			
	$B_{2,i-3}$		
$B_{1,i-3}$		$B_{3,i-2}$	
	$B_{2,i-2}$		$B_{4,i-1}$
0		$B_{3,i-1}$	
	0		$B_{4,i}$
0		0	
	0		
0			

The numerical computation of the B-splines will now become more simpler.

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