COLLEGE OF NATURAL AND COMPUTATIONAL SCIENCES

Department of Mathematics

Growth estimate of composition operator on Hilbert space of Dirichlet series

Marta Ayele Basazin

Advisor: *Hunduma Legesse (PhD)*

Thesis Submitted to the Department of Mathematics in Partial Fulfillment for the Degree of Master of Mathematics in Analysis stream.

Addis Ababa, Ethiopia

September 2024 G.C

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Acknowledgments

Acknowledgement First of all, let me give a continual and great full gratitude to my God who gave me long life, strength and help me to reach this precise time and though only my name appears on the cover of this thesis, many great people have contributed to its success. I owe my gratitude to all those people who made this thesis possible and because of whom my graduate experience has been one that I will value forever. My deepest gratitude is to my advisor, Hunduma Legesse (PhD). I am deeply grateful to him for helping me sorting out the technical details of the research work, giving insightful comments and constructive criticisms, encouraging the use of correct grammar and consistent notation in my writings and for carefully reading and commenting countless revisions of this document. His patience and support helped me overcome many difficult situations and finish this thesis work. I hope one day I would become as good advisor and teacher to my students as he has been to me. None one of this would have been possible without the love and patience of my family and friends.

Abstract

Motivated by a theorem of Gordon and Hedenalm (1), the study of composition operators acting on various scales of function spaces of Dirichlet series has arisen intensive interest. In this thesis, we characterize the composition operators in Hilbert space and growth estimate of the composition operator in Hilbert space of Dirichlet series.

Table of content

Introduction

The study of Dirichlet series, specifically those of the form $\sum_{n=1}^{+\infty} a_n \, n^{-s}$ has a rich history dating back to the nineteenth century. This interest largely stems from the central role these series play in analytic number theory. The general theory of Dirichlet series was significantly developed prominent mathematicians such as Hadamard, Landau, Hardy, Riesz, Schnee and Bohr. Despite their foundational contributions, many of the major results were established before Functional Analysis became a standard tool for every analyst. Incorporating modern Functional Analysis techniques into the study of Dirichlet series is a promising avenue for further research.

Recent efforts have already begun to integrate these modern perspectives. For instance Hedenmalm, Lindquist, and Seip have explored the application of Hilbert spaces to Dirichlet series in their work. This thesis aims to build on such advancements by highlighting several growth estimate of composition operator on Hilbert space of Dirichlet series depends on the transformation φ and its effect on this series.

Let $S = S(G)$ be the class of all analytic self-maps of a domain G of the complex plane C and $H(G)$ denote the space of all analytic functions on G. Each $\varphi \in S$ induces a composition operator

$$C\varphi \; : \; H(G) \to H(G) \; .$$

Defined as follows

$$C\varphi f \;=\; f o \varphi$$

With regard to the theory of composition operators acting on analytic c function spaces, Gordon and Hedenmalm initiated his study of composition operators on the Hardy Space of Dirichlet series. By Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ the Cauchy–Schwarz inequality, we observe that the elements of H^2 are analytic in the half-plane $C_{\frac{1}{2}}$, where for any real number θ,

$$C_\theta := \{s \in C: Res > \theta\}.$$

Definition 1.1. The Gordon– Hedenmalm (1) class, denoted G, is the class of analytic functions $\varphi: C_{\frac{1}{2}} \to C_{\frac{1}{2}}$ which can be expressed as $\varphi(s) = c_o s + \psi(s)$, Where c_o is a non-negative integer, the Dirichlet series ψ converges uniformly in C_ϵ for every $\epsilon > 0$ and has the following properties

i. If $C_O > 0$ then either $\psi \equiv 0$ or $\psi(c_o) \subseteq c_o$

ii. If $c_o = 0$, then $\psi(C0) \subseteq C_{\frac{1}{2}}$

Theorem1.1. Let $\alpha > 0$ and $\varphi \in C_O$. If $\mathrm{Im}\varphi$ is bounded on C_0 and $M\varphi(s) = O(Res - \frac{1}{2})^{2^\infty}$ for $Res \to \left(\frac{1}{2}\right)^+$, and then $C\varphi$ is bounded from $\mathcal{D}_\alpha$ to $\mathcal{H}^2$.

Theorem1.2. *Let* $\alpha > 0$ and $\varphi \in G_0$. *If* $C\varphi$ is bounded from $\mathcal{D}_\alpha$ *to* $\mathcal{H}^2$.

Then $M_\varphi(s) = O(Res - \frac{1}{2})^{2^\propto}$ *for* $Res \to \left(\frac{1}{2}\right)^+$.

Notations Throughout this thesis, we use the letter C to denote absolute constants which may change at every appearance but do not depend on the essential parameters. We write

$A \lesssim B$ or equivalently $B \gtrsim A$ ifthere exist in essential constant C such that $A \leq CB$.Similarly, we use the notation $A \approx B$ if both $A \lesssim B$ and $B \lesssim A$ hold.

Chapter 1

Review of Dirichlet series

1.1 Dirichlet series

A Dirichlet series is a series of the form

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = a_1 + \frac{a_2}{2^s} + \frac{a_3}{3^s} + \frac{a_4}{4^s} + \ldots, \qquad (1)$$

Which we will often abbreviate to $\sum_{n=1}^{+\infty} a_n n^{-s}$. The prototypical Dirichlet series is the series $\sum n^{-s}$ representing the zeta function. Dirichlet series are important in number theory. They were used by Dirichlet to study the distribution of prime numbers in arithmetic progressions.

The modulus

$\left|a_n n^{-(\sigma+it)}\right| = |a_n| n^{-\sigma}$ Of the n^{th} term of the series depends only on the real part σ of $s = \sigma + it$.The modulus decreases as σ increases. Thus if the series (1) converges absolutely for $s = s_o$, then it converges absolutely for all satisfying $R\varepsilon\, s \geq R\varepsilon s_0$.

By the Weierstrass M-test, it also converges uniformly in the half plane $\{R\varepsilon\, s \geq R\varepsilon s_0\}$, by comparison with the numerical series $\sum |a_n|\, n^{-R_\varepsilon S}$, proceeding in analogy with the definition of radius of convergence of power series, we define σ_0 to be the infimum of σ for which $\sum |a_n|\, n^{-\sigma}$ converges, and we have the following.

Theorem 1.3 For each Dirichlet series (1) there is unique extended real number $\sigma_0, -\infty \leq \sigma_0 \leq +\infty$, such that the series converges absolutely if Res $> a_0$,and the series does not converge absolutely if Rεs $<\sigma_0$. For any $\sigma_0 > a_0$, the series converges uniformly on the half plane $\{R\varepsilon s \geq \sigma_0\}$. The sum $f(s) = \sum a_n n^{-s}$ is analytic for Rεs $>\sigma_0$.The extended real number σ_0 is called the abscissa of absolute convergence of the series (1).

Example 1.1 The series $\sum n^{-s}$ for $\zeta(s)$ has abscissa of absolute convergence σ= 1.

It does not converge at $s = 1$.

Suppose the function $f(s)$ is represented by a Dirichlet series say,

$f(s) = \sum a_n n^{-s}$ for Rεs $> \sigma_0$.

Since the series converges uniformly in some half-plane, and each of the terms

$a_n n^{-s} \to 0$ as $\sigma \to \infty$ for $n \geq 2$,

We have

$\sum \frac{a_n}{n^s} \to 0$ as σ =Rεs $\to \infty$.

Consequently, f(s) $f \to a_1$ as $\sigma \to \infty$.

Similarly, we can capture the other coefficients of the series from $f(s)$. We write

$$n^s \left[f(s) - a_1 - \frac{a_2}{2^s} - \cdots - \frac{a_{n-1}}{(n-1)^s}\right] = a_n + \sum_{k=n+1}^{\infty} a_k \frac{n^s}{k^s}. \qquad (2)$$

To estimate the series on the right, choose σ_0 large so that

$$\sum |a_k|\, k^{\sigma_0} = M < \infty.$$

Then

$$\left|\sum_{k=n+1}^{\infty} a_k \frac{n^s}{k^s}\right| \leq \sum_{k=n+1}^{\infty} \frac{n^{\sigma}}{k^{\sigma-\sigma_0}} \frac{|a_k|}{k^{\sigma_0}}$$

$$\leq \frac{n^{\sigma}}{(n+1)^{\sigma-\sigma_0}}M.$$

Since

$$n^{\sigma}/(n+1)^{\sigma-\sigma_0} \to 0 \text{ as } \sigma \to \infty,$$

The series on the right-hand side as (2) tends to 0 as $\sigma \to \infty$, and the left-hand side of tends to a_n as $\sigma \to \infty$.Thus starting witha_1, we can determine in succession the coefficients a_n from $f(s)$.

A Dirichlet series $\sum_{n=1}^{+\infty} n^{-s}$ represent the zeta function. Dirichlet series important in number theory. Over the prime powers excluding1.(Not counting 1 as a prime power in that notation is reasonable in light of the way Dirichlet series that run over prime powers arise in practice, without a constant term.

Example 1.2 If $a_n = 1$ for all n then $f(s) = \zeta(s)$, which converges for $\sigma > 1$.It does not converge at $s = 1$.

Example 1.3 If $a_n = \chi(n)$ for a Dirichlet character χ , $f(s)$ is the L-function $L(s,\chi)$ and converges absolutely for $\sigma > 1$. Note $L(s,\chi4)$ converges for real

$s > 0$ since the Dirichlet series is then an alternating series. A general Dirichlet character does not take alternating values ± 1 , but we'll set as long as χ is not a trivial Dirichlet character,

$L(s,\chi)$ also converges (though not absolutely) when $0 < Re(s) \leq 1$.

Example1.4. The series $\sum \chi 4(p)P^{-S}$, more over the primes, converges for $\sigma > 1$. Although χ_4 is an alternating function on consecutive odd integers, it is not alternating on consecutive odd primes, so it is not clear whether or not it converges if

$0 < \sigma < 1$ .Convergences at $s = 1$ is known for any real $s < 1$.

Example1.5 For a Dirichlet series $\sum_{n=1}^{+\infty} a_n n^{-s}$ we can consider $\chi(n) = \sum_{n=1}^{+\infty} a_n n^{-s}$for some Dirichlet character χ . We call the latter function a twist of the formed by χ. So $L(s,\chi)$is a twist of the zeta-function.

Example 1.6 If a_n=n^k for an integer *k*, then $f(s) = \zeta(s-k)$ converges for $\sigma > k+\ 1$.

Example 1.7 If $a_n = \frac{1}{n}$ then $f(s)$ converges for all s . Unlike power series, no Dirichlet series that arises naturally converges on the whole complex plane, so you should not regard examples of this sort as important.

Example 1.8 A constant polynomial function $c_0 + c_1 + \cdots + c_m s^m (m > 0)$ is *not* expressible as a Dirichlet series

1.2 Multiplier of Dirichlet series

This is for instance a useful way to describe the point wise multipliers in H^2. In particular, since 1∈ H^2, then any multiplier belongs to H^2. Hedenmalm, Lindquist and Seip described the multipliers. For this we need the following definitions. We say that $f \in H^\infty$ if $f \in H^2$ and more over f extends analytically to a bounded function in C_0^+.

Theorem 1.4 The collection of multipliers on H^2 equal the space H^∞.

The above theorem is analogous to the following well-known result for Hardy spaces the (point wise) multipliers of H^2 are the functions in H^∞. A noteworthy difference, however, is that the multipliers in the Dirichlet series case are defined as bounded and analytic on a bigger half-plane than the functions in the space. It should be mentioned that the proof of the above theorem in is based on modeling H^2 as the Hardy space on the infinite-dimensional poly disk D^∞, an idea which goes back to a 1913 paper of Bohr

Theorem1.5 The multiplier of H^2 are exactly H^∞ and the multiplier norm of g coincides with $sup_{Co^+} |g|$. This can be seen because the function in $H^\infty(T^\infty)$. These are functions $f \in L^\infty(T^\infty)$, Such that $\int_{T^\infty} f(z)z^{-\propto}\, d\mu(z) = 0$ for all positive multiindice. By the result of $H.$ Hedenmalm(3) Hilbert space of Dirichlet series.

Theorem1.6 The multipliers of Hwb are contractively contained in the multipliers of H_w.

Corollary 2.1 For $0 < \alpha < 2$ the function φ is a multiplier of H_σ if and only if φ is in D n $D\ n\ H^\infty(\Omega_O)$ .The measure $[\varphi'(s)]^2 d_{N\propto-2}(\delta)\, dt$.

1.3. Convergence issues of Dirichlet series

The series may be convergent for all valves of S or for none or for some only in the last case there is a number σ_o such that the series is convergent for $\sigma > \sigma_o$ and divergent or oscillatory for $\sigma < \sigma_o$.

In other words, the region of convergence is a half plan. We shall call σ_o the abscissa of convergence and the $\sigma = \sigma_o$ the line of convergence. It is convenient to write $\sigma_o = -\infty$ or $= \infty$. When the series is convergent for all or no value of s on the line of convergence the question of the convergence of the question of the convergence of the series remains open and requires consideration of a much more delicate character.

Example

a. The series $\sum_{n=1}^{+\infty} a_n\, n^{-s}$, Where $|a_n| < 1,\ is$ convergent for all values of s.

b. The series $\sum_{n=1}^{+\infty} a_n\, n^{-s}$, Where $|a_n| > 1, is$ convergent for no values of s .

c. The series $\sum_{n=1}^{\infty} a^{-s}$ has $\sigma = 1$ as its lines of convergence it is not convergence at any point of the line of convergence, diverging to $+\infty$ for $s = 1$ and oscillating finitely at all other points of the lines.

d. The series $\sum_{n=1}^{\infty} (\log n)^{-2}\, n^{-s}$ has the same line of convergence as the last series, but it convergent (Indeed absolutely convergent) at all points of the line.

e. The series $\sum_{n=1}^{+\infty} a_n\, n^{-s}$, Where $a_n = (-1)^n + (\log n)^{-2}$ has the same line of convergent and it is convergent (through not absolutely) at all point of it.

Theorem 1.7 Let $\sum_{n=1}^{+\infty} |a_n|^2 < +\infty$, then the series $\sum_{n=1}^{+\infty} a_n\, n^{-\frac{1}{2}+it}$ Converge for almost every $t \in \mathbb{R}$.

Theorem 1.8 Let $f \in H^2$ be of the form $f(s) = \sum_{n=1}^{+\infty} a_n\, n^{-s}$ and $f_x \in H^2$, be defined by

$f_x(s) = \sum_{n=1}^{+\infty} a_n x(n)\, n^{-s}$. Then the series

$$f_x(s) = \sum_{n=1}^{+\infty} a_n x(n)\, n^{-it}$$

Converges for almost all character x and almost all real t .

$\sum_{n=1}^{+\infty} a_n\, n^{-s}$ Converge absolutely for $Re\ (s) > L$, but not $Re(s) \geq L$.

More over for any $\delta > 0$ Convergence is uniform on $Re(s) \geq L + \delta$, so the series represents a holomorphic function on all of $Re(s) > L$.

Theorem 1.9 Let $f(s) = \sum_{n=1}^{+\infty} a_n\, n^{-s}$. Be a Dirichlet series with non-negative real coefficient. Suppose $L \in \mathbb{R}$ is the abscissa of an absolute convergence for $f(s)$ then f can not be extended a holomorpic function on a neighborhood of $S = L$.

1.4 Integral means of Dirichlet series

Suppose $f(s) = \sum_{n=1}^{+\infty} a_n\, n^{-s}$, Where the sum is finite but finitely many of the a_n 's are zero. We might call such functions Dirichlet polynomials. Then

$$\frac{1}{2T}\int_{-T}^{T} |f(\sigma + it)|^{2}\, dt \quad \rightarrow \quad \sum_{n=1}^{+\infty} \frac{|a_n|^2}{n^{2\sigma}} \text{ as } T \rightarrow \infty$$

For each real σ, in the above formula contain the composition of two things that are

A. Plancherel formula The genuine Plancherel formula involves the character. We met earlier

$$\int_{\equiv} \left|f_{x(\sigma)}\right|^2 d_{\omega}\,(x) = \sum_{\mathrm{n}=1}^{+\infty} \frac{|\mathrm{a_n}|^2}{\mathrm{n}^{2\sigma}}$$

Where, I recall the notation

$$f_x\,(s)\ = \textstyle\sum_{n=1}^{+\infty} a_n x(n)\, n^{-s},$$

For the vertical limit function associated with the character x .The character of the form

$$x_n(n) = n^{-st}\boldsymbol{t} \in \mathbb{R}$$

B. **The general ergodic theorem** the theorem says that the time average along the flow of a continuous function equal the space average that is the integral and the limit.

$$\lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} |f(\sigma+it)|^2\ dt = \lim_{T\to+\infty} \int_{-T}^{T} |f_{xt}(\sigma)|^2\, dt$$

Chapter 2

Hilbert space of composition operator

2.1 Inner product space

Definition 2.1 A complex inner product space (or per-Hilbert space) is a complex vector space X together with an inner product a function from $X \times X$ into C (denoted by $\langle y, x \rangle$) satisfying:

(1) $(\forall\, x \in X\,)\ \langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0.$ If and only if $x = 0$

(2) $(\forall \alpha, \beta \in C)$ $(\forall x, y, z \in X)$, $\langle z, \alpha x + \beta y \rangle = \alpha \langle z, x \rangle + \beta \langle z, y \rangle$.

(3) $(\forall x, y \in X)$ $\langle y, x \rangle = \overline{\langle x, y \rangle}$

Remark (2) Says the inner product is linear in the second variable and sesquilinear ;(2) and (3) imply $\langle \alpha x + \beta y, z \rangle = \bar{\alpha} \langle x, z \rangle + \bar{\beta} \langle y, z \rangle$,so the inner product is conjugate linear in the first variable.

Definition 2.2 For $x \in X$, let $\|X\| = \sqrt{\langle x, x \rangle}$.

Cauchy-Schwartz Inequality $\forall x, y \in X$,$|\langle x, y \rangle| \leq \|x\| + \|y\|$ with equality iff x and y are linearly dependent.

Definition 2.3 An inner product space which is complete with respect to the norm induced by the inner product is called a Hilbert space.

Parallelogram law Let X is an inner product space. Then ($\forall$x, y$\in$ X)

$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$.

Polarization Identity is a crucial concept in the study of inner product spaces in mathematics. It allows us to recover the inner product from the norm of a vector space. Specially, for an inner product space V with inner product $\langle .,. \rangle$ and $\|.\|$, the polarization identity can be given by

1. In real inner product space

$$\langle u, v\rangle = \frac{1}{4}(\|u+v\|^2 - \|u-v\|^2)$$

2. In complex inner product space

$$\langle u, v\rangle = \frac{1}{4}(\|u+v\|^2 - \|u-v\|^2 + i\|u+iv\|^2 - i\|u-iv\|^2)$$

Here $\|.\|$ denotes the norm derived from inner product.

2.2 Hilbert space

A Hilbert space is a complete inner product space, which means it is a vector space equipped with an inner product and is complete with respect to the norm induced by the inner product.

Vector Space Let H be a vector space over the field of real or complex numbers. A vector space H is a set equipped with two operations: vector addition and scalar multiplication, satisfying specific axioms such as associativity, commutative, and distributive.

Inner Product An inner product on H is a function $\langle\cdot,\cdot\rangle : H \times H \to F$ (where F is R or C) that satisfies

- **Linearity**: $\langle \alpha u + \beta v, w\rangle = \alpha\langle u, w\rangle + \beta\langle v, w\rangle$ for all $v, u, w \in H$ and $\alpha, \beta \in F$
- **Conjugate Symmetry** : $\langle u, v\rangle = \overline{\langle v, u\rangle}$ for $v, u \in H$
- **Positive Definiteness:** $\langle u, u\rangle \geq 0$ with equality $\langle u, u\rangle = 0$ if and only if $u = 0$.

Norm and Completeness The norm induced by the inner product is defined as $\|u\|=\sqrt{\langle u, u\rangle}$. A Hilbert space is complete with respect to this norm, meaning that every Cauchy sequence in H converges to a limit in H .

Key Properties of Hilbert Spaces

- **Orthogonality and Orthonormal Sets** Orthogonality in Hilbert spaces generalizes the concept from finite-dimensional spaces. An orthonormal set is a set of vectors where each vector is orthogonal to the others and has unit norm.
- **Projection Theorem** The projection theorem states that for any closed subspace M of a Hilbert space H and any vector $x \in H$ there exists a unique vector $p \in M$ such that $\| x - p \|$ is minimized.
- **Completeness and Basis** A Hilbert space can have a countable orthonormal basis (in the case of separable Hilbert spaces) or an uncountable one. The existence of a basis allows for the expansion of any vector in terms of the basis vectors.
- **Riesz Representation Theorem** The Riesz representation theorem provides a one-to-one correspondence between continuous linear functional on a Hilbert space and vectors in the space. This theorem is crucial for understanding dual spaces and functional analysis.

Applications of Hilbert Spaces

1. Quantum Mechanics Hilbert spaces form the mathematical foundation of quantum mechanics, where the state space of a quantum system is described by a Hilbert space, and observables correspond to operators on this space.

2 Signal Processing In signal processing, Hilbert spaces are used to analyze and manipulate signals, with concepts like Fourier transforms being applicable within the framework of Hilbert spaces.

4.3 Functional Analysis Hilbert spaces are integral to functional analysis, providing a context for studying linear operators and their properties, such as boundedness and spectrum.

A Hilbert Space H is a complex inner product space that is complete under the associated norm. Two Hilbert spaces H_1 , H_2 $\in$ H are said to be isomorphic, if there exists a map

U: $H_1 \rightarrow H_2$ that is

i. U is on to

ii. U is one to one

iii. U is linear

Inner product preserving (meaning that $\langle U_x, U_y \rangle_{H_2} = \langle x, y \rangle_{H_1}$ for all $x, y \epsilon$ H_1.Such map is called unitary.

Examples of Hilbert spaces:

- Any finite dimensional inner product space
- $l^2=\{(x_1,x_2,x_3,\ldots):x_k\in C, \langle x,x\rangle=\sum_{k=1}^{\infty}|x_k|^2<\infty.$
- $L^2(A)$For any measurable $A\subset R^n$, with inner product
- $\langle g, f\rangle=\int_A \overline{g(x)}f(x)dx.$

In complete inner product space $C([a,b])$ with $\langle g,f\rangle=\int_A \overline{g(x)}f(x)dx$

$C([a,b])$ With this inner product is *not* complete; it is dense in $L^2([a,b])$,which *is* Complete.

2.2 Hilbert space of Dirichlet series

A Hilbert space of Dirichlet series is a complete inner product space consisting of Dirichlet series, which are formal power series of the form

$$f(s)=\sum_{n=1}^{\infty}a_n n^{-s}$$

Where s is complex variable and a_n are complex number. These spaces are typically denoted as H and consist of Dirichlet series with square summable coefficients

$$\left(\sum|a_n|^2<\infty\right).$$

In a Hilbert space of Dirichlet series H, the space is defined such that Dirichlet series are square-summable with respect to some norm. Specifically, H can be described as:

$$H = \{f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \ : \ \sum_{n=1}^{\infty} |a_n|^2 < \infty\} .$$

This definition implies that the norm in H is: $\|f\|_H = (\sum_{n=1}^{\infty} |a_n|^2)^{\frac{1}{2}}$

Weighted Hilbert Space

Now, consider a weight function $\varphi : N \to [0, +\infty]$. The weighted Hilbert space consists of Dirichlet series where the norm is defined with respect to this weight:

$$\|f\|_{H^2(\varphi)} = \left(\sum_{n=1}^{\infty} |a_n|^2 \varphi(n) \right)^{\frac{1}{2}}$$

The weight function $\varphi(n)$ impacts the space in several ways:

1. **Convergence Condition**: For $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ to be in $H^2(\varphi)$, the Dirichlet series must converge in some half-plane $Re(s) > \sigma_0$, where σ_0 depends on φ. This is because if the norm is finite, it implies that

   $|a_n|^2 \varphi(n)$ Converges, which in turn ensures that $f(s)$ converges in a certain region.

2. **Weighted Norm**: The weight function φ adjusts the norm to take into account how the coefficients a_n are distributed. For example, if $\varphi(n)$ grows too quickly, the weighted space could become trivial because the norm condition might be too restrictive.

Bergman-Type Spaces and Reproducing Kernels

In the context of weighted Hilbert spaces of Dirichlet series, there are two notable areas of study:

1. **Bergman-Type Spaces**: These are spaces where the functions are defined as having integral representations with weights. The study of Bergman-type spaces of Dirichlet series involves understanding how these spaces generalize the classical Bergman spaces.
2. **Reproducing Kernel**: For some weighted spaces, a reproducing kernel Hilbert space (RKHS) structure can be defined, which allows the use of kernel functions to characterize and study the space.

we only consider weights φ with the property that the norm boundedness implies that the Dirichlet series converges in some half plane $Re\ s > \sigma_o$.These spaces $\mathcal{H}^2(\varphi)$ are called weighted Hilbert space of Dirichlet series.

It is then of interest to study the phenomena found for $\mathcal{H}^2$ in this much wider class of space $\mathcal{H}^2(\varphi)$.

A first attempt in this direction has been made McCarthy, where he studied first Bergman-type space of Dirichlet series and second, a space with the reproducing kernel.

Chapter 3

Bounded Linear Operators

Bounded linear operator is a fundamental concept in functional analysis, specifically in the study of vector spaces and their linear mappings. Here's a detailed description,

Definition A linear operator T between vector spaces X and Y (both over the same field, usually the real numbers R or complex numbers C is a function that preserves the operations of vector addition and scalar multiplication. Formally, $T: X \to Y$ is a linear operator if it satisfies the following two properties for all $x_1, x_2 \in X$ and scalars α:

Additivty $T(x_1 + x_2) = T(x_1) + T(x_2)$ for all $x_1, x_2 \in X$

Homogeneity $T(\alpha x) = \alpha T(x)$ for all $x \in X$ and α in the field.

When X and Y are a normed vector space (i.e. they have a norm). We call T abounded linear operator if it additionally satisfies the following condition.

Bounded Linear Operators A linear operator $T \quad X \to Y$ is called bounded if there exists a constant $c \geq 0$ such that for all $x \in X$,

$$\|T(x)\|_Y \leq c\|x\|_X .$$

Here, $\|.\|_X$ and $\|.\|_Y$ are the norms X and Y respectively. The smallest such constant c is called the operator norm of T.

Normed Vector Spaces

When X and Y are normed vector spaces, they are equipped with norms $\|.\|_X$ and $\|.\|_Y$ respectively. A norm is a function that assigns a positive real number to each vector in such a way that it satisfies certain properties (positivity, scalability, and the triangle inequality).

Operator Norm:

The **operator norm** $\| T \|$ of T is the smallest such constant c formally, it is defined as:

$$\|T\| = \sup_{x \neq 0} \frac{\|T(x)\|_Y}{\|x\|_X}$$

.

Equivalently, this can be written as:

$$\|T\| = \sup_{\|x\|_{X \leq 1}} \|T(x)\|_Y$$

Properties and Significance

Equivalence of Norms: The definition of boundedness through the operator norm is equivalent to the existence of a constant c such that

$$\|T(x)\|_Y \leq c\|x\|_X \text{ , for all } x \in X.$$

This is because the supremum of

$$\frac{\|T(x)\|_Y}{\|x\|_X}$$

over all non- zero x is the least constant satisfying the inequality.

- **Continuity:** A linear operator T between normed spaces is continuous if and only if it is bounded. This is because, in normed spaces, boundedness implies that small changes in the input lead to small changes in the output, which is precisely the definition of continuity.
- **Duality:** The concept of boundedness is crucial in understanding the dual space X^* of a normed space X. Each bounded linear functional (a linear operator from X to R or C) in X has a corresponding operator norm that helps in analyzing and characterizing these functionals.
- **Compact Operators:** A related concept is that of compact operators, which a special class of bounded operators are where the image of a bounded set is relatively compact. These operators play a key role in many areas of analysis.

Examples

1. **Finite-Dimensional Spaces:** In finite-dimensional vector spaces, every linear operator is bounded. For example, any matrix A acting on R^n defines a linear operator, and its operator norm is equivalent to the matrix norm.
2. **Integral Operators:** Consider the integral operator T defined by

   $T(f)(x) = \int_a^b \mathrm{k(x,t)f(t)\,dt}$ Where $\mathrm{k(x,t)}$ is a continuous kernel function. For appropriate choices of function spaces (like L^2 spaces), T can be bounded.

3. **Differentiation Operator:** In the space of continuously differentiable functions, the differentiation operator D (where $Df = f'$) can be bounded under certain conditions. However, in general function spaces, such as L^2 spaces, differentiation might not be bounded.

In summary, bounded linear operators are fundamental in functional analysis and provide a framework to study linear mappings between normed vector spaces with control over their growth. Understanding their properties and norms is crucial for deeper insights into both pure and applied mathematics.

Chapter4

Composition operator on Hilbert space of Dirichlet series

Let $f \in H^2$ be of the form

$$f(s) = \sum_{n=1}^{+\infty} a_n\, n^{-s}, \quad s \in C_{\frac{1}{2}}.$$

Fix $k = 1,2,3,\cdots$, Then

$$f_x = f(k\,s) = \sum_{n=1}^{+\infty} a_n\, n^{-ks}, \; s \in C_{\frac{1}{2}}.$$

In other function in H^2 , of the same norm as f. In other words, if $\varphi(s) = ks$ and C_φ is the associated composition operators

$$C_\varphi(s) = f \circ \varphi(s) \quad , \quad s \in C_{\frac{1}{2}}.$$

Then C_φ is an isometry on H^2. One would tend to ask what other kind of composition operator might be around .Recently, Golden and Hedenmalm found a complete answer to this question. The space D consists of somewhere convergent Dirichlet series.

Theorem 4.1 An analytic function $\varphi : C_{\frac{1}{2}} \to C_{\frac{1}{2}}$ generates abounded composition operator C_ϕ ; $H^2 \to H^2$, if and only if It is of the form $\varphi(s) = ks + \emptyset(s)$.

Where $k \in \{0,1,2,3,\ldots\}$, $\varphi \in D$ and φ has an analytic extension to C_+ also denote

by φ. Such as

$$\varphi(C_+) \subset C_+ \quad \text{if} \quad k > 0$$

And

$$\varphi(C_+) \subset C_{\frac{1}{2}} \quad \text{if} \quad k = 0.$$

Theorem 4.2 Let $C_O \in N_O$, the set of compact composition operators with characteristic equal to C_O be arc wise connected.

Proof we fix $\varphi \in \delta$ with $\text{char}(\varphi) = C_O$ such that $C\varphi$ is compact. We first show that there exists an arc of compact composition operators between $C\varphi$ and $C_{\widetilde{\varphi}}$ for some $\widetilde{\varphi} \in \delta$ with $\widetilde{\varphi}(C+) \subseteq C_{\frac{1}{2}+\varepsilon}$ for some $\varepsilon > 0$, $char(\widetilde{\varphi}) = C_O$ and, writing $\widetilde{\varphi} = C_O S + \widetilde{\psi,}$ $\widetilde{\psi}(C_+)$ is bounded. Define $\varphi_\sigma(.) = \varphi_\sigma(. + \sigma)$.We claim that $\sigma \rightarrow C_{\varphi\sigma}$ is continuous C for the operation norm topology. The choice $\widetilde{\varphi} = \varphi_1$ will answer the problem C_O has restricted range. So that in particular, the symbols $\varphi(s) = c + rp_j^{-s}$ generate composition operators of strictly maximal norm in the class of affine symbols with the same mapping properties. Muthu Kumar, Ponnusany and Queffelec (7) have recently investigated the norm of these operators. It is of course sufficient to only consider the case $\varphi(s) = c + r2^{-s}$. They established the estimates

$$\varsigma(2\text{Re}) \leq \|\vartheta_\varphi\| \leq \varsigma(1 + \varepsilon).$$

Where

$$\varepsilon: -\left(\text{Rec} - \frac{1}{2}\right) + \sqrt{\left(\text{Rec} - \frac{1}{2}\right)^2 - r^2}$$

The lower bounded is actually a general lower bounded which holds for Dirichlet series $\varphi \in \vartheta$

$$\left\|\vartheta_{\varphi}\right\|^{2} \geq \varsigma(2\mathrm{Re})$$

Let H be the Hilbert space of Dirichlet series with square summable coefficients:

$$\mathrm{H} = \mathrm{f(s)} = \left\{\sum_{1}^{+\infty} a_n n^{-s} \ : \ \|f\|_2 = \left(\sum_{1}^{+\infty} |a_n|^2 \right)^{\frac{1}{2}} < +\infty\right\}.$$

According to the Cauchy-Schwarz inequality, the functions within the set H are holomorphic in the half-plane $C_{\frac{1}{2}}$ (where, for real θ, $C_\theta = \{s \in C: R(s) > \theta\}$ and $C_+ = C_0$). By selecting a_n = 1/(n1/2 log n)), it can be demonstrated that the functions in H are typically not defined over a broader domain. For which analytic mappings $\varphi: C_{\frac{1}{2}} \rightarrow C_{\frac{1}{2}}$ is the composition operator?

$C\varphi\ f = fo\varphi$ a bounded linear operator on H.

Theorem 4.3 An analytic function $\varphi: \ C_{\frac{1}{2}} \rightarrow C_{\frac{1}{2}}$ defines abounded composition operators

$C\varphi: \ H \rightarrow H$ if and only if it is of the form

$$\varphi(s) = c_0 s + \varphi(C_+)(s),$$

φ has an analytic extension to C_+, also denoted by φ, such that:

a) $\varphi(C_+) \subset C_+$ if $C_0 \geq 1.$

b) $\varphi(C+) \subset C_{\frac{1}{2}}$ if $C_0 = 0.$

In this statement, conditions (a) and (b) have two different meanings: Condition (a) is an arithmetic condition ($f\ o\ \varphi$ must be a Dirichlet series), whereas (b) Is an analytic condition ($fo\varphi$ must be in H).The next step in the study of composition operators on a Banach space of analytic functions is to compare the properties of the operator $C\varphi$ and of its symbol φ. We began this comparison, for example, we characterized completely the Fredholm composition operators on H: $C\varphi$ is Fredholm if and only if

$$\varphi(s) = s + i\tau, \ \tau \in R.$$

Here we consider the compactness question: What conditions should we impose on φ for $C\varphi$ to be a compact operator? We gave some sufficient conditions: If $\varphi(C_+)$ is strictly smaller than it can be, $C\varphi$ is compact. More precisely, if

- $\varphi(C_+) \subset C_\varepsilon$, $\varepsilon > 0$. For $C_0 \geq 1$. Or
- $\varphi(C_+) \subset C_{\frac{1}{2}+\varepsilon}$, $\varepsilon > 0$, for $C_0 = 0$,

Then $C\varphi$ is compact. One of our aims is to obtain less trivial sufficient conditions, and to give necessary conditions.

4.1 Growth estimate of composition operator on Hilbert space of Dirichlet series

Growth estimate of composition operator on Hilbert space of Dirichlet series depends on the transformation φ and its effect on the series. For simple cases, the estimation can be straight forward, while for complex function, a more detailed analysis is required. If you have a specific function φ or type of transformation in mind, that could help narrow down the growth estimate further.

Definition 4.4 In Hilbert space$\mathcal{H}$, the reproducing kernel at a in $C_{\frac{1}{2}}$, is give by

$$\mathcal{K}_a(s) = \sum_{n=1}^{\infty} n^{\overline{-a}} n^{-s}.$$

In other words, for a point a in $C_{\frac{1}{2}}$, the reproducing Kernel function $\mathcal{K}_a$ is the function in $\mathcal{H}^2$.

such that for f in $\mathcal{H}^2$, we have $\langle f, \mathcal{K}_a \rangle = f(a)$, f and $\mathcal{K}_a$ are in $\mathcal{H}^2$.

So,

$$f(s)= f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \quad \text{and} \quad \mathcal{K}_a(s) = f(s) = \sum_{n=1}^{\infty} b_n n^{-s},$$

For some coefficients.

Thus, for f in $\mathcal{H}^2$,

$$\sum_{n=1}^{\infty} a_n \overline{b_n} = \langle f, \mathcal{K}_a \rangle = f(a) = \sum_{n=1}^{\infty} a_n n^{-a},$$

$$b_n = \overline{n^{-a}} = n^{\bar{a}}.$$

Therefore,

$$\mathcal{K}_a(s) = \sum_{n=1}^{\infty} n^{\overline{-a}} n^{-s}.$$

Lemma 4.1 For a composition operator $C_\varphi \mathcal{H}^2 \to \mathcal{H}^2$ and a reproducing Kernel $\mathcal{K}_a$.

We have

$$C_\varphi^* \mathcal{K}_a = \mathcal{K}_{C_\varphi}.$$

Problem Suppose $\sigma = \varphi(+\infty) \in C_{\frac{1}{2}}$ and $\varphi(C_+) \subset C_{\frac{1}{2}}$. Try to estimate from above the norm $\|C_\varphi\|^2$ in terms of σ. Note that it is clear that

$$\zeta\big(2(\mathrm{Re}\sigma + 1)\big) \leq \|C_\varphi\|^2.$$

Solution we try to proof the lower and upper estimates given in this problem. For the lower bounded, let use lemma (2.4). Then it follows that

$$\left\|\mathcal{K}_{\varphi(a)}\right\|_{\mathcal{H}^2} = \left\|C_\varphi^*\mathcal{K}_a\right\|_{\mathcal{H}^2} \leq \left\|C_\varphi^*\right\|\|\mathcal{K}_a\|_{\mathcal{H}^2}$$

$$= \left\|C_\varphi\right\|\|\mathcal{K}_a\|_{\mathcal{H}^2} \text{ ....(1)}$$

The norm of the reproducing Kernel is given by

$$\|\mathcal{K}_a\|_{\mathcal{H}^2}^2 = \langle f, \mathcal{K}_a \rangle = \mathcal{K}_a(a) = \sum_{n=1}^{\infty} n^{\overline{-a}} n^{-a}$$

$$\mathcal{K}_a(a) = \sum_{n=1}^{\infty} n^{\overline{-a}} n^{-a} = \sum_{n=1}^{\infty} n^{-(\bar{a}+a)} = \zeta(\bar{a} + a) = \zeta\big(2\mathrm{Re}\,(a)\big).$$

Letting a go to infinity yields

$$\|\mathcal{K}_a\|_{\mathcal{H}^2}^2 = 1.$$

$$\left(\text{i.e.} \lim_{a\to\infty} \zeta\left(2\mathrm{Re}\,(\,a\,)\right)\right) = \lim_{a\to\infty}\sum_{n=1}^{\infty} n^{-2\mathrm{Re}\,a} = \sum_{n=1}^{\infty} \lim_{a\to\infty} n^{-2\mathrm{Re}\,a}$$

$$= 1 + 2^{-\infty} + 3^{-\infty} + \cdots = 1.$$

Now from (1) we have that

$$\left\|C_\varphi\right\| \geq \left\|\mathcal{K}_{\varphi(a)}\right\|_{\mathcal{H}^2}.$$

But $\left\|\mathcal{K}_{\varphi(a)}\right\|_{\mathcal{H}^2}^2 = \left\|\sum_{n=1}^{\infty} n^{\overline{-\varphi(a)}} n^{-a}\right\|_{\mathcal{H}^2}^2$

$$= \sum_{n=1}^{\infty} \left|n^{\overline{-\varphi(a)}}\right|^2$$

$$= \zeta\big(2\mathrm{Re}\ (\varphi(a)\big)$$

$$= \zeta\big(2\mathrm{Re}\ (\varphi(+\infty)\big).$$

Where $a = +\infty$.

$$\|C_\varphi\| \geq \|\mathcal{K}_{\varphi(a)}\|_{\mathcal{H}^2} = \sqrt{\zeta\big(2\mathrm{Re}\ (\varphi(+\infty)\big)}$$

$$\sqrt{\zeta\big(2\mathrm{Re}\ (\varphi)\big)} \quad \leq \|C_\varphi\|, \quad \text{Where} \quad a = \varphi(+\infty)$$

$$\zeta\big(2\mathrm{Re}\ (\varphi)\big) \quad \leq \|C_\varphi\|^2.$$

To estimate the upper bounded

Let $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ and $\varphi(s) = u(s) + ⅈv(s)$

Let $u(s) = \mathrm{Re}\ (\varphi(s))$

$$\|C_\varphi\| \quad = \quad \sup|C_\varphi f| = \sup|f o \varphi|$$

$$= \sup\left|\sum_{n=1}^{\infty} a_n n^{-\varphi(s)}\right|$$

$$\leq \sup \sum_{n=1}^{\infty} |a_n| |n^{-\varphi(s)}|$$

$$= \sum_{n=1}^{\infty} |a_n|^2\, n^{-\mathrm{Re}\ (\varphi(s))}$$

$$= \sum_{n=1}^{\infty} |a_n|^2\, n^{-2s} = \sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2s}} = M$$

$$\|C_\varphi\| \leq M,$$

Where $M = \sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2s}} = \varphi(+\infty)$

The finally $\|C_\varphi\| \leq \varphi(+\infty)$.

There for, we will proof the lower and upper estimation of the above problem.

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