Superposition of Squeezed Laser Light Beams

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by

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Dedication

This work dedicated to the memory of my late father Masresha Alemu and my late mother W/o Enani Kahssay who were always longing to make a man of me.

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Abstract

In this research we have studied the squeezing and statistical properties of a pair of superposed output light beams generated by three-level lasers, with the injected atoms prepared initially in a coherent superposition of the top and bottom levels. In particular, applying the density operator for the superposed output laser light beams together with the Q functions, we have calculated the global and local mean and variance of the photon number as well as the quadrature squeezing. In addition, we have determined the mean and variance of the photon number sum and difference and the photon numbers correlation of a pair of superposed two-mode output light beams.

It is found that both the mean photon number and quadrature variance of the superposed output laser light beams is the sum of the mean photon numbers and the quadrature variances of the constituent output light beams. However, the variance of the photon number of the superposed output laser light beams is not the sum of the variances of the photon numbers of the component output laser light beams. Moreover, the global and local quadrature squeezing of a pair of superposed output laser light beams turned out to be the average of the global and local quadrature squeezing of the constituent output light beams. We have also noticed that the squeezing of the superposed output laser light beams is 61.5% below the vacuum state-level. On the other hand, we have seen that the local quadrature squeezing is greater than the global quadrature squeezing. It is found that the maximum local quadrature squeezing is 92.5% and occurs in the frequency $\lambda = \pm 0.01$. Furthermore, our results have shown that the local quadrature squeezing, unlike the local mean of the phonon number, quadrature variance, and photon number variance does not increase as the value of λ increases. In addition, our analysis indicates that the quadrature squeezing of the superposition of squeezed laser light beam and a light beam in vacuum state turned out to be just half of the quadrature squeezing of the three-level laser output light beam. Finally, we would like to mention that the photon numbers of a pair of superposed two-mode light beams are correlated.

Contents

1

Introduction

There are several optical systems that could generate light with non classical features. One such optical systems is a three-level laser, which produces a squeezed light under certain conditions. Squeezing is one of the interesting features of light that has been studied by several authors [1-14, 19-25]. This is because, in addition to exhibiting a nonclassical feature, squeezed light has applications in the detection of weak signals and in low-noise communications. In squeezed light the fluctuations in one quadrature is below the vacuum-state level level at the expense of enhanced fluctuations in the other quadrature, with the product of the uncertainties in the two quadratures satisfying the uncertainty relation [1-4, 12-19, 24-25].

It has been established that a three-level laser under certain conditions generates squeezed light [1-9, 20-21, 26-27, 32-34]. In a three-level laser, three-level atoms in a cascade configuration are injected into a cavity coupled to a vacuum reservoir via a single-port mirror. The injected atoms may initially be prepared in a coherent superposition of the top and bottom levels or these levels may be coupled by strong coherent light after they are injected into the cavity. The coherent superposition of the top and bottom levels or the coupling of these levels by coherent light is responsible for the nonclassical features of the generated laser light. When a three-level atom in a cascade configuration makes a transition from the top to the bottom level via the intermediate level, two photons are generated. If the two photons have the same frequency, the three-level atom is called degenerate otherwise it is called nondegenerate.

Some authors have studied the squeezing [28, 32-33] and statistical properties of the light produced by three-level lasers [1-7, 29-30] in which the crucial role is played by the superposition of the top and bottom levels [1-7, 31-34]. Ansari [7] has predicted that a degenerate three-level laser can generate squeezed light under certain conditions. Furthermore, Lu and

Zhu [2] have considered a nondegenerate three-level laser with the atoms initially prepared in a coherent superposition of the top and bottom levels. They have predicted a maximum of 50% intracavity two-mode squeezing.

A three-level laser in which the top and bottom levels of the atoms injected into the cavity are coupled by a strong coherent light has also been studied by different authors [7-10]. Ansari et al [10] have considered a degenerate three-level laser, with the atoms initially in the upper level and with the top and bottom levels of the atoms coupled by coherent light. They have shown that this system behaves like a parametric oscillator for sufficiently strong coherent light. They have also predicted that such a system can generate squeezed light over large range of the amplitude of the coherent light

Some authors [20-21, 27] have studied the squeezing and statistical properties of superposed coherent and squeezed light beams. Using the usual definition for quadrature squeezing of a pair of superposed light beams, these authors have found that the coherent light does not affect the quadrature squeezing of the superposed light beams. However, according to Fesseha [22], applying the usual definition for quadrature squeezing of superposed light beams leads to physically unjustifiable results. This is because one usually calculates the quadrature squeezing of a pair of superposed light beams relative to the quadrature variance of a single coherent light beam. In view of this, Fesseha [22] has modified the definition for quadrature squeezing of superposed light beams. According to this modified definition, the quadrature squeezing of a pair of superposed light beams is calculated relative to the quadrature variance of a pair of superposed coherent light beams.

In this PhD dissertation, we seek to investigate the squeezing and statistical properties of a pair of superposed output light beams generated by three-level lasers whose cavities are coupled to vacuum reservoirs. We arrange our system in such a way that the output light beam (LB1) from one of the lasers is incident on a side of a perfectly transmitting mirror, while the output light beam (LB2) from the other laser is incident on a side of a perfectly reflecting mirror as depicted in the Fig. 1.1.

In order to carry out our analysis, we first derive c-number Langevin equations for the cavity light produced by one laser using the pertinent master equation. Applying the solutions of the resulting c-number Langevin equations along with the properties of the noise forces, we obtain the antinormally ordered characteristic function, with the aid of which the Q function is determined. We then obtain the density operator for a pair of superposed output laser light

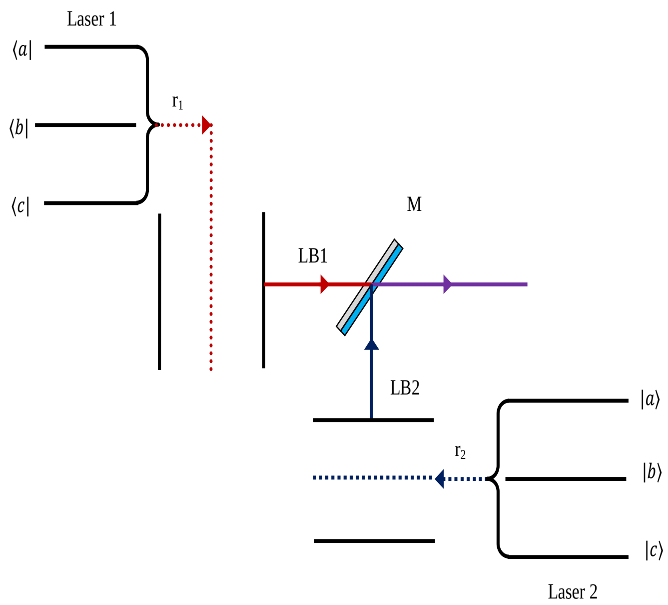

Fig. 1.1: Schematic representation of a pair of superposed laser output light beams.

beams in terms of the Q functions. The resulting density operator is then used to calculate the global and local mean and variance of the photon number as well as the quadrature squeezing. In addition, we have determined the mean and variance of the photon number sum and difference, the quadrature squeezing, and the photon number correlation for the superposed two-mode output laser light beams.

2

A Pair of Superposed Single-Mode Laser Light Beams

In this chapter we seek to investigate the squeezing and statistical properties of a pair of superposed output light beams produced by degenerate three-level lasers whose cavities are coupled to vacuum reservoirs. In order to analyze the squeezing and statistical properties of the superposed output laser light beams, we first arrange the system in such a way the output light beam from one of the lasers is incident on side of a perfectly transmitting mirror and the output light beam from the other laser is incident on side of a perfectly reflecting mirror as shown in Fig. 1.1. Then we derive c-number Langevin equations for the cavity light produced by one of the degenerate three-level laser whose cavity coupled to vacuum reservoir using the pertinent master equation. Employing the solutions of the resulting c-number Langevin equations together with the antinormally ordered characteristic function defined in Hiesenberg picture, we derive the Q function for the cavity light. Furthermore, using this Q function together with the input-output relation and with the input mode operators associated with vacuum reservoir put in normal order, we obtain the output laser light beam Q function, with the aid of which we determine the Q function for the other output laser light beam having different cavity damping constant and linear gain coefficient. Next we obtain the density operator for superposed output light beams in terms of the Q functions of the component output light beams. Finally, applying the resulting density operator together with the Q functions of the output laser light beams, we calculate the global and local quadrature variances as well as the global and local mean and the variance of the photon numbers.

2.1 c-number Langevin equations

In this section we first derive in the linear and large-time approximation schemes [3, 23, 26, 35] the equation of evolution of the density operator for the cavity mode of a degenerate three-level laser whose cavity is coupled to a vacuum reservoir. With the aid of this equation, we obtain c-number Langevin equations associated with the normal ordering.

The Master equation

Here we consider a degenerate three-level laser into which a three-level atoms in a cascade configuration and initially prepared in a coherent superposition of the top and bottom levels are injected at constant rate r_1 and removed from the laser cavity after some time τ. We denote the top, intermediate, and bottom levels of a three-level atom by $|\ a\rangle$, $|\ b\rangle$, and $|\ c\rangle$, respectively. We assume the cavity mode to be at resonance with the two transitions $|\ a\rangle \rightarrow |\ b\rangle$ and $|\ b\rangle \rightarrow |\ c\rangle$, and with direct transition between levels $|\ a\rangle$ and $|\ c\rangle$ to be dipole forbidden.

The Hamiltonian describing the interaction of a three-level atom with the cavity mode in the rotating approximation is expressible as

$$\hat{H} = ig\big[(|a\rangle\langle b| + |b\rangle\langle c|)\hat{a} - \hat{a}^{\dagger}(|b\rangle\langle a\ | + |c\rangle\langle b|)\big], \tag{2.1}$$

where g is the coupling constant and $\hat{a}$ is the annihilation operator for the cavity mode. We consider a single three-level atom to be initially in the state

$$|\psi_A(0)\rangle = C_a(0)|a\rangle + C_c(0)|c\rangle \tag{2.2}$$

and hence the density operator for the single atom has the form

$$\hat{\rho}_A(0) = \rho_{aa}^{(0)}|a\rangle\langle a| + \rho_{ac}^{(0)}|a\rangle\langle c| + \rho_{ca}^{(0)}|c\rangle\langle a| + \rho_{cc}^{(0)}|c\rangle\langle c|, \tag{2.3}$$

where

$$\rho_{aa}^{(0)} = C_a(0)C_a^*(0) \tag{2.4}$$

$$\rho_{cc}^{(0)} = C_c(0)C_c^*(0) \tag{2.5}$$

are the probabilities for the atom to be initially in the upper and lower levels, respectively, and

$$\rho_{ac}^{(0)} = \rho_{ca}^{(0)*} = C_a(0)C_c^*(0) = |\rho_{ac}^{(0)}|e^{i\theta} \tag{2.6}$$

represents the initial atomic coherence of the atom.

Suppose $\hat{\rho}_{AR}(t, t_i)$ is the density operator for a single atom plus the cavity mode at time t, with the atom injected into the cavity at an earlier time t_i, such that $(t-\tau) \leq t_i \leq t$. The density operator for all atoms plus the cavity mode at time t can be written as

$$\hat{\rho}_{AR}(t) = r_1 \sum_i \hat{\rho}_{AR}(t, t_i)\Delta t_i, \tag{2.7}$$

where r_1 represents the number of atoms injected into the cavity at a time Δt_i. Now converting the summation into integration in the limit $\Delta t_i \rightarrow 0$, we have

$$\hat{\rho}_{AR}(t) = r_1 \int_{t-\tau}^{t} \hat{\rho}_{AR}(t, t')dt' \tag{2.8}$$

and on differentiating with respect to t, and applying the relation

$$\frac{d}{dx}\int_{x+x_o}^{x} f(x,y)dy = f(x,x) - f(x, x+x_o) + \int_{x+x_o}^{x} \frac{\partial}{\partial x} f(x,y)dy \tag{2.9}$$

we find

$$\frac{d}{dt}\hat{\rho}_{AR}(t) = r_1[\hat{\rho}_{AR}(t,t) - \hat{\rho}_{AR}(t, t-\tau)] + r_1\int_{t-\tau}^{t}\frac{\partial}{\partial t}\hat{\rho}_{AR}(t,t')dt'. \tag{2.10}$$

We note that $\hat{\rho}_{AR}(t,t)$ is the density operator for the cavity mode plus an atom injected at time t and $\hat{\rho}_{AR}(t, t-\tau)$ is the density operator for the cavity mode at time t plus a single atom injected at time $t-\tau$ and being removed at time t. At the instant the atom is injected or removed from the cavity, the atomic and cavity mode variables are uncorrelated. Thus we can write these density operators as

$$\hat{\rho}_{AR}(t,t) = \hat{\rho}_A(t,t)\hat{\rho}(t) \tag{2.11}$$

$$\hat{\rho}_{AR}(t, t-\tau) = \hat{\rho}_A(t, t-\tau)\hat{\rho}(t), \tag{2.12}$$

where $\hat{\rho}(t)$ is the density operator for the cavity mode and $\hat{\rho}_A(t,t')|_{[t'=t,\ or\ \ t'=(t-\tau)]}$ is the density operator at time t for the atom injected at time t'. Since $\hat{\rho}_A(t,t)$ is the initial density for the atom, we can designate it by

$$\hat{\rho}_A(t,t) = \hat{\rho}_A(0). \tag{2.13}$$

In view of (2.11), (2.12), and (2.13), the time evolution of the density operator described by Eq. (2.10) can be written as

$$\frac{d}{dt}\hat{\rho}_{AR}(t) = r_1[\hat{\rho}_A(0) - \hat{\rho}_A(t, t-\tau)]\hat{\rho}(t) + r_1\int_{t-\tau}^{t}\frac{\partial}{\partial t}\hat{\rho}_{AR}(t,t')dt'. \tag{2.14}$$

In the absence of damping, the density operator $\hat{\rho}_{AR}(t, t^{'})$ evolves in time according to

$$\frac{d}{dt}\hat{\rho}_{AR}(t,t') = -i[\hat{H}, \hat{\rho}_{AR}(t,t')], \tag{2.15}$$

so that using this relation along with (2.8), one can put Eq. (2.14) in the form

$$\frac{d}{dt}\hat{\rho}_{AR}(t) = r_1[\hat{\rho}_A(0) - \hat{\rho}_A(t, t-\tau)]\hat{\rho}(t) - i[\hat{H}, \hat{\rho}_{AR}(t)]. \tag{2.16}$$

Furthermore, tracing over the atomic variables, we see that

$$\frac{d}{dt}\hat{\rho}(t) = -iTr_A[\hat{H}, \hat{\rho}_{AR}(t)], \tag{2.17}$$

where we have used the fact that

$$Tr_A(\hat{\rho}_{AR}(t)) = \hat{\rho}(t) \tag{2.18}$$

and

$$Tr_A(\hat{\rho}_A(t)) = Tr_A(\hat{\rho}_A(t, t-\tau)) = 1. \tag{2.19}$$

With the damping of the cavity mode by vacuum reservoir included, the equation of evolution of the density operator for the cavity mode to be

$$\frac{d}{dt}\hat{\rho}(t) = -iTr_A[\hat{H}, \hat{\rho}_{AR}(t)] + \frac{\kappa_1}{2}(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}), \tag{2.20}$$

where κ_1 is the cavity damping constant.
Employing (2.1) the master equation for the cavity mode (2.20) can be put in the form

$$\begin{aligned}\frac{d}{dt}\hat{\rho}(t) \quad &= \quad g\Big(\hat{\rho}_{ab}\hat{a}^\dagger - \hat{a}^\dagger\hat{\rho}_{ab} + \hat{\rho}_{bc}\hat{a}^\dagger - \hat{a}^\dagger\hat{\rho}_{bc} + \hat{a}\hat{\rho}_{ba} - \hat{\rho}_{ba}\hat{a} + \hat{a}\hat{\rho}_{cb} - \hat{\rho}_{cb}\hat{a}\Big) \\ &\quad + \frac{\kappa_1}{2}(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}),\end{aligned} \tag{2.21}$$

in which the matrix element $\hat{\rho}_{\alpha\beta}$ is defined by

$$\hat{\rho}_{\alpha\beta} = \langle\alpha|\hat{\rho}_{AR}|\beta\rangle, \tag{2.22}$$

with $\alpha, \beta = a, b, c$.

On the other hand, we see from (2.16) that

$$\frac{d}{dt}\hat{\rho}_{\alpha\beta}(t) = r_1\Big(\langle\alpha|\hat{\rho}_A(0)|\beta\rangle - \langle\alpha|\hat{\rho}_A(t, t-\tau) \mid \beta\rangle\Big)\hat{\rho}(t) - i\langle\alpha|[\hat{H}, \hat{\rho}_{AR}(t)]|\beta\rangle - \gamma\hat{\rho}_{\alpha\beta}. \tag{2.23}$$

The last term is included to account for the cavity of the atoms due to spontaneous emission, with atomic decay rate γ considered to be the same for all the three-levels. Assuming that the atoms are removed from the cavity after they have decayed to a level other than the middle and the bottom level, we then see that

$$\langle\alpha|\hat{\rho}_A(t, t-\tau)|\beta\rangle = 0 \tag{2.24}$$

and hence Eq. (2.23) reduces to

$$\frac{d}{dt}\hat{\rho}_{\alpha\beta}(t) = r_1\langle\alpha|\hat{\rho}_A(0)|\beta\rangle\hat{\rho}(t) - i\langle\alpha|\hat{H}\hat{\rho}_{AR}(t)|\beta\rangle + i\langle\alpha|\hat{\rho}_{AR}(t)\hat{H}|\beta\rangle - \gamma\hat{\rho}_{\alpha\beta}. \tag{2.25}$$

Now in view of (2.1) and (2.3), Eq. (2.25) takes the form

$$\begin{aligned}\frac{d}{dt}\hat{\rho}_{\alpha\beta}(t) &= r_1\Big(\rho^{(0)}_{aa}\delta_{\alpha a}\delta_{a\beta} + \rho^{(0)}_{ac}\delta_{\alpha a}\delta_{c\beta} + \rho^{(0)}_{ca}\delta_{\alpha c}\delta_{a\beta} + \rho^{(0)}_{cc}\delta_{\alpha c}\delta_{c\beta}\Big)\hat{\rho}(t)\\ &\quad +g\Big(\hat{a}\hat{\rho}_{b\beta}\delta_{\alpha a} + \hat{a}\hat{\rho}_{c\beta}\delta_{\alpha b} - \hat{a}^\dagger\hat{\rho}_{a\beta}\delta_{\alpha b} - \hat{a}^\dagger\hat{\rho}_{b\beta}\delta_{\alpha c}\\ &\quad -\hat{\rho}_{\alpha a}\hat{a}\delta_{b\beta} - \hat{\rho}_{\alpha b}\hat{a}\delta_{c\beta} + \hat{\rho}_{\alpha b}\hat{a}^\dagger\delta_{a\beta} + \hat{\rho}_{\alpha c}\hat{a}^\dagger\delta_{b\beta}\Big) - \gamma\hat{\rho}_{\alpha\beta},\end{aligned} \tag{2.26}$$

from which follows

$$\frac{d}{dt}\hat{\rho}_{aa}(t) = r_1\rho^{(0)}_{aa}\hat{\rho}(t) + g\Big(\hat{\rho}_{ab}\hat{a}^\dagger + \hat{a}\hat{\rho}_{ba}\Big) - \gamma\hat{\rho}_{aa}, \tag{2.27}$$

$$\frac{d}{dt}\hat{\rho}_{bb}(t) = g\Big(\hat{\rho}_{bc}\hat{a}^\dagger + \hat{a}\hat{\rho}_{cb} - \hat{a}^\dagger\hat{\rho}_{ab} - \hat{\rho}_{ba}\hat{a}\Big) - \gamma\hat{\rho}_{bb}, \tag{2.28}$$

$$\frac{d}{dt}\hat{\rho}_{cc}(t) = r_1\rho^{(0)}_{cc}\hat{\rho}(t) - g\Big(\hat{a}^\dagger\hat{\rho}_{bc} + \hat{\rho}_{cb}\hat{a}\Big) - \gamma\hat{\rho}_{cc}, \tag{2.29}$$

$$\frac{d}{dt}\hat{\rho}_{ab}(t) = g\Big(\hat{\rho}_{ac}\hat{a}^\dagger + \hat{a}\hat{\rho}_{bb} - \hat{\rho}_{aa}\hat{a}\Big) - \gamma\hat{\rho}_{ab}, \tag{2.30}$$

$$\frac{d}{dt}\hat{\rho}_{ac}(t) = r_1\rho^{(0)}_{ac}\hat{\rho}(t) + +g\Big(\hat{a}\hat{\rho}_{bc} - \hat{\rho}_{ab}\hat{a}\Big) - \gamma\hat{\rho}_{ac}, \tag{2.31}$$

$$\frac{d}{dt}\hat{\rho}_{bc}(t) = g\Big(\hat{a}\hat{\rho}_{cc} - \hat{\rho}_{bb}\hat{a} - \hat{a}^\dagger\hat{\rho}_{ac}\Big) - \gamma\hat{\rho}_{bc}, \tag{2.32}$$

We confine ourselves to linear analysis and this can be achieved by dropping the g-terms in Eqs. (2.27), (2.28), (2.29), and (2.31) and applying the large-time approximation scheme. Thus upon dropping the g-terms and applying the large-time approximation scheme, we get

$$\hat{\rho}_{aa} = \frac{r_1\rho^{(0)}_{aa}}{\gamma}\hat{\rho}(t), \tag{2.33}$$

$$\hat{\rho}_{bb} = 0, \tag{2.34}$$

$$\hat{\rho}_{cc} = \frac{r_1\rho^{(0)}_{cc}}{\gamma}\hat{\rho}(t), \tag{2.35}$$

$$\hat{\rho}_{ac} = \frac{r_1\rho^{(0)}_{ac}}{\gamma}\hat{\rho}(t). \tag{2.36}$$

Now in view of these results, Eqs. (2.30) and (2.32) can be put in the form

$$\frac{d}{dt}\hat{\rho}_{ab}(t) = \frac{gr_1}{\gamma}\big(\rho^{(0)}_{ac}\hat{\rho}\hat{a}^\dagger - \rho^{(0)}_{aa}\hat{\rho}\hat{a}\big) - \gamma\hat{\rho}_{ab}, \tag{2.37}$$

$$\frac{d}{dt}\hat{\rho}_{bc}(t) = \frac{gr_1}{\gamma}\big(\rho^{(0)}_{cc}\hat{a} - \rho^{(0)}_{ac}\hat{a}^\dagger\big)\hat{\rho} - \gamma\hat{\rho}_{bc}. \tag{2.38}$$

Applying the large-time approximation scheme once more, we see from Eqs. (2.37) and (2.38) that

$$\hat{\rho}_{ab} = \frac{gr_1}{\gamma^2}\big(\rho_{ac}^{(0)}\hat{\rho}\hat{a}^\dagger - \rho_{aa}^{(0)}\hat{\rho}\hat{a}\big), \tag{2.39}$$

$$\hat{\rho}_{bc} = \frac{gr_1}{\gamma^2}\big(\rho_{cc}^{(0)}\hat{a} - \rho_{ac}^{(0)}\hat{a}^\dagger\big)\hat{\rho}. \tag{2.40}$$

Finally, on account of Eqs. (2.39), (2.40) and their complex conjugate, the master equation described by (2.21) turns out to be

$$\begin{aligned}\frac{d\hat{\rho}}{dt} &= \frac{1}{2}A_1\rho_{aa}^{(0)}\big(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{a}^\dagger - \hat{a}\hat{a}^\dagger\hat{\rho}\big) + \frac{1}{2}\big(A_1\rho_{cc}^{(0)} + \kappa_1\big)\big(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}\big)\\ &+\frac{1}{2}A_1\rho_{ac}^{(0)}\big(\hat{\rho}\hat{a}^{\dagger 2} + \hat{a}^{\dagger 2}\hat{\rho} - 2\hat{a}^\dagger\hat{\rho}\hat{a}^\dagger\big) + \frac{1}{2}A_1\rho_{ca}^{(0)}\big(\hat{\rho}\hat{a}^2 + \hat{a}^2\hat{\rho} - 2\hat{a}\hat{\rho}\hat{a}\big),\end{aligned} \tag{2.41}$$

where

$$A_1 = 2r_1g^2/\gamma^2 \tag{2.42}$$

is the linear gain coefficient.

It is worth mentioning that the quantum properties of the light generated by the three-level laser are determined by the master equation. It is easy to observe that with $\rho_{aa}^{(0)} = 1$ and $\rho_{ac}^{(0)} = \rho_{cc}^{(0)} = 0$, this equation reduces to the master equation for a two-level laser operating below threshold.

c-number Langevin equations

We next seek to obtain, using the master equation, the c-number Langevin equations associated with the normal ordering. The equation of evolution for the expectation value of an operator in the Schrödinger picture is expressible as

$$\frac{d}{dt}\langle\hat{O}\rangle = Tr\Big(\frac{d\hat{\rho}}{dt}\hat{O}\Big). \tag{2.43}$$

To this end, employing the relation (2.43) along with Eq. (2.41), we easily find

$$\frac{d}{dt}\langle\hat{a}(t)\rangle = -\frac{1}{2}\mu_1\langle\hat{a}(t)\rangle, \tag{2.44}$$

$$\frac{d}{dt}\langle\hat{a}(t)\hat{a}(t)\rangle = -\mu_1\langle\hat{a}^2(t)\rangle + A_1\rho_{ac}^{(0)}, \tag{2.45}$$

$$\frac{d}{dt}\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle = -\mu_1\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle + A_1\rho_{aa}^{(0)}, \tag{2.46}$$

in which

$$\mu_1 = A_1(\rho_{cc}^{(0)} - \rho_{aa}^{(0)}) + \kappa_1. \tag{2.47}$$

We see that the c-number equation corresponding to Eqs. (2.44-2.46) in the normal order are

$$\frac{d}{dt}\langle\alpha(t)\rangle = -\frac{1}{2}\mu_1\langle\alpha(t)\rangle, \tag{2.48}$$

$$\frac{d}{dt}\langle\alpha(t)\alpha(t)\rangle = -\mu_1\langle\alpha^2(t)\rangle + A_1\rho_{ac}^{(0)}, \tag{2.49}$$

$$\frac{d}{dt}\langle\alpha^*(t)\alpha(t)\rangle = -\mu_1\langle\alpha^*(t)\alpha(t)\rangle + A_1\rho_{aa}^{(0)}. \tag{2.50}$$

On the basis of Eq. (2.48), one can write

$$\frac{d}{dt}\alpha(t) = -\frac{1}{2}\mu_1\alpha(t) + f_1(t), \tag{2.51}$$

where $f_1(t)$ is a noise force the properties of which remain to be determined.

We note that Eq. (2.48) and the expectation value of Eq.(2.51) will have identical forms if

$$\langle f_1(t)\rangle = 0. \tag{2.52}$$

Moreover, it can be readily verified using (2.51) that

$$\frac{d}{dt}\langle\alpha(t)\alpha(t)\rangle = -\mu_1\langle\alpha^2(t)\rangle + 2\langle\alpha(t)f_1(t)\rangle. \tag{2.53}$$

Comparison of Eqs. (2.49) and (2.53) shows that

$$\langle\alpha(t)f_1(t)\rangle = \frac{1}{2}A_1\rho_{ac}^{(0)}. \tag{2.54}$$

A formal solution of Eq. (2.51) can be written as

$$\alpha(t) = \alpha(0)e^{-\mu_1 t/2} + \int_0^t e^{-\mu_1(t-t')/2}f_1(t')dt'. \tag{2.55}$$

We then note that

$$\langle\alpha(t)f_1(t)\rangle = \langle\alpha(0)f_1(t)\rangle e^{-\mu_1 t/2} + \int_0^t e^{-\mu_1(t-t')/2}\langle f_1(t)f_1(t')\rangle dt'. \tag{2.56}$$

On account of the assertion that a noise force at time t should not affect the system variables at earlier times, we have

$$\langle\alpha(t)f_1(t)\rangle = \int_0^t e^{-\mu_1(t-t')/2}\langle f_1(t)f_1(t')\rangle dt', \tag{2.57}$$

so that in view of (2.54), there follows

$$\int_0^t e^{-\mu_1(t-t')/2}\langle f_1(t)f_1(t')\rangle dt' = \frac{1}{2}A_1\rho_{ac}^{(0)}. \tag{2.58}$$

Now in view of the property of Dirac-delta function

$$\int_0^t g(t')\delta(t-t')dt' = \int_t^{t_1} g(t')\delta(t-t')dt' = \frac{1}{2}g(t), \tag{2.59}$$

one can put Eq. (2.58) in the form

$$\int_0^t e^{-\mu_1(t-t')/2}\langle f_1(t)f_1(t')\rangle dt' = A_1\rho_{ac}^{(0)}\int_0^t e^{-\mu_1(t-t')/2}\delta(t-t')dt'. \tag{2.60}$$

It then follows that

$$\langle f_1(t)f_1(t')\rangle = A_1\rho_{ac}^{(0)}\delta(t-t'). \tag{2.61}$$

Furthermore, it can be established applying (2.51) and its complex conjugate that

$$\frac{d}{dt}\langle\alpha^*(t)\alpha(t)\rangle = -\mu_1\langle\alpha^*(t)\alpha(t)\rangle + \langle\alpha^*(t)f_1(t)\rangle + \langle\alpha(t)f_1^*(t)\rangle. \tag{2.62}$$

On comparing this with (2.50), we have

$$\langle\alpha^*(t)f_1(t)\rangle + \langle\alpha(t)f_1^*(t)\rangle = A_1\rho_{aa}^{(0)}, \tag{2.63}$$

In addition, using (2.55) and its complex conjugate, we easily get

$$\langle\alpha(t)f_1^*(t)\rangle = \int_0^t e^{-\mu_1(t-t')/2}\langle f_1^*(t)f(t')\rangle dt' \tag{2.64a}$$

$$\langle\alpha^*(t)f_1(t)\rangle = \int_0^t e^{-\mu_1(t-t')/2}\langle f_1(t)f_1^*(t')\rangle dt'. \tag{2.64b}$$

Now taking into account (2.63), (2.64a), and (2.64b), and assuming that

$$\langle f_1^*(t)f_1(t')\rangle = \langle f_1(t)f_1^*(t')\rangle, \tag{2.65}$$

we arrive at

$$\int_0^t e^{-\mu_1(t-t')/2}\langle f_1^*(t)f_1(t')\rangle dt' = \int_0^t e^{-\mu_1(t-t')/2}\langle f_1(t)f_1^*(t')\rangle dt' = \frac{1}{2}A_1\rho_{aa}^{(0)}. \tag{2.66}$$

Therefore, in view of (2.59), we see that

$$\langle f_1^*(t)f_1(t')\rangle = \langle f_1(t)f_1^*(t')\rangle = A_1\rho_{aa}^{(0)}\delta(t-t'). \tag{2.67}$$

It is worth mentioning that (2.52), (2.61), and (2.67) describe the correlation properties of the noise force $f_1(t)$ associated with normal ordering.

2.2 The Q functions for the output laser light beams

Various quantum distribution functions are widely used in quantum optics, because they provide convenient means of describing the quantum properties of light modes. Here we confine our discussion to the Q function which is best suited to the evaluation of the expectation value of antinormally ordered operators [3, 15-18]. To this end, the Q function for a single-mode light can be expressed as

$$Q(\alpha^*, \alpha, t) = \frac{1}{\pi^2} \int d^2 z \phi_a(z^*, z, t) \exp{(z^* \alpha - z \alpha^*)}, \tag{2.68}$$

where

$$\phi_a(z^*, z, t) = Tr\Big(\hat{\rho}(0) e^{-z^* \hat{a}(t)} e^{z \hat{a}^\dagger(t)}\Big) \tag{2.69}$$

is the antinormally ordered characteristic function defined in Heisenberg picture. Applying the identity

$$e^{\hat{A}} e^{\hat{B}} = e^{\hat{B}} e^{\hat{A}} e^{[\hat{A}, \hat{B}]}, \tag{2.70}$$

one can write Eq. (2.69) as

$$\phi_a(z^*, z, t) = e^{-z^* z} Tr\Big(\hat{\rho}(0) e^{z \hat{a}^\dagger(t)} e^{-z^* \hat{a}(t)}\Big). \tag{2.71}$$

Although we are interested in the case for which the cavity mode is initially in a vacuum state, we also need the Q function in which the cavity mode is initially in arbitrary state. Thus the density operator, $\hat{\rho}(0)$ that appears in Eq. (2.71) can be expressible in terms of the P function as [3, 8, 15-17]

$$\hat{\rho}(0) \quad = \quad \int d^2 \lambda_o P_j(\lambda_o^*, \lambda_o) |\lambda_o\rangle\langle\lambda_o|. \tag{2.72}$$

In view of this fact, one can put Eq. (2.71) in the form

$$\phi_a(z^*, z, t) = e^{-z^* z} \int d^2 \lambda_o P_j(\lambda_o^*, \lambda_o) Tr\Big(|\lambda_o\rangle\langle\lambda_o| e^{z \hat{a}^\dagger(t)} e^{-z^* \hat{a}(t)}\Big). \tag{2.73}$$

We note that the dynamics of a cavity mode coupled to a vacuum reservoir can also be described using quantum Langevin equation [3]. Therefore, on the basis of Eq. (2.44), one can write

$$\frac{d}{dt} \hat{a}(t) = -\frac{1}{2} \mu_j \hat{a}(t) + F_j(t), \;\; j = 1, 2, \tag{2.74}$$

where $\hat{F}_j(t)$ is a noise operator with vanishing mean. Hence a formal solution of this equation is expressible as

$$\hat{a}(t) = \hat{a}(0) e^{-\mu_j t/2} + \hat{a}'(t), \tag{2.75}$$

where

$$\hat{a}'(t) = \int_0^t e^{-\mu_j(t-t')/2}\hat{F}_j(t')dt'. \tag{2.76}$$

Furthermore, taking into account of Eq. (2.75) along with the fact that

$$[\hat{a}(0), \hat{a}'(t)] = [\hat{a}^\dagger(0), \hat{a}'^\dagger(t)] = 0, \tag{2.77}$$

we readily establish that

$$\begin{aligned}\phi_a(z^*, z, t) \;&=\; \int d^2\lambda_o P_j(\lambda_o^*, \lambda_o)\exp\big[-z^*z + (\lambda_o^* z - \lambda_o z^*)e^{-\mu_j t/2}\big]\\ &\times\Big\langle \exp\big(z\hat{a}'^\dagger(t) - z^*\hat{a}'(t)\big)\Big\rangle.\end{aligned} \tag{2.78}$$

We note that Eq. (2.78) can be expressed in terms of c number variables associated with the normal ordering as

$$\begin{aligned}\phi_a(z^*, z, t) \;&=\; \int d^2\lambda_o P_j(\lambda_o^*, \lambda_o)\exp\big[-z^*z + (\lambda_o^* z - \lambda_o z^*)e^{-\mu_j t/2}\big]\\ &\times\Big\langle \exp\big(z\alpha'^*(t) - z^*\alpha'(t)\big)\Big\rangle,\end{aligned} \tag{2.79}$$

where

$$\alpha'(t) = \int_0^t e^{-\mu_j(t-t')/2} f_j(t')dt' \tag{2.80}$$

is the c number equation corresponding to Eq. (2.76) in normal order. With the aid of Eq. (2.80), the equation of evolution of α' can be written as

$$\frac{d}{dt}\alpha'(t) = -\frac{1}{2}\mu_j\alpha'(t) + f_j(t), \;\; j = 1, 2. \tag{2.81}$$

We observe that Eq. (2.81) is a linear differential equation. In addition, using (2.52), we see that

$$\langle \alpha'(t)\rangle = 0. \tag{2.82}$$

Hence from Eqs. (2.81) and (2.82), one can observe that α' is Gaussian variable with a vanishing mean. Thus in view of this Eq. (2.79) can be put in the form

$$\begin{aligned}\phi_a(z^*, z, t) \;&=\; \int d^2\lambda_o P_j(\lambda_o^*, \lambda_o)\exp\big[-z^*z + (\lambda_o^* z - \lambda_o z^*)e^{-\mu_j t/2}\big]\\ &\times \exp\Big(\frac{1}{2}\Big\langle \big[z\alpha'^*(t) - z^*\alpha'(t)\big]^2\Big\rangle\Big).\end{aligned} \tag{2.83}$$

It then follows that

$$\begin{aligned}\phi_a(z^*, z, t) \;&=\; \int d^2\lambda_o P_j(\lambda_o^*, \lambda_o)\exp\Big[-z^*z\Big(1 + \langle\alpha'^*(t)\alpha'(t)\rangle\Big) + (\lambda_o^* z - \lambda_o z^*)e^{-\mu_j t/2}\\ &+\frac{1}{2}\Big(z^2\langle\alpha'^{*2}(t)\rangle + z^{*2}\langle\alpha'^2(t)\rangle\Big)\Big].\end{aligned} \tag{2.84}$$

Next we proceed to determine the various expectation values involved in Eq. (2.84). To this end, taking Eq. (2.80) into account, one can write

$$\langle\alpha'^*(t)\alpha'(t)\rangle = \int_0^t e^{-\mu_j(t-t')/2}\langle\alpha'^*(t)f_j(t')\rangle dt'. \tag{2.85}$$

Moreover, taking the complex conjugate of Eq. (2.80), it is also possible to see that

$$\langle\alpha'^*(t)f_j(t')\rangle = \int_0^t e^{-\mu_j(t-t'')/2}\langle f_j^*(t'')f_j(t')\rangle dt''. \tag{2.86}$$

Applying (2.67), one can write Eq. (2.86) as

$$\langle\alpha'^*(t)f_j(t')\rangle = A_j\rho_{aa}^{(0)}\int_0^t e^{-\mu_j(t-t'')/2}\delta(t'-t'')dt'', \tag{2.87}$$

from which we obtain

$$\langle\alpha'^*(t)f_j(t')\rangle = A_j\rho_{aa}^{(0)}e^{-\mu_j(t-t')/2}. \tag{2.88}$$

Substituting (2.88) to Eq. (2.85), one can readily gets

$$\langle\alpha'^*(t)\alpha'(t)\rangle = \frac{A_j\rho_{aa}^{(0)}}{\mu_j}(1-e^{-\mu_j t}). \tag{2.89}$$

Following a similar calculations, it is not difficult to show that

$$\langle\alpha'^2(t)\rangle = \frac{A_j \mid \rho_{ac}^{(0)} \mid e^{i\theta}}{\mu_j}(1-e^{-\mu_j t}), \tag{2.90a}$$

$$\langle\alpha'^{*2}(t)\rangle = \frac{A_j \mid \rho_{ac}^{(0)} \mid e^{-i\theta}}{\mu_j}(1-e^{-\mu_j t}). \tag{2.90b}$$

It proves to be more convenient to introduce a new parameter defined by

$$\eta = \rho_{cc}^{(0)} - \rho_{aa}^{(0)}. \tag{2.91}$$

Thus in view of Eqs. (2.3) and (2.19), we see that

$$\rho_{cc}^{(0)} + \rho_{aa}^{(0)} = 1. \tag{2.92}$$

Using this relation along with Eq. (2.91), we get

$$\rho_{aa}^{(0)} = \frac{1-\eta}{2}, \tag{2.93a}$$

$$\rho_{cc}^{(0)} = \frac{1+\eta}{2}, \tag{2.93b}$$

$$\rho_{ac}^{(0)} = \frac{\sqrt{1-\eta^2}}{2}e^{i\theta}. \tag{2.93c}$$

Since the value $\rho_{aa}^{(0)}$ is between 0 and 1, it is not difficult to see that the value of η lies in the interval $-1 \leq \eta \leq 1$.

On account of (2.89) and (2.90) along with the relation (2.93) together with the assumption that $\rho_{ac}^{(0)}$ is real, the characteristic function described by Eq. (2.84) takes the form

$$\phi_a(z^*, z, t) = \int d^2\lambda_o P_j(\lambda_o^*, \lambda_o) \exp\Big[- a_j z^* z + (\lambda_o^* z - \lambda_o z^*) e^{-\mu_j t/2} + b_j(z^2 + z^{*2})/2 \Big], \quad (2.94)$$

in which

$$a_j = 1 + \frac{A_j(1-\eta)}{2(A_j\eta + \kappa_j)}(1 - e^{-(A_j\eta+\kappa_j)t}), \quad (2.95a)$$

$$b_j = \frac{A_j\sqrt{1-\eta^2}}{2(A_j\eta + \kappa_j)}(1 - e^{-(A_j\eta+\kappa_j)t}), \;\; j = 1, 2. \quad (2.95b)$$

With the aid of Eq. (2.94), the Q function for the cavity light described by the Eq. (2.68) is expressible as

$$Q(\alpha^*, \alpha, t) = \frac{1}{\pi}\int d^2\lambda_o P_j(\lambda_o^*, \lambda_o) \int \frac{d^2 z}{\pi} \exp\Big[- a_j z^* z + (\lambda_o^* e^{-\mu_j t/2} - \alpha^*) z - (\lambda_o e^{-\mu_j t/2} - \alpha) z^* + b_j(z^2 + z^{*2})/2 \Big]. \quad (2.96)$$

Employing the relation

$$\int \frac{d^2 z}{\pi} \exp\Big[- a z^* z + b z + c z^* + A z^2 + B z^{*2} \Big] = \Big[\frac{1}{a^2 - 4AB}\Big]^{1/2} \exp\Big[\frac{abc + Ac^2 + Bb^2}{a^2 - 4AB}\Big], \;\; a > 0 \quad (2.97)$$

the integration over z can be evaluated to give

$$Q_j(\alpha^*, \alpha, t) = \frac{[u_j^2 - v_j^2]^{1/2}}{\pi} \int d^2\lambda_o P_j(\lambda_o^*, \lambda_o) R_j(\lambda_o^*, \lambda_o, t) \exp\Big[- u_j \alpha^* \alpha + \big[(u_j\lambda_o^* - v_j\lambda_o)\alpha + (u_j\lambda_o - v_j\lambda_o^*)\alpha^*\big] e^{-\mu_j t/2} + v_j(\alpha^2 + \alpha^{*2})/2 \Big], \quad (2.98)$$

where

$$u_j = \frac{a_j}{a_j^2 - b_j^2}, \quad (2.99a)$$

$$v_j = \frac{b_j}{a_j^2 - b_j^2}, \quad (2.99b)$$

with

$$R_j(\lambda_o^*, \lambda_o, t) = \exp\Big[\big[- u_j\lambda_o^*\lambda_o + v_j(\lambda_o^2 + \lambda_o^{*2})/2\big] e^{-\mu_j t}\Big]. \quad (2.100)$$

The expression described by Eq. (2.98) represents the normalized Q function for the cavity light in terms of arbitrary P function.

We next wish to obtain the normalized Q function for the output laser light beam. Thus according to the input-output relation, the output mode operator $\hat{a}^{out}(t)$ can be written in terms of the cavity mode operator $\hat{a}(t)$ and input mode operator $\hat{a}_{in}(t)$ as

$$\hat{a}^{out}(t) = \sqrt{\kappa_j}\hat{a}(t) - \hat{a}_{in}(t), \quad (2.101)$$

where κ_j is the cavity damping constant, with $j = 1, 2$. However, for a cavity mode coupled to a vacuum reservoir and with the input mode operator associated with vacuum reservoir put in normal order, one can write the output operator as

$$\hat{a}^{out}(t) = \sqrt{\kappa_j}\hat{a}(t). \tag{2.102}$$

The c-number equation corresponding to Eq. (2.102) associated with the normal ordering is

$$\alpha^{out}(t) = \sqrt{\kappa_j}\alpha(t). \tag{2.103}$$

Hence upon replacing $\alpha(t)$ by $\alpha(t)/\sqrt{\kappa_j}$ in the Eq. (2.98), the Q function for the j^{th} output laser light beam found to be

$$\begin{aligned} Q_j(\alpha^*, \alpha, t) \;\; = \;\; & \frac{[u_j^2 - v_j^2]^{1/2}}{\kappa_j \pi} \int d^2\lambda_o P_j(\lambda_o^*, \lambda_o) R_j(\lambda_o^*, \lambda_o, t) \exp\Big[- u_j \alpha^* \alpha/\kappa_j \\ & + \big[(u_j\lambda_o^* - v_j\lambda_o)\alpha/\sqrt{\kappa_j} + (u_j\lambda_o - v_j\lambda_o^*)\alpha^*/\sqrt{\kappa_j}\big] e^{-\mu_j t/2} \\ & + v_j(\alpha^2 + \alpha^{*2})/2\kappa_j \Big], \;\; j = 1, 2. \end{aligned} \tag{2.104}$$

We note that

$$\int d^2\alpha Q_j(\alpha^*, \alpha, t) = \int d^2\lambda_o P_j(\lambda_o^*, \lambda_o) = 1. \tag{2.105}$$

Thus we see that Q the function described by Eq. (2.104) is normalized .

Suppose the cavity mode is initially in a vacuum state $|0\rangle$. Then using the fact that

$$P_j(\lambda_o^*, \lambda_o) = \delta_j^{(2)}(\lambda_o), \tag{2.106}$$

we easily arrive at

$$Q_j(\alpha^*, \alpha, t) = \frac{[u_j^2 - v_j^2]^{1/2}}{\kappa_j \pi} \exp\Big[- u_j \alpha^* \alpha/\kappa_j + v_j(\alpha^2 + \alpha^{*2})/2\kappa_j \Big]. \tag{2.107}$$

2.3 Quadrature squeezing

In this section we wish to study the squeezing properties of the superposed output laser light beams using the density operator. To this end, we first derive the density operator for the superposed laser light beams.

Suppose $\hat{\rho}'(t)$ is the density operator for a certain light beam, say for the first laser light beam. Then upon expanding this density operator in the normal order and introducing the completeness relation for coherent state , we have

$$\hat{\rho}' = \int \frac{d^2\beta}{\pi} \sum_{l,m} C_{lm}(t)\beta^{*l}|\beta\rangle\langle\beta|\hat{a}^m. \tag{2.108}$$

Now applying the identity

$$|\beta\rangle\langle\beta|\hat{a}^n = \big(\beta + \frac{\partial}{\partial\beta^*}\big)^n|\beta\rangle\langle\beta|, \tag{2.109}$$

we see that

$$\begin{aligned}\hat{\rho}' &= \int \frac{d^2\beta}{\pi} \sum_{l,m} C_{lm}(t)\beta^{*l}\big(\beta + \frac{\partial}{\partial\beta^*}\big)^m|\beta\rangle\langle\beta| &(2.110)\\ &= \int d^2\beta Q_1\big(\beta^*, \beta + \frac{\partial}{\partial\beta^*}, t\big)|\beta\rangle\langle\beta|. &(2.111)\end{aligned}$$

This expression can be rewritten as

$$\hat{\rho}' = \int d^2\beta Q_1\big(\beta^*, \beta + \frac{\partial}{\partial\beta^*}, t\big)\hat{D}(\beta)\hat{\rho}(0)\hat{D}(-\beta), \tag{2.112}$$

where $\hat{D}(\beta)$ is the displacement operator and

$$\hat{\rho}(0) = |0\rangle\langle 0| \tag{2.113}$$

represents the density operator for the light initially in the cavity.

We now realize that the density operator for the superposition of the first light beam and another one is expressible as

$$\hat{\rho}(t) = \int d^2\gamma Q_2\big(\gamma^*, \gamma + \frac{\partial}{\partial\gamma^*}, t\big)\hat{D}(\gamma)\hat{\rho}'\hat{D}(-\gamma), \tag{2.114}$$

so that in view of (2.111), we have

$$\hat{\rho}(t) = \int d^2\beta d^2\gamma Q_1\big(\beta^*, \beta + \frac{\partial}{\partial\beta^*}, t\big)Q_2\big(\gamma^*, \gamma + \frac{\partial}{\partial\gamma^*}, t\big)\hat{D}(\gamma)|\beta\rangle\langle\beta|\hat{D}(-\gamma). \tag{2.115}$$

On account of the fact that

$$\hat{D}(\gamma)|\beta\rangle = \exp\Big[\frac{1}{2}\big(\gamma\beta^* - \gamma^*\beta\big)\Big]|\gamma + \beta\rangle, \tag{2.116}$$

the density operator (2.115) takes the form

$$\hat{\rho}(t) = \int d^2\beta d^2\gamma Q_1\big(\beta^*, \beta + \frac{\partial}{\partial\beta^*}, t\big)Q_2\big(\gamma^*, \gamma + \frac{\partial}{\partial\gamma^*}, t\big)|\gamma + \beta\rangle\langle\beta + \gamma|. \tag{2.117}$$

2.3.1 Global quadrature squeezing

Here we calculate the global (in the entire frequency interval) quadrature squeezing of the superposed output light beams. To this end, the quadrature operators for the superposed output light beams is defined by

$$\hat{a}_{\pm} = \sqrt{\pm 1}(\hat{a}^{\dagger} \pm \hat{a}). \tag{2.118}$$

The global variance of the quadratures represented by (2.118) is expressible as

$$\begin{aligned}(\Delta a_{\pm}^2)_s &= \pm\langle \hat{a}^{\dagger} \pm \hat{a}, \hat{a}^{\dagger} \pm \hat{a}\rangle \\ &= \pm[\langle(\hat{a}^{\dagger} \pm \hat{a})^2\rangle - \langle \hat{a}^{\dagger} \pm \hat{a}\rangle^2],\end{aligned} \tag{2.119}$$

where the subscript ”s” stands for superposed.
Now the expectation value of the operator $\hat{a}(t)$ can be written as

$$\langle \hat{a}(t)\rangle = Tr[\hat{\rho}(t)\hat{a}(0)]. \tag{2.120}$$

Applying (2.117) in Eq. (2.120), we have

$$\langle \hat{a}(t)\rangle = \int d^2\beta d^2\gamma Q_1\big(\beta^*, \beta + \frac{\partial}{\partial\beta^*}, t\big) Q_2\big(\gamma^*, \gamma + \frac{\partial}{\partial\gamma^*}, t\big)\big(\beta + \gamma\big). \tag{2.121}$$

In view of the fact that

$$\langle \hat{a}_1(t)\rangle = \int d^2\beta Q_1\big(\beta^*, \beta + \frac{\partial}{\partial\beta^*}, t\big)\beta \tag{2.122}$$

$$\langle \hat{a}_2(t)\rangle = \int d^2\gamma Q_2\big(\gamma^*, \gamma + \frac{\partial}{\partial\gamma^*}, t\big)\gamma, \tag{2.123}$$

one can write

$$\langle \hat{a}(t)\rangle = \langle \hat{a}_1(t)\rangle + \langle \hat{a}_2(t)\rangle. \tag{2.124}$$

Making use of the Q function described by (2.104), with (α, α^*) replaced once by (β, β^*) and more (γ, γ^*), one can readily establish that

$$\langle \hat{a}_j(t)\rangle = \sqrt{\kappa_j} \int d^2\lambda_o P_j(\lambda_o^*, \lambda_o)\lambda_o e^{-\mu_j t/2}, \;\; j = 1, 2. \tag{2.125}$$

Moreover, on account of (2.125), the equation of evolution for the expectation value of the output operators of the component light beams can be written as

$$\frac{d}{dt}\langle \hat{a}_j(t)\rangle = -\frac{1}{2}\mu_j\langle \hat{a}_j(t)\rangle, \;\; j = 1, 2. \tag{2.126}$$

On the basis of this relations, one can write

$$\frac{d}{dt}\hat{a}_j(t) = -\frac{1}{2}\mu_j\hat{a}_j(t) + \hat{F}_j(t), \;\; j = 1, 2, \tag{2.127}$$

with $\hat{F}_j(t)$ being a noise operator with a vanishing mean. The solution of this equation can be written as

$$\hat{a}_j(t) = \hat{a}_j(t_o)e^{-\mu_j(t-t_o)/2} + \int_{t_o}^{t} e^{-\mu_j(t-t')/2}\hat{F}_j(t')dt', \;\; j = 1, 2 \tag{2.128}$$

In view of this result, one can write

$$\hat{a}(t) = \hat{a}_1(t) + \hat{a}_2(t). \tag{2.129}$$

This represents the operator for the superposed output laser light beams.

The expression represented by Eq. (2.124) can also be written in explicit form

$$\langle\hat{a}(t)\rangle = \langle\hat{a}_1(t_o)\rangle e^{-\mu_1(t-t_o)/2} + \langle\hat{a}_2(t_o)\rangle e^{-\mu_2(t-t_o)/2}. \tag{2.130}$$

In view of (2.129), one can show that

$$[\hat{a}(t), \hat{a}^\dagger(t)] = \kappa_1 + \kappa_2, \tag{2.131}$$

with

$$[\hat{a}_i(t), \hat{a}_j^\dagger(t)] = \kappa_i\delta_{ij}. \tag{2.132}$$

Using the commutation relation (2.131), the global quadrature variance of a pair of superposed output light beams can then be rewritten in the form

$$(\Delta a_\pm^2)_s = \kappa_1 + \kappa_2 + (:\Delta a_\pm^2:)_s, \tag{2.133}$$

where

$$\begin{aligned}(:\Delta a_\pm^2:)_s &= \langle :\hat{a}_\pm, \hat{a}_\pm: \rangle \\ &= \pm[\langle\hat{a},\hat{a}\rangle + \langle\hat{a}^\dagger,\hat{a}^\dagger\rangle \pm 2\langle\hat{a}^\dagger,\hat{a}\rangle]\end{aligned} \tag{2.134}$$

is the normally-ordered global quadrature variance.

Moreover, suppose the cavity mode is initially in a vacuum state. Thus applying (2.106) in Eq. (2.125), we obtain

$$\langle\hat{a}_j(t)\rangle = \langle\hat{a}_j^\dagger(t)\rangle = 0, \;\; j = 1, 2. \tag{2.135}$$

Therefore, one can write

$$\langle\hat{a}(t)\rangle = 0. \tag{2.136}$$

In view of (2.136), we see that

$$(\Delta a_{\pm}^2)_s \quad = \quad \kappa_1 + \kappa_2 \pm [\langle \hat{a}^2 \rangle + \langle \hat{a}^{\dagger 2} \rangle \pm 2\langle \hat{a}^{\dagger} \hat{a} \rangle]. \tag{2.137}$$

Now the expectation value of the operator $\hat{a}^{\dagger}(t)\hat{a}(t)$ is expressible as

$$\langle \hat{a}^{\dagger}(t)\hat{a}(t) \rangle = Tr[\hat{\rho}(t)\hat{a}^{\dagger}(0)\hat{a}(0)]. \tag{2.138}$$

Applying (2.117) in Eq. (2.138), we have

$$\begin{aligned} \langle \hat{a}^{\dagger}(t)\hat{a}(t) \rangle \quad &= \quad \int d^2\beta d^2\gamma Q_1\big(\beta^*, \beta + \frac{\partial}{\partial \beta^*}, t\big) Q_2\big(\gamma^*, \gamma + \frac{\partial}{\partial \gamma^*}, t\big) \\ &\quad \times \big(\beta^*\beta + \beta^*\gamma + \gamma^*\gamma + \gamma^*\beta\big). \end{aligned} \tag{2.139}$$

This can be put in the form

$$\langle \hat{a}^{\dagger}(t)\hat{a}(t) \rangle \quad = \quad \langle \hat{a}_1^{\dagger}(t)\hat{a}_1(t) \rangle + \langle \hat{a}_1^{\dagger}(t) \rangle \langle \hat{a}_2(t) \rangle + \langle \hat{a}_2^{\dagger}(t)\hat{a}_2(t) \rangle + \langle \hat{a}_2^{\dagger}(t) \rangle \langle \hat{a}_1(t) \rangle. \tag{2.140}$$

in which

$$\langle \hat{a}_1^{\dagger}(t)\hat{a}_1(t) \rangle \quad = \quad \int d^2\beta Q_1\big(\beta^*, \beta + \frac{\partial}{\partial \beta^*}, t\big)\beta^*\beta \tag{2.141a}$$

$$\langle \hat{a}_2^{\dagger}(t)\hat{a}_2(t) \rangle \quad = \quad \int d^2\gamma Q_2\big(\gamma^*, \gamma + \frac{\partial}{\partial \gamma^*}, t\big)\gamma^*\gamma. \tag{2.141b}$$

Furthermore, on account of (2.135), Eq. (2.140) reduces to

$$\langle \hat{a}^{\dagger}(t)\hat{a}(t) \rangle \quad = \quad \langle \hat{a}_1^{\dagger}(t)\hat{a}_1(t) \rangle + \langle \hat{a}_2^{\dagger}(t)\hat{a}_2(t) \rangle. \tag{2.142}$$

Next we seek to evaluate the expectation values involved in this expression. To this end, we note that the Q function described by (2.107) can be rewritten as

$$Q_1(\beta^*, \beta + \frac{\partial}{\partial \beta^*}, t) \quad = \quad Q_1(\beta^*, \beta, t) \exp\Big[\Big(v_1\beta - u_1\beta^*\Big)\frac{\partial}{\partial \beta^*} + \frac{v_1\kappa_1}{2}\frac{\partial^2}{\partial \beta^{*2}}\Big], \tag{2.143}$$

where

$$Q_1(\beta^*, \beta, t) \quad = \quad \frac{\big(u_1^2 - v_1^2\big)^{1/2}}{\kappa_1 \pi} \exp\Big[- u_1\beta^*\beta/\kappa_1 + v_1(\beta^2 + \beta^{*2})/2\kappa_1\Big] \tag{2.144}$$

is the Q function for the first output laser light beam, with u_1 and v_1 defined by (2.99a) and (2.99b). Now making use of the Q function described by Eq. (2.143), one can rewrite Eq. (2.141a) as

$$\begin{aligned} \langle \hat{a}_1^{\dagger}(t)\hat{a}_1(t) \rangle \quad &= \quad \frac{\big(u_1^2 - v_1^2\big)^{1/2}}{\kappa_1} \int \frac{d^2\beta}{\pi} \exp\Big[- u_1\beta^*\beta/\kappa_1 + v_1(\beta^2 + \beta^{*2})/2\kappa_1 \\ &\quad + \Big(v_1\beta - u_1\beta^*\Big)\frac{\partial}{\partial \beta^*} + \frac{v_1\kappa_1}{2}\frac{\partial^2}{\partial \beta^{*2}}\Big]\beta^*\beta. \end{aligned} \tag{2.145}$$

$$= \quad \int d^2\beta Q_1(\beta^*, \beta, t) \exp\Big[\Big(v_1\beta - u_1\beta^*\Big)\frac{\partial}{\partial \beta^*} + \frac{v_1\kappa_1}{2}\frac{\partial^2}{\partial \beta^{*2}}\Big]\beta^*\beta. \tag{2.146}$$

This can be put in the form

$$\langle \hat{a}_1^\dagger(t)\hat{a}_1(t)\rangle = \int d^2\beta \frac{\partial}{\partial\lambda}\Big[Q_1(\beta^*,\beta,t)d_1(\lambda,\beta^*,\beta)\Big]\Big|_{\lambda=0}, \tag{2.147}$$

where

$$d_1(\lambda,\beta^*,\beta) = \exp\left[\Big(v_1\beta - u_1\beta^*\Big)\frac{\partial}{\partial\beta^*} + \frac{v_1\kappa_1}{2}\frac{\partial^2}{\partial\beta^{*2}}\right]\exp(\lambda\beta^*\beta). \tag{2.148}$$

Now considering a differential operator $\hat{A}$ satisfying an eigenvalue equation

$$\hat{A}f(x) = af(x) \tag{2.149}$$

and using power series one can easily see that

$$e^{\hat{A}}f(x) = e^a f(x). \tag{2.150}$$

Using this result, we see that

$$d_1(\lambda,\beta^*,\beta) = \exp\left[\lambda\Big(v_1\beta - u_1\beta^*\Big)\beta + \lambda\beta^*\beta + \frac{v_1\kappa_1\beta^2\lambda^2}{2}\right]. \tag{2.151}$$

Now employing (2.151) in Eq. (2.147) along with differentiating and applying the condition $\lambda = 0$, yields

$$\langle \hat{a}_1^\dagger(t)\hat{a}_1(t)\rangle = (1-u_1)\int d^2\beta Q_1(\beta^*,\beta,t)\beta^*\beta + v_1\int d^2\beta Q_1(\beta^*,\beta,t)\beta^2. \tag{2.152}$$

We can also put Eq. (2.152) in the form

$$\begin{aligned}\langle \hat{a}_1^\dagger(t)\hat{a}_1(t)\rangle &= \big(u_1^2 - v_1^2\big)^{1/2}\Big[(u_1-1)\frac{\partial}{\partial u_1} + 2\frac{\partial}{\partial\lambda_1}\Big]\\ &\quad\times\int\frac{d^2\beta}{\pi}\exp\Big[-u_1\beta^*\beta/\kappa_1 + v_1(\lambda_1\beta^2+\beta^{*2})/2\kappa_1\Big]\Big|_{\lambda_1=1},\end{aligned} \tag{2.153}$$

from which follows

$$\begin{aligned}\langle \hat{a}_1^\dagger(t)\hat{a}_1(t)\rangle &= \kappa_1\big(u_1^2 - v_1^2\big)^{1/2}\Big[(u_1-1)\frac{\partial}{\partial u_1} + 2\frac{\partial}{\partial\lambda_1}\Big]\\ &\quad\times\big(u_1^2 - v_1^2\lambda_1\big)^{-1/2}\Big|_{\lambda_1=1}.\end{aligned} \tag{2.154}$$

Upon performing differentiation and applying the condition $\lambda_1 = 1$, one gets

$$\begin{aligned}\langle \hat{a}_1^\dagger(t)\hat{a}_1(t)\rangle &= \kappa_1\Big[\frac{u_1}{u_1^2 - v_1^2} - 1\Big]\\ &= \kappa_1(a_1 - 1).\end{aligned} \tag{2.155}$$

Making use of similar calculations, it is not difficult to show that

$$\begin{aligned}\langle \hat{a}_2^\dagger(t)\hat{a}_2(t)\rangle &= \kappa_2\Big[\frac{u_2}{u_2^2 - v_2^2} - 1\Big]\\ &= \kappa_2(a_2 - 1).\end{aligned} \tag{2.156}$$

With the aid of Eqs. (2.155) and (2.156), one can write

$$\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle = \kappa_1(a_1 - 1) + \kappa_2(a_2 - 1). \tag{2.157}$$

Following the same procedure, we find

$$\begin{aligned}\langle \hat{a}^2\rangle = \langle \hat{a}^{\dagger 2}\rangle &= \frac{\kappa_1 v_1}{u_1^2 - v_1^2} + \frac{\kappa_2 v_2}{u_2^2 - v_2^2} \\ &= \kappa_1 b_1 + \kappa_2 b_2. \end{aligned} \tag{2.158}$$

Employing (2.157) and (2.158) in Eq. (2.137), we readily obtain

$$(\Delta a_\pm^2)_s = \kappa_1 + \kappa_2 + 2\kappa_1(a_1 - 1) + 2\kappa_2(a_2 - 1) \pm 2(\kappa_1 b_1 + \kappa_2 b_2). \tag{2.159}$$

Hence in view of (2.95a) and (2.95b), along with Eq. (2.159), the quadrature variance is found to be

$$\begin{aligned}(\Delta a_\pm^2)_s &= \kappa_1 + \left[\frac{\kappa_1 A_1(1-\eta) \pm \kappa_1 A_1\sqrt{1-\eta^2}}{A_1\eta + \kappa_1}\right]\left[1 - e^{-(A_1\eta+\kappa_1)t}\right] \\ &+\kappa_2 + \left[\frac{\kappa_2 A_2(1-\eta) \pm \kappa_2 A_2\sqrt{1-\eta^2}}{A_2\eta + \kappa_2}\right]\left[1 - e^{-(A_2\eta+\kappa_2)t}\right]. \end{aligned} \tag{2.160}$$

We notice that for $t = 0$, the quadrature variance (2.160) reduces to

$$(\Delta a_\pm^2)_v = \kappa_1 + \kappa_2, \tag{2.161}$$

which is the output quadrature variances of a pair of superposed vacuum states. On account of this, it may not be difficult to observe that until the lasing system able to generate strong light, the effect of the vacuum fluctuations dominates and later, undoubtedly, the effect of the lasing system begins to overtake it.

At steady state, Eq. (2.160) takes the form

$$\begin{aligned}(\Delta a_\pm^2)_s &= \kappa_1 + \frac{\kappa_1 A_1}{A_1\eta + \kappa_1}\left[1 - \eta \pm \sqrt{1-\eta^2}\right] \\ &+\kappa_2 + \frac{\kappa_2 A_2}{A_2\eta + \kappa_2}\left[1 - \eta \pm \sqrt{1-\eta^2}\right]. \end{aligned} \tag{2.162}$$

We observe that the global quadrature variance given by (2.160) or (2.162) is the sum of the quadrature variances of the constituent output laser light beams. We can also get the same result as of (2.161) upon setting $A_1 = A_2 = 0$ in the expression described by Eq. (2.162).

It is interesting to examine some special cases. First we consider the case in which either κ_1 or κ_2 is equal to zero. In this case, the quadrature variance (2.162) turns out to be

$$\begin{aligned}(\Delta a_\pm^2)_s &= \kappa_j + \frac{\kappa_j A_j}{A_j\eta + \kappa_j}\left[1 - \eta \pm \sqrt{1-\eta^2}\right] \\ &= (\Delta a_\pm^2)_j, \;\; j = 1, 2, \end{aligned} \tag{2.163}$$

which is exactly the same as the quadrature variance of one output light beam.

It is easy to observe that for $\rho_{aa}^{(0)} = 1$ and $\rho_{cc}^{(0)} = \rho_{ac}^{(0)} = 0$ (or $\eta = -1$), Eq. (2.162) goes over into

$$(\Delta a_{\pm}^2)_s = \kappa_1(1 + 2\bar{N}_1) + \kappa_2(1 + 2\bar{N}_2), \tag{2.164}$$

where

$$\bar{N}_j = \frac{x_j}{1 - x_j}, \;\; j = 1, 2 \tag{2.165}$$

is the steady-state mean photon number of the component light beams. We see from Eq. (2.164) that the output component of a light beam generated by two-level lasers operating below threshold $(A_j/\kappa_j = x_j < 1)$ separately exhibits chaotic nature.

Furthermore, it is worth considering the case in which when one of the component output light beam is either vacuum fluctuation or thermal (chaotic) light. For the case in which one of the output beam is in a vacuum state. This corresponds to no atoms are injected into the cavity coupled to vacuum reservoir. Hence upon setting $A_2 = 0$ in Eq. (2.162), we see that

$$(\Delta a_{\pm}^2)_s = \kappa_2 + \left[\kappa_1 + \frac{\kappa_1 A_1}{A_1\eta + \kappa_1}\left[1 - \eta \pm \sqrt{1 - \eta^2}\right]\right]. \tag{2.166}$$

We notice that the first term in Eq. (2.166) is due to the output light beam in the absence of injected atoms into the second cavity and the second one represents the contribution from the first cavity output light beam.

On the other hand, when one of the component output light beam is chaotic light generated by two-level laser operating below threshold $(A_2/\kappa_2 = x_2 < 1)$ while the other one squeezed laser, Eq. (2.162) turns out to be

$$\begin{aligned}(\Delta a_{\pm}^2)_s &= \kappa_1 + \frac{\kappa_1 A_1}{A_1\eta + \kappa_1}\left[1 - \eta \pm \sqrt{1 - \eta^2}\right] \\ &\quad + \kappa_2(1 + 2\bar{N}_2),\end{aligned} \tag{2.167}$$

which represents the global quadrature variance of the superposed squeezed laser and chaotic output light beam. It is worth mentioning that the global quadrature variance of a pair of superposed identical output laser light beams is twice that of the global quadrature variance of one output laser light beam.

We calculate the global quadrature squeezing of the superposed output light beams relative to the global quadrature variance of the output light beams in a pair of superposed vacuum states. We then define the global quadrature squeezing of the superposed output laser light beams by [22]

$$S = \frac{(\Delta a_-^2)_v - (\Delta a_-^2)_s}{(\Delta a_-^2)_v}. \tag{2.168}$$

In view of (2.161) and (2.162), the global quadrature squeezing is found to be

$$S = \frac{1}{\kappa_1 + \kappa_2}\left(\frac{\kappa_1 A_1}{A_1\eta + \kappa_1} + \frac{\kappa_2 A_2}{A_2\eta + \kappa_2}\right)\left[\sqrt{1-\eta^2} - (1-\eta)\right]. \tag{2.169}$$

This can be put in the form

$$S = \frac{\kappa_1 S_1 + \kappa_2 S_2}{\kappa_1 + \kappa_2}, \tag{2.170}$$

where

$$S_j = \frac{A_j}{A_j\eta + \kappa_j}\left[\sqrt{1-\eta^2} - (1-\eta)\right], \quad j = 1, 2 \tag{2.171}$$

is the global quadrature squeezing of the j^{th} output laser light beam relative to the global quadrature variance of a light beam in a vacuum state, which is exactly the same as the intra-cavity light.

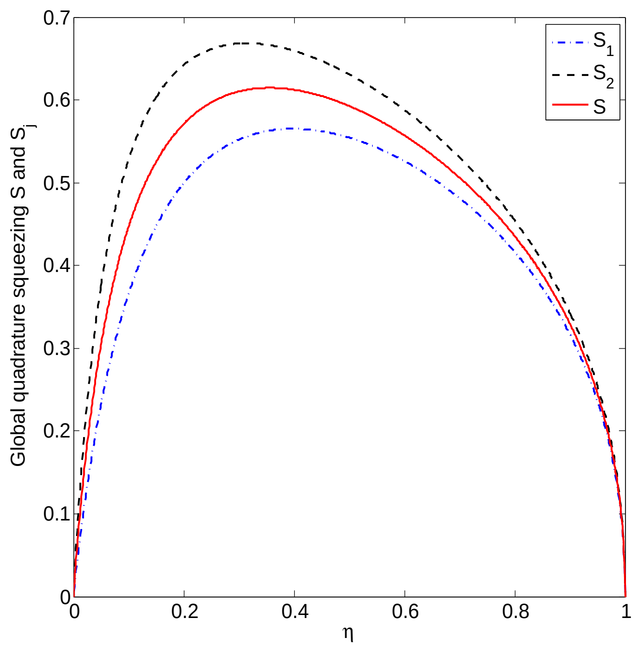

Fig. 2.1: Plots of the global quadrature squeezing (S) and (S_j) of the output laser light beams [Eq. (2.172) and Eq. (2.171)] versus η for $\kappa_1 = \kappa_2 = 0.8$, $A_1 = 5$, and $A_2 = 10$.

It is easy to observe that for $\kappa_1 = \kappa_2 = \kappa$, Eq. (2.169) reduces to

$$S = \frac{S_1 + S_2}{2}, \tag{2.172}$$

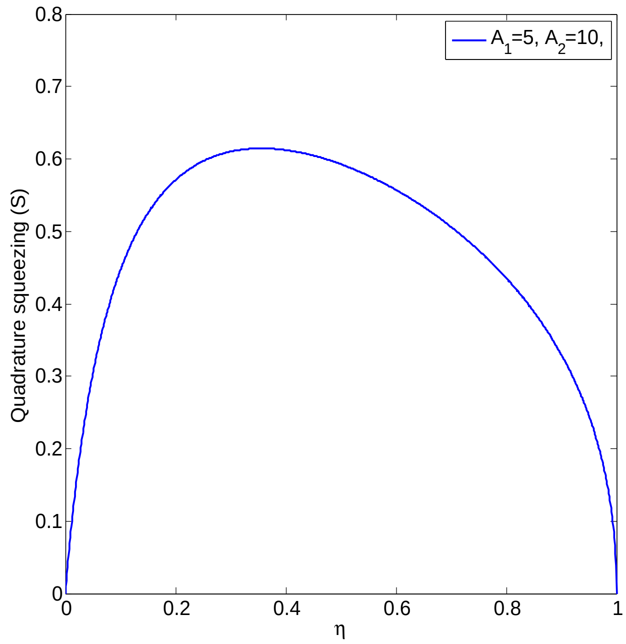

Fig. 2.2: Plot of the global quadrature squeezing (S) of the superposed output laser light beams [Eq. (2.172)] versus η for $\kappa_1 = \kappa_2 = 0.8$, $A_1 = 5$, and $A_2 = 10$.

in which

$$S_j = \frac{A_j}{A_j\eta + \kappa}\left[\sqrt{1-\eta^2} - (1-\eta)\right], \;\; j = 1, 2. \tag{2.173}$$

It is not difficult to see from Eq. (2.172) and Fig. 2.1 that the quadrature squeezing of the superposed laser light beams is the mean of the quadrature squeezing of the constituent output laser light beams. It is worth mentioning that the global quadrature squeezing of the output light beam is the same as the global quadrature squeezing of the intracavity light beam.

Moreover, on the basis of the definition of the parameter η Eq. (2.91), it can be realized for $\eta = 0$ that $\rho_{aa}^{(0)} = \rho_{cc}^{(0)} = \rho_{ac}^{(0)} = 1/2$, which corresponds to a maximum possible initial atomic coherence. But a case for which $\eta = 1$, that $\rho_{aa}^{(0)} = \rho_{ac}^{(0)} = 0$, and $\rho_{cc}^{(0)} = 1$, is related to the absence of injected atomic coherence at the beginning. One can readily see from Eq. (2.172) that there is no global squeezing when the atoms are initially prepared with a maximum or minimum atomic coherence. However, as shown in Fig. 2.2, the maximum global squeezing occurs when the atoms are prepared with initial coherence close to the maximum possible value. As it can be seen from Fig. 2.2 that the superposed output light beams is in a squeezed state for all values of η between zero and one. In particular, the maximum value of quadrature squeezing for $A_1 = 5$, $A_2 = 10$, and $\kappa_1 = \kappa_2 = 0.8$ is found to be 61.5% below the vacuum-level and occurs at $\eta = 0.35$.

Furthermore, for the case in which either κ_1 or $\kappa_2 = 0$, the global quadrature squeezing of the superposed output laser light beams turns out to be

$$S = S_j, \tag{2.174}$$

which is the same as the global quadrature squeezing of one output laser light beam. In addition, it is worth mentioning that the global quadrature squeezing of a pair of superposed identical output laser light beams is the same as the global quadrature squeezing of one output laser light beam. This can be achieved when both the linear gain coefficients and cavity damping constants of the component light beams are equal.

On the other hand, for the case in which either A_1 or $A_2 = 0$, the global quadrature squeezing of the superposed output laser light beams (2.169) takes the form

$$S = \frac{\kappa_2 S_2}{\kappa_1 + \kappa_2}, \quad A_1 = 0 \tag{2.175a}$$
$$= \frac{\kappa_1 S_1}{\kappa_1 + \kappa_2}, \quad A_2 = 0. \tag{2.175b}$$

This result corresponds to one of the component output light beams is in vacuum state.

However, for the case in which $\kappa_1 = \kappa_2$, the global quadrature squeezing described by Eq. (2.175) goes over into

$$S = \frac{1}{2} S_j, \quad j = 1, 2, \tag{2.176}$$

which is just half of the global quadrature squeezing of the laser output light beam. This result is consistent with the quadrature squeezing of the superposed coherent (vacuum) and squeezed light obtained in [22]. Thus we observe that the effect of the vacuum is to reduce the quadrature squeezing by a factor of one-half.

Moreover, for the case in which one of the constituent output light beam is chaotic light, we see that

$$S = \frac{\kappa_1 S_1 - 2\kappa_2 \bar{N}_2}{\kappa_1 + \kappa_2}. \tag{2.177}$$

This can be achieved when $x_2 < 1$. It is not difficult to notice from Eq. (2.177) for $\kappa_1 = 0$ that the squeezing does not exist, which indicates that the squeezing in the superposed output laser light beams is directly attributed to the squeezed laser light beam, since the quadrature squeezing of the squeezed laser light beam is found to be maximum for initial coherence close to the maximum possible value. In addition, as clearly shown in Fig. 2.1 that the degree of squeezing of the output light beam produced by squeezed laser light beam increases with the linear gain coefficient. On the other hand, as we have seen in Eq. (2.167), the spontaneously emitted radiation from an ensemble of two-level lasers exhibits chaotic nature. As a result, unfortunately, the superposition of the two-level and squeezed laser light beams destroys the existing squeezing.

2.3.2 Local quadrature squeezing

We next seek to calculate the local (in a given frequency interval) quadrature squeezing of the superposed output light beams. To this end, the spectrum of quadrature fluctuations for a given output light beam with central frequency ω_o is defined by

$$S_{\pm}(\omega) = \frac{1}{\pi} Re \int_0^{\infty} \langle \hat{a}_{\pm}(t), \hat{a}_{\pm}(t+\tau)\rangle_{ss} e^{i(\omega-\omega_o)\tau} d\tau, \tag{2.178}$$

where the subscript "ss" stands for steady state. Upon integrating both sides of (2.178) over ω, we easily get

$$\int_{-\infty}^{\infty} S_{\pm}(\omega) d\omega = (\Delta a_{\pm}^2)_s, \tag{2.179}$$

in which

$$(\Delta a_{\pm}^2)_s = \langle \hat{a}_{\pm}(t), \hat{a}_{\pm}(t)\rangle_{ss} \tag{2.180}$$

is the global quadrature variance of the output light beam at steady-state. On the basis of the result given by Eq. (2.179), we assert that $S_{\pm}(\omega)d\omega$ is the steady-state quadrature variance of the output light in the interval between ω and $\omega + d\omega$. In view of of Eq. (2.136) along with the definition for quadrature operators described by Eq. (2.118), we see that

$$\begin{aligned} \langle \hat{a}_{\pm}(t), \hat{a}_{\pm}(t+\tau)\rangle &= \langle \hat{a}_{\pm}(t)\hat{a}_{\pm}(t+\tau)\rangle \\ &= \pm\Big[\langle \hat{a}(t)\hat{a}(t+\tau)\rangle + \langle \hat{a}^{\dagger}(t)\hat{a}^{\dagger}(t+\tau)\rangle \\ &\quad \pm\langle \hat{a}(t)\hat{a}^{\dagger}(t+\tau)\rangle \pm \langle \hat{a}^{\dagger}(t)\hat{a}(t+\tau)\rangle\Big]. \end{aligned} \tag{2.181}$$

We note that the two-time correlation functions for the superposed output light beams that appears in the expression described by (2.181) can be calculated using the quantum regression theorem [3, 26, 31, 36, 38-42]. To this end, the expression for $\langle \hat{a}(t+\tau)\rangle$ can be obtained from (2.130) upon replacing t_o by t and t by $t+\tau$. We thus see that

$$\langle \hat{a}(t+\tau)\rangle = \langle \hat{a}_1(t+\tau)\rangle + \langle \hat{a}_2(t+\tau)\rangle, \tag{2.182}$$

where

$$\langle \hat{a}_j(t+\tau)\rangle = \langle \hat{a}_j(t)\rangle e^{-\mu_j\tau/2}, \ j=1,2. \tag{2.183}$$

Now applying the quantum regression theorem to (2.182), one finds

$$\langle \hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle = \langle \hat{a}^\dagger(t)\hat{a}_1(t+\tau)\rangle + \langle \hat{a}^\dagger(t)\hat{a}_2(t+\tau)\rangle. \tag{2.184}$$

Substituting the adjoint of Eq. (2.129) in the right hand side of Eq. (2.184), we see that

$$\begin{aligned}\langle \hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle &= \langle \hat{a}_1^\dagger(t)\hat{a}_1(t+\tau)\rangle + \langle \hat{a}_2^\dagger(t)\hat{a}_1(t+\tau)\rangle \\ &\quad + \langle \hat{a}_1^\dagger(t)\hat{a}_2(t+\tau)\rangle + \langle \hat{a}_2^\dagger(t)\hat{a}_2(t+\tau)\rangle. \end{aligned} \tag{2.185}$$

Moreover, the operator $\hat{a}_1(t)$ and $\hat{a}_2(t)$ are not correlated since they are independent variables, so that

$$\langle \hat{a}_j^\dagger(t)\hat{a}_i(t+\tau)\rangle = \langle \hat{a}_j^\dagger(t)\rangle\langle \hat{a}_i(t+\tau)\rangle, \ i\neq j, \ i,j=1,2. \tag{2.186}$$

On account of Eq. (2.135), one can readily find

$$\langle \hat{a}_j^\dagger(t)\hat{a}_i(t+\tau)\rangle = \langle \hat{a}_j^\dagger(t)\rangle\langle \hat{a}_i(t+\tau)\rangle = 0, \ i\neq j, \ i,j=1,2. \tag{2.187}$$

In view of this result, one can write Eq. (2.185) as

$$\langle \hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle = \langle \hat{a}_1^\dagger(t)\hat{a}_1(t+\tau)\rangle + \langle \hat{a}_2^\dagger(t)\hat{a}_2(t+\tau)\rangle. \tag{2.188}$$

Next we seek to obtain the two-time correlation functions for the component output light beams that appears in the right hand side of the expression (2.188). Again applying the quantum regression theorem to Eq. (2.183), we see that

$$\langle \hat{a}_j^\dagger(t)\hat{a}_j(t+\tau)\rangle = \langle \hat{a}_j^\dagger(t)\hat{a}_j(t)\rangle e^{-\mu_j\tau/2}, \ j=1,2. \tag{2.189}$$

On account of (2.189), one can easily find

$$\langle \hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle = \langle \hat{a}_1^\dagger(t)\hat{a}_1(t)\rangle e^{-\mu_1\tau/2} + \langle \hat{a}_2^\dagger(t)\hat{a}_2(t)\rangle e^{-\mu_2\tau/2}. \tag{2.190}$$

In a similar manner, it is also possible to verify that

$$\langle \hat{a}(t)\hat{a}^\dagger(t+\tau)\rangle = \langle \hat{a}_1(t)\hat{a}_1^\dagger(t)\rangle e^{-\mu_1\tau/2} + \langle \hat{a}_2(t)\hat{a}_2^\dagger(t)\rangle e^{-\mu_2\tau/2}, \tag{2.191}$$

$$\langle \hat{a}(t)\hat{a}(t+\tau)\rangle = \langle \hat{a}_1(t)\hat{a}_1(t)\rangle e^{-\mu_1\tau/2} + \langle \hat{a}_2(t)\hat{a}_2(t)\rangle e^{-\mu_2\tau/2}, \tag{2.192}$$

$$\langle \hat{a}^\dagger(t)\hat{a}^\dagger(t+\tau)\rangle = \langle \hat{a}_1^\dagger(t)\hat{a}_1^\dagger(t)\rangle e^{-\mu_1\tau/2} + \langle \hat{a}_2^\dagger(t)\hat{a}_2^\dagger(t)\rangle e^{-\mu_2\tau/2}. \tag{2.193}$$

Taking into account of (2.190)-(2.193), Eq. (2.181) can be written as

$$\begin{aligned}\langle \hat{a}_\pm(t), \hat{a}_\pm(t+\tau)\rangle_{ss} &= \langle \hat{a}_{1\pm}(t)\hat{a}_{1\pm}(t)\rangle e^{-\mu_1\tau/2} + \langle \hat{a}_{2\pm}(t)\hat{a}_{2\pm}(t)\rangle e^{-\mu_2\tau/2} \\ &= (\Delta a_\pm^2)_1 e^{-\mu_1\tau/2} + (\Delta a_\pm^2)_2 e^{-\mu_2\tau/2},\end{aligned} \tag{2.194}$$

where $(\Delta a_\pm^2)_j$ is the global quadrature variance of the j^{th} output laser light beam at steady-state given by Eq. (2.163).

Now on introducing (2.194) into Eq. (2.178), we find the spectrum of the minus quadrature fluctuations for the superposed output light beams to be

$$S_-(\omega') = \frac{1}{\pi} Re \int_0^\infty \left[(\Delta a_-^2)_1 e^{-(\mu_1/2 - i\omega')\tau} + (\Delta a_-^2)_2 e^{-(\mu_2/2 - i\omega')\tau}\right] d\tau. \tag{2.195}$$

Upon integrating over τ, we obtain

$$S_-(\omega') = (\Delta a_-^2)_1 \frac{\mu_1/2\pi}{[\mu_1/2]^2 + \omega'^2} + (\Delta a_-^2)_2 \frac{\mu_2/2\pi}{[\mu_2/2]^2 + \omega'^2}\Bigg]. \tag{2.196}$$

We observe that the spectrum of quadrature fluctuations of the superposed output laser light beams given by Eq. (2.196) is the sum of the spectra of quadrature fluctuations of the constituent output laser light beams.

Following Ref. [26], the variance of the minus quadrature in the frequency interval between $\omega' = -\lambda$ and $\omega' = \lambda$ can be expressed as

$$(\Delta a_-^2)_{s\pm\lambda} = \int_{-\lambda}^{\lambda} S_-(\omega') d\omega', \tag{2.197}$$

in which $\omega' = \omega - \omega_o$.

Hence taking Eq. (2.196) and using the fact that

$$\int_{-\lambda}^{\lambda} \frac{d\omega'}{a^2 + \omega'^2} = \frac{2}{a}\tan^{-1}\Big(\frac{\lambda}{a}\Big), \tag{2.198}$$

we obtain

$$(\Delta a_-^2)_{s\pm\lambda} = z_1(\lambda)(\Delta a_-^2)_1 + z_2(\lambda)(\Delta a_-^2)_2, \tag{2.199}$$

in which

$$z_j(\lambda) = \frac{2}{\pi}\tan^{-1}\left(\frac{2\lambda}{A_j\eta + \kappa_j}\right), \;\; j = 1, 2. \tag{2.200}$$

Moreover, we can also rewrite Eq. (2.199) in terms of the local quadrature variance of the component output light beams as

$$(\Delta a_-^2)_{s\pm\lambda} = (\Delta a_-^2)_{1\pm\lambda} + (\Delta a_-^2)_{2\pm\lambda}, \tag{2.201}$$

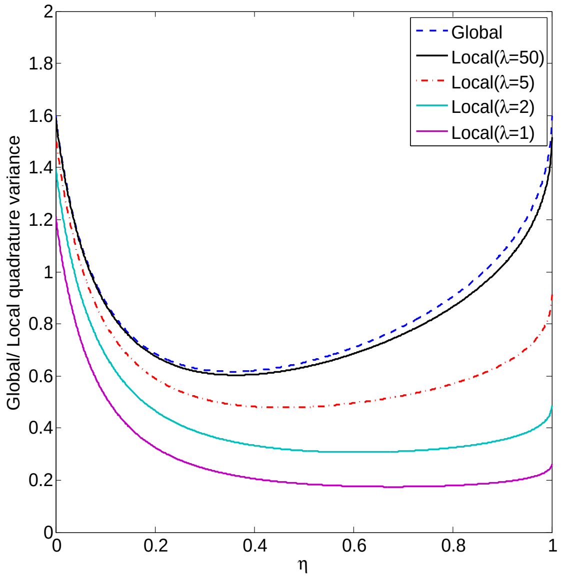

Fig. 2.3: Plots of the global and local quadrature variance of the superposed output laser light beams [Eq. (2.162) and Eq. (2.199)] versus η for $\kappa_1 = \kappa_2 = 0.8$, $A_1 = 5$, and $A_2 = 10$.

in which

$$(\Delta a_-^2)_{j\pm\lambda} \quad = \quad z_j(\lambda)(\Delta a_-^2)_j, \quad j = 1, 2 \tag{2.202}$$

is the local quadrature variance of the component output light beams. We see that the local quadrature variance of a pair of superposed output laser light beams is the sum of the local quadrature variances of the constituent output laser light beams. It is not difficult to see from Fig. 2.3 and 2.4 that the quadrature variance in a relatively small frequency interval is less than the global quadrature variance. A closer inspection of these plots clearly indicates that the local quadrature variance of the output laser light beams increases with λ and approaches to the global quadrature variance for sufficiently large values of λ. In addition, we observe that a large part of the total quadrature variance is confined in a relatively small frequency interval.

It is important to note that for the case in which identical component output light beams. In this case the local quadrature variance of the superposed output laser light beams is twice that of the local quadrature variance of one output laser light beam.

However, when ether κ_1 or κ_2 goes to zero, the local quadrature variance given by Eq.

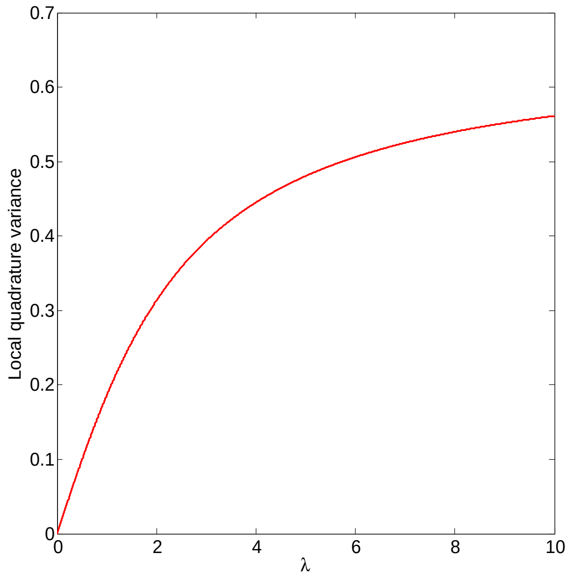

Fig. 2.4: Plot of the local quadrature variance of the superposed output laser light beams [Eq. (2.199)] versus λ for $\kappa_1 = \kappa_2 = 0.8$, $\eta = 0.5$, $A_1 = 5$, and $A_2 = 10$.

(2.199) reduces to

$$\begin{aligned}(\Delta a_-^2)_{s\pm\lambda} &= z_j(\lambda)(\Delta a_-^2)_j \\ &= (\Delta a_-^2)_{j\pm\lambda}, \quad j = 1, 2, \end{aligned} \tag{2.203}$$

which is exactly the same as the local quadrature variance of one output laser light beam.

It is worth considering the case that when $A_1 = A_2 = 0$, which corresponds the constituent output light beams is a vacuum state. Hence upon setting $A_1 = A_2 = 0$ in Eqs. (2.200) and (2.163), the local quadrature variance described by Eq. (2.199) takes the form

$$(\Delta a_-^2)_{v\pm\lambda} = \kappa_1 z_{1v}(\lambda) + \kappa_2 z_{2v}(\lambda), \tag{2.204}$$

where

$$z_{jv}(\lambda) = \frac{2}{\pi}\tan^{-1}\left(\frac{2\lambda}{\kappa_j}\right), \quad j = 1, 2, \tag{2.205}$$

with

$$(\Delta a_-^2)_j = \kappa_j \tag{2.206}$$

is the global quadrature variance of a light beam in a vacuum state.

We then seek to calculate the quadrature squeezing of the superposed output laser light beams relative to the quadrature variance of the output light beam in a pair of superposed vacuum states in the interval between $\omega' = -\lambda$ and $\omega' = \lambda$. Thus we define the quadrature squeezing of the superposed output laser light beams in the aforementioned frequency interval by

$$S_{\pm\lambda} = \frac{(\Delta a_-^2)_{v\pm\lambda} - (\Delta a_-^2)_{s\pm\lambda}}{(\Delta a_-^2)_{v\pm\lambda}}. \tag{2.207}$$

Hence with the aid of Eqs. (2.199) and (2.204), we see that

$$\begin{aligned} S_{\pm\lambda} \;=\; & \frac{1}{\kappa_1 z_{1v}(\lambda) + \kappa_2 z_{2v}(\lambda)}\Big[\kappa_1\big(z_{1v}(\lambda) - z_1(\lambda)\big) + \kappa_2\big(z_{2v}(\lambda) - z_2(\lambda)\big) \\ & + \bigg(\frac{z_1(\lambda)\kappa_1 A_1}{A_1\eta + \kappa_1} + \frac{z_2(\lambda)\kappa_2 A_2}{A_2\eta + \kappa_2}\bigg)\Big[\sqrt{1-\eta^2} - (1-\eta)\Big]\Big]. \end{aligned} \tag{2.208}$$

This can be put in the form

$$S_{\pm\lambda} = \frac{\kappa_1\big(z_{1v}(\lambda) - z_1(\lambda)\big) + \kappa_2\big(z_{2v}(\lambda) - z_2(\lambda)\big) + z_1(\lambda)\kappa_1 S_1 + z_2(\lambda)\kappa_2 S_2}{\kappa_1 z_{1v}(\lambda) + \kappa_2 z_{2v}(\lambda)}. \tag{2.209}$$

This represents the local quadrature squeezing of the superposed output laser light beams.

It is worth considering some special cases. First we consider the case in which equal damping constants. Hence upon setting $\kappa_1 = \kappa_2 = \kappa$ in Eqs. (2.200), (2.205), and (2.208) the local quadrature squeezing described by Eq. (2.208) turns out to be

$$S_{\pm\lambda} = \frac{\big(z_v(\lambda) - z_1(\lambda)\big) + \big(z_v(\lambda) - z_2(\lambda)\big) + z_1(\lambda)S_1 + z_2(\lambda)S_2}{2z_v(\lambda)}. \tag{2.210}$$

where

$$z_j(\lambda) = \frac{2}{\pi}\tan^{-1}\bigg(\frac{2\lambda}{A_j\eta + \kappa}\bigg), \tag{2.211}$$

$$z_v(\lambda) = \frac{2}{\pi}\tan^{-1}\bigg(\frac{2\lambda}{\kappa}\bigg), \tag{2.212}$$

One can also rewrite Eq. (2.210) in the form

$$S_{\pm\lambda} = \frac{S_{\pm\lambda 1} + S_{\pm\lambda 2}}{2}, \tag{2.213}$$

with

$$S_{\pm\lambda j} = \frac{\big(z_v(\lambda) - z_j(\lambda)\big) + z_j(\lambda)S_j}{z_v(\lambda)} \tag{2.214}$$

is the local quadrature squeezing of the j^{th} squeezed laser output light beam relative to the local quadrature variance of a light beam in a vacuum state. We see from Eq. (2.213) that the

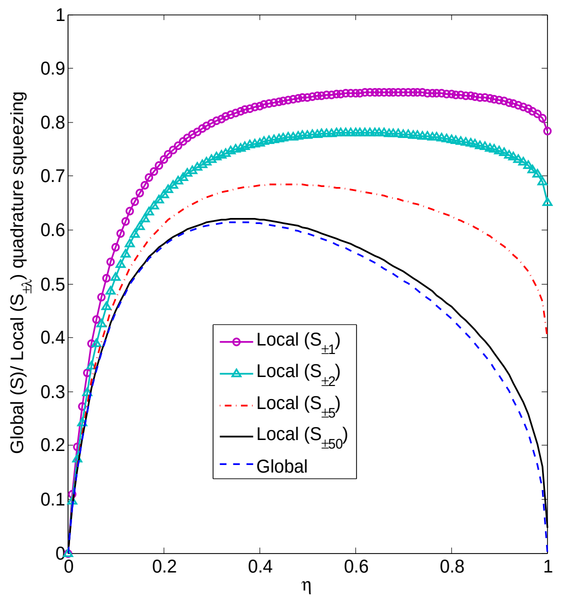

Fig. 2.5: Plots of the global (S) and local $(S_{\pm\lambda})$ quadrature squeezing of the superposed output laser light beams [Eq. (2.172) and Eq. (2.213)] versus η for $\kappa_1 = \kappa_2 = 0.8$, $A_1 = 5$ and $A_2 = 10$.

local quadrature squeezing of the superposed output laser light beams is equal to the mean of the local quadrature squeezing of the constituent output laser light beams.

Moreover, we notice that from the plots 2.5- 2.9 that the local quadrature squeezing is greater than the global quadrature squeezing. A closer inspection of the plots Figs. 2.5-2.9 shows that as the value of λ increases, the local quadrature squeezing decreases and approaches to the global quadrature squeezing for sufficiently large value of λ. In other words, the degree of quadrature squeezing increases as the frequency interval decreases. The maximum squeezing of the superposed output laser light beams is found to be 92.5% and occurs at $\lambda = \pm 0.01$ for $\kappa_1 = \kappa_2 = 0.8$, $\eta = 0.5$, $A_1 = 5$, and $A_2 = 10$. It appears that almost perfect squeezing can be achieved for sufficiently small values of λ.

Furthermore, we see that for the case in which $A_1 = A_2$ as well as $\kappa_1 = \kappa_2$, the local quadrature squeezing (2.208) goes over into

$$\begin{aligned} S_{\pm\lambda} &= \frac{\big(z_{jv}(\lambda) - z_j(\lambda)\big) + z_j(\lambda) S_j}{z_{jv}(\lambda)} \\ &= S_{\pm\lambda_j}, \ \ j = 1, 2, \end{aligned} \tag{2.215}$$

which is the local quadrature squeezing of a pair of superposed identical output laser light beams and is the same as the local quadrature squeezing of the j^{th} component of squeezed

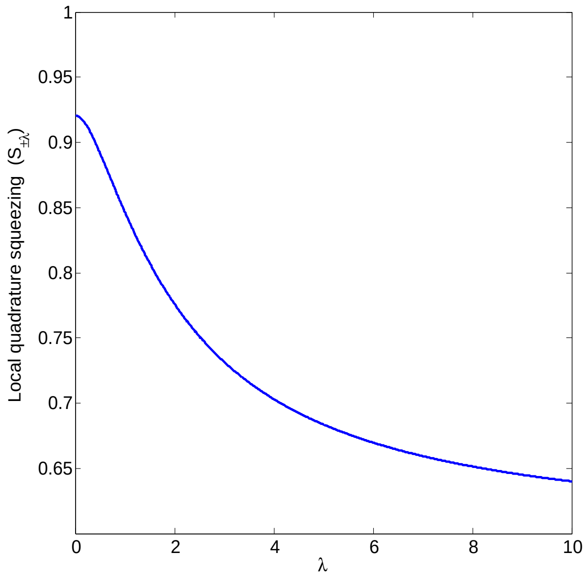

Fig. 2.6: Plot of the local quadrature squeezing ($S_{\pm\lambda}$) of the superposed output laser light beams [Eq. (2.213)] versus λ for $\kappa_1 = \kappa_2 = 0.8$, $\eta = 0.5$ $A_1 = 5$, and $A_2 = 10$.

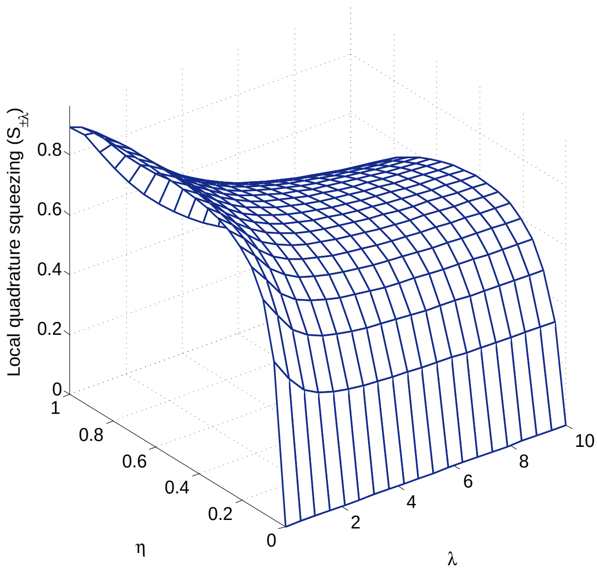

Fig. 2.7: Plot of the local quadrature squeezing ($S_{\pm\lambda}$) of the superposed output laser light beams [Eq. (2.213)] for $\kappa_1 = \kappa_2 = 0.8$, $A_1 = 5$, and $A_2 = 10$.

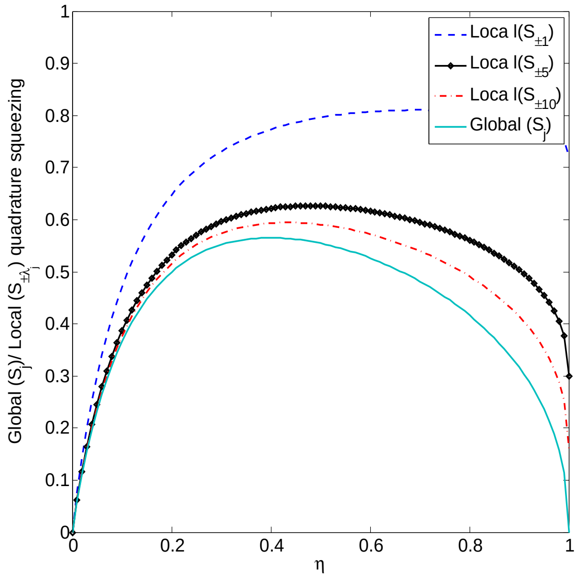

Fig. 2.8: Plots of the global (S_j) and local ($S_{\pm\lambda_j}$) quadrature squeezing of the j^{th} output laser light beam [Eq. (2.171) and Eq. (2.215)] versus η for $\kappa_j = 0.8$ and $A_j = 5$.

laser output light beam.

On the other hand, for the case in which either A_1 or $A_2 = 0$, the local quadrature squeezing (2.208) reduces to

$$S_{\pm\lambda} = \frac{\kappa_2\big(z_{2v}(\lambda) - z_2(\lambda)\big) + z_2(\lambda)\kappa_2 S_2}{\kappa_1 z_{1v}(\lambda) + \kappa_2 z_{2v}(\lambda)}, \quad A_1 = 0 \tag{2.216a}$$

$$= \frac{\kappa_1\big(z_{1v}(\lambda) - z_1(\lambda)\big) + z_1(\lambda)\kappa_1 S_1}{\kappa_1 z_{1v}(\lambda) + \kappa_2 z_{2v}(\lambda)}, \quad A_2 = 0. \tag{2.216b}$$

It is quite interesting to note that the result given by Eq. (2.216) corresponds to the case no atoms are injected into either one of the cavity.

Moreover, for the case in which $\kappa_1 = \kappa_2$, the local quadrature squeezing of the superposed output laser light beams described by Eq. (2.216) turns out to be

$$S_{\pm\lambda} = \frac{\big(z_{jv}(\lambda) - z_j(\lambda)\big) + z_j(\lambda)S_j}{2z_{jv}(\lambda)}$$

$$= \frac{1}{2}S_{\pm\lambda_j}, \quad j = 1, 2, \tag{2.217}$$

which is just half of the local quadrature squeezing of the squeezed laser output light beam.

However, when either κ_1 or κ_2 goes to zero [or $A_1 = A_2$ and $\kappa_1 = \kappa_2$], the local quadrature

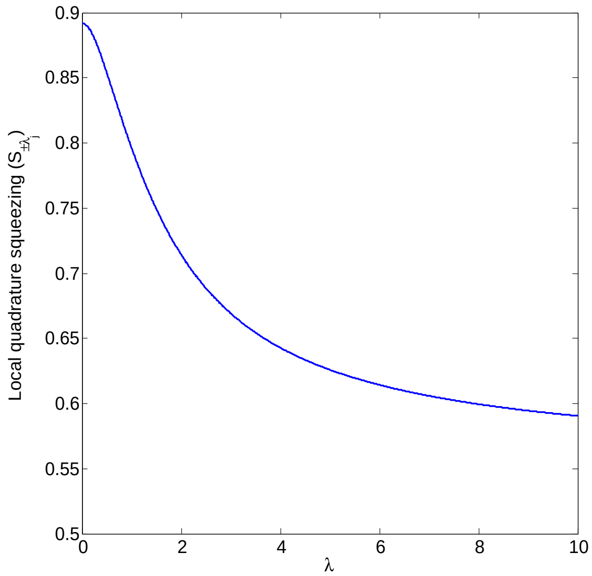

Fig. 2.9: Plot of the local quadrature squeezing ($S_{\pm\lambda_j}$) of the j^{th} output laser light beam [Eq. (2.215)] versus λ for $\kappa_j = 0.8$, $\eta = 0.5$, and $A_j = 5$.

squeezing given by Eq. (2.208) goes over into

$$S_{\pm\lambda} = \frac{\big(z_{jv}(\lambda) - z_j(\lambda)\big) + z_j(\lambda)S_j}{z_{jv}(\lambda)} \tag{2.218}$$

$$= S_{\pm\lambda_j}, \quad j = 1, 2 \tag{2.219}$$

which is the same as the local quadrature squeezing of one output laser beam.

Finally, we consider the case that either one of the component output light beam is a chaotic light generated by a two-level laser operating below threshold ($A_2/\kappa_2 = x_2 < 1$). In such a case, one can readily establish that

$$S_{\pm\lambda} = \frac{1}{\kappa_1 z_{1v}(\lambda) + \kappa_2 z_{2v}(\lambda)}\Big[\kappa_1\big(z_{1v}(\lambda) - z_1(\lambda)\big) + \kappa_2\big(z_{2v}(\lambda) - z_c(\lambda)\big) + z_1(\lambda)\kappa_1 S_1 - 2z_c(\lambda)\kappa_2\bar{N}_2\Big], \tag{2.220}$$

where

$$z_c(\lambda) = \frac{2}{\pi}\tan^{-1}\left(\frac{2\lambda}{\kappa_2 - A_2}\right). \tag{2.221}$$

We note that the expression described by Eq. (2.220) represents the local quadrature squeezing of the superposed squeezed laser and chaotic output light beams.

2.4 Photon statistics

In this section we study the statistical properties of the output light beams produced by the system under consideration. We determine the global mean and variance of the photon number as well as the local mean and variance of the photon number for the superposed output laser light beams. In order to determine the local mean and variance of the photon number, we first calculate the power spectrum and the spectrum of intensity fluctuations respectively.

2.4.1 The global mean and variance of the photon number

Here we seek to determine the global mean and variance of the photon number for the superposed output laser light beams. To this end, we define the operator representing the photon number for the superposed output laser light beams by

$$\hat{n}_s = \hat{a}^\dagger(t)\hat{a}(t). \tag{2.222}$$

Thus the global mean photon number for the superposed output laser light beams can be

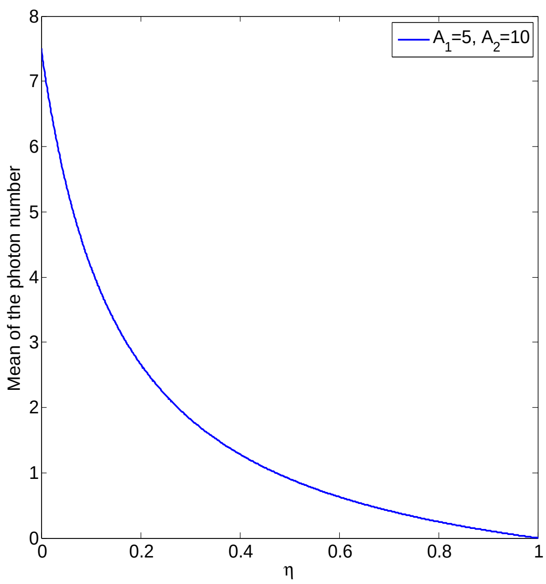

Fig. 2.10: Plot of the global mean photon number of the superposed output light beams [Eq. (2.224)] versus η for $\kappa_1 = \kappa_2 = 0.8$, $A_1 = 5$, and $A_2 = 10$.

expressible as

$$\bar{n}_s = \langle\hat{a}^\dagger(t)\hat{a}(t)\rangle. \tag{2.223}$$

In view of (2.157) along with (2.95a), the steady-state global mean of the photon number of the superposed output laser light beams is found to be

$$\bar{n}_s \;=\; \bar{n}_1 + \bar{n}_2, \tag{2.224}$$

where

$$\bar{n}_j \;=\; \frac{\kappa_j A_j(1-\eta)}{2(A_j\eta + \kappa_j)}, \;\; j = 1,2 \tag{2.225}$$

is the steady-state global mean photon number of the j^{th} squeezed laser output light beam. We observe that the global mean photon number of the superposed output laser light beams given by Eq. (2.224) is the sum of the global mean photon numbers of the constituent output laser light beams. A closer inspection of the plot Fig. 2.10 and Eq. (2.224) shows that the global mean of the photon number of the superposed output laser light beams is equal to the average of the constituent linear gain coefficients for $\eta = 0$ and for any values of $\kappa_1 = \kappa_2$, A_1, and A_2. Moreover, it worth considering the case that when the component output light beams are identical. Hence upon setting $\bar{n}_1 = \bar{n}_2$ in Eq. (2.224), the global mean photon number of the superposed output laser light beams reduces to

$$\bar{n}_s \;=\; 2\bar{n}_j, \tag{2.226}$$

which is twice that of the global mean photon number of one output laser light beam.

Next we proceed to calculate the global variance of the photon number of the output laser light beams. The global variance of the photon number for the superposed output laser light beams is defined by

$$\Delta n_s^2 = \langle(\hat{a}^\dagger\hat{a})^2\rangle - \bar{n}_s^2. \tag{2.227}$$

Applying the commutation relation described by (2.131), one can write Eq. (2.227) as

$$\Delta n_s^2 =: \Delta n_s^2 : +(\kappa_1 + \kappa_2)\bar{n}_s, \tag{2.228}$$

where

$$: \Delta n_s^2 := \langle\hat{a}^{\dagger 2}\hat{a}^2\rangle - \bar{n}_s^2 \tag{2.229}$$

is the normally ordered global variance of the photon number for the superposed output laser light beams. To this end, the expectation value of the operator $\hat{a}^{\dagger 2}(t)\hat{a}^2(t)$ is expressible as

$$\langle\hat{a}^{\dagger 2}(t)\hat{a}^2(t)\rangle = Tr\big[\hat{\rho}(t)\hat{a}^{\dagger 2}(0)\hat{a}^2(0)\big]. \tag{2.230}$$

With the aid of Eq. (2.117), one can readily obtain

$$\begin{aligned}\langle\hat{a}^{\dagger 2}(t)\hat{a}^{2}(t)\rangle &= \int d^2\beta d^2\gamma Q_1\big(\beta^*,\beta+\frac{\partial}{\partial\beta^*},t\big)Q_2\big(\gamma^*,\gamma+\frac{\partial}{\partial\gamma^*},t\big)\Big[(\beta^*\beta)^2+(\gamma^*\gamma)^2\\ &+4(\beta^*\beta)(\gamma^*\gamma)+(\beta^*\gamma)^2+(\beta\gamma^*)^2+2(\beta^*\beta)(\gamma^*\beta)+2(\beta^*\beta)(\beta^*\gamma)\\ &+2(\gamma^*\gamma)(\gamma^*\beta)+2(\gamma^*\gamma)(\beta^*\gamma)\Big]. \end{aligned}\tag{2.231}$$

This can be rewritten in the form

$$\begin{aligned}\langle\hat{a}^{\dagger 2}(t)\hat{a}^{2}(t)\rangle &= \langle\hat{a}_1^{\dagger 2}(t)\hat{a}_1^{2}(t)\rangle+\langle\hat{a}_2^{\dagger 2}(t)\hat{a}_2^{2}(t)\rangle+4\langle\hat{a}_1^{\dagger}(t)\hat{a}_1(t)\rangle\langle\hat{a}_2^{\dagger}(t)\hat{a}_2(t)\rangle+\langle\hat{a}_1^{\dagger 2}(t)\rangle\langle\hat{a}_2^{2}(t)\rangle\\ &+\langle\hat{a}_1^{2}(t)\rangle\langle\hat{a}_2^{\dagger 2}(t)\rangle+2\langle\hat{a}_1^{\dagger 2}(t)\hat{a}_1(t)\rangle\langle\hat{a}_2(t)\rangle+2\langle\hat{a}_2^{\dagger 2}(t)\hat{a}_2(t)\rangle\langle\hat{a}_1(t)\rangle\\ &+2\langle\hat{a}_1^{\dagger}(t)\hat{a}_1^{2}(t)\rangle\langle\hat{a}_2^{\dagger}(t)\rangle+2\langle\hat{a}_2^{\dagger}(t)\hat{a}_2^{2}(t)\rangle\langle\hat{a}_1^{\dagger}(t)\rangle. \end{aligned}\tag{2.232}$$

In view of (2.135), one can also write Eq. (2.232) as

$$\begin{aligned}\langle\hat{a}^{\dagger 2}(t)\hat{a}^{2}(t)\rangle &= \langle\hat{a}_1^{\dagger 2}(t)\hat{a}_1^{2}(t)\rangle+\langle\hat{a}_2^{\dagger 2}(t)\hat{a}_2^{2}(t)\rangle+\langle\hat{a}_1^{\dagger 2}(t)\rangle\langle\hat{a}_2^{2}(t)\rangle+\langle\hat{a}_1^{2}(t)\rangle\langle\hat{a}_2^{\dagger 2}(t)\rangle\\ &+4\langle\hat{a}_1^{\dagger}(t)\hat{a}_1(t)\rangle\langle\hat{a}_2^{\dagger}(t)\hat{a}_2(t)\rangle. \end{aligned}\tag{2.233}$$

On the other hand, using (2.111), we readily establish that

$$\langle\hat{a}_1^{\dagger 2}(t)\hat{a}_1^{2}(t)\rangle = \int d^2\beta Q_1\big(\beta^*,\beta+\frac{\partial}{\partial\beta^*},t\big)(\beta^*\beta)^2. \tag{2.234}$$

This can be put in the form

$$\langle\hat{a}_1^{\dagger 2}(t)\hat{a}_1^{2}(t)\rangle = \int d^2\beta\frac{\partial^2}{\partial\lambda^2}\Big[Q_1(\beta^*,\beta,t)d_1(\lambda,\beta^*,\beta)\Big]\Big|_{\lambda=0}, \tag{2.235}$$

Furthermore, upon combining Eqs. (2.151) and (2.235) along with differentiating and applying the condition $\lambda=0$, we arrive at

$$\langle\hat{a}_1^{\dagger 2}(t)\hat{a}_1^{2}(t)\rangle = I_1+I_2+I_3+I_4, \tag{2.236}$$

in which

$$I_1 = (u_1-1)^2\int d^2\beta Q_1(\beta^*,\beta,t)\beta^{*2}\beta^2, \tag{2.237a}$$

$$I_2 = \kappa_1 v_1\int d^2\beta Q_1(\beta^*,\beta,t)\beta^2, \tag{2.237b}$$

$$I_3 = 2v_1(1-u_1)\int d^2\beta Q_1(\beta^*,\beta,t)\beta^*\beta^3, \tag{2.237c}$$

$$I_4 = v_1^2\int d^2\beta Q_1(\beta^*,\beta,t)\beta^4. \tag{2.237d}$$

Applying (2.144), one can write Eq. (2.237a) in the form

$$I_1 = \kappa_1(u_1^2-v_1^2)^{1/2}(u_1-1)^2\frac{\partial^2}{\partial u_1^2}\int\frac{d^2\beta}{\pi}\exp\Big[-\frac{u_1}{\kappa_1}\beta^*\beta+\frac{v_1}{2\kappa_1}(\beta^2+\beta^{*2})\Big], \tag{2.238}$$

so that on carrying out the integration, we obtain

$$I_1 \quad = \quad \kappa_1^2(u_1^2 - v_1^2)^{1/2}(u_1 - 1)^2 \frac{\partial^2}{\partial u_1^2}\left[\frac{1}{(u_1^2 - v_1^2)^{1/2}}\right], \tag{2.239}$$

there follows

$$I_1 \quad = \quad \frac{\kappa_1^2(2u_1^2 + v_1^2)(u_1 - 1)^2}{(u_1^2 - v_1^2)^2}. \tag{2.240}$$

Following the same procedure, one can also show that

$$I_2 \quad = \quad \frac{\kappa_1^2 v_1^2}{u_1^2 - v_1^2}, \tag{2.241a}$$

$$I_3 \quad = \quad \frac{6\kappa_1^2 u_1 v_1^2(1 - u_1)}{(u_1^2 - v_1^2)^2}, \tag{2.241b}$$

$$I_4 \quad = \quad \frac{3\kappa_1^2 v_1^4}{(u_1^2 - v_1^2)^2}. \tag{2.241c}$$

Therefore, taking into account of Eqs. (2.240) and (2.241), one gets

$$\langle \hat{a}_1^{\dagger 2}(t)\hat{a}_1^2(t)\rangle \quad = \quad 2\kappa_1^2 + \frac{\kappa_1^2(2u_1^2 + v_1^2)}{(u_1^2 - v_1^2)^2} - \frac{4\kappa_1^2 u_1}{u_1^2 - v_1^2} \tag{2.242}$$

$$= \quad 2\kappa_1^2(a_1 - 1)^2 + \kappa_1^2 b_1^2. \tag{2.243}$$

Furthermore, in view of (2.95) along with (2.99, Eq. (2.243) reduces to

$$\langle \hat{a}_1^{\dagger 2}(t)\hat{a}_1^2(t)\rangle = 2\bar{n}_1^2 + \kappa_1^2 b_1^2. \tag{2.244}$$

Making use of similar calculations, it is not difficult to show that

$$\langle \hat{a}_j^{\dagger 2}(t)\hat{a}_j^2(t)\rangle \quad = \quad 2\bar{n}_j^2 + \kappa_j^2 b_j^2, \tag{2.245a}$$

$$\langle \hat{a}_j^{\dagger 2}(t)\rangle = \langle \hat{a}_j^2(t)\rangle \quad = \quad \kappa_j b_j, \tag{2.245b}$$

Now with the aid of Eqs. (2.155), (2.156), and(2.245), Eq. (2.233) can be put in the form

$$\langle \hat{a}^{\dagger 2}(t)\hat{a}^2(t)\rangle \quad = \quad 2\bar{n}_s^2 + \left[\kappa_1 b_1 + \kappa_2 b_2\right]^2, \tag{2.246}$$

where $\bar{n}_s$ is the mean of the photon number of the superposed output laser light beams given by Eq. (2.224). Employing (2.246) in Eq. (2.229), we obtain

$$:\Delta n_s^2 := \bar{n}_s^2 + \left[\kappa_1 b_1 + \kappa_2 b_2\right]^2. \tag{2.247}$$

Furthermore, in view of this, the global variance of the photon number of the superposed output laser light beams found to be

$$\Delta n_s^2 \quad = \quad \bar{n}_s^2 + (\kappa_1 + \kappa_2)\bar{n}_s + \left[\kappa_1 b_1 + \kappa_2 b_2\right]^2. \tag{2.248}$$

This can be put in the form

$$\Delta n_s^2 = \Delta n_1^2 + \Delta n_2^2 + 2(\bar{n}_1\bar{n}_2 + \kappa_1\kappa_2 b_1 b_2) + \kappa_2\bar{n}_1 + \kappa_1\bar{n}_2, \tag{2.249}$$

where

$$\Delta n_j^2 = \bar{n}_j^2 + \kappa_j\bar{n}_j + \kappa_j^2 b_j^2. \tag{2.250}$$

is the global variance of the photon number for the j^{th} squeezed laser output light beam.

Suppose the constituent light beams are chaotic (thermal) output light generated by a two-level laser operating below threshold ($x_j < 1$). Then using the the fact that $\rho_{aa}^{(0)} = 1$ and $\rho_{cc}^{(0)} = \rho_{ac}^{(0)} = 0$ (or $\eta = -1$), we easily arrive at

$$\Delta n_s^2 = \Delta n_1^2 + \Delta n_2^2 + 2\bar{n}_1\bar{n}_2 + \kappa_2\bar{n}_1 + \kappa_1\bar{n}_2, \tag{2.251}$$

where

$$\bar{n}_j = \kappa_j\bar{N}_j, \tag{2.252}$$

$$\Delta n_j^2 = \bar{n}_j^2 + \kappa_j\bar{n}_j, \ \ j = 1, 2 \tag{2.253}$$

is the global mean and variance of the photon number of the component output light beams, with $\bar{N}_j$ is the mean photon number for the chaotic light given by Eq. (2.165).

The expression described by Eq. (2.251) can also be rewritten in the form

$$\Delta n_s^2 = \kappa_1^2\Delta N_1^2 + \kappa_2^2\Delta N_2^2 + \kappa_1\kappa_2[2\bar{N}_1\bar{N}_2 + \bar{N}_1 + \bar{N}_2], \tag{2.254}$$

in which

$$\Delta N_j^2 = \bar{N}_j^2 + \bar{N}_j, \ \ j = 1, 2 \tag{2.255}$$

is the variance of the photon number for the j^{th} component of the chaotic light generated by two-level laser operating below threshold ($x_j < 1$). We note that Eq. (2.251) or (2.254) represents the global variance of the photon number of a pair of superposed chaotic output light beams. In addition, we see from Eqs. (2.249) and (2.251) that the global variance of the photon number of the superposed output laser light beams is not the sum of the global variances of the photon numbers of the constituent output laser light beams.

We next consider the case in which the atoms are not injected into one of the cavity coupled to vacuum reservoir. Hence upon setting $A_2 = 0$ in the expression given by Eq. (2.249), we find

$$\Delta n_s^2 = \Delta n_1^2 + \kappa_2\bar{n}_1. \tag{2.256}$$

This indicates the global variance of the photon number of the squeezed laser light and a light beam in vacuum state.

On the other hand, for the case in which no output light beams from one of the cavity, the variance of the photon number of the superposed output laser light beams is reduces to

$$\Delta n_s^2 \;=\; \Delta n_j^2,\;\; j = 1, 2, \tag{2.257}$$

which is the global variance of the photon number for the j^{th} output laser light beam. This can be achieved upon setting either κ_1 or $\kappa_2 = 0$.

$\Delta n_j^2 > \bar{n}_j$ and $\Delta n_s^2 > \bar{n}_s$, so that the photon statistics of the single light beam as well as that of the superposed light beams is super-Poissonian.

It is worth mentioning that the global variance of the photon number of a pair of superposed identical output laser light beams is four times that of the global variances of the photon number of one output laser light beam.

2.4.2 The local mean and variance of the photon number

We know proceed to obtain the local (in a given frequency interval) mean and variance of the photon number of the superposed output laser light beams. To this end, we define the power spectrum of the output light with central frequency ω_o by

$$P(\omega) = \frac{1}{\pi} Re \int_0^{\infty} \langle \hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle_{ss} e^{i(\omega-\omega_o)\tau} d\tau. \tag{2.258}$$

Upon integrating both sides of (2.258) over ω, we easily get

$$\int_{-\infty}^{\infty} P(\omega) d\omega = \langle \hat{a}^\dagger(t)\hat{a}(t)\rangle_{ss}, \tag{2.259}$$

where $\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle_{ss} = \bar{n}_s$ is the steady state global mean photon number of a pair of superposed output laser light beams. On the basis of the relation (2.259), the mean photon number can then be written in the interval between $\omega' = -\lambda$ and $\omega' = \lambda$ as

$$\bar{n}_{s\pm\lambda} = \langle \hat{a}^\dagger(t)\hat{a}(t)\rangle_{\pm\lambda} = \int_{-\lambda}^{\lambda} P(\omega') d\omega'. \tag{2.260}$$

Now in view of (2.190), the power spectrum described by Eq. (2.258) can be put in the form

$$P(\omega') \;=\; \frac{1}{\pi} Re \int_0^{\infty} [\bar{n}_1 e^{-(\mu_1/2 - i\omega')\tau} + \bar{n}_2 e^{-(\mu_2/2 - i\omega')\tau}] d\tau, \tag{2.261}$$

where $\bar{n}_j$ is the steady state global mean photon number of the j^{th} laser output laser light beam given by (2.225).

Now carrying out the integration over τ, the power spectrum of the superposed output laser light beams turns out to be

$$P(\omega') = \frac{\bar{n}_1\mu_1/2\pi}{[\mu_1/2]^2+\omega'^2}+\frac{\bar{n}_1\mu_2/2\pi}{[\mu_2/2]^2+\omega'^2}. \tag{2.262}$$

We observe that the power spectrum of the superposed output laser light beams given by Eq. (2.262) is the sum of the power spectra of the constituent output laser light beams.

Furthermore, upon substituting (2.262) into Eq. (2.260) and carrying out the integration, applying (2.198), we arrive at

$$\bar{n}_{s_{\pm\lambda}} = \bar{n}_{1_{\pm\lambda}}+\bar{n}_{2_{\pm\lambda}}, \tag{2.263}$$

in which

$$\bar{n}_{j_{\pm\lambda}} = z_j(\lambda)\bar{n}_j, \ \ j=1,2 \tag{2.264}$$

is the local mean photon number of the j^{th} squeezed output laser light beam. We see from Eq. (2.263) that the local mean photon number of the superposed output laser light beams is the sum of the local mean photon numbers of the component output laser light beams. Moreover, we see from Figs. 2.11 and 2.14 that the local mean photon number is less than the

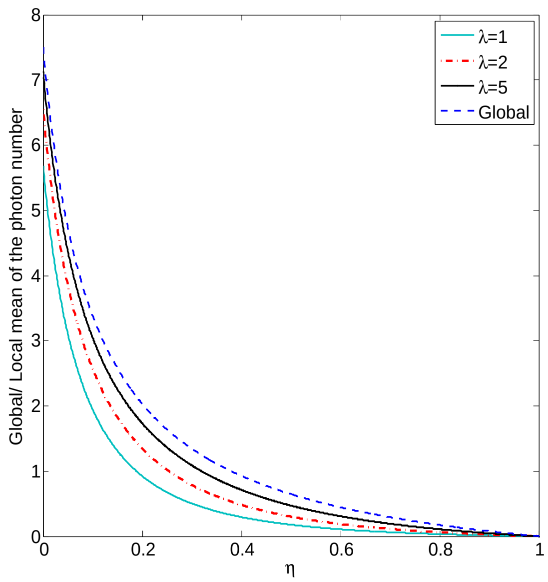

Fig. 2.11: Plots of the global and local mean of the photon number of the superposed output laser light beams [Eq. (2.224) and Eq. (2.263)] versus η for $\kappa_1=\kappa_2=0.8$, $A_1=5$, and $A_2=10$.

global mean photon number. A closer inspection of the plots 2.11-2.15 clearly indicates that

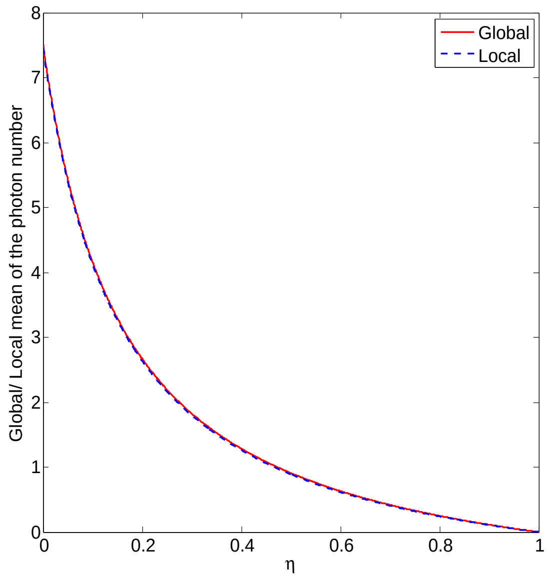

Fig. 2.12: Plots of the global and local $(\lambda = 50)$ mean of the photon number of the superposed output laser light beams [Eq. (2.224) and Eq. (2.263)] versus η for $\kappa_1 = \kappa_2 = 0.8$, $A_1 = 5$, and $A_2 = 10$.

the local mean photon number increases with λ and approaches to the global mean photon number for sufficiently large values of λ. In addition, we see form the plots Figs. 2.11 and 2.14 that a large part of the total mean photon number is confined in a relatively small frequency interval. It is quite interesting to note that the local mean photon number, unlike the local quadrature squeezing does not increase as the value of λ decreases. Furthermore, for the case in which equal linear gain coefficients as well as damping constants, the local mean photon number (2.263) turns out to be

$$\bar{n}_{s_{\pm\lambda}} = 2\bar{n}_{j_{\pm\lambda}}. \tag{2.265}$$

This represents the local mean photon number of a pair of superposed identical output laser light beams and is twice that of the local mean photon number of one output laser light beam.

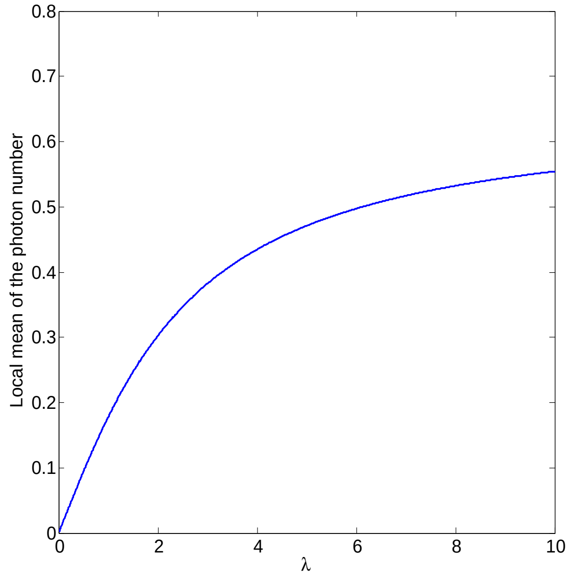

Fig. 2.13: Plot of the local mean of the photon number of the superposed output laser light beams [Eq. (2.263)] versus λ for $\kappa_1 = \kappa_2 = 0.8$, $\eta = 0.5$, $A_1 = 5$, and $A_2 = 10$.

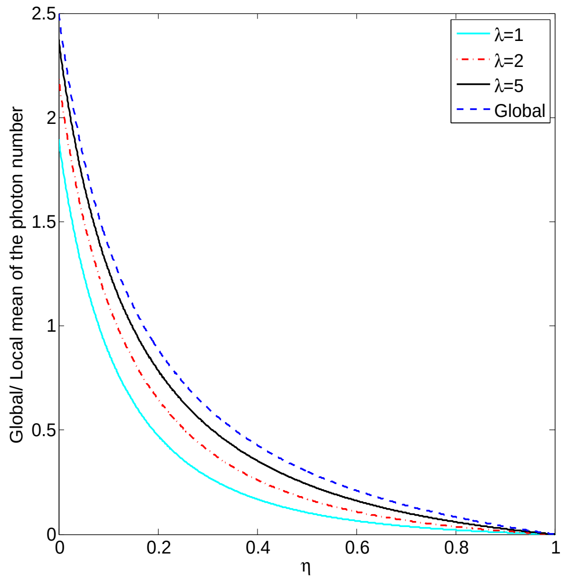

Fig. 2.14: Plots of the global and local mean of the photon number of the j^{th} output laser light beams [Eq. (2.225) and Eq. (2.264)] versus η for $\kappa_j = 0.8$, and $A_j = 5$

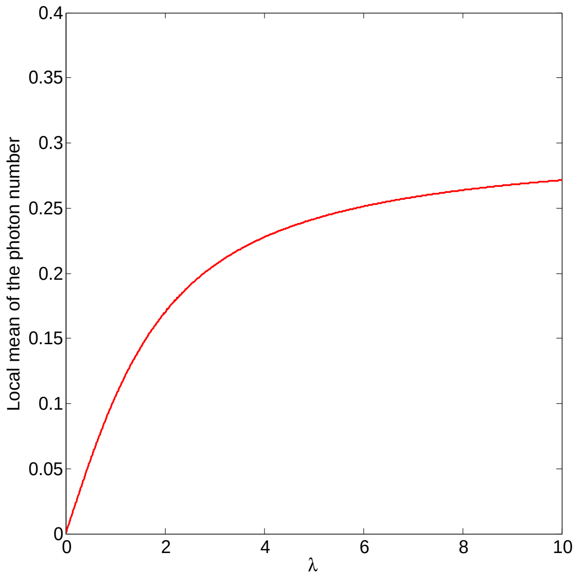

Fig. 2.15: Plot of the local mean of the photon number of the j^{th} output laser light beams [Eq. (2.264)] versus λ for $\kappa_j = 0.8$, $\eta = 0.5$, and $A_j = 5$

We finally wish to obtain the local variance of the photon number of the superposed output laser light beams. Thus the spectrum of intensity fluctuations for the output light with central frequency ω_o defined by

$$I(\omega) \quad = \quad \frac{1}{\pi} Re \int_0^\infty \langle \hat{a}^\dagger(t)\hat{a}(t+\tau), \hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle_{ss} e^{i(\omega-\omega_0)\tau} d\tau, \tag{2.266}$$

in which

$$\langle \hat{a}^\dagger(t)\hat{a}(t+\tau), \hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle_{ss} \quad = \quad \left\langle \left(\hat{a}^\dagger(t)\hat{a}(t+\tau)\right)^2\right\rangle_{ss} - \langle \hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle_{ss}^2. \tag{2.267}$$

It is not hard to establish that

$$\int_{-\infty}^{\infty} I(\omega) d\omega \quad = \quad \Delta n_s^2, \tag{2.268}$$

where

$$\Delta n_s^2 \quad = \quad \langle \hat{a}^\dagger(t)\hat{a}(t), \hat{a}^\dagger(t)\hat{a}(t)\rangle_{ss} \tag{2.269}$$

is the steady-state global variance of the photon number of a pair of superposed output laser light beams.

Now applying the quantum regression theorem to (2.182), one gets

$$\begin{aligned}\langle\hat{a}(t+\tau)\hat{a}^\dagger(t)\rangle &= e^{-\mu_1\tau/2}[\langle\hat{a}_1(t)\hat{a}_1^\dagger(t)\rangle+\langle\hat{a}_1(t)\hat{a}_2^\dagger(t)\rangle]\\ &\quad+e^{-\mu_2\tau/2}[\langle\hat{a}_2(t)\hat{a}_2^\dagger(t)\rangle+\langle\hat{a}_2(t)\hat{a}_1^\dagger(t)\rangle].\end{aligned}\tag{2.270}$$

Furthermore, applying the quantum regression theorem to (2.270) leads to

$$\begin{aligned}\langle\hat{a}^\dagger(t)\hat{a}(t+\tau)\hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle &= e^{-\mu_1\tau/2}[\langle\hat{a}^\dagger(t)\hat{a}_1(t)\hat{a}_1^\dagger(t)\hat{a}(t+\tau)\rangle+\langle\hat{a}^\dagger(t)\hat{a}_1(t)\hat{a}_2^\dagger(t)\hat{a}(t+\tau)\rangle]\\ &\quad+e^{-\mu_2\tau/2}[\langle\hat{a}^\dagger(t)\hat{a}_2(t)\hat{a}_2^\dagger(t)\hat{a}(t+\tau)\rangle+\langle\hat{a}^\dagger(t)\hat{a}_2(t)\hat{a}_1^\dagger(t)\hat{a}(t+\tau)\rangle].\end{aligned}\tag{2.271}$$

We note that the expression for $\hat{a}(t+\tau)$ can be obtained from (2.129) upon replacing t_o by t and t by $t+\tau$. We thus see that

$$\hat{a}(t+\tau) = \hat{a}_1(t+\tau)+\hat{a}_2(t+\tau),\tag{2.272}$$

in which

$$\hat{a}_j(t+\tau) = \hat{a}_j(t)e^{-\mu_j\tau/2}+\int_0^\tau e^{-\mu_j(\tau-\tau')/2}\hat{F}_j(t+\tau')d\tau',\ \ j=1,2.\tag{2.273}$$

Substituting (2.272) and the adjoint of Eq. (2.129) in the right hand side of (2.271), one can write

$$\begin{aligned}\langle\hat{a}^\dagger(t)\hat{a}(t+\tau)\hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle &= e^{-\mu_1\tau/2}[\langle\hat{a}_1^\dagger(t)\hat{a}_1(t)\hat{a}_1^\dagger(t)\hat{a}_1(t+\tau)\rangle+\langle\hat{a}_1^\dagger(t)\hat{a}_1(t)\hat{a}_1^\dagger(t)\hat{a}_2(t+\tau)\rangle\\ &\quad+\langle\hat{a}_2^\dagger(t)\hat{a}_1(t)\hat{a}_1^\dagger(t)\hat{a}_1(t+\tau)\rangle+\langle\hat{a}_2^\dagger(t)\hat{a}_1(t)\hat{a}_1^\dagger(t)\hat{a}_2(t+\tau)\rangle\\ &\quad+\langle\hat{a}_1^\dagger(t)\hat{a}_1(t)\hat{a}_2^\dagger(t)\hat{a}_1(t+\tau)\rangle+\langle\hat{a}_1^\dagger(t)\hat{a}_1(t)\hat{a}_2^\dagger(t)\hat{a}_2(t+\tau)\rangle\\ &\quad+\langle\hat{a}_2^\dagger(t)\hat{a}_1(t)\hat{a}_2^\dagger(t)\hat{a}_1(t+\tau)\rangle+\langle\hat{a}_2^\dagger(t)\hat{a}_1(t)\hat{a}_2^\dagger(t)\hat{a}_2(t+\tau)\rangle]\\ &\quad+e^{-\mu_2\tau/2}[\langle\hat{a}_1^\dagger(t)\hat{a}_2(t)\hat{a}_2^\dagger(t)\hat{a}_1(t+\tau)\rangle+\langle\hat{a}_1^\dagger(t)\hat{a}_2(t)\hat{a}_2^\dagger(t)\hat{a}_2(t+\tau)\rangle\\ &\quad+\langle\hat{a}_2^\dagger(t)\hat{a}_2(t)\hat{a}_2^\dagger(t)\hat{a}_1(t+\tau)\rangle+\langle\hat{a}_2^\dagger(t)\hat{a}_2(t)\hat{a}_2^\dagger(t)\hat{a}_2(t+\tau)\rangle\\ &\quad+\langle\hat{a}_1^\dagger(t)\hat{a}_2(t)\hat{a}_1^\dagger(t)\hat{a}_1(t+\tau)\rangle+\langle\hat{a}_1^\dagger(t)\hat{a}_2(t)\hat{a}_1^\dagger(t)\hat{a}_2(t+\tau)\rangle\\ &\quad+\langle\hat{a}_2^\dagger(t)\hat{a}_2(t)\hat{a}_1^\dagger(t)\hat{a}_1(t+\tau)\rangle+\langle\hat{a}_2^\dagger(t)\hat{a}_2(t)\hat{a}_1^\dagger(t)\hat{a}_2(t+\tau)\rangle].\end{aligned}\tag{2.274}$$

Since the operators $\hat{a}_1(t)$ and $\hat{a}_2(t)$ are uncorrelated. Therefore, these operators can be decoupled, so that

$$\langle\hat{a}_i^\dagger(t)\hat{a}_i(t)\hat{a}_i^\dagger(t)\hat{a}_j(t+\tau)\rangle = \langle\hat{a}_i^\dagger(t)\hat{a}_i(t)\hat{a}_i^\dagger(t)\rangle\langle\hat{a}_j(t+\tau)\rangle = 0 \tag{2.275a}$$

$$\langle\hat{a}_j^\dagger(t)\hat{a}_i(t)\hat{a}_i^\dagger(t)\hat{a}_i(t+\tau)\rangle = \langle\hat{a}_j^\dagger(t)\rangle\langle\hat{a}_i(t)\hat{a}_i^\dagger(t)\hat{a}_i(t+\tau)\rangle = 0 \tag{2.275b}$$

$$\langle\hat{a}_i^\dagger(t)\hat{a}_i(t)\hat{a}_j^\dagger(t)\hat{a}_j(t+\tau)\rangle = \bar{n}_i\langle\hat{a}_j^\dagger(t)\hat{a}_j(t+\tau)\rangle \tag{2.275c}$$

$$\langle\hat{a}_i^\dagger(t)\hat{a}_j(t)\hat{a}_j^\dagger(t)\hat{a}_i(t+\tau)\rangle = \langle\hat{a}_j(t)\hat{a}_j^\dagger(t)\rangle\langle\hat{a}_i^\dagger(t)\hat{a}_i(t+\tau)\rangle \tag{2.275d}$$

$$\langle\hat{a}_i^\dagger(t)\hat{a}_j(t)\hat{a}_i^\dagger(t)\hat{a}_j(t+\tau)\rangle = \langle\hat{a}_i^{\dagger 2}(t)\rangle\hat{a}_j(t)\hat{a}_j(t+\tau)\rangle, \;\; i \neq j, \tag{2.275e}$$

where $\bar{n}_i = \langle\hat{a}_i^\dagger(t)\hat{a}_i(t)\rangle$ with $i, j = 1, 2$. In view of this result, Eq. (2.274) reduces to

$$\begin{aligned}\langle\hat{a}^\dagger(t)\hat{a}(t+\tau)\hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle = \; & e^{-\mu_1\tau/2}\big\{\langle\hat{a}_1^\dagger(t)\hat{a}_1(t)\hat{a}_1^\dagger(t)\hat{a}_1(t+\tau)\rangle \\ & +\langle\hat{a}_2^\dagger(t)\hat{a}_2(t+\tau)\rangle[\langle\hat{a}_1(t)\hat{a}_1^\dagger(t)\rangle + \bar{n}_1] \\ & +\langle\hat{a}_2^{\dagger 2}(t)\rangle\langle\hat{a}_1(t)\hat{a}_1(t+\tau)\rangle\big\} \\ & +e^{-\mu_2\tau/2}\big\{\langle\hat{a}_2^\dagger(t)\hat{a}_2(t)\hat{a}_2^\dagger(t)\hat{a}_2(t+\tau)\rangle \\ & +\langle\hat{a}_1^\dagger(t)\hat{a}_1(t+\tau)\rangle[\langle\hat{a}_2(t)\hat{a}_2^\dagger(t)\rangle + \bar{n}_2] \\ & +\langle\hat{a}_1^{\dagger 2}(t)\rangle\langle\hat{a}_2(t)\hat{a}_2(t+\tau)\rangle\big\}. \end{aligned} \tag{2.276}$$

We now proceed to determine the two-time correlation functions that appears in the expression described by Eq. (2.276). To this end, applying the quantum regression theorem to Eq. (2.183), we see that

$$\langle\hat{a}_j^\dagger(t)\hat{a}_j(t)\hat{a}_j^\dagger(t)\hat{a}_j(t+\tau)\rangle = e^{-\mu_j\tau/2}\langle(\hat{a}_j^\dagger(t)\hat{a}_j(t))^2\rangle \tag{2.277}$$

$$\langle\hat{a}_j(t)\hat{a}_j(t+\tau)\rangle = e^{-\mu_j\tau/2}\langle\hat{a}_j^2(t)\rangle, \;\; j = 1, 2. \tag{2.278}$$

Now with the aid of Eqs. (2.189), (2.277), and (2.278), Eq. (2.276) can be put in the form

$$\begin{aligned}\langle\hat{a}^\dagger(t)\hat{a}(t+\tau)\hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle = \; & e^{-\mu_1\tau}\langle(\hat{a}_1^\dagger(t)\hat{a}_1(t))^2\rangle + e^{-\mu_2\tau}\langle(\hat{a}_2^\dagger(t)\hat{a}_2(t))^2\rangle \\ & +e^{-\mu_2\tau}\langle\hat{a}_1^{\dagger 2}(t)\rangle\langle\hat{a}_2^2(t)\rangle + e^{-\mu_1\tau}\langle\hat{a}_2^{\dagger 2}(t)\rangle\langle\hat{a}_1^2(t)\rangle \\ & +2e^{-(\mu_1+\mu_2)\tau/2}\langle\hat{a}_1^\dagger(t)\hat{a}_1(t)\rangle\langle\hat{a}_2^\dagger(t)\hat{a}_2(t)\rangle \\ & +e^{-(\mu_1+\mu_2)\tau/2}\langle\hat{a}_2^\dagger(t)\hat{a}_2(t)\rangle\langle\hat{a}_1(t)\hat{a}_1^\dagger(t)\rangle \\ & +e^{-(\mu_1+\mu_2)\tau/2}\langle\hat{a}_1^\dagger(t)\hat{a}_1(t)\rangle\langle\hat{a}_2(t)\hat{a}_2^\dagger(t)\rangle. \end{aligned} \tag{2.279}$$

Applying the commutation relation (2.132) on the last two expressions of Eq. (2.279), one can rewrite Eq. (2.279) in the form

$$
\begin{aligned}
\langle \hat{a}^\dagger(t)\hat{a}(t+\tau)\hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle \;&=\; e^{-\mu_1\tau}\langle(\hat{a}_1^\dagger(t)\hat{a}_1(t))^2\rangle + e^{-\mu_2\tau}\langle(\hat{a}_2^\dagger(t)\hat{a}_2(t))^2\rangle.\\
&+e^{-(\mu_1+\mu_2)\tau/2}\big[\kappa_2\langle\hat{a}_1^\dagger(t)\hat{a}_1(t)\rangle + \kappa_1\langle\hat{a}_2^\dagger(t)\hat{a}_2(t)\rangle\big]\\
&+e^{-\mu_1\tau}\langle\hat{a}_2^{\dagger 2}(t)\rangle\langle\hat{a}_1^2(t)\rangle + e^{-\mu_2\tau}\langle\hat{a}_1^{\dagger 2}(t)\rangle\langle\hat{a}_2^2(t)\rangle\\
&+4e^{-(\mu_1+\mu_2)\tau/2}\langle\hat{a}_1^\dagger(t)\hat{a}_1(t)\rangle\langle\hat{a}_2^\dagger(t)\hat{a}_2(t)\rangle. \qquad (2.280)
\end{aligned}
$$

Now in view of (2.190), one can write

$$
\begin{aligned}
\langle\hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle^2 \;&=\; e^{-\mu_1\tau}\langle\hat{a}_1^\dagger(t)\hat{a}_1(t)\rangle^2 + e^{-\mu_2\tau}\langle\hat{a}_2^\dagger(t)\hat{a}_2(t)\rangle^2\\
&+2e^{-(\mu_1+\mu_2)\tau/2}\langle\hat{a}_1^\dagger(t)\hat{a}_1(t)\rangle\langle\hat{a}_2^\dagger(t)\hat{a}_2(t)\rangle. \qquad (2.281)
\end{aligned}
$$

Moreover, taking Eqs. (2.280) and (2.281) into account, we readily establish that

$$
\begin{aligned}
\langle\hat{a}^\dagger(t)\hat{a}(t+\tau),\hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle \;&=\; e^{-\mu_1\tau}\big[\langle(\hat{a}_1^\dagger(t)\hat{a}_1(t))^2\rangle - \langle\hat{a}_1^\dagger(t)\hat{a}_1(t)\rangle^2 + \langle\hat{a}_2^{\dagger 2}(t)\rangle\langle\hat{a}_1^2(t)\rangle\big]\\
&+e^{-\mu_2\tau}\big[\langle(\hat{a}_2^\dagger(t)\hat{a}_2(t))^2\rangle - \langle\hat{a}_2^\dagger(t)\hat{a}_2(t)\rangle^2 + \langle\hat{a}_1^{\dagger 2}(t)\rangle\langle\hat{a}_2^2(t)\rangle\big]\\
&+e^{-(\mu_1+\mu_2)\tau/2}\big[\kappa_2\langle\hat{a}_1^\dagger(t)\hat{a}_1(t)\rangle + \kappa_1\langle\hat{a}_2^\dagger(t)\hat{a}_2(t)\rangle\\
&+2\langle\hat{a}_1^\dagger(t)\hat{a}_1(t)\rangle\langle\hat{a}_2^\dagger(t)\hat{a}_2(t)\rangle\big]. \qquad (2.282)
\end{aligned}
$$

This can also be rewritten as

$$
\begin{aligned}
\langle\hat{a}^\dagger(t)\hat{a}(t+\tau),\hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle \;&=\; e^{-\mu_1\tau}\big[\Delta n_1^2 + \kappa_1\kappa_2 b_1 b_2\big] + e^{-\mu_2\tau}\big[\Delta n_2^2 + \kappa_1\kappa_2 b_1 b_2\big]\\
&+e^{-(\mu_1+\mu_2)\tau/2}\big[\kappa_2\bar{n}_1 + \kappa_1\bar{n}_2 + 2\bar{n}_1\bar{n}_2\big], \qquad (2.283)
\end{aligned}
$$

where $\bar{n}_j$ and Δn_j^2 is the global mean and variance of the photon number for the j^{th} component of output laser light beams and $\langle\hat{a}_j^2(t)\rangle = \langle\hat{a}_j^{\dagger 2}(t)\rangle = \kappa_j b_j$. Employing (2.283) in Eq.(2.266), we obtain

$$
\begin{aligned}
I(\omega') \;&=\; \frac{1}{\pi}Re\int_0^\infty \Big[\big[\Delta n_1^2 + \kappa_1\kappa_2 b_1 b_2\big]e^{-(\mu_1 - i\omega')\tau} + \big[\Delta n_2^2 + \kappa_1\kappa_2 b_1 b_2\big]e^{-(\mu_2 - i\omega')\tau}\\
&+\big[\kappa_2\bar{n}_1 + \kappa_1\bar{n}_2 + 2\bar{n}_1\bar{n}_2\big]e^{-((\mu_1+\mu_2)/2 - i\omega')\tau}\Big]d\tau.
\end{aligned}
\qquad (2.284)
$$

Furthermore, carrying out the integration over τ, the spectrum of intensity fluctuations for the superposed output laser light beams turns out to be

$$
\begin{aligned}
I(\omega') \;&=\; \frac{\Delta n_1^2\mu_1/\pi}{\mu_1^2 + \omega'^2} + \frac{\Delta n_2^2\mu_2/\pi}{\mu_2^2 + \omega'^2} + \frac{(\kappa_2\bar{n}_1 + \kappa_1\bar{n}_2)(\mu_1+\mu_2)/2\pi}{[(\mu_1+\mu_2)/2]^2 + \omega'^2}\\
&+\frac{\bar{n}_1\bar{n}_2(\mu_1+\mu_2)/\pi}{[(\mu_1+\mu_2)/2]^2 + \omega'^2} + \frac{\kappa_1\kappa_2 b_1 b_2\mu_1/\pi}{\mu_1^2 + \omega'^2} + \frac{\kappa_1\kappa_2 b_1 b_2\mu_2/\pi}{\mu_2^2 + \omega'^2}. \qquad (2.285)
\end{aligned}
$$

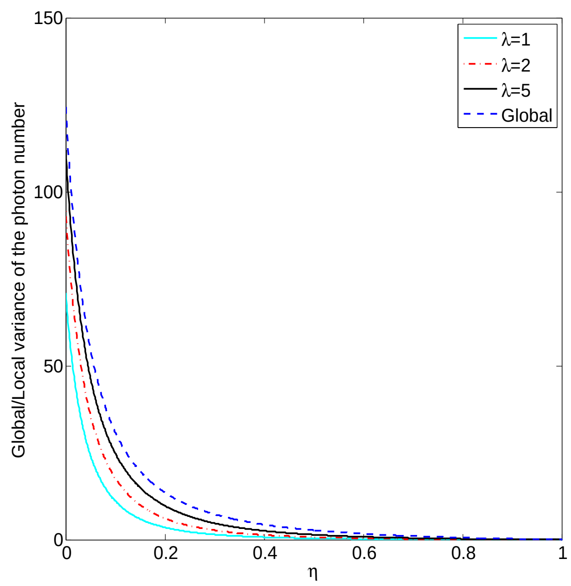

Fig. 2.16: Plots of the global and local variance of the photon number of the superposed output laser light beams [Eq. (2.249) and Eq. (2.287)] versus η for $\kappa_1 = \kappa_2 = 0.8$, $A_1 = 5$, and $A_2 = 10$.

We observe that the spectrum of intensity fluctuations for the superposed output laser light beams is not the sum of the spectra of intensity fluctuations of the constituent output laser light beams. Now upon performing the integration, applying (2.198), we arrive at

$$\int_{-\lambda}^{\lambda} I(\omega')d\omega' \quad = \quad \Delta n^2_{s_{\pm\lambda}}, \tag{2.286}$$

where

$$\begin{aligned}\Delta n^2_{s_{\pm\lambda}} \quad &= \quad z_{11}(\lambda)\Delta n_1^2 + z_{22}(\lambda)\Delta n_2^2 + z_{12}(\lambda)[\kappa_2\bar{n}_1 + \kappa_1\bar{n}_2] \\ &\quad +2z_{12}(\lambda)\bar{n}_1\bar{n}_2 + \kappa_1\kappa_2 b_1 b_2[z_{11}(\lambda) + z_{22}(\lambda)],\end{aligned} \tag{2.287}$$

with

$$z_{ij}(\lambda) \quad = \quad \frac{2}{\pi}\tan^{-1}\left(\frac{2\lambda}{(A_i + A_j)\eta + \kappa_i + \kappa_j}\right), \quad i, j = 1, 2. \tag{2.288}$$

We note that the local variance of the photon number of a pair of superposed output laser light beams described by (2.287) is not the sum of the local variances of the photon numbers of the component output laser light beams. We see from the plots Figs. 2.16 - 2.19 that the local variance of the photon number, as of the local mean photon number and quadrature variance is less than the global variance of the photon number. Moreover, we observe

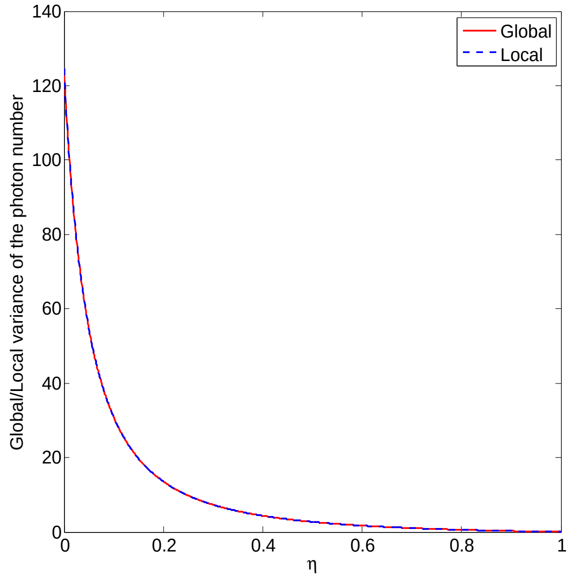

Fig. 2.17: Plots of the global and local ($\lambda = 50$) variance of the photon number of the superposed output laser light beams [Eq. (2.249) and Eq. (2.287)] versus η for $\kappa_1 = \kappa_2 = 0.8$, $A_1 = 5$, and $A_2 = 10$.

that a large part of the total variance of the photon number is confined in a relatively small frequency interval. In addition, for sufficiently large values of λ, the local variance of the photon number turns out to be the global variance of the photon number. Furthermore, we see that for the case in which $A_1 = A_2$ and $\kappa_1 = \kappa_2$, the variance of the photon number of the superposed output laser light beams in a certain frequency interval is found to be

$$\Delta n^2_{s_{\pm\lambda}} = 4\Delta n^2_{j_{\pm\lambda}}, \tag{2.289}$$

where

$$\Delta n^2_{j_{\pm\lambda}} = z_{jj}(\lambda)\Delta n^2_j, \;\; j = 1, 2 \tag{2.290}$$

is the local variance of the photon number of one output laser light beam. We note that Eq. (2.289) represents the local variances of the photon number of a pair of superposed identical output laser light beams and is fourfold of the local variance of the photon number of one output laser light beam.

On the other hand, for the case in which either κ_1 or $\kappa_2 = 0$, the local variance of the photon number of the superposed output laser light beams (2.287) goes over into

$$\Delta n^2_{s_{\pm\lambda}} = z_{jj}(\lambda)\Delta n^2_j, \tag{2.291}$$

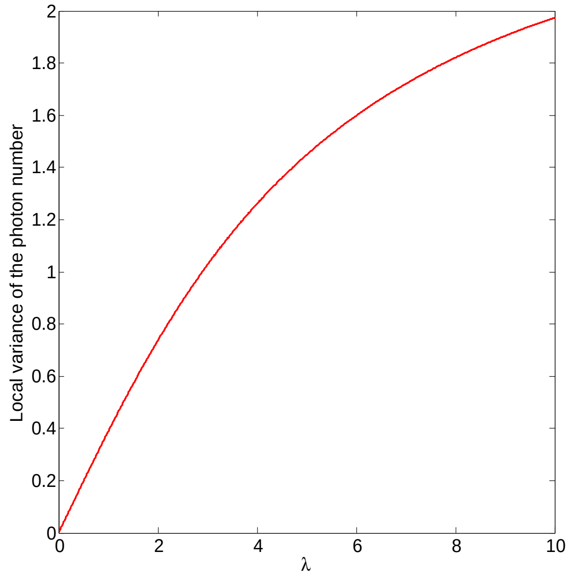

Fig. 2.18: Plots of the local variance of the photon number of the superposed output laser light beams [Eq. (2.287)] versus λ for $\kappa_1 = \kappa_2 = 0.8$, $\eta = 0.5$, $A_1 = 5$, and $A_2 = 10$

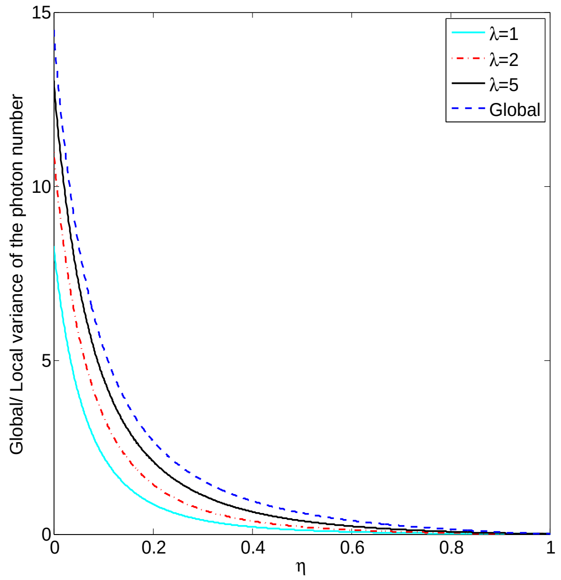

Fig. 2.19: Plots of the global and local variance of the photon number of the j^{th} output laser light beams [Eq. (2.250) and Eq. (2.290)] versus η for $\kappa_j = 0.8$, $A_j = 5$.

which is exactly the same as the local variance of the photon number of one output laser light beam.

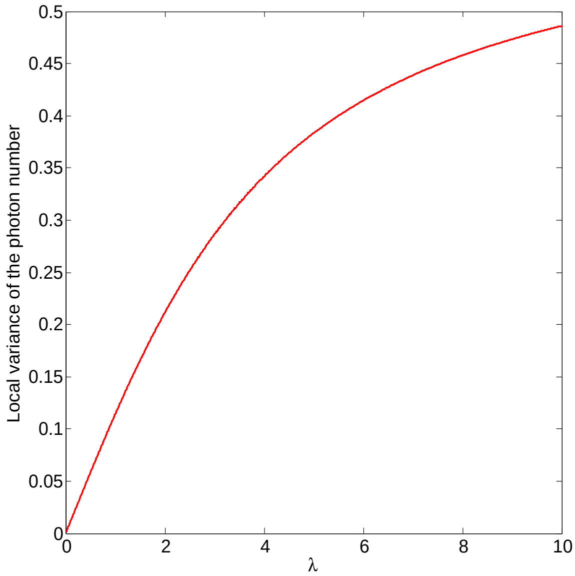

Fig. 2.20: Plot of the local variance of the photon number of the j^{th} output laser light beams [Eq. (2.290)] versus λ for $\kappa_j = 0.8$, $\eta = 0.5$, and $A_j = 5$.

3

A Pair of Superposed Two-Mode Laser Light Beams

Here we wish to explore the squeezing and statistical properties of the output light beams produced by a pair of superposed two-mode three-level cascade lasers whose cavities are coupled to a vacuum reservoirs. We arrange our system in such a way that the output light beam (LB1) from one of the two-mode lasers is incident on a side of a perfectly transmitting mirror, while the output light beam (LB2) from the other laser is incident on a side of a perfectly reflecting mirror as shown in Fig. 1.1. In order to analyze the squeezing and statistical properties of the output light for the superposed two-mode laser light beams, we first derive c-number Langevin equations for the cavity light produced by one of the two-mode laser whose cavity coupled to vacuum reservoir using the pertinent master equation. Employing the solutions of the resulting c-number Langevin equations together with the antinormally ordered characteristic function defined in Hiesenberg picture, we derive the Q function for the two-mode cavity light. By inspection, we also write the Q function for the other two-mode light with different cavity damping constant and linear gain coefficient. Following similar procedures as in the previous chapter, we obtain the Q functions for two-mode output laser light beams. We next derive the density operator for the superposed two-mode output light in terms of the Q functions of the component output light beams.

Using the density operator for the superposed two-mode output light beams together with the Q functions of the two-mode output laser light beams, we determine the quadrature squeezing, the mean and the variances of the photon number sum and difference, and the normalized photon number correlation.

3.1 c-number Langevin equations

In this section we wish to obtain the equation of evolution of the cavity operator for a nondegenerate three-level laser whose cavity is coupled to a two-mode vacuum reservoir. In order to obtain the equation of evolution of the cavity operator, we first drive the time evolution of the reduced density operator for the system under consideration.

The Master equation

A coherently prepared three-level laser may be defined as a quantum optical system in which three level atoms in a cascade configuration and initially in a coherent superposition of the top and bottom levels are injected at constant rate into a cavity coupled to a vacuum reservoir via single port mirror. The three-level atoms in a cascade configuration with its top, intermediate, and bottom levels denoted by $|a\rangle$, $|b\rangle$, and $|c\rangle$ are injected into the cavity at a constant rate r_a and removed after time τ. When the atom decays from the level $|a\rangle$ to $|c\rangle$ via $|b\rangle$, two photons of different frequencies are generated. Direct transition from $|a\rangle$ to $|c\rangle$ is dipole forbidden. We first carry out our analysis for the case in which the transitions $|a\rangle$ to $|b\rangle$ and $|b\rangle$ to $|c\rangle$ are at resonance with the cavity modes. The Hamiltonian describing the interaction of a three-level atom with the cavity modes is expressible as [3, 28]

$$\hat{H} = ig\Big(\hat{a}^\dagger|b\rangle\langle a| - \hat{a}|a\rangle\langle b| + \hat{b}^\dagger|c\rangle\langle b| - \hat{b}|b\rangle\langle c|\Big), \tag{3.1}$$

where $\hat{a}$ and $\hat{b}$ are the annihilation operators for the two cavity modes and g is the coupling constant, assumed to be the same for the two modes. We consider a single three-level atom initially in the state

$$|\psi_A(0)\rangle = C_a(0)|a\rangle + C_c(0)|c\rangle, \tag{3.2}$$

and hence the density operator for a single atom before it is injected into the cavity has the form

$$\hat{\rho}_A(0) = \rho_{aa}^{(0)}|a\rangle\langle a| + \rho_{ac}^{(0)}|a\rangle\langle c| + \rho_{ca}^{(0)}|c\rangle\langle a| + \rho_{cc}^{(0)}|c\rangle\langle c|, \tag{3.3}$$

where

$$\rho_{aa}^{(0)} = C_a(0)C_a^*(0), \tag{3.4}$$

$$\rho_{cc}^{(0)} = C_c(0)C_c^*(0), \tag{3.5}$$

are the probabilities for the the atom to be initially in the upper and lower levels respectively, and

$$\rho_{ac}^{(0)} = \rho_{ca}^{*(0)} = C_a(0)C_c^*(0) = |\rho_{ac}^{(0)}|e^{i\theta} \tag{3.6}$$

represents the initial coherence of the atom.

With the aid of Eqs. (2.17) and (3.1), the equation of evolution of the density operator for the cavity modes can be put in the form

$$\frac{d}{dt}\hat{\rho}(t) = g\Big(\hat{a}^\dagger\hat{\rho}_{ab} - \hat{a}\hat{\rho}_{ba} + \hat{b}^\dagger\hat{\rho}_{bc} - \hat{b}\hat{\rho}_{cb} - \hat{\rho}_{ab}\hat{a}^\dagger + \hat{\rho}_{ba}\hat{a} - \hat{\rho}_{bc}\hat{b}^\dagger + \hat{\rho}_{cb}\hat{b}\Big), \tag{3.7}$$

where

$$\hat{\rho}_{\alpha\beta} = \langle\alpha|\hat{\rho}_{AR}(t)|\beta\rangle, \tag{3.8}$$

with $\alpha, \beta = a, b, c$. Taking the damping of the cavity modes by a two-mode vacuum reservoir into account, the equation of evolution of the density operator for the cavity modes to be

$$\begin{aligned}\frac{d}{dt}\hat{\rho}(t) &= g\Big(\hat{a}^\dagger\hat{\rho}_{ab} - \hat{a}\hat{\rho}_{ba} + \hat{b}^\dagger\hat{\rho}_{bc} - \hat{b}\hat{\rho}_{cb} - \hat{\rho}_{ab}\hat{a}^\dagger + \hat{\rho}_{ba}\hat{a} - \hat{\rho}_{bc}\hat{b}^\dagger + \hat{\rho}_{cb}\hat{b}\Big)\\ &\quad + \frac{\kappa}{2}(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}) + \frac{\kappa}{2}(2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{b}^\dagger\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^\dagger\hat{b}),\end{aligned} \tag{3.9}$$

where κ is the cavity damping constant and is assumed to be the same for both modes.

We next proceed to determine the density matrix element $\hat{\rho}_{\alpha\beta}$ involved in Eq. (3.9). Thus Taking into account of Eqs. (2.23), (2.24), (3.1), and (3.3), one can write

$$\begin{aligned}\frac{d}{dt}\hat{\rho}_{\alpha\beta}(t) &= r_a\Big(\rho_{aa}^{(0)}\delta_{\alpha a}\delta_{a\beta} + \rho_{ac}^{(0)}\delta_{\alpha a}\delta_{c\beta} + \rho_{ca}^{(0)}\delta_{\alpha c}\delta_{a\beta} + \rho_{cc}^{(0)}\delta_{\alpha c}\delta_{c\beta}\Big)\hat{\rho}(t)\\ &\quad + g\Big(\hat{a}^\dagger\delta_{\alpha b}\hat{\rho}_{a\beta} - \hat{a}\delta_{\alpha a}\hat{\rho}_{b\beta} + \hat{b}^\dagger\delta_{\alpha c}\hat{\rho}_{b\beta} - \hat{b}\delta_{\alpha b}\hat{\rho}_{c\beta} - \hat{\rho}_{\alpha b}\hat{a}^\dagger\delta_{a\beta}\\ &\quad + \hat{\rho}_{\alpha a}\hat{a}\delta_{a\beta} - \hat{\rho}_{\alpha c}\hat{b}^\dagger\delta_{b\beta} + \hat{\rho}_{\alpha b}\hat{b}\delta_{c\beta}\Big) - \gamma\hat{\rho}_{\alpha\beta},\end{aligned} \tag{3.10}$$

where γ is the spontaneous decay rate assumed to be the same for all the three levels. Using (3.10), we readily find

$$\frac{d}{dt}\hat{\rho}_{aa}(t) = r_a\rho_{aa}^{(0)}\hat{\rho}(t) - g\Big(\hat{a}\hat{\rho}_{ba} + \hat{\rho}_{ab}\hat{a}^\dagger\Big) - \gamma\hat{\rho}_{aa}, \tag{3.11}$$

$$\frac{d}{dt}\hat{\rho}_{bb}(t) = g\Big(\hat{a}^\dagger\hat{\rho}_{ab} - \hat{b}\hat{\rho}_{cb} + \hat{\rho}_{ab}\hat{a} - \hat{\rho}_{bc}\hat{b}^\dagger\Big) - \gamma\hat{\rho}_{bb}, \tag{3.12}$$

$$\frac{d}{dt}\hat{\rho}_{cc}(t) = r_a\rho_{cc}^{(0)}\hat{\rho}(t) + g\Big(\hat{b}^\dagger\hat{\rho}_{bc} + \hat{\rho}_{cb}\hat{b}\Big) - \gamma\hat{\rho}_{cc}, \tag{3.13}$$

$$\frac{d}{dt}\hat{\rho}_{ab}(t) = g\Big(-\hat{a}\hat{\rho}_{bb} + \hat{\rho}_{aa}\hat{a} - \hat{\rho}_{ac}\hat{b}^\dagger\Big) - \gamma\hat{\rho}_{ab}, \tag{3.14}$$

$$\frac{d}{dt}\hat{\rho}_{ac}(t) = r_a\rho_{ac}^{(0)}\hat{\rho}(t) + g\Big(\hat{a}\hat{\rho}_{bc} + \hat{\rho}_{ab}\hat{b}\Big) - \gamma\hat{\rho}_{ac}, \tag{3.15}$$

$$\frac{d}{dt}\hat{\rho}_{bc}(t) = g\Big(\hat{a}^\dagger\hat{\rho}_{ac} - \hat{b}\hat{\rho}_{cc} + \hat{\rho}_{bb}\hat{b}\Big) - \gamma\hat{\rho}_{bc}. \tag{3.16}$$

We see that Eqs. (3.11-3.16) are nonlinear differential equations and hence it is not possible to obtain exact time-dependent solutions of these equations. Thus upon dropping the g-terms and applying the large-time approximation scheme, we get from Eqs. (3.11), (3.12), (3.13), and (3.15) the approximately valid relation

$$\hat{\rho}_{aa} = \frac{r_a\rho_{aa}^{(0)}}{\gamma}\hat{\rho}(t), \tag{3.17}$$

$$\hat{\rho}_{bb} = 0, \tag{3.18}$$

$$\hat{\rho}_{cc} = \frac{r_a\rho_{cc}^{(0)}}{\gamma}\hat{\rho}(t), \tag{3.19}$$

$$\hat{\rho}_{ac} = \frac{r_a\rho_{ac}^{(0)}}{\gamma}\hat{\rho}(t). \tag{3.20}$$

Substituting the above results into Eqs. (3.14) and (3.16) leads to

$$\frac{d}{dt}\hat{\rho}_{ab}(t) = \frac{gr_a}{\gamma}\left(\rho_{aa}^{(0)}\hat{\rho}\hat{a} - \rho_{ac}^{(0)}\hat{\rho}\hat{b}^{\dagger}\right) - \gamma\hat{\rho}_{ab}, \tag{3.21}$$

$$\frac{d}{dt}\hat{\rho}_{bc}(t) = \frac{gr_a}{\gamma}\left(\hat{a}^{\dagger}\rho_{ac}^{(0)} - \hat{b}\rho_{cc}^{(0)}\right)\hat{\rho} - \gamma\hat{\rho}_{bc}. \tag{3.22}$$

Applying the large-time approximation once more, we obtain from Eqs. (3.21) and (3.22)

$$\hat{\rho}_{ab} = \frac{gr_a}{\gamma^2}\left(\rho_{aa}^{(0)}\hat{\rho}\hat{a} - \rho_{ac}^{(0)}\hat{\rho}\hat{b}^{\dagger}\right), \tag{3.23}$$

$$\hat{\rho}_{bc} = \frac{gr_a}{\gamma^2}\left(\hat{a}^{\dagger}\rho_{ac}^{(0)} - \hat{b}\rho_{cc}^{(0)}\right)\hat{\rho}. \tag{3.24}$$

On account of Eqs.(3.23), (3.24) and their complex conjugate, the master equation described by (3.7) turns out to be

$$\begin{aligned}\frac{d}{dt}\hat{\rho}(t) = {} & \frac{\kappa}{2}(2\hat{a}\hat{\rho}\hat{a}^{\dagger} - \hat{a}^{\dagger}\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^{\dagger}\hat{a}) + \frac{1}{2}A\rho_{aa}^{(0)}(2\hat{a}^{\dagger}\hat{\rho}\hat{a} - \hat{a}\hat{a}^{\dagger}\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^{\dagger}) \\ & + \frac{1}{2}(A\rho_{cc}^{(0)} + \kappa)(2\hat{b}\hat{\rho}\hat{b}^{\dagger} - \hat{b}^{\dagger}\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^{\dagger}\hat{b}) \\ & + \frac{1}{2}A\rho_{ac}^{(0)}(\hat{\rho}\hat{a}^{\dagger}\hat{b}^{\dagger} - 2\hat{a}^{\dagger}\hat{\rho}\hat{b}^{\dagger} + \hat{a}^{\dagger}\hat{b}^{\dagger}\hat{\rho}) + \frac{1}{2}A\rho_{ca}^{(0)}(\hat{\rho}\hat{a}\hat{b} - 2\hat{b}\hat{\rho}\hat{a} + \hat{a}\hat{b}\hat{\rho}),\end{aligned} \tag{3.25}$$

where

$$A = 2r_a g^2/\gamma^2 \tag{3.26}$$

is the linear gain coefficient.

The existence of the terms $\hat{a}\hat{b}$ and $\hat{a}^{\dagger}\hat{b}^{\dagger}$ in the master equation (3.25) is a signature that the nondegenerate three-level laser can be a source of squeezed light.

In order to understand the existing situation in depth, one needs to revisit the master equation (3.25). From the general form of the master equation, it is not difficult to infer that the second term stands for the gain of mode a, the third term for the lose of mode b and the first term for the lose of mode a. It is also noteworthy that the first term and the part involving κ in the third term comes from the contribution of the coupling of the cavity with external vacuum modes (reservoir).

c-number Langevin equations

Now we derive the c-number Langevin equations associated with the normal ordering, using the aforementioned master equation. Here we first derive the equation of evolution for the expectation values of the cavity mode operators in the normal order and with the aid of the resulting equations we get the c-number Langevin equations. To this end, employing the relation (2.43) along with the (3.25), we readily find

$$\frac{d}{dt}\langle\hat{a}(t)\rangle = -\frac{1}{2}\mu_a\langle\hat{a}(t)\rangle + \frac{1}{2}\nu_-\langle\hat{b}^\dagger(t)\rangle, \tag{3.27}$$

$$\frac{d}{dt}\langle\hat{b}(t)\rangle = -\frac{1}{2}\mu_b\langle\hat{b}(t)\rangle + \frac{1}{2}\nu_+\langle\hat{a}^\dagger(t)\rangle, \tag{3.28}$$

$$\frac{d}{dt}\langle\hat{a}^2(t)\rangle = -\mu_a\langle\hat{a}^2(t)\rangle + \nu_-\langle\hat{b}^\dagger(t)\hat{a}(t)\rangle, \tag{3.29}$$

$$\frac{d}{dt}\langle\hat{b}^2(t)\rangle = -\mu_b\langle\hat{b}^2(t)\rangle + \nu_+\langle\hat{a}^\dagger(t)\hat{b}(t)\rangle, \tag{3.30}$$

$$\frac{d}{dt}\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle = -\mu_a\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle + \frac{1}{2}\nu_-\langle\hat{a}^\dagger(t)\hat{b}^\dagger(t)\rangle + \frac{1}{2}\nu_-^*\langle\hat{a}(t)\hat{b}(t)\rangle + A\rho_{aa}^{(0)}, \tag{3.31}$$

$$\frac{d}{dt}\langle\hat{b}^\dagger(t)\hat{b}(t)\rangle = -\mu_b\langle\hat{b}^\dagger(t)\hat{b}(t)\rangle + \frac{1}{2}\nu_+\langle\hat{a}^\dagger(t)\hat{b}^\dagger(t)\rangle + \frac{1}{2}\nu_+^*\langle\hat{a}(t)\hat{b}(t)\rangle, \tag{3.32}$$

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}(t)\hat{b}(t)\rangle &= -\frac{1}{2}(\mu_a+\mu_b)\langle\hat{a}(t)\hat{b}(t)\rangle + \frac{1}{2}\nu_+\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle \\ &\quad + \frac{1}{2}\nu_-\langle\hat{b}^\dagger(t)\hat{b}(t)\rangle + \frac{1}{2}\nu_+,\end{aligned} \tag{3.33}$$

$$\frac{d}{dt}\langle\hat{a}^\dagger(t)\hat{b}(t)\rangle = -\frac{1}{2}(\mu_a+\mu_b)\langle\hat{a}^\dagger(t)\hat{b}(t)\rangle + \frac{1}{2}\nu_+\langle\hat{a}^{\dagger 2}(t)\rangle + \frac{1}{2}\nu_-^*\langle\hat{b}^2(t)\rangle. \tag{3.34}$$

in which

$$\mu_a = \kappa - A\rho_{aa}^{(0)}, \tag{3.35}$$

$$\mu_b = \kappa + A\rho_{cc}^{(0)}, \tag{3.36}$$

$$\nu_\pm = \pm A\rho_{ac}^{(0)}. \tag{3.37}$$

We see that Eqs. (3.27)- (3.34) are in the normal order. Replacing $\hat{a}$ and $\hat{a}^\dagger$ by α and α^* and $\hat{b}$ and $\hat{b}^\dagger$ by β and β^*. Thus, the corresponding c-number equation are

$$\frac{d}{dt}\langle\alpha(t)\rangle = -\frac{1}{2}\mu_a\langle\alpha(t)\rangle + \frac{1}{2}\nu_-\langle\beta^*(t)\rangle, \tag{3.38}$$

$$\frac{d}{dt}\langle\beta(t)\rangle = -\frac{1}{2}\mu_b\langle\beta(t)\rangle + \frac{1}{2}\nu_+\langle\alpha^*(t)\rangle, \tag{3.39}$$

$$\frac{d}{dt}\langle\alpha^2(t)\rangle = -\mu_a\langle\alpha^2(t)\rangle + \nu_-\langle\beta^*(t)\alpha(t)\rangle, \tag{3.40}$$

$$\frac{d}{dt}\langle\beta^2(t)\rangle = -\mu_b\langle\beta^2(t)\rangle + \nu_+\langle\alpha^*(t)\beta(t)\rangle, \tag{3.41}$$

$$\frac{d}{dt}\langle\alpha^*(t)\alpha(t)\rangle = -\mu_a\langle\alpha^*(t)\alpha(t)\rangle + \frac{1}{2}\nu_-\langle\alpha^*(t)\beta^*(t)\rangle + \frac{1}{2}\nu_-^*\langle\alpha(t)\beta(t)\rangle + A\rho_{aa}^{(0)}, \tag{3.42}$$

$$\frac{d}{dt}\langle\beta^*(t)\beta(t)\rangle = -\mu_b\langle\beta^*(t)\beta(t)\rangle + \frac{1}{2}\nu_+\langle\alpha^*(t)\beta^*(t)\rangle + \frac{1}{2}\nu_+^*\langle\alpha(t)\beta(t)\rangle, \tag{3.43}$$

$$\begin{aligned}\frac{d}{dt}\langle\alpha(t)\beta(t)\rangle &= -\frac{1}{2}(\mu_a+\mu_b)\langle\alpha(t)\beta(t)\rangle + \frac{1}{2}\nu_+\langle\alpha^*(t)\alpha(t)\rangle \\ &\quad + \frac{1}{2}\nu_-\langle\beta^*(t)\beta(t)\rangle + \frac{1}{2}\nu_+,\end{aligned} \tag{3.44}$$

$$\frac{d}{dt}\langle\alpha^*(t)\beta(t)\rangle = -\frac{1}{2}(\mu_a+\mu_b)\langle\alpha^*(t)\beta(t)\rangle + \frac{1}{2}\nu_+\langle\alpha^{*2}(t)\rangle + \frac{1}{2}\nu_-^*\langle\beta^2(t)\rangle. \tag{3.45}$$

On the basis of Eqs. (3.38) and (3.39), one can write

$$\frac{d}{dt}\alpha(t) = -\frac{1}{2}\mu_a\alpha(t) + \frac{1}{2}\nu_-\beta^*(t) + f_\alpha(t) \tag{3.46}$$

$$\frac{d}{dt}\beta(t) = -\frac{1}{2}\mu_b\beta(t) + \frac{1}{2}\nu_+\alpha^*(t) + f_\beta(t). \tag{3.47}$$

where $f_\alpha(t)$ and $f_\beta(t)$ are the noise forces whose properties remain to be determined. If we compare Eqs. (3.38) and (3.39) with the expectation value of Eqs. (3.46) and (3.47), we can see that

$$\langle f_\alpha(t)\rangle = \langle f_\beta(t)\rangle = 0. \tag{3.48}$$

Furthermore, upon using (3.46) and (3.47) one readily finds

$$\frac{d}{dt}\langle\alpha^2(t)\rangle = -\mu_a\langle\alpha^2(t)\rangle + \nu_-\langle\beta^*(t)\alpha(t)\rangle + 2\langle\alpha(t)f_\alpha(t)\rangle, \tag{3.49}$$

$$\frac{d}{dt}\langle\beta^2(t)\rangle = -\mu_b\langle\beta^2(t)\rangle + \nu_+\langle\alpha^*(t)\beta(t)\rangle + 2\langle\beta(t)f_\beta(t)\rangle, \tag{3.50}$$

$$\begin{aligned}
\frac{d}{dt}\langle\alpha^*(t)\alpha(t)\rangle &= -\mu_a\langle\alpha^*(t)\alpha(t)\rangle + \frac{1}{2}\nu_-\langle\alpha^*(t)\beta^*(t)\rangle + \frac{1}{2}\nu_-^*\langle\alpha(t)\beta(t)\rangle \\
&\quad +\langle\alpha(t)f_\alpha^*(t)\rangle + \langle\alpha^*(t)f_\alpha(t)\rangle, && (3.51)\\
\frac{d}{dt}\langle\beta^*(t)\beta(t)\rangle &= -\mu_b\langle\beta^*(t)\beta(t)\rangle + \frac{1}{2}\nu_+\langle\alpha^*(t)\beta^*(t)\rangle + \frac{1}{2}\nu_+^*\langle\alpha(t)\beta(t)\rangle \\
&\quad +\langle\beta(t)f_\beta^*(t)\rangle + \langle\beta^*(t)f_\beta(t)\rangle, && (3.52)\\
\frac{d}{dt}\langle\alpha(t)\beta(t)\rangle &= -\frac{1}{2}(\mu_a+\mu_b)\langle\alpha(t)\beta(t)\rangle + \frac{1}{2}\nu_+\langle\alpha^*(t)\alpha(t)\rangle \\
&\quad +\frac{1}{2}\nu_-\langle\beta^*(t)\beta(t)\rangle + \langle\alpha(t)f_\beta(t)\rangle + \langle\beta(t)f_\alpha(t)\rangle, && (3.53)\\
\frac{d}{dt}\langle\alpha^*(t)\beta(t)\rangle &= -\frac{1}{2}(\mu_a+\mu_b)\langle\alpha^*(t)\beta(t)\rangle + \frac{1}{2}\nu_+\langle\alpha^{*2}(t)\rangle + \frac{1}{2}\nu_-^*\langle\beta^2(t)\rangle \\
&\quad +\langle\alpha^*(t)f_\beta(t)\rangle + \langle\beta(t)f_\alpha^*(t)\rangle. && (3.54)
\end{aligned}$$

We observe that Eq. (3.40) and (3.49), (3.41) and (3.50), (3.42) and (3.51), (3.43) and (3.52), (3.44) and (3.53) as well as (3.45) and (3.54) will have the same forms if

$$\langle\alpha(t)f_\alpha(t)\rangle = 0, \tag{3.55}$$

$$\langle\beta(t)f_\beta(t)\rangle = 0, \tag{3.56}$$

$$\langle\alpha(t)f_\alpha^*(t)\rangle + \langle\alpha^*(t)f_\alpha(t)\rangle = A\rho_{aa}^{(0)}, \tag{3.57}$$

$$\langle\beta(t)f_\beta^*(t)\rangle + \langle\beta^*(t)f_\beta(t)\rangle = 0, \tag{3.58}$$

$$\langle\alpha(t)f_\beta(t)\rangle + \langle\beta(t)f_\alpha(t)\rangle = \frac{1}{2}\nu_+, \tag{3.59}$$

$$\langle\alpha^*(t)f_\beta(t)\rangle + \langle\beta(t)f_\alpha^*(t)\rangle = 0. \tag{3.60}$$

The formal solution of Eqs.(3.46) and (3.47) can be written as

$$\begin{aligned}
\alpha(t) &= \alpha(0)e^{-\mu_a t/2} + \int_0^t e^{-\mu_a(t-t')/2}\Big[\frac{1}{2}\nu_-\beta^*(t') + f_\alpha(t')\Big]dt' && (3.61)\\
\beta(t) &= \beta(0)e^{-\mu_b t/2} + \int_0^t e^{-\mu_b(t-t')/2}\Big[\frac{1}{2}\nu_+\alpha^*(t') + f_\beta(t')\Big]dt'. && (3.62)
\end{aligned}$$

With aid of (3.61), we can write

$$\langle\alpha(t)f_\alpha(t)\rangle = \langle\alpha(0)f_\alpha(t)\rangle e^{-\mu_a t/2} + \int_0^t e^{-\mu_a(t-t')/2}\Big[\frac{1}{2}\nu_-\langle\beta^*(t')f_\alpha(t)\rangle + \langle f_\alpha(t')f_\alpha(t)\rangle\Big]dt'. \tag{3.63}$$

On account of (3.55) and the fact that the noise force at time t does not affect the the cavity mode variable at earlier time t, we see that

$$\langle f_\alpha(t')f_\alpha(t)\rangle = 0. \tag{3.64}$$

Following the same line of reasoning, one can also show that

$$\langle f_\beta(t')f_\beta(t)\rangle = 0. \tag{3.65}$$

Moreover, taking into account of (3.61) and its complex conjugate along with (3.57), we get

$$\int_0^t e^{-\mu_a(t-t')/2}\big[\langle f_\alpha(t')f_\alpha^*(t)\rangle + \langle f_\alpha^*(t')f_\alpha(t)\rangle\big]dt' = A\rho_{aa}^{(0)}, \tag{3.66}$$

so that assuming $\langle f_\alpha(t')f_\alpha^*(t)\rangle = \langle f_\alpha^*(t')f_\alpha(t)\rangle$, we have

$$\int_0^t e^{-\mu_a(t-t')/2}\langle f_\alpha(t')f_\alpha^*(t)\rangle dt' = \frac{1}{2}A\rho_{aa}^{(0)}. \tag{3.67}$$

On the basis of this result, one can obtain

$$\langle f_\alpha(t')f_\alpha^*(t)\rangle = \langle f_\alpha^*(t')f_\alpha(t)\rangle = A\rho_{aa}^{(0)}\delta(t-t'). \tag{3.68}$$

Following the same procedure, one can also show that

$$\langle f_\beta(t')f_\beta^*(t)\rangle = \langle f_\beta^*(t')f_\beta(t)\rangle = 0, \tag{3.69}$$

$$\langle f_\alpha(t')f_\beta^*(t)\rangle = \langle f_\beta^*(t')f_\alpha(t)\rangle = 0, \tag{3.70}$$

$$\langle f_\alpha(t')f_\beta(t)\rangle = \langle f_\beta(t')f_\alpha(t)\rangle = \frac{1}{2}\nu_+\delta(t-t'). \tag{3.71}$$

It is worth mentioning that Eqs.(3.48), (3.64), (3.65), (3.68), (3.69), (3.70), and (3.71) describe the correlation properties of the noise forces $f_\alpha(t)$ and $f_\beta(t)$ associated with normal ordering.

We now proceed to obtain the time dependent solutions of the coupled differential equations described by Eqs. (3.46) and (3.47). To solve such a coupled differential equations, we rewrite these equation in a matrix equation form as

$$\frac{d}{dt}\mathbf{U}(t) = \mathbf{M}\mathbf{U}(t) + \mathbf{N}(t), \tag{3.72}$$

where

$$\mathbf{U}(t) = \begin{pmatrix} \alpha^*(t) \\ \beta(t) \end{pmatrix}, \tag{3.73}$$

$$\mathbf{M} = -\frac{1}{2}\begin{pmatrix} \mu_a & -\nu_-^* \\ -\nu_+ & \mu_b \end{pmatrix}, \tag{3.74}$$

$$\mathbf{N}(t) = \begin{pmatrix} f_\alpha^*(t) \\ f_\beta(t) \end{pmatrix}. \tag{3.75}$$

In order to obtain the solution Eq. (3.72), we need to find the eigenvalues of $\hat{\mathbf{M}}$ such that

$$\mathbf{M}\mathbf{V} = \lambda\mathbf{V}, \tag{3.76}$$

where **V** defined by

$$\mathbf{V} = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}. \tag{3.77}$$

Eq.(3.76) has a nontrivial solution provide that

$$|\mathbf{M} - \lambda\mathbf{I}| = 0. \tag{3.78}$$

This implies that

$$4\lambda^2 + 2(\mu_a + \mu_b)\lambda + (\mu_a\mu_b - \nu_+\nu_-^*) = 0. \tag{3.79}$$

Solving this quadratic equation, the eigenvalues of the Matrix **M** found to be

$$\lambda_\pm = -\frac{1}{4}\Big\{(\mu_a + \mu_b) \mp \sqrt{(\mu_a - \mu_b)^2 + 4\nu_+\nu_-^*}\Big\}. \tag{3.80}$$

Taking into account Eqs. (3.35), (3.36), (3.37), and the relation

$$|\rho_{ac}^{(0)}|^2 = \rho_{aa}^{(0)}\rho_{cc}^{(0)}, \tag{3.81}$$

Eq. (3.80) can be rewritten as

$$\lambda_\pm = -\frac{1}{4}\big[2\kappa + A\eta(1 \mp 1)\big], \tag{3.82}$$

where

$$\eta = \rho_{cc}^{(0)} - \rho_{aa}^{(0)}. \tag{3.83}$$

We consider the case in which the atoms are initially in a coherent superposition of the top and bottom levels. Thus, in view of Eqs.(3.3) and (2.19), the following condition hold

$$\rho_{aa}^{(0)} + \rho_{cc}^{(0)} = 1. \tag{3.84}$$

Using this relation along with Eq. (3.83), we get

$$\rho_{aa}^{(0)} = \frac{1-\eta}{2}, \tag{3.85}$$

$$\rho_{cc}^{(0)} = \frac{1+\eta}{2}, \tag{3.86}$$

$$\rho_{ac}^{(0)} = \frac{\sqrt{1-\eta^2}}{2}e^{i\theta}. \tag{3.87}$$

Since the value $\rho_{aa}^{(0)}$ is between 0 and 1, then it is not difficult to see that the value of η lies in the interval $-1 \leq \eta \leq 1$. We note that the formal solution of the expression described by (3.72), can be written as

$$\mathbf{U}(t) = e^{\mathbf{M}t}\mathbf{U}(0) + \int_0^t e^{\mathbf{M}(t-t')}\mathbf{N}(t')dt'. \tag{3.88}$$

We also note that $\lambda_+ \neq \lambda_-$. Employing the relation [24]

$$f(\mathbf{A}) = \sum_{i=1}^{n} f(\lambda_i) \prod_{i \neq j} \frac{\mathbf{A} - \lambda_j \mathbf{I}}{\lambda_i - \lambda_j}, \tag{3.89}$$

we readily establish that

$$e^{\mathbf{M}t} = \frac{1}{\lambda_+ - \lambda_-} \Big[(\mathbf{M} - \lambda_- \mathbf{I}) e^{\lambda_+ t} - (\mathbf{M} - \lambda_+ \mathbf{I}) e^{\lambda_- t} \Big], \tag{3.90}$$

from which follows

$$e^{\mathbf{M}t} = \frac{1}{\lambda_+ - \lambda_-} \begin{pmatrix} m_{11}(t) & m_{12}(t) \\ m_{21}(t) & m_{22}(t) \end{pmatrix}, \tag{3.91}$$

and

$$e^{\mathbf{M}(t-t')} = \frac{1}{\lambda_+ - \lambda_-} \begin{pmatrix} m_{11}(t-t') & m_{12}(t-t') \\ m_{21}(t-t') & m_{22}(t-t') \end{pmatrix}, \tag{3.92}$$

where

$$m_{11}(t) = -\frac{1}{2} \Big[(\mu_a + 2\lambda_-) e^{\lambda_+ t} - (\mu_a + 2\lambda_+) e^{\lambda_- t} \Big], \tag{3.93}$$

$$m_{22}(t) = -\frac{1}{2} \Big[(\mu_b + 2\lambda_-) e^{\lambda_+ t} - (\mu_b + 2\lambda_+) e^{\lambda_- t} \Big], \tag{3.94}$$

$$m_{12}(t) = \frac{\nu_-^*}{2} \Big(e^{\lambda_+ t} - e^{\lambda_- t} \Big), \tag{3.95}$$

$$m_{21}(t) = \frac{\nu_+}{2} \Big(e^{\lambda_+ t} - e^{\lambda_- t} \Big), \tag{3.96}$$

Furthermore, using Eqs. (3.35), (3.36), (3.37), and (3.82) together with the relations (3.85), (3.86), and (3.87), one can put Eqs. (3.93),(3.94), (3.95), and (3.96)in the form

$$m_{11}(t) = \frac{A}{4} \Big[(\eta - 1) e^{-(A\eta + \kappa)t/2} + (\eta + 1) e^{-\kappa t/2} \Big], \tag{3.97}$$

$$m_{22}(t) = \frac{A}{4} \Big[(\eta + 1) e^{-(A\eta + \kappa)t/2} + (\eta - 1) e^{-\kappa t/2} \Big], \tag{3.98}$$

$$m_{12}(t) = \frac{\sqrt{1 - \eta^2}}{4} e^{-i\theta} \Big[e^{-(A\eta + \kappa)t/2} - e^{-\kappa t/2} \Big], \tag{3.99}$$

$$m_{21}(t) = -\frac{\sqrt{1 - \eta^2}}{4} e^{i\theta} \Big[e^{-(A\eta + \kappa)t/2} - e^{-\kappa t/2} \Big], \tag{3.100}$$

With the aid of Eqs.(3.73),(3.75),(3.91),(3.92), and(3.82) we readily obtain

$$e^{\mathbf{M}t} \mathbf{U}(0) = \frac{2}{A\eta} \begin{pmatrix} m_{11}(t) \alpha^*(0) + m_{12}(t) \beta(0) \\ m_{21}(t) \alpha^*(0) + m_{22}(t) \beta(0) \end{pmatrix} \tag{3.101}$$

and

$$\int_0^t e^{\mathbf{M}(t-t')} \mathbf{N}(t') dt' = \frac{2}{A\eta} \begin{pmatrix} \int_0^t \big[m_{11}(t-t') f_\alpha^*(t') + m_{12}(t-t') f_\beta(t') \big] dt' \\ \int_0^t \big[m_{21}(t-t') f_\alpha^*(t') + m_{22}(t-t') f_\beta(t') \big] dt' \end{pmatrix}. \tag{3.102}$$

Hence upon combining Eqs. (3.88),(3.101), and (3.102), we get

$$\alpha^*(t) = \frac{2}{A\eta}\Big[m_{11}(t)\alpha^*(0) + m_{12}(t)\beta(0) + \int_0^t \big[m_{11}(t-t')f_\alpha^*(t') + m_{12}(t-t')f_\beta(t')\big]dt'\Big] \tag{3.103}$$

and

$$\beta(t) = \frac{2}{A\eta}\Big[m_{21}(t)\alpha^*(0) + m_{22}(t)\beta(0) + \int_0^t \big[m_{21}(t-t')f_\alpha^*(t') + m_{22}(t-t')f_\beta(t')\big]dt'\Big]. \tag{3.104}$$

Moreover, taking into account of Eqs. (3.97), (3.98), (3.99), and (3.100), and setting $\theta = 0$, one can put Eqs. (3.103) and (3.104) in the form

$$\alpha(t) = A_+(t)\alpha(0) + B_+(t)\beta^*(0) + C_+(t) + D_+(t) \tag{3.105}$$

$$\beta(t) = A_-(t)\beta(0) + B_-(t)\alpha^*(0) + C_-(t) + D_-(t), \tag{3.106}$$

where

$$A_\pm(t) = \frac{1}{2\eta}\Big[(\eta \mp 1)e^{-(A\eta+\kappa)t/2} + (\eta \pm 1)e^{-\kappa t/2}\Big], \tag{3.107}$$

$$B_\pm(t) = \pm\frac{\sqrt{1-\eta^2}}{2\eta}\Big[e^{-(A\eta+\kappa)t/2} - e^{-\kappa t/2}\Big], \tag{3.108}$$

$$C_+(t) = \frac{1}{2\eta}\int_0^t \Big[(\eta-1)e^{-(A\eta+\kappa)(t-t')/2} + (\eta+1)e^{-\kappa(t-t')/2}\Big]f_\alpha(t')dt', \tag{3.109}$$

$$C_-(t) = \frac{1}{2\eta}\int_0^t \Big[(\eta+1)e^{-(A\eta+\kappa)(t-t')/2} + (\eta-1)e^{-\kappa(t-t')/2}\Big]f_\beta(t')dt', \tag{3.110}$$

$$D_+(t) = \frac{\sqrt{1-\eta^2}}{2\eta}\int_0^t \Big[e^{-(A\eta+\kappa)(t-t')/2} - e^{-\kappa(t-t')/2}\Big]f_\beta^*(t')dt', \tag{3.111}$$

$$D_-(t) = -\frac{\sqrt{1-\eta^2}}{2\eta}\int_0^t \Big[e^{-(A\eta+\kappa)(t-t')/2} - e^{-\kappa(t-t')/2}\Big]f_\alpha^*(t')dt'. \tag{3.112}$$

3.2 The Q functions for the output laser light beams

Here we wish to obtain the Q function for the cavity modes produced by the system under consideration. The Q function for a two-mode light can be expressed as

$$Q_1(\alpha_1, \beta_1, t) = \frac{1}{\pi^4}\int d^2z d^2\eta \phi_a(z, \eta, t) \exp\big(z^*\alpha - z\alpha^* + \eta^*\beta - \eta\beta^*\big), \tag{3.113}$$

where

$$\phi_a(z, \eta, t) = Tr\Big(\hat{\rho}(0)e^{-z^*\hat{a}(t)}e^{z\hat{a}^\dagger(t)}e^{-\eta^*\hat{b}(t)}e^{\eta\hat{b}^\dagger(t)}\Big) \tag{3.114}$$

is the antinormally ordered characteristic function defined in Heisenberg picture [3, 15-18]. To this end, using the identity described by (2.70), the characteristic function can be written in the normal order as

$$\phi_a(z, \eta, t) = e^{-z^*z-\eta^*\eta}Tr\Big(\hat{\rho}(0)e^{z\hat{a}^\dagger(t)}e^{-z^*\hat{a}(t)}e^{\eta\hat{b}^\dagger(t)}e^{-\eta^*\hat{b}(t)}\Big), \tag{3.115}$$

so that the corresponding c-number equation is

$$\phi_a(z, \eta, t) = e^{-z^*z-\eta^*\eta}\Big\langle \exp\big(z\alpha^*(t) - z^*\alpha(t) + \eta\beta^*(t) - \eta^*\beta(t)\big)\Big\rangle. \tag{3.116}$$

If we assume the cavity modes are initially in a vacuum state, then we see from Eqs. (3.105), (3.106), along with (3.48) that

$$\langle\alpha(t)\rangle = \langle\beta(t)\rangle = 0. \tag{3.117}$$

On the other hand, we note that Eqs. (3.38) and (3.39) are linear differential equations for $\alpha(t)$ and $\beta(t)$. Hence from Eqs. (3.38), (3.39), and (3.117), one can observe that α and β are the Gaussian variables with a vanishing mean. Thus in view of this Eq. (3.116) can be put in the form

$$\phi_a(z, \eta, t) = e^{-z^*z-\eta^*\eta}\exp\left[\frac{1}{2}\bigg\langle\Big(z\hat{\alpha}^*(t) - z^*\alpha(t) + \eta\beta^*(t) - \eta^*\beta(t)\Big)^2\bigg\rangle\right]. \tag{3.118}$$

It then follows that

$$\begin{aligned}\phi_a(z, \eta, t) \;=\; & \exp\left[\frac{1}{2}\big(z^2\langle\alpha^{*2}(t)\rangle + z^{*2}\langle\alpha^2(t)\rangle + \eta^2\langle\beta^{*2}(t)\rangle + \eta^{*2}\langle\beta^2(t)\rangle\big)\right]\\ & \times\exp\Big[\big(z\eta + z^*\eta^*\big)\langle\alpha(t)\beta(t)\rangle - z^*\eta\langle\alpha(t)\beta^*(t)\rangle - z\eta^*\langle\alpha^*(t)\beta(t)\rangle\Big]\\ & \times\exp\Big[-z^*z\big(1 + \langle\alpha^*(t)\alpha(t)\rangle\big) - \eta^*\eta\big(1 + \langle\beta^*(t)\beta(t)\rangle\big)\Big].\end{aligned} \tag{3.119}$$

The various expectation values in Eqs. (3.119) can be evaluated applying Eqs. (3.105) and (3.106). Furthermore, employing (3.105), one can also write

$$\begin{aligned}\langle\alpha^2(t)\rangle \;=\; & A_+(t)\langle\alpha(t)\alpha(0)\rangle + B_+(t)\langle\alpha(t)\beta^*(0)\rangle\\ & +\langle\alpha(t)C_+(t)\rangle + \langle\alpha(t)D_+(t)\rangle.\end{aligned} \tag{3.120}$$

Making use of (3.105) once more, we see that

$$\begin{aligned}\langle\alpha(t)\alpha(0)\rangle &= A_+(t)\langle\alpha^2(0)\rangle + B_+(t)\langle\alpha(0)\beta^*(0)\rangle \\ &\quad +\langle\alpha(0)C_+(t)\rangle + \langle\alpha(0)D_+(t)\rangle.\end{aligned} \tag{3.121}$$

Based on the fact that the noise force at time t does not affect the cavity mode variables at earlier times and taking the cavity modes to be initially in a vacuum state, one can get

$$\langle\alpha(t)\alpha(0)\rangle = 0. \tag{3.122}$$

A similar calculations show that

$$\langle\alpha(t)\beta^*(0)\rangle = 0, \tag{3.123}$$

$$\langle\alpha(t)C_+(t)\rangle = \langle C_+(t)C_+(t)\rangle + \langle D_+(t)C_+(t)\rangle, \tag{3.124}$$

$$\langle\alpha(t)D_+(t)\rangle = \langle C_+(t)D_+(t)\rangle + \langle D_+(t)D_+(t)\rangle. \tag{3.125}$$

Hence upon substituting Eqs. (3.122), (3.123), (3.124), and (3.125) into (3.120), one obtains

$$\begin{aligned}\langle\alpha^2(t)\rangle &= \langle C_+(t)C_+(t)\rangle + \langle D_+(t)C_+(t)\rangle \\ &\quad +\langle C_+(t)D_+(t)\rangle + \langle D_+(t)D_+(t)\rangle.\end{aligned} \tag{3.126}$$

Taking into account of (3.109) along with (3.64), one can readily find

$$\langle C_+(t)C_+(t)\rangle = 0. \tag{3.127}$$

It can also verified in a similar manner that

$$\langle D_+(t)D_+(t)\rangle = 0, \tag{3.128}$$

$$\langle C_+(t)D_+(t)\rangle = \langle D_+(t)C_+(t)\rangle = 0. \tag{3.129}$$

In view of these result, the expression described by Eq. (3.126) turns out to be

$$\langle\alpha^2(t)\rangle = 0. \tag{3.130}$$

Making use of similar calculations, it is not difficult to show that

$$\langle\beta^2(t)\rangle = \langle\beta^{*2}(t)\rangle = \langle\alpha^{*2}(t)\rangle = 0. \tag{3.131}$$

Now with the aid of the complex conjugate of Eq. (3.106), one can write

$$\begin{aligned}\langle\alpha(t)\beta^*(t)\rangle &= A_-(t)\langle\alpha(t)\beta^*(0)\rangle + B_-(t)\langle\alpha(t)\alpha(0)\rangle \\ &\quad +\langle\alpha(t)C^*_-(t)\rangle + \langle\alpha(t)D^*_-(t)\rangle. \end{aligned} \tag{3.132}$$

Again employing (3.105), we have

$$\begin{aligned}\langle\alpha(t)\beta^*(0)\rangle &= A_+(t)\langle\alpha(0)\beta^*(0)\rangle + B_+(t)\langle\beta^{*2}(0)\rangle \\ &\quad +\langle\beta^*(0)C_+(t)\rangle + \langle\beta^*(0)D_+(t)\rangle. \end{aligned} \tag{3.133}$$

In view of the fact that the noise force at time t does not affect the cavity mode variables at earlier times and taking the cavity modes to be initially in a vacuum state, one can get

$$\langle\alpha(t)\beta^*(0)\rangle = 0. \tag{3.134}$$

It is also possible to verify that in a similar manner that

$$\langle\alpha(t)\alpha(0)\rangle = 0, \tag{3.135}$$

$$\langle\alpha(t)C^*_-(t)\rangle = \langle C_+(t)C^*_-(t)\rangle + \langle D_+(t)C^*_-(t)\rangle, \tag{3.136}$$

$$\langle\alpha(t)D^*_-(t)\rangle = \langle C_+(t)D^*_-(t)\rangle + \langle D_+(t)D^*_-(t)\rangle. \tag{3.137}$$

Upon substituting Eqs. (3.134), (3.135), (3.136), and (3.137) into (3.132), we have

$$\begin{aligned}\langle\alpha(t)\beta^*(t)\rangle &= \langle C_+(t)C^*_-(t)\rangle + \langle D_+(t)C^*_-(t)\rangle \\ &\quad +\langle C_+(t)D^*_-(t)\rangle + \langle D_+(t)D^*_-(t)\rangle. \end{aligned} \tag{3.138}$$

Now we proceed to evaluate the correlation functions in Eq.(3.138). To this end, it is possible to write using Eq. (3.109) and the complex conjugate of (3.110) that

$$\begin{aligned}\langle C_+(t)C^*_-(t)\rangle &= \frac{1}{4\eta^2}\int_0^t\int_0^t\left[(\eta-1)e^{-(A\eta+\kappa)(t-t')/2}+(\eta+1)e^{-\kappa(t-t')/2}\right] \\ &\quad\times\left[(\eta+1)e^{-(A\eta+\kappa)(t-t'')/2}+(\eta-1)e^{-\kappa(t-t'')/2}\right]\langle f^*_\alpha(t')f_\beta(t'')\rangle dt'dt''. \end{aligned} \tag{3.139}$$

In view of (3.70), there follows

$$\langle C_+(t)C^*_-(t)\rangle = 0. \tag{3.140}$$

It is not hard to verify in a similar that

$$\langle D_+(t)D_-^*(t)\rangle = 0, \tag{3.141}$$

$$\langle D_+(t)C_-^*(t)\rangle = \langle C_+(t)D_-^*(t)\rangle = 0. \tag{3.142}$$

Therefore, taking into account of Eqs.(3.140), (3.141), and (3.142), one can get

$$\langle \alpha(t)\beta^*(t)\rangle = 0. \tag{3.143}$$

Making use of similar calculations, it is not difficult to show that

$$\langle \alpha^*(t)\beta(t)\rangle = 0. \tag{3.144}$$

On account of (3.130), (3.131), (3.143), and (3.144), characteristic function described by Eq. (3.119) reduces to

$$\phi_a(z,\eta,t) \quad = \quad \exp\Big[-a_1 z^* z - b_1\eta^*\eta + c_1(z\eta + z^*\eta^*)\Big], \tag{3.145}$$

where

$$a_1 = 1 + \langle \alpha^*(t)\alpha(t)\rangle, \tag{3.146}$$

$$b_1 = 1 + \langle \beta^*(t)\beta(t)\rangle, \tag{3.147}$$

$$c_1 = \langle \alpha(t)\beta(t)\rangle. \tag{3.148}$$

Moreover, upon taking Eq. (3.105) into account, one can write

$$\begin{aligned}\langle \alpha^*(t)\alpha(t)\rangle \quad &= \quad A_+(t)\langle \alpha^*(t)\alpha(0)\rangle + B_+(t)\langle \alpha^*(t)\beta^*(0)\rangle \\ &\quad + \langle \alpha^*(t)C_+(t)\rangle + \langle \alpha^*(t)D_+(t)\rangle.\end{aligned} \tag{3.149}$$

In addition, taking the complex conjugate of Eq. (3.105), it is also possible to see that

$$\begin{aligned}\langle \alpha^*(t)\alpha(0)\rangle \quad &= \quad A_+(t)\langle \alpha^*(0)\alpha(0)\rangle + B_+(t)\langle \alpha(0)\beta(0)\rangle \\ &\quad + \langle \alpha(0)C_+^*(t)\rangle + \langle \alpha(0)D_+^*(t)\rangle.\end{aligned} \tag{3.150}$$

Based on the fact that the noise force at time t does not affect the cavity mode variables at earlier times and taking the cavity modes to be initially in a vacuum state, one can get

$$\langle \alpha^*(t)\alpha(0)\rangle = 0. \tag{3.151}$$

It is also possible to verify that in a similar manner that

$$\langle\alpha^*(t)\beta^*(0)\rangle = 0, \tag{3.152}$$

$$\langle\alpha^*(t)C_+(t)\rangle = \langle C_+^*(t)C_+(t)\rangle + \langle D_+^*(t)C_+(t)\rangle, \tag{3.153}$$

$$\langle\alpha^*(t)D_+(t)\rangle = \langle C_+^*(t)D_+(t)\rangle + \langle D_+^*(t)D_+(t)\rangle. \tag{3.154}$$

Hence upon substituting Eqs.(3.151), (3.152), (3.153), and (3.154) into (3.149), one obtains

$$\begin{aligned}\langle\alpha^*(t)\alpha(t)\rangle &= \langle C_+^*(t)C_+(t)\rangle + \langle D_+^*(t)D_+(t)\rangle \\ &+\langle C_+^*(t)D_+(t)\rangle + \langle D_+^*(t)C_+(t)\rangle.\end{aligned} \tag{3.155}$$

Next the remaining expectation values in Eq. (3.155) would be evaluated. To this end, it is possible to write having Eq. (3.109) that

$$\begin{aligned}\langle C_+^*(t)C_+(t)\rangle &= \frac{1}{4\eta^2}\int_0^t\int_0^t\Big[(\eta-1)e^{-(A\eta+\kappa)(t-t')/2} + (\eta+1)e^{-\kappa(t-t')/2}\Big] \\ &\times\Big[(\eta-1)e^{-(A\eta+\kappa)(t-t'')/2} + (\eta+1)e^{-\kappa(t-t'')/2}\Big]\langle f_\alpha(t'')f_\alpha^*(t')\rangle dt'dt''.\end{aligned} \tag{3.156}$$

From which follows using Eq. (3.68) together with Eq. (3.85), we have

$$\begin{aligned}\langle C_+^*(t)C_+(t)\rangle &= \frac{A(1-\eta)}{\eta^2}\Big[\frac{(\eta+1)^2}{8\kappa}(1-e^{-\kappa t}) + \frac{(\eta-1)^2}{8(A\eta+\kappa)}(1-e^{-(A\eta+\kappa)t}) \\ &+\frac{(\eta^2-1)}{2(A\eta+2\kappa)}(1-e^{-(A\eta+2\kappa)t/2})\Big].\end{aligned} \tag{3.157}$$

It is not difficult to demonstrate in a similar manner that

$$\langle D_+^*(t)D_+(t)\rangle = 0, \tag{3.158}$$

$$\begin{aligned}&\langle C_+^*(t)D_+(t)\rangle = \langle D_+^*(t)C_+(t)\rangle = -\frac{A(1-\eta^2)}{\eta^2}\Big[\frac{(\eta+1)}{16\kappa}(1-e^{-\kappa t}) \\ &-\frac{(\eta-1)}{16(A\eta+\kappa)}(1-e^{-(A\eta+\kappa)t}) - \frac{1}{4(A\eta+2\kappa)}(1-e^{-(A\eta+2\kappa)t/2})\Big].\end{aligned} \tag{3.159}$$

Therefore, in view of Eqs. (3.157), (3.158), and (3.159), one can readily establish

$$\begin{aligned}\langle\alpha^*(t)\alpha(t)\rangle &= -\frac{A(1-\eta)^2}{4\eta(A\eta+\kappa)}(1-e^{-(A\eta+\kappa)t}) \\ &+\frac{A(1-\eta^2)}{2\eta(A\eta+2\kappa)}(1-e^{-(A\eta+2\kappa)t/2}).\end{aligned} \tag{3.160}$$

Furthermore, upon taking Eq. (3.106) into account

$$\begin{aligned}\langle\beta^*(t)\beta(t)\rangle &= A_-(t)\langle\beta^*(t)\beta(0)\rangle + B_-(t)\langle\beta^*(t)\alpha^*(0)\rangle \\ &+\langle\beta^*(t)C_-(t)\rangle + \langle\beta^*(t)D_-(t)\rangle.\end{aligned} \tag{3.161}$$

In connection, using the complex conjugate of Eq. (3.106), we have

$$\begin{aligned}\langle\beta^*(t)\beta(0)\rangle &= A_-(t)\langle\beta^*(0)\beta(0)\rangle + B_-(t)\langle\alpha(0)\beta(0)\rangle \\ &+\langle\beta(0)C_-(t)\rangle + \langle\beta(0)D_-(t)\rangle. \end{aligned} \tag{3.162}$$

In view of the fact that the noise force at time t does not affect the cavity mode variables at earlier times and taking the cavity modes to be initially in a vacuum state, one can get

$$\langle\beta^*(t)\beta(0)\rangle = 0. \tag{3.163}$$

Following the same procedure, one can see that

$$\langle\beta^*(t)\alpha^*(0)\rangle = 0, \tag{3.164}$$

$$\langle\beta^*(t)C_-(t)\rangle = \langle C_-^*(t)C_-(t)\rangle + \langle D_-^*(t)C_-(t)\rangle, \tag{3.165}$$

$$\langle\beta^*(t)D_-(t)\rangle = \langle C_-^*(t)D_-(t)\rangle + \langle D_-^*(t)D_-(t)\rangle. \tag{3.166}$$

Upon substituting Eqs. (3.163), (3.164), (3.165), and (3.166), into Eq. (3.161), one obtains

$$\begin{aligned}\langle\beta^*(t)\beta(t)\rangle &= \langle C_-^*(t)C_-(t)\rangle + \langle D_-^*(t)C_-(t)\rangle \\ &+\langle C_-^*(t)D_-(t)\rangle + \langle D_-^*(t)D_-(t)\rangle. \end{aligned} \tag{3.167}$$

Next we proceed to evaluate the remaining correlation functions in Eq. (3.167). To this end, it is possible to write having Eq. (3.112) that

$$\begin{aligned}\langle D_-^*(t)D_-(t)\rangle &= \frac{(1-\eta^2)}{4\eta^2}\int_0^t\int_0^t\Big[e^{-(A\eta+\kappa)(t-t')/2} - e^{-\kappa(t-t')/2}\Big] \\ &\times\Big[e^{-(A\eta+\kappa)(t-t'')/2} - e^{-\kappa(t-t'')/2}\Big]\langle f_\alpha(t')f_\alpha^*(t'')\rangle dt'dt'', \end{aligned} \tag{3.168}$$

from which follows using Eq. (3.68), we obtain

$$\begin{aligned}\langle D_-^*(t)D_-(t)\rangle &= \frac{A(1-\eta^2)(1-\eta)}{\eta^2}\Big[\frac{1}{8\kappa}(1-e^{-\kappa t}) + \frac{1}{8(A\eta+\kappa)}(1-e^{-(A\eta+\kappa)t}) \\ &-\frac{1}{2(A\eta+2\kappa)}(1-e^{-(A\eta+2\kappa)t/2})\Big]. \end{aligned} \tag{3.169}$$

It is not difficult to demonstrate in a similar manner that

$$\langle C_-^*(t)C_-(t)\rangle = 0, \tag{3.170}$$

$$\begin{aligned}\langle C_-^*(t)D_-(t)\rangle = \langle D_-^*(t)C_-(t)\rangle &= \frac{A(1-\eta^2)}{\eta^2}\Big[\frac{(\eta-1)}{16\kappa}(1-e^{-\kappa t}) \\ -\frac{(\eta+1)}{16(A\eta+\kappa)}(1-e^{-(A\eta+\kappa)t}) &+ \frac{1}{4(A\eta+2\kappa)}(1-e^{-(A\eta+2\kappa)t/2})\Big]. \end{aligned} \tag{3.171}$$

Therefore, in view of Eqs. (3.169), (3.170), and (3.171), one can readily establish

$$\begin{aligned}\langle\beta^*(t)\beta(t)\rangle &= -\frac{A(1-\eta^2)}{4\eta(A\eta+\kappa)}(1-e^{-(A\eta+\kappa)t})\\ &+\frac{A(1-\eta^2)}{2\eta(A\eta+2\kappa)}(1-e^{-(A\eta+2\kappa)t/2}).\end{aligned}\tag{3.172}$$

Making use of a similar calculations, we have

$$\begin{aligned}\langle\alpha(t)\beta(t)\rangle &= \frac{A(\eta-1)\sqrt{1-\eta^2}}{4\eta(A\eta+\kappa)}(1-e^{-(A\eta+\kappa)t})\\ &+\frac{A\sqrt{1-\eta^2}}{2\eta(A\eta+2\kappa)}(1-e^{-(A\eta+2\kappa)t/2}).\end{aligned}\tag{3.173}$$

In view of (3.160), (3.172), and (3.173) along with replacing A and κ by A_1 and κ_1 in these equations, one can put Eqs. (3.146), (3.147), and (3.148) in the form

$$\begin{aligned}a_1 &= 1+\frac{A_1(1-\eta^2)}{2\eta(A_1\eta+2\kappa_1)}(1-e^{-(A_1\eta+2\kappa_1)t/2})\\ &-\frac{A_1(1-\eta)^2}{4\eta(A_1\eta+\kappa_1)}(1-e^{-(A_1\eta+\kappa_1)t}),\end{aligned}\tag{3.174}$$

$$\begin{aligned}b_1 &= 1+\frac{A_1(1-\eta^2)}{2\eta(A_1\eta+2\kappa_1)}(1-e^{-(A_1\eta+2\kappa_1)t/2})\\ &-\frac{A_1(1-\eta^2)}{4\eta(A_1\eta+\kappa_1)}(1-e^{-(A_1\eta+\kappa_1)t}),\end{aligned}\tag{3.175}$$

$$\begin{aligned}c_1 &= \frac{A_1(\eta-1)\sqrt{1-\eta^2}}{4\eta(A_1\eta+\kappa_1)}(1-e^{-(A_1\eta+\kappa_1)t})\\ &+\frac{A_1\sqrt{1-\eta^2}}{2\eta(A_1\eta+2\kappa_1)}(1-e^{-(A_1\eta+2\kappa_1)t/2}).\end{aligned}\tag{3.176}$$

With the aid of Eq. (3.145), the Q function for the two-mode light described by the Eq. (3.113) can be expressible as

$$\begin{aligned}Q_1(\alpha_1,\beta_1,t) &= \frac{1}{\pi^2}\int\frac{d^2z}{\pi}\exp\big[-a_1z^*z+z^*\alpha_1-z\alpha_1^*\big]\\ &\times\int\frac{d^2\eta}{\pi}\exp\big(-b_1\eta^*\eta+(c_1z-\beta_1^*)\eta+(c_1z^*+\beta_1)\eta^*\big).\end{aligned}\tag{3.177}$$

Upon employing (2.97), the integration over η can be evaluated to give

$$\begin{aligned}Q_1(\alpha_1,\beta_1,t) &= \frac{e^{-\beta^*\beta/b_1}}{b_1\pi^2}\int\frac{d^2z}{\pi}\exp\Big[-(a_1-c_1^2/b_1)z^*z+(c_1\beta_1/b_1-\alpha_1^*)z\\ &+(\alpha_1-c_1\beta_1^*/b_1)z^*\Big]\end{aligned}\tag{3.178}$$

and performing the integration over z, we obtain

$$Q_1(\alpha_1,\beta_1,t) = \frac{u_1v_1-r_1^2}{\pi^2}\exp\left[-v_1\alpha_1^*\alpha_1-u_1\beta_1^*\beta_1+r_1(\alpha_1\beta_1+\alpha_1^*\beta_1^*)\right],\tag{3.179}$$

where

$$u_1 = \frac{a_1}{a_1b_1 - c_1^2}, \tag{3.180}$$

$$v_1 = \frac{b_1}{a_1b_1 - c_1^2}, \tag{3.181}$$

$$r_1 = \frac{c_1}{a_1b_1 - c_1^2}. \tag{3.182}$$

Next we proceed to obtain the two-mode output laser light Q function satisfying the input-output relations. To this end, according to the input-output relations for a two-mode light beam, we have

$$\hat{a}^{out}(t) = \sqrt{\kappa}\hat{a}(t) - \hat{a}_{in}(t) \tag{3.183}$$

$$\hat{b}^{out}(t) = \sqrt{\kappa}\hat{b}(t) - \hat{b}_{in}(t). \tag{3.184}$$

For a cavity mode coupled to a two-mode vacuum reservoir and with the input mode operators corresponding to vacuum reservoirs put in normal ordering, one can then write

$$\hat{a}_1^{out}(t) = \sqrt{\kappa_1}\hat{a}_1(t) \tag{3.185}$$

$$\hat{b}_1^{out}(t) = \sqrt{\kappa_1}\hat{b}_1(t). \tag{3.186}$$

The c-number equations corresponding to Eqs. (3.185) and (3.186) associated with the normal ordering is

$$\alpha_1^{out}(t) = \sqrt{\kappa_1}\alpha_1(t) \tag{3.187}$$

$$\beta_1^{out}(t) = \sqrt{\kappa_1}\beta_1(t). \tag{3.188}$$

On account of this, replacing $\alpha_1(t)$ and $\beta_1(t)$ by $\alpha_1(t)/\sqrt{\kappa_1}$ and $\beta_1(t)/\sqrt{\kappa_1}$, the normalized Q function for output light beam is expressible as

$$Q_1(\alpha_1, \beta_1, t) = \frac{u_1v_1 - r_1^2}{\kappa_1^2\pi^2} \exp\left[-\frac{v_1}{\kappa_1}\alpha_1^*\alpha_1 - \frac{u_1}{\kappa_1}\beta_1^*\beta_1 + \frac{r_1}{\kappa_1}(\alpha_1\beta_1 + \alpha_1^*\beta_1^*)\right] \tag{3.189}$$

This can be put in the form

$$Q_i(\alpha_i, \beta_i, t) = \frac{u_iv_i - r_i^2}{\kappa_i^2\pi^2} \exp\left[-\frac{v_i}{\kappa_i}\alpha_i^*\alpha_i - \frac{u_i}{\kappa_i}\beta_i^*\beta_i + \frac{r_i}{\kappa_i}(\alpha_i\beta_i + \alpha_i^*\beta_i^*)\right], \quad i = 1, 2. \tag{3.190}$$

We note that the normalized Q function for the output mode a is expressible as

$$Q_i(\alpha_i^*, \alpha_i, t) = \int d^2\beta_i Q_i(\alpha_i, \beta_i, t). \tag{3.191}$$

Taking into account of (3.190), this can be put in the form

$$Q_i(\alpha_i^*, \alpha_i, t) = \frac{u_i v_i - r_i^2}{\kappa_i^2 \pi} \exp\Big[-\frac{v_i}{\kappa_i}\alpha_i^*\alpha_i\Big] \int \frac{d^2\beta_i}{\pi} \exp\Big[-\frac{u_i}{\kappa_i}\beta_i^*\beta_i + \frac{r_i}{\kappa_i}\big(\alpha_i\beta_i + \alpha_i^*\beta_i^*\big)\Big] \tag{3.192}$$

so that on carrying out the integration over β_i, we obtain

$$Q_i(\alpha_i^*, \alpha_i, t) = \frac{u_i v_i - r_i^2}{\kappa_i u_i \pi} \exp\bigg[-\bigg(\frac{u_i v_i - r_i^2}{\kappa_i u_i}\bigg)\alpha_i^*\alpha_i\bigg]. \tag{3.193}$$

It can also verified in a similar manner that

$$Q_i(\beta_i^*, \beta_i, t) = \frac{u_i v_i - r_i^2}{\kappa_i v_i \pi} \exp\bigg[-\bigg(\frac{u_i v_i - r_i^2}{\kappa_i v_i}\bigg)\beta_i^*\beta_i\bigg]. \tag{3.194}$$

This represents the output Q function for mode b.

3.3 Quadrature squeezing

In this section, the squeezing properties of the two-mode output light produced by the superposition of two-mode three-level cascade laser is analyzed.

In order to obtain the quadrature squeezing of the output light beams produced by the superposition of two-mode light beams produced by the non degenerate three-level lasers, we first derive the density operator for the superposed two-mode laser light beams.

Consider $\hat{\rho}'(t) = \hat{\rho}'(\hat{a}^\dagger, \hat{b}^\dagger, \hat{a}, \hat{b})$ is the density operator for the first two-mode light beam. Then upon expanding this density operator in the normal order and applying the completeness relation for two-mode coherent states,

$$\hat{I} = \int \frac{d^2\eta_a}{\pi}\frac{d^2 z_b}{\pi}|\eta_a, z_b\rangle\langle z_b, \eta_a| \tag{3.195}$$

we have

$$\hat{\rho}'(t) = \int \frac{d^2\eta_a}{\pi}\frac{d^2 z_b}{\pi}\sum_{klmn} C_{klmn}|\eta_a, z_b\rangle\langle z_b, \eta_a|\hat{a}^{\dagger k}\hat{b}^{\dagger l}\hat{a}^m\hat{b}^n. \tag{3.196}$$

From which follows

$$\hat{\rho}'(t) \;=\; \int \frac{d^2\eta_a}{\pi}\frac{d^2 z_b}{\pi}\sum_{klmn} C_{klmn}\eta_a^{*k} z_b^{*l}|\eta_a, z_b\rangle\langle z_b, \eta_a|\hat{a}^m\hat{b}^n, \tag{3.197}$$

in view of the fact that

$$|\eta_a, z_b\rangle\langle z_b, \eta_a|\hat{a}^m\hat{b}^n = \left(\eta_a + \frac{\partial}{\partial\eta_a^*}\right)^m\left(z_b + \frac{\partial}{\partial z_b^*}\right)^n|\eta_a, z_b\rangle\langle z_b, \eta_a|, \tag{3.198}$$

one can put Eq. (3.197) in the form

$$\hat{\rho}'(t) \;=\; \int d^2\eta_a d^2 z_b Q_1\Big(\eta_a^*, z_b^*, \eta_a + \frac{\partial}{\partial\eta_a^*}, z_b + \frac{\partial}{\partial z_b^*}, t\Big)|\eta_a, z_b\rangle\langle z_b, \eta_a|. \tag{3.199}$$

Again taking into account of the relation

$$|\eta_a, z_b\rangle\langle z_b, \eta_a| = \hat{D}(\eta_a)\hat{D}(z_b)|0_a, 0_b\rangle\langle 0_b, 0_a|\hat{D}(-z_b)\hat{D}(-\eta_a), \tag{3.200}$$

the density operator described by Eq.(3.199) can be rewritten as

$$\begin{aligned}\hat{\rho}'(t) \;=\; & \int d^2\eta_a d^2 z_b Q_1\Big(\eta_a^*, z_b^*, \eta_a + \frac{\partial}{\partial\eta_a^*}, z_b + \frac{\partial}{\partial z_b^*}, t\Big)\\ & \times\hat{D}(\eta_a)\hat{D}(z_b)\hat{\rho}_{0_a,0_b}\hat{D}(-z_b)\hat{D}(-\eta_a),\end{aligned} \tag{3.201}$$

where

$$\hat{\rho}_{0_a,0_b} = |0_a, 0_b\rangle\langle 0_b, 0_a|. \tag{3.202}$$

Following the same procedure the density operator for another two-mode light can be written as

$$\hat{\rho}(t) \;=\; \int d^2\lambda_a d^2\chi_b Q_2\Big(\lambda_a^*, \chi_b^*, \lambda_a + \frac{\partial}{\partial\lambda_a^*}, \chi_b + \frac{\partial}{\partial\chi_b^*}, t\Big)|\lambda_a, \chi_b\rangle\langle\chi_b, \lambda_a|. \tag{3.203}$$

We now realize that the density operator for the superposition of the the first two-mode light beams and another is expressible as

$$\begin{aligned}\hat{\rho}(t) \;=\; & \int d^2\eta_a d^2 z_b d^2\lambda_a d^2\chi_b Q_1\Big(\eta_a^*, z_b^*, \eta_a + \frac{\partial}{\partial\eta_a^*}, z_b + \frac{\partial}{\partial z_b^*}, t\Big)\\ & \times Q_2\Big(\lambda_a^*, \chi_b^*, \lambda_a + \frac{\partial}{\partial\lambda_a^*}, \chi_b + \frac{\partial}{\partial\chi_b^*}, t\Big)\\ & \times\big|\eta_a + \lambda_a, z_b + \chi_b\big\rangle\big\langle z_b + \chi_b, \eta_a + \lambda_a\big|.\end{aligned} \tag{3.204}$$

We now proceed to obtain the density operators for a pair of superposed output light mode a and b in terms of arbitrary Q functions. We note that the density operators for the output light mode a and b is expressible as

$$\hat{\rho}_m(t) = Tr_l(\hat{\rho}(t)), \;\; m \neq l, \tag{3.205}$$

with $m, l = a, b$. Thus on account of Eq. (3.204), we find the density operator for a pair of superposed output light mode a and b to be

$$\begin{aligned}\hat{\rho}_a(t) \;=\; & \int d^2\eta_a d^2 z_b d^2\lambda_a d^2\chi_b Q_1\Big(\eta_a^*, z_b^*, \eta_a + \frac{\partial}{\partial\eta_a^*}, z_b, t\Big)\\ & \times Q_2\Big(\lambda_a^*, \chi_b^*, \lambda_a + \frac{\partial}{\partial\lambda_a^*}, \chi_b, t\Big)\big|\eta_a + \lambda_a\big\rangle\big\langle\lambda_a + \eta_a\big|.\end{aligned} \tag{3.206}$$

$$\begin{aligned}\hat{\rho}_b(t) \;=\; & \int d^2\eta_a d^2 z_b d^2\lambda_a d^2\chi_b Q_1\Big(\eta_a^*, z_b^*, \eta_a, z_b + \frac{\partial}{\partial z_b^*}, t\Big)\\ & \times Q_2\Big(\lambda_a^*, \chi_b^*, \lambda_a, \chi_b + \frac{\partial}{\partial\chi_b^*}, t\Big)\big|z_b + \chi_b\big\rangle\big\langle\chi_b + z_b\big|.\end{aligned} \tag{3.207}$$

We note that the Q function described by Eq. (3.190) can be rewritten with $(\alpha_1, \alpha_1^*, \beta_1, \beta_1^*)$ and $(\alpha_2, \alpha_2^*, \beta_2, \beta_2^*)$ replaced by $(\eta_a, \eta_a^*, z_b, z_b^*)$ and $(\lambda_a, \lambda_a^*, \chi_b, \chi_b^*)$ respectively, as

$$Q_1(\eta_a, z_b, t) \;=\; \frac{u_1 v_1 - r_1^2}{\kappa_1^2\pi^2}\exp\big[-\frac{v_1}{\kappa_1}\eta_a^*\eta_a - \frac{u_1}{\kappa_1}z_b^* z_b + \frac{r_1}{\kappa_1}(\eta_a z_b + \eta_a^* z_b^*)\big] \tag{3.208}$$

$$Q_2(\lambda_a, \chi_b, t) \;=\; \frac{u_2 v_2 - r_2^2}{\kappa_2^2\pi^2}\exp\big[-\frac{v_2}{\kappa_2}\lambda_a^*\lambda_a - \frac{u_2}{\kappa_2}\chi_b^*\chi_b + \frac{r_2}{\kappa_2}(\lambda_a\chi_b + \lambda_a^*\chi_b^*)\big]. \tag{3.209}$$

Moreover, applying (3.208) and (3.209) in Eqs. (3.206) and carrying out the integration over z_b and χ_b, we readily establish that

$$\begin{aligned}\hat{\rho}_a(t) \;=\; & \int d^2\eta_a d^2\lambda_a Q_{1a}\Big(\eta_a^*, \eta_a + \frac{\partial}{\partial\eta_a^*}, t\Big)Q_{2a}\Big(\lambda_a^*, \lambda_a + \frac{\partial}{\partial\lambda_a^*}, t\Big)\\ & \times\big|\eta_a + \lambda_a\big\rangle\langle\eta_a + \lambda_a\big|,\end{aligned} \tag{3.210}$$

where

$$Q_{1a}(\eta_a^*, \eta_a + \frac{\partial}{\partial \eta_a^*}, t) = Q_{1a}(\eta_a^*, \eta_a, t) \exp\Big[-\Big(\frac{u_1 v_1 - r_1^2}{u_1}\Big)\eta_a^* \frac{\partial}{\partial \eta_a^*}\Big] \tag{3.211}$$

$$Q_{2a}(\lambda_a^*, \lambda_a + \frac{\partial}{\partial \lambda_a^*}, t) = Q_{2a}(\lambda_a^*, \lambda_a, t) \exp\Big[-\Big(\frac{u_2 v_2 - r_2^2}{u_2}\Big)\lambda_a^* \frac{\partial}{\partial \lambda_a^*}\Big], \tag{3.212}$$

with

$$Q_{1a}(\eta_a^*, \eta_a, t) = \frac{u_1 v_1 - r_1^2}{\kappa_1 u_1 \pi} \exp\Big[-\Big(\frac{u_1 v_1 - r_1^2}{\kappa_1 u_1}\Big)\eta_a^* \eta_a\Big] \tag{3.213}$$

$$Q_{2a}(\lambda_a^*, \lambda_a, t) = \frac{u_2 v_2 - r_2^2}{\kappa_2 u_2 \pi} \exp\Big[-\Big(\frac{u_2 v_2 - r_2^2}{\kappa_2 u_2}\Big)\lambda_a^* \lambda_a\Big] \tag{3.214}$$

are the Q functions for the output laser light beams of mode a.

It can also verified in a similar manner that

$$\hat{\rho}_b(t) = \int d^2 z_b d^2 \chi_b Q_{1b}\Big(z_b^*, z_b + \frac{\partial}{\partial z_b^*}, t\Big) Q_{2b}\Big(\chi_b^*, \chi_b + \frac{\partial}{\partial \chi_b^*}, t\Big) \times \big|z_b + \chi_b\big\rangle\langle z_b + \chi_b\big|. \tag{3.215}$$

where

$$Q_{1b}(z_b^*, z_b + \frac{\partial}{\partial z_b^*}, t) = Q_{1b}(z_b^*, z_b, t) \exp\Big[-\Big(\frac{u_1 v_1 - r_1^2}{v_1}\Big) z_b^* \frac{\partial}{\partial z_b^*}\Big] \tag{3.216}$$

$$Q_{2b}(\chi_a^*, \chi_a + \frac{\partial}{\partial \chi_b^*}, t) = Q_{2b}(\chi_b^*, \chi_b, t) \exp\Big[-\Big(\frac{u_2 v_2 - r_2^2}{v_2}\Big)\chi_b^* \frac{\partial}{\partial \chi_b^*}\Big], \tag{3.217}$$

with

$$Q_{1b}(z_b^*, z_b, t) = \frac{u_1 v_1 - r_1^2}{\kappa_1 v_1 \pi} \exp\Big[-\Big(\frac{u_1 v_1 - r_1^2}{\kappa_1 v_1}\Big) z_b^* z_b\Big] \tag{3.218}$$

$$Q_{2b}(\chi_b^*, \chi_b, t) = \frac{u_2 v_2 - r_2^2}{\kappa_2 v_2 \pi} \exp\Big[-\Big(\frac{u_2 v_2 - r_2^2}{\kappa_2 v_2}\Big)\chi_b^* \chi_b\Big] \tag{3.219}$$

are the Q functions for the output laser light beams of mode b.

It is worth mentioning that Eqs. (3.210) and (3.215) represents the density operator for a pair of superposed laser output light mode a and b. To this end, the quadrature operators for a pair of superposed two-mode output light beams are defined by

$$\hat{c}_{\pm} = \sqrt{\pm 1}(\hat{c}^\dagger \pm \hat{c}), \tag{3.220}$$

where

$$\hat{c} = \hat{a} + \hat{b}, \tag{3.221}$$

with

$$\hat{a} = \hat{a}_1 + \hat{a}_2 \tag{3.222a}$$

$$\hat{b} = \hat{b}_1 + \hat{b}_2. \tag{3.222b}$$

Using (3.221) and (3.222), one easily gets

$$[\hat{c}, \hat{c}^{\dagger}] = 2\kappa_1 + 2\kappa_2, \tag{3.223}$$

with

$$[\hat{a}, \hat{a}^{\dagger}] = [\hat{b}, \hat{b}^{\dagger}] = \kappa_1 + \kappa_2. \tag{3.224}$$

Applying Eq. (3.221) along with the commutation relation (3.223), the variance of quadrature of a pair of superposed two-mode laser light beams is expressible as

$$\Delta c_{\pm}^2 = 2\kappa_1 + 2\kappa_2 + :\Delta c_{\pm}^2:, \tag{3.225}$$

where

$$:\Delta c_{\pm}^2: \;=\; 2\langle \hat{c}^{\dagger}, \hat{c}\rangle \pm \left(\langle \hat{c}, \hat{c}\rangle + \langle \hat{c}^{\dagger}, \hat{c}^{\dagger}\rangle\right) \tag{3.226}$$

is the normally-ordered quadrature variance. Taking into account of (3.221), we see that

$$\begin{aligned} :\Delta c_{\pm}^2: \;=\; & 2\langle \hat{a}^{\dagger}(t), \hat{a}(t)\rangle + 2\langle \hat{b}^{\dagger}(t), \hat{b}(t)\rangle + 2\langle \hat{a}^{\dagger}(t), \hat{b}(t)\rangle + 2\langle \hat{b}^{\dagger}(t), \hat{a}(t)\rangle \\ & \pm\big(\langle \hat{a}(t), \hat{a}(t)\rangle + \langle \hat{b}(t), \hat{b}(t)\rangle + 2\langle \hat{a}(t), \hat{b}(t)\rangle + \langle \hat{a}^{\dagger}(t), \hat{a}^{\dagger}(t)\rangle \\ & + \langle \hat{b}^{\dagger}(t), \hat{b}^{\dagger}(t)\rangle + 2\langle \hat{a}^{\dagger}(t), \hat{b}^{\dagger}(t)\rangle\big). \end{aligned} \tag{3.227}$$

Making use of (3.210), one can write the expectation value of the operator $\hat{a}(t)$ as

$$\langle \hat{a}(t)\rangle \;=\; \int d^2\eta_a d^2\lambda_a Q_{1a}\Big(\eta_a^*, \eta_a + \frac{\partial}{\partial \eta_a^*}, t\Big) Q_{2a}\Big(\lambda_a^*, \lambda_a + \frac{\partial}{\partial \lambda_a^*}, t\Big)(\eta_a + \lambda_a). \tag{3.228}$$

This can be put in the form

$$\langle \hat{a}(t)\rangle \;=\; \langle \hat{a}_1(t)\rangle + \langle \hat{a}_2(t)\rangle. \tag{3.229}$$

where

$$\langle \hat{a}_1(t)\rangle \;=\; \int d^2\eta_a Q_{1a}\Big(\eta_a^*, \eta_a + \frac{\partial}{\partial \eta_a^*}, t\Big)\eta \tag{3.230a}$$

$$\langle \hat{a}_2(t)\rangle \;=\; \int d^2\lambda_a Q_{2a}\Big(\lambda_a^*, \lambda_a + \frac{\partial}{\partial \lambda_a^*}, t\Big)\lambda. \tag{3.230b}$$

Applying (3.211) in Eq. (3.230a), we have

$$\langle \hat{a}_1(t)\rangle \;=\; \int d^2\eta_a Q_{1a}(\eta_a^*, \eta_a, t) \exp\Big[-m_1\eta_a^*\frac{\partial}{\partial \eta_a^*}\Big]\eta_a, \tag{3.231}$$

in which

$$m_1 \;=\; \frac{u_1 v_1 - r_1^2}{u_1}. \tag{3.232}$$

Since η_a and η_a^* are independent variables, we see that

$$\langle \hat{a}_1(t)\rangle = 0. \tag{3.233}$$

On account of the fact that the Q functions expressed by Eqs. (3.211) and (3.212) have exactly the same form, we also observe that

$$\langle \hat{a}_i(t)\rangle = 0, \;\; i = 1, 2. \tag{3.234}$$

In view of this result, we readily get

$$\langle \hat{a}(t)\rangle = 0. \tag{3.235}$$

Following the same procedure, one can also show that

$$\langle \hat{b}(t)\rangle \;\; = \;\; 0 \tag{3.236}$$

$$\langle \hat{a}^2(t)\rangle = \langle \hat{b}^2(t)\rangle \;\; = \;\; 0 \tag{3.237}$$

$$\langle \hat{a}^\dagger(t)\hat{b}(t)\rangle \;\; = \;\; 0. \tag{3.238}$$

Furthermore, on account of Eqs. (3.235-3.238), and their complex conjugate, one can write Eq. (3.227) as

$$: \Delta c_\pm^2 : \;\; = \;\; 2\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle + 2\langle \hat{b}^\dagger(t)\hat{b}(t)\rangle \pm 2\big(\langle \hat{a}(t)\hat{b}(t)\rangle + \langle \hat{a}^\dagger(t)\hat{b}^\dagger(t)\rangle\big). \tag{3.239}$$

Next we seek to obtain the expectation values involved in this expression. To this end, the expectation value of the operator $\hat{a}^\dagger(t)\hat{a}(t)$ can be expressible in terms of the Q functions as

$$\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle \;\; = \;\; \int d^2\eta_a d^2\lambda_a Q_{1a}\Big(\eta_a^*, \eta_a + \frac{\partial}{\partial \eta_a^*}, t\Big) Q_{2a}\Big(\lambda_a^*, \lambda_a + \frac{\partial}{\partial \lambda_a^*}, t\Big)$$
$$(\eta_a^*\eta_a + \eta_a^*\lambda_a + \lambda_a^*\lambda_a + \lambda_a^*\eta_a). \tag{3.240}$$

In view of Eqs. (3.230) and (3.234), this can also be rewritten in the form

$$\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle \;\; = \;\; \langle \hat{a}_1^\dagger(t)\hat{a}_1(t)\rangle + \langle \hat{a}_2^\dagger(t)\hat{a}_2(t)\rangle, \tag{3.241}$$

where

$$\langle \hat{a}_1^\dagger(t)\hat{a}_1(t)\rangle \;\; = \;\; \int d^2\eta_a Q_{1a}\Big(\eta_a^*, \eta_a + \frac{\partial}{\partial \eta_a^*}, t\Big)\eta_a^*\eta_a \tag{3.242a}$$

$$\langle \hat{a}_2^\dagger(t)\hat{a}_2(t)\rangle \;\; = \;\; \int d^2\lambda_a Q_{2a}\Big(\lambda_a^*, \lambda_a + \frac{\partial}{\partial \lambda_a^*}, t\Big)\lambda_a^*\lambda_a. \tag{3.242b}$$

Now applying (3.211) in Eq. (3.242a), we readily establish that

$$\langle \hat{a}_1^\dagger(t)\hat{a}_1(t)\rangle \;\; = \;\; \int d^2\eta_a \frac{\partial}{\partial \lambda}\Big[Q_{1a}(\eta_a^*, \eta_a, t)\exp\big(-m_1\eta_a^*\frac{\partial}{\partial \eta_a^*}\big)\exp\left(\lambda\eta_a^*\eta_a\right)\Big]\Big|_{\lambda=0}. \tag{3.243}$$

This can also be rewritten as

$$\langle\hat{a}_1^\dagger(t)\hat{a}_1(t)\rangle = \int d^2\eta_a \frac{\partial}{\partial\lambda}\Big[Q_{1a}(\eta_a^*,\eta_a,t)\exp\big[-(m_1-1)\lambda\eta_a^*\eta_a\big]\Big]\Big|_{\lambda=0}. \tag{3.244}$$

Furthermore, upon performing differentiation and applying the condition $\lambda = 0$, leads to

$$\langle\hat{a}_1^\dagger(t)\hat{a}_1(t)\rangle = (1-m_1)\int d^2\eta_a Q_{1a}(\eta_a^*,\eta_a,t)\eta_a^*\eta_a. \tag{3.245}$$

On the other hand, employing the Q function (3.213) in Eq. (3.245), we have

$$\langle\hat{a}_1^\dagger(t)\hat{a}_1(t)\rangle = \frac{(1-m_1)m_1}{\kappa_1}\int \frac{d^2\eta_a}{\pi}\exp\big[-\frac{m_1}{\kappa_1}\eta_a^*\eta_a\big]\eta_a^*\eta_a. \tag{3.246}$$

There follows

$$\langle\hat{a}_1^\dagger(t)\hat{a}_1(t)\rangle = -(1-m_1)m_1\frac{\partial}{\partial m_1}\int \frac{d^2\eta_a}{\pi}\exp\big[-\frac{m_1}{\kappa_1}\eta_a^*\eta_a\big], \tag{3.247}$$

so that on carrying out the integration, yields

$$\langle\hat{a}_1^\dagger(t)\hat{a}_1(t)\rangle = -\kappa_1(1-m_1)m_1\frac{\partial}{\partial m_1}\bigg(\frac{1}{m_1}\bigg). \tag{3.248}$$

From which, we obtain

$$\langle\hat{a}_1^\dagger(t)\hat{a}_1(t)\rangle = \frac{\kappa_1(1-m_1)}{m_1} \tag{3.249}$$

$$= \kappa_1\bigg(\frac{u_1}{u_1v_1-r_1^2}-1\bigg). \tag{3.250}$$

With the aid of (3.180), (3.181), and (3.182), one can readily find

$$\langle\hat{a}_1^\dagger(t)\hat{a}_1(t)\rangle = \kappa_1(a_1-1) \tag{3.251}$$

Moreover, in view of the fact that the Q functions for the laser light beam have the same form, we observe that

$$\langle\hat{a}_2^\dagger(t)\hat{a}_2(t)\rangle = \kappa_2(a_2-1). \tag{3.252}$$

Making use of Eqs.(3.251) and (3.252), one can find

$$\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle = \kappa_1(a_1-1)+\kappa_2(a_2-1). \tag{3.253}$$

Following similar procedure, it is not difficult to verify that

$$\langle\hat{b}^\dagger(t)\hat{b}(t)\rangle = \kappa_1(b_1-1)+\kappa_2(b_2-1). \tag{3.254}$$

Again we proceed to evaluate the remaining correlation functions in Eq.(3.239). To this end, it is possible to write having Eq.(3.204) that

$$\begin{aligned}\langle\hat{a}^\dagger(t)\hat{b}^\dagger(t)\rangle &= \int d^2\eta_a d^2z_b d^2\lambda_a d^2\chi_b Q_1\Big(\eta_a^*, z_b^*, \eta_a+\frac{\partial}{\partial\eta_a^*}, z_b+\frac{\partial}{\partial z_b^*}, t\Big)\\ &\times Q_2\Big(\lambda_a^*, \chi_b^*, \lambda_a+\frac{\partial}{\partial\lambda_a^*}, \chi_b+\frac{\partial}{\partial\chi_b^*}, t\Big)\\ &\times[\eta_a^* z_b^* + \eta_a^*\chi_b^* + \lambda_a^*\chi_b^* + \lambda_a^* z_b^*]. \end{aligned} \tag{3.255}$$

This can also be expressible as

$$\langle\hat{a}^\dagger(t)\hat{b}^\dagger(t)\rangle = \langle\hat{a}_1^\dagger(t)\hat{b}_1^\dagger(t)\rangle + \langle\hat{a}_1^\dagger(t)\rangle\langle\hat{b}_2^\dagger(t)\rangle + \langle\hat{a}_2^\dagger(t)\hat{b}_2^\dagger(t)\rangle + \langle\hat{a}_2^\dagger(t)\rangle\langle\hat{b}_1^\dagger(t)\rangle \tag{3.256}$$

On account of (3.234) and (3.236), one can write

$$\langle\hat{a}^\dagger(t)\hat{b}^\dagger(t)\rangle = \langle\hat{a}_1^\dagger(t)\hat{b}_1^\dagger(t)\rangle + \langle\hat{a}_2^\dagger(t)\hat{b}_2^\dagger(t)\rangle. \tag{3.257}$$

Furthermore, the expectation value of the operator represented by $\hat{a}_1^\dagger(t)\hat{b}_1^\dagger(t)$ can be expressed as

$$\langle\hat{a}_1^\dagger(t)\hat{b}_1^\dagger(t)\rangle = Tr\Big(\hat{\rho}'(t)\hat{a}_1^\dagger\hat{b}_1^\dagger\Big). \tag{3.258}$$

Applying (3.199), one can get

$$\langle\hat{a}_1^\dagger(t)\hat{b}_1^\dagger(t)\rangle = \int d^2\eta_a d^2z_b Q_1\Big(\eta_a^*, z_b^*, \eta_a+\frac{\partial}{\partial\eta_a^*}, z_b+\frac{\partial}{\partial z_b^*}, t\Big)\eta_a^* z_b^*. \tag{3.259}$$

We note that the Q function given by Eq. (3.208) is expressible as

$$\begin{aligned}Q_1\Big(\eta_a^*, z_b^*, \eta_a+\frac{\partial}{\partial\eta_a^*}, z_b+\frac{\partial}{\partial z_b^*}, t\Big) &= Q_1(\eta_a, z_b, t)\exp\Big[-\big(v_1\eta_a^* - r_1 z_b\big)\frac{\partial}{\partial\eta_a^*}\\ &-\big(u_1 z_b^* - r_1\eta_a\big)\frac{\partial}{\partial z_b^*} + r_1\kappa_1\frac{\partial^2}{\partial\eta_a^*\partial z_b^*}\Big]\end{aligned} \tag{3.260}$$

where the Q function $Q_1(\eta_a, z_b, t)$ is given by Eq. (3.208).

Now Employing (3.260) in Eq. (3.259), we have

$$\begin{aligned}\langle\hat{a}_1^\dagger(t)\hat{b}_1^\dagger(t)\rangle &= \int d^2\eta_a d^2z_b Q_1(\eta_a, z_b, t)\exp\Big[-\big(v_1\eta_a^* - r_1 z_b\big)\frac{\partial}{\partial\eta_a^*}\\ &-\big(u_1 z_b^* - r_1\eta_a\big)\frac{\partial}{\partial z_b^*} + r_1\kappa_1\frac{\partial^2}{\partial\eta_a^*\partial z_b^*}\Big]\eta_a^* z_b^*.\end{aligned} \tag{3.261}$$

This can be put in the form

$$\begin{aligned}\langle\hat{a}_1^\dagger(t)\hat{b}_1^\dagger(t)\rangle &= \int d^2\eta_a d^2z_b\frac{\partial^2}{\partial\lambda\partial\mu}\Big[Q_1(\eta_a, z_b, t)\exp\Big[-\big(v_1\eta_a^* - r_1 z_b\big)\frac{\partial}{\partial\eta_a^*}\\ &-\big(u_1 z_b^* - r_1\eta_a\big)\frac{\partial}{\partial z_b^*} + r_1\kappa_1\frac{\partial^2}{\partial\eta_a^*\partial z_b^*}\Big]\exp\left(\lambda\eta_a^* + \mu z_b^*\right)\Big]\Big|_{\lambda=\mu=0}.\end{aligned} \tag{3.262}$$

Moreover, this can also be rewritten as

$$\begin{aligned}\langle\hat{a}_1^\dagger(t)\hat{b}_1^\dagger(t)\rangle &= \int d^2\eta_a d^2z_b Q_1(\eta_a,z_b,t)\frac{\partial^2}{\partial\lambda\partial\mu}\exp\Big[(1-v_1)\lambda\eta_a^* + (1-u_1)\mu z_b^* \\ &\quad + r_1(\mu\eta_a + \lambda z_b + \kappa_1\lambda\mu)\Big]\Big|_{\lambda=\mu=0}. \end{aligned} \tag{3.263}$$

Now differentiating and applying the condition $\lambda=\mu=0$, one can readily establish that

$$\langle\hat{a}_1^\dagger(t)\hat{b}_1^\dagger(t)\rangle = \kappa_1 r_1 + (1-u_1)(1-v_1)I_{11} + r_1^2 I_{11}^* + r_1(1-v_1)I_{22} + r_1(1-u_1)I_{33}, \tag{3.264}$$

in which

$$I_{11} = \int d^2\eta_a d^2z_b Q_1(\eta_a,z_b,t)\eta_a^* z_b^* \tag{3.265}$$

$$I_{22} = \int d^2\eta_a Q_{1a}(\eta_a^*,\eta_a,t)\eta_a^*\eta_a \tag{3.266}$$

$$I_{33} = \int d^2z_b Q_{1b}(z_b^*,z_b,t)z_b^* z_b. \tag{3.267}$$

Applying (3.208) in Eq. (3.265), one can write

$$I_{11} = \frac{u_1v_1-r_1^2}{\kappa_1^2}\int\frac{d^2\eta_a}{\pi}\frac{d^2z_b}{\pi}\exp\Big[-\frac{v_1}{\kappa_1}\eta_a^*\eta_a - \frac{u_1}{\kappa_1}z_b^*z_b + \frac{r_1}{\kappa_1}(\eta_a^*z_b^* + \eta_a z_b)\Big]\eta_a^* z_b^*. \tag{3.268}$$

This can be put in the form

$$\begin{aligned} I_{11} &= \frac{u_1v_1-r_1^2}{r_1\kappa_1}\int\frac{d^2\eta_a}{\pi}\exp\Big[-\frac{v_1}{\kappa_1}\eta_a^*\eta_a\Big]\eta_a^* \\ &\quad\times\frac{\partial}{\partial\eta_a^*}\int\frac{d^2z_b}{\pi}\exp\Big[-\frac{u_1}{\kappa_1}z_b^*z_b + \frac{r_1}{\kappa_1}(\eta_a^*z_b^* + \eta_a z_b)\Big], \end{aligned} \tag{3.269}$$

so that on carrying out the integration over z_b, yields

$$I_{11} = \frac{m_1}{r_1}\int\frac{d^2\eta_a}{\pi}\exp\Big[-\frac{v_1}{\kappa_1}\eta_a^*\eta_a\Big]\eta_a^*\frac{\partial}{\partial\eta_a^*}\exp\Big[\frac{r_1^2}{u_1\kappa_1}\eta_a^*\eta_a\Big]. \tag{3.270}$$

Furthermore differentiating, leads to the expression

$$I_{11} = \frac{r_1}{u_1}I_{22} \tag{3.271}$$

On the other hand, with the aid of (3.213), one can put Eq. (3.266) in the form

$$I_{22} = \frac{m_1}{\kappa_1}\int\frac{d^2\eta_a}{\pi}\exp\Big[-\frac{m_1}{\kappa_1}\eta_a^*\eta_a\Big]\eta_a^*\eta_a. \tag{3.272}$$

This can also be rewritten as

$$I_{22} = -m_1\frac{\partial}{\partial m_1}\int\frac{d^2\eta_a}{\pi}\exp\Big[-\frac{m_1}{\kappa_1}\eta_a^*\eta_a\Big]. \tag{3.273}$$

Furthermore, carrying out the integration, we obtain

$$I_{22} = -\kappa_1 m_1 \frac{\partial}{\partial m_1}\Big(\frac{1}{m_1}\Big), \tag{3.274}$$

from which, we find

$$I_{22} = \frac{\kappa_1 u_1}{u_1 v_1 - r_1^2} \tag{3.275}$$

Employing (3.275) in Eq. (3.271), we readily get

$$I_{11} = \frac{\kappa_1 r_1}{u_1 v_1 - r_1^2} \tag{3.276}$$

Making use of similar calculations, it is not difficult to show that

$$I_{33} = \frac{\kappa_1 v_1}{u_1 v_1 - r_1^2} \tag{3.277}$$

Substituting (3.275-3.277) in Eq. (3.264), we find

$$\langle \hat{a}_1^\dagger(t)\hat{b}_1^\dagger(t)\rangle = \frac{\kappa_1 r_1}{u_1 v_1 - r_1^2} \tag{3.278}$$

$$= \kappa_1 c_1. \tag{3.279}$$

Moreover, in view of the fact that the Q functions (3.208) and (3.209) have the same form, we see that

$$\langle \hat{a}_i^\dagger(t)\hat{b}_i^\dagger(t)\rangle = \langle \hat{a}_i(t)\hat{b}_i(t)\rangle = \kappa_i c_i,\ \ i = 1, 2. \tag{3.280}$$

In view of these result, we observe that

$$\langle \hat{a}^\dagger(t)\hat{b}^\dagger(t)\rangle = \langle \hat{a}(t)\hat{b}(t)\rangle = \kappa_1 c_1 + \kappa_2 c_2. \tag{3.281}$$

Now taking into account of Eqs. (3.253), (3.254), and (3.281) together with (3.239), one can readily establish that

$$\Delta c_\pm^2 = 2\kappa_1 + 2\kappa_2 + (:\Delta c_\pm^2:)_1 + (:\Delta c_\pm^2:)_2, \tag{3.282}$$

where

$$(:\Delta c_\pm^2:)_i = 2\kappa_i\big[(a_i - 1) + (b_i - 1) \pm 2c_i\big],\ \ i = 1, 2. \tag{3.283}$$

Furthermore, employing (3.174-3.176) in Eq. (3.283), the variance of the quadratures found to be

$$\begin{aligned}\Delta c_\pm^2 = {} & 2\kappa_1 + \frac{\kappa_1 A_1(\eta - 1)}{\eta(A_1\eta + \kappa_1)}\Big[1 \pm \sqrt{1-\eta^2}\Big](1 - e^{-(A_1\eta+\kappa_1)t}) \\ & \pm \frac{2\kappa_1 A_1\sqrt{1-\eta^2}}{\eta(A_1\eta + 2\kappa_1)}\Big[1 \pm \sqrt{1-\eta^2}\Big](1 - e^{-(A_1\eta+2\kappa_1)t/2}). \\ & +2\kappa_2 + \frac{\kappa_2 A_2(\eta - 1)}{\eta(A_2\eta + \kappa_2)}\Big[1 \pm \sqrt{1-\eta^2}\Big](1 - e^{-(A_2\eta+\kappa_2)t}) \\ & \pm \frac{2\kappa_2 A_2\sqrt{1-\eta^2}}{\eta(A_2\eta + 2\kappa_2)}\Big[1 \pm \sqrt{1-\eta^2}\Big](1 - e^{-(A_2\eta+2\kappa_2)t/2}).\end{aligned} \tag{3.284}$$

At steady state, Eq. (3.284) takes the form

$$\Delta c_{\pm}^2 = 2\kappa_1 + \frac{\kappa_1 A_1(1-\eta)(A_1+2A_1\eta+2\kappa_1) \pm \kappa_1 A_1\sqrt{1-\eta^2}(A_1+A_1\eta+2\kappa_1)}{(A_1\eta+\kappa_1)(A_1\eta+2\kappa_1)}$$
$$+2\kappa_2 + \frac{\kappa_2 A_2(1-\eta)(A_2+2A_2\eta+2\kappa_2) \pm \kappa_2 A_2\sqrt{1-\eta^2}(A_2+A_2\eta+2\kappa_2)}{(A_2\eta+\kappa_2)(A_2\eta+2\kappa_2)}. \tag{3.285}$$

We observe that the quadrature variance given by (3.284) or (3.285) is the sum of the quadrature variance of the constituent output laser light beams.

It is interesting to examine some special cases. First we consider the case in which $A_2 = 0$. In this case, the quadrature variance (3.285) turns out to be

$$\Delta c_{\pm}^2 = 2\kappa_2 + 2\kappa_1 + \frac{\kappa_1 A_1(1-\eta)(A_1+2A_1\eta+2\kappa_1) \pm \kappa_1 A_1\sqrt{1-\eta^2}(A_1+A_1\eta+2\kappa_1)}{(A_1\eta+\kappa_1)(A_1\eta+2\kappa_1)}. \tag{3.286}$$

This represents the quadrature variance of the superposed output two-mode squeezed laser light beam and a light beam in a two-mode vacuum state.

However, for the case in which κ_2 is equal to zero, the the quadrature variance described by Eq. (3.285) reduces to

$$\Delta c_{\pm}^2 = 2\kappa_1 + \frac{\kappa_1 A_1(1-\eta)(A_1+2A_1\eta+2\kappa_1) \pm \kappa_1 A_1\sqrt{1-\eta^2}(A_1+A_1\eta+2\kappa_1)}{(A_1\eta+\kappa_1)(A_1\eta+2\kappa_1)}, \tag{3.287}$$

which is the two-mode quadrature variance of one output light beam.

It is worth mentioning that the quadrature variance of a pair of superposed identical output laser light beams is twice that of the quadrature variance of one output laser light beam.

We define the quadrature squeezing of the two-mode superposed output light by

$$S = \frac{(\Delta c_-^2)_v - (\Delta c_-^2)}{(\Delta c_-^2)_v}, \tag{3.288}$$

where $(\Delta c_{\pm}^2)_v = 2\kappa_1 + 2\kappa_2$ is the output quadrature variance of a pair of superposed two-mode vacuum states.

On account of (3.285), the quadrature squeezing is found to be

$$S = \frac{\kappa_1 S_1 + \kappa_2 S_2}{\kappa_1 + \kappa_2}, \tag{3.289}$$

where

$$S_i = \frac{A_i\left[\sqrt{1-\eta^2}(A_i+A_i\eta+2\kappa_i) - (1-\eta)(A_i+2A_i\eta+2\kappa_i)\right]}{2(A_i\eta+\kappa_i)(A_i\eta+2\kappa_i)}, \quad i=1,2 \tag{3.290}$$

is the two-mode quadrature squeezing of the i^{th} laser output light beam.

We now proceed to consider some special cases. First we consider the case in which $\kappa_1 = \kappa_2$. In this case, we see that Eq. (3.289) takes the form

$$S = \frac{S_1 + S_2}{2}. \tag{3.291}$$

We see from Eq. (3.291) and Fig. (3.1) that the quadrature squeezing of the superposed laser light beams is the average of the quadrature squeezing of the component output laser light beams for all values of η between zero and one. And the degree of squeezing of the component output laser light beams increases with the linear gain coefficient for small values of η and almost perfect squeezing can be obtained for large values of the linear gain coefficient. Furthermore, for the case in which $A_1 = A_2$ and $\kappa_1 = \kappa_2$, the quadrature squeezing of the

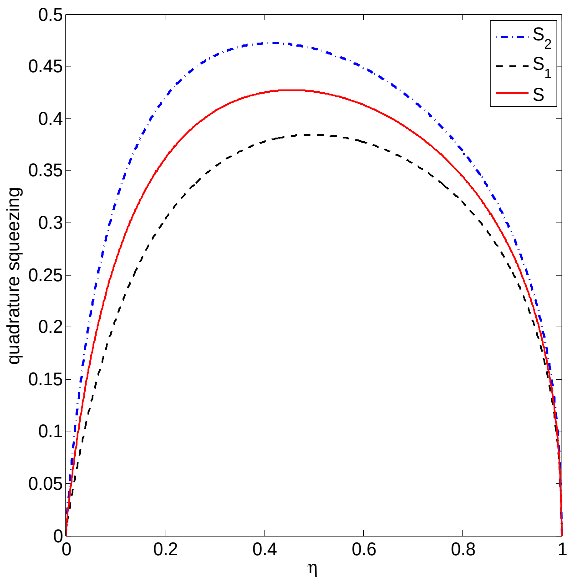

Fig. 3.1: Plots of the quadrature squeezing (S and S_i) [Eq. (3.291) and Eq. (3.290)] versus η for $\kappa_1 = \kappa_2 = 0.8$, $A_1 = 5$, and $A_2 = 10$.

superposed output laser light beams (3.289) turns out to be

$$S = S_i, \;\; i = 1, 2. \tag{3.292}$$

We see from (3.292) that the quadrature squeezing of a pair of superposed identical two-mode laser output light beams is exactly the same as the quadrature squeezing of one output laser light beam. In addition, for the case in which either κ_1 or $\kappa_2 = 0$, the two-mode quadrature squeezing becomes

$$S = S_i, \tag{3.293}$$

which is the identical to the quadrature squeezing of one output laser light beam.

On the other hand, for the case in which $A_2 = 0$, the quadrature squeezing described by (3.289) takes the form

$$S = \frac{\kappa_1 S_1}{\kappa_1 + \kappa_2}. \tag{3.294}$$

However, for the case in which either $\kappa_1 = \kappa_2$, the quadrature squeezing described by (3.294) reduces to

$$S = \frac{1}{2} S_i. \tag{3.295}$$

This represents the quadrature squeezing of the superposed two-mode squeezed laser light beam and a light beam in a vacuum state. Thus we observe that the effect of the vacuum is to reduce the quadrature squeezing by a factor of one-half.

3.4 Photon statistics

In this section, we seek to analyze the statistical properties of the superposed two-mode output light beams.

3.4.1 The mean of the photon number sum and difference

We now seek to determine the mean of the photon number sum and difference of a pair of superposed two-mode output light beams. To this end, we define the operators representing the photon number for the superposed two-mode output laser light beams by

$$\hat{n}_a(t) = \hat{a}^\dagger(t)\hat{a}(t) \tag{3.296a}$$

$$\hat{n}_b(t) = \hat{b}^\dagger(t)\hat{b}(t). \tag{3.296b}$$

Taking the expectation value of (3.296) and applying Eqs. (3.253), (3.254), (3.174), and (3.175), the mean of the photon numbers of the output modes a and b at steady have the form

$$\bar{n}_a = \bar{n}_{1a} + \bar{n}_{2a} \tag{3.297a}$$

$$\bar{n}_b = \bar{n}_{1b} + \bar{n}_{2b}, \tag{3.297b}$$

where

$$\bar{n}_{ia} = \frac{\kappa_i A_i(1-\eta)(A_i + 3A_i\eta + 4\kappa_i)}{4(A_i\eta + \kappa_i)(A_i\eta + 2\kappa_i)} \tag{3.298a}$$

$$\bar{n}_{ib} = \frac{\kappa_i A_i^2(1-\eta^2)}{4(A_i\eta + \kappa_i)(A_i\eta + 2\kappa_i)}, \quad i = 1, 2 \tag{3.298b}$$

is the mean of the photon numbers for the i^{th} laser output light beams associated to mode a and b respectively.

We see that the mean photon number of mode a(b) of a pair of superposed output laser light beams given by Eqs. (3.297a) and (3.297b) is the sum of the mean photon numbers of the constituent output laser light beam corresponding to each mode. It is worth mentioning that the mean photon numbers of mode a(b) of a pair of superposed identical output laser light beams is twice that of the mean photon number of mode a(b) of one output laser light beam. Fig. (3.2) represents plots of the mean photon number of mode a [Eq. (3.297a)] and mode b [Eq. (3.297b)] of a pair of superposed output laser light beams versus η. We see from the Fig.(3.2) that the mean photon number of mode a is greater than that of mode b.

Next, we seek to calculate the mean of the photon number sum and difference of a pair of superposed two-mode output laser light beams. To this end, we define the operators repre-

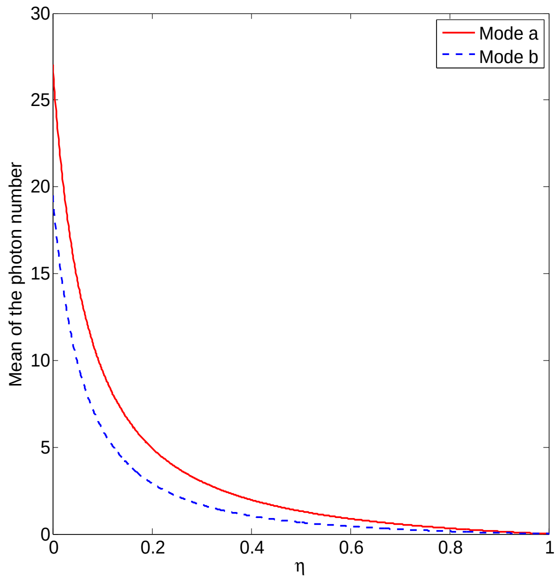

Fig. 3.2: Plots of the mean of the photon number of mode a [Eq. (3.297a)] and mode b [Eq. (3.297b)] of a pair of superposed output laser light beams versus η for $\kappa_1 = \kappa_2 = 0.8$, $A_1 = 5$, and $A_2 = 10$.

senting the photon number sum and difference of the two-mode output light by

$$\hat{n}_{\pm} = \hat{n}_a \pm \hat{n}_b. \tag{3.299}$$

Upon taking the expectation value of (3.299) and applying Eqs.(3.297a) and (3.297b), the mean of the photon number sum and difference of a pair of superposed two-mode output laser light beams found to be

$$\bar{n}_{\pm} \;=\; \bar{n}_{1\pm} + \bar{n}_{2\pm}, \tag{3.300}$$

in which

$$\bar{n}_{i\pm} \;=\; \bar{n}_{ia} \pm \bar{n}_{ib}, \;\; i = 1, 2 \tag{3.301}$$

is the mean of the photon number sum and difference of the constituent two-mode output laser light beams. Furthermore, applying Eqs.(3.298a) and (3.298b) in Eq.(3.301), one can readily find

$$\bar{n}_{i\pm} \;=\; \frac{2\kappa_i A_i(1-\eta)(A_i\eta + 2\kappa_i) + \kappa_i A_i^2(1-\eta^2)(1\pm 1)}{4(A_i\eta + \kappa_i)(A_i\eta + 2\kappa_i)} \tag{3.302}$$

We see from Eq. (3.300) that the mean of the photon number sum and difference of the superposed two-mode output laser light beams is the sum of the mean of the photon numbers

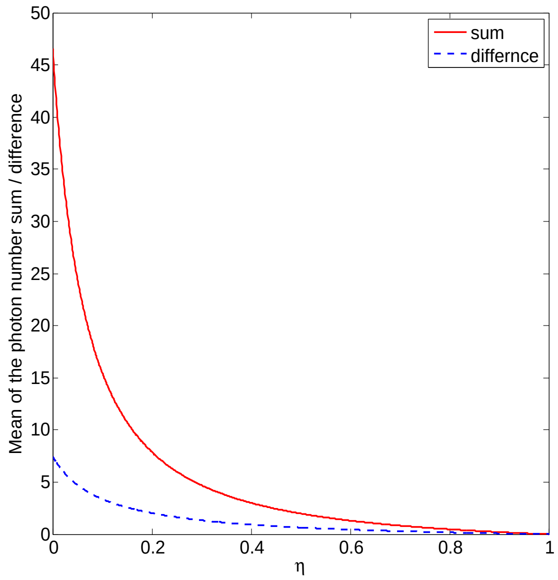

Fig. 3.3: Plots of the mean of the photon number sum and difference [Eq.(3.300)] of a pair of superposed two-mode output laser light beams versus η for $\kappa_1 = \kappa_2 = 0.8$, $A_1 = 5$, and $A_2 = 10$.

sum and difference of the constituent two-mode output laser light beams. We also observe from Eq. (3.300) and Figs. (3.2) and (3.3) that the mean of the photon number difference of the output modes is positive. This also shows that the mean of the photon number of mode a is greater than that of mode b. And the increase in the mean photon number of mode a must be due to the decay of some atoms from the intermediate level to levels other than lower level spontaneously.

It is important to note that for the case in which the component of the output light beams have equal linear gain coefficients as well as damping constants. In this case the mean of the photon number sum and difference of a pair of superposed two-mode output laser light beams is twice that of the mean of the photon number sum and difference of one output laser light beam.

3.4.2 The variances of the photon number sum and difference

Here we calculate the variances of the photon number sum and difference for the two-mode superposed output laser light beams. The variances of the photon number sum and difference defined by

$$\Delta n_{\pm}^2 = \langle(\hat{a}^\dagger\hat{a} \pm \hat{b}^\dagger\hat{b})^2\rangle - \langle\hat{a}^\dagger\hat{a} \pm \hat{b}^\dagger\hat{b}\rangle^2 \tag{3.303}$$

can be expressed as

$$\Delta n_{\pm}^2 = \Delta n_a^2 + \Delta n_b^2 \pm 2\langle \hat{n}_a, \hat{n}_b \rangle, \tag{3.304}$$

in which

$$\Delta n_a^2 = \langle (\hat{a}^\dagger \hat{a})^2 \rangle - \bar{n}_a^2 \tag{3.305}$$

and

$$\Delta n_b^2 = \langle (\hat{b}^\dagger \hat{b})^2 \rangle - \bar{n}_b^2 \tag{3.306}$$

are the variance of the superposed output light beams for mode a and mode b respectively, and

$$\langle \hat{n}_a, \hat{n}_b \rangle = \langle \hat{n}_a \hat{n}_b \rangle - \bar{n}_a \bar{n}_b. \tag{3.307}$$

Using the commutation relation (3.224), the variance of the photon number for the superposed output light defined by Eqs. (3.305) can also be expressed as

$$\Delta n_a^2 = \langle \hat{a}^{\dagger 2} \hat{a}^2 \rangle + (\kappa_1 + \kappa_2)\bar{n}_a - \bar{n}_a^2, \tag{3.308}$$

Now using Eq. (3.210), one can readily obtain

$$\begin{aligned} \langle \hat{a}^{\dagger 2}(t)\hat{a}^2(t)\rangle &= \int d^2\eta_a d^2\lambda_a Q_{1a}\Big(\eta_a^*, \eta_a + \frac{\partial}{\partial \eta_a^*}, t\Big) Q_{2a}\Big(\lambda_a^*, \lambda_a + \frac{\partial}{\partial \lambda_a^*}, t\Big) \\ &\quad \times \Big\{(\eta_a^*\eta_a)^2 + (\lambda_a^*\lambda_a)^2 + 4(\eta_a^*\eta_a)(\lambda_a^*\lambda_a) \\ &\quad + \big[(\eta_a\lambda_a^*)^2 + 2\eta_a^*\eta_a^2\lambda_a^* + 2\lambda_a^*\lambda_a^2\eta_a^* + c.c\big]\Big\}. \end{aligned} \tag{3.309}$$

This can be rewritten as

$$\begin{aligned} \langle \hat{a}^{\dagger 2}(t)\hat{a}^2(t)\rangle &= \langle \hat{a}_1^{\dagger 2}(t)\hat{a}_1^2(t)\rangle + \langle \hat{a}_2^{\dagger 2}(t)\hat{a}_2^2(t)\rangle + \langle \hat{a}_1^2(t)\rangle\langle \hat{a}_2^{\dagger 2}(t)\rangle \\ &\quad + \langle \hat{a}_2^2(t)\rangle\langle \hat{a}_1^{\dagger 2}(t)\rangle + 2\langle \hat{a}_2^\dagger(t)\rangle\langle \hat{a}_1^\dagger(t)\hat{a}_1^2(t)\rangle \\ &\quad + 2\langle \hat{a}_1^\dagger(t)\rangle\langle \hat{a}_2^\dagger(t)\hat{a}_2^2(t)\rangle + 2\langle \hat{a}_2(t)\rangle\langle \hat{a}_1^{\dagger 2}(t)\hat{a}_1(t)\rangle \\ &\quad + 2\langle \hat{a}_1(t)\rangle\langle \hat{a}_2^{\dagger 2}(t)\hat{a}_2(t)\rangle + 4\langle \hat{a}_1^\dagger(t)\hat{a}_1(t)\rangle\langle \hat{a}_2^\dagger(t)\hat{a}_2(t)\rangle. \end{aligned} \tag{3.310}$$

Using the Q function (3.211), one can easily verify that

$$\langle \hat{a}_1^2(t)\rangle = 0. \tag{3.311}$$

Moreover, since the Q functions (3.211) and (3.212) have the same form, we observe that

$$\langle \hat{a}_i^2(t)\rangle = 0, \;\; i = 1, 2. \tag{3.312}$$

Taking into account of Eqs. (3.234) and (3.312), the expression described by Eq. (3.310) reduces to

$$\langle\hat{a}^{\dagger 2}(t)\hat{a}^2(t)\rangle = \langle\hat{a}_1^{\dagger 2}(t)\hat{a}_1^2(t)\rangle + \langle\hat{a}_2^{\dagger 2}(t)\hat{a}_2^2(t)\rangle + 4\langle\hat{a}_1^\dagger(t)\hat{a}_1(t)\rangle\langle\hat{a}_2^\dagger(t)\hat{a}_2(t)\rangle. \tag{3.313}$$

We note that the first term in Eq. (3.313) is expressible as

$$\langle\hat{a}_1^{\dagger 2}(t)\hat{a}_1^2(t)\rangle = \int d^2\eta_a Q_{1a}(\eta_a^*, \eta_a + \frac{\partial}{\partial\eta_a^*}, t)\eta_a^{*2}\eta_a^2. \tag{3.314}$$

Applying (3.211) in Eq. (3.314), we have

$$\langle\hat{a}_1^{\dagger 2}(t)\hat{a}_1^2(t)\rangle = \int d^2\eta_a \frac{\partial^2}{\partial\lambda^2}\Big[Q_{1a}(\eta_a^*, \eta_a, t)\exp\Big(-m_1\eta_a^*\frac{\partial}{\partial\eta_a^*}\Big)e^{\lambda\eta_a^*\eta_a}\Big]\Big|_{\lambda=0}. \tag{3.315}$$

There follows

$$\langle\hat{a}_1^{\dagger 2}(t)\hat{a}_1^2(t)\rangle = \frac{m_1}{\kappa_1}\int\frac{d^2\eta_a}{\pi}\frac{\partial^2}{\partial\lambda^2}\exp\Big[-\frac{m_1}{\kappa_1}\eta_a^*\eta_a - (m_1-1)\lambda\eta_a^*\eta_a\Big]\Big|_{\lambda=0}. \tag{3.316}$$

Furthermore, upon performing differentiation and applying the condition $\lambda = 0$, leads to

$$\langle\hat{a}_1^{\dagger 2}(t)\hat{a}_1^2(t)\rangle = \frac{(1-m_1)^2 m_1}{\kappa_1}\int\frac{d^2\eta_a}{\pi}\exp\Big[-\frac{m_1}{\kappa_1}\eta_a^*\eta_a\Big]\eta_a^{*2}\eta_a^2. \tag{3.317}$$

This can be rewritten in the form

$$\langle\hat{a}_1^{\dagger 2}(t)\hat{a}_1^2(t)\rangle = \kappa_1(1-m_1)^2 m_1\frac{\partial^2}{\partial m_1^2}\int\frac{d^2\eta_a}{\pi}\exp\Big[-\frac{m_1}{\kappa_1}\eta_a^*\eta_a\Big] \tag{3.318}$$

so that on carrying out the integration, yields

$$\langle\hat{a}_1^{\dagger 2}(t)\hat{a}_1^2(t)\rangle = \kappa_1^2(1-m_1)^2 m_1\frac{\partial^2}{\partial m_1^2}\Big(\frac{1}{m_1}\Big), \tag{3.319}$$

from which, we obtain

$$\begin{aligned}\langle\hat{a}_1^{\dagger 2}(t)\hat{a}_1^2(t)\rangle &= 2\kappa_1^2\Big(\frac{u_1}{u_1v_1 - r_1^2} - 1\Big)^2 &(3.320)\\ &= 2\kappa_1^2(a_1-1)^2. &(3.321)\end{aligned}$$

Moreover, on account of the fact that the Q functions expressed by Eqs. (3.211) and (3.212) have exactly the same form, we also see that

$$\langle\hat{a}_i^{\dagger 2}(t)\hat{a}_i^2(t)\rangle = 2\kappa_i^2(a_i-1)^2,\ \ i = 1, 2. \tag{3.322}$$

In view of (3.251) and (3.252), this can be put in the form

$$\langle\hat{a}_i^{\dagger 2}(t)\hat{a}_i^2(t)\rangle = 2\bar{n}_{ia}^2,\ \ i = 1, 2. \tag{3.323}$$

where $\bar{n}_{ia}$ is the mean of the photon number of mode a for the i^{th} laser output light beam given by Eq. (3.298a). Now taking Eqs. (3.251), (3.252), and (3.323) into account, one can put Eq. (3.313) in the form

$$\langle \hat{a}^{\dagger 2}(t)\hat{a}^{2}(t)\rangle = 2\bar{n}_{1a}^{2} + 2\bar{n}_{2a}^{2} + 4\bar{n}_{1a}\bar{n}_{2a} = 2\bar{n}_{a}^{2}, \tag{3.324}$$

where $\bar{n}_a$ is the mean of the photon number of mode a for a pair of superposed laser output light beams given by (3.297a). With the aid of (3.324), one can readily find

$$\Delta n_a^2 = \bar{n}_a^2 + (\kappa_1 + \kappa_2)\bar{n}_a. \tag{3.325}$$

It is not difficult to demonstrate in a similar manner that

$$\Delta n_b^2 = \bar{n}_b^2 + (\kappa_1 + \kappa_2)\bar{n}_b \tag{3.326}$$

$$\langle \hat{n}_a, \hat{n}_b\rangle = (\kappa_1 c_1 + \kappa_2 c_2)^2. \tag{3.327}$$

On the other hand, we can also write Eqs. (3.325) and (3.326) in terms of the mean and variance of the photon numbers of mode a and b of the component output light beams as

$$\Delta n_m^2 = \Delta n_{1m}^2 + \Delta n_{2m}^2 + 2\bar{n}_{1m}\bar{n}_{2m} + \kappa_2\bar{n}_{1m} + \kappa_1\bar{n}_{2m}, \quad m = a, b, \tag{3.328}$$

where

$$\Delta n_{im}^2 = \bar{n}_{im}^2 + \kappa_i\bar{n}_{im}, \quad i = 1, 2 \tag{3.329}$$

is the variances of the photon number for the i^{th} component output light modes $m = a, b$. We see from Eq. (3.328) that the variance of the photon number of a pair of superposed output light beams corresponding to mode a and b is not the sum of the variances of the photon number of the constituent output light beams. Moreover, based on Eq. (3.329), we certainly identify that the i^{th} component output light mode a and b are separately in chaotic state and hence both exhibit supper-Poissonian statistics.

Now combination of Eqs. (3.304), (3.325), (3.326), and(3.327) results in

$$\Delta n_\pm^2 = \bar{n}_a^2 + (\kappa_1 + \kappa_2)\bar{n}_a + \bar{n}_b^2 + (\kappa_1 + \kappa_2)\bar{n}_b \pm 2\big(\kappa_1 c_1 + \kappa_2 c_2\big)^2. \tag{3.330}$$

This can be expressible in the form

$$\begin{aligned}\Delta n_\pm^2 = {} & \Delta n_{1a}^2 + \Delta n_{2a}^2 + 2\bar{n}_{1a}\bar{n}_{2a} + \kappa_2\bar{n}_{1a} + \kappa_1\bar{n}_{2a} \\ & + \Delta n_{1b}^2 + \Delta n_{2b}^2 + 2\bar{n}_{1b}\bar{n}_{2b} + \kappa_2\bar{n}_{1b} + \kappa_1\bar{n}_{2b} \\ & \pm 2\big(\kappa_1^2 c_1^2 + \kappa_2^2 c_2^2 + 2\kappa_1\kappa_2 c_1 c_2\big),\end{aligned} \tag{3.331}$$

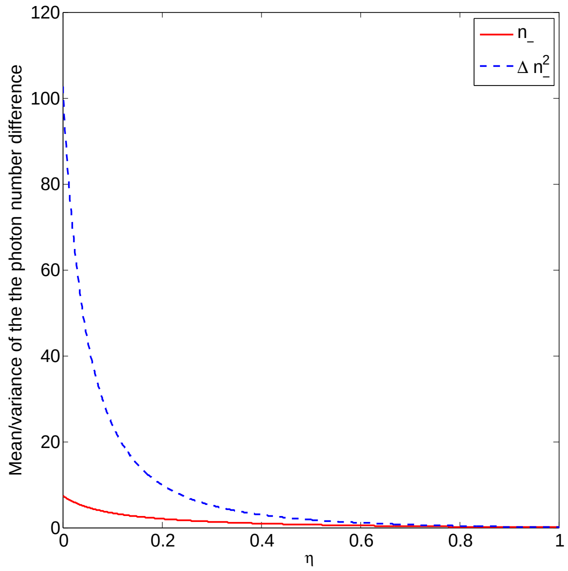

Fig. 3.4: Plots of the mean of photon number difference [Eq.(3.300)] versus η (solid curve) and the variance of the photon number difference [Eq.(3.330)] versus η (dashed curve) of the superposed output laser light beams for $\kappa_1 = \kappa_2 = 0.8$, $A_1 = 5$, and $A_2 = 10$.

where Δn_{im}^2 is given by (3.329).On the other hand, this can also be rewritten as

$$\begin{aligned}\Delta n_\pm^2 &= (\Delta n_\pm^2)_1 + (\Delta n_\pm^2)_2 + 2(\bar{n}_{1a}\bar{n}_{2a} + \bar{n}_{1b}\bar{n}_{2b}) \\ &\quad + \kappa_2(\bar{n}_{1a} + \bar{n}_{1b}) + \kappa_1(\bar{n}_{2a} + \bar{n}_{2b}) \pm 4\kappa_1\kappa_2 c_1 c_2, \end{aligned} \tag{3.332}$$

where

$$(\Delta n_\pm^2)_i = \bar{n}_{ia}^2 + \kappa_i \bar{n}_{ia} + \bar{n}_{ib}^2 + \kappa_i \bar{n}_{ib} \pm 2\kappa_i^2 c_i^2, \;\; i = 1, 2. \tag{3.333}$$

is the variances of the photon number sum and difference of the constituent output laser light beams. It is worth mentioning that the variances of the photon number sum and difference of a pair of superposed two-mode output laser light beams is not the sum of the variances of the photon numbers sum and difference of the constituent two-mode output laser light beams. In addition, we notice from Eq. (3.332)and Fig. 3.4 that $\Delta n_\pm^2 > \bar{n}_\pm$ and thus photon statistics of photon number sum and difference of a pair of superposed two-mode output laser light beams is supper-Poissonian.

It is easy to observe that for the case in which $A_1 = A_2$ and $\kappa_1 = \kappa_2$, the variance of the photon number sum and difference of a pair of superposed two-mode output laser light beams given by Eq. (3.332) leads to

$$\Delta n_{\pm}^{2} \;=\; 4(\Delta n_{\pm}^{2})_{i},\;\; i=1,2. \tag{3.334}$$

The normalized photon number correlation

Finally, in order to determine whether the photon numbers of mode a and mode b are correlated or not, we must examine the normalized photon numbers correlation. Thus the photon numbers correlation of the superposed two-mode output laser light beams can be defined as [3, 28, 37]

$$g_{a,b}^{(2)}(t)=\frac{\langle\hat{n}_a\hat{n}_b\rangle}{\langle\hat{n}_a\rangle\langle\hat{n}_b\rangle}. \tag{3.335}$$

Comparison of Eqs. (3.307) and (3.327) shows that

$$\langle\hat{n}_a\hat{n}_b\rangle \;=\; (\kappa_1 c_1+\kappa_2 c_2)^2+\bar{n}_a\bar{n}_b. \tag{3.336}$$

Now applying (3.336) in Eq. (3.335), we obtain

$$g_{a,b}^{(2)}(t)=1+\frac{\left(\kappa_1 c_1+\kappa_2 c_2\right)^2}{\bar{n}_a\bar{n}_b}. \tag{3.337}$$

This result indicates that the photon numbers correlation is different from one. Thus the photon numbers of mode a and mode b of a pair of superposed two-mode output laser light beams are correlated. Moreover, we can see from Eq. (3.336) that the expectation value of the product of number operators, $\langle\hat{n}_a\hat{n}_b\rangle$ is different from $\langle\hat{n}_a\rangle\langle\hat{n}_b\rangle$. This implies that there is an intermode correlation. Thus this intermode correlation must be due to the atomic coherence induced by initially preparing atoms in coherent superposition of the top and bottom levels.

4

Conclusion

In this PhD dissertation, we have studied the squeezing and statistical properties of a pair of superposed output light beams generated by three-level laser whose cavities are coupled to a vacuum reservoirs. Making use of the density operators of the superposed output laser light beams together with Q functions, we have determined the global and local mean and variance of the photon number as well as the global and local quadrature squeezing. Moreover, we have calculated the mean and variance of the photon number sum and difference and the photon numbers correlation of a pair of superposed two-mode output light beams.

We have found that both the mean photon number and quadrature variance of a pair of superposed output laser light beams is the sum of the mean photon numbers and quadrature variances of the constituent output light beams. However, the variance of the photon number of the superposed output laser light beams is not the sum of the variances of the photon numbers of the component output laser light beams. Moreover, we have established that the global and local quadrature squeezing of the superposed output laser light beams is the average of the global and local quadrature squeezing of the constituent output laser light beams. The output light produced by the pair of superposed laser light beams is in a squeezed state for all values of η between zero and one. We have also noticed that the superposed output laser light beams is squeezed 61.5% below the vacuum state-level. On the other hand, we have seen that the local quadrature squeezing is greater than the global quadrature squeezing. Besides, the maximum local quadrature squeezing of the superposed output laser light beams is found to be 92.5% and occurs in the frequency $\lambda = \pm 0.01$. It is found that superposing a pair of identical squeezed laser output light beams considerably increases the mean and variance of the photon number, without affecting the quadrature squeezing. This makes the system under consideration to be a source of bright squeezed light. Furthermore, our results have shown that the local quadrature squeezing, unlike the mean photon number, the quadrature variance, and the photon number variance does not increase as the value of λ increase. In addition, our analysis indicates that the quadrature squeezing of the superposition of squeezed laser light beam and a light beam in vacuum state turned out to be just half of the quadrature squeezing of the three-level laser output light beam. This result is consistent with the quadrature squeezing of the superposed coherent and squeezed light obtained in [22]. On the other hand, in the absences of the output light beam from one of the cavity, the mean photon number, the quadrature squeezing (variance), and the photon number variance of the superposed squeezed laser light beams is identical to the mean photon number, the quadrature squeezing (variance), and the photon number variance of one output light beam. Finally, we would like to mention that the photon numbers of a pair of superposed two-mode light beams are correlated.

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DECLARATION

I hereby declare that this PhD dissertation is my original work and has not been presented for a degree in any other universities, and that all sources of material used for the dissertation have been duly acknowledged.

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This PhD dissertation has been submitted to for examination with my approval as university advisor.

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