# ADDIS ABABA UNIVERSITY

## SCHOOL OF GRADUATE STUDIES

## DEPARTMENT OF MATHEMATICS



GRADUATE SEMINAR REPORT

on

# Duality in Vector Optimization Problem

(submitted in partial fulfillment of M.Sc. degree in mathematics)

by

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CONTENTS

## Preface

Human beings have been confronted with multiple criteria decision making problems. Our food should test good, smell good, look good and be nutritious. We want to have a good life, which may mean more wealth, more power, more respect and more time for our selves, together with a good health and a good second generation, e.t.c. Indeed, in the records of human culture, all important political, economical and cultural events have involved multiple criteria in their evolution.

Unlike single objective optimization problems in solving multi-objective optimization problem, we have solution set that is called efficient set. It is from this set decision is made by taking elements of efficient set as alternatives, which is given by analysts.

This graduate seminar report contains seminar I and II together for qualification of M.Sc. programs in mathematics. Fundamental notions in multi-objective decision making and its historical back ground are briefly explained in chapter one. Furthermore the mathematical theories in multi-objective optimization of existence necessary and sufficient condition of efficient solutions and duality are explained in three chapters.

I want to express my deepest gratitude to Dr. Semu Mitiku, my advisor for his very useful advice, suggestion and material support. Finally I am grateful to my friends and colleagues for their indispensable help in preparing this graduate seminar report. Particularly I would like to express my thanks to my family for their financial support; to Nega Dubre and Dereje Bekele for their friendship and active cooperation.

# CHAPTER ONE

## 1. Introduction and Problem Formulation.

Every day we encounter various kinds of decision making problems as managers, designers, administrative officers, mere individuals, and so on. In these problems, the final decision is usually made through several steps; the structure model, the impact model, and the evaluation model even though they sometimes might not be perceived explicitly.

By structure modeling, we mean constructing a model in order to know the structure of the problem, what the problem is, which factors comprise the problem, how they interrelate, and so on. Through the process, the objective of the problem and alternatives to perform it are specified. Hereafter, we shall use the notation O for the objective and X for the set of alternatives, which is supposed to be a subset of an n–dimensional vector space.

In order to solve our decision making problem by systems –analytical methods, we usually require that degrees of objectives be represented in numerical terms, which may be of multiple kinds even for one objective. We restrict numerical terms to physical measures (for example money, weight, length, time,…).As a performance index, for the objective $O_i$ an objective function $f_i : X \to R^1$ is introduced. The values of $f_i$(x) indicate how much impact is given on the objective $O_i$, by performing an alternative x. In this report we assume that a smaller value for each objective function is preferred to a larger one.

Now we formulate our decision making problems as a multi-objective (vector) optimization problem:

(P) Minimize $f(x) = (f_1(x), f_2(x),..., f_p(x))$ over $x \in X$.

In some cases, some of the constraint functions are required to minimize other objective functions. We denote these objective functions $g_j(x) \leq 0$, for j= 1,…m. We will consider the problem (P) itself or (P) accompanied by the constraint $g_j(x) \leq 0$, j = 1,…m.

Of coarse, an equality constraint $h_k(x) = 0$ can be embedded within two inequalities $h_k(x) \leq 0$ and $-h_k(x) \leq 0$,and, hence it doesn't appear in this report.

Unlike the traditional mathematical programming with a single objective function, an optimal solution in the sense of one that minimizes all the objective functions simultaneously does not necessarily exist in multi-objective optimization problems, and, hence, we are in trouble of conflicts among objectives in decision making problems with multiple objectives, The final decision should be made by taking the total balance of objectives in to account. Here we assume a decision maker who is responsible for the final decision. The decision maker's value is usually represented by saying whether or not an alternative x is preferred to another alternative $x'$, or equivalently whether or not f(x) is preferred to f($x'$). In other words, the decision maker's value is represented by some binary relation over X or f(X). Since such a binary relation representing the decision maker's preference usually becomes an order, it is called a *preference order* and it is supposed to be defined on the so called criteria space Y, which include the set f(X). Several kind of preference orders will be possible, sometimes, the decision maker cannot judge whether or not f(x) is preferred to $f(x')$. Such an order that admits incomparability for a pair of objects is called partial order, where as the order requiring the comparability for every pair of objects is called a weak order (or total order). In practice, we often observe a partial order for the decision maker's preference. Unfortunately, however, an optimal solution in the sense of one that is more preferred with respect to the order, hence the notion of optimality does not necessarily exist for partial orders. Instead of strict optimality, we introduce in multi-objective optimization the notion of efficiency. A vector $f(x')$ is said to be efficient if there is no f(x), $x \in X$ preferred to $f(x')$ with respect to the preference order. The final decision is made among the set of efficient solutions.

This report is mainly concerned with some of the theoretical aspects in vector optimization problems; in particular we will focus on existence, necessary and sufficient conditions of efficient solutions.

Chapter two is devoted to mathematical notions and preliminaries. The first section gives a review of convex sets, cones, convex functions and other properties related to convexity, which have important role in multi-objective optimization .The second section, introduce point- to- set map that play a very important role, since efficient solutions usually constitutes a set. The third section is concerned with a brief explanation of preference order. These concepts are fundamental for existence and necessary/sufficient condition for efficient solutions.

Chapter three begins with the introduction of several possible concepts for solutions in multi-objective optimization. Above all, efficient solutions will be the subject of primary consideration in subsequent theories. Next, some properties of efficient solutions, such as existence and external stability will be discussed.

Chapter four will be devoted to the duality theory in multi-objective optimization. Duality is a fruitful result in traditional mathematical programming and is very useful both theoretically and practically. Consequently, it is quit interesting to extend the duality theory to the case of multi-objective optimization. In first section the duality theory in nonlinear cases will be discussed in parallel with the case of ordinary convex programming. Given a convex multi-objective programming problem, some new concepts such as the primal map, the dual map, and the vector valued Lagrangian will be defined. The Lagrange multiplier theorem, the saddle point theorem, and the duality theorem will be obtained. The second section will be devoted in deriving first order necessary and sufficient conditions for unconstrained cone d.c. programming problems. These conditions are given in terms of directional derivative and subdifferentials of component functions. Moreover, conjugate duality for cone d.c. optimization is discussed and weak duality theorem is proved in a more general partially ordered linear topological vector space.

# CHAPTER TWO

## 2. Mathematical Preliminaries

This chapter is devoted to mathematical preliminaries. First, some fundamental results in convex analysis, second, the concepts of continuity and convexity of vector valued point –to –set maps are introduced. Finally the concepts of preference order, domination structure, and none dominated solutions are introduced to provide a way of specifying solutions for multi-objective optimization.

## 2.1. Elements of Convex Analysis

As it is well known that, convexity plays a fundamental role in the theory of optimization with single objective. Since they are also fundamental in theory multi-objective optimization problems, some elementary results are summarized below. All spaces considered are finite-dimensional Euclidean spaces.

### 2.1.1 Convex Sets

In this subsection, we will first look at convex set, which are fundamental in convex analysis.

A set $D \subset R^n$ is said to be convex if $\alpha x_1 + (1-\alpha) x_2 \in D$ for any $x_1, x_2 \in D$ and for any $\alpha \in [0, 1]$. The intersection of all convex sets containing a given subset D of $R^n$ is called the convex hull of D and is denoted by co D. D is also called a cone if $\alpha x \in D$, for $x \in D$ and $\alpha>0$. Further more, D is a convex cone when it is also convex.

Proposition: a set D in $R^n$ is a convex cone iff

(i). $x^1 \in D$ and $\alpha > 0$ imply $\alpha x^1 \in D$,

(ii). $x^1, x^2 \in D$ imply $x^1 + x^2 \in D$

Proof: ($\Rightarrow$) suppose D in $R^n$ be convex cone then (i) follows from definition of a cone.

Since D is a convex cone for any $x^1, x^2 \in D$, $\frac{1}{2}x^1 + \frac{1}{2}x^2 \in D$ and also

$2(\frac{1}{2}x^1 + \frac{1}{2}x^2) = x^1 + x^2 \in D$.

($\Leftarrow$) suppose (i) and (ii) holds true. Then for arbitrary $x^1, x^2 \in D$ and $\alpha \in [0,1]$ from (i) we have that $\alpha x^1 \in D$ and $(1-\alpha)x^2 \in D$. Using (ii) also we get $\alpha x^1 + (1-\alpha)x^2 \in D$. Hence D is convex cone.

A cone $D \subset R^n$ is said to be pointed if $-x \notin D$ when $x \neq 0$ and $x \in D$. That is, $D \cap (-D) = \{0\}$. More over D is said to be acute if there is an open half space $\overset{o}{H^+} = \{x \in R^n : \langle x, x^* \rangle > 0\}$ $x^* \neq 0$ such that $clD \subset \overset{o}{H^+} \cap \{0\}$. The positive polar and strict positive polar of D, denoted by $D^0$ and $D^{so}$ are defined by

$$D^0 = \{x^* \in R^n : \langle x, x^* \rangle \geq 0 \text{ for any } x \in D\};$$

$$D^{so} = \{x^* \in R^n : \langle x, x^* \rangle > 0 \text{ for any non zero } x \in D\} \text{ respectively.}$$

A set D in $R^n$ is said to be a polyhedral convex set if it can be expressed as intersection of finite collection of closed half spaces i. e if

$$D = \{x : \langle b^i, x \rangle \leq \beta_i \ (i = 1,\ldots, m)\}, \text{ Where } b^i \in R^n \text{ and } \beta_i \in R \ (i=1,\ldots,m).$$

Further more, if $\beta_i = 0$ for all $i=1,\ldots, m$ in the above expression, D is said to be a polyhedral convex cone.

Note that given a set D and convex cone K in $R^p$, D is said to be K- convex if X + K is a convex set. Note also that a set D is convex if and only if D is $\{0\}$-convex. Moreover, if D is a convex set, it is K-convex for an arbitrary, nonempty convex cone K.

## 2.1.2 Convex Functions

In the following, we consider an extended real valued function f from $R^n$ to $[-\infty,+\infty]$ Let f be a function from $X \subset R^n$ to $[-\infty, +\infty]$. The set $\{(x, \alpha) \in X \times R : f(x) \leq \alpha\}$ is called the epigraph of f and is denoted by epif. A function f is said to be a convex function on X if epif is convex as a subset of $R^{n+1}$. A concave function on X is a function whose negative is convex. An affine function on X is a function which is finite, convex, and concave.

The effective domain of a convex function f on X, denoted by dom f is given by $\{x \in X$: there exist $\alpha \in R$ such that $(x, \alpha) \in \text{epi } f\} = \{x \in X: f(x) < +\infty\}$

A convex function f on X is said to be proper if $f(x) < (+\infty)$ for at least one $x \in X$, and if $f(x) > (-\infty)$ every where. Let X be convex set in $R^n$, f be function from X in to $R^p$ and D be a convex cone in $R^p$. Then f is said to be D-convex if for $x^1, x^2 \in X$ and $\alpha \in [0, 1]$

$$\alpha f(x^1)+(1-\alpha)f(x^2) - f(\alpha x^1 +(1-\alpha) x^2) \in D$$

Proposition: let X be a convex set in $R^n$, f be a function from $R^n$ in to $R^p$ and D be a convex cone in $R^n$. If the function f is D-convex, then the set f(X) is D-convex.

Proposition: let X be a convex set in $R^n$, and $f= (f_1,\ldots,f_p)$ be a function from $R^n$ in to $R^p$. The function f is $R^p_+$-convex if and only if each $f_i$ is convex, and in this case f(X) is $R^P_+$-convex.

Proof: ($\Rightarrow$) Let f be $R^p_+$-convex function.

If $x, y \in R^n$, then $\alpha f(x) +(1-\alpha)f(y)-f[\alpha x+(1-\alpha)y] \in R^p_+$, for $\alpha \in [0,1]$,which implies

$\alpha(f_1(x), f_2(x),\ldots, f_p(x))+(1-\alpha)( f_1(y), f_2(y),\ldots, f_p(y))-( f_1[\alpha x+(1-\alpha)y], f_2 [\alpha x+(1-\alpha)y],\ldots\ldots, f_p [\alpha x+(1-\alpha)y]) \in R^p_+$

$\Rightarrow(\alpha f_1(x) + (1-\alpha)f_1(y),\ldots, \alpha f_p(x)+ (1-\alpha) f_p(y))-( f_1[\alpha x+(1-\alpha)y], f_2 [\alpha x+(1-\alpha)y],\ldots, f_p [\alpha x+(1-\alpha)y]) \in R^p_+$

$\Rightarrow \alpha f_i(x)+ (1-\alpha)f_i(y) - f_i[\alpha x+(1-\alpha)y] \in R^p_+$.

Hence $f_i$ is convex for each i.

($\Leftarrow$) Suppose $f_i$ is convex for each i.

$\alpha f(x) +(1-\alpha)f(y)-f[\alpha x+(1-\alpha)y]= (\alpha f_1(x)+ (1-\alpha)f_1(y),\ldots, \alpha f_p(x)+ (1-\alpha) f_p(y))-( f_1[\alpha x+(1-\alpha)y], f_2 [\alpha x+(1-\alpha)y],\ldots, f_p [\alpha x+(1-\alpha)y])$ .And since $f_i$ is convex for each i

$\alpha f_i(x) +(1-\alpha)f(y)-f_i[\alpha x+(1-\alpha)y] \in R^p_+$ for each i.

Therefore $\alpha f(x) + (1-\alpha)f(y)-f[\alpha x+(1-\alpha)y] \in R^p_+$.

## 2.1.3 Conjugate Functions

Let f be a convex function from $R^n$ to [-∞, +∞]. The function $f^*$ on $R^n$ defined by

$$f^*(x^*) = \sup\{\langle x, x^*\rangle - f(x): x \in R^n\},\ x^* \in R^n$$

is called the conjugate function of f. The conjugate of $f^*$, i.e the function $f^{**}$ on $R^n$ defined by

$$f^{**}(x) = \sup\{\langle x, x^*\rangle - f^*(x^*): x^* \in R^n\},\ x \in R^n$$

is called the biconjugate function of f.

Proposition2.1.3 (Fenchel's inequality)

Let f be a proper convex function. Then $f(x) + f^*(x^*) \geq \langle x, x^*\rangle$ for any x and x*.

Proof: directly follows from the definition.

## 2.1.4 Subgradients of Convex Functions

Let f be a function from $R^n$ to [-∞, +∞], and let x be a point at which f is finite. The one-side directional derivative of f at x with respect to a vector d is defined to be the limit

$$f'(x, d) = \lim_{t \to 0^+} \frac{f(x+td) - f(x)}{t} \quad \text{if it exist.}$$

If f is actually differentiable at x, then $f'(x, d) = \langle \nabla f(x), d\rangle$ for any d. ∇f(x) is the gradient of f at x.

Let f be a convex function from $R^n$ to [-∞, +∞].A vector $x^* \in R^n$ is said to be a subgradient of f at x if

$$f(x') \geq f(x) + \langle x^*, x' - x\rangle \quad \text{for any } x^* \in R^n$$

The set of all subgradients of f at x is called the subdifferential of f at x and is denoted by $\partial f(x)$. If $\partial f(x)$ is not empty, f is said to be subdifferentiable at x.

Proposition: Let f be a convex function from $R^n$ to [-∞, +∞].Then

$f(x') = \text{Min}\{f(x): x \in R^n\}$ iff $0 \in \partial f(x')$.

Proposition: Let f be a convex function and suppose that f(x) is finite .Then x* is a subgradient of f at x iff

$f'(x, d) \geq \langle x^*, d \rangle$ for any d.

## 2.2. Point –to –set Maps

A point-to-set map F from a set X into a set Y is a map that associates a subset of Y with each point of X. Equivalently, F can be viewed as a function from the set X into the power set of Y ($2^Y$).

In multi-objective optimization problem, it is difficult to obtain a unique optimal solution. Solving the problem often leads to a solution set. Thus, if the problem has a parameter, the solution set defines a point-to-set map from parameter space into the objective space.

Deffinition2.2.1: If F is a point-to-set map from a set X into a set Y, then F is said to be

(i) Lower semicontinuous (l.s.c) at a point $x \in X$ if $\{x^k\} \subset X$, $x^k \to x$, and $y \in F(x)$ all implies the existence of an integer m and a sequence $\{y^k\} \subset Y$ such that $y^k \in F(x^k)$ for $k \geq m$ and $y^k \to y$.

(ii) Upper semicontinuous (u.s.c) at a point $x \in X$ if $\{x^k\} \subset X$, $x^k \to x$, $y^k \in F(x^k)$, and $y^k \to y$ imply that $y \in F(x)$.

(iii). Continuous at a point $x \in X$ if it is both l.s.c and u.s.c at x; and

(iv). L.s.c. (resp.u.s.c, continuous) on $X' \subset X$ if it is l.s.c. (resp.u.s.c, continuous) at every $x \in X'$.

Deffinition2.2.2 Let F be a point-to-set map from $R^n$ into $R^p$ and D be a convex cone in $R^p$ .The set $\{(x,y): x \in R^n, y \in R^p, y \in F(x)+D\}$is called the D-epigraph of F and is denoted by D-epi F. Note that for a convex cone D in $R^p$, F is said to be D-convex if D-epi F is convex.

## 2.3 Preference Order and Domination Structure

In ordinary single- objective optimization problems, the meaning of optimality is clear. I usually mean maximization or minimization of a certain objective function under given constraints. In multi objective optimization problems, on the other hand, it is not clear. Let us consider the case in which there is finite number of objective functions each of which is to be minimized. If there exist a feasible solution, action or alternative that minimizes all of the objective functions simultaneously, we will have an objection to adopting it as the optimal solution. However, we can rarely expect the existence of such an optimal solution, since the objectives usually conflict with one another. Thus in multi-objecting optimization problems, the preference attitudes of the decision maker play an essential role that specifies the meaning of optimality or desirability. They are very often represented as binary relations on the objective space and are called preference orders.

### 2.3.1 Preference Order

A preference order represents preference attitude of the decision maker in the objective space. It is a binary relation on a set Y=f(X) $\subset R^p$ where f is the vector- valued objective function, and X is feasible decision set.

The basic binary relation $\succ$ means strict preference, i.e. $y \succ z$ for y, z $\in$Y means objective value y is preferred to z. We define two binary relation $\sim$ and $\succeq$ in relation with strict preference ($\succ$), as

$$y \sim z \text{ iff not } y \succ z \text{ and not } z \succ y,$$

$$y \succeq z \text{ iff } y \succ z \text{ or } z \sim y$$

The relation $\sim$ is called indifference(y $\sim$ z as y is indifference to z), and $\succeq$ is called preference –indifference(y $\succeq$ z as z is not preferred to y).

The binary relation used as preference or indifference orders have interesting properties. Some of these properties of binary relations are listed below.

A binary relation R on a set Y is :

(1). Reflexive if y R y for every y $\in$Y

(2). Irrefilexive if not y R y for every y $\in$Y

(3). Symmetric if y R z$\Rightarrow$z R y, for every y, z $\in$Y

(4). Asymmetric if y R z$\Rightarrow$ not z R y, for every y, z $\in$Y

(5). AntiSymmetric if (y R z, z R y) $\Rightarrow$y = z, for every y, z $\in$Y

(6). Transitive if (y R z, z R w) $\Rightarrow$y R w, for every y, z, w $\in$Y

(7). Negatively transitive if (not y R z, not z R w) $\Rightarrow$ not y R w, for every y, z, w $\in$Y

(8). Connected or complete if y R z or z R y (possibly both) for every y, z$\in$Y

(9). Weakly connected if y $\neq$ z $\Rightarrow$ (y R z or z R y) throughout Y

The preference order is usually assumed to be at least a strict partial order, i.e. irreflexiveity of preference (y is not preferred to itself) and transitivity of preference.

Lemma 2.3.1 Let R be a binary relation on Y.

(i) If R is transitive and irreflexive, it is asymmetric.

(ii) If R is negatively transitive and asymmetric, it is transitive

(iii) If R is transitive, irrefilexive, and weakly connected, it is negatively transitive

Definition 2.3.1 a binary relation R on a set Y is said to be

(i). a strict partial order if R is irrefilexive and transitive

(ii). A weak order if R is asymmetric and negatively transitive, and

(iii). A total order if R is irrefilexive, transitive and weakly connected

From Lemma 2.3.1 and Definition 2.3.1 one can see that a binary relation R is:

(i). a strict partial order if it is asymmetric

(ii). A weak order if it is transitive

(iii). A total order if it is negatively transitive

Definition2.3.2 **:** Let Y be feasible set in the objective space $R^p$ and let $\succ$ be a preference order on Y then the element $y^{'}$ $\in$ Y is said to be an efficient element of Y with respect to the order $\succ$ if there is no y $\in$ Y such that y $\succ$ $y^{'}$ .

The set of all efficient element is denoted by $\xi$(Y, $\succ$) and

$\xi$(Y, $\succ$) = { $y^{'}$ $\in$ Y : there is no y $\in$ Y such that y$\succ$ $y^{'}$ }.

Our aim in multi-objective optimization problem is to find the set of efficient elements (usually not singleton)

### 2.3.2 Denomination Structure

Preference orders (and more generally, binary relationships) on a set Y can be represented by a point -to -set map form Y in to Y. In fact, a binary relationship may be considered to be a subset of the product set Y x Y, and so it can be regarded as a graph of a point -to -set map form Y in to Y.That means we identify the preference order $\succ$ with the graph of the set valued map P:

$$P(y) = \{ y' \in Y : y \succ y' \}$$

[P(y) is the set of the elements in Y less preferred to y]

Another way of representing a preference order by point- to -set maps is the concept of domination structure (ordering cone)

For each $y \in Y \subset R^p$ we define the set of domination factor by

$$D(y) = \{d \in R^p : y \succ y + d\} \cup \{0\}$$

i.e. deviation d from y is less preferred to original y .

Then the point –to –set map D: $Y \rightarrow R^p$ represents preference order and we can call D domination structure (ordering cone).

Given a set Y in $R^p$ and a domination structure D (.). The set of all efficient elements is defined by $\xi(y, D) = \{ y' \in Y$: there is no $y \neq y' \in Y$ such that $y' \in y + D(y)\}$.

The set $\xi(y, D)$ is called the efficient set.

## Remark 2.3.1

We can introduce a domination structure $D'$ on a given domination structure D on Y as follows:

$$D'(x) = \{ d' \in R^n : f(x + d') \in f(x) + D(f(x)) \setminus \{0\} \} \cap \{0\}$$

If we denote $D' = f^{-1}(D)$ and the set of efficient solution in the decision space is

$$\{x : f(x) \in \xi(y, D)\} = \xi(x, f^{-1}(D))$$

The most important and interesting special case of domination structures is when D(.) is a constant point – to- set map, particularly when D(y) is a constant cone for all y. When D(y) = a cone D.

D as preference order is :

(i). Asymmetry iff $d \in D, d \neq 0 \Rightarrow -d \notin D$ iff D is pointed

(ii). Transitive iff $d, d' \in D \Rightarrow d + d' \in D$ iff D is convex.

Pointed convex cones are often used to define domination structure and write

$y \leq_D y'$ iff $y' - y \in D$ for $y, y' \in R^p$ and

$y <_D y' \Rightarrow y' - y \in D$ but $y - y' \notin D$ for a convex cone D in $R^p$.

If D is pointed $y <_D y'$ iff $y' - y \in D\backslash\{0\}$.

If $D = R_p^+$ the subscript D is omitted and

$y \leq y'$ iff $y_i \leq y'_i$ for all i =1,…,p

$y < y'$ iff $y \leq y'$ and $y \neq y'$

i.e $y_i \leq y'_i$ for all i =1,…,p

$y_i < y'_i$ for some i =1,…,p

Moreover we write

$y < y'$ iff $y_i < y'_i$ for all i = 1,..,p.

Lemma2.3.1: Let Y be a set and D be a pointed convex cone in $R^p$. Then

$y^1 \leq_D y^2$ and $y^2 <_D y^3$ imply $y^1 <_D y^3$,

$y^1 <_D y^2$ and $y^2 \leq_D y^3$ imply $y^1 <_D y^3$,

Or in the form of contraposition,

$y^1 \not<_D y^3$ and $y^1 \leq_D y^2$ imply $y^2 \not<_D y^3$

$y^1 \not<_D y^3$ and $y^2 \leq_D y^3$ imply $y^1 \not<_D y^2$

Proof: The result is immediate from the fact that

$D + D\backslash\{0\} = D\backslash\{0\}$ for a pointed convex cone D.

# CHAPTER THREE

## 3. Solution Concepts and Some Properties of Solutions.

In this section we discuss solution concepts for multi-objective optimization problems and investigate some properties of solution. Efficiency and proper efficiency are introduced as solution concepts in the first section. In the second section, existence and external stability of efficient solution are discussed.

## 3.1 Solution concepts

The concepts of optimal solutions to multi-objective optimization problems are closely related to the preference attitudes of the decision makers. The most fundamental solution concept is that of efficient solutions with respect to the domination structure of the decision maker.

### 3.1.1 Efficient Solutions

We consider the multi-objective optimization problem

(P) Minimize $f(x)=(f_1(x), f_2(x),..., f_p(x))$ subjected to x $\in$ X $\subset$ $R^n$ and

$Y = f(X) = \{y : y = f(x), x \in X\}$

A domination structure representing a preference attitude of the decision maker is supposed to be given as point –to- set map D from Y to $R^n$.

Definition 3.1.1 A decision vector $x'$ $\in$ X is said to be an efficient solution to the multi-objective optimization problem (P) with respect to the domination structure D if $f(x') \in \xi(Y, D)$. This means if there is no x ε X such that $f(x') \in f(x) + D$ and $f(x') \neq f(x)$ (i.e. such that $f(x') \in f(x) + D \setminus \{0\}$.

A decision vector $x' \in$ X is *pareto optimal* (D= $R^p_+$) if there does not exist another decision vector x $\in$ X such that $f_i(x) \leq f_i(x')$ for all i=1,2,...p and $f_i(x) < f_i(x')$ for at least one objective function . In this case, $(f(x')-R_+^p)\cap(Y)=\{f(x')\}$ or equivalently $(Y-f(x'))\cap(-R_+^p)=\{0\}$. $x' \in$ X is also said to be *weak pareto optimal* solution to the problem (P) if there does not exist another decision vector x $\in$ X such that $f(x)<f(x')$.

This means, $(Y-f(x'))\cap(-\operatorname{int} D)=(Y-f(x'))\cap(-D\setminus\{0\}=(Y-f(x')\cap\left(-\overset{o}{R_+^p}\right)=\varnothing$

(or if there is no another x$\in$X such that $f_i(x)<f_i(x')$ for all i=1,…,p).

Proposition3.1.1: Let $D_1$ and $D_2$ be domination structures. Then $D_1$ is said to be included by $D_2$ if $D_1(y)\subset D_2(y)$ for all y $\in$Y in this case $\xi$(Y, $D_2$) $\subset \xi$(Y, $D_1$).
When D is a constant set valued map (whose value is a convex cone), we identify the map (domination structure) with the cone D. Then $x' \in$X is an efficient solution to the problem (P) iff there is no x$\in$X such that f($x'$) - f(x) $\in D\setminus\{0\}$.

Proposition3.1.2 Let Y and Z be two sets in $R^p$, and let D be a constant ordering cone on $R^p$ then $\xi$(Y+Z,D) $\subset \xi$(Y,D) +$\xi$(Z,D).

Proof: Let y* $\in \xi$(Y+Z,D),then y*=y + z for some y$\in$Y, z$\in$Z. We have to show y$\in \xi$(Y, D) and z$\in \xi$(Z,D).

Suppose not, there is $y' \in$Y and a nonzero d$\in$D such that y = $y'$ + d.

Then y* = $y'$ + z + d and $y'$ + z$\in$ Y+Z which contradicts the supposition y* $\in \xi$(Y+Z, D).
Similar for Z.

Note that the converse inclusion of Proposition 3.1.2. is not always true.

Example:

Let Y=Z= $\{(y^1, y^2): y^1+y^2\leq 1\}\subset R^2$ and $D=R^2_+$,then y=(-1,0) $\in \xi$(Y,D) and z=(0,-1) $\in \xi$(Z,D).However y + z =(-1,-1)> $(-\sqrt{2},-\sqrt{2})=(-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2})+(-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2})\in Y+D$.

Proposition3.1.3 Let Y be a set in $R^p$, D be a cone in $R^p$ and $\alpha$ be a positive number, then $\xi(\alpha Y,D) = \alpha\, \xi(Y,D)$.

Proof: If $\alpha = 1$ the statement holds true trivially.

If $\alpha \neq 1$, then $y \in \xi(\alpha Y,D)$ implies there is $y' \in Y$ such that $y = \alpha\, y'$ it follows that $\alpha\, y' \in \alpha\, \xi(Y,D)$ implies $y' \in \xi(Y,D)$

## 3.1.2 PROPERLY EFFICIENT SOLUTIONS

Recall that ξ(y, D) is a set that means, the decision maker has to choose an alternative among infinitely many optimal solutions. So we need to have a relatively smaller size optimal solution set. Although pareto optimal solutions are important for theoretical considerations, they are not always useful in practice because of its big size, therefore it needs a more restricted concept than efficient (pareto optimal) solutions, which are properly efficient solutions.

The concept of properly efficient solutions is studied and developed by different scholars such as Borwein, Benson, Henig, Geofrion, Kuhn –Tucker and others

DEFINITION3.1.4 (Kuhn –Tucker proper efficiency)

Consider multi-objective programming problem

(P) Min $f(x) = (f_1(x)\ldots\ldots\ldots f_p(x))$

S.t $x \in X = \{x: g(x) = (g_1(x)\ldots\ldots g_m(x) \leq 0\}$

Assume that all $f_i$ and $g_i$ are continuously differentiable, $x'$ is properly efficient solution of (P) if it is efficient and there is no $h \in R^n$ such that

$< \nabla f_i(x'), h > \leq 0$ for any $i = 1,\ldots,p$

$< \nabla f_i(x'), h > < 0$ for some i

$< \nabla g_i(x'), h > \leq 0$ for any $j \in J(x') = \{ j: g_j(x') = 0 \}$

## 3.2 Existence and External Stability of Efficient Solution

Recall that in ordinary optimization problem,

Minimize f(x) Subject to $x \in X \subset R^n$

The existence of optimal solution x' is guaranteed if X is compact and the objective function f is lower semicontinuous.

This idea can be extended to vector optimization problem (P).

### 3.2.1 Existence of Efficient Solution

Existence of efficient solution requires the property of acylicity of the domination structure.

Definition 3.2.1 A domination structure D is said to be acyclic if it has no cycle. That means for n= 1,2,.... it never occurs that

$y^1 \in y^2+D(y^2)\backslash\{0\}, y^2 \in y^3+D(y^3)\backslash\{0\}, \ldots, y^n \in y^1+D(y^1)\backslash\{0\}$

In other words,( $y^1 \prec y^2 \prec \ldots \prec y^n \prec y^1$ should not hold.)

Note that a domination structure D is asymmetric if it is acyclic. Conversely, every transitive and asymmetric domination structure is acyclic.

Theorem 3.2.1: If a domination structure D on Y is acyclic, the set $D(y)\backslash\{0\}$ are open and Y is nonempty and compact, then $\xi(Y,D) \neq \phi$.

Proof: Suppose the contrary, that is $\xi(Y, D) = \phi$, then for any $y \in Y$ there exist $y' \in Y$ such that $y \in y' + D(y')\backslash\{0\}$. Thus $Y \subset \bigcup_{y \in Y} (y + D(y)\backslash\{0\})$

Thus, the family of the sets$\{y + D(y)\backslash\{0\}\}$ forms an open cover of Y. Since Y is compact, there is a finite subcover $\{ D(y^i)\backslash\{0\}$ (i=1,...,n).Then, for any $i \in \{1,\ldots,n\}$

$y^i \in y^j + D(y^j)\backslash\{0\}$ for some $j \in \{1,\ldots,n\}$

However, this contradicts the assumption that D is acyclic. Hence, $\xi(Y, D) \neq \phi$.

In ordinary scalar optimization problem ($D= R^1_+$), the existence of minimal element is guaranteed under the condition that Y is bounded from below and $Y + R^1_+$ is closed. That means, under a kind of semicompactness condition. We extend this condition to multiobjective optimization problems.

**Definition 3.2.2:** Let D be a cone and Y be a set in $R^p$ .Y is said to be D-semicompact if every open cover of Y of the form $\{(y^\gamma - clD)^c : y^\gamma \in Y, \gamma \in \tau\}$ has a finite subcover. Where $\tau$ is some index set and the superscript c denotes the complement of a set.

**Theorem 3.2.2:** If D is an acute convex cone and Y is nonempty D-semicompact set in $R^p$, then $\xi(Y, D) \neq \phi$.

**Proof:** $D \subset clD$ implies $\xi(Y, clD) \subset \xi(Y, D)$. (by Proposition 3.1.1).

It is enough to show the case in which D is a pointed closed convex cone. In this case, D defines a particular order $\leq_D$ on Y as $y^1 \leq_D y^2$ iff $y^2 - y^1 \in D$

An element in $\xi(Y, D)$ is a minimal element with respect to $\leq_D$. Therefore we can show that Y is inductively ordered and applying Zorn's lemma to establish the existence of minimal element.

Now, suppose the contrary that Y is not inductively ordered. Then there exist a totally ordered set $\bar{Y} = \{y^\gamma : \gamma \in \tau\}$ in Y which has no lower bound in Y. Thus

$$\bigcap_{\gamma \in \tau} \left[(y^\gamma - D) \cap Y\right] = \varnothing$$

Otherwise any element of this intersection is lower bound of $\bar{Y}$ in Y. Now it follows that for any $y \in Y$ there exist $y^\gamma \in \bar{Y}$ such that $y \notin y^\gamma$ -D. Since $y^\gamma$ -D is closed, the family of $\{(y^\gamma\text{-D})^c : \gamma \in \tau\}$ forms an open cover of Y. Moreover, $y^\gamma$-D$\subset y^{\gamma'} - D$ iff $y^\gamma \leq_D y^{\gamma'}$, and so they are totally ordered by inclusion.

Since Y is D-semicompact, the cover has finite subcover, and hence there exist a single $y^{\bar{\gamma}} \in Y$ such that $Y \subset (y^{\bar{\gamma}}\text{-D})^c$.

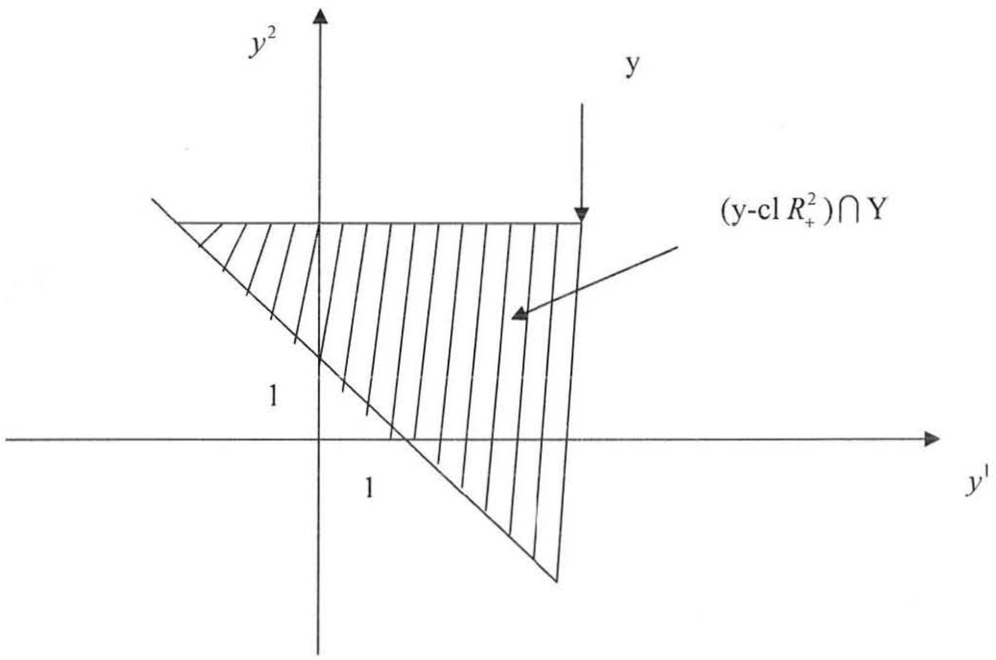
However this contradicts the fact that $y^{\bar{\gamma}} \in Y$ .Therefore, Y is inductively ordered by $\leq_D$ and $\xi(Y, D) \neq \phi$ by Zorn's lemma.

It is difficult to check whether Y is D-semicompact or not, we shall introduce a more stronger concept that is called cone compactness.

**Definition 3.2.3** Let D be a cone and Y be a set in $R^p$ Y is said to be D-compact if the set (y- clD) $\cap$ Y is compact for any $y \in Y$.

Remark3.2.1

1) If Y is D-compact, then Y is D-semicompact.
2) A compact set is D-compact and so D-semicompact. However D-compact set is not necessarily compact.
3) Example: let D= $R_+^2$ and Y={ y:$R^2$:$y^1 + y^2 \geq 1$}




Here Y is not bounded, which implies not compact. But Y is D-compact because of the compactness of the set (y-cl $R_+^2$)∩Y for any y $\in$ Y.

Theorem 3.2.3 Let D be an acute convex cone in $R^p$. If Y $\subset$ $R^p$ is nonempty and D-compact then $\xi$(y, D) $\neq \phi$.

Proof: Follows immediately from Theorem 3.2.2 and Remark 3.2.1.

### 3.2.2 External Stability of Efficient Solution

In this subsection we introduce a new concept, external stability of the efficient set.
In section 3.1 we have defined the efficient set, which is the set of all nondominated points in the objective space. Each point outside the efficient set is, therefore, dominated by some other points in the feasible set.
However, is it also dominated by a point in the efficient set? If this is the case, the efficient set is said to be externally stable.

Definition 3.2.4 Let Y be a set of feasible points in $R^p$, S be a subset of Y, and D be a domination structure on Y. S is said to be externally stable if ,for each y $\in$ Y\S there exist some $y^{'}$ $\in$ S such that y $\in$ $y^{'}$ + D( $y^{'}$ ).

Remark3.2.2.Since each D(y) is assumed to contain the zero vector, the external stability condition can be rewritten as follows: For each y $\in$ Y, there exist $y^{'}$ $\in$ S such that y $\in$ $y^{'}$ + D ( $y^{'}$ ). Hence, if D(y) =D (constant) for all y, this can also be rewritten as $Y \subset S + D$.

DEFINITION 3.2.5 A set S $\subset$ Y is said to be internally stable if y $\in$ $y^{'}$ + D ( $y^{'}$ )\ {0} (i.e. y $\not\prec$ $y^{'}$ ) whenever y, $y^{'}$ $\in$ S.
It is clear that the efficient set of Y is internally stable.

Definition 3.2.5 Let Y be a set in $R^p$ and D be a domination structure on Y. A subset of Y is called a kernel of Y with respect to D, denoted by $K(Y,D)$ if it is both externally stable and internally stable.

Proposition3.2.4: If D is transitive, asymmetric, and a kernel exists, then it is unique.
Proof: assume to the contrary that there are two different kernels $K(Y,D)$ and $K^{'}(Y,D)$, then we have $y \in K(Y,D)$ but $y \notin K^{'}(Y,D)$ for some $y \in Y$. From $y \notin K^{'}(Y,D)$, there exist $y^{'} \in K^{'}(Y,D)$ such that $y \in y^{'} + D(y^{'})\backslash\{0\}$. Since $K(Y,D)$ is internally stable and $y \in K(Y,D)$, then $y^{'} \notin K(Y,D)$ and so there exist $y^{''} \in K(Y,D)$ such that $y^{'} \in y^{''}$ + D( $y^{''}$ )\{0}. Since D is transitive and asymmetric, y $\in$ $y^{''}$ +D(y'')\{0}, which leads to contradiction to the internal stability of $K(Y,D)$.

**Proposition3.2.5**: Suppose that D is transitive and asymmetric. If $K(Y,D)$ exist, then $K(Y,D) = \xi(Y,D)$.

**Proof**: Let $\hat{y} \in K(Y,D)\setminus\xi(Y,D)$. Then there exist y $\in$ Y such that $\hat{y} \in$ y+ D(y)\{0}. From the external stability of $K(Y,D)$, there exist y' $\in K(Y,D)$ such that y $\in y'$+D(y'). Since D is transitive and asymmetric, $\hat{y} \in y'$+D(y')\{0},which contradicts the internal stability of $K(Y,D)$, there exist y $\in K(Y,D) \subset$ Y such that $\hat{y} \in$y+D(y)\{0}. However, this is a contradiction.

**Proposition3.2.6**Let Y be a nonempty set and D be a transitive asymmetric domination structure on Y. Then $K(Y,D)$ exists iff $\xi(Y,D)$ is externally stable.

**Proof**: In view of Proposition 3..2.5, $K(Y,D) = \xi(Y,D)$ if $K(Y,D)$ exist. Hence $\xi(Y,D)$ is externally stable. Conversely if $\xi(Y,D)$ is externally stable, then it becomes a kernel because it is internally stable.

**Theorem 3.2.4** Let Y be a nonempty compact set. Suppose that a domination structure D is transitive and upper semicontinuous (as a point-to-set map) on Y. Moreover, for each compact subsct $Y'$ of Y, $\xi(Y',D)$ is assumed to be nonempty. Then $\xi(Y,D)$ is externally stable and so is the kernel of Y.

**Proof**: Let y be an arbitrary point in Y, and define a set

$$Y' = \{y' \in Y : y \in y' + D(y')\}.$$

This implies, the set $Y'$ consists of y and all points in Y that dominate y. We must show that $Y' \cap \xi(Y,D) \neq \varnothing$.It suffices for this to prove that

(i) $\xi(Y',D) \neq \varnothing$ and that (ii) $\xi(Y',D) \subset \xi(Y,D)$.

(i). We prove the compactness of $Y'$, Since it implies that $\xi(Y',D) \neq \varnothing$ from the assumption. Since $Y' \subset$ Y and Y is bounded, $Y'$ is also bounded. To prove the closedness of $Y'$, let $\{y^k\} \subset Y'$ and $y^k \to$y. Then y$\in y^k$ + D($y^k$),

i.e. y - $y^k \in$D ($y^k$) and y - $y^k \to$y - $y'$.

Since D is upper semicontnuous, y- $y' \in$D ($y'$), hence $y' \in Y'$.

(ii). A vector $\hat{y} \in R^p$ is supposed not to be contained in $\xi(Y, D)$. We suppose $\hat{y} \in Y'$, since otherwise it is clear that $\hat{y} \notin \xi(Y', D)$. Then $\hat{y} \in$ Y, and there exist $y'' \in$ Y such that $\hat{y} \in y'' +$ D($y''$)\{0}. Since D is transitive and $\hat{y} \in Y'$, y$\in$ $y''$+D($y''$), which implies that $y'' \in Y'$.Hence , $\hat{y} \notin \xi(Y', D)$, as was to be proved. This completes the proof.

**Theorem 3.2.5** Let D(y) =D be a pointed closed convex cone and Y be a nonempty D-compact set .Then $\xi(Y, D)$ is externally stable; that is Y $\subset$ $\xi(Y, D)$+D.

**Proof:** The proof is similar to that of Theorem 3.2.4. In this case

$$Y' = (y - D) \cap Y,$$

is D-compact from the D-compactness of Y. Therefore, in view of Theorem 3.2.3 $\xi(Y', D) \neq \varnothing$. On the other hand, since D is a pointed convex cone we can immediately establish that $\xi(Y', D)` \subset \xi(Y, D)$. Thus $\xi(Y, D)$ is externally stable.

# CHAPTER FOUR

# 4. Duality in Vector Optimization.

## 4.1: Lagrange Duality in Nonlinear Vector Optimization

In this section we present lagrange duality for efficient solution in multiobjective (vector) optimization where the domination structure is supposed to be a pointed closed convex cone. As was seen in the previous chapter, efficient solution corresponds to minimal (maximal) solutions in ordinary mathematical programming. Therefore, for convenience in this chapter we define:

D-Minimizing as finding efficient solutions with respect toD,

In particular, we define minimizing for cases with the cone ordering $R_+^p$,

Similarly we use the notation $Min_D^Y$ for representing the set of all efficient elements $\xi$(Y,D), in particular, we define MinY as $\xi$(Y, $\leq_{R_+^p}$). D-Maximization and $Max_D$ are used in similar fashion for the cone ordering $\leq_{-D}$.

Conceder nonlinear multi-objective optimization problem formulated as follows:

(P) D-Minimize $\{f(x):x \in X\}$

Where $X=\{x \in X': g(x) \leq_Q 0 : X' \subset R^n\}$.

Throughout this section, we impose the following assumptions:

(i). $X'$ is nonempty compact convex set.

(ii). D and Q are pointed closed convex cones with non-empty interiors of $R^p$ and $R^m$ respectively.

(iii). f is continuous and D- convex.

(iv). g is continuous and Q-convex.

Under the assumption it can be readily shown that for every $u \in R^m$, both sets

$X(u) = \{x \in X' : g(x) \leq_Q u\}$ and

$$Y(u) = f[X(u)] = \{y \in R^p : y = f(x), x \in X', g(x) \leq_Q u\} \quad (4.1.1)$$

are compact, X(u) is convex, and Y(u) is D-convex. Although these conditions might seem too strong, something like these conditions would be more or less inevitable as long as we consider efficient solutions.

### 4.1.1 Perturbation (or Primal) Map and Lagrange Multiplier Theorem

Let us consider the primal problem (P) by embedding it in a family of perturbed problems with Y(u) given by (4.1.1):

$(P_u)$ D-Minimize Y(u)

Clearly primal problem (P) is identical to problem $(P_u)$ with u = 0.

Now define the set $\Gamma$ as

$$\Gamma = \{u \in R^m : X(u) \neq \varnothing\}.$$

It is easy to show that the set $\Gamma$ is convex.

The point- to- set map $W : \Gamma \to R^p$ defined by

$$W(u) = \mathrm{Min}_D Y(u)$$

is called perturbation (or primal) map.Observe that the perturbation map corresponds to the perturbation (or primal) function

$$w(u) = \min\{f(x): x \in X' : g(x) \leq u\}$$

in ordinary mathematical programming.

Obviously the original problem (P) can be regarded as determining the set W (0) and $f^{-1}[W(0)] \cap X$.

Proposition4.1.1: For any $u \in \Gamma$,

$$W(u) + D = Y(u) + D.$$

Proof: Note first that

$$W(u) = \mathrm{Min}_D Y(u) \subset Y(u).$$

This implies that $W(u) + D \subset Y(u) + D$.

On the other hand since Y(u) is D-compact, Theorem 3.2.5 yields

$$Y(u) \subset W(u) + D.$$

From which Y(u) +D $\subset$ W (u) + D + D= W (u) + D, because D is a convex cone.
Hence W (u) +D =Y(u) +D.

Proposition4.1.2: For each $u \in \Gamma$, W(u) is a D-convex set in $R^p$.

Proof: Immediate from Proposition 4.1.1 and the D- convexity of Y(u).

Proposition4.1.3: The map W is D-monotone on $\Gamma$, namely, W ($u^1$) $\subset$ W ($u^2$) +D for any $u^1, u^2 \in \Gamma$ *such that* $u^1 \leq_Q u^2$.

Proof: obviously, $Y(u^1) \subset Y(u^2)$ where $u^1 \leq_Q u^2$.Hence

$$W(u^1) \subset Y(u^1) \subset Y(u^2) \subset W(u^2) + D.$$

Proposition4.1.4: W is a D-convex point-to-set map on $\Gamma$

Proof: In view of Preposition 4.1.1, it suffices to show that

$\alpha$ Y($u^1$)+(1- $\alpha$)Y($u^2$)$\subset$Y( $\alpha$ $u^1$+(1- $\alpha$) $u^2$ )+D for any $u^1, u^2 \in \Gamma$ and $\alpha \in [0,1]$.

If we suppose that

$$y \in \alpha Y(u^1)+(1-\alpha)Y(u^2),$$

then there exist $x^1, x^2 \in X'$ such that

$$g(x^1) \leq_Q u^1, \quad g(x^2) \leq_Q u^2, and\ y = \alpha f(x^1) + (1-\alpha) f(x^2).$$

since $X'$ is a convex set $\alpha x^1 + (1-\alpha) x^2 \in X'$. Furthermore, from the Q-convexity of g, $g(\alpha x^1 + (1-\alpha)x^2) \leq_Q \alpha g(x^1) + (1-\alpha) g(x^2) \leq_Q \alpha u^1 + (1-\alpha)u^2$ ,which implies that

$\alpha x^1 + (1-\alpha)x^2 \in X(\alpha u^1 + (1-\alpha)u^2)$ and, thus, $f(\alpha x^1 + (1-\alpha)x^2) \in Y(\alpha u^1 + (1-\alpha)u^2)$.

On the other hand, from the D-convexity of f

$$\alpha f(x^1) + (1-\alpha) f(x^2) \in f(\alpha x^1 + (1-\alpha)x^2) + D$$

Finally we have

$$y \in Y(\alpha u^1 + (1-\alpha)u^2) + D.$$

This completes the proof of the Proposition.

Remark4.1.1: Proposition 4.1.3 and 4.1.4 correspond to the fact that the primal function w in ordinary mathematical programming is monotonically non-increasing and convex. It is well known that in scalar convex optimization, the convexity of w ensures that by adding an appropriate linear functional $<\lambda, u>$ to w(u), the resulting combination w(u) + $<\lambda, u>$ is minimized at u = 0. In analogous manner, the D-convexity of point-to-set map W ensures that if an appropriate linear vector valued functional $\Lambda u$ is added to W(u) there will exist no point of W(u) + $\Lambda u$ that dominates a given point of W(0).

Theorem4.1.1 (Lagrange Multiplier Theorem)

If $x^{'}$ is Properly efficient solution to (P) and if Slater's constraint qualification holds (i.e there exist $x^{'} \in$X such that $g(x^{'}) <_Q 0$ ) then there exist p x m matrix $\Lambda^{'}$ such that

$\Lambda^{'}Q \subset$ D and

f($x^{'}$)$\in$Min D{f(x) + $\Lambda^{'}$g (x): x$\in X^{'}$ }and

$$\Lambda^{'} g(x^{'}) = 0$$

Proof: Let $X = \{x \in R^n : g(x) \leq_Q 0\} \cap X^{'}$. Since $x^{'}$ is properly efficient solution of f(X) with respect to $\leq_D$, there exist a vector $\mu^{'} \in \text{int}\, D^o$ such that

$$<\mu^{'}, f(x^{'})> \leq <\mu^{'}, f(x)> \text{ for any } x \in X.$$

Note here that $<\mu^{'}, f(x)>$ is a convex function on $X^{'}$. In fact, due to the D-convexity of f, since $\alpha$ f($x^1$)+(1-$\alpha$)f($x^2$)-f($\alpha$ $x^1$+(1-$\alpha$)$x^2$)$\in$D for any $x^1, x^2 \in X^{'}$ and any $\alpha \in$[0, 1], we have:

$$\alpha <\mu^{'}, f(x^{'})> + (1-\alpha) <\mu^{'}, f(x^2)> - <\mu^{'}, f(\alpha x^1 + (1-\alpha) x^2) \geq 0.$$

Therefore, the well- known Lagrange multiplier theorem in scalar convex optimization leads to the existence of a vector $\lambda^{'}$ such that:

$$<\mu^{'}, f(x^{'})> + <\lambda^{'}, g(x^{'})> \leq <\mu^{'}, f(x)> + <\lambda^{'}, g(x)> \qquad 4.1.2$$

$$\text{for any } x \in X^{'} \text{ and}$$

$$<\lambda^{'}, g(x^{'})> = 0$$

Now for such $\mu^{'}$ *and* $\lambda^{'}$ take $\Lambda^{'}$ with $\Lambda^{'T}\mu^{'} = \lambda^{'}$ in such a way that

$$\Lambda^{'} = (\lambda_1^{'} e, \lambda_2^{'} e, ..., \lambda_m^{'} e)$$

where e is a vector of D with $<\mu^{'}, e> = 1$. Then clearly $\Lambda^{'}Q \subset D$ *and* $\Lambda^{'}g(x^{'}) = 0$.

If we suppose that for this $\Lambda'$ there exist

$\bar{x} \in X'$ *such that*

$$f(x') - f(\bar{x}) - \Lambda' g(\bar{x}) \in D \setminus \{0\}$$

*Hence*

$$< \mu', f(x') >< \mu', f(\bar{x}) > + < \mu', \Lambda' g(\bar{x}) >=< \mu', f(\bar{x}) > + < \lambda', g(\bar{x}) > .$$

This contradicts the relation (4.1.2) Therefore,

$f(x') \in Min_D\{f(x) + \Lambda' g(x) : x \in X'\}$

Note that in the above case where $(1,\ldots,1)^T \in D$, by normalizing $\mu'$ such that a particular way that $\sum_{i=1}^{p} \mu_i' = 1$,we can take $e = (1,1,\ldots,1)^T$ in the proof of Theorem 4.1.1. We then have $\Lambda g(x) = (< \lambda, g(x) >, \ldots, < \lambda, g(x) >)^T$.

## 4.1.2 VECTOR-VALUED LAGRANGIAN FUNCTION AND ITS SADDLE POINT

Hereafter, we shall denote by $\Pi$ a family of all p x m matrices $\Lambda$ such that $\Lambda Q \subset D$. Such matrices are said to be positive in some literature.

Note: for given $\mu \in D^0 \setminus \{0\}$ and $\lambda \in Q^0$ there exist $\Lambda \in \Pi$ such that

$$\Lambda^T \mu = \lambda$$

For some vector $e \in D$ with $< \mu, e >=1$

$\Lambda = (\lambda_1 e, \lambda_2 e, \ldots, \lambda_m e)$ is the desired p x m matrix.

DEFINITION4.1.1 A vector valued Lagrangian function for problem (P) is defined on $X' \times \Pi$ by $L(x, \Lambda) = f(x) + \Lambda g(x)$.

DEFINITION4.1.2 A pair $(x', \Lambda') \in X' \times \Pi$ is said to be saddle point for the vector valued lagrangian function $L(x, \Lambda)$ if

$$L(x', \Lambda') \in Min_D\{L(x, \Lambda') = x \in X'\} \cap Max_D\{L(x', \Lambda) : \Lambda \in \Pi\}.$$

Theorem4.1.2 The following conditions are necessary and sufficient for a pair $(x^{'}, \Lambda^{'})$ $\in X^{'} \times \Pi$ to be a saddle point for a vector valued lagrangian function L(x, $\Lambda$)

i. $L(x^{'}, \Lambda^{'}) \in \mathrm{Min}_D \{L(x, \Lambda^{'}) : x \in X\}$

ii. $g(x^{'}) \leq_Q 0$

iii. $\Lambda^{'} g(x^{'}) = 0$

Proof:

($\Rightarrow$) suppose that $(x^{'}, \Lambda^{'})$ is a saddle point of L (x, $\Lambda$). Then

(i). is the same as part of the definition of saddle point.

(ii). $L(x^{'}, \Lambda^{'}) \in \mathrm{Max}_D\{ f(x^{'}) + \Lambda g(x^{'}),\ \Lambda \in \Pi \}$

$\Rightarrow f(x^{'}) + \Lambda^{'} g(x') \not\leq_D f(x^{'}) + \Lambda g(x^{'})$ for any $\Lambda \in \Pi$ (4.1.3)

From which we have

$< \mu^{'}, \Lambda g(x^{'}) - \Lambda^{'} g(x^{'}) > \leq 0$ (4.1.4) for

some $\mu^{'} \in D^0 \backslash \{0\}$ and for any $\Lambda \in \Pi$.

Suppose that $g(x^{'}) \not\leq_Q 0$.

Then there exists $\lambda^{'} \in Q^0$ such that $<\lambda^{'}, g(x^{'})> > 0$.

Making $\| \lambda^{'} \|$ sufficiently large and taking $\Lambda \in \Pi$ such that $\mu^{'T} \Lambda = \lambda^{'T}$, we obtain the relation $< \mu^{'}, \Lambda\ g(x^{'})> - <\mu^{'}, \Lambda^{'} g(x^{'})> > 0$

This contradicts relation (4.1.4). Thus $g(x^{'}) \leq_Q 0$

(iii). Using this result $\Lambda^{'} g(x^{'}) \not\leq_D 0$ for $\Lambda^{'} \in \Pi$.

On the other hand substituting $\Lambda = 0$ in to Eq (4.1.3) yields $\Lambda^{'} g(x^{'}) \not\leq_D 0$.

Finally we have $\Lambda^{'} g(x) = 0$

($\Leftarrow$) since $\Lambda g(x^{'}) \in -D$ for any $\Lambda \in \Pi$ as long as $g(x^{'}) \leq_Q 0$, it follows that

$\mathrm{Max}_D \{ \Lambda g(x^{'}) : \Lambda \in \Pi\} = \{0\}$. Thus from $\Lambda^{'} g(x^{'}) = 0$, we have

$L(x^{'}, \Lambda^{'}) \in \mathrm{Max}_D\{ f(x^{'}) + \Lambda g(x^{'}) : \Lambda \in \Pi \}$

This result and condition (i) implies the pair $(x^{'}, \Lambda^{'})$ is a saddle point.

Corollary 4.1.1 Suppose $x'$ is a properly efficient solution to problem (P) and let Slater's constraint qualification is satisfied. Then there exist a p x m matrix $\Lambda' \in \Pi$ s. t $(x', \Lambda')$ is a saddle point for the vector valued Lagrangian function L(x, $\Lambda$)

Proof : Immediate from Theorems4.1.1 and 4.1.2 .

Thus, we have verified that properly efficient solutions to the problem (P) together with a matrix give a saddle point for the vector-valued Lagrangian function under the convexity assumptions and the appropriate regularity conditions. Conversely, the saddle point provides a sufficient condition for optimality of problem (P).

Theorem 4.1.3 If $(x', \Lambda') \in X'$ x $\Pi$ is a saddle point for one vector valued Lagrangian function L (x, $\Lambda$) then $x'$ is an efficient solution to problem (P).

Proof:- Suppose that $x'$ is not a solution to problem (P) this implies there exist $x_o \in X'$ such that. $f(x_o) \leq_D f(x')$. Since $g(x') \leq_Q 0$ and $\Lambda' \in \Pi$ yield $\Lambda' g(x_o) \in$-D, we finally have

$f(x_o)+ \Lambda' g(x_o) \leq_D f(x')$, which contradicts $L(x', \Lambda') \in \text{Min}_D \{L(x, \Lambda'): x \in X'\}$. Thus $x'$ is an efficient solution to problem (P)

## 4.1.3 DUAL MAP AND DUALITY THEORY

Recall that the dual function in ordinary optimization is defined by

$$\phi(\lambda) = \inf \{f(x) + \langle\lambda, g(x)\rangle : x \in X'\}$$

Definition 4.1.2 Define for any $\Lambda \in \Pi$

$\Omega(\Lambda) = \{L(x, \Lambda): x \in X'\} = \{f(x) + \Lambda g(x) : x \in X'\}$ and

$\Phi(\Lambda) = \text{Min}_D\, \Omega(\Lambda)$.

A point -to -set map $\Phi : \Pi \to R^p$ is called a *dual map.*

The dual problem associated with the primal problem (P) is

(Dts) D-Maximize $\bigcup_{\Lambda \in \Pi} \Phi(\Lambda)$.

Proposition4.1.5 For each $\Lambda \in \Pi$, $\Phi(\Lambda)$ is a D-convex set in $R^p$.

Proof: Since the map f and g are D-convex and Q-convex respectively, the map L $(., \Lambda)$ is D-convex over $X^{'}$ for each fixed $\Lambda \in \Pi$. Hence $\Omega(\Lambda)$ is a compact set in $R^p$, for $X^{'}$ is a compact convex set. Therefore,

$$\Phi(\Lambda) + D = \Omega(\Lambda) + D.$$

By Theorem 3.2.5 and thus $\Phi(\Lambda)$ is also D-convex.

Proposition4.1.6 $\Phi$ is a D-convex point-to-set map on $\Gamma$. Namely, for any $\Lambda^1, \Lambda^2 \in \Pi$ and any $\alpha \in [0,1]$ $\Phi(\alpha \Lambda^1 + (1-\alpha)\Lambda^2) \subset \alpha \Phi(\Lambda^1) + (1-\alpha)\Phi(\Lambda^2) + D$

Proof: Note that Theorem 3.2.5 yields $Min_D A \subset A \subset B \subset Min_D B + D$, for any sets A and B such that $A \subset B$, and B is compact. From this and the relation

$$\{\alpha (f(x) + \Lambda^1 g(x)) + (1-\alpha)(f(x) + \Lambda^2 g(x)) : x \in X^{'}\}$$
$$\subset \alpha \{f(x) + \Lambda^1 g(x) : x \in X^{'}\} + (1-\alpha)\{ f(x) + \Lambda^2 g(x) : x \in X^{'}\},$$

We have

$$\Phi(\alpha \Lambda^1 + (1-\alpha)\Lambda^2) = Min_D \{f(x) + (\alpha \Lambda^1 + (1-\alpha)\Lambda^2 g(x) : x \in X^{'}\}$$
$$+Min_D\{\alpha ( f(x) + \Lambda^1 g(x)) + (1-\alpha)(f(x) + \Lambda^2 g(x)) : x \in X^{'}\}$$
$$\subset Min_D[\alpha \{f(x) + \Lambda^1 g(x) : x \in X^{'}\}] + D.$$

Hence, in view of Proposition3.1.3

$$\Phi(\alpha \Lambda^1 + (1-\alpha)\Lambda^2) \subset Min_D \alpha [f(x) + \Lambda^1 g(x) : x \in X^{'}\}$$
$$+ Min_D(1-\alpha)f(x) + \Lambda^2 g(x) : x \in X^{'}\} + D$$
$$= \alpha Min_D\{f(x) + \Lambda^1 g(x) : x \in X^{'}\}$$
$$+(1-\alpha)Min_D\{f(x) + \Lambda^2 g(x) : x \in X^{'}\} + D$$
$$= \alpha \Phi(\Lambda^1) + (1-\alpha)\Phi(\Lambda^2) + D.$$

Note that Proposition 4.1.6 is an extension of the fact that the dual function $\Phi(\lambda)$ is concave. We can now establish the following relationship between the dual map $\Phi(\Lambda)$ and the primal map W(u), which is an extension of the following relation between the dual function $\Phi(\lambda)$ and the primal function w(u):

$$\Phi(\lambda) = \inf\{w(u) + \langle \lambda, u \rangle : u \in \Gamma\}.$$

Proposition4.1.7 The following relation holds:

$$\Phi(\Lambda) = \mathrm{Min_D} \bigcup_{u \in \Gamma} (W(u) + \Lambda U)$$

Proof: Let $y^1 = f(x^1) + \Lambda g(x^1)$ for any $x^1 \in X^{'}$.

Then letting $u^1 = g(x^1)$

$$y^1 = f(x^1) + \Lambda u^1.$$

Note here that

$$f(x^1) \in W(u^1) + D, \text{ because } f(x^1) \in Y(u^1).$$

Hence

$$y^1 \in W(u^1) + \Lambda u^1 + D,$$

from which $\Omega(\Lambda) \subset \bigcup_{u \in \Gamma} (W(u) + \Lambda U) + D$

$$\Omega(\Lambda) + D \subset \bigcup_{u \in \Gamma} (W(u) + \Lambda U) + D.$$

Conversely suppose that

$y^1 \in W(u^1) + \Lambda u^1$ for some $u^1 \in \Gamma$, this implies that $y^1 - \Lambda u^1 \in \mathrm{Min_D} Y(u^1)$.

Thus

$y^1 - \Lambda u^1 = f(x^1)$ for some $x^1 \in X^{'}$ such that $g(x^1) \leq_Q u^1$.

Then for $\Lambda \in \Pi$

$y^1 = f(x^1) + \Lambda u^1 \geq_D f(x^1) + \Lambda g(x^1)$, and hence

$$y^1 \in L(x^1, \Lambda) + D \subset \Omega(\Lambda) + D.$$

Therefore,

$\bigcup_{u \in \Gamma} (W(u) + \Lambda U) \subset \Omega(\Lambda) + D$, and thus

$$\bigcup_{u \in \Gamma} (W(u) + \Lambda U) + D \subset \Omega(\Lambda) + D.$$

Finally, we have

$\Omega(\Lambda) + D = \bigcup_{u \in \Gamma} (W(u) + \Lambda U) + D$ and hence

$\mathrm{Min_D}(\Omega(\Lambda) + D) = \mathrm{Min_D}(\bigcup_{u \in \Gamma} (W(u) + \Lambda U) + D)$. Which establishes the proposition, because in general $\mathrm{Min_D} A = \mathrm{Min_D}(A + D)$ whenever D is a pointed convex cone.

Recall that Theorem 4.1.1 implies that, given a properly efficient solution $\hat{x}$, then $f(\hat{x}) \in \Omega(\Lambda')$ for some $\Lambda' \in \Pi$. Hence we may see from Proposition 4.1.7 that

$$f(\hat{x}) \in \mathrm{Min}_D \bigcup_{u \in \Gamma} (W(u) + \Lambda' u) \text{ for some } \Lambda' \in \Pi.$$

This is an extension of property of primal function w in ordinary convex optimization problem stated in subsection 4.1.1.

The following Theorem represents some properties of efficient solutions to primal problem (P) in connection with the dual map $\Phi$ and might be considered as duality theorem for multi-objective optimization problem.

Theorem 4.1.4(Weak Duality)

For any $x \in X$ and any $y \in \Phi(\Lambda)$

$$y \not\geq_D f(x)$$

Proof: For any y with the property

$$y \not\geq_D f(x) + \Lambda g(x) \text{ for all } x \in X',$$

the result of the theorem follows immediately from Lemma 2.2.1 because for any $x \in X$ and $\Lambda \in \Pi$, $\Lambda g(x) \leq_D 0$.

Theorem 4.1.5 (a) suppose that $x' \in X$, $\Lambda \in \Pi$ and $f(x') \in \Phi(\Lambda')$. Then $y' = f(x')$ is an efficient point to the primal problem (P) and also to the dual problem (Dts)

(b).Suppose that x' is properly efficient solution to (P) and that Slater's constraint qualification is satisfied. Then $f(x') \in \mathrm{Max}_D \bigcup_{\Lambda \in \Pi} \Phi(\Lambda)$

Proof: (a) if f (x') is not an efficient point to (P), then there exist $x \in X$ such that $f(x) \leq_D f(x')$. Since $g(x) \leq_Q 0$ and $\Lambda' \in \Pi$ yield $\Lambda' g(x) \in -D$, we finally have $f(x) + \Lambda' g(x) \leq_D f(x')$, which contradicts to the fact that $f(x') \in \mathrm{Min}_D\{f(x) + \Lambda' g(x) : x \in X'\}$. Hence $f(x')$ is an efficient point to problem (P).

Furthermore, suppose that $f(x') \leq_D y$ fore some $y \in \bigcup_{\Lambda \in \Pi} \Phi(\Lambda)$.

Let $\hat{\Lambda} \in \Pi$ be such that $y \in \Phi(\hat{\Lambda})$.Then, since $\hat{\Lambda} g(x') \leq_D 0$ we have $f(x') + \hat{\Lambda} g(x') \leq_D y$, which contradicts $y \in \mathrm{Min}_D \{f(x) + \hat{\Lambda} g(x) : x \in X'\}$

Therefore f (x') is also a solution to (Dts)

(b). Theorem 4.1.1 ensures that there exist a pxm matrix $\Lambda \in \Pi$ such that $f(x^{'}) \in \Phi(\Lambda)$, which leads to the conclusion via result (a)

# 4.2. Conjugate Duality in Vector Optimization.

## 4.2.1 Some Definitions and Preliminary Concepts.

Let Y be a real topological vector space which is partially ordered by a pointed, closed, convex cone D with nonempty interior in Y.

Let Y* denote the extended space of Y (i.e. $Y^* = Y \cap \{\pm \infty\}$).

Given a set $Z \subset Y^*$ we define the set A(Z) of all points above Z, and the set B(Z) of all points below Z by:

$A(Z)=\{y \in Y^* \mid y^{'} <_D y \text{ for some } y^{'} \in Z\}$

And

$B(Z) = \{y \in Y^* \mid y^{'} >_D y \text{ for some } y^{'} \in Z\}$

Clearly $A(Z) \subseteq Y \cup \{+\infty\}$ and $B(Z) \subseteq Y \cup \{-\infty\}$

Definition 4.2.1 a POINT $z \in Z$ is said to be a D-minimal point of Z if there is no $y \in Z$ with $y <_D z$. The set of all D-minimal points of Z is called the D-minimum of Z and is denoted by D-MinZ. The D-maximum of Z, D-Max Z, is defined similarly

Definition 4.2.2: Let z be a nonempty subset of Y* such that $Z \neq \{+\infty\}$. A point $p \in Y^*$ is said to be a D-infimal point of a set Z ,if there is no $y \in Z$ such that $y <_D p$ and if the relation $y^{'} >_D p$ implies the existence of some $z \in Z$ such that $y^{'} >_D z$.

The set of all D-infimal point of Z is called a D-infimum of Z and is denoted by D-Inf Z. D-Sup Z is defined similarly.

If $Z = \phi$ and $Z= \{+\infty\}$,D-Inf $Z = \{+\infty\}$ .

As an easy consequence from the definition

( i ): –D-Max(-Z)=D-Min Z and –D-Inf (-Z)= D-Sup Z

(ii): D-Max $\phi = \phi$ and D-Sup $\phi = \{-\infty\}$.

Note that in the above definition the D-infimum and the D-minimum are defined for the weak pareto minimality (by using the order $>_D$). The following proposition shows a set can be partitioned by using the above defined concepts.

Proposition4.2.1 $Y^* = (\text{D-Sup } Z) \cup A\, (\text{D-Sup } Z) \cup B\, (\text{D-Sup } Z)$ and the above three sets in the right hand side are disjoint.

Let X and Y be real locally convex topological vector spaces and let L(X,Y) denote the space of all linear continuous operators from X into Y. Then we define the conjugate mapping and the subdiferential of a mapping $F : X \to Y^*$. Hereafter by domain of F, domf we mean the effective domain of F which is given by

dom $F=\{x \in X : F(x) \neq \phi, F(x)\neq\{+\infty\}\}$, for a set-valued mapping F and by

domf$=\{x \in X: f(x) <_D +\infty\}$, for a vector valued function f.

Definition4.2.3: For a vector valued function $f : X \to Y^*$ its conjugate map is a set valued mapping $f^* : L(X,Y) \to Y^*$ defined by

$$f^*(T) = \text{D-Sup } \{Tx - f(x) : x \in X\} \text{ for } T \in L(X,Y).$$

Moreover, its biconjugate mapping is a set valued mapping $f^{**} : X \to Y^*$ defined by

$$f^{**}(x) = \text{D-Sup} \bigcup_{T \in L(X,Y)} [Tx - f^*(T)] \text{ for } x \in \text{dom } f.$$

Definition4.2.4: For a set-valued mapping $F: X \to Y^*$, its conjugate map is a set valued mapping $F^*: L(X, Y) \to Y^*$ defined by

$$F^*(T) = \text{D-Sup} \bigcup_{x \in X} \{Tx - F(x) : x \in X\} \quad \text{for } T \in L(X, Y).$$

Moreover, its biconjugate mapping is a set valued mapping $F^{**}: X \to Y^*$ defined by

$$F^{**}(x) = \text{D-Sup} \bigcup_{T \in L(X,Y)} [Tx - F^*(T)] \qquad \text{for } x \in \text{dom } F.$$

Assume that dom $f \neq \phi$

Definition4.2.5: Let $x_o \in X$. An operator $T \in L(X, Y)$ is said to be a subgradient of a vector valued function f at $x_0$ if

$$f(x) \geq_D f(x_0) + T(x - x_0) \text{ for all } x \in X$$

Or equivalently

$$f(x_0) - Tx_0 \in \text{D-Min } \{f(x) - Tx : x \in X\}$$

The set of all subgradients of f at $x_o$ is called the subdifferential of f at $x_o$ and is denoted by $\partial f(x_o)$

More precisely

$\partial f(x_o)=\{T \in L(X,Y) : f(x)- f(x_o) \geq_D +T(x-x_o) \text{ for all } x \in X\}$

For a set-valued mapping F: X $\rightarrow$ Y*, let $x_o \in X$ and $y_o \in F(x_o)$. An operator T $\in L(X,Y)$ is called a subgradient of F at $(x_o, y_o)$ if

$T x_o - y_o \in \text{D-Max} \bigcup_{x \in X} [Tx-F(x)]$

The subdifferential of F at $(x_o, y_o)$ is denoted by $\partial F(x_o, y_o)$.

Moreover, we let $\partial F(x_o) = \bigcup_{y \in F(x_0)} \partial F(x_o, y)$

If the subdifferential of f at $x_o$ is nonempty (or if $\partial F(x_o, y_o) \neq \phi$ for every $y_o \in F(x_o)$ when F set-valued mapping ),then f is said to be subdifferentiable at $x_0$.

Lemma4.2.1: Suppose that f is a subdifferentiable function, then

(i). $T \in \partial f(x)$ iff $Tx - f(x) \in f^*(T)$

(ii). $f(x^*) \in$ D-Min $\{f(x) : x \in X)$ iff $0 \in \partial f(x^*)$.

Proof: (i). Suppose $T \in \partial f(x) \Rightarrow T(\bar{x} - x) \leq_D f(\bar{x})-f(x)$ for all $\bar{x} \in X$

$\Rightarrow T(\bar{x}) - f(\bar{x}) \leq_D T(x)- f(x)$ for all $\bar{x} \in X$

$\Rightarrow T(x)- f(x) \in$ D-sup$\{ T(\bar{x}) - f(\bar{x})$ for all $\bar{x} \in X\}$

$\Rightarrow T(x)- f(x) \in f^*(T)$.

The converse follows easily.

(ii). Suppose $f(x^*) \in$ D-Min $\{f(x) : x \in X)$

$\Rightarrow f(x^*) \leq_D f(x)$ for all $x \in X$

$\Rightarrow 0 \leq_D f(x) - f(x^*)$ for all $x \in X$

$\Rightarrow 0 \in \partial f(x^*)$.

The converse follows easily.

## 4.2.2 Optimality Conditions.

Let X be a real topological vector space and assume Y= $R^p$. which is partially ordered by a pointed, closed, convex cone D with nonempty interior in Y.

Now consider the following unconstrained vector minimization problem.

(P1) Min f (x) such that x $\in$ X

Definition4.2.6: A function f: X$\rightarrow$Y is said to be locally D-Lipschitz if for every x $\in$ X there exist a neighborhood B (x, $\delta$) of x with radius $\delta$>0 and L $\in$ D such that

$-L \|x - y\| \leq_D f(x) - f(y) \leq_D L \| x - y\| \quad \forall x, y \in B(x, \delta)$

Here, L is called Lipschitz constant.

For a vector valued function f: X $\rightarrow$ Y* the directional derivative of f at a point $x_o \in$ X in the direction u $\in$ X is given by

$$f'(x_o, u) = \lim_{t \to 0^+} \frac{f(x_o + tu) - f(x_o)}{t}$$ , if the limit exist.

Note that every D-convex functions are both directionally differentiable and locally D-Lipschitz.

Lemma 4.2.2: Suppose that f is a directionally differentiable vector -valued function. Then T $\in \partial f(x_0)$ iff $f'(x_o, u) \geq_D Tu$ for any u $\in$ X.

Proof: Let T$\in \partial f(x_o)$ .Then we have t f' $(x_o, u)$ + o (t) = f $(x_o$+ tu) -f $(x_o) \geq_D$ T(tu) for all t >0 and for all u $\in$X, where o(t)/t$\rightarrow$ 0 as t$\rightarrow$ 0,or f' $(x_o, u) + \frac{o(t)}{t} \geq_D$ Tu for all t > o, which in turn implies that $f'(x_0, u) \geq_D$ Tu for all u $\in$ X.

The proof of the other side is obvious.

Lemma4.2.3: for a D-convex function f, we have a stronger condition

(i). $f'(x_o, u) = \lim_{t \to 0^+} \frac{f(x_o + tu) - f(x_o)}{t} \leq_D f(x_o + u) - f(x_o)$ for all u $\in$ X.

(ii). $f'(x_o, u) \in$D-Max {Tu: T $\in \partial f(x_o)$ }

Proof: Let f be a D-convex function.Then for all $x_o$, u $\in$ X and t $\in$ [0,1]

$$f(x_o + tu) - f(x_o) = f(x_o - t x_o + t x_o + tu) - f(x_o)$$

$\Rightarrow f((1-t)x_o + t(x_o + u)) - f(x_o) \leq_D (1-t) f(x_o) + t f(x_o + u) - f(x_o)$

$= t[f(x_o + u) - f(x_o)]$

Dividing by t and taking a limit as $t \to 0^+$ we get

$$f'(x_o, u) = \lim_{t \to 0^+} \frac{f(x_o + tu) - f(x_o)}{t} \leq_D f(x_o + u) - f(x_o) \text{ for all } u \in X$$

Hence (i) is proved.

D- Convexity of f implies that f is directional differentiable hence using Lemma 4.2.2 we have $Tu \leq_D f'(x_o, u)$ iff $T \in \partial f(x_o)$ for all $u \in X$. This implies that

$f'(x_o, u) \in$ D-Max $\{Tu: T \in \partial f(x_o)\}$ which is (ii).

A point $x_o \in$ domf is said to be a local efficient point for (P1) if there exist a neighborhood of $x_o$ such that $f(x) \geq_D f(x_o)$ $\forall$ x in the neighborhood.

Proposition 4.2.2: Let f be directionally differentiable at $x_o \in X$. If $x_o$ is a local efficient point for (P1), then $f'(x_o, u) \geq_D 0$ for all $u \in X$

Proof: Since f is directionally differentiable at $x_o$, we have

$f(x_o + tu) = f(x_o) + t f'(x_o, u) + o(t)$, $\forall\, t \geq 0$ $\forall u \in X$, where $o(t) = o(t, x_o, u)$ and

$\lim_{t \to o^+} t^{-1}[o(t, x_o, u)] = 0$ $\forall u \in X$.

Then $f'(x_o, u) = t^{-1}[f(x_o + tu) - f(x_o)] - o(t)/t$ .Since $x_o$ is a local minimum point of f ,the assertion of the proof follows.

Proposition4.2.2: Let f be a function which is directionally differentiable at a point $x_o \in X$. If f is locally D-lipschitz in a neighborhood of $x_o$ and if $f'(x_o, u) \geq_D 0$ $\forall$ $u \in X$ and $u \neq 0$, then $x_o$ is local efficient point of f.

Proof: Suppose to the contrary. Then there exist a net of vectors $\{u_i\}$ with $||u_i||=1$, $\forall i$ in a neighborhood of $x_o$ and a sequence $\{t_i\}$ with $t_i \geq 0$, $t_i \to 0$ and $u_i \to u$ as $i \to \infty$ such that

$f(x_o + t_i u_i) <_D f(x_o)$ .

But $f(x_o + t_i u_i) - f(x_o) = f(x_o + t_i u) - f(x_o) + f(x_o + t_i u_i) - f(x_o + t_i u) <_D 0$.

Since f is locally D-Lipschitz, there exists $L \in D$ such that for sufficiently large i,

$-L\, t_i\, ||u_i - u|| \leq_D f(x_o + t_i u_i) - f(x_o + t_i u) \leq_D L\, t_i\, ||u_i - u||$

Then $[f(x_o + t_i u)- f(x_o)]/ t_i <_D L\ ||u_i-u||$ for sufficiently large i. Hence

$$f'(x_o, u) = \lim_{i\to\infty} [\frac{f(x_o + t_i u)- f(x_o)}{t_i}] <_D \lim_{i\to\infty} L\ ||u_i-u|| = 0$$

i.e. $f'(x_o, u) <_D 0$ contradicting the assumption. Therefore the conclusion of the proposition is true.

A D-convex function f is said to be proper if $f(x) >_D -\infty\ \forall\ x \in X$

Definition4.2.7: A vector valued function f: X → Y* is said to be a D-d.c. function iff it can be written as a difference of two proper Cone-convex functions i.e, f(x)= g(x)- h(x) where g and h are D-convex and proper vector valued functions.

Note: Let g, h: X → Y* be D-convex proper vector valued functions. Then the function f: X → Y* given by f(x)= g(x)- h(x) is a D-d.c. function on X and it is easy to verify that f is locally D-Lipschitz at each point of X and is directionally differentiable on X with $f'(x_o,u) = g'(x_o,u) - h'(x_o,u)$ for all $u, x_o \in X$.

Consider the following D-d.c optimization problem.

(P) Min f (x) s. t $x \in X$

Where f = g - h and g and h are as above.

Convention $+\infty - (+\infty) = +\infty$

To state the necessary condition for minimality we first define the strong subdifferential of a D-convex vector valued function f at a point $x_o$, denoted by $\partial_s f(x_o)$

$\partial_s f(x_o) = \{ T \in L(x, y): T(x-x_o) \leq_D f(x)-f(x_o)\quad \forall\, x \in X \}$

Note that $\partial_s f(x_o) \subseteq \partial f(x_o)$ for any $x_o$ at which f is subdifferentiable

Lemma4.2.4 For D-convex vector valued function f, we have

(i). $T \in \partial_s f(x_o)$ iff $Tu \leq_D f'(x_o,u)$ for all $u \in X$

(ii). For D-convex f, $\partial_s f(x_o)$ is nonempty as it at least contains the directional derivative of f at the point $x_o$.

Proof: (i). Suppose $T \in \partial_s f(x_o)$ and f is D-convex implies that it is directional differentiable. Thus

$t f'(x_o,u) + o(t) = f(x_o + tu) - f(x_o) \geq_D tTu$

$\Rightarrow f'(x_o,u) + \lim_{x\to 0^+} \frac{o(t)}{t} \geq_D$ Tu where $\lim_{x\to 0^+} \frac{o(t)}{t} = o$.

$$f'(x_o,u) \geq_D Tu \text{ for all } u \in X.$$

The converse follows easily.

(ii). Since f is D-convex which implies directional differentiable from (ii) of Lemma4.2.3 we have

$f'(x_o,u) \leq_D f(x_o+u) - f(x_o)$ for all u $\in$ X . This implies that

$$f'(x_o,u) \in \partial_s f(x_o) \text{ for all } u \in X. \text{ Hence } \partial_s f(x_o) \neq \emptyset$$

Theorem4.2.1: For f = g -h to attain its local D-minimal value at a point $x_o \in X$ it is necessary that $\partial_s h(x_o) \subseteq \partial g(x_o)$.

Proof: If $x_o$ is local minimum point for f, then there exists s a neighborhood $\theta$ of $x_o$ such that $f(x_o) \leq_D f(x)$ for all $x \in \theta$, which implies $g(x_o) - h(x_o) \leq_D g(x) - h(x)$, or $h(x) - h(x_o) \leq_D g(x) - g(x_o)$ for all $x \in \theta$.

But for $T \in \partial_s h(x_o)$ we have $T(x - x_o) \leq_D h(x) - h(x_o)$ for all x $\in$X . Then one can conclude that $g(x) - g(x_o) \leq_D T(x - x_o)$ or $T \in \partial g(x_o)$.

Theorem4.2.2: If $\partial_s h(x_o) \subseteq \partial g(x_o)$, then the criterion vector $x_o$ is a local efficient point for (P).

Proof: Let f(x) = g(x) - h(x) for all x $\in$ X. Then clearly f is directionally differentiable on X and it is locally D-Lipschitz. From the assumption of the theorem, we have $\partial_s h(x_o) \subseteq \partial g(x_o)$, and from the relation (ii) of Lemma4.2.3) it follows immediately that

$$g'(x_o, u) \geq_D h'(x_o, u) \ \forall\ u \in X.\ u \neq 0.$$

Which implies $f'(x_o,u) = g'(x_o, u) - h'(x_o, u) \geq_D 0$.

Hence using Proposition (4.2.3) we have the conclusion of the theorem.

## 4.2.3 Conjugate Duality in Cone d.c Optimization:

Let X be a real topological vector space and Y be a locally convex linear topological vector space. Assume that Y is partially ordered by a pointed, closed, convex cone D which has a nonempty interior in Y. Let g and h be proper vector valued D-convex functions from X in to Y*.

Now consider the D-d. c optimization problem

(P) Min f (x) s.t $x \in X$

Solving this problem means to find the set

D-Inf (P) =D-Inf $\{g(x) - h(x): x \in X\}$

Let $U \subseteq X$ be another locally convex linear topological vector space. We introduce the special perturbation function

$\Psi : X \times U \rightarrow Y^*$ such that $\Psi(x, u) = h(x+u) - g(x)$ for all $(x, u) \in X \times U$

Then clearly $\Psi(x, 0) = -f(x)$ for all $x \in X$.

For $\Lambda \in M = L(U, Y)$, the space of all linear continuous operators from U to Y, let the Lagrangian of problem (P) be given by

$-L(x, \Lambda) = \text{D-Sup}\{\Lambda u - \Psi(x, u) : u \in U\}$

$= \text{D-Sup}\{\Lambda(x+u) + g(x) - h(x+u) - \Lambda x : u \in U\}$

$= h^*(\Lambda) + g(x) - \Lambda x$

Where $h^*(\Lambda)$ denotes the conjugate map of h.

Now we put $-J(\Lambda) = \text{D-Sup} \bigcup_{x \in X} L(x, \Lambda) = g^*(x) - h^*(x)$.

Then the dual optimization problem for (P) is written as

**(Dc) D-Inf** $\bigcup_{\Lambda \in \Pi} h^*(\Lambda) - g^*(\Lambda)$

We can also observe the symmetry between (P) and (Dc). But since both h*(.) and g*(.) are set valued maps the (Dc) is not a vector optimization problem. However it can be understood as determining the set D-Inf $\bigcup_{\Lambda \in \Pi} h^*(\Lambda) - g^*(\Lambda)$

On the other hand,

$\text{D-Sup} \bigcup_{x \in X} [-L(x, \Lambda)] = \text{D-Sup} \bigcup_{x \in X} \text{D-Sup}\{\Lambda u - \Psi(x, u) : u \in U\}$

$= \text{D-Sup} \bigcup_{(x,u)\in XxU} \{\Lambda u + 0x - \Psi\ (x, u): u \in U\}$

$= \Psi\ ^*(0,\Lambda)$

Therefore, $\Psi\ ^*(0, \Lambda) = h^*(\Lambda) - g^*(\Lambda)$

**Theorem 4.2.3:** For any $x \in X$ and $\Lambda \in M$, $f(x) \notin A\ (h^*(\Lambda) - g^*(\Lambda))$ and thus D-Inf (P) $\cup$ A (D-Inf (Dc)) = $\phi$.

**Proof:** Suppose the contrary. Then there exist $y \in h^*(\Lambda) - g^*(\Lambda)$ such that $y <_D f(x)$ .But

since $h^*(\Lambda) - g^*(\Lambda) = \text{D-Sup} \bigcup_{(x,u)\in XxU} \{\Lambda u + 0x - \Psi\ (x, u)\}$

$y \geq_D (\Lambda u - \Psi\ (x, u))$ for all $u \in U$. In particular , if we put u = 0 and noting that

$f(x) = -\Psi\ (x, 0)$ it follows that $y \geq_D -\Psi\ (x, 0) = f(x)\ \forall\ y \in h^*(\Lambda) - g^*(\Lambda)$ which

contradicts our assumption. Hence the theorem is proved.

The above theorem assures us that for any x, $f(x) \in \text{Inf} \bigcup_{\Lambda\in M} [h^*(\Lambda) - g^*(\Lambda)]$.

If we can find $\Lambda_o \in M$ such that $f(x_o) \in h^*(\Lambda_o) - g^*(\Lambda_o)$ for some $x_o$ , then it means that

$\Lambda_o$ solves (Dc), the next theorem reflects this fact.

**Theorem 4.2.4:** If $x_o$ solves (P), then there exist some $\Lambda_o \in M$ which solves (Dc).

**Proof:** If $x_o$ solves (P), then since $f(x) = -\Psi\ (x, 0)$ the same $x_o$ solves

$(P')$ Min- $\Psi\ (x, 0)$ $x \in X$.

Then there exist some $\Lambda_o \in M$ such that $(0, \Lambda_o) \in \partial\ \Psi\ (x, 0)$.But this in turn implies that

$(0, \Lambda_o)(x_o, 0)^T - \Psi\ (x, 0) \in \Psi\ ^*(0, \Lambda_o)$, which means that

$f(x_o) = -\Psi\ (x_o, 0) \in \Psi\ ^*(0, \Lambda_o) = h^*(\Lambda) - g^*(\Lambda)$.

Now assume that $\Lambda_o$ does not solve ($D_C$) .Then there exist $\Lambda \in M$ such that

$h^*(\Lambda) - g^*(\Lambda) \cup B(h^*(\Lambda_o) - g^*(\Lambda_o)) \neq \varnothing$.

Since $f(x_o) \in h^*(\Lambda_o) - g^*(\Lambda_o)$ there exist $y \in h^*(\Lambda) - g^*(\Lambda)$ such that $y <_D f(x_o)$.

This implies $f(x_o) \in A(h^*(\Lambda) - g^*(\Lambda))$ but this contradicts the statement in Theorem 4.2.3

Hence $\Lambda_o$ solves (Dc).

Corollary 4.2.1: If $x_o$ solves (P) and $\Lambda_o \in \partial_s h(x_o)$, then $\Lambda_o$ solves (Dc).

Proof: From the assumption we have $\partial_s h(x_o) \subseteq \partial g(x_o)$, and the relation $T \in \partial f(x_o)$ iff $Tx_o - f(x_o) \in f^*(T)$ gives

$g(x_o) - h(x_o) = (\Lambda_o x_o - h(x_o)) - (\Lambda_o x_o - g(x_o)) \in h^*(\Lambda_o) - g^*(\Lambda_o)$

That is $f(x_o) \in h^*(\Lambda_o) - g^*(\Lambda_o)$ hence $\Lambda_o$ solves (Dc).

Proposition 4.2.2: Let $x_o \in$ dom f. If $f(x_o) \in h^*(\Lambda_o) - g^*(\Lambda_o)$ for $\Lambda_o \in M$, then $x_o$ solves (P) at least locally.

Proof: By the assumption we have

$-\Psi(x_o, 0) = f(x_o) \in h^*(\Lambda_o) - g^*(\Lambda_o) = \Psi^*(0, \Lambda_o)$

Which is equivalent to $(0, \Lambda_o)(x_o, 0) - \Psi(x_o, 0) \in \Psi^*(0, \Lambda_o)$ and by Lemma 4.2.1 we have $(0, \Lambda_o) \in \partial \Psi(x_o, 0)$ (4.2.1)

Then from the relation $-\Psi(x_o, 0) = f(x_o)$ and (4.2.1), we can see that $0 \in \partial f(x_o)$, or $x_o$ is a local minimum of f(x).

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