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On Commutativity of Prime Near-Rings By Generalized Derivations

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A thesis submitted in partial fulfillment of the requirement of the degree of Master of Science in Mathematics to the Department of Mathematics, College of Natural and Computational Sciences, Addis Ababa University

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Abstract

Given a nonempty set N together with two binary operations, called addition "+" and multiplication ".", if $(N,+)$ is a group, $(N,.)$ is a semigroup and multiplication is left or right distributive over addition, the algebraic structure $(N,+,.)$ is called a near-ring. If $(N,+,.)$ is a near ring, a function $D: N \longrightarrow N$ is said to be a derivation on N if $D(xy) = D(x)y + xD(y)$ for all $x, y \in N$ and an additive mapping $F: N \longrightarrow N$ satisfying $F(xy) = F(x)y + xD(y)$ for all $x, y \in N$, is called generalized derivation on N associated with the derivation D. The aim of this thesis is to study the commutativity of a near-ring using properties of generalized derivations on the given near ring. If F is a generalized derivation on a near-ring N associated with a derivation D and if either $F[x,y] + [x,y] = 0$ for all $x, y \in N$, $F[x,y] - [x,y] = 0$ for all $x, y \in N$, $F(xoy) - (xoy) = 0$ for all $x, y \in N$ or $F(xoy) + (xoy) = 0$ for all $x, y \in N$, then it is proved that N is commutative, where $[x,y] = xy - yx$ and $xoy = xy + yx$ for all $x, y \in N$. In this thesis, the commutativity of 3-torsion free prime near-ring N involving generalized derivation F associated with non-zero idempotent derivation D on N, satisfying

$$F^2[x,y] - [x,y] = 0 \text{ for all } x, y \in N \text{ and } F^2(xoy) - (xoy) = 0 \text{ for all } x, y \in N$$

and commutativity of 5-torsion free prime near ring N involving generalized derivation F associated with non-zero idempotent derivation D on N, satisfying

$$F^2[x,y] + [x,y] = 0 \text{ for all } x, y \in N \text{ and } F^2(xoy) + (xoy) = 0 \text{ for all } x, y \in N$$

are proved. These results can be used to further study the commutativity of prime near-rings using generalized derivations defined on a near-ring.

Introduction

In recent years, interest has arisen in algebraic systems with two binary operations called addition and multiplication, satisfying all the ring axioms except possibly one of the distributive laws and the commutativity of addition. Such systems are called near-rings. The first step towards near-rings was the axiomatic research done by Dikson in 1905 [6]. He showed that there do exist fields with only one distributive law (called near-fields).
Given a nonempty set N together with two binary operations, called addition "+" and multiplication ".", if $(N, +)$ is a group, $(N, .)$ is a semigroup and multiplication is left or right distributive over addition, the algebraic structure $(N, +, .)$ is called a near-ring. A near-ring N is called

(i) a prime near-ring if $xNy = \{0\}$ for $x, y \in N$,then $x = 0$ or $y = 0$;

(ii) a zero symmetric right near-ring if $x0 = 0$ for all $x \in N$ and

(iii) an $n-$torsion free if $nt = 0$ for $t \in N$, then $t = 0$ (for a positive integer n).

The concept of generalized derivation on rings was introduced by B.Hvala in 1998 [5]. Given a near-ring $(N, +, .)$, an additive mapping $D : N \longrightarrow N$ is called a derivation of N if

$$D(xy) = xD(y) + D(x)y \text{ holds for all } x, y \in N.$$

An additive mapping $F : N \longrightarrow N$ is said to be a right generalized derivation associated with a derivation D if

$$F(xy) = F(x)y + xD(y) \text{ for all } x, y \in N.$$

and is said to be a left generalized derivation associated with D if

$$F(xy) = xF(y) + D(x)y \text{ for all } x, y \in N.$$

A mapping $F : N \longrightarrow N$ is said to be a generalized derivation associated with a derivation D if it is a right as well as a left generalized derivation associated with D. Oznur Golbasi in 2006,**[8]** worked on generalized derivations on prime rings and showed the commutativity 2-torsion free prime near-ring using generalized derivations. In 2013, Asma Ali, Howard E. Bell and Phool Miyan **[2]** worked on the generalized Derivations on prime near-rings and Mohd Rais Khan and Mohd Mueenul Hasnain in 2013 **[11]** worked on the commutativity of prime near-rings and in their results they have proved that, if F is a generalized derivation on a near-ring N associated with a derivation D and if either $F[x, y]+[x, y] = 0$ for all $x, y \in N$, $F[x, y] - [x, y] = 0$ for all $x, y \in N$, $F(xoy) - (xoy) = 0$ for all $x, y \in N$ or $F(xoy) + (xoy) = 0$ for all $x, y \in N$, then N is commutative, where $[x, y] = xy - yx$ and $xoy = xy + yx$ for all $x, y \in N$.

The commutativity of prime near-rings N if F is generalized derivation associated with non-zero idempotent derivation D on N, satisfying

$$F^n[x,y] - [x,y] = 0 \text{ for all } x,y \in N \text{ and } F^n(xoy) - (xoy) = 0 \text{ for all } x,y \in N$$

for a positive integer $n > 1$ and the commutativity of prime near-ring N involving generalized derivation F associated with non-zero idempotent derivation D on N, satisfying

$$F^n[x,y] + [x,y] = 0 \text{ for all } x,y \in N \text{ and } F^n(xoy) + (xoy) = 0 \text{ for all } x,y \in N$$

are open problems in the literature. In this thesis we want to work on the commutativity of prime near-rings using generalized derivations. In particular, our work will focus on the commutativity of a 3-torsion free prime near-ring N involving generalized derivation F associated with non-zero idempotent derivation D on N satisfying

$$F^2[x,y] - [x,y] = 0 \text{ for all } x,y \in N \text{ and } F^2(xoy) - (xoy) = 0 \text{ for all } x,y \in N$$

and commutativity of 5-torsion free prime near ring N involving generalized derivation F associated with non-zero idempotent derivation D on N, satisfying

$$F^2[x,y] + [x,y] = 0 \text{ for all } x,y \in N \text{ and } F^2(xoy) + (xoy) = 0 \text{ for all } x,y \in N.$$

This thesis consists of three chapters. The first chapter is on preliminaries, the second chapter is on near-rings and chapter three is about the commutativity of prime near-rings and the main results of the thesis.

Chapter 1

Preliminaries

In this chapter basic information that will give a platform for our main discussions are given. In the first section, derivation on rings are discussed and the second section is about generalized derivations on rings. Basic knowledge on rings is assumed to be well understood by the reader.

1.1 Derivations on Rings

Definition 1.1.1. *Let $(R, +, .)$ be a ring. An additive mapping $D : R \longrightarrow R$ is said to be a derivation on R if $D(xy) = D(x)y + xD(y)$ for all $x, y \in R$*

The most natural example of a non-trivial derivation on a ring is the usual differentiation on the ring $F[x]$ of polynomials defined over a field F. That is, if $f(x) \in F[x]$ and $D : F[x] \longrightarrow F[x]$ defined by $D(f) = \frac{d(f)}{d(x)}$ is clearly a non trivial derivation on $F[x]$

Example 1.1. *Let $(R, +, .)$ be a ring. For fixed $a \in R$, define $D : R \longrightarrow R$ by*

$$D(x) = [x, a],$$

for all $x \in R$, where $[x, a] = xa - ax$ is a derivation.

Proof. Let $x, y \in R$. Then

$$D(x + y) = [x + y, a] = [x, a] + [y, a] = D(x) + D(y)$$

and

$$D(xy) = [xy, a] = [x, a]y + x[y, a] = D(x)y + xD(y)$$

□

This implies that D is a derivation on R.

Definition 1.1.2. *Let R be a ring. For fixed $a \in R$ define a map $I_a : R \longrightarrow R$ by*

$$I_a(x) = [x, a]$$

for all $x \in R$ is called an inner derivation on R, which denoted by I_a.

The above example shows that every inner derivation on a ring R is a derivation. But the converse is not in general true as we can see the in following example
Let R=$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \right\}$ be ring of 2×2 matrices over $\mathbb{Z}$ (the ring of integers). Define a map $D : R \longrightarrow R$ by

$$D \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -a \\ c & 0 \end{pmatrix}$$

Then it can be easily shown that D is a derivation on R but not an inner derivation on R.

Definition 1.1.3. *Let R be a ring and D be a derivation on R. Then D is called an idempotent derivation if $D^2 = D$.*

Example 1.2. *Let R=$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}_2 \right\}$ Define a map $D : R \longrightarrow R$ by*

$$D \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & a + b + d \\ c & c \end{pmatrix}$$

Then D is an idempotent derivation because $D^2 = D$.

1.2 Generalized Derivations on Rings

In this section generalized derivations on ring are considered. The notions of generalized derivations on rings was first introduced by Hvala [5] and the topics discussed in this section are based on this work.

Definition 1.2.1. *Let $(R, +, .)$ be a ring. An additive mapping $F : R \longrightarrow R$ is called a generalized derivation on R associated with a derivation D if there exists a derivation $D : R \longrightarrow R$ such that $F(xy) = F(x)y + xD(y)$ holds for all $x, y \in R$.*

Example 1.3. *(i) Every derivation is a generalized derivation.*

(ii) An additive mapping $F : R \longrightarrow R$ such that $F(xy) = F(x)y$ for all $x, y \in R$,called left multiplier is a generalized derivation on R.

(iii) Let $(R, +, .)$ be a ring. For fixed $a, b \in R$ define a map $F_{a,b} : R \longrightarrow R$ by $F_{a,b}(x) = ax + xb$ for all $x \in R$ called a generalized inner derivation is a generalized derivation on R.

Example 1.4. *Let R be a commutative ring and*

$$N = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} : a, b, c \in R \right\}.$$

Define a map $F: \ N \longrightarrow N$ by

$$F \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ a+b & 0 \end{pmatrix}.$$

Then it can be easily shown that F is a generalized derivation associated with the derivation D, where D is the derivation on N given by

$$D \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}$$

1.3 Commutativity of Rings

Definition 1.3.1. *Let $(R, +.)$ be a ring. Then R is said to be a commutative ring if $rs = sr$ for all $r, s \in R$.*

Theorem 1.1. *([1], Corollary 3.2). Let $(R, +, .)$ be a prime ring. If R admits a nonzero generalized derivation F associated with non-zero derivation D such that $[F(x), x] = 0$ for all $x \in R$, then R is commutative.*

Proof. See [1]. □

Theorem 1.2. *([1], Lemma 2.4). Let $(R, +.)$ be a prime ring and I be a nonzero left ideal of R. If R admits a nonzero derivation D such that $[x, D(x)]$ is central for all $x \in I$, then R is commutative.*

Definition 1.3.2. *Let $(R, +.)$ be a ring and U be left ideal of R. Then a mapping F is said to be commuting on U, if $[F(x), x] = 0$ for all $x \in U$.*

Definition 1.3.3. *Let $(R, +.)$ be a ring and U be left ideal of R. Then a mapping F is said to be centralizing on U, if $[F(x), x] \in Z(R)$ for all $x \in U$.*

Theorem 1.3. *([12], Theorem 1). Let $(R, +.)$ be a prime ring. Let $D: \ R \longrightarrow R$ be a nonzero derivation and F be a generalized derivation associated with D on a left ideal U of R. If F is commuting on U, then R is commutative.*

Proof. See [12]. □

Theorem 1.4. *([12], Theorem 2). Let $(R, +.)$ be a prime ring and U be a left ideal of R such that $U \cap Z(R) \neq 0$. Let D be a nonzero derivation and F be a generalized derivation associated with D on R such that F is centralizing on U. Then R is commutative.*

Proof. See [12]. □

Chapter 2

Near-rings

2.1 Definitions and Examples of Near-Rings

In this section, in addition to definitions and examples of near-rings, some properties of near-rings are given. Let us start the section by giving the formal definition of a near-ring.

Definition 2.1.1. *A non-empty set N together with two binary operations "+" and ".", called addition and multiplication respectively, is said to be a right (respectively left) near-ring if it satisfies the following axioms:*

1. $(N, +)$ is a group (not necessarily abelian).
2. $(N, .)$ is a semigroup.
3. "." is right (respectively left) distributive over "+", that is,

$$(n_1 + n_2)n_3 = n_1n_3 + n_2n_3 \quad for \quad all \quad n_1, n_2, n_3 \in N.$$

$$(respectively \quad n_1(n_2 + n_3) = n_1n_2 + n_1n_3 \quad for \quad all \quad n_1, n_2, n_3 \in N).$$

An algebraic structure $(N, +, .)$ satisfying all the ring axioms except possibly any one of the distributive laws (either right or left distributive laws of multiplication over addition) and the commutativity of addition is called a near-ring. Thus, by a near-ring, we mean that it is either a right near-ring or a left near-ring.
In our desiccations, by a near-ring we mean right near-ring, unless stated otherwise.

Example 2.1. *Every ring is near-ring. This is natural example of a near-ring. That is, if $(R, +, .)$ is a ring, then it is true that $(R, +)$ is a group, $(R, .)$ is a semigroup and "." is both right and left distributive over "+" and hence $(R, +, .)$ is a near-ring.*

But the converse in the above example is not true as we can see in the following example.

Example 2.2. *Let $(G, +)$ be a group (not necessarily abelian) and $M(G)$ be the set of all mappings of G into itself. Define addition and multiplication on $M(G)$ by* *(i)* $(f+g)(a) = f(a) + g(a)$ *and*

(ii) $(fg)(a) = f(g(a))$ *for all* $a \in G$, $f, g \in M(G)$ *respectively.*

One can easily see that, $(M(G), +, .)$ *is a right near-ring, but not a ring, if, for example,* $(G, +.)$ *is not abelian which is a requirement for a ring.*

This example can also be used to show that right near-rings which are not left near-rings.

Example 2.3. *Let* $(G, +, .)$ *be a non-trivial abelian group. Then the set of all mappings from* G *into itself,* $M(G)$*, is a right near-ring under the usual addition and composition of mappings (see the previous example).*
In this right near-ring, the left distributive law fails to hold. To verify this, let $a, b, c \in G$ *and* $a \neq 0$.
Define $f_a : G \longrightarrow G$ *by* $f_a(t) = a$ *for all* $t \in G$, $f_b : G \longrightarrow G$ *by* $f_b(t) = b$ *for all* $t \in G$ *and* $f_c : G \longrightarrow G$ *by* $f_c(t) = c$ *for all* $t \in G$.
Clearly f_a, f_b *and* f_c *are elements of* $M(G)$. *For* $t \in G$,

$$[f_a \circ (f_b + f_c)](t) = f_a((f_b + f_c)(t) = f_a[f_b(t) + f_c(t)] = f_a(b + c) = a.$$

and

$$[(f_a \circ f_b) + (f_a \circ f_c)](t) = (f_a \circ f_b)(t) + (f_a \circ f_c)(t) = f_a(f_b(t)) + f_a(f_c(t)) = f_a(b) + f_a(c) = a + a = 2a.$$

From our assumption $a \neq 0$ *which implies that* $2a \neq a$. *Thus,*

$$[f_a \circ (f_b + f_c)] \neq (f_a \circ f_b) + (f_a \circ f_c)$$

and hence, multiplication is not left distributive over addition.

Given an abelian group $(R, +)$, if for any two elements $a, b \in R$, multiplication is defined by $ab = 0$, then $(R, +, ,)$ is a ring, which is also a near ring. The multiplication defined here is called trivial multiplication.

Definition 2.1.2. *Let* $(N, +.)$ *be a near ring. An element* $e \in N$ *is said to be*

1. *a left identity element, if* $en = n$, *for all* $n \in N$.
2. *a right identity element, if* $ne = n$, *for all* $n \in N$.
3. *an identity element, if it is both right identity and left identity.*

In any near-ring, if an identity element exists, then is unique and the unique identity element is usually denoted by 1 and we say that the near ring is a near-ring with identity.

Definition 2.1.3. *Let* $(N, +, .)$ *be a near-ring with identity 1. An element* $n \in N$ *is said to be*

1. *right invertible if there exists an element $m \in N$ such that $nm = 1$.*

2. *left invertible if there exists an element $m \in N$ such that $mn = 1$.*

3. *invertible if it is both right and left invertible, that is, if there exists an element $m \in N$ such that $mn = 1 = nm$. In this case the element m is called an inverse of n.*

By the associativity of multiplication, if an element is invertible, then its inverse is unique.

Now, let us define different types of near-rings and study their properties.

Definition 2.1.4. *Let $(N, +, .)$ be a near-ring. Then*

(i) N is said to be prime near-ring, if $nNm = \{0\}$ for $n, m \in N$, then $n = 0$ or $m = 0$, where $nNm = \{ntm : t \in N\}$.

(ii) N is said to be semi-prime near-ring, if $nNn = \{0\}$ for $n \in N$, then $n = 0$, where $nNn = \{ntn : t \in N\}$.

(iii) N is said to be an abelian near-ring, if $n + m = m + n$ for all $n, m \in N$.

(iv) N is said to be $n-$torsion free, if $nt = 0$ for $t \in N$, then $t = 0$ (for some positive integer n).

Recall that In the definition of a near-ring, addition need not be commutative. The following theorem gives a sufficient condition for a near-ring to be abelian.

Theorem 2.1. *Let $(N, +, .)$ be a near-ring with identity. If $n(-1) = -n$ for all $n \in N$, then $(N, +)$ is abelian. (In this notation, $-n$ denotes the additive inverse of n in N.)*

Proof. Let $(N, +, .)$ be a near-ring with identity 1. Suppose $n(-1) = -n$ for all $n \in N$. We want to show that for any $n, m \in N$, $n + m = m + n$.
Let $n, m \in N$. Then $n + m - m - n = 0$ implies $(n + m) + (m(-1) + n(-1)) = 0$ (by assumption.)
By the same reasoning, we have $(n+m)+(m+n)(-1) = 0$ and hence $(n+m)-(m+n) = 0$. This implies $n + m = m + n$ and hence N is abelian near-ring. □

Theorem 2.2. *Let $(N, +, .)$ be a near-ring.*

(a) $0n = 0$, for all $n \in N$.

(b) $-(n + m) = -m - n$, for all $n, m \in N$.

Proof.

(a) Let $n \in N$. Then $0n = (0+0)n = 0n+0n$, since multiplication is right distributive over addition and $0 + 0n = 0n + 0n$, since 0 is the additive identity. Thus, by cancelation law in groups, we get $0n = 0$.

(b) Let $m, n \in N$. Then $n + m + (-m - n) = n + (m - m) - n = n - n = 0$ and hence $-(n + m) = -m - n$ (by the uniqueness of additive inverse in a group).

□

Theorem 2.3. *Let $(N, +, .)$ be a near-ring. The two sets $\{n \in N : n0 = n\}$ and $\{n \in N : \; nm = n, \; for\ all\ m \in N\}$ are equal. That is,*

$$\{n \in N : \; n0 = n\} = \{n \in N : \; nm = n, \; for\ all\ m \in N\}.$$

Proof. Let $a \in \{n \in N : \; n0 = n\}$. Then we want to show that $a \in \{n \in N : \; nm = n, \; \text{for all } m \in N\}$. By assumption $a0 = a$ If $m \in N$, then

$$am = (a0)m = a(0m) = a0 = a.$$

This implies $a \in \{n \in N : \; nm = n, \; \text{for all } m \in N\}$ and hence $\{n \in N : \; n0 = n\}$ is a subset of $\{n \in N : \; nm = n, \text{for all } m \in N\}$.
The reverse inclusion is immediate.
Therefore, we have the equality of the two sets, that is,

$$\{n \in N : \; n0 = n\} = \{n \in N : \; nm = n, \; \text{for all } m \in N\}.$$

□

Definition 2.1.5. *Let $(N, +.)$ be a near-ring.*

(a) The set $N_0 = \{n \in N : \; n0 = 0\}$ is called the zero-symmetric part of N and

(b) the set $N_c = \{n \in N : \; n0 = n\} = \{n \in N : \; nm = n\}$ for all $m \in N$ is called the constant part of N.

Remark 2.1. *Let $(N, +.)$ be a near-ring , $n \in N$ and $n \neq 0$. In general,*

(i) $n0$ is not be equal to 0.

(ii) $n(-m)$ is not be equal to $-nm$.

Proof.

(i) Let G be a nonzero group. Consider $N = M(G)$ the near-ring of all functions from G into G under addition and composition of function (as multiplication). Let $a \in G$ and $a \neq 0$. Define $f_a : G \longrightarrow G$ by $f_a(t) = a$ for all $t \in G$.

$$(f_a 0)(t) = f_a(0(t) = f_a(0) = a \neq 0.$$

(where 0 in this case is the zero function on G.)

(ii) Let (G,+) be a group containing a non-zero element a with $a + a \neq 0$. $N = M(G)$ the near-ring of all functions from G into G under addition and composition of function (as multiplication). Then for $t \in G$,

$$(f_a(-f_a))(t) = f_a(-f_a(t)) = f_a(-a) = a.$$

On the other hand,

$$-(f_a f_a)(t) = -f_a(f_a(t)) = -f_a(a) = -a.$$

This implies, $f_a(-f_a) \neq -(f_a f_a)$ (since $a \neq -a$ by assumption).

□

Definition 2.1.6. *Let $(N, +, .)$ be a near-ring.*

1. *An element $n \in N$ is said to be left cancelable (respectively right cancelable) if for $a, b \in N$, $na = nb$ implies $a = b$ (respectively $an = bn$ implies $a = b$).*
2. *A nonzero $n \in N$ is said to be a right zero divisor (respectively left zero divisor), if there exists a nonzero element $a \in N$ such that $an = 0$ (respectively $na = 0$).*

Theorem 2.4. *Let $(N, +, .)$ be a near ring. An element $n \in N$ is right cancelable if and only if it is not a right zero divisor.*

Proof. ($\Rightarrow$) Let $n \in N$ be a right cancelable. Suppose $mn = 0$ for $n \in N$. Then, since N is near-ring, we have $0n = 0$ This implies $mn = 0n$ and hence $m = 0$ (using right cancelation law).
Therefore n is not right zero divisor.
($\Leftarrow$) Suppose that $n \in N$ is not a right zero divisor. Let $a, b \in N$ and $an = bn$. This implies $an - bn = 0$ and hence $(a - b)n = 0$ (since multiplication is right distributive over addition). Since n is not a right zero divisor, we obtain that $a - b = 0$. This implies that $a = b$ and hence n is right cancelable. □

Theorem 2.5. *If $n \in N_0$ is left cancelable, then it is not a left zero divisor.*

Proof. Let $n \in N_0$ be left cancelable. Suppose $nm = 0$ for some $m \in N$. Since N is zero-symmetric, we have $n0 = 0$. This implies $nm = 0 = n0$ and hence $m = 0$ (since n is left cancelable by assumption) which shows that n is not left zero divisor. □

Definition 2.1.7. *Let $(N, +, .)$ be a near-ring. An element $a \in N$ is said to be a distributive element, if $a(x + y) = ax + ay$ for all $x, y \in N$. The set of all distributive elements of a near-ring N is denoted by*

$$N_d = \{a \in N : a(x + y) = ax + ay, \text{ for all } x, y \in N\}.$$

Remark 2.2.

(i) If a near ring N has an identity 1, then $1 \in N_d$.

(ii) If N is a ring, then $N = N_d$.

Definition 2.1.8. *Let* $(N, +, .)$ *be a ring. An element* $e \in N$ *such that* $e^2 = e$ *is called an idempotent element in N.*

Theorem 2.6. *Let* $(N, +, .)$ *be a near-ring. If an element* $e \in N$ *is idempotent, then for any* $n \in N$ *there corresponds exactly one* $n_0 \in \{x \in N : xe = 0\}$ *and there corresponds exactly one* $n_1 \in Ne = \{xe : x \in N\}$ *such that* $n = n_0 + n_1$.

Proof. First, let us proof the existence. Let $e \in N$ be an idempotent element and $n \in N$. Then $n = (n - ne) + ne$. Consider $(n - ne)e = ne - nee = ne - ne = 0$. This implies $(n - ne) \in \{x \in N : xe = 0\}$ and also $ne \in Ne$. This proves the exitance.
Now let us prove the uniqueness. Let $n \in N$ and $n = n_0 + n_1 = n_0^{'} + n_1^{'}$, for some $n_0, n_0^{'} \in \{x \in N : xe = 0\}$ and $n_1, n_1^{'} \in Ne$. Then

$$ne = (n_0 + n_1)e = (n_0^{'} + n_1^{'})e$$

which implies

$$n_0e + n_1e = n_0^{'}e + n_1^{'}e.$$

This implies $n_1e = n_1^{'}e$ (because $n_0e = n_0^{'}e = 0$).
For $n_1, n_1^{'} \in Ne$, there exist $x, y \in N$ such that $n_1 = xe$ and $n_1^{'} = ye$. Then

$$n_1e = (xe)e = x(ee) = xe = n_1$$

(since e is idempotent) and

$$n_1^{'}e = (ye)e = y(ee) = ye = n_1^{'}.$$

This implies, $n_1 = n_1^{'}$ and from the relation $n_0 + n_1 = n_0^{'} + n_1^{'}$, we get $n_0 = n_0^{'}$.
This proves the uniqueness.

□

Corollary 2.1. *Let (N,+,.) be a near-ring, then for any* $n \in N$*, there exist unique elements* $n_0 \in N_0$ *and* $n_c \in N_c$ *such that* $n = n_0 + n_c$*. That is,* $(N, +) = (N_0, +) + (N_c, +)$ *and* $N_0 \cap N_c = \{0\}$.

Proof. Let $n \in N$. Then by the above theorem, there exist unique $n_0 \in \{x \in N | x0 = 0\}$ and $n_1 \in N0$ such that $n = n_0 + n_1$.
Now, we to want show that $N0 = N_c$.
Let $n \in N_c$. Then $n0 = n$ which implies that $n = n0 \in N0$. Therefore,

$$N_c \subseteq N0. \tag{2.1}$$

To prove the other inclusion, let $n \in N0$. Then there exists some $n_1 \in N$ such that $n = n_1 0$. Now

$$n0 = (n_1 0)0 = n_1(00) = n_1 0 = n.$$

This implies $n \in N_c$ and hence

$$N_c \subseteq N0. \tag{2.2}$$

From (2.1)and (2.2), we have $N_c = N0$ and

$$(N_0, +) + (N_c, +) = (N, +).$$

It remains to show that $N_0 \cap N_c = \{0\}$. Since $00 = 0$, then $\{0\} \subseteq N_0$ and $\{0\} \subseteq N_0$. This implies

$$\{0\} \subseteq N_0 \cap N_c. \tag{2.3}$$

Let $n \in N_0 \cap N_c$. Then $n \in N_0$ and $n \in N_c$, which implies that $n0 = 0$ and $n0 = n$ and hence $n = 0$. This implies

$$N_0 \cap N_c \subseteq \{0\}. \tag{2.4}$$

From(2.3) and (2.4)

$$N_0 \cap N_c = \{0\}.$$

□

Definition 2.1.9. *Let $(N, +.)$ be a near-ring.*

(i) N is said to be a commutative near-ring if $nm = mn$ for all $n, m \in N$.

(ii) N is said to be a distributive near-ring if $N = N_d$.

(iii) If all nonzero elements of N are left (respectively right) cancelable, then we say that N fulfills the left (respectively right) cancellation law.

(iv) N is said to be an integral near-ring if N has no zero divisors.

2.2 Substructures of Near Rings.

Almost every algebraic system has its own subsystem(s). Subgroups are subsystems in the theory of groups; subfields are subsystems of fields; subspaces are subsystems of vector spaces; sub-rings and ideals are the two different types of subsystems of rings and submodules are the subsystems of modules. In a similar way, we study two different types of substructures of near rings, namely, sub-near-rings and ideals. In this section, we present the results related to these two subsystems.

Definition 2.2.1. *Let $(N, +.)$ be a near-ring.*

(i) A subgroup M of a near-ring $(N, +)$ is called a sub-near-ring of N if $MM \subseteq M$. If M is a sub-near-ring of N, then this relation is denoted by $M \leq N$.

(ii) A sub-near-ring M of N is called left invariant (respectively right invariant,) if $MN \subseteq M$, (respectively $NM \subseteq M$). If N is both left invariant and right invariant, then we say that it is invariant.

Given a near-ring $(N, +, .)$, the trivial sub-near-rings of N are $\{0\}$ and N. The other common examples of sub-near-rings are the zero-symmetric part of N, N_0, and the constant part of N, N_c and we give proofs for these two sub-near-rings in the following theorem.

Theorem 2.7. *Let $(N, +, .)$ be a near-ring. Then the zero symmetric part of N, N_0, and the constant part of N, N_c, are sub-near-rings of N.*

Proof. (i) First, let us show that N_0 is a sub-near-rings of N.
Let $x, y \in N_0$ Then $x0 = 0$ and $y0 = 0$. Now, using the right distributivity of multiplication on addition,

$$(x - y)0 = x0 - y0 = 0 - 0 = 0.$$

This implies, $x - y \in N_0$ and hence $(N_0, +)$ is a subgroup of $(N, +)$.
Let $n, m \in N_0$. Using the associativity of multiplication and definition of N_0,, we have $(nm)0 = n(m0) = n0 = 0$. This implies, $nm \in N_0$ and so $N_0N_0 \subseteq N_0$. Hence, N_0 is a sub-near ring of N.

(ii) Next, let us show that N_c is a sub-near-ring of N.
Let $x, y \in N_c$. Then, using the right distributivity of multiplication over addition and definition of N_c, we have $(x - y)0 = x0 - y0 = x - y$. This implies $x - y \in N_c$ and then $(N_c, +)$ is a subgroup of $(N, +)$.
Let $n, m \in N_c$. Then $(nm)0 = n(m0) = nm$, which implies that $nm \in N_c$. Hence, $N_cN_c \subseteq N_c$.
Therefore, N_c is a sub-near-ring of N.

□

Definition 2.2.2. *A normal subgroup I of (N,+) is called*

(i) a right ideal if $IN \subseteq I$.

(ii) a left ideal if $n(m + i) - nm \in I$ for all $n, m \in N$ and for all $i \in I$.

(iii) an ideal if it is both right and left ideal.

Two examples of ideal of a near-ring N are N itself and the trivial ideal (denoted by 0), which consists only of the zero element.

Example 2.4. *Let $N = \{0, 1, 2, 3, 4, 5\}$. Define addition, "+", and multiplication, ".", on N by the following two tables.*

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	0	3	2	5	4
2	2	4	0	5	1	3
3	3	5	1	4	0	2
4	4	2	5	0	3	1
5	5	3	4	1	2	0

and

.	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	1	1	1	1	1
2	1	1	1	2	1	2
3	0	0	0	3	0	3
4	0	0	0	4	0	4
5	1	1	1	5	1	5

Then $(N, +, .)$ is a near-ring and $I = \{0, 3, 4\}$ is an ideal of N.

Every ideal of a near ring is a sub-near-ring, but the converse is not true in general as we can see in the following example.

Example 2.5. *Consider the two near rings $(\mathbb{Z}, +,)$ and $(\mathbb{Q}, +,)$, where $\mathbb{Z}$ and $\mathbb{Q}$ are the set of integers and the set of rational numbers, respectively. $\mathbb{Z}$ is a sub-near ring of $\mathbb{Q}$. But $\mathbb{Z}$ is not an ideal of the near-ring $\mathbb{Q}$.*

Definition 2.2.3. *A near-ring N is called simple if N has no non-trivial ideals.*

2.3 Derivations in Near-Rings

In this section, some basic results concerning the study of derivation in near-rings are discussed and these are obtained from the results studied by Mohammed Ashraf, Abdelkarim Bous, and Abderrahmane Haji [10] and Malik Rahsid Jamal [9].

Definition 2.3.1. *Let $(N, +, .)$ be a near-ring. An additive mapping $D : N \longrightarrow N$ is said to be a derivation on N if $D(xy) = D(x)y + xD(y)$ for all $x, y \in N$*

Lemma 2.1. *([10]). Let $(N, +, .)$ be a near-ring and D be a derivation on N. Then N satisfies the following partial distributive law:*

(i) $z(xD(y) + D(x)y) = zxD(y) + zD(x)y$, for all $x, y, z \in N$.

(ii) $z(D(x)y + xD(y)) = zD(x)y + zxD(y)$, for all $x, y, z \in N$.

Proof. (i) Using the definition of a derivation and associativity of multiplication on a near-ring, we have

$$D(z(xy)) = zD(xy) + D(z)xy = z(xD(y) + D(x)y) + D(z)xy$$

and also

$$\begin{aligned} D((zx)y) &= zxD(y) + D(zx)y \\ &= zxD(y) + (zD(x) + D(z)x)y \\ &= zxD(y) + zD(x)y + D(z)xy. \end{aligned}$$

Equating the last parts of the above two expressions gives us

$$z(xD(y) + D(x)y) = zxD(y) + zD(x)y,$$

for all $x, y, z \in N$.

(ii) Again using the definition of a derivation and associativity of multiplication on a near-ring, we have

$$D(z(xy) = D(z)xy + zD(xy) = D(z)xy + z(D(x)y + xD(y))$$

and

$$\begin{aligned} D((zx)y) &= D(zx)y + zxD(y) \\ &= (D(z)x + zD(x))y + zxD(y) \\ &= D(z)xy + zD(x)y + zxD(y). \end{aligned}$$

Equating the last parts of the above two expressions gives us

$$z(D(x)y + xD(y)) = zD(x)y + zxD(y),$$

for all $x, y, z \in N$ and this completes the proof.

□

Definition 2.3.2. *Let $(N, +, .)$ be near ring. Then the set $\{x \in N : xn = nx \text{ for all } n \in N\}$ is called the center of N denoted by $Z(N)$*

Lemma 2.2. *([10], Lemma 1). Let $(N, +, .)$ be a prime near ring. If N admits a non-zero derivation D for which $D(N) \subseteq Z(N)$, then N is a commutative ring.*

Proof. Let N be prime near ring and D be a non-zero derivation on N so that $D(N) \subseteq Z(N)$. That is, $D(x)n = nD(x)$ for all $x, n \in N$. If we replace x by yx, then we have $D(xy)n - nD(xy) = 0$ for all $x, n \in N$. Then by Lemma 2.1, we have the following expression.

$$\begin{aligned} 0 =& (D(x)y + xD(y))n - n(D(x)y + xD(y)) \\ =& D(x)yn + xD(y)n - nD(x)y - nxD(y) \text{ for } y \in Z(N) \\ =& xD(y)n - nxD(y) \\ =& xnD(y) - nxD(y) \\ =& (xn - nx)D(y) = [x, n]D(y). \end{aligned}$$

Now, replacing n by nx gives us $[x, n]xD(y) = 0$ for all $x, n \in N$, $y \in Z(N)$. This implies $[x, n]ND(y) = 0$ and from the primeness of N and $D \neq 0$, we get $[x, n] = 0$ for all $x, n \in N$. Therefore, N is commutative. □

Theorem 2.8. *([10], Lemma 3). Let $(N, +, .)$ be a prime near-ring. If N admits a non zero derivation D such that for all $x, y \in N$, $D([x, y]) = 0$, then N a commutative ring.*

Proof. Let N be a prime near-ring and D be a non-zero derivation on N. Suppose $D([x,y]) = 0$ for all $x, y \in N$.

$$0 = D([x,y]) = D([x,yx]) = D([x,y]x)$$
$$= D([x,y])x + [x,y]D(x) = [x,y]D(x).$$

This implies $[x,y]D(x) = 0$ and hence $xyD(x) = yxD(x)$ for all $x, y \in N$. Now, replacing yz for y gives us $xyzD(x) = yzxD(x)$. This implies $xyzD(x) = yxzD(x)$ and then $xyzD(x) - yxzD(x) = 0$.

$$(xy - yx)zD(x) = 0 \text{ for all } z \in N \Longrightarrow [x,y]ND(x) = 0.$$

Therefore, the primeness of N and D is non-zero implies $[x,y] = 0$ and hence N is a commutative ring. □

Theorem 2.9. *([10],Theorem 1) Let N be prime near-ring which admits a nonzero derivation D. Then the following assertion are equivalent.*

(i) $D([x,y]) = [D(x), y]$ *for all* $x, y \in N$*.*

(ii) $[D(x), y] = [x, y]$ *for all* $x, y \in N$*.*

(iii) N *is a commutative ring.*

Proof. It can be easily shown that $(iii) \Rightarrow (i)$ and $(ii) \Rightarrow (i)$. This is because, N is commutative implies $D([x,y]) = 0$ and $[D(x), y] = 0$ and also we have $[D(x), y] = 0$ and $[x,y] = 0$.
$(i) \Rightarrow (iii)$: Suppose $D([x,y]) = [D(x), y]$ for all $x, y \in N$, By replacing y by yx, we get $[D(x), yx] = D([x,y]x)$ for all $x, y \in N$. This implies, $xyD(x) = yD(x)x$ for all $x, y \in N$. Then if we substitute yz for y

$$[x,y]zD(x) = 0 \text{ for all } x,y,z \in N \Rightarrow [x,y]ND(x) = \{0\} \text{ for all } x,y,z \in N.$$

By the primeness of N and because D is non-zero, we have that $[x,y] = 0$.
Therefore N is commutative.
To prove that $(ii) \Rightarrow (iii)$, suppose $[D(x), y] = [x, y]$ for all $x, y \in N$. Replacing x by xy and since $[xy, y] = [x, y]y$, we get,

$$[D(xy), y] = [(xy), y] = [x,y]y = [D(x), y]y$$
$$= (D(x)y - yD(x))y$$
$$= D(x)yy - yD(x)y.$$

And also we have the following

$$[D(xy), y] = [D(x)y + xD(y), y]$$
$$= (D(x)y + xD(y))y - y(D(x)y + xD(y))$$
$$= (D(x)y + xD(y))y - yD(x)y + yxD(y)$$
$$= D(x)yy + xD(y)y - yD(x)y + yxD(y).$$

If we compare the two expressions, we get

$$xyD(x) = yD(x)x \text{ for all } x, y \in N.$$

Substituting yz for y gives us $[x, y]zD(x) = 0$ for all $x, y, z \in N$ and this implies $[x, y]ND(x) = \{0\}$ for all $x, y, z \in N$. Then using the primeness of N and D is non-zero implies $[x, y] = 0$ for all $x, y \in N$.
hence N is commutative □

Theorem 2.10. *([12],Theorem 2) Let N be a 2-torsion free prime near-ring which admits a non-zero derivation D. Then the following assertion are equivalent.*

(i) $D([x, y]) \in Z(N)$ *for all* $x, y \in N$.

(ii) N *is a commutative ring .*

Proof. See [12] □

Lemma 2.3. *([12],Lemma 2.2) Let N be a prime near-ring.*

(i) If $z \in Z(N)\backslash\{0\}$*, then* z *is not a zero divisor.*

(ii) If $Z(N)$ *contains a non-zero element* z *for which* $z + z \in Z(N)$*, then* $(N, +)$ *is abelian.*

(iii) Let D *be a non-zero derivation on* N*. Then* $xD(N) = \{0\}$ *implies* $x = 0$ *and* $D(N)x = \{0\}$ *implies* $x = 0$.

(iv) If N *is 2-torsion free and* D *is a derivation on* N *such that* $D^2 = 0$*, then* $D = 0$.

Proof.

(i) Suppose $z \in Z(N)$, Let $zx = 0$ replace x by yx then we get $zyx = 0$ for all $x, y \in N$ $zNx = 0$ for all $x, \in N$. Since N is prime near ring and $z \neq 0$ we have $x = 0$. Hence z is not left zero divisor. Similarly we can show that z is not right zero divisor. There fore z is not zero divisor.

(ii) Let $x, y \in N$. Then we claim that $x + y = y + x$. Now $z + z \in Z(N)$, since $z + z \in Z(N)$ $z + z \in Z(N)$ $(x + y)(z + z) = (z + z)(x + y)$.

$$\begin{aligned}(x + y)(z + z) &= x(z + z) + y(z + z)\\ &=(z + z)x + (z + z)y\\ &=zx + zx + zy + zy\\ &=xz + xz + yz + yz \quad since \quad z \in Z(N)\\ &=(x + x + y + y)z \quad since \quad N \quad is \quad near - ring.\end{aligned}$$

On the other hand we have

$$\begin{aligned}(z+z)(x+y) &= z(x+y)+z(x+y)\\ &= (x+y)z+(x+y)z\\ &= xz+yz+xz+yz\\ &= (x+y+x+y)z.\end{aligned}$$

Now, by comparing the two expressions, we get $(x+x+y+y)z = (x+y+x+y)z$ which implies $((x+x+y+y)-(x+y+x+y))z = 0$. But z is not zero divisor implies $x+x+y+y = x+y+x+y$ and hence by using cancelation law in a group, we have $x+y = y+x$ for all $x, y \in N$. Therefore $(N, +)$ is abelian.

(iii) Let $xD(N) = \{0\}$, and let r, s be arbitrary elements of N. Then

$$\begin{aligned}0 &= xD(rs)\\ &= x(D(r)s + rD(s))\\ &= xD(r)s + xrD(s)\\ &= xrD(s)\\ &= xND(s).\end{aligned}$$

Thus $xNd(N) = \{0\}$ and since $d(N) \neq \{0\}$, $x = 0$. A similar argument works if $d(N)x = \{0\}$.

(iv) For arbitrary $x, y \in N$,we have

$$\begin{aligned}0 &= D^2(xy)\\ &= D(D(xy))\\ &= D(D(x)y + xD(y))\\ &= D(D(x)y) + D(xD(y))\\ &= D^2(x)y + D(x)D(y) + D(x)D(y) + xD^2(y)\\ &= 2D(x)D(y).\end{aligned}$$

Then we have

$$\begin{aligned}D(x)D(y) &= 0. \text{ for all } x, y \in N \text{ since N is 2-torsion free}\\ &\Rightarrow D(x)D(N) = 0. \text{ for all } x \in N\\ &\Rightarrow D(x) = 0 \text{ for all } x \in N\\ &\Rightarrow D = 0.\end{aligned}$$

□

Chapter 3

Generalized Derivations on Near-Rings

3.1 Generalized Derivations on Near-Rings

The concept of generalized derivation is introduced by Hvala [5]. Throughout this chapter N will denote a zero symmetric abelian near-ring

Definition 3.1.1. *Let $(N, +, .)$ be a near-ring. An additive mapping $F: N \longrightarrow N$ is said to be*

(a) a right generalized derivation associated with a derivation D if

$$F(xy) = F(x)y + xD(y) \text{ for all } x, y \in N.$$

(b) a left generalized derivation associated with the derivation D if

$$F(xy) = xF(y) + D(x)y \text{ for all } x, y \in N.$$

(c) a generalized derivation on N associated with a derivation D if it is either a right or a left generalized derivation associated with the derivation D.

The most common example on generalized derivations comes from the derivation on near-rings,

Example 3.1. *Every derivation in a near-ring is a generalized derivation associated with itself.*

Definition 3.1.2. *Let N be a near-ring. An additive mapping $g : N \longrightarrow N$ satisfying*

$$g(xy) = g(x)y$$

is called left multiplier

Example 3.2. *Every left multiplier on a near-ring N is generalized derivation .*

Example 3.3. *Every map of the form*

$$F(x) = cx + D(x),$$

where c is fixed element of N and D is a derivation of N, is a generalized derivation. Since

$$F(xy) = cxy + D(xy) = cxy + D(x)y + xD(y) = (cx + D(x))y + xD(y) = F(x)y + xD(y)$$

Now let us consider some properties of generalized derivations on near-rings.

Lemma 3.1. *Let* $(N, +, .)$ *be a near-ring.*

(i) If F *is a right generalized derivation on* N *associated with a derivation D, then*

$$F(xy) = xD(y) + F(x)y$$

for all $x, y \in N$*.*

(ii) If F *is a left generalized derivation on* N *associated with a derivation D, then*

$$F(xy) = xF(y) + D(x)y$$

for all $x, y \in N$*.*

Proof.

(i) Suppose F is a right generalized derivation on N associated with a derivation D. Then for any $x, y \in N$, we have

$$F((x + x)y) = F(x + x)y + (x + x)D(y) = F(x)y + F(x)y + xD(y) + xD(y).$$

On the other hand, $F(xy + xy) = F(xy) + F(xy) = F(x)y + xD(y) + F(x)y + xD(y)$. If we compare these two expressions, because $F((x + x)y) = F(xy + xy)$, we

$$F(x)y + F(x)y + xD(y) + xD(y) = F(x)y + xD(y) + F(x)y + xD(y).$$

Using the cancellation law in a group we obtain

$$F(x)y + xD(y) = xD(y) + F(x)y$$

and the right side of this equation is $F(xy)$. Therefore, $F(xy) = xD(y) + F(x)y$ for all $x, y \in N$.

(ii) Suppose F is a left generalized derivation of N associated with a derivation D. Then for any $x, y \in N$, we have

$$\begin{aligned} F(x(y+y)) = D(x(y+y)) + xF(y+y) \\ =& D(x)(y+y) + xD(y+y) + x(F(y)+F(y)) \\ =& D(x)y + D(x)y + xF(y) + xF(y). \end{aligned}$$

On the other hand,

$$F(xy+xy) = F(xy) + F(xy) = D(x)y + xF(y) + D(x)y + xF(y).$$

If we compare these two expressions, because $F(x(y+y)) = F(xy+xy)$, we have

$$D(x)y + D(x)y + xF(y) + xF(y) = D(x)y + xF(y) + D(x)y + xF(y).$$

Using the cancellation law in a group we obtain

$$D(x)y + xF(y) = xF(y) + D(x)y$$

and the right side of this equation is $F(xy)$.
Therefore, $F(xy) = xF(y) + D(x)y$ for all $x, y \in N$. □

Lemma 3.2. *([11],Lemma A]). Let $(N, +, .)$ be a prime near-ring. If F a generalized derivation on N associated with D of N, then*

$$a(bF(c) + D(b)c) = abF(c) + aD(b)c,$$

for all $a, b, c \in N$

Proof. For $a, b, c \in N$, we have $F(a(bc)) = aF(bc) + D(a)bc = a(bF(c) + D(b)c) + D(a)bc$ and

$$F(ab(c)) = abF(c) + D(ab)c = abF(c)) + (aD(b) + D(a)b)c = abF(c)) + aD(b)c + D(a)bc.$$

Then compare the two expressions gives us

$$a(bF(c) + D(b)c) + D(a)bc = abF(c)) + aD(b)c + D(a)bc.$$

This implies

$$a(bF(c) + D(b)c) = abF(c) + aD(b)c.$$

□

3.2 Commutativity of Prime Near-Rings

Definition 3.2.1. *Let $(N, +, .)$ be a near-ring a nonempty subset U of N is called a semigroup right ideal (respectively semigroup left ideal) of N if $UN \subset U$ (respectively$NU \subset U$).*

Remark 3.1. *Let $(N, +, .)$ be a near-ring a nonempty subset U of N is called a semigroup ideal if it is both a semigroup right ideal and a semigroup left ideal.*

Lemma 3.3. *([11],Lemma C]). Let $(N, +, .)$ be a prime near-ring and let $U \neq \{0\}$ be a semigroup ideal of N. If $m \in N$ such that $mU = 0$ or $Um = 0$, then $m = 0$.*

Proof. Suppose $mu = 0$ for all $u \in U$ and for $m \in N$. For nay $n \in N$ and $u \in U$, we have $mnu = 0$. This implies, $mNU = 0$. Since N is a prime near-ring and $U \neq \{0\}$. we get $m = 0$. The case of $Um = 0$ implies $m = 0$ can be proved similarly. □

Lemma 3.4. *([11],Lemma B]). Let $(N, +, .)$ be a prime near-ring and $U \neq \{0\}$ be a semigroup ideal of N. If $U \subseteq Z(N)$, then N is commutative.*

Proof. Let N be a prime near-ring and $U \neq \{0\}$ be a semigroup ideal of N.
Suppose $U \subseteq Z(N)$. Then, $nmu = unm$ for all $u \in U$ and $n, m \in N$, because $U \subseteq Z(N)$. Again, if $u \in U$, then $un = nu$ for all $n \in N$. So for $m \in N$ $nmu = unm$ implies $nmu = mun = mnu$ (because $u, un \in U$). This implies

$$(nm - mn)u = 0 \text{ for all } u \in U \text{ and for all } n, m \in N.$$

That is, $[n, m]U = 0$ and $U \neq \{0\}$. Now, using Lemma 3.3 we have, $nm = mn$ for all $n, m \in N$ and hence N is commutative. □

Lemma 3.5. *([9],Lemma 3). Let $(N, +, .)$ be a prime near-ring and let $U \neq \{0\}$ be a semigroup ideal of N. If n, m are elements of N such that $mUn = 0$, then $m = 0$ or $n = 0$.*

Proof. Suppose $mUn = 0$ for $n, m \in N$. Now if we replace m by mx, then we have $mNUn = 0$. Then either $m = 0$ or $Un = 0$, since N is prime. If $m \neq 0$, then $Un = 0$ and thus we have $n = 0$. □

Lemma 3.6. *([11],Lemma D]). Let $(N, +, .)$ be a prime near-ring and $U \neq \{0\}$ a semigroup ideal of N. If D is a derivation on N such that $D(U) = 0$, then $D = 0$.*

Proof. Suppose $D(U) = 0$. Then $D(un) = 0$ for all $u \in U, n \in N$ and hence $uD(n)+D(u)n = 0$. This implies $uD(n) = 0$ for all $u \in U$ and thus $UD(n) = 0$ which implies $D(n) = 0$ for all $n \in N$ (by Lemma 3.3).
Therefore, $D = 0$. □

Lemma 3.7. *let $(N, +, .)$ be a prime near-ring and $U \neq \{0\}$ be a semigroup ideal of N. Let F be a non-zero generalized derivation such that F is left multiplier. If $F(x) = x$ for all $x \in U$ then $F(n) = n$ for all $n \in N$.*

Proof. Suppose $F(x) = x$ for all $x \in U$. Then we have $F(nx) = nx$ and $F(nx) = F(n)x$ for all $x \in U$ and $n \in N$, because $nx \in U$ and F is a left multiplier). By comparing these two expressions, we get $nx = F(n)x$. This implies $nx - F(n)x = 0$ for all $x \in U$ and $n \in N$ and then $(n - F(n))x = 0$ for all $x \in U$ and $n \in N$. That is, $(n - F(n))U = 0$ for all $x \in U, n \in N$ and henc $n - F(n) = 0$ for all $n \in N$ (by Lemma 3.3). Therefore, $F(n) = n$ for all $n \in N$. □

Theorem 3.1. *([11], Theorem 1). Let $(N, +, .)$ be a non-commutative prime near-ring, U be a nonzero semigroup ideal of N and $F \neq 0$ be a generalized derivation associated with D of N. If $F[x, y] - [x, y] = 0$ for all $x, y \in U$, then F is the identity mapping on N.*

Proof. Suppose $F[x, y] = [x, y]$, for all $x, y \in U$. If we replace y by yx, we have $F[x, yx] = [x, yx] = [x, y]x$ and

$$F[x, yx] = F([x, y]x) = (F[x, y])x + [x, y]D(x)] = ([x, y])x + [x, y]D(x).$$

This implies

$$xyD(x) = yxD(x) \text{ for all } x, y \in U. \tag{3.1}$$

Now, if we replace y with ny for $n \in N$, we get

$$xnyD(x) = nyxD(x) = nxyD(x) \text{ for all } x, y \in U, n \in N$$

. This implies $(xn - nx)yD(x) = 0$ for all $n \in N, x, y \in U$

$$[x, n]UD(x) = 0 \text{ for all } n \in N, x \in U$$

since N is non commutative prime near-ring and using (by Lemma 3.5) $D(x) = 0$, for all $x \in U$. this implies $D(U) = 0$.

$$D = 0, \text{ (by Lemma 3.6)} \tag{3.2}$$

Now our hypothesis $F[x, y] - [x, y] = 0$ becomes $(F(x) - x)y = (F(y) - y)x$

let $B(x) = F(x) - x$ and so

$$B(x)y = B(y)x, \text{ for all } x, y \in U. \tag{3.3}$$

Replace x by xm then we get

$$B(xm)y = B(y)xm \text{ for all } x, y \in U, m \in N.$$

By using (3.3) and adding on both sides $-B(x)ym$ we get

$$B(x)my - B(x)ym = B(y)xm - B(x)ym = B(y)xm - B(y)xm = 0.$$

By using(3.3) we have

$$B(x)my - B(x)ym = 0 = B(my)x - B(ym)x \text{ for all } x, y \in U, m \in N$$

This implies $B(ym - my)x = 0 = B(x)(ym - my)$ and hence $B(x)[y, m] = 0$. Now replace x by xn $B(xn)[y, m] = B(x)n[y, m] = 0$ for all $x, y \in U, n, m \in N$ from this we have $B(x)N[y, m] = 0$ for all $x, y \in U, , m \in N$

$$B(x) = 0 \text{ or } [y, m] = 0 \text{ since N is prime .} \tag{3.4}$$

For if $B(x) \neq 0$,then $[y, m] = 0$, This implies $y \in Z(N)$ for all $y \in U$ i.e $U \subseteq Z(N)$ and hence N is commutative. This contradicts non-commutative of N. There fore $B(x) = 0$ for all $x \in U$,
By using (Lemma 3.7) and $B(x) = F(x) - x$, F is identity map. □

Corollary 3.1. *Let N is a prime near-ring.*

(i) Let U be a nonzero semigroup ideal of N. If N admits a generalized derivation F associated with D such that $F[x, y] - [x, y] = 0$ for all $x, y \in U$, then N is commutative or F is identity map.

(ii) If N admits a generalized derivation F associated with $D \neq 0$ such that $F[x, y] - [x, y] = 0$ for all $x, y \in N$, then N is commutative.

Proof. (i) Suppose the hypothesis is true.
Then by using the same argument with Theorem 3.1((3.4)
i.e $B(x) = 0$ or $[y, m] = 0$ for all $y, x \in U$ and $m \in N$. If $B(x) = 0$ for all $x \in U$, then $F(x) - x = 0$ for all $x \in U$ and by using lemma3.7, we have $F(n) = n$ for all $n \in N$ and hence F is an identity map. If $B(x) \neq 0$,then $[y, m] = 0$. This implies $y \in Z(N)$ for all $y \in U$ and we have $U \subseteq Z(N)$ and hence N is commutative.

(ii) If $F = 0$ it is clear. Now $F \neq 0$ from the hypothesis we have $F[x, y] = [x, y]$. Replace y by yx we get $F[x, yx] = [x, y]x$. And also we have $F([x, y]x) = F([x, y])x + [x, y]D(x)$ by comparing the two we have

$$xyD(x) = yxD(x) \text{ for all } x, y \in N. \tag{3.5}$$

Now replace y by yz in (3.5) we have

$$xyzD(x) = yzxD(x) = yxzD(x) \text{ for all } x, y, z \in N.$$

$$[x, y]zD(x) = 0 \text{ for all } x, y, z \in N$$

$$[x, y]ND(x) = 0 \text{ for all } x, y \in N.$$

As $D \neq 0$ and primeness of N $[x, y] = 0$ this implies $xy = yx$ for all $x, y \in N$ and hence N is commutative .

□

Remark 3.2. *In Corollary 3.1(i)and (ii), the hypothesis of primeness may be weakened by assuming that $D(x) \in N$ is not a right as well left zero divisor of N, where N is a near-ring. Then the same proof will lead to the conclusion that N is commutative.*

Theorem 3.2. *Let N be a non commutative prime near-ring, U a nonzero semigroup ideal of $N, F \neq 0$ a generalized derivation associated with a derivation D on N such that $F[x, y] + [x, y] = 0$. for all $x, y \in U$, then $F(n) = -n$ for all $n \in N$.*

Proof. Suppose the hypothesis is true. Then we have $F[x, y] = -[x, y]$,for all $x, y \in U$ replace y by yx.

$$F[x, yx] = -[x, yx] = -[x, y]x \text{ for all} \tag{3.6}$$

and

$$F[x, yx] = F([x, y]x) = (F[x, y])x + [x, y]D(x)] = (-[x, y])x + [x, y]D(x) \tag{3.7}$$

This implies

$$xyD(x) = yxD(x) \text{ for all } x, y \in U. \tag{3.8}$$

By using the same argument with Theorem 3.1 we get theorem 3.1(3.2) i.e $D = 0$. Now our hypothesis $F[x, y] + [x, y] = 0$ becomes $(F(x) + x)y = (F(y) + y)x$.

Let $B(x) = F(x)+x$ and so $B(x)y = B(y)x$, for all $x, y \in U$. And also by using the same argument with Theorem 3.1 we get 3.1(3.4) i.e

$$B(x) = 0 \text{ or } [y, m] = 0 \tag{3.9}$$

For if $B(x) \neq 0$, then $[y, m] = 0$, this implies that $y \in Z(N)$ for all $y \in U$ and hence $U \subseteq Z(N)$. Therefore N is commutative by lemma3.4. This contradicts non-commutative of N. Then we have

$$B(x) = 0 \text{ for all } x \in U.$$

This implies

$$F(x) = -x \text{ for all } x \in U. \tag{3.10}$$

Now replace x by nx in (3.10), then we have $F(nx) = -nx$ for all $x \in U, n \in N$. This implies

$$(F(n) + n)x = 0 \text{ for all } x \in U, n \in N. \text{ And hence}$$

$$(F(n) + n)U = 0 \text{ for all } n \in N.$$

There fore

$$F(n) = -n \text{ for all } n \in N. \text{ using lemma 3.3}$$

This completes the proof □

Corollary 3.2. *i. Suppose N is a prime near-ring and U a nonzero semigroup ideal of N. If N admits a generalized derivation F associated with a derivation D on N such that $F[x,y]+[x,y]=0$ for all $x,y \in U$, then N is commutative or $F(n)=-n$.*

ii. Suppose N is a prime near-ring and If N admits a generalized derivation F associated with $D \neq 0$ such that $F[x,y]+[x,y]=0$ for all $x,y \in N$,then N is commutative.

Proof. i. Suppose the hypothesis is true.
Then by using the same argument with theorem 3.2 we reach (3.9)
i.e, $B(x)=0$ or $[y,m]=0$ for all $x,y \in U$, $m \in N$.

For if $[y,m] \neq 0$ $B(x)=0$ for all $z \in U$, and using the same argument with theorem 3.2 we get (3.9)

$$(F(n)=-n \text{ for all } n \in N.$$

other wise $[y,m]=0$ for all $y \in U$,$m \in N$ and there fore N is commutative.

ii. If $F=0$,then it is clear .Now suppose $F \neq 0$. From the hypothesis we have $F[x,y]=-[x,y]$,replace y by yx we get

$$F[x,yx]=-[x,y]x \text{ for all } x,y \in N.$$

And

$$-[x,y]x=F[x,yx]=F([x,y]x)=F[x,y]x+[x,y]D(x) \text{ for all } x,y \in N.$$

This implies

$$xyD(x)=yxD(x) \text{ for all } x,y \in N \tag{3.11}$$

By replacing y by zy in (3.11)

$$xzyD(x)=zyxD(x)=zxyD(x) \text{ for all } x,y \in N.$$

$$[x,z]ND(x)=0 \text{ for all } x,z \in N$$

Using the hypothesis $D \neq 0$ and primeness of N $[x,z]=0$ this implies $xz=zx$ for all $x,z \in N$ and hence N is commutative.

□

Remark 3.3. *In Corollary 3.2(i)and (ii), the hypothesis of primeness may be weakened by assuming that $D(x) \in N$ is not a right as well left zero divisor of N, where N is a near-ring. Then the same proof will lead to the conclusion that N is commutative.*

Below we construct an example to demonstrates that the above result do not hold for arbitrary rings.

Example 3.4. *Let* N=$\{\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$: $a, b, c \in R\}$ *where* R *is a commutative ring and* U= $\{\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$: $a, b \in R\}$. *Define a map* $F : N \to N$ *by*

$$F \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ a+b & 0 \end{pmatrix}.$$

Then it is easy to check that F *is a generalized derivation associated with D, where* $D : N \longrightarrow N$ *defined as*

$$D \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}$$

on N. F satisfies the properties of Theorems 3.1 and 3.2 and corollary 3.1(i),(ii) and corollary 3.2(i),(ii), but neither N *is commutative, nor* F *is identity map, nor* $F(n) = -n$.

Theorem 3.3. *([11], Theorem 3). Let* N *be a non commutative prime near-ring,* U *a nonzero semigroup ideal of* $N, F \neq 0$ *a generalized derivation associated with* D *of* N*. If* $n(-1) = -n$ *for all* $n \in N$ *and* $F(xoy) - (xoy) = 0$ *for all* $x, y \in U$*, then* F *is identity map.*

Proof. Suppose the hypothesis is true.Thus we have

$$0 = F(xoy) - (xoy) = F(xy + yx) - (xoy) = F(xy) + F(yx) - (xoy).$$

This implies

$$F(x)y + xD(y) + F(y)x + yD(x) - (xoy) = 0 \text{ for all } x, y \in U \tag{3.12}$$

Now replace y by yx in (3.12) and since F is generalized derivation then we get

$0 = F(x)yx+xD(yx)+F(yx)x+yxD(x)-(xoyx) = F(x(yx))+F((yx)x)-(xoyx)$ for all $x, y \in U$.

By associativity of N and additive map F we have

$0 = F((xy)x) + F((yx)x) - (xoyx) = F((xy)x + (yx)x) - (xoyx) = F((xy + yx)x) - (xoyx)$.

Then we get

$$F((xy + yx))x + (xy + yx)D(x) - (xoyx) = 0 \text{ for all } x, y \in U.$$

By using the hypothesis we have

$$xyD(x) = -yxD(x) \text{ for all } x, y \in U. \tag{3.13}$$

Now replace y by ny, in (3.13), then

$$xnyD(x) = -nyxD(x) = -n(-xy)D(x) = nxyD(x) \text{ for all } x, y \in U, n \in N.$$

This implies

$$xnyD(x) - nxyD(x) = 0 \text{ for all } x, y \in U, n \in N.$$

By using right distributivity and commutator of x and y we have

$$[x, n]UD(x) = 0 \text{ for all } x \in U, n \in N.$$

By non commutativity of N and lemma 3.5 we get

$$D = 0 \tag{3.14}$$

Now our hypothesis $F(xoy) - (xoy) = 0$ becomes $(F(x) - x)y + (F(y) - y)x = 0$

Let $B(x) = F(x) - x$ and so

$$B(x)y + B(y)x = 0 \text{ for all } x, y \in U. \tag{3.15}$$

Replace x by xm in (3.15) then we get

$$B(xm)y = -B(y)xm = B(x)ym \text{ for all } x, y \in U, m \in N.$$

By using (3.15) we get

$$B(x)my = B(x)ym \text{ for all } x, y \in U, m \in N.$$

Then we have

$$B(my)x = B(ym)x \text{ for all } x, y \in U, m \in N.$$

This implies

$$0 = B(ym - my)x = B(x)(ym - my) = B(x)[y, m]$$

Now replace x by xn

$$B(xn)[y, m] = B(x)n[y, m] = 0 \text{ for all } x, y \in U, n, m \in N$$

from this we have

$$B(x)N[y, m] = 0 \text{for all } x, y \in U, m \in N.$$

$$B(x) = 0 \text{ or } [y, m] = 0 \text{ since N is prime .} \tag{3.16}$$

For if $B(x) \neq 0$, then $[y, m] = 0$, This implies $y \in Z(N)$ for all $y \in U$ (i.e $U \subseteq Z(N)$). And hence N is commutative. This contradicts non-commutative of N. There fore $B(x) = 0$ for all $x \in U$.

By using (Lemma 3.6) and $B(x) = F(x) - x$, F is an identity map. □

Corollary 3.3. *Let N is a prime near-ring*

i. *Suppose U be a nonzero semigroup ideal of N . If N admits a generalized derivation F associated with D such that $n(-1) = -n$ for all $n \in N$ and $F(xoy) - (xoy) = 0$ for all $x, y \in U$, then N is commutative or F is an identity map.*

ii. *Which admits a generalized derivation F associated with $D \neq 0$ such that $n(-1) = -n$ for all $n \in N$ and $F(xoy) - (xoy) = 0$ for all $x, y \in N$, then N is commutative.*

Proof. i. Suppose the hypothesis is true by using The same argument with theorem 3.3 we get (3.16) B(x)=0 or [y,m]=0 $x, y \in U$ and $m \in N$ is prime. For if $B(x) \neq 0$, then $[y, m] = 0$, This implies $y \in Z(N)$ for all $y \in U$ i.e $U \subseteq Z(N)$ and hence N is commutative. Other wise $B(x) = 0$ for all $x \in U$.
By using (Lemma 3.6) and $B(x) = F(x) - x$, F is identity map.

ii. From the hypothesis we have $F(xoy) = (xoy)$.replace y by yx we get $F(xoyx) = (xoyx)$.and also we have $F((xoy)x) = (F(xoy))x + (xoy)D(x)$ by comparing the two then we have $xyD(x) = -yxD(x)$ for all $x, y \in N$ By replacing y by zy

$$xzyD(x) = -zyxD(x) = zxyD(x) \text{ for all } x, y \in N.$$

This implies

$$[x, z]yD(x) = 0 \text{ for all } x, y, z \in N \text{ means that}$$
$$[x, z]ND(x) = 0 \text{ for all } x, y \in N.$$

As $D \neq 0$ and primeness of N, $[x, z] = 0$ this implies $xz = zx$ for all $x, z \in N$ and hence N is commutative.

□

Remark 3.4. *In corollary 3.3 (i)and (ii), the hypothesis of primeness may be weakened by assuming that $D(x) \in N$ is not a right as well left zero divisor of N, where N is a nearring.Then the same proof will lead to the conclusion that N is commutative.*

Theorem 3.4. *Let N be a non-commutative prime near-ring, U a nonzero semigroup ideal of N, $F \neq 0$ a generalized derivation associated with D of N. If $n(-1) = -n$ for all $n \in N$ and $F(xoy) + (xoy) = 0$ for all $x, y \in U$, then $F(n) = -n$ for all $n \in N$.*

Proof. Suppose the hypothesis is true

$$F(xoy) + (xoy) = 0 \text{ for all } x, y \in U.$$

This implies $F(x)y + xD(y) + F(y)x + yD(x) + (xoy) = 0$ for all $x, y \in U$. Now replace y by yx Then we have

$$F(x)yx + xD(yx) + F(yx)x + yxD(x) + (xoyx) = 0 \text{ for all } x, y \in U.$$

By the associativity of N and additive map F we have

$0 = F(x(yx)) + F((yx)x) + (xoyx) = F((xy)x) + F((yx)x) + (xoyx) = F((xy)x + (yx)x) + (xoyx)$ for all x, y

$0 = F((xy+yx))x + (xy+yx)D(x) + (xoyx) = F((xy+yx))x + (xoyx) + (xy+yx)D(x) = 0 + (xy+yx)D(x).$

Then we have

$$xyD(x) = -yxD(x) \text{ for all } x, y \in U. \tag{3.17}$$

Replace y by ny

$xnyD(x) - nxyD(x) = -nyxD(x) - nxyD(x).$

$$\Rightarrow xnyD(x) - nxyD(x) = -nyxD(x) + nyxD(x).$$
$$\Rightarrow xnyD(x) - nxyD(x) = 0.$$
$$\Rightarrow [x, n]yD(x) = 0 \text{ for all } x, y \in U. \tag{3.18}$$
$$\Rightarrow [x, n]UD(x) = 0.$$

From this we have $D = 0$. (By lemma3.5 and non commutativity of N.
Now our hypothesis $F(xoy) + (xoy) = 0$ for all $x, y \in U$ becomes $(F(x) + x)y + (F(y) + y)x = 0$. Let $B(x) = F(x) + x$. Then $B(x)y = -B(y)x$ for all $x, y \in U$. Replace x by xm then we get $B(xm)y = -B(y)xm$ for all $x, y \in U$, $m \in N$.

$$B(x)my = -B(y)xm = B(x)ym \text{ for all } x, y \in U, m \in N.$$

$B(my)x = B(ym)x$ this implies

$0 = B(ym)x - B(my)x = (B(ym) - B(my))x = B(ym - my)x = B(ym - my)x = B(x)(ym - my).$

Replace x by xn, we get

$$B(xn)[y, m] = B(x)n[y, m] = 0 \text{ for all } x, y \in Un, m \in N.$$

Then we have

$$B(x)N[y, m] = 0 \text{ for all } x, y \in U, m \in N.$$
$$B(x) = 0 \text{ or } [y, m] = 0 \text{ by the primness of N} \tag{3.19}$$

This implies

$$B(z) = 0 \text{ as N is non commutative.}$$

There fore $F(n) = -n$ □

Corollary 3.4. *Let N be a prime near-ring.*

(i) If U is a nonzero semigroup ideal of N and if N admits a generalized derivation F associated with D such that $n(-1) = -n$ for all $n \in N$ and $F(xoy) + (xoy) = 0$ for all $x, y \in U$, then N is commutative or $F(n) = -n$ for all $n \in N$.

(ii) *If N admits a generalized derivation F associated with $D \neq 0$ such that $n(-1) = -n$ for all $n \in N$ and $F(xoy) + (xoy) = 0$ for all $x, y \in N$, then N is commutative.*

Proof. (i) From Theorem 3.4, we get (3.19) i.e $B(z) = 0$ or $[y, m] = 0$.
If $B(x) \neq 0$, then $[y, m] = 0$ for all $y \in U$ $m \in N$. This implies $U \subseteq Z(N)$ and hence N is commutative. Otherwise $B(x) = 0$ for all $x \in U$ and hence $F(n) = -n$ for all $n \in N$

(ii) From the hypothesis $F(xoy) + (xoy) = 0$ for all $x, y \in N$ from Theorem 3.4 we reach (3.18).i.e $[x, n]yD(x) = 0$ for all $x, n, y \in N$.This implies [x,n]ND(x)=0 for all $x, n \in N$ as $D \neq 0$ then $xn = nx$ for all $x, n \in N$

□

Remark 3.5. *In Corollary 3.4 (i)and (ii), the hypothesis of primeness may be weakened by assuming that $D(x) \in N$ is not a right as well left zero divisor of N, where N is a near-ring.Then the same proof will lead to the conclusion that N is commutative. The following example demonstrates that the above results do not hold for arbitrary rings.*

Example 3.5. *Let* $N=\{\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} : a, b, c \in R\}$ *and let* $U= \{\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} : a, b \in R\}$ *be a non zero semigroup ideals of N, where R is a non commutative ring with condition $a^2 = 0$ for all $a \in R$.*
Define a map $F : N \to N$ by

$$F\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ a+b & 0 \end{pmatrix}$$

Then it is easy to check that F is a generalized derivation associated with D, where $D : N \to N$ defined as

$$D\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}$$

on N. However, F satisfies the properties of Theorems 3.3 and 3.4 and Corollary 3.3(i),(ii) and Corollary 3.4(i),(ii), but neither N is commutative, nor F is identity map, nor $F(n) = -n$.

Theorem 3.5. *Let N be 3-torsion free prime near-ring which admits a generalized derivation F associated with a non-zero idempotent derivation D of N. If $F^2[x, y] - [x, y] = 0$ for all $x, y \in N$, then N is commutative.*

Proof. Let N be a 3-torsion free prime near-ring which admits a generalized derivation F associated with a nonzero idempotent derivation D of N.
Suppose $F^2[x, y] = [x, y]$ for $x, y \in N$. Now, replacing y by yx gives us

$$F^2[x, yx] = [x, y]x = F^2[x, y]x \text{ (since } [x, yx] = [x, y]x). \tag{3.20}$$

and from the right side of Equation (3.20) we have

$$\begin{aligned} F^2[x, yx] =& F^2([x, y]x) = F(F([x, y]x)) = F(F[x, y]x + [x, y]D(x)) \\ =& F(F[x, y]x) + F([x, y]D(x)) \text{ (since F is additive)} \\ =& F^2[x, y]x + F[x, y]D(x) + F[x, y]D(x) + [x, y]D^2(x). \end{aligned} \tag{3.21}$$

This implies

$$F[x, y]D(x) + F[x, y]D(x) + [x, y]D^2(x) = 0. \tag{3.22}$$

On the other hand, the derivation D is idempotent, which implies that Equation (3.22) becomes $F[x, y]D(x)+F[x, y]D(x)+[x, y]D(x) = 0$ and using the right distributive property over multiplication on addition we have the following equation.

$$(F[x, y] + F[x, y] + [x, y])D(x) = 0 \tag{3.23}$$

Apply the additive function F on both sides of Equation (3.23) gives us

$$F((F[x, y] + F[x, y] + [x, y])D(x)) = F(0) = 0.$$

This implies $F(F[x, y] + F[x, y] + [x, y])D(x) + (F[x, y] + F[x, y] + [x, y])D^2(x) = 0$ and the derivation D is idempotent. Thus,

$$([x, y] + [x, y] + F[x, y])D(x) + (F[x, y] + F[x, y] + [x, y])D(x) = 0.$$

From Equation (3.23) we have $(F[x, y] + F[x, y] + [x, y])D(x) = 0$, which implies that

$$[x, y]D(x) + [x, y]D(x) + F[x, y]D(x) = 0. \tag{3.24}$$

Now, equating Equations (3.23) and (3.24), we have the following expression.

$$F[x, y]D(x) + F[x, y]D(x) + [x, y]D(x) = [x, y]D(x) + [x, y]D(x) + F[x, y]D(x).$$

This implies $F([x, y])D(x) = [x, y]D(x)$ and hence

$$F[x, y]D(x) - [x, y]D(x) = 0 \text{ for all } x, y \in N. \tag{3.25}$$

Now equating Equations (3.24) and (3.25) since both of them are equal to zero for all $x, y \in N$, we have

$$[x, y]D(x) + [x, y]D(x) + F[x, y]D(x) = F[x, y]D(x) - [x, y]D(x)$$

which implies $3[x, y]D(x) = 0$. But from our assumption, N is a 3-torsion free prime near ring. This implies $[x, y]D(x) = 0$, Thus

$$xyD(x) = yxD(x). \tag{3.26}$$

Now replacing y by yz in Equation (3.26) gives us $xyzD(x) = yzxD(x)$. Again by using Equation (3.26), we have $xyzD(x) = yxzD(x)$ and hence $[x, y]zD(x) = 0$ for all $x, y, z \in N$. This implies $[x, y]ND(x) = 0$ for all $x, y, \in N$. From the primeness of N and because D is a non-zero derivation on N, we have

$$[x, y] = 0 \text{ for all } x, y, \in N$$

and hence $xy = yx$ for all $x, y, \in N$, that is, N is commutative. This completes the proof of the theorem. □

Theorem 3.6. *Let N be a 5-torsion free prime near-ring which admits a generalized derivation F associated with non-zero idempotent derivation D of N. If*

$$F^2[x, y] + [x, y] = 0$$

for all $x, y \in N$, then N is commutative.

Proof. Let N be a 5-torsion free prime near-ring and F be a generalized derivation on N associated with a non-zero idempotent derivation D.
Suppose $F^2[x, y] = -[x, y]$ for $x, y \in N$. Now, replacing y by yx gives us

$$F^2[x, yx] = -[x, y]x, \text{ (since } [x, yx] = [x, y]x) \tag{3.27}$$

and we also have

$$\begin{aligned} F^2[x, yx] =& F^2([x, y]x) = F(F([x, y]x)) \\ =& F(F[x, y]x + [x, y]D(x)) \\ =& F(F[x, y]x) + F([x, y]D(x)) \text{ (since F is additive)} \end{aligned}$$

and $F(F[x, y]x + F([x, y]D(x)) = F^2[x, y]x + F[x, y]D(x) + F[x, y]D(x) + [x, y]D^2(x)$.
This implies $F^2[x, yx] = F^2[x, y]x + F[x, y]D(x) + F[x, y]D(x) + [x, y]D^2(x)$ and D idempotent. Thus, if we use the cancellation law in a group we have,

$$(F[x, y] + F[x, y] + [x, y])D(x) = 0. \tag{3.28}$$

Applying the additive function F on Equation (3.28) and considering the give assumption $F^2[x, y] = -[x, y]$ we get

$$(-[x, y] - [x, y] + F[x, y])D(x) + (F[x, y] + F[x, y] + [x, y])D^2(x) = 0. \tag{3.29}$$

But D is idempotent, from our assumption, and using Equation (3.28) we have the equation $(-[x, y] - [x, y] + F[x, y])D(x) = 0$, which implies

$$F[x, y]D(x) = ([x, y] + [x, y])D(x). \tag{3.30}$$

Applying the additive function F on both sides of Equation (3.30) gives us

$$F^2[x,y]D(x)+F[x,y]D(x)=(F[x,y]+F[x,y])D(x)+([x,y]+[x,y])D^2(x). \tag{3.31}$$

Using the given assumptions $F^2[x,y]=-[x,y]$ and D is idempotent, we have

$$-[x,y]D(x)+F[x,y]D(x)=(F[x,y]+F[x,y])D(x)+([x,y]+[x,y])D(x).$$

Using the cancellation law in a group we have,

$$F[x,y]D(x)=-[x,y]D(x)-[x,y]D(x)-[x,y]D(x). \tag{3.32}$$

If we compare Equations (3.30) and (3.32) we have the following equation

$$[x,y]D(x)+[x,y]D(x)+[x,y]D(x)+[x,y]D(x)+[x,y]D(x)=5[x,y]D(x)=0. \tag{3.33}$$

Since N is a 5-torsion free near-ring, by our assumption, we have that $[x,y]D(x)=0$ which implies

$$xyD(x)=yxD(x). \tag{3.34}$$

Now, replacing y by yz in Equation (3.34) gives us $xyzD(x)=yzxD(x)$. Again by using Equation (3.34) we have that $xyzD(x)=yxzD(x)$ which implies that

$$[x,y]zD(x)=0 \text{ for all } x,y,z\in N.$$

This implies $[x,y]ND(x)=0$ for all $x,y,\in N$. From the primeness of N and since D is a non-zero derivation on N, we have

$$[x,y]=0 \text{ for all } x,y,\in N$$

and hence $xy=yx$ for all $x,y,\in N$. That is, N is commutative and this completes the proof of the theorem. □

Theorem 3.7. *Let N be a 3-torsion free prime near-ring with identity which admits a generalized derivation F associated with non zero idempotent derivation D of N. If $n(-1)=-n$ for all $n\in N$ and $F^2(xoy)-(xoy)=0$ for all $x,y\in N$, then N is commutative*

Proof. Let N be a 3-torsion free prime near-ring with identity and F be generalized derivation associated with non-zero idempotent derivation D on N.
Suppose $F^2(xoy)-(xoy)=0$ for all $x,y\in N$. Then $F^2(xoy)=(xoy)$. Now, if we replace y by yx, then we get

$$F^2(xo(yx))=(xoy)x \tag{3.35}$$

and on the other hand

$$\begin{aligned} F^2(xoyx)=&F^2((xoy)x)=F(F((xoy)x)) \\ =&F(F(xoy)x+(xoy)D(x))=F(F(xoy)x+F((xoy)D(x)) \\ =&F^2(xoy)x+F(xoy)D(x)+F(xoy)D(x)+(xoy)D^2(x) \end{aligned} \tag{3.36}$$

This implies

$$F(xoy)D(x) + F(xoy)D(x) + (xoy)D^2(x) = 0 \tag{3.37}$$

Since D is idempotent, Equation (3.37) becomes

$$F(xoy)D(x) + F(xoy)D(x) + (xoy)D(x) = 0$$

and by the right distributivity of multiplication on addition we have,

$$(F(xoy) + F(xoy) + (xoy))D(x) = 0. \tag{3.38}$$

Applying the additive function F on Equation (3.38) gives us

$$F((F(xoy) + F(xoy) + (xoy))D(x)) = F(0).$$

$$\begin{aligned} 0 =& F((F(xoy) + F(xoy) + (xoy))D(x)) \\ =& F(F(xoy) + F(xoy) + (xoy))D(x) + (F(xoy) + F(xoy) + (xoy))D^2(x) \end{aligned} \tag{3.39}$$

and D is idempotent implies that

$$(F^2(xoy) + F^2(xoy) + F(xoy))D(x) + (F(xoy) + F(xoy) + (xoy))D(x) = 0.$$

From our assumption, $F^2(xoy) = xoy$ and this implies that

$$((xoy) + (xoy) + F(xoy))D(x) + (F(xoy) + F(xoy) + (xoy))D(x) = 0.$$

Using Equation (3.38) we get $((xoy) + (xoy) + F(xoy))D(x) = 0$ and again using our assumption, $F^2(xoy) = xoy$, we have

$$[(xoy) + (xoy) + (xoy)]D(x) = 3(xoy)D(x) = 0$$

and N is a 3-torsion free prime near-ring. This implies $(xoy)D(x) = 0$ and hence

$$xyD(x) = -yxD(x). \tag{3.40}$$

Now replacing y by yz in Equation (3.40) gives us

$$xyzD(x) = -yzxD(x) = yxzD(x)$$

and hence $[x, y]zD(x) = 0$ for all $x, y, z \in N$. This implies

$$[x, y]ND(x) = 0 \text{ for all } x, y \in N.$$

From the primeness of N and since D is a non-zero derivation on N, by assumption, we have

$$[x, y] = 0 \text{ for all } x, y \in N.$$

This implies $xy = yx$ for all $x, y, \in N$.
Therefore, N is commutative. □

Theorem 3.8. *Let N be a 5-torsion free prime near-ring with identity and F be a generalized derivation on N associated with non-zero idempotent derivation D on N. If $n(-1) = -n$ for all $n \in N$ and $F^2(xoy) + (xoy) = 0$ for all $x, y \in N$, then N is commutative.*

Proof. Let N be a 5-torsion free prime near-ring F be a generalized derivation on N associated with non-zero idempotent derivation D of N.
Suppose $F^2(xoy) + (xoy) = 0$ for all $x, y \in N$. Then $F^2(xoy) = -(xoy)$.
Now we replace y by yx and get

$$F^2(xoyx) = -(xoy)x. \tag{3.41}$$

On the other hand,

$$\begin{aligned} F^2((xoy)x) =& F^2((xoy)x) = F(F((xoy)x)) \\ =& F(F(xoy)x + (xoy)D(x)) \\ =& F(F(xoy)x + F((xoy)D(x)) \text{ (since F is additive)} \end{aligned} \tag{3.42}$$

and

$$\begin{aligned} F(F(xoy)x) + F((xoy)D(x)) =& F^2(xoy)x + F(xoy)D(x) + F(xoy)D(x) + (xoy)D^2(x) \\ =& -(xoy)x + F(xoy)D(x) + F(xoy)D(x) + (xoy)D^2(x) \\ =& -(xoy)x + F(xoy)D(x) + F(xoy)D(x) + (xoy)D(x) \end{aligned}$$

(since D is idempotent). This implies that

$$F^2(xoy)x) = -(xoy)x + F(xoy)D(x) + F(xoy)D(x) + (xoy)D(x). \tag{3.43}$$

Using the relation $F^2(xoyx) = -(xoy)x$, the cancellation law in a group and the right distributivity of multiplication over addition, we have the equation

$$F(xoy)D(x) + F(xoy)D(x) + (xoy)D(x) = (F(xoy) + F(xoy) + (xoy))D(x) = 0. \tag{3.44}$$

Applying the additive function F on Equation (3.44) and the assumption that D is idempotent gives us that $(-(xoy) - (xoy) + F(xoy))D(x) = 0$. This implies

$$F(xoy)D(x) = ((xoy) + (xoy))D(x). \tag{3.45}$$

Again applying the additive function on both sides of Equation (3.45) gives us

$$F^2(xoy)D(x) + F(xoy)D^2(x) = (F(xoy) + F(xoy))D(x) + ((xoy) + (xoy))D^2(x). \tag{3.46}$$

By our assumption, $F^2(xoy) = -(xoy)$ and D is idempotent implies that

$$-(xoy)D(x) + F(xoy)D(x) = (F(xoy) + F(xoy))D(x) + ((xoy) + (xoy))D(x). \tag{3.47}$$

Using the cancellation law in a group in Equation (3.47), we get

$$F(xoy)D(x) = (-(xoy) - (xoy) - (xoy))D(x). \tag{3.48}$$

By considering Equations (3.45) and (3.48), we have

$$(xoy)D(x) + (xoy)D(x) + (xoy)D(x) + (xoy)D(x) + xoy)D(x) = 5xoy)D(x) = 0. \tag{3.49}$$

Since N is 5-torsion free near-ring we have

$$(xoy)D(x) = 0 \tag{3.50}$$

Now replacing y by yz in Equation (3.50) gives us

$$xyzD(x) = -yzxD(x) = yxzD(x)$$

and hence $[x, y]zD(x) = 0$ for all $x, y, z \in N$. This implies

$$[x, y]ND(x) = 0 \text{ for all } x, y \in N.$$

From the primeness of N and since D is a non-zero derivation on N, by assumption, we have

$$[x, y] = 0 \text{ for all } x, y \in N.$$

This implies $xy = yx$ for all $x, y, \in N$.
Therefore, N is commutative.

□

Conclusion and Recommendations

If $(N, +, .)$ is a near-ring and $F : N \longrightarrow N$ satisfying $F(xy) = F(x)y + xD(y)$ for all $x, y \in N$ is generalized derivation on N associated with the derivation D we proved the commutativity of N using some addition coditions on F. The results that we proved can be summarized as follows.

(i) Let $N, +.)$ be a 3-torsion free prime near ring and F be a generalized derivation associated with a nonzero idempotent derivation D.

   (a) If $F^2[x, y] - [x, y] = 0$ for all $x, y \in N$, then N is commutative;

   (b) If $n(-1) = -n$ for all $n \in N$ and $F^2(xoy) - (xoy) = 0$ for all $x, y \in N$, then N is commutative.

(ii) Let N be a 5-torsion free prime near ring and F be a generalized derivation with non-zero idempotent derivation D.

   (a) If $F^2([x, y]) + [x, y] = 0$ for all $x, y \in N$, then N is commutative;

   (b) If $n(-1) = -n$ for all $n \in N$ and $F^2(xoy) + (xoy) = 0$ for all $x, y \in N$, then N is commutative.

All the above cases for a positive integer $n > 2$ are still open for further investigation. That is, we have the following open problems.

(i) Suppose N is a 3-torsion free prime near-ring which admit a generalized derivation F associated with a non zero idempotent derivation D.
Does the condition that $F^n[x, y] + [x, y] = 0$ for all $x, y \in N$ and $F^n[x, y] - [x, y] = 0$ for all $x, y \in N$ imply that N is commutative?

(ii) Suppose N is a 5-torsion free prime near-ring which admit a generalized derivation F associated with a non zero idempotent derivation D.
Does the condition $F^n(xoy) + (xoy) = 0$ for all $x, y \in N$ and $F^n(xoy) - (xoy) = 0$ for all $x, y \in N$ imply that N is commutative?

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