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Thermodynamics and Quantum Properties of Ultra-Relativistic Degenerate Electron Gas in White Dwarf and Neutron Stars

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This is to certify that the thesis prepared by **Ibrahim Ali**, entitled **"Thermodynamics and Quantum Properties of Ultra-Relativistic Degenerate Electron Gas in White Dwarf and Neutron Stars"** and submitted in partial fulfillment of the requirements for the degree of **Master of Science in Physics (Statistical Physics)** complies with the regulations of the University and meets the accepted standards with respect to originality.

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Abstract

We study the problem of the relativistic free electron gas at arbitrary degenerate electron gas. The specific heat at constant volume and particle number c_v and the specific heat at constant pressure and particle number c_p are calculated. The equation of state is also studied. Non degenerate and degenerate electron limits are considered. We generalize the formulas obtained in the non-relativistic and ultra-relativistic degenerate electron gas.

Neutron stars are much denser than white dwarf stars, which, once again, causes the core of the stars to collapse. The compression of neutrons in the contracting core, however, creates a neutron degenerate pressure. This pressure, analogous to the electron degenerate pressure in white dwarf stars, combats the gravitational collapse of the star. If, however, the neutron star is too massive (more than three solar masses), the neutron degenerate pressure fails and the neutron star collapses into a black hole.

We now see that the role of both the neutron degenerate pressure, and the electron degenerate pressure is crucial to the maintained stability of a star The energy corresponding to this momentum, called the Fermi energy which we will discuss in the next section will also increase. So, with a decreasing volume and an increasing particle momentum, we can say that a pressure formed inside of the core of the star, and continues to increase as long as the volume continues to increase, and that there are degenerate neutron energy states. Now that we know where the pressure comes from, we can finally derive a mathematical expression for the neutron degeneracy pressure by non-relativistic neutrons inside of a neutron star.

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Contents

List of Figures

Chapter 1

Introduction

Compact stars, i.e. white dwarf and neutron stars [1] are the final stages in the evolution of ordinary stars. After hydrogen and helium burning, the helium reservoir in the star's core is burned up to carbon and oxygen. The nuclear processes in the core will stop and the star's temperature will decrease. Consequently, the star shrinks and the pressure in the core increases.

As long as the star's mass is above a certain value, it will be able to initialize new fusion processes to heavier elements. The smaller the mass, the more the core has to be contracted to produce the required heat for the next burning process to start. If the star's initial mass is below eight solar masses, the gravitational pressure is too weak to reach the required density and temperature to initialise carbon fusion[2],[3]. Due to the high core temperature, the outer regions of the star swell and are blown away by stellar winds. What remains is a compact core mainly composed of carbon and oxygen. The interior consists of a degenerate electron gas which, as we will discuss, is responsible for the intrinsic high pressure. The compact remnant starts to emit this thermal energy and a white dwarf is born.[4]

As there are no fusion processes taking place in the interior of the white dwarf, a different force than thermal pressure is needed to keep the star in hydrostatic equilibrium.[5] Due to the Pauli principle, two fermions cannot occupy the same quantum state, hence, more than two electrons (with different spin) cannot occupy the same phase space. With increasing density, the electrons fill up the phase space from the lowest energy state, i.e. the smallest momentum. Consequently, the remaining electrons sit in physical states with increasing momentum. The resulting large velocity leads to an adequate pressure of the electrons the degenerate pressure counterbalancing gravity's pull and stabilising the white dwarf[4].

The electrons are the first particles to become degenerate in dense matter due to their small mass. White dwarfs consist also of carbon and oxygen[6] but the contribution of nuclei to the pressure is negligible for the density region of interest here. Consequently, a larger degenerate pressure, i.e. a greater density, is needed for stability for more massive white dwarfs[5]. There is a mass limit for white dwarf beyond which even the degenerate electron gas cannot prevent the star from collapsing[6]. This mass limit is around 1.4 solar masses the famous Chandrasekhar mass. For greater masses, the white dwarf collapses to a neutron star or a black hole.[9] The radii of white dwarfs are in the range of 10,000 km about the size of our planet earth. Similarly, a neutron star is stabilized by the degenerate pressure also. The difference is that for neutron stars the degenerate pressure originates mainly from neutrons, not from electrons. If the initial mass of an ordinary star exceeds eight solar masses, carbon and oxygen burning starts in the core. Around the core is a layer of helium and a layer of hydrogen, both taking part in fusion processes.

For even larger density there will be just a dense and incompressible core of neutrons with a small fraction of electrons and protons. Neutrons in neutron stars are then degenerate like electrons in white dwarfs. There will be an outgoing shock wave generated by the proto–neutron star due to the high incomprehensibility of neutron star matter. The falling outer layers of the ordinary star bounce and move outwards interacting with the other still collapsing layers and generate an overall outward expansion a core collapse supernova arises[6].

The neutron rich remnant of the core collapse will become a neutron star as long as his mass is less than about two to three solar masses, otherwise it proceeds to shrink and becomes finally a black hole. The radii of neutron stars are typically 10 km the size of the city of Frankfurt.

With in the core of a star we find a direct application of a quantum mechanical phenomenon, the degenerate pressure. The purpose of the degenerate pressure to combat the collapse of the star by its own gravity. Using an expression for the Fermi energy, we will construct a mathematical expression for the degenerate pressure. We will then find the gravitational pressure due to the binding energy of a star. Next, we will examine an equilibrium state in a neutron star by balancing the pressures. .

Thermonuclear fusion, the thermally-induced combining of nuclei as they tunnel through the Coulomb barrier, is initially responsible for supporting stars against gravitational contraction. The ultimate fate of the star depends upon its remaining mass once thermonuclear fusion can no longer provide the pressure required to counteract gravity.

Thermonuclear fusion drives stars through many stages of combustion; the hot center of the star allows hydrogen to fuse into helium. Once the core has burned all available hydrogen, it will contract until another source of support becomes available.

As the core contracts and heats, transforming gravitational energy into kinetic (or thermal) energy, the burning of the helium ashes begins. For stars to burn heavier elements, higher temperatures are necessary to overcome the increasing Coulomb repulsion and allow fusion through quantum-mechanical tunneling. Thermonuclear burning continues until the formation of an iron core. Once iron the most stable of nuclei is reached, fusion becomes an endothermic process.

However,combustion to iron is only possible for the most massive of stars. When thermonuclear fusion can no longer support the star against gravitational collapse, either because they are not massive enough (like our Sun) or because they have developed an iron core, the star dies and a compact object is ultimately formed.

The three final possible stages a star can take is a white dwarf, a neutron star, or a black hole. we focus exclusively on the physics of white dwarf stars. The fascinating topic of neutron stars requires the use of general relativity and is therefore reserved for a more advanced forthcoming publication. Our Sun will die as a white dwarf star once all of the hydrogen and helium in the core has been burned.

Towards the final stages of burning, the star will expand and expel most of the outer matter to create a platary nebula. At the beginning, the non-degenerate core contracts and heats up through conversion of gravitational energy into thermal kinetic energy.However, at some point the Fermi pressure of the degenerate electrons begins to dominate, the contraction is slowed up, and the core becomes a compact object known as a white dwarf, cooling steadily towards the ultimate cold, dark, static black dwarf state. On the other hand, neutron stars result from one of the most cataclysmic events in the universe, the death of a star with a mass much greater than that of our Sun[16].

Electrons in these stars behave ultra-relativistically, and as pointed out by Chandrasekhar, hydrostatic equilibrium as a cold body becomes impossible to achieve when $M > M_{ch}$. However, during the collapse of the core, a supernovae shock develops ejecting most of the mass of the star into the interstellar space and leaving behind an extremely dense core the neutron star.[14]

As the star collapses, it becomes energetically favorable for electrons to be captured by protons, making neutrons and neutrinos. The neutrinos carry away 99% of the gravitational binding energy of the compact object, leaving neutrons behind to support the star against further collapse.

The pressure provided by the degenerate neutrons, like degenerate electron pressure for white dwarf stars, has a limit on the mass it can bear. Beyond this limiting mass, no source of pressure exists that can prevent gravitational contraction. If such is the case, then the star will continue to collapse into an object of zero radius: a black hole.

There is a large number of excellent textbooks on the birth, life, and death of stars. The following are some references used in this work [13], [14], [15], [16], [17], [18]

1.1 Background of Study

To begin our examination, we must discuss one of the most fundamental principles in quantum mechanics. The Pauli Exclusion Principle, was formulated by Wolfgang Pauli in 1925 while he was considering Bohr's atomic model, and in order to attempt to explain the results of his experiments on the Zeeman Effect in atomic spectroscopy. This principle states that no two identical fermions (spin 1/2 particles) can occupy the same quantum state at the same time. The application of this principle to atoms, in the case of Neils Bohr, has allowed physicists to better explain how electrons fill orbitals around a nucleus of an atom, satisfying the different selection rules. This paved the way for physicists to delve deeper into the study of the atom and examine other interactions such as the spin-spin interaction of two electrons or the spin-orbit interaction. Solving these problems gave atomic physicists a closer approximation of the energy, for example, of non-Hydrogen like atoms. This idea of energy state filling becomes crucial when discussing degeneracy pressure.[9],[10]

The Pauli Exclusion Principle can also be applied to further our understanding of stars, black holes, supernovae, and many other celestial objects and phenomena studied in astrophysics. A vital application is the degeneracy pressure found in the interior of a white dwarf or neutron star. The degeneracy pressure is greatly needed in the interior of these stars to counteract the pressure due to gravity. The balance of the gravitational pressure and degeneracy pressure is what keeps these star stable. Without the degeneracy pressure in neutron stars and white dwarf stars, the pull of gravity would become too great for the thermal pressure to counteract, forcing the star to implode, which could possibly result in the formation of a black hole.[12]

Let us briefly look at some properties of white dwarf and neutron stars. The two types of stars mentioned above are directly dependent on the degeneracy pressure because they are stars that no longer fuse elements, and thus do not burn as hot as larger, younger stars. This results in cooler temperatures at the core, causing the thermal pressure to decrease. The white dwarf star is a small, dense star with a mass less than the Chandradekhar mass limit of 1.44 solar masses. Many white dwarf stars are born out of red giant stars that have burned their fuel, and lost a large fraction of their mass. These stars have fused all of their hydrogen and helium, leaving an abundance of carbon and, in many cases, oxygen in the core.[13],[14]

At this point, the star no longer has enough energy to fuse carbon into a heavier element so the white dwarf star contracts. This contraction compresses the electrons in the core into degenerate energy levels, forming the electron degeneracy pressure. A typical white dwarf with a mass less than 1.44 solar masses will become stable after balancing the gravitational pressure due to the contraction, and the electron degeneracy pressure more massive stars, typically greater than eight times the mass of our sun, however, are energetic enough to fuse elements up to iron.[15]

Fusion can no longer naturally occur in stars past iron, however, because it becomes an endothermic process, thus requiring energy. When iron is formed, it is deposited in the core, causing the density to rapidly increase and the core to begin to contract inward. This causes temperatures to rise in the core to help resist collapse. The rise in temperature and density allows for electron capture in the core by the reaction[16]

$$p^+ + e^- \rightarrow n + v_e \tag{1.1}$$

Both neutrinos and neutron rich matter are produced at the core of these large stars. Eventually, the core of these larger stars will become too massive, causing a gravitational core collapse supernova which, in many cases, leave behind a neutron star. These neutron stars are neutron rich due to reaction (1), and can weigh up to three solar masses. Neutron stars are much denser than white dwarf stars, which, once again, causes the core of the stars to collapse. The compression of neutrons in the contracting core, however, creates a neutron degeneracy pressure. This pressure, analogous to the electron degeneracy pressure in white dwarf stars, combats the gravitational collapse of the star. If, however, the neutron star is too massive (more than three solar masses), the neutron degeneracy pressure fails and the neutron star collapses into a black hole. We now see that the role of both the neutron degeneracy pressure, and the electron degeneracy pressure is crucial to the maintained stability of a star. We must now ask what neutron or electron degeneracy is, and how it forms a pressure in a star. Since these pressures are analogous, we will examine neutron degeneracy pressure in a neutron star's core. We have discussed earlier that the Pauli Exclusion Principle prevents two identical fermions from occupying the same state at the same time. Also, we know that when all of the lowest energy states of the neutrons in a neutron star are filled, the neutron star is at its lowest possible total energy. When a star contracts, the free neutrons get pushed closer together, and thus, by the Pauli Exclusion Principle, cannot all remain in their lowest energy states. This, therefore, forces the neutrons to occupy increasingly higher energy states, which, in turn, creates a pressure.

In one application, Heisenberg uncertainty principle asserts that complementary properties of a system, such as position and momentum, cannot be determined simultaneously, or mathematically, that $\Delta x \Delta p_x \geq \hbar/2$. Therefore, we see that there is a momentum, called the Fermi momentum, associated with the neutrons. As the core collapses and the energy states fill to higher levels, the Fermi momentum of the neutrons increases. The energy corresponding to this momentum, called the Fermi energy which we will discuss in the next section will also increase. So, with a decreasing volume and an increasing particle momentum, we can say that a pressure formed inside of the core of the star, and continues to increase as long as the volume continues to increase, and that there are degenerate neutron energy states. Now that we know where the pressure comes from, we can finally derive a mathematical expression for the neutron degeneracy pressure by non-relativistic neutrons inside of a neutron star.[9],[10],[11]

1.2 Objectives of the Study

The aim is to establish equations of state for white dwarfs and neutron stars for computing mass–radius relations as well as corresponding maximum masses. First, white dwarfs are described by a Fermi gas model of degenerate electrons and neutrons and effects from general relativity are examined. For neutron star matter, the influences of a finite fraction of protons and electrons and of strong nucleon–nucleon interactions are studied. Finally, masses and radii of neutron stars are computed for given central pressure.

The thesis is organized as follows. In Chap. 2 we present the non relativistic degenerate electron gas general formalism and deduce the equation governing the gravitational equilibrium. Further we show that, in the limit of weak gravitational fields, we recover the Fermi gas in Newtonian gravity. In Chap. 3 we focus our attention on the relativistic degenerate electron gas fermionic configurations, by studying the gravitational equlibrium and the dynamical stability of such configurations. In Chap. 4 we consider the white dwarf and neutron stars limit of Fermi-Dirac statistics (i.e. Botzmann statistics), which allows us to extend to GR the Newtonian models considered in 4.2 and complete the investigations discussed in Sec.4.3 . In Chap. 5, finally, conclusion

Chapter 2

Non Relativistic Degenerate Electron Gas

2.1 Non Relativistic Degenerate Electron Gas at $T = 0$

At the beginning, the non-degenerate core contracts and heats up through conversion of gravitational energy into thermal kinetic energy.however, at some point the Fermi pressure of the degenerate electrons begins to dominate, the contraction is slowed up, and the core becomes a compact object known as a white dwarf, cooling steadily towards the ultimate cold, dark, static black dwarf state Properties of Fermi gas are different from those of Bose gas because exclusion principle prevents multi-occupancy of quantum states.

In an ordinary fermion gas in which thermal effects dominate, most of the available electron energy levels are unfilled and the electrons are free to move to these states. As particle density is increased, electrons progressively fill the lower energy states and additional electrons are forced to occupy states of higher energy even at low temperatures. Degenerate gases strongly resist further compression because the electrons cannot move to already filled lower energy levels due to the Pauli exclusion principle. Since electrons cannot give up energy by moving to lower energy states, no thermal energy can be extracted.

At $T = 0$ we can deduce P using

$$E = \frac{\hbar^2 k^2}{2m}, non - relativistic, P = \left(\frac{\partial U}{\partial V}\right)_N \tag{2.1}$$

which is exact at $T = 0$. So it is just a matter of deducing the average internal energy of the Fermi gas at finite T

$$U = \int \frac{d^3k}{(\pi/L)^3} \frac{2E}{exp\beta(E - E_F)} \tag{2.2}$$

Where U -average internal energy

where $P = \hbar k$ is momentum

Where $\rho = \frac{d^3k}{(\pi/L)^3}$ is density of states and $\beta = 1/k_BT$. The factor $(\pi/L)^3$ is the volume of a single particle state in k-space and the factor 2 is to account for the spin in the case of an electron gas.L is length of cube at $T = 0$ the Fermi-Dirac distribution becomes a Heaviside Theta function, so that

$$U = \int_0^{k_F} \frac{4\pi L^3 k^2}{8\pi^3} 2E dk \tag{2.3}$$

Where U-average internal energy The Fermi wavenumber can itself be established in terms of the number of particles, N :

$$N = \int_0^{k_F} \frac{4\pi L^3 k^2}{8\pi^3} 2dk = \frac{V k_F^3}{3\pi^2} \tag{2.4}$$

Where N-number of particles with $V = L^3$.

we have

$$U = \int_0^{k_F} \frac{V\hbar c}{\pi^2} k^2 \sqrt{k^2 + m^2c^2/\hbar^2} dk - - - nonrelativistic \tag{2.5}$$

Let $y = \hbar k/(mc)$ and $x = \hbar k_F/(mc)$ then we can write

$$U = V \frac{mc^2}{\pi^2} \left(\frac{mc}{\hbar}\right)^3 \int_0^x y\sqrt{y^2 + 1} dy - - - relativistic \tag{2.6}$$

The integral gives

$$\int_0^x y\sqrt{y^2+1}dy = \frac{1}{8}[(x+2x^2)\sqrt{1+x^2} - sinh^{-1}x] \tag{2.7}$$

Noting that $sinh^{-1}x = ln(x+\sqrt{1+x^2})$ we can then write our final expression for U :

$$U = V\frac{mc^2}{8\pi^2}\left(\frac{mc}{\hbar}\right)^3 \left[(x+2x^2)\sqrt{1+x^2} - ln(x+\sqrt{1+x^2})\right] \tag{2.8}$$

Now we need to take the derivative with respect to V at fixed N . We will do this by differentiating the integral directly, *i.e.*

$$U = -\left(\frac{U}{V} + \frac{\partial x}{\partial V}\right)_N V\frac{mc^2}{\pi^2}\left(\frac{mc}{\hbar}\right)^3 x^2\sqrt{x^2+1} \tag{2.9}$$

Putting $(\frac{\partial x}{\partial V})_N = -x/(3V)$ gives

$$\begin{aligned}
P &= -\frac{mc^2}{8\pi^2}\left(\frac{mc}{\hbar}\right)^3 \left[(x+2x^2)\sqrt{1+x^2} - ln(x+\sqrt{1+x^2}) - \frac{1}{3}\frac{mc^2}{\pi^2}\left(\frac{mc}{\hbar}\right)^3 x^2\sqrt{x^2+1}\right] \\
&= -\frac{mc^2}{8\pi^2}\left(\frac{mc}{\hbar}\right)^3 \left[(x+2x^2)\sqrt{1+x^2} - ln(x+\sqrt{1+x^2}) - \frac{8}{3}x^2\sqrt{x^2+1}\right] \\
&= -\frac{mc^2}{8\pi^2}\left(\frac{mc}{\hbar}\right)^3 \left[x\left(\frac{2}{3}x^2-1\right)\sqrt{1+x^2} + ln\left(x+\sqrt{1+x^2}\right)\right]
\end{aligned}$$

We can write this so that we can see explicitly the deviation from the ultra-relativistic limit, *i.e.* we write

$$P = Cn^{4/3}I(x), \tag{2.10}$$

where

$$C = \frac{\pi\hbar c}{3}\left(\frac{3}{8\pi}\right)^3 \tag{2.11}$$

and (using $n = k_F^3/(3\pi^2) = (mcx/\hbar)^3/(3\pi^2)$)

$$I(x) = \frac{3}{2x^4}\left[x\left(\frac{2}{3}x^2-1\right)\sqrt{1+x^2} + ln\left(x+\sqrt{1+x^2}\right)\right]$$

the function $I(x) \to 1$ as $x \to \infty$, which is the ultra-relativistic limit. And $I(x) \to 0, 4x/5, as, x \to 0$, which gives the non-relativistic result, *i.e.* putting $x = (3\pi^2 n)^{1/3}(\hbar/mc)$ gives

$$P = C_{NR} n^{5/3} \tag{2.12}$$

with

$$C_{NR} = \frac{4}{5} C (3\pi^2)^{1/3} \frac{\hbar}{mc} = \frac{\pi\hbar^2}{mc} (9\pi)^{1/3} \tag{2.13}$$

2.2 Non Relativistic Degenerate Electron Gas T≠0

So far we have been studying the quantum mechanical ground state of the N -fermion system. The particles occupy the lowest energy states available. Those states with energy below E_F have unit probability to be occupied, those with energies above E_F have unit probability to be empty. Now we consider what happens when such a system is brought into thermal equilibrium at a temperature T . These fermions can exchange energy with the heat bath. It is most convenient to describe the thermodynamics of this system in a framework where the number of particles in the system is regarded as variable and is known only as a thermal, or "ensemble" average.

Properties of Fermi gas are different from those of Bose gas because exclusion principle prevents multi-occupancy of quantum states. As a result - no condensation at ground state occurs at low temperatures For macroscopic system chemical potential can be found at all temperatures using

$$N = \int_0^\infty d\epsilon \rho(\epsilon) f(\epsilon) = \int_0^\infty d\epsilon \frac{\rho(\epsilon)}{e^{\beta(\epsilon-\mu)} + 1} \tag{2.14}$$

Where N-fermion system

This is a nonlinear equation for μ that in general can be solved only numerically
In limit $T \to 0$ fermions fill certain number of low-lying energy levels to minimize total energy while obeying exclusion principle Chemical potential of fermions is positive at low temperatures ($\mu > 0$)

For $T \to 0$ ($i.e., \beta \to \infty$) it follows that

$$\begin{cases} e^{\beta(\epsilon-\mu)} & \to 0 \text{ if } \epsilon < \mu \\ e^{\beta(\epsilon-\mu)} & \to \infty \text{ if } \ \epsilon > \mu \end{cases} \tag{2.15}$$

yielding

$$\begin{cases} 1, & \epsilon < \mu \\ 0, & \epsilon > \mu \end{cases} \tag{2.16}$$

2.3 Fermi Temperature

Zero-temperature value μ_0 defined

$$N = \int_0^{\mu_0} d\epsilon \rho(\epsilon) \tag{2.17}$$

Whereρ-density of states

Fermions are mainly electrons having spin $\frac{1}{2}$ and correspondingly degeneracy because of two states of spin.

In three dimensions

$$N = \frac{2V}{(2\pi)^2}\left(\frac{2m}{\hbar}\right)^{3/2}\int_0^{\mu_0} d\epsilon\sqrt{\epsilon} = \frac{2V}{(2\pi)^2}\left(\frac{2m}{\hbar}\right)^{3/2}\frac{2}{3}\mu_0^{3/2} \tag{2.18}$$

It follows that

$$\mu_0 = \frac{\hbar^2}{2m}\left(3\pi^2 n\right)^{2/3} = \epsilon_F \tag{2.19}$$

$\epsilon_F \to$ Fermi energy

Convenient to introduce Fermi temperature

$$k_B T_F = \epsilon_F \tag{2.20}$$

In typical metals $T_F \approx 10^5 K$ so that at room temperatures $\rightarrow T \ll T_F$ and electron gas is degenerate

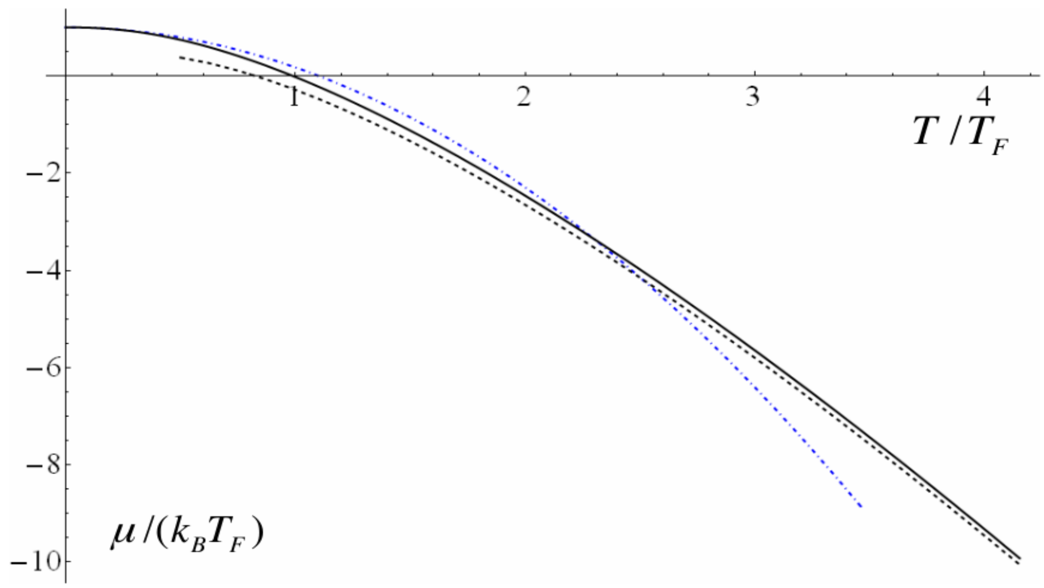

Figure 2.1: Chemical potential μ(T) of the Fermi-Dirac gas. Dashed line: High-temperature asymptote corresponding to the Boltzmann statistics, Dashed-dotted line: Low-temperature asymptote

Dashed line: High-temperature asymptote corresponding to Boltzmann statistics.

Dashed-dotted line: line:Low-temperature statistics.

2.4 Internal Energy and Pressure

Convenient to express density of states in terms of ϵ_F

$$\rho(F) = \frac{3}{2} N \frac{\sqrt{\epsilon}}{\epsilon_F^{3/2}} \tag{2.21}$$

Internal energy at $T = 0$

$$U = \int_0^{\mu_0} d\epsilon \rho(\epsilon)\epsilon = \frac{3}{2}\frac{N}{\epsilon_F^{3/2}} \int_0^{\epsilon_F} d\epsilon \epsilon^{3/2} = \frac{3}{2}\frac{N}{\epsilon_F^{3/2}} \frac{2}{5}\epsilon_F^{5/2} = \frac{3}{5} N\epsilon_F \tag{2.22}$$

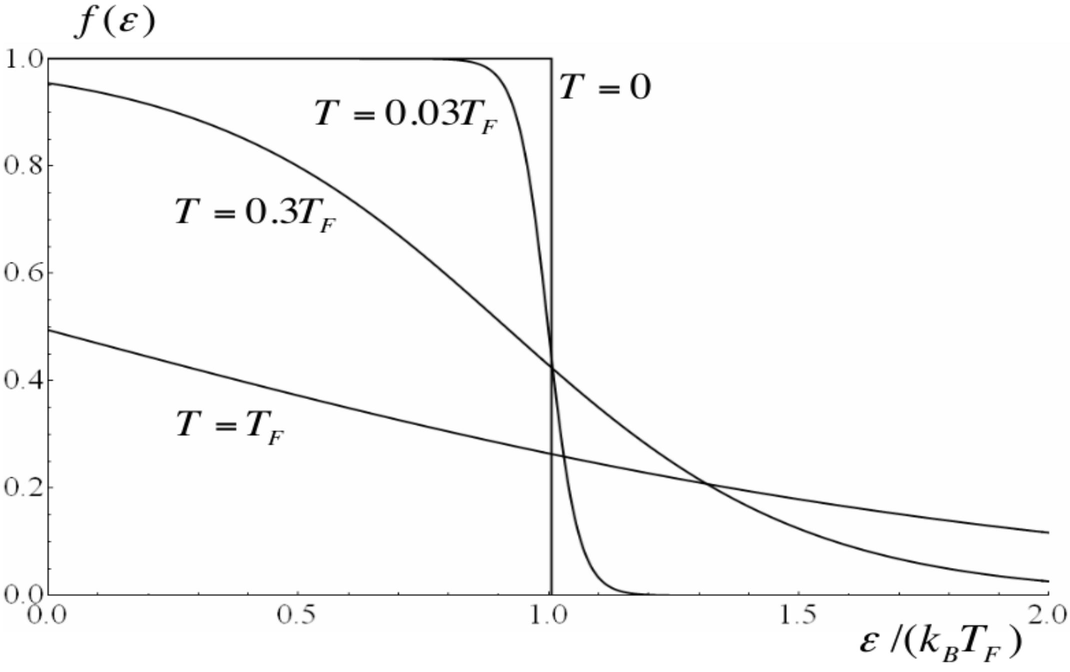

Figure 2.2: In the limit $T \to 0$ fermions fill the certain number of low-lying energy levels to minimize the total energy while obeying the exclusion principle. As we will see, the chemical potential of fermions is positive at low temperatures,$\mu > 0$.
For $T \to 0$ (i. e.,$\beta \to \infty$) one has $e^{\beta(\epsilon-\mu)} \to 0$ if $\epsilon < \mu$ and $e^{\beta(\epsilon-\mu)} \to \infty$ if $\epsilon > \mu$.
Chemical potential Fermi-Dirac Distribution Function

We cannot calculate heat capacity C_V from next chapter as it requires taking into account small temperature- dependent corrections in U.

We can calculate pressure at low temperatures since S should be small and

$$F = U - TS \approxeq U \tag{2.23}$$

Where F-free energy

$$\begin{aligned} P &= -\left(\frac{\partial F}{\partial V}\right)_{T=0} \approxeq -\left(\frac{\partial U}{\partial V}\right)_{T=0} = -\frac{3}{5}N\frac{\partial \epsilon_F}{\partial V} \\ &= -\frac{3}{5}N\left(-\frac{2}{3}\frac{\epsilon_F}{V}\right) = \frac{2}{5}n\epsilon_F = \frac{\hbar}{2m}\frac{2}{5}(3\pi^2)^{2/3}n^{5/3} \end{aligned}$$

we will need integral of a general type

$$M_\eta = \int_0^\infty d\epsilon \epsilon^\eta f(\epsilon) = \int_0^\infty d\epsilon \frac{\epsilon^\eta}{e^{(\epsilon-\mu)/(k_BT)}+1} \tag{2.24}$$

From (2.20) it follows that

$$N = \frac{3}{2}\frac{N}{\epsilon_F^{3/2}} M_{1/2} \tag{2.25}$$

$$U = \frac{3}{2}\frac{N}{\epsilon_F^{3/2}} M_{3/2} \tag{2.26}$$

It is easily seen that for $k_BT \ll \mu$ expansion of M_η up to quadratic terms has form

$$M_\eta = \frac{\epsilon^{\eta+1}}{\eta+1}\left[1+\frac{\pi^2\eta(\eta+1)}{6}\left(\frac{k_BT}{\mu}\right)^2\right] \tag{2.27}$$

2.5 Chemical Potential

Eq.(2.25) becomes

$$\epsilon_F^{3/2} = \mu^{3/2}\left[1+\frac{\pi^2}{8}\left(\frac{k_BT}{\mu}\right)^2\right] \tag{2.28}$$

that defines $\mu(T)$ up to terms of order T^2

$$\mu = \epsilon_F\left[1+\frac{\pi^2}{8}\left(\frac{k_BT}{\mu}\right)^2\right]^{-3/2} \approxeq \epsilon_F\left[1-\frac{\pi^2}{12}\left(\frac{k_BT}{\mu}\right)^2\right] \approxeq \epsilon_F\left[1-\frac{\pi^2}{12}\left(\frac{k_BT}{\epsilon_F}\right)^2\right] \tag{2.29}$$

or using (2.20)

$$\mu = \epsilon_F\left[1-\frac{\pi^2}{12}\left(\frac{T}{T_F}\right)^2\right] \tag{2.30}$$

It is not surprising that chemical potential decreases with temperature because at high temperatures it takes large negative values (2.27) becomes

$$U = \frac{3}{2}\frac{N}{\epsilon_F^{3/2}}\frac{\mu^{5/2}}{(5/2)}\left[1 + \frac{5\pi^2}{8}\left(\frac{k_B T}{\mu}\right)^2\right] \approxeq \frac{3}{5}N\frac{\mu^{5/2}}{\epsilon_F^{3/2}}\left[1 + \frac{5\pi^2}{8}\left(\frac{k_B T}{\mu}\right)^2\right] \tag{2.31}$$

Using Eq.(2.31)

$$\begin{aligned} U =& \frac{3}{5}N\epsilon_F\left[1 - \frac{\pi^2}{12}\left(\frac{T}{T_F}\right)^2\right]^{5/2}\left[1 + \frac{5\pi^2}{8}\left(\frac{T}{T_F}\right)^2\right] \\ \approxeq& \frac{3}{5}N\epsilon_F\left[1 - \frac{5\pi^2}{24}\left(\frac{T}{T_F}\right)^2\right]\left[1 + \frac{5\pi^2}{8}\left(\frac{T}{T_F}\right)^2\right] \end{aligned}$$

that yields

$$U = \frac{3}{5}N\epsilon_F\left[1 + \frac{5\pi^2}{12}\left(\frac{T}{T_F}\right)^2\right] \tag{2.32}$$

At $T = 0$ this formula reduces to (2.22) Heat capacity:

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V = Nk_B T\frac{\pi^2}{2}\frac{T}{T_F}$$

is small at $T \ll T_F$

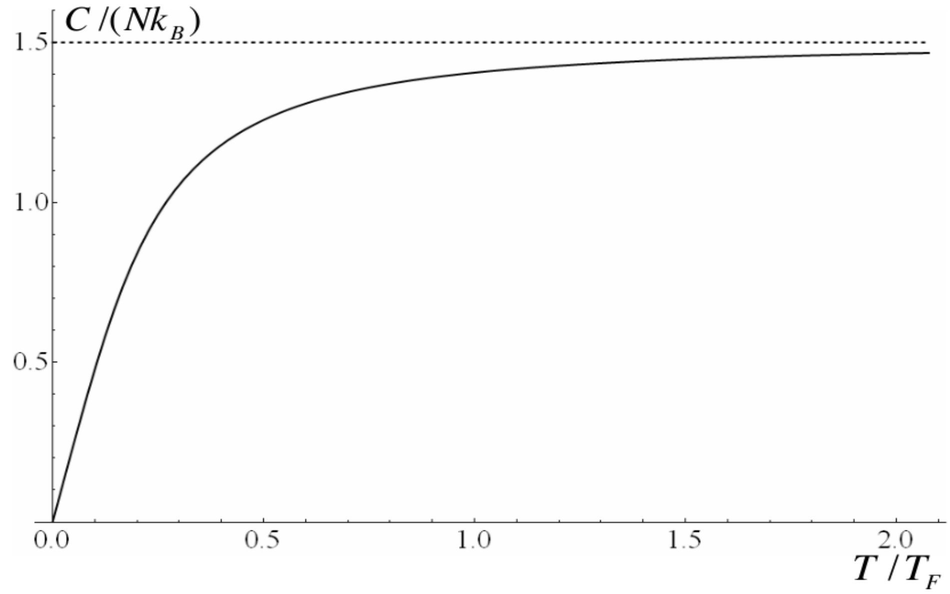

Figure 2.3: Heat capacity of Ideal Fermi-Dirac gas

Chapter 3

Relativistic Degenerate Electron Gas

3.1 Physics of neutron stars

The direct Durca process:In this section we describe the decay processes that play a vital role in cooling of a neutron star. As introduced in the first section, in cores heavier than the Chandrasekhar limit (typically 1.4M), electron degeneracy pressure is generated as in a white dwarf, but the electrons will become ultra-relativistic. Ultra-relativistic electrons generate a pressure of the same footing and scaling as the gravitational pressure when the star collapses. Therefore they fail to achieve any equilibrium marking the formation of a neutron star. The core still continues to collapse beyond the white dwarf, and thus the release of gravitational potential energy allows the matter within to heat up. This makes available some free energy enough for the following inverse beta decay reaction to occur:

$$p^+ + e^- + 1.36MeV \rightarrow n + \bar{v}_e \tag{3.1}$$

The energy trapped in a neutron star is given away as neutrino radiation. Three essential processes are useful in the creation of neutrinos in a neutron star:

- the direct Urca (Durca) process,
- the modified Urca (Murca) process, and available electron states have now been filled by

the degenerate electron (Fermi) gas in the star hindering the formation of any electron with energy less than 1.36 MeV. Neutrons, protons and electrons are all fermions. Thus inside an atomic nucleus exists such an environment of quantized energy states. In these nuclei there are zero unoccupied low energy states for protons created in beta decay of neutrons. This makes the neutrons stable. With further increase in energy inverse beta decay continues to take place within remaining atomic nuclei. A peak with iron is reached

- neutrino bremsstrahlung

Neutron decay and its inverse comprise the direct Urca process. Normally, the neutrons formed from the above process, being more massive than protons would, within their half life of about 10 minutes, undergo the process of beta decay by releasing electrons and protons. But all the

$$Fe^{56} + 3.7MeV \rightarrow Mn^{56} + \bar{v}_e \tag{3.2}$$

This way a huge number of neutrons are generated in the core.
The flux outflow of neutrinos is sufficient enough and the surrounding material is sufficiently dense so that the neutrino pressure slows down the collapse of the star from a free-fall time.

The Durca process, a fast neutrino process, is effective in the extremely dense cores of the most massive neutron stars (2.5 M) where proton density is slightly higher than neutron density. But in the outer layers of a massive neutron star or in the cores of lighter neutron stars (1.5 M) slow neutrino processes occur, helping in cooling the neutron stars. The slow processes are modified Urca and neutrino bremsstrahlung. The modified Urca process is a neutron (n) turning into a proton (p) and this again becoming a neutron, the process being catalysed by a nucleon (N). The presence of this extra nucleon facilitates the process by helping conserve momentum and energy which would be violated in the Durca process.

$$n + N \rightarrow p + N + e + v_e \tag{3.3}$$

$$p + N + e \rightarrow n + N + v_e \tag{3.4}$$

Just like photon bremsstrahlung in which an electron emits light as it passes by an atomic nucleus, in neutrino bremsstrahlung two nucleons collide with each other to create an electron neutrino anti-neutrino pair.

$$N + N \rightarrow N + N + v_e + \bar{v}_e \tag{3.5}$$

The temperature inside a neutron star is uniform as the matter inside, composed of degenerate fermi gases is extremely good conducting. Whereas in the outer layer temperature is radially varying as it is composed of atomic nuclei and thus is insulating. It is of much less temperature (few million K degrees) as compared to the core (a billion degrees). So by generating neutrinos a neutron star cools faster than by emitting X-rays from its photosphere. Radiative cooling dominates over neutrino cooling only when the neutron star is cold. The direct Urca process plays a bigger role than the two other processes in cooling of a neutron star. If allowed it can be relatively very fast when the proton fraction runs past a critical value x_{DU} [13] given by

$$x_{DU} = \frac{1}{1 + (1 + {x_e}^1/3)} \tag{3.6}$$

where x_e is the electron leptonic fraction $n_e/(n_e+n_\mu$. Here n e is the electron number density andn_μ is the muon number density.

3.2 Relativistic Degenerate Fermi Gas

We will compute the $T = 0$ approximation for the pressure P using correct relativistic formula,

$$E^2 = p^2c^2 + m_0^2c^4 \tag{3.7}$$

At very high density, degenerate fermions become so energetic that they no longer obey $E = p^2/2m$. Relativistic energy $E = \sqrt{p^2c^2 + m_0^2c^4}$, when $m_0c^2 << pc$, then $E = pc$ is the

correct expression for the energy. Such a gas is called a "relativistic Fermi gas", and the pressure takes a different form than for a the non- relativistic Fermi gas that we have already studied in detail. The relativistic Fermi gas is relevant to white dwarfs and neutron stars. In this problem you will study such a gas. Assume that electrons are providing the pressure.
(a) Assuming $E = pc$, obtain the Fermi energy in terms of the electron number density n_e. The Fermi momentum is as before:

$$p_f = \left(3\pi^2\right)^{1/3} \hbar n_e^{1/3}$$

The Fermi energy is:

$$E_f = p_f c = \left(3\pi^2\right)^{1/3} \hbar c n_e^{1/3}$$

(b) Beginning with the appropriate phase space integral, calculate the average particle energy. Proceed as we did in class, without introducing the confusing density of states. Because the system is degenerate, the distribution function is unity up to the Fermi momentum, and then drops quickly to zero. The total energy of the system is

$$E = \frac{2}{\hbar^3} \int d^3r \int d^3p pc \tag{3.8}$$

where the factor of two comes from the two possible spin states of an electron. The integral is only up to the Fermi surface of radius p_f , so

$$E = \frac{8\pi V c}{\hbar^3} \int_0^{p_f} dp^3 = \frac{\left(3\pi^2\right)^{4/3}}{4\pi^2} V \hbar c n_e^{4/3} \tag{3.9}$$

where n_e is the number density of electrons. The average energy per particle is

$$\overline{E} = \frac{E}{N} = \frac{\left(3\pi^2\right)^{4/3}}{4\pi^2 N} V \hbar c n_e^{4/3} \tag{3.10}$$

where $n_e = N/V$ was used.

(c) Use thermodynamics to get the pressure in terms of the electron number density.

Let's rewrite the energy as

$$E = \frac{(3\pi^2)^{4/3}}{4\pi^2} V\hbar c \left(\frac{N}{V}\right)^{4/3} \tag{3.11}$$

The pressure is

$$P = -\left(\frac{\partial E}{\partial N}\right)_N = \frac{(3\pi^2)^{4/3}}{12\pi^2} \hbar c n_e^{4/3} \tag{3.12}$$

We can account for composition by writing

$$n_e = \frac{\rho}{\mu_e M} \tag{3.13}$$

where ρ is the mass density, $\mu_e = A/Z$ is the molecular weight per electron, and M is the mass of a nucleon. Hence, the pressure is

$$P = \frac{(3\pi^2)^{4/3}}{12\pi^2} \hbar c \left(\frac{\rho}{\mu_e M}\right)^{4/3} \tag{3.14}$$

(d) Assume we have iron with free electrons. Calculate the density, in gcm^{-3} at which relativistic pressure begins to dominate non-relativistic pressure. In reality, there is a continuous transition from one to the other; equating the two limits provides a good estimate for the transition density to a relativistic gas.

For non-relativistic electron degeneracy pressure, we found:

$$P_{nr} = \frac{1}{5m_e} \left(3\pi^2\right)^{2/3} \hbar^2 \left(\frac{\rho}{\mu_e M}\right)^{5/3} \tag{3.15}$$

Equating to the relativistic pressure found above gives the density at which the system begins to make the transition from non-relativistic to relativistic:

$$\rho = \mu_e M \left(3\pi^2\right)^2 \left(\frac{5}{12\pi^2}\right)^3 \hbar^2 \left(\frac{m_e c}{\hbar}\right)^3 \tag{3.16}$$

For iron, $\mu_e = 56/26$, giving a transition density of

$$\rho = 4 \times 10^6 \ g \ cm^{-3}.$$

This density is approximately that of a white dwarf.

(e) Calculate the Fermi energy in MeV using the relativistic expression. Estimate the temperature below which the gas is degenerate.

The Fermi energy at the above density is

$$E_f = \left(3\pi^2\right)^{1/3} \hbar c \left(\frac{\rho}{\mu_e M}\right)^{1/3} = 0.6MeV$$

comparable to the rest mass of the electron, because the electrons are starting to become relativistic. At higher densities, such as in a neutron star, the electrons become fully relativistic.

The requirement that the gas be degenerate is

$$E_f \gg kT$$

$0.6MeV$ is $7 \times 10^9 K$. The electrons will be degenerate for $T \leq 10K$. Such temperatures are found only in supernovae. The assumption from the beginning that the electrons are degenerate was thus a very good assumption.

3.3 Equation of state of the relativistic free electron gas at arbitrary degenerate

We consider a free electron gas in thermodynamic equilibrium at temperature T in the volume V.

The grand potential of this system reads[12, 13]

$$\Omega = -\frac{2V}{\beta \int \frac{dp}{(2\pi\hbar)^3}} \ln[1 + e^{-\beta(\epsilon-\mu)}] \tag{3.17}$$

where $\beta = 1/k_B T$ and $\hbar$ is the reduced Planck constant. Taking the electron rest-mass energy mc^2 as the reference of energies,

$$\epsilon = \sqrt{p^2c^2 + m^2c^4} - mc^2 \tag{3.18}$$

With this convention, the chemical potential μ does not contain the electron rest mass energy and $\epsilon = 0$ when $p = 0$. In the non-relativistic regime, $\epsilon \approx p^2/2m$ and in the ultra-relativistic regime, $\epsilon \approx pc$. The question is to see what happens in the intermediate regime and to study the transition between the non-relativistic and the ultra-relativistic regimes. From the grand potential ω, one has access to the electron number N , the kinetic energy U , and the pressure. One has the following identities

$$N = -\frac{\partial \Omega}{\partial \mu}|_{V,T} \tag{3.19}$$

$$U = \frac{\partial(\beta\Omega)}{\partial\beta}|_{V,\mu+\mu N}, \tag{3.20}$$

$$and \tag{3.21}$$

$$\Omega = -PV \tag{3.22}$$

To calculate N and U , one uses Eq. (18). One finds that

$$c^2 p dp = (\epsilon + mc^2) d\epsilon \tag{3.23}$$

and

$$pc = \sqrt{\epsilon}\sqrt{\epsilon + 2mc^2} \tag{3.24}$$

For N , one finds that

$$N = \frac{\sqrt{2}}{V} m^3 c^3 \theta^3 / 2\pi^2 \hbar^3 [F_1/2(\eta, \theta) + \theta F_3/2(\eta, \theta)] \tag{3.25}$$

where$F_{k(\eta,\theta)}$

$$F_K(\eta, \theta) = \int_0^{+\infty} d_x \frac{x^k \sqrt{1 + \theta_x}}{1 + e^{x-\eta}} \tag{3.26}$$

For U , one finds that

$$U = \frac{\sqrt{2} V m^4 c^5 \eta^5 / 2}{\pi^2 \hbar^2} [F_3/2(\eta, \theta) + \eta F_{5/2(\eta,\theta)}] \tag{3.27}$$

When θ is small, one recovers the non-relativistic expressions and when θ is large the ultra-relativistic expressions for N and U . The calculation of the pressure from Eqs. (26) and (27) is not so straightforward. Making the change of variable between p and ϵ, one arrives at

$$P = \frac{\sqrt{2} m^4 c^5 \eta^{5/2}}{\pi^2 \hbar^3} \left[\int_0^{+\infty} dx x^{1/2} \sqrt{1 + \frac{\eta_x}{2}} \ln(1 + e^{-x+\eta}) + \eta \int_0^{+\infty} dx x^3/2 \sqrt{1 + \frac{\eta_x}{2}} \ln(1 + e^{-x+\eta}) \right] \tag{3.28}$$

The idea is to do as in the non-relativistic and ultra-relativistic regimes, *i.e.*, an integration by parts by differentiating the logarithm term and integrating the factor in front of this logarithm term. How ever,finding a primitive $F^{int}_{3/2}(x, \theta)$ to $f^{int}_{3/2}(x, \eta)$ with respect to x where

$$f^{int}_k(x, \theta) = x^k \sqrt{1 + \frac{\theta_x}{2}} \tag{3.29}$$

is not so easy. 'int' is for integrand. Indeed, using Mathematical [29], one finds that

$$F^{int}_{1/2} = \int_0^x dy f^{int}_{1/2}(y, \eta) = \frac{\sqrt{x}\sqrt{1 + \frac{\eta_x}{2}}(1 + \eta_x)}{2\eta} - arcsinh \frac{\left(\sqrt{\frac{\eta_x}{2}}\right)}{\sqrt{2}\eta^3/2} \tag{3.30}$$

and

$$F_{3/2}^{int} = \int_0^x dy f_{3/2}^{int} = \frac{\sqrt{x}\sqrt{1+\frac{\theta x}{2}}(-3+\theta x)+2\eta^2 x^2}{6\theta^2} + \frac{arcsinh(\frac{\sqrt{\theta x}}{2})}{\sqrt{2}\theta^5/2} \tag{3.31}$$

ejecting these results in Eq. (30), one can see that the terms in arcsinh cancel each other and that the other ones combine to give

$$P = \frac{2\sqrt{2}m^4c^5\theta^5/2}{3\pi\hbar}[F_{3/2(\eta,\theta)} + \frac{\theta}{2}F_{5/2(\eta,\theta)}] \tag{3.32}$$

Our results are consistent with the ones obtained by Miralles and Van Riper [9]. From these results, we can get other thermodynamic functions. For instance, from Eqs. (25) and (31), one finds that the enthalpy $H = U + PV$ is given by

$$H = \frac{\sqrt{2}Vm^4c^5\theta^5/2}{\pi^2\hbar^3}[5/3F_{3/2(\eta,\theta)+4/3\theta F_{5/2}(\eta,\theta)}] \tag{3.33}$$

Now, using Eq. (21) and the thermodynamic identity $\Omega = F - \mu$N where F is the free energy, one finds that

$$F = \frac{\sqrt{2}Vm^4c^5\theta^5/2}{\pi^2\hbar^3}\left\{\eta\left[F_{1/2(\eta,\theta)} + \theta F_{3/2}(\eta,\theta) - 2/3F_{3/2}(\eta,\theta) + \frac{\theta}{2}F_{5/2}(\eta,\theta)\right]\right\} \tag{3.34}$$

It is possible to obtain more compact expressions by dividing by Nk_BT . From Eqs. (27), (28), (32), (33), (34), one finds that

$$\frac{U}{NK_BT} = \frac{F_{3/2}(\eta,\theta) + \theta F_{5/2}(\eta,\theta)}{F_{1/2}(\eta,\theta) + \theta F_{3/2}(\eta,\theta)} \tag{3.35}$$

$$\frac{PV}{NK_BT} = \frac{2/3F_{3/2(\eta,\theta)+\frac{\theta}{3}F_{5/2}(\eta,\theta)}}{F_{1/2}(\eta,\theta) + \theta F_{3/2}(\eta,\theta)} \tag{3.36}$$

$$\frac{H}{NK_BT} = \frac{5/3F_{3/2}(\eta,\theta) + 4/3\theta F_{5/2}(\eta,\theta)}{F_{1/2}(\eta,\theta) + \theta F_{3/2}(\eta,\theta)} \tag{3.37}$$

and

$$\frac{F}{NK_BT} = \eta - \frac{2/3F_{3/2}(\eta,\theta) + \frac{\theta}{3}F_{5/2}(\eta,\theta)}{F_{1/2}(\eta,\theta) + \theta F_{3/2}(\eta,\theta)} \tag{3.38}$$

Now, since $F = U - TS$, one finds also that

$$\frac{S}{NK_B} = \frac{-\eta + 5/3F_{3/2}(\eta,\theta) + 4/3\theta F_{5/3}(\eta,\theta)}{F_{1/2}(\eta,\theta) + \theta F_{3/2}(\eta,\theta)} \tag{3.39}$$

As expected,$S/Nk_B = -\eta + H/Nk_BT$. We have the principal dimensionless equation of state quantities expressed as universal functions of η and θ. Obviously,

$$\frac{PV}{U} = 2/3\frac{F_{3/2}(\eta,\theta) + \frac{\theta}{2}F_{5/2}(\eta,\theta)}{F_{3/2}(\eta,\theta) + \theta F_{5/2}(\eta,\theta)} \tag{3.40}$$

Let us now calculate the specific heat at constant volume and particle number c_v and the specific heat at constant pressure and particle number c_p. The method is very similar to the one employed in the non-relativistic and ultra-relativistic but the formulas obtained here are more cumbersome. Let us begin with c_v . By definition,

$$c_v = \frac{\partial U}{\partial T}|_{V,N} \tag{3.41}$$

Since U depends on η and θ, there is an implicit dependence with respect to temperature in η and explicit dependence in θ. This fact makes the calculations more complicated. $\frac{\partial \eta}{\partial T}|_{V,N}$ is found by differentiating the constant electron density $N_e = \frac{N}{V}$ obtained from Eq. (27). One finds that

$$G_o(\eta,\theta) = \frac{3}{2}\frac{F_{1/2}(\eta,\theta) + \theta F_{3/2}(\eta,\theta)}{\partial\eta F_{1/2}(\eta,\theta) + \theta\partial\eta F_{3/2}(\eta,\theta) + \theta\partial\theta F_{1/2}(\eta,\theta)} + \frac{\theta F_{3/2}(\eta,\theta) + \theta^2\partial\theta F_{3/2}(\eta,\theta)}{\partial\eta F_{1/2}(\eta,\theta) + \theta\partial\eta F_{3/2}(\eta,\theta)} \tag{3.42}$$

$$\partial_\eta F_k(\eta,\theta) \cong \frac{\partial F_K(\eta,\theta)}{\partial_\eta} = \int_0^{+\infty} \frac{dx x^k\sqrt{1+\frac{\theta_x}{2}}}{(1+e^{x-\eta})(1+e^{-x+\eta})} \tag{3.43}$$

and

$$\partial_\theta F_K(\eta,\theta) \cong \frac{\partial F_K(\eta,\theta)}{\partial\theta} = \int_0^{+\infty} \frac{dx x^k\sqrt{1+\frac{\theta_x}{2}}}{4\sqrt{1+\frac{\theta}{2}}}(1+e^{x-\eta}) \tag{3.44}$$

Knowing$\frac{\partial_\eta}{\partial T}|_{V,N}$ one can differentiate U with respect to T [9] leading for the specific heat at constant volume and particle number

$$\frac{C_V}{NK_B} = \frac{5/2F_{3/2}(\eta,\theta)+\theta\partial_\theta F_{3/2}(\eta,\theta)+7/2\theta F_{5/2}(\eta,\theta)+\theta^2\partial_\theta F_{5/2}}{F_{1/2}(\eta,\theta)+\theta F_{3/2}(\eta,\theta)}$$

$$-\Bigg[\frac{3/2F_{1/2}(\eta,\theta)+\theta\partial_\theta F_{1/2}(\eta,\theta)+5/2\theta F_{3/2}(\eta,\theta)+\theta^2\partial_\theta F_{3/2}(\eta,\theta)}{F_{1/2}(\eta,\theta)+\theta F_{3/2}(\eta,\theta)} \times\left(\frac{\partial_\eta F_{3/2}\eta,\theta+\theta\partial_\eta F_{5/2}(\eta,\theta)}{\partial_\eta F_{1/2}(\eta,\theta)+\theta\partial_\eta F_{3/2}(\eta,\theta)}\right)\Bigg]$$

$\frac{C_V}{NK_B}$ is thus a universal function of η and θ. This result is consistent with what was found by Miralles and Van Riper[9]. Let us now calculate C_P. By definition

$$C_P = \frac{\partial H}{\partial T}|_{P,N} \tag{3.45}$$

where the enthalpy H is given by Eq. (32). To obtain C_P one needs to know $\frac{\partial\eta}{\partial T}|_{P,N}$and $\frac{\partial V}{\partial T}|_{P,N}$. The first quantity is found from Eq. (31) by differentiating with respect to T the constant pressure P and the second quantity is found from Eq. (27) by differentiating with respect to T the constant particle number N and using the result previously obtained for $\frac{\partial\eta}{\partial T}|_{P,N}$. One finds respectively

$$\frac{\partial\eta}{\partial T}|_{P,N} = -\frac{1}{T}G_1(\eta,\theta) \tag{3.46}$$

where

$$G_1(\eta,\theta) = \frac{5}{2}\frac{F_{3/2}(\eta,\theta)+\frac{\theta}{2}F_{5/2}(\eta,\theta)}{2\partial_\eta F_{3/2}(\eta,\theta)+\frac{\theta}{2}\partial_\eta F_{5/2(\eta,\theta)}} + \frac{\theta\partial_\theta F_{3/2}(\eta,\theta)+\frac{\theta}{2}F_{5/2}(\eta,\theta)+\frac{\theta^2}{2}\partial_\theta F_{5/2}(\eta,\theta)}{\partial_\eta F3/2(\eta,\theta)+\frac{\theta}{2}\partial_\eta F_{5/2}(\eta,\theta)} \tag{3.47}$$

and

$$\frac{\partial V}{\partial T}|_{P,N} = -\frac{V}{T} G_2(\eta, \theta) \tag{3.48}$$

where

$$G_2(\eta, \theta) = 3/2 - G_1(\eta, \theta) \frac{\partial_\eta F_{1/2}(\eta, \theta) + \theta \partial_\eta F_{3/2}(\eta, \theta)}{F_{1/2}(\eta, \theta) + \theta F_{3/2}(\eta, \theta)} + \frac{\theta \partial_\theta F_{1/2}(\eta, \theta) + \theta F_{3/2}(\eta, \theta) + \theta^2 \partial_\theta F_{3/2}(\eta, \theta)}{F_{1/2}(\eta, \theta) + \theta F_{3/2}(\eta, \theta)} \tag{3.49}$$

Using these results when we differentiate the enthalpy (32) with respect to , one arrives at

$$\begin{aligned} \frac{C_p}{N k_B} = \left[\frac{5}{2} - G_2(\eta, \theta)\right] \frac{\frac{5}{2} F_{3/2}(\eta, \theta) + \frac{4}{3} \theta F_{5/2}(\eta, \theta)}{F_{1/2}(\eta, \theta) + \theta F_{3/2}(\eta, \theta)} \\ - G_1(\eta, \theta) \frac{\frac{5}{2} \partial_\eta F_{3/2}(\eta, \theta) + \frac{4}{3} \theta \partial_\eta F_{5/2}(\eta, \theta)}{F_{1/2}(\eta, \theta) + \theta F_{3/2}(\eta, \theta)} \\ + \frac{\frac{5}{2} \theta \partial_\theta F_{3/2}(\eta, \theta) + \frac{4}{3} \theta F_{5/2}(\eta, \theta) + \frac{4}{3} \theta^2 \partial_\theta F_{5/2}(\eta, \theta)}{F_{1/2}(\eta, \theta) + \theta F_{3/2}(\eta, \theta)} \end{aligned}$$

C_P/Nk_B is also a universal function of η and θ as well as the ratio C_P/C_V . Indeed, C_P and C_V are not independent. They satisfy the following thermodynamic identity[15]

$$C_p - C_v = T \frac{\partial P}{\partial T}|_{V,N} \frac{\partial V}{\partial T}|_{P,N} \tag{3.50}$$

The second term is known from Eq.(41). To calculate the first term, we differentiate Eq. (15) with respect to T at constant V and N while using Eq.(25). We find that

$$\frac{\partial P}{\partial T}|_{V,N} = N_e k_B G_3(\eta, \theta) \tag{3.51}$$

where

$$\begin{aligned} G_3(\eta, \theta) = \frac{2}{3} \left[\frac{5}{3} \frac{F_{3/2}(\eta, \theta) + \frac{1}{2} \theta F_{5/2}(\eta, \theta)}{F_{1/2}(\eta, \theta) + \theta F_{3/2}(\eta, \theta)} - G_0(\eta, \theta) \frac{\partial_\eta F_{3/2}(\eta, \theta) + \frac{\theta}{2} \partial_\eta F_{5/2}(\eta, \theta)}{F_{1/2}(\eta, \theta) + \theta F_{3/2}(\eta, \theta)} \right. \\ \left. + \frac{\theta \partial_\theta F_{3/2}(\eta, \theta) + \frac{\theta}{2} F_{5/2}(\eta, \theta) + \frac{\theta^2}{2} \partial_\theta F_{5/2}(\eta, \theta)}{F_{1/2}(\eta, \theta) + \theta F_{3/2}(\eta, \theta)} \right] \end{aligned}$$

Consequently,

$$\frac{C_p - C_v}{Nk_B} = -G_2(\eta,\theta)G_3(\eta,\theta) \tag{3.52}$$

It can be checked[14] that Eq. (39) is consistent with Eqs. (29) and (35). The general proof is quite involved. These universal functions are far more complicated than the ones obtained in the two limit non-relativistic and ultra-relativistic cases. In these cases, only appear the Fermi-Dirac integrals[16, 17] which are functions of η, i.e.,

$$I_n(\eta) = \int_0^{+\infty} dx \frac{x^n}{1+e^{x-\eta}} \tag{3.53}$$

with $I'_n(\eta) = nIn - 1(\eta)$. we obtain Fermi-Dirac integrals of integer order. Thanks to the work of Gong et al.[4] expressions (29) and (35) can be computed with high accuracy. Eqs. (25) and (31), one have access to the non-relativistic and ultra-relativistic limits of G_0 and G_1 , respectively. For G_2 and G_3 , we find that in the non-relativistic regime

$$G_2 = \frac{3}{2} - \frac{\frac{5}{6} I_{3/2}(\eta) I_{-1/2}(\eta)}{I_{1/2}(\eta) I_{1/2}(\eta)} \tag{3.54}$$

and

$$G_3 = \frac{5}{3}\frac{I_{3/2}(\eta)}{I_{1/2}(\eta)} - \frac{3I_{1/2}(\eta)}{I_{-1/2}(\eta)} \tag{3.55}$$

whereas in the ultra-relativistic regime,

$$G_2 = 3 - \frac{8I_3(\eta)I_1(\eta)}{3I_2(\eta)I_2(\eta)} \tag{3.56}$$

and

$$G_3 = \frac{4}{3}\frac{I_3(\eta)}{I_2(\eta)} - \frac{3I_2(\eta)}{2I_1(\eta)} \tag{3.57}$$

These results are consistent with Eq. (39). They have been obtained using the expressions of $F_k(\eta,\theta), \partial_\eta F_k(\eta,\theta)$, and $\partial_\theta F_k(\eta,\theta)$ in the non-relativistic and ultra-relativistic regimes for $k = 1/2, 3/2$, and $5/2$. the general expressions of these quantities in the two extreme situations for $k \neq 0$.

Chapter 4

White Dwarf and Neutron stars

4.1 Introduction

White dwarfs are stars of about one solar mass with characteristic radii of about $5,000km$ and mean densities of around $10^6 gcm^{-3}$. These stars no longer burn nuclear fuel. Instead, they are slowly cooling as they radiate away their residual thermal energy. We know today that white dwarfs support themselves against gravity by the pressure of degenerate electrons. This fact was not always clear to astronomers, although the compact nature of white dwarfs was readily apparent from early observations.

A star with mass greater than the Chandrasekhar limit cannot form a white dwarf. Either it expels enough mass during some cataclysmic event (a nova) or it collapses beyond the white dwarf domain to higher density regime. The possibility that there might be another regime of stable stars at density for greater than that of a white dwarf was perceived by Landau shortly after the discovery of the neutron. He apparently applied the stability argument he had developed for white dwarfs to a cold, degenerate neutron gas. Credit for the idea of neutron stars goes, however, to Baade and Zwicky, who in 1934 not only described their properties in detail but suggested that they would be formed in supernova explosions. Once again we quote liberally the historical background given in Shapiro and Teukolsky: In 1934 Baade and Zwicky proposed the idea of neutron stars, pointing out that they would be at very high density and small radius, and would be much more gravitationally bound than ordinary stars. They also made the remarkably prescient suggestion that neutron stars would be formed in supernova explosions. The first calculation of neutron star models was performed by Oppen- heimer and Volkoff (1939), who assumed matter to be composed of an ideal gas of free neutrons at high density.

The stars in the sky may seem ageless and unchanging, but eventually most of them will turn into white dwarfs, the last observable stage of evolution for low- and medium-mass stars. These dim stellar corpses dot the galaxy, leftovers of stars that once burned bright.Refs[6,7,8,]

4.2 White Dwarf Stars

Consider mass $M \approx 10^{33}g$ of helium at nuclear densities of $\eta \approx 10^{7}g/cm^{3}$ and temperature $T \approx 10^{7}K$

This temperature is much larger than ionization energy of ${}^{4}He$, hence we may safely assume that all helium atoms are ionized.

If there are N electrons - number of α particles ($i.e.{}^{4}He$ nuclei) must be $\frac{1}{2}N$. Mass of α particle $m_{\alpha} \approx 4m_{p}$.

Total stellar mass M is almost completely due to α particle cores.

using

$$M = Nm_{e} + \frac{1}{2}N4m_{p} \tag{4.1}$$

electron density

$$n = \frac{N}{V} = \frac{2M/(4m_{p})}{V} = \frac{\rho}{2m_{p}} \approx 10^{30}cm^{-3} \tag{4.2}$$

Since electrons are degenerate we estimate p to be order of uncertainty in momentum Δp Δx is order of average distance between electrons - approximately $n^{1/3}$

$$\Delta p \Delta x \approx \hbar$$

from number density n we find Fermi momentum of electron gas

$$p_F = \hbar(3\pi^2 n)^{1/3} \approx 2.26x10^{-17} gcm/s$$
$$mc = (9.1x10^{-28} g)(3x10^9 m/s) = 2.7x10^{-19} gm/s$$

Since $p_F \approx mc$ -electrons are relativistic.
Fermi temperature will then be $T_F \approx mc^2 \approx 10^6 eV \approx 10^{12} K$.
$T \ll T_F$ -electron gas is degenerate and considered to be at $T \approx 0$, So we need to understand ground state properties of relativistic electron gas.
kinetic energy

$$\epsilon(\overrightarrow{p}) = \sqrt{p^2c^2 + m^2c^4} - mc^2$$

velocity

$$\overrightarrow{v} = \frac{\partial \epsilon}{\partial \overrightarrow{p}} = \frac{pc^2}{\sqrt{p^2c^2 + m^2c^4}}$$

4.3 Newtonian Stars

Let us start by addressing Newtonian stars. For these stars we assume that corrections due to Einstein's greatest triumph the Theory of General Relativity may be safely ignored. White dwarf stars, with escape velocities of only 3% of the speed of light, fall into this category. Not so neutron stars typical escape velocities of half the speed of light cause extreme sensitivity to these corrections.

We start by considering the radial force acting on a small mass element ($\Delta m = \rho(r)\Delta V$) located at a distance r from the center of the star (see Fig. 1):

$$F_r = -\frac{GM(r)\Delta m}{r^2} - P(r + \Delta r)\Delta A + P(r)\Delta A = \Delta m \frac{d^2 r}{dt^2} \tag{4.3}$$

Here $\rho(r)$ is the mass density of the star, $M(r)$ denotes the enclosed mass within a radius r, and P is the pressure. Expanding the above equation to lowest order in Δr one obtains

$$-\frac{GM(r)\rho(r)}{r^2} - \frac{dP}{dr} = \rho(r)\frac{d^2r}{dt^2} \tag{4.4}$$

Assuming hydrostatic equilibrium ($\ddot{r} = \dot{r} \equiv 0$), one arrives at the fundamental equations describing the structure of Newtonian stars. That is,

$$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2}, \quad P(r=0) \equiv P_0 \tag{4.5}$$

$$\frac{dM}{dr} = 4\pi^4\rho(r), \quad M(r=0) \equiv 0 \tag{4.6}$$

where $Eq.$(4.4) defines the enclosed mass.

It is simple to see that in hydrostatic equilibrium, the pressure of the star is a decreasing (or at least not increasing) function of r; otherwise the star collapses. Note that the radius of the star R is defined as the value of r at which the pressure goes to zero, *i.e.*, $P(R) = 0$. Similarly, the mass of the star corresponds to the value of the enclosed mass at $r = R$, when $M = M(R)$.

4.3.1 Equation of State

The above set of equations, together with their associated boundary conditions, must be completed by an equation of state (EoS), namely a relation $P = P(\epsilon)$ between the density and pressure. For simplicity, we limit ourselves to the EoS of a zero-temperature Fermi gas composed of constituents (*e.g.*, electrons or neutrons) having a rest mass m. The main assumption behind the Fermi gas hypothesis is that no correlations (or interactions) are relevant to the system other than those generated by the Pauli exclusion principle. For some standard references on the equation of state of a free Fermi gas at both zero and finite temperatures see Refs. [18, 19].

To start, the Fermi wavenumberk_F is defined;k_F represents the momentum of the fastest

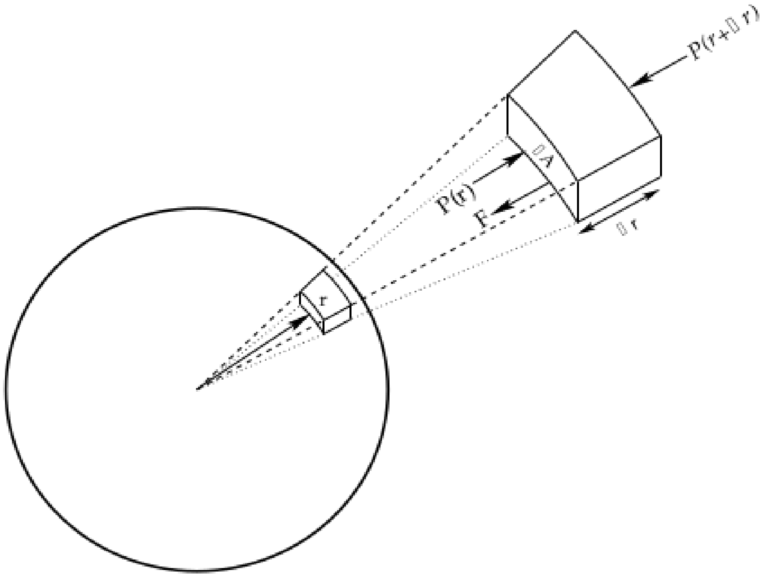

Figure 4.1: The radial force acting on a small mass element a distance r from the center of the star.

moving fermion and is solely determined by the number density ($n \equiv N/V$) of the system, where N is the total number of particles in our system, and V is the enclosed volume. That is,

$$N = 2\sum_{k} \Theta\left(k_F - |k|\right) = 2\int \frac{V}{(2\pi)^3} d^3k \Theta\left(k_F - |k|\right) = V\frac{k_F^3}{3\pi^2} \tag{4.7}$$

or equivalently

$$k_F = \left(3\pi^2 n\right)^{1/3} \tag{4.8}$$

In Eq.(4.7), $\Theta(x)$ represents the Heaviside (or step) function. Having defined the Fermi wavenumber k_F , the energy density of the system is obtained from a configuration in which all single-particle momentum states are progressively filled in accordance with the Pauli exclusion principle. For a degenerate (spin-1/2) Fermi gas at zero temperature, exactly two fermions occupy each single-particle state below the Fermi momentum $p_F = \hbar k_F$; all remaining states above the Fermi momentum are empty. In this manner we obtain the following expression for the energy density:

$$\epsilon = E/V = 2\int \frac{d^3k}{(2\pi)^3}\Theta\left(k_F - |k|\right)\epsilon(k) \tag{4.9}$$

where $\epsilon(k)$ is the single-particle energy of a fermion with momentum k. In what follows, the most general free-particle dispersion (energy vs. momentum) relation is assumed, namely,one consistent with the postulates of special relativity. That is,

$$\epsilon(k) = \sqrt{(\hbar kc)^2 + (mc^2)^2} = mc^2\sqrt{1+x^2} \tag{4.10}$$

with

$$x = \left(\frac{\hbar kc}{m^2c^2}\right)$$

In spite of its slightly intimidating form, the integral in Eq. (4.9) may be performed in closed form. We obtain

$$\epsilon = \epsilon_0\bar{\epsilon}(x_F) \tag{4.11}$$

where ϵ_0 is a dimensionful constant that may be written using dimensional analysis

$$\epsilon_0 = \frac{(mc^2)^4}{(\hbar c)^3}, \tag{4.12}$$

and $\bar{\epsilon}(x_F)$ is a dimensionless function of the single variable $x_F = \hbar k_F c/mc^2$ given by

$$\bar{\epsilon}(x_F) = \frac{1}{\pi^2}\int_0^{x_F} x^2\sqrt{1+x^2}dx = \frac{1}{8\pi^2}\left[x_F(1+2x_F^2)\sqrt{1+x_F^2} - ln(x_F + \sqrt{1+x_F^2})\right] \tag{4.13}$$

The pressure of the system may now be directly obtained from the energy density by using the following thermodynamic relation - which is only valid at zero temperature:

$$P = \left(\frac{\partial E}{\partial V}\right)_{N,T\equiv 0} = -\left(\frac{\partial (V\epsilon)}{\partial V}\right)_{N,T\equiv 0} = P_0\bar{P} \tag{4.14}$$

In analogy to the energy density, dimensionful and dimensionless quantities for the pressure have been defined:

$$P_0 = \epsilon_0 = \frac{(mc^2)^4}{(\hbar c)^3}, \tag{4.15}$$

$$P_{x_F} = \left[\frac{x_F}{3}\bar{\epsilon}'_{(x_F)} - \bar{\epsilon}_{(x_F)}\right] \tag{4.16}$$

It may be surprising to find that a gas of particles at zero-temperature may still generate a non-zero pressure. It is quantum statistics, in the form of the Pauli exclusion principle not temperature - that is responsible for generating the pressure. It is nevertheless surprising that quantum pressure, a purely microscopic phenomenon, should be ultimately responsible for supporting compact stars against gravitational collapse.

With an expression for the pressure in hand, we are finally in a position to compute its derivative with respect to x_F (a quantity that we label as η). As we shall see in the next section, η - a function closely related to the zero-temperature incompressibility is the only property of the EoS that Newtonian stars are sensitive to [19]. We obtain

$$\eta = \frac{dP}{x_F} = P_0\left[\frac{x_F}{3}\bar{\epsilon}''_{(x_F)} - \frac{2}{3}\bar{\epsilon}'_{(x_F)}\right] = \frac{P_0}{3\pi^2}\frac{x_F^4}{\sqrt{1+x_F^2}} \tag{4.17}$$

The above expression has a surprisingly simple form that depends on the energy density only through its derivatives. Alternatively, one could have bypassed the above derivation in favor of the following general relation valid for a zero-temperature Fermi gas:

$$\eta = \frac{dP}{x_F} = n\frac{d\epsilon_{(F)}}{x_F} \tag{4.18}$$

In view of Eq. (4.17), the attentive reader may be asking why go through the trouble of computing the energy density and the corresponding pressure if all that is required is the dependence of the Fermi energy on x_F . The answer is general relativity. While Newtonian stars depend exclusively on η, the structure of relativistic stars (such as neutron stars) are highly sensitive to corrections from general relativity. These corrections depend on both the energy density and the pressure and will be treated in detail in a future publication.

4.4 Ground State Pressure

Pressure in ground state is

$$P_0 = \frac{1}{3} n \langle \overrightarrow{v} . \overrightarrow{p} \rangle \tag{4.19}$$

$$\begin{aligned} P_0 =& \frac{1}{3\pi^2\hbar^3} \int_0^{p_F} dp p^2 . \frac{pc^2}{\sqrt{p^2c^2 + m^2c^4}} \\ =& \frac{m^4c^5}{3\pi^2\hbar^3} \int_0^{\theta_F} d\theta sinh^4\theta \\ P_0 =& \frac{m^4c^5}{96\pi^2\hbar^3} \left(sinh\left(4\theta_F\right) - 8 sinh\left(2\theta_F\right) + 12\theta_F \right) \end{aligned}$$

To compute the equation of state at zero temperature, one can use the present development based on energy or keep the one based on momentum. They are equivalent but the formulas obtained using the momentum can be more readily be put under a compact form at zero temperature[12]. However, using the energy form leads to expressions that involve the chemical potential. They are better adapted to calculate the expansion of the quantities of interest in function of T at given θ_F .

We used substitution

$$p = mcsinh\theta, v = ctanh\theta \tag{4.20}$$

$$\Rightarrow \theta = \frac{1}{2} ln \left(\frac{c+v}{c-v} \right), \quad p_F = \hbar(3\pi^2 n)^{1/3} \tag{4.21}$$

$$n = \frac{M}{2m_pV} \Rightarrow 3\pi^2 n = \frac{9\pi}{8} \frac{M}{R^3 m_p} \tag{4.22}$$

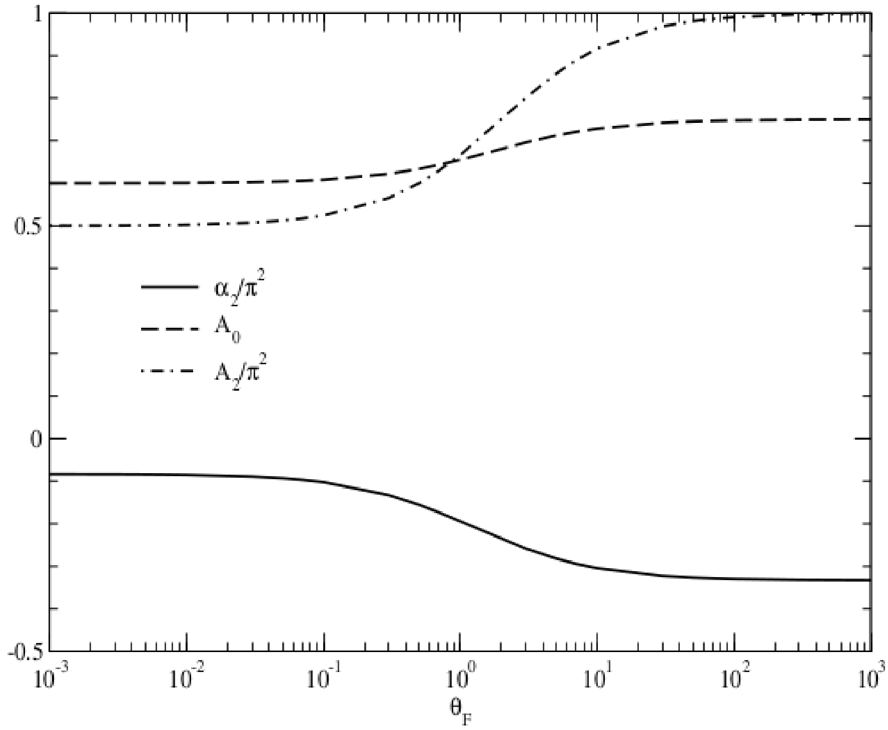

Figure 4.2: (Color online) α^2/π^2, A_0, and $A^2\ /\pi^2$as a function of θ_F .

4.5 Balance Equation

In equilibrium pressure:

$$dU_0 = -P_0 dV = -P_0(R).4\pi R^2 dR \tag{4.23}$$

is balanced by gravitational pressure;

$$dU_g = \gamma.\frac{GM^2}{R^2}dR \tag{4.24}$$

Gamma is depends on radial mass distribution.

Equilibrium then implies;

$$P_0(R) = \frac{\gamma}{4\pi}\frac{GM^2}{R^2} \tag{4.25}$$

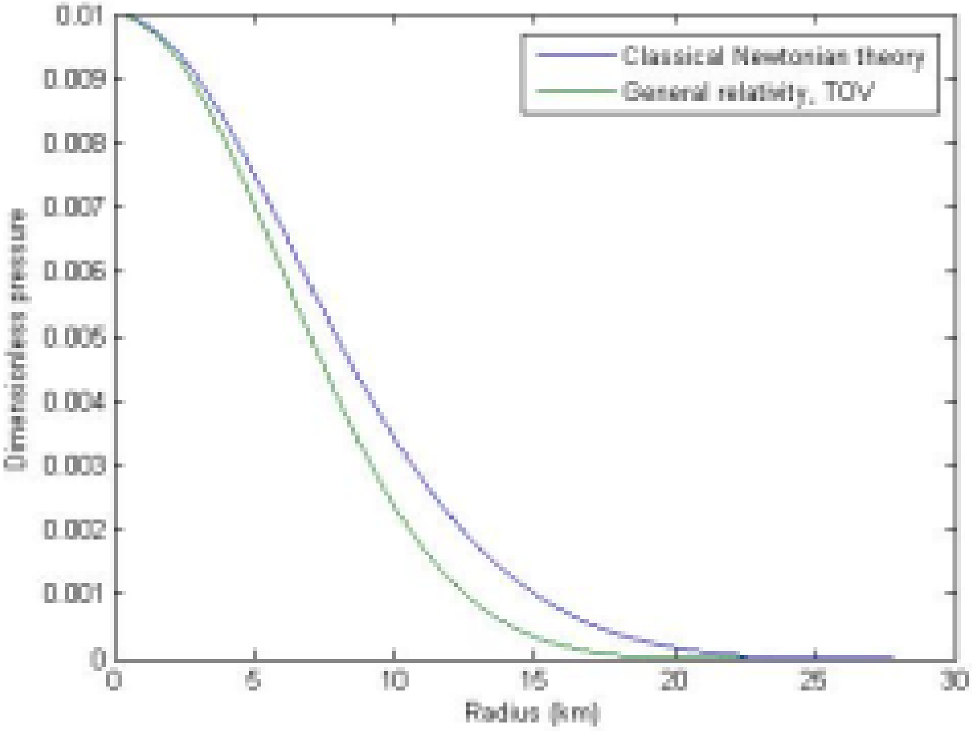

Figure 4.3: The pressure $P(r)$ as a function of the radius r in km for a pure neutron star with central pressure$P(0)$ using the Fermi equation of state for arbitrary relativity

To find relation $R = R(M)$ we must solve

$$\frac{\gamma}{4\pi}\frac{GM^2}{R^2} = \frac{m^4c^5}{96\pi^2\hbar^3}\left(sinh\left(4\theta_F\right) - 8sinh\left(2\theta_F\right) + 12\theta_F\right) \tag{4.26}$$

Note that

$$sinh\left(4\theta_F\right) - 8sinh\left(2\theta_F\right) + 12\theta_F = \begin{cases} \frac{96}{15}\theta_F^5 & \theta_F \to 0 \\ \frac{1}{2}e^{4\theta_F} & \theta_F \to \infty \end{cases} \tag{4.27}$$

4.6 Chandrasekhar Limit

we may write

$$P_0(R) = \frac{\gamma}{4\pi}\frac{GM^2}{R^2} = \begin{cases} \frac{\hbar^2}{15\pi^2 m}\left(\frac{9\pi}{8}\frac{M}{R^3 m_p}\right)^{5/3} & \theta_F \to 0 \\ \frac{\hbar c}{12\pi^2}\left(\frac{9\pi}{8}\frac{M}{R^3 m_p}\right)^{5/3} & \theta_F \to \infty \end{cases} \tag{4.28}$$

This point defines the radius of the star, while the mass of the star is the value of the enclosed mass at this last point (this point also corresponds to when the pressure goes to zero). Up to this point, both radius and mass are obtained in dimensionless units. To convert back to physical units we simply multiplied these dimensionless quantities by the dimensionful parameters (R_0 and M_0) defined in Eq. (4.28). Of course, there is no need to repeat the full calculation for a different value of Y e as we scaled R_0 and M_0 appropriately. In order to create the complete mass-radius relation, we repeated the same procedure for a large number of central densities. Here a step size of Δr = 0.0001 was used throughout and scaled central densities ranged from 0.1 to 100, moving in steps of 0.1 at first, then to increments of 1. Doing so produces Fig. 4.4 and, in particular, a Chandrasekhar limit very close to 1.4 M_0 . However, as alluded in the toy-model problem and confirmed in this numerical calculation, there is no Chandrasekhar limit if one uses a non-relativistic dispersion relation.
In limit $\theta_F \to 0$ - we solve for $R(M)$ and find

$$R = \frac{3}{40\gamma}(9\pi)^{2/3}\frac{\hbar^2}{GM_p^{5/3} m M^{1/2}} \approx M^{-1/3} \tag{4.29}$$

In limit $\theta_F \to \infty$- $R(M)$ factors divide out and we obtain

$$M = M_0 = \frac{9\pi}{8}\left(\frac{3\pi}{\gamma^3}\right)^{1/2}\left(\frac{\hbar c}{G}\right)^{3/2}\frac{1}{m_p^2} \tag{4.30}$$

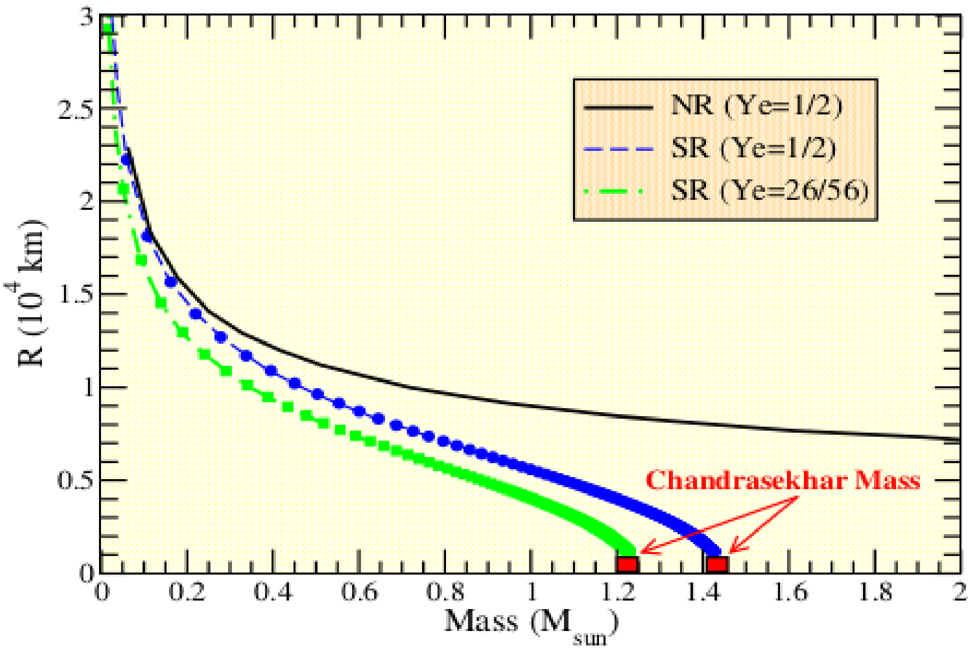

Figure 4.4: Mass-vs-radius relation for white dwarf stars obtained using Excel (lines) and Maple (symbols).

To find R dependence- we must go beyond lowest order expansion of Eq.(4.29)

$$\text{We obtain} \quad R = \left(\frac{9\pi}{8}\right)^{1/3} \left(\frac{\hbar}{mc}\right) \left(\frac{M}{m_p}\right)^{1/3} \left[1 - \left(\frac{M}{M_0}\right)^{2/3}\right]^{1/2} \tag{4.31}$$

Value M_0 is limiting size for a white dwarf. It is called **Chandrasekharlimt**.

For a fully relativistic treatment, the equation of state used interpolates between the equations $P = K^1\rho^{5/3}$ for small ρ and $P = K^2\rho^{4/3}$ for large ρ. When this is done, the model radius still decreases with mass, but becomes zero at M_{limit}. This is the Chandrasekhar limit.[9] The curves of radius against mass for the non-relativistic and relativistic models are shown in the graph. They are colored blue and green, respectively. μ_e has been set equal to 2. Radius is measured in standard solar radii[10] or kilometers, and mass in standard Msolar

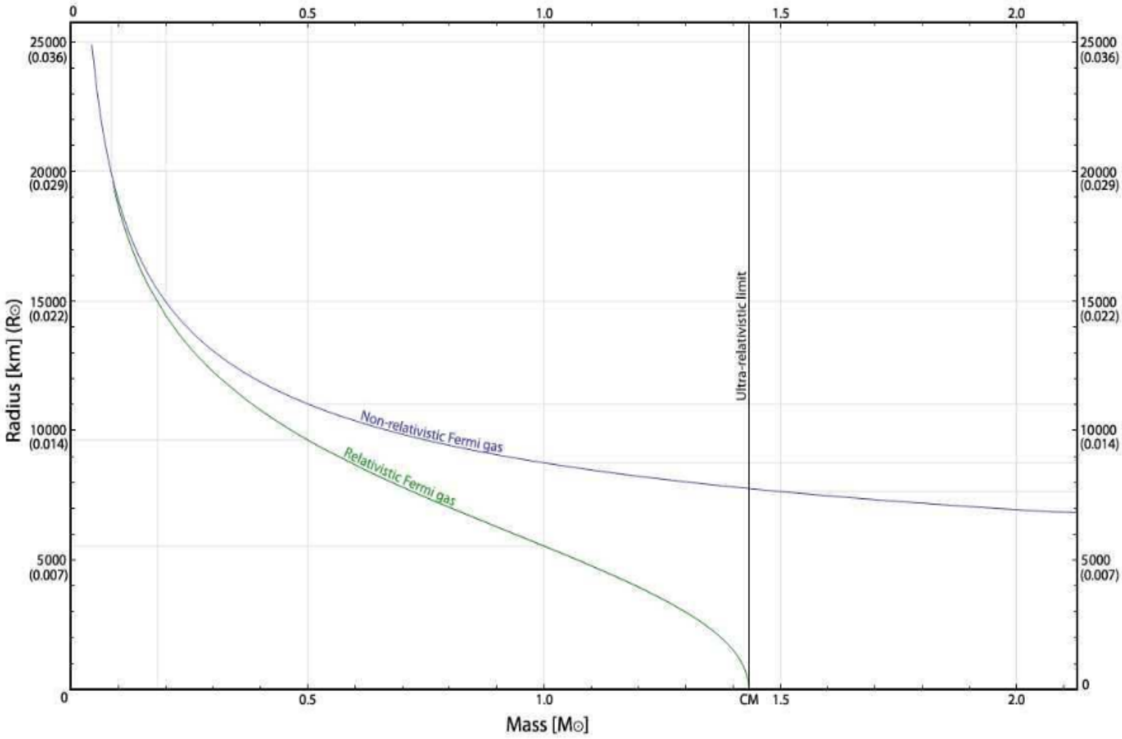

Figure 4.5: Mass-Radius Relationship for White Dwarf Stars

Chapter 5

Conclusion

The free electron gas is widely used to describe the equation of state of electrons in the astrophysical context where extreme conditions can be encountered or in inertial confinement fusion. If many works exist in the non-relativistic (NR) and ultra-relativistic (UR) approximations, less works exist for arbitrarily degeneracy and relativity situation. Indeed, degeneracy and relativistic effects of the free electron gas can be found in the interior of massive stars, collapsing stars, neutron star envelopes, or white dwarfs.

In the case of neutron star envelopes, the classical gas is encountered on the surface whereas extremely degenerate Fermi gas is found in the interior. In practice, the main problem is to compute with sufficient accuracy the various generalized Fermi-Dirac integrals that appear in this problem to calculate the equation of state of interest.

We have studied the relativistic free electron gas at arbitrarily degeneracy. We have calculated the specific heat at constant volume and particle number C_V and the specific heat at constant pressure and particle number C_P as well as the equation of state. We have tried to find expressions as compact as possible. The present results interpolate between the non-relativistic and the ultra-relativistic regimes. Non degenerate and degenerate limits were examined. New formulas based on the Sommerfeld expansion were found for the equation of state that generalize the non-relativistic and the ultra-relativistic regimes. The energy corresponding to this momentum, called the Fermi energy which increase. So, with a decreasing volume and an increasing particle momentum, we can say that a pressure formed inside of the core of the star, and continues to increase as long as the volume continues to increase, and that there are degenerate neutron energy states. Now that we know where the pressure comes from, we can finally derive a mathematical expression for the neutron degeneracy pressure by non-relativistic neutrons inside of a neutron star.

Declaration of Authorship

This project is my original work, has not been presented for a degree in any other University and that all the sources of material used for the project have been dully acknowledged.

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This project has been submitted for examination with my approval as University advisor

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Place and time of submission:

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