Addis Ababa University

Fixed Point Approximation for Some Nonlinear Mappings in Banach and Modular Metric Spaces with Graph Structure

by

Tibebu Worku Hunde

Research Advisors:

Prof. Habtu Zegeye Hailu, Botswana International University of Science and Technology

Dr. Mengistu Goa Sangago, Addis Ababa University

Thesis Submitted to the Department of Mathematics in Fulfillment of the Requirements for Doctor of Philosophy in Mathematics

Addis Ababa, Ethiopia

May 24, 2018.

Approval

This thesis has been examined and approved as meeting the requirements for the fulfillment of Doctor of Philosophy in Mathematics.

Student's Name	Signature	Date
Supervisor	Signature	Date
Internal Examiner	Signature	Date
External Examiner	Signature	Date
Head, Department	Signature	Date

Declaration

I, **Tibebu Worku**, with student ID $GSR/2562/06$, hereby declare that this thesis has been composed by myself and that it has never been submitted for completion of graduate qualification at any higher learning institution. I confirm that the work contained in this thesis is my own original research work under the supervision of Prof. Habtu Zegeye and Dr. Mengistu Goa. Any work done by others has been acknowledged and referenced accordingly.

____________________ ____________________

Signature Date

Addis Ababa

Ethiopia

Dedication

To my let father Worku Hunde.

Abstract

In this thesis, we study fixed point approximation methods for some nonlinear mappings in Banach and Modular metric spaces with graph structure. We review the contemporary literature in these directions. Then we establish some weak and strong convergence results for finite Noor iterative method to approximate common fixed point of a finite family of G-nonexpansive self mappings in uniformly convex Banach spaces with graph structure. Moreover, we introduce a new class of self mappings in Banach spaces called G-asymptotically nonexpansive mappings, which is more general than the class of G-nonexpansive mappings and obtain some weak and strong convergence of the modified Noor iterative method to a common fixed point of a finite family of such class of mappings in the setting of uniformly convex Banach spaces with graph structure under some mild conditions. Furthermore, we define a class of self mappings in modular metric spaces called G-monotone generalized quasi contraction mappings and establish necessary and sufficient conditions for the convergence of the Picard iteration process to fixed point of such class of mappings on the setting of modular metric spaces with graph structure. Our results improve and extend these results in the contemporary literature.

Acknowledgement

I express my deepest gratitude to my supervisors Prof. Habtu Zegeye of Botswana International University of Science and Technology and Dr. Mengistu Goa of Addis Ababa University, Department of Mathematics for their tremendous and continuous guidance throughout the entire course of research work and the foundation they laid down for me during the course work of Functional Analysis and Real and complex Analysis.

I have to be grateful to Simons Foundation which was based at University of Botswana (UB) and later moved to Botswana International University of Science and Technology (BIUST) for its partial financial support. I am also grateful to the International Science Program(ISP)- Sweden for its partial financial support. Moreover, I would like to express my gratitude to Adama Science and Technology University (ASTU) for the study leave permit with full salary.

My deep appreciation goes to all the staff of Mathematics Department, Adama Science and Technology University. Particularly, Dr. Shiferaw Feyissa, Dr. Legesse Lemecha, Dr. Lemi Guta, Dr. Nathea Hunde, Bullo Hindebu and Sileshi Mebrate for their kind assistant and encouragement in various ways. I also grateful to my colleagues and officials at Department of Mathematics, Addis Ababa University for their encouragement and shared materials.

Very special thanks go to my colleague Dr. Wondimu Woldie, Mesfin

Dejene, Mamo Mengesha and Negasho Bedhaso for all the time we have had together and shared materials.

I would also like to say a heartfelt thank you to my mother Sidise Tullu, my sisters and my brothers for supporting me in whatever way they could during this challenging period.

Thank you so much !!

Tibebu Worku Hunde

Addis Ababa, Ethiopia

April, 2018.

Contents

Chapter 1

Introduction

1.1 Background

The most essential nonlinear problems in applied sciences reduce to solving a given equation which in turn may reduce to finding fixed point of a certain nonlinear mappings. Practically, it is important not only to know its existence (possibly, is unique), but also requires to obtain the fixed pint. In most cases, it is difficult and even impossible to get the exact value of the fixed point. Consequently, the existence of fixed point and its approximation methods for different class of nonlinear mappings have been studied by several authors (see, e.g., [8, 30, 38, 48, 64, 71]).

Among different types of iterative methods: the Mann, Ishikawa and Noor are the most commonly used fixed point approximation methods to find common fixed point of a family of nonexpansive mappings in Banach spaces. Likewise, the modified Mann, modified Ishikawa and modified Noor iterative methods are the commonly used iterative methods to approximate common fixed point of a family of asymptotically nonexpansive mappings in Banach spaces. It is discussed in [10] that the Noor method performs better than Mann and Ishikawa iterations for solving variational inequality problems. Moreover, it is shown in [48] that the method due to Noor is a natural generalization of the splitting method in solving partial differential equations. Because of these facts, the Noor method and its modified version, in the context of one and more mappings has been extensively studied by several authors (see, e.g., [10, 28, 49, 63, 79]).

Since Banach spaces with Opial's condition are very limited, researchers in the area try to obtain fixed point results for different class of nonlinear operators defined in Banach spaces without of it. Chang et.al [14] obtained some fixed point results for nonexpansive and asymptotically nonexpansive self mappings in Banach spaces without assuming Opial's condition on it. Many authors have studied fixed point results for some nonlinear mappings in Banach spaces without assuming Opial's condition on it (see, e.g., [17, 61]).

Moreover, the study of fixed point results in modular function spaces attracts several authors (see, e.g., [26, 37, 41, 42]). However, to solve several nonlinear problems, for instance supper position (or Nemytskii) operators, the notion of a modular on a linear space or on a space with an additional algebraic structure are too restrictive. So, the notion of a modular on an arbitrary set X is proposed. In this regard, Chistyakov [20, 22], introduced a modular metric space and study some fixed point theorems for ω-contraction mappings defined in it. Several authors are attracted to study fixed point results for some monotone nonlinear mappings in modular metric space (see, e.g., [1, 5, 6, 56, 81]).

To deal with fixed point results for a certain class of nonlinear mappings, researchers have imposed partial ordering on the space under considerations (see, e.g., [47, 57, 76]). Jachymski [32] replace the partial ordering with graph structure and obtain more general fixed point results for contraction mappings. After then, several authors have investigated several fixed point results for nonlinear mappings defined in spaces with graph structure (see, e.g., [5, 7, 19, 75, 77]). This opens the way forward to deal with approximation of fixed points for some nonlinear mappings in spaces with graph structure.

Our purpose in this thesis is, therefore, to introduce and study iterative methods for approximating a common fixed point of a family of G-nonexpansive and G-asymptotically nonexpansive self mappings in uniformly convex Banach spaces with graph structure and to establish some convergence theorems for G-monotone generalized quasi contraction mappings in the setting of modular metric spaces with graph structure.

The remaining part of our thesis is organized in the following manner. Section 1.2 presents some preliminary definitions and fundamental properties of spaces. Here some basic terminologies from graph theory and preliminary definitions of nonlinear mappings were also presented. Chapter 2, deals with literature review on approximation of fixed points of nonlinear mappings in Banach and modular metric spaces. Under this chapter in Section 2.1, literature on approximation of common fixed points for a family of G-nonexpansive and

G-asymptotically nonexpansive self mappings in Banach spaces with graph structure are presented. In Section 2.2, literature on approximation of fixed points for G-monotone generalized quasi contractions mappings in the setting of modular metric spaces with graph structure are described. In Section 2.3, statement of the problem and in Section 2.4, research questions which addresses the stated problem are stated. In Section 2.5, the general objective and specific objectives of the research are stated. In Section 2.6, the scope of the research has described. To this end, in Section 2.7, the research methodology has described.

In Chapter 3, some auxiliary results from the literature which will be used directly in the proof of our results were stated.

In Chapter 4, we present the main results of the thesis. Section 4.1, deals with some weak and strong theorems on the convergence of Noor iterations to common fixed points of a finite family of G-nonexpansive self-mappings on uniformly convex Banach spaces with graph structure. Section 4.2 deals with some weak and strong convergence theorems for the modified Noor iteration process to common fixed points of a finite family of G-asymptotically nonexpansive mappings in uniformly convex Banach spaces with graph structure. In the last section of this chapter, Section 4.3, the necessary and sufficient conditions for the convergence of Picard iterations to fixed points of the class of G-monotone generalized quasi contraction mappings in the setting of modular metric spaces with graph structure is established.

The last chapter of the thesis, Chapter 5, deals on the discussion of our main results, justifications to how and what these results extend the results in the contemporary literature. Furthermore, some open problems related to our results are identified for future research works. Bibliography is given at the end of the thesis.

1.2 Preliminaries

1.2.1 Basic Properties of Banach Spaces

Let X be a real Banach space. The set of all bounded and linear functionals on X, i.e., $\{T : X \to \mathbb{R} : T$ is linear and bounded $\}$ is denoted by X^* and it is called dual space of X. It is well known that X^* is a Banach space and hence we can consider its dual, X^{**}, which is called the conjugate of X. Let $x \in X$ be fixed. Then a functional $\varphi_x : X^* \to \mathbb{R}$ given by

$$\varphi_x(f) = \langle f, x \rangle = f(x), \ \forall f \in X^*.$$

x is bounded linear functional. $\varphi_x = x \in X^{**}$. The function φ is called the canonical (or natural) embedding of X in to X^{**}.

Definition 1.1. A Banach space X is called reflexive if the canonical embedding $\varphi : X \to X^{**}$ is bijective.

Example 1.1. *l_p and L_p spaces, for $1 < p < \infty$ are reflexive Banach spaces.*

Example 1.2. *Neither of $l_1, \ L_1, \ l_\infty$ and L_∞ spaces is reflexive.*

Definition 1.2. Let X be a Banach space and $S_X := \{x \in X : ||x|| = 1\}$. X is said to be strictly convex if $x, y \in S_X$ with $x \neq y$ implies that $||\lambda x + (1-\lambda)y|| < 1$, for all $\lambda \in (0,1)$.

Definition 1.3. A Banach space X is called uniformly convex if for any $\epsilon \in (0,2]$, there exists a $\delta = \delta(\epsilon)$ such that if $x, y \in X$ with $||x|| = 1,\ ||y|| = 1$ and $||x-y|| \geq \epsilon$, then $||\frac{x+y}{2}|| \leq 1 - \delta$.

It is shown in [2] that every uniformly convex Banach space is strictly convex and reflexive.

Definition 1.4. A sequence $\{x_n\}$ in a Banach space X is said to converge weakly to $x \in X$ if $f(x_n) \to f(x)$ for all $f \in X^*$. In this case, we write $x_n \rightharpoonup x$ or weak$-\lim\limits_{n\to\infty} x_n = x$.

In any Banach space, the weak-limit of a sequence is unique. We note that, strong convergence (norm convergence) implies weak convergence but the converse is not true in general. It is discussed in [69] that any bounded sequence in a reflexive Banach space has a weakly convergent subsequence.

Definition 1.5. [2] Let X be a Banach space. Then a function $\delta_X : [0,2] \to [0,1]$ is called the modulus of convexity of X if

$$\delta_X(\epsilon) = \inf\{1 - \frac{||x+y||}{2} : ||x|| \leq 1, ||y|| \leq 1,\ ||x-y|| \geq \epsilon\}.$$

It is stated in [69] that, for a uniformly convex Banach space X with modulus of convexity δ_X, if $r > 0$ and $x, y \in X$ with $||x|| \leq r,\ ||y|| \leq r$ then

$$||tx + (1-t)y|| \ \leq\ r[1 - 2\min\{t, 1-t\}\delta_X(\frac{||x-y||}{r})],\ \text{for all } t \in (0,1).$$

Definition 1.6. [2] A Banach space X is said to be smooth if the limit

$$\lim_{t \to 0} \frac{||x + ty|| - ||x||}{t} \tag{1.1}$$

exists for each $x, y \in S_X$. In this case, the norm of X is said to be Gateaux differentiable. If for each $y \in S_X$, the limit in (1.1) is attained uniformly for $x \in S_X$, then X is said to have a uniformly Gateaux differentiable norm.

It is well known that the L_p spaces with $1 < p < \infty$ has a uniform Gateaux differentiable norm.

Definition 1.7. [51] A Banach space X is said to satisfy Opial's condition if the following inequality holds for any distinct elements x and y in X and for each sequence $\{x_n\}$ weakly convergent to x:

$$\limsup_{n \to \infty} ||x_n - x|| < \limsup_{n \to \infty} ||x_n - y||.$$

Every Hilbert space satisfies the Opial's condition. It is shown in [2] that $L_p[0, 2\pi]$ spaces for $p \neq 2$ does not satisfy the Opial's condition.

1.2.2 Fundamental Properties in Modular Metric Spaces

In this section, we discuss some basic concepts and fundamental properties of modular metric spaces.

Informally speaking, whereas a metric on a set represents nonnegative finite distances between any two points of the set, a modular metric on a set attributes a nonnegative (possibly, infinite valued) field of (generalized) velocities: to each time $\lambda > 0$ (the absolute value of) an average velocity $\omega_\lambda(x,y)$ is associated in such a way that in order to cover the distance between points $x, y \in X$ it takes time λ to move from x to y with velocity $\omega_\lambda(x,y)$. But the way we approach the concept of modular metric spaces is different. Indeed, the modular metric spaces were considered as the nonlinear version of the classical modular spaces introduced by Nakano [46] on vector spaces and modular function spaces introduced by Orlicz [44] and Musielak and Orlicz [45].

Let X be a nonempty set. Throughout, for a function $\omega : (0,\infty) \times X \times X \to [0,\infty]$, we will write

$$\omega_\lambda(x,y) \;=\; \omega(\lambda,x,y) \text{ for all } \lambda > 0 \text{ and } x,y \in X.$$

Definition 1.8. [20] A function $\omega : (0,\infty) \times X \times X \to [0,\infty]$ is said to be a modular metric on X if it satisfies for each $x, y, z \in X$ and $\lambda, \mu \in (0,\infty)$:

(i) $x = y$ if and only if $\omega_\lambda(x,y) = 0$;

(ii) $\omega_\lambda(x,y) = \omega_\lambda(y,x)$; and

(iii) $\omega_{\lambda+\mu}(x,y) \leq \omega_\lambda(x,z)+\omega_\mu(z,y)$.

If instead of (i) we have only the condition:

(i') $\omega_\lambda(x,x) = 0$, for all $\lambda > 0,\ x \in X$, then ω is said to be pseudo-modular (metric) on X.

A modular metric ω on X is said to be regular(or strict) if the following weaker version of (i) is satisfied:

$$x = y \text{ if and only if } \omega_\lambda(x,y) = 0 \text{ for some } \lambda > 0.$$

ω is said to be convex if for all $\lambda, \mu > 0$ and $x, y, z \in X$, it satisfies the inequality:

$$\omega_{\lambda+\mu}(x,y) \leq \frac{\lambda}{\lambda+\mu}\omega_\lambda(x,z) + \frac{\mu}{\lambda+\mu}\omega_\mu(z,y).$$

Remark 1.1. For a pseudomodular metric ω on a set X, and any $x, y \in X$, the function $\lambda \mapsto \omega_\lambda(x,y)$ is nonincreasing on $(0,\infty)$, and so the limit from the right $\omega_{\lambda+0}(x,y)$ and the limit from the left $\omega_{\lambda-0}(x,y)$ exists in $[0,\infty]$ and satisfy the inequalities:

$$\omega_{\lambda+0}(x,y) \leq \omega_\lambda(x,y) \leq \omega_{\lambda-0}(x,y).$$

Definition 1.9. [20] Let ω be a pseudomodular on X. Fix $x_0 \in X$. The two sets:

$$X_\omega = X_\omega(x_0) = \{x \in X : \omega_\lambda(x,x_0) \to 0 \text{ as } \lambda \to \infty\} \text{ and}$$

$$X^*_\omega(x_0) = \{x \in X : \exists \lambda = \lambda(x) > 0 \text{ such that } \omega_\lambda(x,x_0) < \infty\}$$

are said to be modular spaces around x_0.

It is clear that $X_\omega \subseteq X^*_\omega$ and this inclusion may be proper in general.

It is proved in [20] that if ω is a metric modular on X, then the modular space X_ω is a metric space with metric given by:

$$d_\omega(x,y) = \inf\{\lambda > 0 : \omega_\lambda(x,y) \leq \lambda\},$$

for any $x, y \in X_\omega$.

Remark 1.2. Let ω be a convex modular on a set X. Given $x, y \in X$,

(i) $\widehat{\omega}_\lambda(x,y) = \lambda\omega_\lambda(x,y)$, is a modular metric on X.

(ii) the functions $\lambda \mapsto \omega_\lambda(x,y)$ and $\lambda \mapsto \widehat{\omega}_\lambda(x,y)$ are nonincreasing on $(0,\infty)$.

Note that if ω is convex modular on X then the two modular spaces coincide, i.e., $X_\omega = X^*_\omega$, and this common set can be endowed with the metric d^*_ω given by:

$$d^*_\omega(x, y) = \inf\{\lambda > 0 : \omega_\lambda(x, y) \leq 1\}, \text{ for any } x, y \in X_\omega.$$

We observe that, every metric space (X, d) is a modular metric space but the converse may not be true in general. Indeed, if we define a functional

$$\omega : (0, \infty) \times X \times X \to [0, \infty) \text{ by}$$

$$\omega_\lambda(x, y) = \frac{d(x, y)}{\lambda},$$

then, we see that ω is a convex modular on X. So,

$$d(x, y) = \lambda\omega_\lambda(x, y) = \widehat{\omega}_\lambda(x, y)$$

is a modular on X. On the other hand, a modular metric ω on a set X is a metric if ω assumes only finite values and is independent of the parameter λ.

In studying fixed point theory in modular metric spaces, we repeatedly use the notion of ω-convergence. In the following definition we describe this concept and other important terms in the modular metric spaces.

Definition 1.10. [1] Let (X, ω) be a modular metric space.

(1) The sequence $\{x_n\}_{n\in\mathbb{N}}$ in X_ω is said to be ω-convergent to $x \in X$ if and only if $\lim\limits_{n\to\infty} \omega_1(x_n, x) = 0$. x will be called the ω-limit of $\{x_n\}$.

(2) The sequence $\{x_n\}_{n\in\mathbb{N}}$ in X_ω is said to be ω-Cauchy if

$$\lim_{n,m\to\infty} \omega_1(x_n, x_m) = 0.$$

(3) A subset M of X_ω is said to be ω-closed, if the ω-limit of ω-convergent sequence of M always belongs to M.

(4) A subset M of X_ω is said to be ω-complete if any ω-cauchy sequence in M is ω-convergent sequence and its ω-limit is in M.

(5) A subset M of X_ω is said to be ω-bounded if we have

$$\delta_\omega(M) = \sup\{\omega_1(x, y) : x, y \in M\} < \infty.$$

(6) A subset M of X_ω is said to be ω-compact if for any sequence $\{x_n\}_{n\in\mathbb{N}}$ in M, there exists a subsequence $\{x_{k_n}\}_{n\in\mathbb{N}}$ of $\{x_n\}$ and $x \in M$ such that $\omega_1(x_{k_n}, x) \to 0$ as $n \to \infty$.

(7) ω is said to satisfy the Fatou property if and only if for any sequence $\{x_n\}_{n\in\mathbb{N}}$ in X_ω, ω-convergent to x, we have

$$\omega_1(x, y) \le \liminf_{n\to\infty} \omega_1(x_n, y), \text{ for any } y \in X_\omega.$$

In general, if $\lim\limits_{n\to\infty} \omega_\lambda(x_n, x) = 0$ for some $\lambda > 0$, then we may not have $\lim\limits_{n\to\infty} \omega_\lambda(x_n, x) = 0$ for all $\lambda > 0$. But, if it is the case, we have the following definition.

Definition 1.11. [1] Let (X, ω) be a modular metric space. ω is said to satisfy the Δ_2-condition if and only if $\lim\limits_{n\to\infty} \omega_\lambda(x_n, x) = 0$ for some $\lambda > 0$ implies that $\lim\limits_{n\to\infty} \omega_\lambda(x_n, x) = 0$ for all $\lambda > 0$.

If ω satisfies the Δ_2-condition, then ω-convergence and d_ω-convergence are equivalent.

1.2.3 Fundamental Concepts from Graph Theory

Before giving basic notions in graph theory, we recall some definitions and terminologies of partially ordered set. Let X be any nonempty set. Suppose $\preceq$ is a binary relation on X such that the following properties hold:

1. $x \preceq x$ for all $x \in X$ (reflexive);
2. if $x \preceq y$ and $y \preceq x$ then $x = y$ for all $x, y \in X$ (antisymmetric);
3. if $x \preceq y$ and $y \preceq z$ then $x \preceq z$ for all $x, y, z \in X$ (triangle inequality).

Then X is said to be partially ordered by '$\preceq$' and a pair $(X, \preceq)$ is called a partially ordered set. A partially ordered set is complete if every pair is ordered. An element x is an upper (respectively, a lower) bound for a set A in a partially ordered set X if $y \preceq x$ ($x \preceq y$) for each $y \in A$. An element x is the greatest(respectively, the least) element of A if it belongs to A and is an upper (a lower) bound of A. The element x is the least upper (respectively, greatest lower) bound or supremum (infimum) of A if it is the least (greatest) element of the set of upper (lower) bounds of A.

A lattice is a partially ordered set $(X, \preceq)$ where every pair x, y in X has a supremum and an infimum. In the remaining part of this section, we gather some basic terminologies in graph theory which will be found in [25].

A graph is a pair of sets (V, E), where V is the set of vertices and E is the set of edges, formed by pairs of vertices.

Below, few terminologies from graph theory which will be used in the subsequent sections and chapters are listed.

- The two vertices u and v are end vertices of the edge (u, v).
- Edges that have the same end vertices are parallel.
- An edge of the form (v, v) is a loop.
- A graph is simple if it has no parallel edges or loops.
- A graph with no edges (i.e., E is empty) is empty.
- A graph with no vertices (i.e., V and E are empty) is a null graph.

A simple graph that contains every possible edge between all the vertices is called a complete graph.

A directed graph(digraph) is a pair (V, E), where V is a vertex set and E is the set of vertex pairs. The difference is that now the elements of E are ordered pairs: the arc from vertex u to vertex v is written as (u, v) and the other pair (v, u) is the opposite direction arc.

If we assume that G has no parallel Edges, then G can be identified with the pair $G = (V(G), E(G))$.

If x and y are vertices of G, then a path in G from x to y of length $k \in \mathbb{N}$ is a finite sequence $\{x_n\}_{n=0}^{n=k}$ of vertices such that $x_0 = x,\ x_k =$

y and $(x_{i-1}, x_i) \in E(G)$, for $i = 1, 2, 3, ..., k$.

We say that the $E(G)$ is symmetric if $x, y \in V(G)$ and $(x, y) \in E(G)$ implies $(y, x) \in E(G)$.

The conversion of G, which is denoted by G^{-1}, is the graph obtained from G by reversing the direction of edges of G.

The undirected graph obtained from G by ignoring the direction of its edges is denoted by $\tilde{G}$. Actually it is more convenient for us to treat $\tilde{G}$ as a digraph for which the set of its edges is symmetric. Under this convention, $E(\tilde{G}) = E(G) \bigcup E(G^{-1})$.

We say that G is connected if there is a path between any two vertices and we say it is weakly connected if $\tilde{G}$ is connected. If G is such that $E(G)$ is symmetric and x is a vertex in G, then the subgraph G_x consisting of all edges and vertices which are contained in some path beginning at x is called the component of G containing x. In this case $V(G_x) = [x]_G$, where $[x]_G$ is the equivalence class of the following relation R defined on $V(G)$ by the rule: yRz if there is a (directed) path in G from y to z. It is clear that G_x is connected.

Definition 1.12. *[67] Let $x_0 \in V(G)$ and A a subset of $V(G)$. We say that*

(i) A is dominated by x_0 if $(x_0, x) \in E(G)$ for all $x \in A$.

ii A dominates x_0 if for each $x \in A$, $(x, x_0) \in E(G)$.

Definition 1.13. *Let C be a nonempty subset of a normed space X and let $G = (V(G), E(G))$ be a digraph such that $V(G) \subset C$. Then C*

is said to have property A if for each sequence $\{x_n\}$ in C converging weakly to $x \in C$ and $(x_n, x_{n+1}) \in E(G)$, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x) \in E(G)$ for all $k \in \mathbb{N}$.

Remark 1.3. *If G is transitive, then property A is equivalent to the property: if $\{x_n\}$ is a sequence in C with $(x_n, x_{n+1}) \in E(G)$ such that for any subsequence $\{x_{n_j}\}$ of the sequence $\{x_n\}$ converging weakly to x in X, then $(x_n, x) \in E(G)$ for all $n \in \mathbb{N}$.*

1.2.4 Nonlinear Mappings in Banach and Modular Metric Spaces

In this subsection, we give the definition of different classes of nonlinear mappings.

Definition 1.14. Let C be a nonempty subset of a metric space X with metric d. A mapping $T : C \to C$ is called contraction if there exists $k \in [0, 1)$ such that for any $x, y \in C$, we have

$$d(T(x), T(y)) \leq kd(x, y).$$

Definition 1.15. Let C be a nonempty subset of a metric space X with metric d. A mapping $T : C \to C$ is called Kannan contraction if there exists $k < \frac{1}{2}$ such that for any $x, y \in C$, we have

$$d(T(x), T(y)) \leq k[d(x, Ty) + d(y, Tx)].$$

Definition 1.16. [23] Let C be a nonempty subset of a metric space X with metric d. A mapping $T : C \to C$ is called quasi contraction if there exists $k \in [0,1)$ such that for any $x, y \in C$, we have

$$\begin{aligned} d(T(x), T(y)) &\leq k \max\{d(x,y), d(x,T(x)), d(y,T(y)), \\ &\quad d(x,T(y)), d(y,T(x))\}. \end{aligned}$$

Definition 1.17. Let X be a nonempty set endowed with digraph G. A mapping $T : X \to X$ is called G-monotone if T preserves edges of G, i.e., for each $x, y \in X$,

$$(x,y) \in E(G) \;\Rightarrow\; (Tx, Ty) \in E(G).$$

Definition 1.18. [32] Let X be a metric space with metric d and endowed with digraph G. A mapping $T : X \to X$ is called G-contraction if T is G-monotone and decreases weight of edges of G, i.e., there exists $\alpha \in [0,1)$ such that for each $x, y \in X$,

$$(x,y) \in E(G) \Rightarrow d(T(x), T(y)) \leq \alpha d(x,y).$$

Definition 1.19. [6] Let C be a nonempty subset of metric space X with metric d. A mapping $T : C \to C$ is called G-monotone quasi contraction if T is G-monotone and there exists $k \in [0,1)$ such that for each $x, y \in C, (x,y) \in E(G)$ we have

$$\begin{aligned} d(T(x), T(y)) &\leq k \max\{d(x,y), d(x,T(x)), d(y,T(y)), \\ &\quad d(x,T(y)), d(y,T(x)\}. \end{aligned}$$

The definitions of Lipschitzian mappings in modular metric spaces are straightforward generalizations of their metric counterparts.

Definition 1.20. Let (X, ω) be a modular metric space. Let C be a nonempty subset of X_ω. A mapping $T : C \to C$ is called Lipschitzian mapping if there exists a constant $k \geq 0$ such that for each $x, y \in C$,

$$\omega_1(Tx, Ty) \leq k\omega_1(x, y).$$

The smallest such constant k will be known as $Lip_\omega(T)$. If $Lip_\omega(T) < 1$, then T is called a ω-contraction mapping and if $Lip_\omega(T) \leq 1$, then T is called ω-nonexpansive.

Definition 1.21. [6] Let C be a nonempty subset of a modular metric space (X, ω). A mapping $T : C \to C$ is said to be G-monotone ω-quasi contraction if T is G-monotone and there exists $k \in [0, 1)$ such that for any $x, y \in C$, and $(x, y) \in E(G)$, we have

$$\begin{aligned}\omega_1(T(x), T(y)) \leq\ & kmax\{\omega_1(x, y), \omega_1(x, T(x)), \omega_1(y, T(y)), \\ & \omega_1(x, T(y)), \omega_1(T(x), y)\}\end{aligned}$$

For nonlinear contraction mappings, we need a class of comparison functions.

Definition 1.22. Let Φ be the class of all functions $\varphi : [0, \infty) \to [0, \infty)$ such that:

(i) φ is nondecreasing and upper semi-continuous.

(ii) $\varphi(t) < t,\ \forall t > 0$.

In the literature, Φ is called the class of comparison functions. If in addition φ satisfies the condition:

(iii) $\sum_{n=1}^{\infty} \phi^n(t) < \infty$, then Φ is called the class of strong comparison functions .

Lemma 1.1. [65] If $\varphi \in \Phi$, then $\varphi(0) = 0$, and $\lim_{n\to\infty} \varphi^n(t) = 0$

for each $t > 0$, where φ^n is the n-times composition of the function φ with itself.

As a consequence of property (ii), we can see that the sequence $\{\varphi^n(t)\}$ is a nonincreasing sequence, for any $t > 0$.

Definition 1.23. Let X be a metric space with metric d and let C be a nonempty subset of X. A mapping $T : C \to C$ is called nonlinear contraction if there exists a $\varphi \in \Phi$ such that for each $x, y \in C$,

$$d(Tx, Ty) \leq \varphi(d(x, y)).$$

Motivated by the previous definitions, we give the definition of G-monotone generalized quasi-contraction mappings on the setting of metric spaces with graph structure.

Definition 1.24. Let X be a metric space with metric d and let C be a nonempty subset of X endowed with digraph G. A mapping $T : C \to C$ is called G-monotone generalized quasi contraction if T is G-monotone and there exists a $\varphi \in \Phi$ such that for any $x, y \in C$ with $(x, y) \in E(G)$, we have

$$\begin{aligned} d(T(x), T(y)) \ \leq \ & max\{\varphi(d(x, y)), \varphi(d(x, T(x))), \varphi(d(y, T(y))), \\ & \varphi(d(x, T(y))), \varphi(d(T(x), y))\}. \end{aligned}$$

Remark 1.4. If we take $\varphi(t) = kt$, where $k \in [0, 1)$, then the above definition is reduced to the definition due to Alfuraidan [6].

Let X be a metric space with metric d and C be a nonempty subset of X such that $T : C \to C$ is any mapping. For any $x \in C$, we define the orbit

$$O(x) = \{x, T(x), T^2(x), \cdots\}$$

and its diameter by

$$\delta(x) = sup\{d(T^n(x), T^m(x)) : n, m \in \mathbb{N}_0\}.$$

The counter part of the above definition in the setting of modular metric spaces with graph structure is given as follows:

Definition 1.25. Let C be a nonempty subset of a modular metric space (X, ω). A mapping $T : C \to C$ is said to be G-monotone generalized ω-quasi contraction if T is G-monotone and there exists $\varphi \in \Phi$ such that for any $x, y \in C$, and $(x, y) \in E(G)$, we have

$$\begin{aligned} \omega_1(T(x), T(y)) \;\leq\; & max\{\varphi(\omega_1(x, y)), \varphi(\omega_1(x, T(x))), \varphi(\omega_1(y, T(y))), \\ & \varphi(\omega_1(x, T(y))), \varphi(\omega_1(T(x), y))\} \end{aligned}$$

Let C be a nonempty subset of a modular metric (X, ω) and let $T : C \to C$ be any self mapping. For any $x \in C$ we define the ω-diameter of orbit $O(x)$ by

$$\delta_\omega(x) = sup\{\omega_1(T^n(x), T^m(x)) : n, m \in \mathbb{N}_0\}.$$

Definition 1.26. Let C be a nonempty subset of a Banach space X. A mapping $T : C \to C$ is called nonexpansive if

$$||Tx - Ty|| \;\leq\; ||x - y||, \; \forall x, y \in C. \tag{1.2}$$

Definition 1.27. Let C be a nonempty subset of a Banach space X. A mapping $T : C \to C$ is called asymptotically nonexpansive if there exists a sequence $\{u_n\} \subset [0, \infty)$, $u_n \to 0$ as $n \to \infty$ such that for all $x, y \in C$ we have

$$||T^n x - T^n y|| \ \leq \ (1 + u_n)||x - y|| \text{ for all } n \in \mathbb{N}.$$

Definition 1.28. [7] Let C be a nonempty subset of a Banach space X such that C is endowed with digraph G. A mapping $T : C \to C$ is called G-nonexpansive if it satisfies the conditions:

(i) $(x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G)$ i.e., T preserves edges of G;

(ii) $||Tx - Ty|| \leq ||x - y||$ for each $(x, y) \in E(G)$.

Motivated by the above definitions, a new class of mappings in Banach spaces is defined as follows.

Definition 1.29. Let C be a nonempty subset of a Banach space X such that C is endowed with digraph G. A mapping $T : C \to C$ is called G-asymptotically nonexpansive if it satisfies the conditions:

(i) $(x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G)$;

(ii) there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim\limits_{n\to\infty} k_n = 1$ and $||T^n x - T^n y|| \leq k_n ||x - y||$ for each $(x, y) \in E(G)$ and $n \in \mathbb{N}$.

Chapter 2

Review of Literature

2.1 Common Fixed Point Approximation Methods for a Family of Nonlinear Mappings

It was Picard who first applied the technique of successive approximations to solve certain problems which can be formulated as fixed point problem of nonlinear mappings. This method of Picard was precisely formulated by Banach [8] in 1922. This fundamental result is stated as follows:

Theorem 2.1. [8] Let (M, d) be a complete metric space and $T : M \to M$ be a contraction mapping with constant k. Then T has a unique fixed point p in M, and for each $x \in M$, we have

$$\lim_{n\to\infty} T^n x \;=\; p.$$

Moreover, for each $x \in M$, we have

$$d(T^n x, p) \;\leq\; \frac{k^n}{1-k} d(Tx, x).$$

The contraction mapping principle due to Banach [8] is simple and easy to apply and perhaps the most widely applied result in the theory of fixed point problems. This is because, it does not only guarantee the existence of unique fixed point but also provides an approximation technique to find the fixed point under suitable conditions. It only requires the completeness of the metric space for its setting.

Independent of the results due to Banach, in 1927, Tarski and Knaster proved a set theoretical fixed point theorem by which every function on and to the family of all subsets of a set, which is increasing under set theoretical inclusion, has at least one fixed point. In 1955, Tarski [74] obtained the following fundamental theorem on the existence of fixed points for increasing functions defined on a complete lattice:

Theorem 2.2. [74] Let

(i) $\mho = \langle A, \preceq \rangle$ be a complete lattice,

(ii) f be an increasing functions on A to A,

(iii) P be the set of all fixed points of f.

Then the set P is not empty and the system $\langle P, \preceq \rangle$ is a complete lattice.

The results due to Tarski are also widely applicable. Therefore, like Banach contraction mapping principle, Tarski's result is also extended in different directions (see, e.g., [27, 47, 57, 76]). Apart from extending these two independent fundamental theorems in different directions, in 2004, Ran and Ruerings considered the hybrid of the two. In [57], the authors obtained a new fixed point theorems for monotone single valued mappings in metric spaces endowed with partial ordering. The following result is due to Ran and Ruerings

[57]:

Theorem 2.3. [57] Let $\langle X, \preceq \rangle$ be a partially ordered set such that every pair $x, y \in X$ has an upper bound and a lower bound. Furthermore, let d be a metric on X such that (X, d) is a complete metric space. Let $f : X \to X$ be a continuous monotone (either order-preserving or order-reversing) mapping. Suppose that the following conditions hold.

(i) There exists $k \in (0, 1)$ with $d(f(x), f(y)) \leq kd(x, y)$, for all $x \succeq y$.

(ii) There is $x_0 \in X$ with $x_0 \succeq f(x_0)$ or $x_0 \preceq f(x_0)$. Then f is a Picard Operator (PO) that is, there is a unique fixed point x^* of f such that for each $x = x_0 \in X$, the sequence $x_{n+1} = f(x_n)$ converges to x^*.

The results due to Ran and Ruerings have been widely investigated. On the other hand, in 2008, Jachymski [32] investigated a new approach in metric fixed point theory by replacing the order structure considered by Ran and Reurings with graph structure. In this way, the results proved in ordered metric spaces are generalized (see for detail [32] and the reference therein). The result due to Jachymski in [32] is stated as follows:

Theorem 2.4. [32] Let (X, d) be complete, and let the triple (X, d, G) have the following property property P : for any sequence (x_n) in X if $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$, then there is a subsequence $(x_{k_n})_{n \in b\mathbb{N}}$ with $(x_{k_n}, x) \in E(G)$ for $n \in \mathbb{N}$. Let $f : X \to X$ be a G-contraction, and set $X_f := \{x \in X : (x, fx) \in E(G)\}$. Then the following statements hold.

(1) $Card(Fixf) = Card\{[x]_{\tilde{G}} : x \in X_f\}$.

(2) $Fixf \neq \emptyset$ if and only if $X_f \neq \emptyset$.

(3) f has a unique fixed point if and only if there exists $x_0 \in X_f$ such that $X_f \subseteq [x_0]_{\tilde{G}}$.

(4) For any $x \in X_f$, $f|_{[x]_{\tilde{G}}}$ is a PO.

(5) If $X_f \neq \emptyset$and G is weakly connected, then f is a PO.

(6) If $X' := \bigcup\{[x]_{\tilde{G}} : x \in X_f\}$, then $f|_{X'}$ is WPO.

(7) If $f \subseteq E(G)$ then f is a WPO, that is, for each $x = x_0 \in X$, the sequence $x_{n+1} = f(x_n)$ converges to a fixed point of f.

After the appearance of this famous result, many authors (see, e.g., [3–6, 9]) have obtained fixed point results for different class of non-linear maps defined in spaces with graph structure.

However, the Picard iteration scheme fails to converge to a fixed point (if it exists) of a nonexpansive mappings. In an attempt to obtain an iterative scheme which converges to a fixed point of nonexpansive self mappings, many iterative schemes have been introduced. In 1953, Mann [50] introduced the one-step iterative method:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n)Tx_n, \ n \geq 0 \tag{2.1}$$

where the initial guess $x_0 \in C$ is taken arbitrary and the sequence $\{\alpha_n\}$ is in the interval $[0, 1]$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$.

In general, the iterative process (2.1) is referred to as Mann iterative method for finding fixed points of nonexpansive mappings and extensively studied by several authors (see, e.g., [58, 62]) in Banach spaces. However, if T is only a pseudocontractive mapping, then generally the Mann iterative process does not converge to the fixed point (see, [18]). In 1974, Ishikawa [30] constructed an iterative process which strongly converges to a fixed point of Lipschitz pseudocontractive self-mappings as follows:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T((1-\beta_n)x_n + \beta_n T(x_n)),\ n \geq 0$$

where the initial guess $x_0 \in C$ is taken arbitrary and the sequence $\{\alpha_n\}$ and $\{\beta_n\}$ are in the interval $[0,1]$ such that $0 < \alpha_n \leq \beta_n < 1,\ \lim_{n\to\infty} \beta_n = 0,\ \sum_{n=1}^{\infty} \alpha_n\beta_n = \infty$. In its original form, the Ishikawa iteration does not include the Mann iteration. Ishikawa iteration which includes the Mann iteration as a special case is:

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n \\ y_n = (1-\beta_n)x_n + \beta_n T x_n,\ n \geq 0 \end{cases} \tag{2.2}$$

where x_0 is taken arbitrary and $\{\alpha_n\}, \{\beta_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0, \frac{1}{2})$.

Although, Ishikawa iteration process is designed for approximating fixed point of Lipschitz pseudocontractive mappings, several authors (see, e.g., [66, 68, 72, 73, 77]) were used the method for nonexpansive mappings too.

On the other hand, finding common fixed point of a finite family

$\{T_i\}_{i=1}^k$ of nonlinear mappings acting on a Banach space is a problem that often arises in applied sciences. In fact, many iterative process have been employed to find common fixed point of a finite family of nonlinear mappings in the setting of Banach spaces (see, e.g., [62] and references there in). In particular, the Mann-type iterative method is used for approximating common fixed point of a finite family of nonexpansive mappings in Banach spaces setting. On the other hand, Takhashi and Tamura [68] used an Ishikawa-type scheme defined by:

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_1 y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T_2 x_n, \end{cases} \tag{2.3}$$

to approximate a common fixed point of a pair of nonexpansive mappings T_1 and T_2 defined in Banach spaces.

Recently, several authors have studied fixed point results for the class of G-nonexpansive mappings (see, e.g., [4, 67, 75, 77]). The concept of Monotone G-nonexpansive self-mappings in Bach spaces is first introduced by Alfraidan [4] in 2015. In [4], the author study the convergence of Krasnoselskii sequence to fixed points of such class of mappings. Tiammee et al.[75] proved Browders theorem and the convergence of Halpern iteration for a G-nonexpansive mapping in a Hilbert space with a directed graph. Tripak in [77] obtained convergence theorems for Ishikawa-type iterative scheme (2.3) to common fixed point of a pair of G-nonexpansive self mappings in Banach spaces with graph structure. Specifically, he obtained the following weak and strong convergence results for a couple of G-nonexpansive mappings in Banach spaces with graph structure.

Theorem 2.5. [77] Suppose X is a uniformly convex Banach space and $\{\alpha_n\},\ \{\beta_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0, \frac{1}{2}),\ T_i(i = 1, 2)$ satisfy Condition $B,\ F$ dominates $x_0,\ F$ is dominated by x_0 and

$$(x_0, z), (y_0, z), (z, x_0), (z, y_0) \in E(G) \text{ for each } z \in F \text{ and arbitrary } x_0 \in C.$$

Then $\{x_n\}$ converges strongly to a common fixed point of T_i.

For a nonempty closed convex subset of a real uniformly convex Banach space X, a finite family of mappings T_i on C are said to satisfy Condition B if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that, for all $x \in C$,

$$\max\{||x - T_i(x)||\} \geq f(d(x, F))$$

where $F = \bigcap F(T_i)$ and $F(T_i)$ are the sets of fixed points of T_i.

Definition 2.1. *[66] Let C be a subset of a metric space (X, d). A mapping $T : C \to C$ is semi-compact if for a sequence $\{x_n\}$ in C with $\lim\limits_{n\to\infty} d(x_n, Tx_n) = 0$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to p \in C$ as $j \to \infty$.*

Theorem 2.6. [77] Suppose X is a uniformly convex Banach space and $\{\alpha_n\},\ \{\beta_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0, \frac{1}{2})$, one of $T_i(i = 1, 2)$ is semi-compact, F dominates $x_0,\ F$ is dominated by x_0 and $(x_0, z), (y_0, z), (z, x_0), (z, y_0) \in E(G)$ for each $z \in F$ and arbitrary

$x_0 \in C$. Then $\{x_n\}$ converges strongly to a common fixed point of T_i.

Theorem 2.7. [77] Suppose X is uniformly convex Banach space and $\{\alpha_n\}$, $\{\beta_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0, \frac{1}{2})$. If X satisfies Opial's property, $I - T_i$ is G-demiclosed at zero for each i, F dominates x_0, F is dominated by x_0 and $(x_0, z_0), (y_0, z_0), (z_0, x_0), (z_0, y_0) \in E(G)$ for $z_0 \in F$ and arbitrary $x_0 \in C$, then $\{x_n\}$ converges weakly to a common fixed point of T_i.

In an attempt to generalize Ishikawa iteration, in 2000, Noor [48] obtained three-step iterative method:

$$\begin{cases} z_n = (1-\gamma_n)x_n + \gamma_n T x_n, \\ y_n = (1-\beta_n)x_n + \beta_n T z_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \ n = 1, 2, 3, , \cdots \end{cases} \tag{2.4}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences of real numbers in $[0, 1]$ to approximate fixed points of nonexpansive self mappings in Banach spaces.

In addition to including Mann (one-step) and Ishikawa (two-step) iterations as special cases, the theory of three-step iterative scheme is very rich, and this scheme, in the context of one or more mappings, has been extensively investigated by many authors (see, e.g., [10, 48, 49, 63]). It is shown in [10] that three-step method performs better than two-step and one-step methods for solving variational inequalities. The three-step schemes are natural generalizations of the splitting methods to solve partial differential equations(or inclusions). This signifies that Noor three-step method is robust and more efficient than the Mann and Ishikawa iterative methods to solve problems of pure and applied sciences.

One of our purpose in this thesis is to extend the result due to Tripak [77] from a couple of mappings to arbitrary finite family of mappings. However, the Ishikawa-type iteration method is no longer applicable for the approximation of common fixed point of a family of three or more nonexpansive mappings. To resolve this problem, new iterative process that includes the iterative schemes due to Mann, Ishikawa and Noor is introduced.

On the other hand, many research findings have been obtained on iterative methods for the class of asymptotically nonexpansive mappings. Bose [11], in 1978 initiated the study of iterative construction for fixed points of asymptotically nonexpansive mappings and proved that, if K is a bounded closed convex nonempty subset of a uniformly convex Banach space X satisfying Opial's condition and $T : K \to K$ is an asymptotically nonexpansive mapping, then the sequence $\{T^n x\}$ converges weakly to a fixed point of T provided T is asymptotically regular at $x \in K$, i.e., $\lim\limits_{n\to\infty} ||T^n x - T^{n+1}x|| = 0$. For its convergence, Bose imposed strong conditions on the operator and the space under consideration. Since then, a variant of research results were obtained mainly focusing on the weakening of the strong conditions imposed on the operator and the ambient domain. In line to this, Pastry [54] proved that the requirement that X satisfies the

Opial's condition can be replaced by the Frechet differentiable norm. Schu [64], in 1991 introduced the modified Mann iteration process:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \; n = 1, 2, 3, \cdots$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ which is bounded away from 0 and 1, i.e., $a \leq \alpha_n \leq$ for all n and some $0 < a \leq b < 1$ to approximate fixed points of asymptotically nonexpansive self-maps defined on nonempty closed convex and bounded subsets of a Hilbert space. Later on, Tan and Xu [70] proved that the asymptotic regularity of T at x can be replaced by weak asymptotic regularity of T at x, i.e., $\omega\text{-}\lim\limits_{n\to\infty}(T^n x - T^{n+1}x) = 0$. Osilike and Aniagbosor [52] proved that the theorem of Schu [64] remains true without the boundness assumption on K provided that the fixed point set is nonempty. Furthermore, Chang et al. [14] proved convergence theorems for asymptotically nonexpansive mappings and nonexpansive mappings in Banach spaces without assuming any of the conditions (a) X satisfies Opial's condition (b) T is weak-asymptotically regular (c) K is bounded. Following the appearance of Chang et al. [14] results, many known mathematician were obtained different results for nonexpansive and asymptotically nonexpansive self as well as non-self mappings in Banach spaces without assuming the Opial's condition on it (see, e.g., [17, 61]).

As Ishikawa iterations are more general than Mann iteration, it is natural to consider the modified version of Ishikawa iteration which include the iteration process due to Schu [64] to approximate fixed points of asymptotically nonexpansive self-mappings. In 1994, Tan and Xu [71] studied the modified Ishikawa iteration process and used to approximate fixed points for asymptotically nonexpansive mappings:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n((1-\beta_n)x_n + \beta_n T^n x_n)), \ n = 1, 2, 3, \cdots$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$ such that α_n is bounded away from 0 and 1 and β_n is bounded away from 1. Khan and Takahashi [39] have approximated common fixed points of two asymptotically nonexpansive self-mappings by using the modified Ishikawa iteration process:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n((1-\beta_n)x_n + \beta_n S^n x_n)), \ n = 1, 2, 3, \cdots$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$ such that α_n is bounded away from 0 and 1 and β_n is bounded away from 1.

However, there is no research findings regarding fixed point approximation methods for the class of G-asymptotically nonexpansive mappings so far.

2.2 Fixed Point of Nonlinear Mappings in Modular Metric Spaces

Due to its wide range of applications, Banach contraction mapping principle has been extended and generalized in different directions. One of the significant generalization of Banach's principle was carried out by Boyd and Wang [12] in 1969. They introduced a new class of mappings called *nonlinear contraction mappings* using certain functions with some general properties and study fixed point results for such class of mappings. To be specific, they obtain the following results.

Theorem 2.8. [12] Let X be a complete metric space and suppose $T : X \to X$ satisfies

$$d(Tx, Ty) \ \leq \ \phi(d(x, y)), \text{ for each } x, y \in X$$

where $\phi : \mathbb{R}^+ \to [0, \infty)$ is upper semi-continuous from the right and satisfies

$$0 \leq \phi(t) < t, \text{ for all} t > 0.$$

Then T has a unique fixed point p, and $\{T^n x\}$ converges to p for each $x \in X$.

Note that a function $f : X \to (-\infty, \infty)$ on a topological space X is upper semi-continuous at the point $x \in X$ if $x_n \to x \Rightarrow \limsup\limits_{n\to\infty} f(x_n) \leq f(x)$.

Since it is the explicit control over the error term that plays too much role for the wide range applications of Banach's principle, the following variant of the Boyd-Wong theorem due to Browder [13] is also of great interest.

Theorem 2.9. [13] Let X be a complete metric space and M be a bounded subset of X. Suppose $T : M \to M$ satisfies

$$d(Tx, Ty) \ \leq \ \phi(d(x, y)), \text{ for each } x, y \in M$$

where $\phi : [0, \infty) \to [0, \infty)$ is monotone nondecreasing and continuous from the right such that

$$\phi(t) \;<\; t, \quad \text{for all } t > 0.$$

Moreover, if d_0 is the diameter of M, then

$$\begin{aligned} d(T^n x, p) \;&\leq\; \phi^n(d_0) \text{ and} \\ \phi^n(d_0) \to 0 \text{ as } n \to \infty. \end{aligned}$$

In 1975, Matkowski [43] replaces the continuity condition on ϕ with another condition and obtained the following variant of the result due to Boyd and Wong.

Theorem 2.10. [43] Let X be a complete metric space and suppose $T : M \to M$ satisfies

$$d(Tx, Ty) \;\leq\; \phi(d(x, y)), \text{ for each } x, y \in X$$

where $\phi : (0, \infty) \to (0, \infty)$ is monotone nondecreasing and satisfies

$$\lim_{n\to\infty} \phi^n(t) \;=\; 0, \text{ for } t > 0.$$

Then T has a unique fixed point p, and

$$\lim_{n\to\infty} d(T^n x, p) \;=\; 0, \text{ for every } x \in X.$$

The results due to Matkowski [43] is generalized in different directions mainly on relaxing certain properties of the ambient class of functions. It is Arandelovic [31] who gave a simple and unified treatment of fixed points in this direction by using different classes of comparison functions.

Also Banach contraction mapping principle is fundamental and widely applicable, it is forceful, that means the contractive condition forces the operator to become uniformly continuous. So, it is desirable to look for another type of contraction mappings in which the operator under consideration is not necessarily continuous, but the Picard iteration process still converges to the fixed point. In this regard, it was Kannan [34] in 1969, who came up with the following influential result in fixed point theory in which the operator under consideration need not be continuous but the Picard iteration process still converges to the fixed point. This result is stated as follows:

Theorem 2.11. [34] Let (X, d) be metric space. Then a Kannan operator T on X has a fixed point if and only if X is complete.

We see that the contraction mapping due to Kannan is not an extension of Banach contraction mapping principle. Later on, Ciric define a new type of contraction mappings which generalizes these two independent results. In 1974, Ciric [23] introduced quasi contraction mappings which are more general than Banach contraction and Kannan contraction mappings. In [23], the author proved the following most well-known result in which the Picard iteration process converges to the fixed point.

Theorem 2.12. [23] Let the metric space X be T-Orbitally complete and let T be quasi-contraction. Then we have

1. T has a unique fixed point p in X.

2. $\lim T^n x = x,$ for $x \in X$.

3. $d(T^n x, p) \leq \frac{q^n}{1-q} d(x, Tx)$ for all $x \in X$.

Despite of its diverse applicability, metric spaces, particularly Banach spaces are limited to encounter certain problems raised from practical situations. These attracts several researchers in the area of nonlinear analysis to extend Banach spaces, in particular, L^p to a more general abstract spaces. In 1950, Nakanno [46] introduced the concept of modular spaces by replacing particular integral type in the L^p spaces by an abstract functional called a modular. Musielak and Orlicz in [44] generalized the L^p spaces to L^φ spaces called *Orlicz* spaces. The Orlicz space is seen as a vector space endowed with a modular function. It is proved in [41] that this space is a complete normed linear space with norm usually called *Luxemburg* norm. But the linear modular spaces or spaces with algebraic structures are still limited to solve certain problems from set valued analysis, such as supper position operators [21]. Thus, the modular theory on an arbitrary set was proposed. The affirmative response to the problem is appeared in 2010 by Chistyakov [20]. In [20], the author introduced a modular metric space which are nonlinear versions of modular function spaces and is more general than the usual metric spaces. A year later, Chistyakov [22] defined contraction mappings and obtained the following analogue results of the Banach contraction mapping principles in the setting of modular metric spaces.

Theorem 2.13. [22] Let (X, ω)be a modular metric space. Assume that ω is regular. Let C be a nonempty subset of X_ω. Assume that C

is ω-complete and ω-bounded. Let $T : C \to C$ be an ω-contraction. Then T has a unique fixed point x_0. Moreover, the Orbit $\{T^n x\}$ ω-converges to x_0 for each $x \in C$.

The result in [22] opens a new era in the development of fixed point theory and its wide range of applicability. In 2015, Alfraidan [5] considered modular metric spaces with graph structure and extend the results due to Jachymski to the class of modular metric spaces. Very soon, in the same year, the author extended his result in [5] to quasi-contraction mappings in the setting of modular metric spaces with graph structure and obtained the following two results. The fist one is on metric spaces where as the second one is on the setting modular metric spaces.

Theorem 2.14. [6] Let (X, d) be a metric space and G be a reflexive transitive digraph on X. Assume that (X, d) is complete. Let C be a closed nonempty subset of X and $T : C \to C$ be a G-monotone quasi contraction mapping. Let $x \in C$ be such that $(x, T(x)) \in E(G)$ and $\delta(x) < \infty$. Then:

(i) $\{T^n(x)\}$ converges to $P \in C$ which is a fixed point of T and $(x, P) \in E(G)$. Moreover, we have

$$d(T^n(x), P) \leq k^n \delta(x),\ n \geq 0.$$

(ii) if z is a fixed point of T such that $(x, z) \in E(G)$, then $z = P$.

Theorem 2.15. [6] Let (X, ω) be a modular metric space and G be a reflexive transitive digraph on X. Let C be a nonempty subset of X which is ω-complete. Let $T : C \to C$ be a G-monotone ω-quasi contraction mapping. Let $x \in C$ be such that $(x, T(x)) \in E(G)$ and $\delta_\omega(x) < \infty$. Then :

(i) $\{T^n x\}$ ω-converges to $z \in C$ which is a fixed point of T and $(x, z) \in E(G)$, provided $\omega_1(z, Tz) < \infty$ and $\omega_1(x, Tz) < \infty$. Moreover, we have

$$\omega_1(T^n x, z) \leq k^n \delta_\omega(x), \; n \geq 1.$$

(ii) If y is a fixed point of T such that $(x, y) \in E(G)$ and $\omega_1(T^n x, y) < \infty$ for any $n \geq 1$ then $z = y$.

One of our purpose in this thesis is to extend the results due to Alfraidan [6] to a more general class of mappings called G-monotone generalized quasi-contraction mappings which is the nonlinear case of G-monotone quasi contraction mappings considered by the author in [6].

2.3 Statement of the Problem

From reviewed literature, we came up with the following research problems.

(1) In Theorems 2.5 through Theorem 2.7, the space under consideration is very limited because of the Opial's condition assumption on it; if the number of mappings are more than two then the used method is no longer valid; the dominance assumption made among the points is more than necessary. To solve the problem, new iterative method, we call it a finite Noor iterative

method is proposed. This iterative method is defined as follows:

Let X be a Banach space and C be a nonempty subset of X such C is endowed with a digraph G. Let $\{T_i\}_{i=1}^{k}$ be a finite family of G-nonexpansive self mappings on C, define a sequence $\{x_n\}$ as follows:

$$\begin{cases} x_{n+1} = (1-\alpha_{k,n})x_n + \alpha_{k,n}T_k y_{k-1,n} \\ y_{k-1,n} = (1-\alpha_{k-1,n})x_n + \alpha_{k-1,n}T_{k-1} y_{k-2,n} \\ y_{k-2,n} = (1-\alpha_{k-2,n})x_n + \alpha_{k-2,n}T_{k-2} y_{k-3,n} \\ \vdots \\ y_{2,n} = (1-\alpha_{2,n})x_n + \alpha_{2,n}T_2 y_{1,n} \\ y_{1,n} = (1-\alpha_{1,n})x_n + \alpha_{1,n}T_1 x_n \\ y_{0,n} = x_n, \text{ for each } n \geq 0 \end{cases} \tag{2.5}$$

where $\alpha_{i,n} \subset (0, \frac{1}{2})$ for each $n \in \mathbb{N}$ and arbitrary $x_1 \in C$.

(2) The study of fixed point problems for the class of G-asymptotically nonexpansive mappings which are more general than that of G-nonexpansive as well as asymptotically nonexpansive mappings get less emphasis. In particular, the study of common fixed point approximation methods for a finite family of G-asymptotically nonexpansive mappings in uniformly convex Banach spaces with graph structure is not yet investigated. To handle the problem, the modified Noor iterative method defined below is proposed.

Let X be a Banach space and C be a nonempty subset of X such

C is endowed with a digraph G. Let $\{T_i\}_{i=1}^k$ be a finite family of G-asymptotically nonexpansive self-mappings in C, define a sequence $\{x_n\}$ as follows:

$$\begin{cases} x_{n+1} = (1-\alpha_{k,n})x_n + \alpha_{k,n}T_k^n y_{k-1,n}, \\ y_{k-1,n} = (1-\alpha_{k-1,n})x_n + \alpha_{k-1,n}T_{k-1}^n y_{k-2,n}, \\ y_{k-2,n} = (1-\alpha_{k-2,n})x_n + \alpha_{k-2,n}T_{k-2}^n y_{k-3,n}, \\ \vdots \\ y_{2,n} = (1-\alpha_{2,n})x_n + \alpha_{2,n}T_2^n y_{1,n}, \\ y_{1,n} = (1-\alpha_{1,n})x_n + \alpha_{1,n}T_1^n x_n, \end{cases} \tag{2.6}$$

where $y_{0,n} = x_n$ for each $n \in \mathbb{N}$ and arbitrary $x_1 \in C$.

(3) The study of nonlinear contraction mappings in modular metric spaces with graph structure is overlooked. In particular, the existence and approximation of fixed point of G-monotone generalized quasi contraction mappings has not studied so far.

Therefore, the aforementioned research gaps lead us to pose the following research questions.

2.4 Research Questions

1. Can we apply the finite Noor iteration process to find approximate common fixed point of a finite family of G-nonexpansive self mappings in uniformly convex Banach spaces with graph structure?

2. Can we establish a necessary and sufficient conditions for the

convergence of the modified Noor iterations to converge to common fixed points of a finite family of G-asymptotically nonexpansive self mappings in uniformly convex Banach spaces with graph structure?

3. Can we establish a necessary and sufficient conditions for the convergence of the Picard iterations to fixed point of G-monotone generalized quasi contraction mappings in the setting of modular metric spaces with graph structure?

2.5 Objectives

Our objectives in this thesis are

- to construct a finite Noor iterative method which converges to a common fixed point of a finite family of G-nonexpansive self mappings in uniformly convex Banach spaces with graph structure.
- to establish a necessary and sufficient conditions for the convergence of the modified Noor iteration method to a common fixed point of a finite family of G-asymptotically nonexpansive self mappings in uniformly convex Banach spaces with graph structure .
- to establish ω-convergence of the Picard iteration to fixed point of G-monotone generalized quasi contraction mappings in the setting of modular metric spaces with graph structure.

2.6 Scope of the Study

This thesis is limited to the study of convergence theorems:

- for a finite Noor iteration method to find a common fixed point of a finite family of G-nonexpansive self mappings in uniformly convex Banach spaces.
- for the modified Noor iteration method to find a common fixed point of a finite family G-asymptotically nonexpansive self mappings in uniformly convex Banach spaces.
- for Picard iteration method to find fixed point of G-monotone generalized quasi contraction mappings in modular metric spaces satisfying the Δ_2-condition.

2.7 Methodology

At first, a finite Noor iterative method (2.5) which could be used to approximate a common fixed point of a finite family of G-nonexpansive self mappings is defined. In (2.5): if $k = 2$ then we have the Ishikawa-type iterative method (2.3); if $k = 3$ and $T_1 = T_2 = T_3 = T$ then we have the Noor iterative method (2.4).

Secondly, the modified Noor iterative method (2.6) is introduced and employed for the approximation of a common fixed point of a finite family of G-asymptotically nonexpansive self mappings in uniformly convex Banach spaces with graph structure.

Finally, the Picard iteration technique is used for the approximation of a fixed point of G-monotone generalized quasi contraction mappings in modular metric spaces with directed graph. For the uniqueness of fixed points, an additional assumption on edge preserving of the operator under considerations is imposed.

Chapter 3

Some Auxiliary Mathematical Tools

In this chapter, some results from the literature which will be used in the sequel are stated. The results are already proved in the cited references and so we advise interested readers to refer them.

Lemma 3.1. [14] Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers with $\sum_{n=1}^{\infty} b_n < \infty$. If one of the following conditions is satisfied:

(i) $a_{n+1} \leq a_n + b_n,\ n \geq 1,$

(ii) $a_{n+1} \leq (1 + b_n)a_n,\ n \geq 1,$

then $\lim_{n\to\infty} a_n$ exists.

Lemma 3.2. [80] Let X be a Banach space, and $R > 1$ be a fixed number. Then X is uniformly convex if and only if there exists a continuous, strictly increasing, and convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that

$$||\lambda x + (1-\lambda)y||^2 \ \leq\ \lambda||x||^2 + (1-\lambda)||y||^2 - \lambda(1-\lambda)g(||x-y||)$$

for all $x, y \in B_R(0) = \{x \in X : ||x|| \leq R\}$ and $\lambda \in [0, 1]$.

Lemma 3.3. [64] Let X be a uniformly convex Banach space and $\{\alpha_n\}$ a sequence in $[\delta, 1-\delta]$ for some $\delta \in (0, 1)$. Suppose that sequences $\{x_n\}$ and $\{y_n\}$ in X are such that

(i) $\limsup_{n\to\infty} ||x_n|| \le c$,

(ii) $\limsup_{n\to\infty} ||y_n|| \le c$,

(iii) $\lim_{n\to\infty} ||\alpha_n x_n + (1-\alpha_n)y_n|| = c$ for some $c \ge 0$

then $\lim_{n\to\infty} ||x_n - y_n|| = 0$.

Lemma 3.4. ([14]) Let X be a uniformly convex Banach space and C be a nonempty bounded convex subset of X. Then there exists a strictly increasing continuous convex function $\gamma : [0,\infty) \to [0,\infty)$ with $\gamma(0) = 0$ such that, for any Lipschitzian mapping $T : C \to X$ with the Lipschitz constant $L \ge 1$, any finite many elements $\{x_i\}_{i=1}^n$ in C and any finite many nonnegative numbers $\{t\}_{i=1}^n$ with $\sum_{i=1}^n t_i = 1$, the following inequality holds:

$$||T(\sum_{i=1}^n t_i x_i) - \sum_{i=1}^n t_i T x_i|| \le L\gamma^{-1} \max_{1\le i,j\le n} (||x_i - x_j|| - L^{-1}||Tx_i - Tx_j||).$$

Lemma 3.5. [61] Let $\{x_n\}$ be a bounded sequence in a reflexive Banach space X. If for any weakly convergent subsequence $\{x_{n_j}\}$ of $\{x_n\}$, both $\{x_{n_j}\}$ and $\{x_{n_j+1}\}$ converge weakly to the same point in X, then the sequence $\{x_n\}$ is weakly convergent.

Chapter 4

Main Results

4.1 Approximation of a Common Fixed Point of a Family of G-nonexpansive Mappings in Banach Spaces with a Graph

The following proposition is one of our main results and is used in the proof of the subsequent results. Below, we state and prove this fundamental result.

Proposition 4.1. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and $\{T_i\}_{i=1}^{k}$ is a finite family of G-nonexpansive mappings on C. Let $z_0 \in F$ be such that (x_0, z_0) and (z_0, x_0) are in $E(G)$ for arbitrary $x_0 \in C$. Then, for a sequence $\{x_n\}$ generated by x_0 with iterative scheme defined by (2.6), we have $(x_n, z_0), (z_0, x_n), (x_n, y_{i,n}), (y_{i,n}, x_n), (z_0, y_{i,n}), (y_{i,n}, z_0)$ and (x_n, x_{n+1}) are in $E(G)$ for each $i = 1, 2, 3, \cdots, k$ and $n = 0, 1, 2, \cdots$.*

Proof. We proceed by induction. First we let $(x_0, z_0) \in E(G)$. Since T_1 is edge-preserving, we have $(T_1x_0, z_0) \in E(G)$. By the convexity of $E(G)$, we get

$$(1 - \alpha_{1,0})(x_0, z_0) + \alpha_{1,0}(T_1x_0, z_0) \;=\; ((1 - \alpha_{1,0})x_0 + \alpha_{1,0}T_1x_0, z_0)$$

$$= (y_{1,0}, z_0) \in E(G).$$

Since T_2 is edge-preserving, $(T_2 y_{1,0}, z_0) \in E(G)$ and again by the convexity of $E(G)$ we have

$$\begin{aligned}(1-\alpha_{2,0})(x_0, z_0) + \alpha_{2,0}(T_2 y_{1,0}, z_0) &= ((1-\alpha_{2,0})x_0 + \alpha_{2,0} T_2 y_{1,0}, z_0) \\ &= (y_{2,0}, z_0) \in E(G).\end{aligned}$$

Assume that

$$(y_{l,0}, z_0) \in E(G), \ for \ some \ l \in \{1, 2, 3, \cdots, k-2\}.$$

As T_{l+1} is edge-preserving, $(T_{l+1} y_{l,0}, z_0) \in E(G)$ and by using the convexity of $E(G)$, we get

$$\begin{aligned}(1-\alpha_{l+1,0})(x_0, z_0) + \alpha_{l+1,0}(T_{l+1} y_{l,0}, z_0) &= ((1-\alpha_{l+1,0})x_0 + \\ &\quad \alpha_{l+1,0} T_{l+1} y_{l,0}, z_0) \\ &= (y_{l+1,0}, z_0) \in E(G).\end{aligned}$$

Thus

$$(y_{i,0}, z_0) \in E(G), \ \text{for each } i = 1, 2, 3, \cdots, k-1.$$

In particular, for $i = k-1$

$$(y_{k-1,0}, z_0) \in E(G).$$

Since T_k is edge-preserving, we have

$$(T_k y_{k-1,0}, z_0) \in E(G).$$

Using the convexity of $E(G)$, we have

$$(1-\alpha_{k,0})(x_0, z_0) + \alpha_{k,0}(T_k y_{k-1,0}, z_0) = ((1-\alpha_{k,0})x_0 +$$

$$\begin{aligned} & \alpha_{k,0}T_k y_{k-1,0}, z_0) \\ = & (x_1, z_0) \in E(G). \end{aligned}$$

Thus, we obtain

$$(y_{i,0}, z_0) \in E(G) \text{ for } i = 1, 2, 3, \cdots, k-1 \text{ and } (x_1, z_0) \in E(G).$$

By replacing (x_1, z_0) in place of (x_0, z_0) in the above process, we get

$$(y_{i,1}, z_0) \in E(G) \text{ for } i = 1, 2, 3, \cdots, k-1 \text{ and } (x_2, z_0) \in E(G).$$

Assume that $(x_m, z_0) \in E(G)$, for some $m \in \mathbb{N}$.
Since T_1 is edge-preserving, we have $(T_1 x_m, z_0) \in E(G)$ and by using the convexity of $E(G)$, we get

$$\begin{aligned} (1-\alpha_{1,m})(x_m, z_0) + \alpha_{1,m}(T_1 x_m, z_0) &= ((1-\alpha_{1,m})x_m + \alpha_{1,m}T_1 x_m, z_0) \\ &= (y_{1,m}, z_0) \in E(G) \end{aligned}$$

Since T_2 is edge-preserving, $(T_2 y_{1,m}, z_0) \in E(G)$ and $E(G)$ is convex, we obtain

$$\begin{aligned} (1-\alpha_{2,m})(x_m, z_0) + \alpha_{2,m}(T_2 y_{1,m}, z_0) &= ((1-\alpha_{2,m})x_m + \alpha_{2,m}T_2 y_{1,m}, z_0) \\ &= (y_{2,m}, z_0) \in E(G). \end{aligned}$$

By repeating the process, we conclude that

$$(y_{i,m}, z_0) \text{ and } (x_{m+1}, z_0) \text{ are in } E(G) \text{ for all } i = 1, 2, 3, \cdots, k-1.$$

Continuing the process once again for (x_{m+1}, z_0), we have $(y_{i,m+1}, z_0)$ are in $E(G)$ for all $i = 1, 2, 3, \cdots, k-1$.
Therefore, by induction, we conclude that (x_n, z_0) and $(y_{i,n}, z_0)$ are

in $E(G)$ for all $i = 1, 2, 3, \cdots, k-1$ and $n = 0, 1, 2, 3, \cdots$. Using a similar argument, we can show that $(z_0, x_n), (z_0, y_{i,n})$ are in $E(G)$ for all $i = 1, 2, 3, \cdots, k-1$ and $n = 0, 1, 2, \cdots$, under the assumption that $(z_0, x_0) \in E(G)$. The transitivity property of G implies that $(x_n, x_{n+1}), (x_n, y_{i,n}), (y_{i,n}, x_n)$ are in $E(G)$ for all $i = 1, 2, 3, \cdots, k-1$ and $n = 0, 1, 2, \cdots$. This completes the proof. □

Lemma 4.1. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and $\{T_i\}_{i=1}^{k}$ be a finite family of G-nonexpansive mappings on C. If $\{\alpha_{i,n}\} \subset [\delta, 1-\delta]$ for some δ in $(0,1)$ and $(x_0, z_0), (z_0, x_0)$ are in $E(G)$ for arbitrary $x_0 \in C$ and $z_0 \in F$, then for the sequence $\{x_n\}$ generated by (2.5), we have:*

(i) $||x_{n+1} - z_0|| \leq ||x_n - z_0||,$ for each $n \in \mathbb{N}$.

(ii) $\lim\limits_{n\to\infty} ||x_n - T_i x_n|| = 0,$ for each $i = 1, 2, 3, \cdots, k$.

Proof. First, we prove (i). Let $x_0 \in C$ and $z_0 \in F$ be as in the hypothesis and let $\{x_n\}$ be a sequence generated by (2.5). By Proposition 4.2, $(x_n, z_0), (z_0, x_n), (x_n, y_{i,n}), (y_{i,n}, x_n)$ and (x_n, x_{n+1}) are in $E(G)$. By the G-nonexpansiveness of T_1, we have

$$\begin{aligned}
||y_{1,n} - z_0|| &= ||(1-\alpha_{1,n})x_n + \alpha_{1,n}T_1x_n - z_0|| \\
&= ||(1-\alpha_{1,n})(x_n - z_0) + \alpha_{1,n}(T_1x_n - z_0)|| \\
&\leq (1-\alpha_{1,n})||x_n - z_0|| + \alpha_{1,n}||T_1x_n - z_0|| \\
&\leq (1-\alpha_{1,n})||x_n - z_0|| + \alpha_{1,n}||x_n - z_0|| \\
&= ||x_n - z_0||
\end{aligned}$$

Thus

$$||y_{1,n} - z_0|| \ \leq \ ||x_n - z_0||. \tag{4.1}$$

Again by (4.1) and using the G-nonexpansiveness of T_2, we have

$$\begin{aligned}
||y_{2,n} - z_0|| &= ||(1-\alpha_{2,n})x_n + \alpha_{2,n}T_2y_{1,n} - z_0|| \\
&= ||(1-\alpha_{2,n})(x_n - z_0) + \alpha_{2,n}(T_2y_{1,n} - z_0) \\
&\leq (1-\alpha_{2,n})||x_n - z_0|| + \alpha_{2,n}||T_2y_{1,n} - z_0|| \\
&\leq (1-\alpha_{2,n})||x_n - z_0|| + \alpha_{2,n}||y_{1,n} - z_0|| \\
&\leq (1-\alpha_{2,n})||x_n - z_0|| + \alpha_{2,n}||x_n - z_0|| \\
&= ||x_n - z_0||
\end{aligned}$$

which implies,

$$||y_{2,n} - z_0|| \ \leq \ ||x_n - z_0||. \tag{4.2}$$

Assume that

$$||y_{l,n} - z_0|| \leq ||x_n - z_0|| \tag{4.3}$$

for some $l \in \{1, 2, 3, \cdots, k-2\}$. By G-nonexpansiveness of T_{l+1} and (4.3), we have

$$\begin{aligned}
||y_{l+1,n} - z_0|| &= ||(1-\alpha_{l+1,n})(x_n - z_0) + \alpha_{l+1,n}(T_{l+1}y_{l,n} - z_0)|| \\
&\leq (1-\alpha_{l+1,n})||x_n - z_0|| + \alpha_{l+1,n}||T_{l+1}y_{l,n} - z_0|| \\
&\leq (1-\alpha_{l+1,n})||x_n - z_0|| + \alpha_{l+1,n}||x_n - z_0|| \\
&= ||x_n - z_0||
\end{aligned}$$

Therefore, for each $i = 1, 2, 3, \cdots, k-1$, we have

$$||y_{i,n} - z_0|| \ \leq \ ||x_n - z_0||. \tag{4.4}$$

In particular,

$$||y_{k-1,n} - z_0|| \leq ||x_n - z_0||,$$

so that by the G-nonexpansiveness of T_k we get

$$\begin{aligned}
||x_{n+1} - z_0|| &= ||(1-\alpha_{k,n})(x_n - z_0) + \alpha_{k,n}(T_k y_{k-1,n} - z_0)|| \\
&\leq (1-\alpha_{k,n})||x_n - z_0|| + \alpha_{k,n}||y_{k-1,n} - z_0|| \\
&\leq (1-\alpha_{k,n})||x_n - z_0|| + \alpha_{k,n}||x_n - z_0|| \\
&= ||x_n - z_0||
\end{aligned}$$

Therefore, for each $n = 0, 1, 2, \cdots$, we have

$$||x_{n+1} - z_0|| \leq ||x_n - z_0||. \tag{4.5}$$

Next, we prove (ii). From (i), we have $\lim\limits_{n\to\infty} ||x_n - z_0||$ exists and hence $\{x_n - z_0\}$ is bounded. Let

$$\lim_{n\to\infty} ||x_n - z_0|| = c, \text{ for some } c \geq 0. \tag{4.6}$$

If $c = 0$, then for each $l \in \{1, 2, 3, \cdots, k\}$

$$\begin{aligned}
||x_n - T_l x_n|| &\leq ||x_n - z_0|| + ||z_0 - T_l x_n|| \\
&\leq ||x_n - z_0|| + ||z_0 - x_n|| \\
&\to 0 \text{ as } n \to \infty.
\end{aligned}$$

Thus, for each $i = 1, 2, 3, \cdots, k$, we have

$$\lim_{n\to\infty} ||x_n - T_i x_n|| = 0. \tag{4.7}$$

Suppose that $c > 0$.

We claim that

$$\lim_{n\to\infty} ||y_{l,n} - z_0|| = c, \tag{4.8}$$

for each $l \in \{1, 2, 3, \cdots, k-1\}$. Now, from (4.1), we have

$$||y_{l,n} - z_0|| \ \leq \ ||x_n - z_0||$$

and using (4.6), we get

$$\limsup_{n\to\infty} ||y_{l,n} - z_0|| \leq c. \tag{4.9}$$

On the other hand by Lemma 3.2, we have

$$\begin{aligned}
||x_{n+1} - z_0||^2 &= ||(1-\alpha_{k,n})(x_n - z_0) + \alpha_{k,n}(T_k y_{k-1,n} - z_0)||^2 \\
&\leq (1-\alpha_{k,n})||x_n - z_0||^2 + \alpha_{k,n}||T_k y_{k-1,n} - z_0||^2 \\
&\quad -\alpha_{k,n}(1-\alpha_{k,n})g(||x - T_k y_{k-1,n}||) \\
&\leq (1-\alpha_{k,n})||x_n - z_0||^2 + \alpha_{k,n}||y_{k-1,n} - z_0||^2 \\
&\quad -\alpha_{k,n}(1-\alpha_{k,n})g(||x - T_k y_{k-1,n}||) \\
&\leq (1-\alpha_{k,n})||x_n - z_0||^2 + \alpha_{k,n}||x_n - z_0||^2 \\
&\quad -\alpha_{k,n}(1-\alpha_{k,n})g(||x - T_k y_{k-1,n}||) \\
&= ||x_n - z_0||^2 - \alpha_{k,n}(1-\alpha_{k,n})g(||x - T_k y_{k-1,n}||).
\end{aligned}$$

By rearranging, we obtain

$$\alpha_{k,n}(1-\alpha_{k,n})g(||x - T_k y_{k-1,n}||) \ \leq \ ||x_n - z_0||^2 - ||x_{n+1} - z_0||^2$$

which can be simplified to

$$\delta^2 g(||x - T_k y_{k-1,n}||) \ \leq \ ||x_n - z_0||^2 - ||x_{n+1} - z_0||^2. \tag{4.10}$$

Letting $n \to \infty$ in (4.10), we get

$$\lim_{n\to\infty} g(||x_n - T_k y_{k-1,n}||) \ = \ 0. \tag{4.11}$$

Since g is continuous and strictly increasing with $g(0) = 0$, we must have

$$\lim_{n\to\infty} ||x_n - T_k y_{k-1,n}|| \;=\; 0. \tag{4.12}$$

Now, using the G-nonexpansiveness of T_k, we have

$$\begin{aligned} ||x_n - z_0|| &\leq ||x_n - T_k y_{k-1,n}|| + ||T_k y_{k-1,n} - z_0|| \\ &\leq ||x_n - T_k y_{k-1,n}|| + ||y_{k-1,n} - z_0|| \end{aligned}$$

which implies that

$$||x_n - z_0|| \;\leq\; ||x_n - T_k y_{k-1,n}|| + ||y_{k-1,n} - z_0||. \tag{4.13}$$

But

$$\begin{aligned} ||y_{k-1,n} - z_0|| &= ||(1-\alpha_{k-1,n})(x_n - z_0) + \alpha_{k-1,n}(T_{k-1} y_{k-2,n} - z_0)|| \\ &\leq (1-\alpha_{k-1,n})||x_n - z_0|| + \alpha_{k-1,n}||T_{k-1} y_{k-2,n} - z_0|| \\ &\leq (1-\alpha_{k-1,n})||x_n - z_0|| + \alpha_{k-1,n}||y_{k-2,n} - z_0|| \end{aligned}$$

which implies that

$$\begin{aligned} ||y_{k-1,n} - z_0|| \;\leq\; & (1-\alpha_{k-1,n})||x_n - z_0|| + \\ & \alpha_{k-1,n}||y_{k-2,n} - z_0|| \end{aligned} \tag{4.14}$$

Substituting (4.14) in (4.13), we get

$$\begin{aligned} ||x_n - z_0|| \;\leq\; & ||x_n + T_k y_{k-1,n}|| + (1-\alpha_{k-1,n})||x_n - z_0|| + \\ & \alpha_{k-1,n}||y_{k-2,n} - z_0||. \end{aligned}$$

By rearranging and making some simplifications, we obtain that

$$||x_n - z_0|| \;\leq\; \frac{1}{\alpha_{k-1,n}}||x_n - T_k y_{k-1,n}|| + ||y_{k-2,n} - z_0||$$

$$\leq \frac{1}{\delta}||x_n - T_k y_{k-1,n}|| + ||y_{k-2,n} - z_0||.$$

This implies that

$$||x_n - z_0|| \leq \frac{1}{\delta}||x_n - T_k y_{k-1,n}|| + ||y_{k-2,n} - z_0||. \quad (4.15)$$

By similar procedure as we did to get (4.14), we obtain

$$||y_{k-2,n} - z_0|| \leq (1 - \alpha_{k-2,n})||x_n - z_0|| + \alpha_{k-2,n}||y_{k-3,n} - z_0||$$

and substituting in (4.15), we have

$$\begin{aligned} ||x_n - z_0|| \leq\ & \frac{1}{\delta}||x_n - T_k y_{k-1,n}|| + \\ & (1 - \alpha_{k-2,n})||x_n - z_0|| + \\ & \alpha_{k-2,n}||y_{k-3,n} - z_0||. \end{aligned} \quad (4.16)$$

Again, by making some rearrangements and simplifications of (4.16), we get

$$\begin{aligned} \alpha_{k-2,n}||x_n - z_0|| \leq\ & \frac{1}{\delta}||x_n - T_k y_{k-1,n}|| + \\ & \alpha_{k-2,n}||y_{k-3,n} - z_0|| \end{aligned} \quad (4.17)$$

which can be further simplified to

$$||x_n - z_0|| \leq \frac{1}{\delta^2}||x_n - T_k y_{k-1,n}|| + ||y_{k-3,n} - z_0||. \quad (4.18)$$

Continuing the process we can reach on the generalization,

for each $l \in \{1, 2, 3, \cdots, k-1\}$,

$$||x_n - z_0|| \leq \frac{1}{\delta^{k-1-l}}||x_n - T_k y_{k-1,n}|| + ||y_{l,n} - z_0||. \quad (4.19)$$

Taking lim inf of (4.19) and using (4.6) and (4.12), we obtain

$$c \leq \liminf_{n\to\infty} ||y_{l,n} - z_0||. \quad (4.20)$$

By combining (4.4) and (4.20), we get

$$\lim_{n\to\infty} ||y_{l,n} - z_0|| \;=\; c. \tag{4.21}$$

Since

$$\begin{aligned} ||T_l y_{l-1,n} - z_0|| &\leq ||y_{l-1,n} - z_0|| \\ &\leq ||x_n - z_0||, \end{aligned} \tag{4.22}$$

we have

$$\limsup_{n\to\infty} ||T_l y_{l-1,n} - z_0|| \;\leq\; c. \tag{4.23}$$

Using (4.9), (4.23) and applying Lemma 3.3, we get

$$\lim_{n\to\infty} ||x_n - T_l y_{l-1,n}|| \;=\; 0, \tag{4.24}$$

for each $l \in \{2, 3, 4, \cdots, k\}$.

On the other hand by the G-nonexpansiveness of T_l, we have

$$\begin{aligned} ||x_n - T_l x_n|| &\leq ||x_n - T_l y_{l-1,n}|| + ||T_l y_{l-1,n} - T_l x_n|| \\ &\leq ||x_n - T_l y_{l-1,n}|| + ||y_{l-1,n} - x_n|| \\ &= ||x_n - T_l y_{l-1,n}|| + \alpha_{l-1,n}||T_{l-1} y_{l-2,n} - x_n|| \\ &\leq ||x_n - T_l y_{l-1,n}|| + (1-\delta)||T_{l-1} y_{l-2,n} - x_n|| \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Thus, for $l \in \{3, 4, 5, \cdots, k\}$, we have

$$\lim_{n\to\infty} ||x_n - T_l x_n|| \;=\; 0. \tag{4.25}$$

But also we have

$$||y_{1,n} - z_0||^2 \;=\; ||(1-\alpha_{1,n})(x_n - z_0)\alpha_{1,n}(T_1 x_n - z_0)||^2$$

$$\begin{aligned}
&\leq (1-\alpha_{1,n})||x_n - z_0||^2 + \alpha||T_1 x_n - z_0||^2 \\
&\quad -\alpha_{1,n}(1-\alpha_{1,n})g(||x_n - T_1 x_n||) \\
&\leq ||x_n - z_0||^2 - \alpha_{1,n}(1-\alpha_{1,n})g(||x_n - T_1 x_n||).
\end{aligned}$$

After rearranging and making some simplifications, we get

$$\delta^2 g(||x_n - T_1 x_n)|| \leq ||x_n - z_0||^2 - ||y_{1,n} - z_0||^2. \qquad (4.26)$$

Letting $n \to \infty$ in (4.26) and using (4.6) and (4.21), we get

$$\lim_{n\to\infty} g(||x_n - T_1 x_n||) = 0.$$

As g is continuous and strictly increasing with $g(0) = 0$, we must have

$$\lim_{n\to\infty} ||x_n - T_1 x_n|| = 0. \qquad (4.27)$$

By using (4.24), (4.27) and G-nonexpansiveness of T_2, we have

$$\begin{aligned}
||x_n - T_2 x_n|| &\leq ||x_n - T_2 y_{1,n}|| + ||T_2 y_{1,n} - T_2 x_n|| \\
&\leq ||x_n - T_2 y_{1,n}|| + ||y_{1,n} - x_n|| \\
&= ||x_n - T_2 y_{1,n}|| + \alpha_{1,n}||x_n - T_1 x_n|| \\
&\leq ||x_n - T_2 y_{1,n}|| + (1-\delta)||x_n - T_1 x_n|| \\
&\quad \to 0 \; as \; n \to \infty
\end{aligned}$$

which implies that

$$\lim_{n\to\infty} ||x_n - T_2 x_n|| = 0. \qquad (4.28)$$

Therefore, combining (4.25),(4.27) and (4.28), we conclude that

$$\lim_{n\to\infty} ||x_n - T_i x_n|| = 0. \qquad (4.29)$$

This completes the proof. □

The subsequent lemma deals with the G-demiclosed principle.

Lemma 4.2. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and suppose that C has property A. Let $\{T_i\}_{i=1}^k$ be G-nonexpansive mappings on C. Then $I - T_i$ are G-demi-closed at* 0.

Proof. Let $\{x_n\}$ be a sequence in C such that $(x_n, x_{n+1}) \in E(G)$, $x_n \rightharpoonup q \in C$ as $n \to \infty$ and $\lim_{n\to\infty} ||x_n - T_i x_n|| = 0$. By property A, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $(x_{n_j}, q) \in E(G)$ for all $j \in \mathbb{N}$. By Remark 1.3, $(x_n, q) \in E(G)$ for all $n \in \mathbb{N}$. Since $\{x_n\}$ weakly converges in a uniformly convex Banach space X, it is bounded and hence there exists $r \geq 0$ such that $\{x_n\} \subset D =: C \cap B(0, r)$. Then D is nonempty closed convex subset of C. Thus, $T_i : D \to C$ are G-nonexpansive mappings. Let $l \in \{1, 2, 3, \cdots, k\}$. By Mazur's theorem (Cf. [78]), for each positive integer n, there exists a convex combination $y_n = \sum_{i=n}^{m(n)} \lambda_i x_i$ with $\lambda_i \geq 0$ and $\sum_{i=n}^{m(n)} \lambda_i = 1$ such that

$$||y_n - q|| < \frac{1}{n}. \tag{4.30}$$

Since $E(G)$ is convex and $(x_i, q) \in E(G)$ for each $i \in \mathbb{N}$, we must have that

$$\begin{aligned}
\sum_{i=n}^{m(n)} \lambda_i (x_i, q) &= (\sum_{i=n}^{m(n)} \lambda_i x_i, \sum_{i=n}^{m(n)} \lambda_i q) \\
&= (\sum_{i=n}^{m(n)} \lambda_i x_i, q) \\
&= (y_n, q) \in E(G).
\end{aligned}$$

From $\lim_{n\to\infty} ||x_n - T_l x_n|| = 0$, we have that: for every $\epsilon > 0$, there exists a positive integer N such that

$$||x_n - T_l x_n|| < \epsilon \text{ for every } n > N. \tag{4.31}$$

On the other hand, from Lemma 3.4 and (4.31) we have

$$\begin{aligned} ||T_l y_n - y_n|| &= ||T_l y_n - \sum_{i=n}^{m(n)} \lambda_i T_l x_i + \sum_{i=n}^{m(n)} \lambda_i T_l x_i - \sum_{i=n}^{m(n)} \lambda_i x_i|| \\ &\leq ||T_l y_n - \sum_{i=n}^{m(n)} \lambda_i T_l x_i|| + ||\sum_{i=n}^{m(n)} \lambda_i T_l x_i - \sum_{i=n}^{m(n)} \lambda_i x_i|| \\ &\leq ||T_l y_n - \sum_{i=n}^{m(n)} \lambda_i T_l x_i|| + \sum_{i=n}^{m(n)} \lambda_i ||T_l x_i - x_i|| \\ &\leq \gamma^{-1} \max_{n\leq i,j\leq m(n)} (||x_i - x_j|| - ||T_l x_i - T_l x_j||) + \epsilon \\ &\leq \gamma^{-1} \max_{n\leq i,j\leq m(n)} (||x_i - T_l x_i|| + ||x_i - T_l x_j||) \\ &\quad +\epsilon. \end{aligned} \tag{4.32}$$

Therefore, from (4.30),(4.31),(4.32) and G-nonexpansiveness of T_l, we have

$$\begin{aligned} ||q - T_l q|| &\leq ||q - y_n|| + ||y_n - T_l y_n|| + ||T_l y_n - T_l q|| \\ &\leq 2||q - y_n|| + \gamma^{-1} \max_{n\leq i,j\leq m(n)} (||x_i - T_l x_i|| + \\ &\quad ||x_j - T_l x_j||) + \epsilon \\ &\leq \frac{2}{n} + \gamma^{-1}(2\epsilon) + \epsilon, \text{ for } n > N. \end{aligned}$$

Taking the supremum limit as $n \to \infty$, we obtain

$$||q - T_l q \leq \gamma^{-1}(2\epsilon) + \epsilon. \tag{4.33}$$

Since γ is monotonically increasing with $\gamma(0) = 0$ and ϵ is arbitrary, we must have

$$||q - T_l q|| = 0 \tag{4.34}$$

Therefore, $q = T_i q$ for each $i \in \{1, 2, 3, \cdots, k\}$. $\square$

Theorem 4.1. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and suppose that C has property A. Let $\{T_i\}_{i=k}^{k}$ be a finite family of G-nonexpansive mappings on C with the nonempty common fixed points set $F = \bigcap_{i=1}^{k} F(T_i)$. Let $x_0 \in C$ be fixed so that F dominates x_0 and F is dominated by x_0. If $\{x_n\}$ is a sequence generated by x_0 with iterative scheme (2.5) such that $\{\alpha_{i,n}\} \subset [\delta, 1-\delta]$ for some $\delta \in (0, \frac{1}{2})$, then $\{x_n\}$ converges weakly to a common fixed point of the family $\{T_i\}_{i=1}^{k}$.*

Proof. For each $z \in F$, by dominance assumption, we have $(x_0, z), (z, x_0)$ are in $E(G)$. By Lemma 4.1, we have

$$\lim_{n\to\infty} ||x_n - z|| \text{ exists.} \tag{4.35}$$

$$\lim_{n\to\infty} ||x_n - T_i x_n|| = 0, \text{ for each } i = 1, 2, 3, \cdots, k. \tag{4.36}$$

$$\lim_{n\to\infty} ||x_n - T_i y_{i-1,n}|| = 0, \text{ for each } i = 1, 2, 3, \cdots, k. \tag{4.37}$$

From (4.35), we see that $\{x_n\}$ is a bounded sequence in C. Since C is nonempty closed convex subset of a uniformly convex Banach space X, it is weakly compact and hence there exists a subsequence $\{x_{n_j}\}$ of the sequence $\{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to some point $p \in C$. It follows from (4.36)

$$\lim_{j\to\infty} ||T_i x_{n_j} - x_{n_j}|| \ = \ 0. \tag{4.38}$$

By Lemma 4.2, $I - T_i$ are G-demiclosed at 0 so that $p \in F$.

To complete the proof it suffices to show that $\{x_n\}$ converges weakly to p.

To this end we need to show that $\{x_n\}$ satisfies the hypothesis of Lemma 3.5.

Let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ which converges weakly to some $q \in C$.

By similar arguments as above q is in F.

Now, for each $j \geq 1$, we have

$$x_{n_j+1} \;=\; x_{n_j} + \alpha_{k,n_j}(T_k y_{k-1,n_j} - x_{n_j}). \tag{4.39}$$

It follows from (4.37) that

$$\lim_{j\to\infty} ||T_k y_{k-1,n_j} - x_{n_j}|| \;=\; 0. \tag{4.40}$$

Since $\alpha_{k,n_j} \in [\delta, 1-\delta]$ for each $j \in \mathbb{N}$ and $\delta \in (0, \frac{1}{2})$, we have

$$\lim_{j\to\infty} \alpha_{k,n_j} ||T_k y_{k-1,n_j} - x_{n_j}|| \;=\; 0. \tag{4.41}$$

Thus, from (4.39) and (4.41), we conclude that

$$weak - \lim_{j\to\infty} x_{n_j+1} \;=\; q. \tag{4.42}$$

Therefore, the sequence $\{x_n\}$ satisfies the hypothesis of Lemma 3.5 which in turn implies that $\{x_n\}$ weakly converges to q so that $p = q$. This completes the proof. □

Theorem 4.2. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and suppose that C has property A. Let $\{T_i\}_{i=1}^{k}$ be a finite family of G-nonexpansive mappings on C with the nonempty common fixed points set $F = \bigcap_{i=1}^{k} F(T_i)$. Let $x_0 \in C$ be* *fixed so that F dominates x_0 and F is dominated by x_0. If at least one of the mappings in the family is semi-compact, then the iteration $\{x_n\}$ defined by (2.5) with $\{\alpha_{i,n}\} \subset [\delta, 1-\delta]$ for some $\delta \in (0, \frac{1}{2})$, converges strongly to a common fixed point of the family $\{T_i\}_{i=1}^k$.*

Proof. Assume that for some $l \in \{1, 2, 3, \cdots, k\}$, T_l is semi-compact. It follows from (4.36) and (4.37) that $\{x_n\}$ is a bounded sequence in C with

$$\lim_{n\to\infty} ||x_n - T_l x_n|| = 0. \tag{4.43}$$

By the definition of semi-compactness, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that for some $z \in C$,

$$\lim_{j\to\infty} ||x_{n_j} - z|| = 0. \tag{4.44}$$

Since strong convergence implies weak convergence and using Remark 1.3, we have $(x_{n_j}, z) \in E(G)$. Now it is obvious that z is a fixed point of T_l. By the G-nonexpansiveness of T_i for each $i \in \{1, 2, 3, \cdots, k\}$, and using (4.36) and (4.44), we have

$$\begin{aligned} ||T_i z - z|| &\leq ||T_i z - T_i x_{n_j}|| + ||T_i x_{n_j} - x_{n_j}|| + ||x_{n_j} - z|| \\ &\leq 2||z - x_{n_j}|| + ||T_i x_{n_j} - x_{n_j}|| \\ &\to 0 \text{ as } j \to \infty. \end{aligned}$$

Thus z is a common fixed point of the family $\{T_i\}_{i=1}^k$, so that $\lim_{n\to\infty} ||x_n - z||$ exists. Hence it must be the case that

$$\lim_{n\to\infty} ||x_n - z|| = 0. \tag{4.45}$$

This completes the proof. □

4.2 Common Fixed Points for a Finite Family of G-asymptotically Nonexpansive Mappings in UC Banach Spaces

First, we prove the following fundamental result that will be used in the proof of the subsequent results.

Proposition 4.2. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and $\{T_i\}_{i=1}^{k}$ is a family of G-asymptotically nonexpansive mappings on C. Let $z \in F$ be such that (x_1, z) and (z, x_1) are in $E(G)$ for arbitrary $x_1 \in C$. Then, for a sequence $\{x_n\}$ generated by x_1 with iterative scheme defined by (2.6), we have $(x_n, z), (z, x_n), (x_n, y_{i,n}), (y_{i,n}, x_n), (z, y_{i,n}), (y_{i,n}, z)$ and (x_n, x_{n+1}) are in $E(G)$ for each $i = 1, 2, 3, \cdots, k$ and $n = 1, 2, 3, \cdots$.*

Proof. We proceed by induction. First we let $(x_1, z) \in E(G)$. Since T_1 is edge preserving, we have $(T_1 x_1, z) \in E(G)$. By the convexity of $E(G)$, we have

$$\begin{aligned}(1-\alpha_{1,1})(x_1, z) + \alpha_{1,1}(T_1 x_1, z) &= ((1-\alpha_{1,1})x_1 + \alpha_{1,1}T_1 x_1, z) \\ &= (y_{1,1}, z) \in E(G).\end{aligned}$$

Since T_2 is edge preserving, $(T_2 y_{1,1}, z) \in E(G)$ and again by the convexity of $E(G)$ we have

$$\begin{aligned}(1-\alpha_{2,1})(x_1, z) + \alpha_{2,1}(T_2 y_{1,1}, z) &= ((1-\alpha_{2,1})x_1 + \alpha_{2,1}T_2 y_{1,1}, z) \\ &= (y_{2,1}, z) \in E(G).\end{aligned}$$

Assume that

$$(y_{l,1}, z) \in E(G), \text{ for some } l \in \{1, 2, 3, \cdots, k-2\}.$$

As T_{l+1} is edge-preserving, $(T_{l+1}y_{l,1}, z) \in E(G)$ and by using the convexity of $E(G)$, we get

$$\begin{aligned}(1-\alpha_{l+1,1})(x_1, z) + \alpha_{l+1,1}(T_{l+1}y_{l,1}, z) &= ((1-\alpha_{l+1,1})x_1 + \alpha_{l+1,1}T_{l+1}y_{l,1}, z) \\ &= (y_{l+1,1}, z) \in E(G).\end{aligned}$$

Thus

$$(y_{i,1}, z) \in E(G), \text{ for each } i = 1, 2, 3, \cdots, k-1. \tag{4.46}$$

In particular, for $i = k-1$

$$(y_{k-1,1}, z) \in E(G).$$

Since T_k is edge-preserving, we have

$$(T_k y_{k-1,1}, z) \in E(G).$$

Using the convexity of $E(G)$, we have

$$\begin{aligned}(1-\alpha_{k,1})(x_1, z) + \alpha_{k,1}(T_k y_{k-1,1}, z) &= ((1-\alpha_{k,1})x_1 + \alpha_{k,1}T_k y_{k-1,1}, z) \\ &= (x_2, z) \in E(G).\end{aligned}$$

Thus, we obtain

$$(y_{i,1}, z) \in E(G) \text{ for } i = 1, 2, 3, \cdots, k-1 \text{ and } (x_2, z) \text{ is in } E(G).$$

Since $\{T_i\}_{i=1}^k$ are edge-preserving, $\{T_i^2\}_{i=1}^k$ are also edge-preserving. Thus, repeating the previous process for (x_2, z) in place of (x_1, z) and using the operators T_i^2 in place of T_i, we obtain

$$(y_{i,2}, z) \in E(G) \text{ for } i = 1, 2, 3, \cdots, k-1,$$

so that (x_3, z) is also in $E(G)$.

Assume that $(x_m, z) \in E(G)$ for some $m \in \mathbb{N}$.

Since T_i is edge-preserving, we have T_i^m are also edge-preserving and hence, we have $(T_1^m x_m, z) \in E(G)$ and by using the convexity of $E(G)$, we get

$$\begin{aligned}(1-\alpha_{1,m})(x_m, z) + \alpha_{1,m}(T_1^m x_m, z) &= ((1-\alpha_{1,m})x_m + \alpha_{1,m}T_1^m x_m, z) \\ &= (y_{1,m}, z) \in E(G).\end{aligned}$$

As T_2^m is edge-preserving, $(T_2^m y_{1,m}, z) \in E(G)$, as $E(G)$ is convex, we have

$$\begin{aligned}(1-\alpha_{2,m})(x_m, z) + \alpha_{2,m}(T_2^m y_{1,m}, z) &= ((1-\alpha_{2,m})x_m + \alpha_{2,m}T_2^m y_{1,m}, z) \\ &= (y_{2,m}, z) \in E(G).\end{aligned}$$

By repeating the process, we conclude that

$$(y_{i,m}, z) \textit{ and } (x_{m+1}, z) \text{ are in } E(G) \text{ for all } i = 1, 2, 3, \cdots, k-1.$$

Continuing the process once again for (x_{m+1}, z), we have $(y_{i,m+1}, z)$ are in $E(G)$ for all $i = 1, 2, 3, \cdots, k-1$.

Therefore, by induction, we conclude that (x_n, z) and $(y_{i,n}, z)$ are in $E(G)$ for all $i = 1, 2, 3, \cdots, k-1$ and $n = 1, 2, 3, \cdots$. Using a similar argument, we can show that $(z, x_n), (z, y_{i,n})$ are in $E(G)$ for all $i = 1, 2, 3, \cdots, k-1$ and $n = 1, 2, 3, \cdots$, under the assumption that $(z, x_1) \in E(G)$. The transitivity property of G implies that $(x_n, x_{n+1}), (x_n, y_{i,n}), (y_{i,n}, x_n)$ are in $E(G)$ for all $i = 1, 2, 3, \cdots, k-1$ and $n = 1, 2, 3, \cdots$. This completes the proof. $\square$

Lemma 4.3. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and $\{T_i\}_{i=1}^k$ be a finite family of G-asymptotically nonexpansive mappings on C, i.e., $||T_i^n x - T_i^n y|| \leq (1+u_{i,n})||x-y||$ for all $x, y \in C$ and $(x,y) \in E(G)$, where $\{u_{i,n}\} \subset [0,\infty)$ with $\sum_{n=1}^{\infty} u_{i,n} < \infty$ for each $i \in \{1,2,3,\cdots,k\}$. If (x_1, z) and (z, x_1) are in $E(G)$ for arbitrary $x_1 \in C$ and $z \in F$, then the sequence $\{x_n\}$ generated by (2.6) with $\{\alpha_{i,n}\} \subset [\delta, 1-\delta]$ for some δ in $(0,1)$, we have*

(i) $\lim_{n\to\infty} ||x_n - z||$ *exists.*

(ii) $\lim_{n\to\infty} ||x_n - T_i^n y_{i-1,n}|| = 0$, *for each* $i = 2,3,4,\cdots,k$.

(iii) $\lim_{n\to\infty} ||x_n - T_i^n x_n|| = 0$, *for each* $i = 1,2,3,\cdots,k$.

(iv) $\lim_{n\to\infty} ||x_n - T_i x_n|| = 0$, *for each* $i = 1,2,3,\cdots,k$.

Proof. First we prove (i). Let $x_1 \in C$ and $z \in F$ be as in the hypothesis and let $\{x_n\}$ be a sequence generated by (2.6). By Proposition 4.2, $(x_n, z), (z, x_n), (x_n, y_{i,n}), (y_{i,n}, x_n)$ and (x_n, x_{n+1}) are in $E(G)$. Set $v_n = \max_{1\leq i\leq k} u_{i,n}$, for all n. Since $\sum_{n=1}^{\infty} u_{i,n} < \infty$, for each i, we mus have

$$\sum_{n=1}^{\infty} v_n < \infty. \tag{4.47}$$

Now by the G-asymptotically nonexpansiveness of T_1 and (2.6), we have

$$\begin{aligned} ||y_{1,n} - z|| &\leq (1-\alpha_{1,n})||x_n - z|| + \alpha_{1,n}||T_1^n x_n - z|| \\ &\leq (1-\alpha_{1,n})||x_n - z|| + \alpha_{1,n}(1+u_{1,n})||x_n - z|| \\ &= (1+\alpha_{1,n}u_{1,n})||x_n - z_0|| \end{aligned}$$

$$\leq \ (1+v_n)||x_n - z||.$$

Thus

$$||y_{1,n} - z|| \ \leq \ (1+v_n)||x_n - z||. \tag{4.48}$$

Assume that, for some $m \in \{1, 2, 3, \cdots, k-2\}$,

$$||y_{m,n} - z|| \leq (1+v_n)^m ||x_n - z||. \tag{4.49}$$

By G-asymptotically nonexpansiveness of T_{m+1} and using (2.6) and (4.49), we have

$$\begin{aligned}
||y_{m+1,n} - z|| \ &\leq \ (1-\alpha_{m+1,n})||x_n - z|| + \alpha_{m+1,n}||T^n_{m+1}y_{m,n} - z|| \\
&\leq \ (1-\alpha_{m+1,n})||x_n - z|| + \\
&\quad\ \alpha_{m+1,n}(1+u_{m+1,n})||y_{m,n} - z|| \\
&\leq \ (1-\alpha_{m+1,n})||x_n - z|| + \\
&\quad\ \alpha_{m+1,n}(1+u_{m+1,n})(1+v_n)^m||x_n - z|| \\
&\leq \ (1-\alpha_{m+1,n})||x_n - z|| + \\
&\quad\ \alpha_{m+1,n}(1+v_{m,n})^{m+1}||x_n - z|| \\
&= \ [1-\alpha_{m+1,n} + \alpha_{m+1,n}(1+ \\
&\quad\ \sum_{j=1}^{m+1}\binom{m+1}{j}v_n^j\ )]||x_n - z|| \\
&= \ (1+\alpha_{m+1,n}\sum_{j=1}^{m+1}\binom{m+1}{j}v_n^j\ )||x_n - z|| \\
&\leq \ (1+\sum_{j=1}^{m+1}\binom{m+1}{j}v_n^j\ )||x_n - z|| \\
&= \ (1+v_n)^{m+1}||x_n - z||,
\end{aligned}$$

where

$$\binom{r}{s} = \frac{r!}{(r-s)!s!}.$$

Thus, for each $i = 1, 2, 3, \cdots, k-1$, we have

$$||y_{i,n} - z|| \;\leq\; (1+v_n)^i ||x_n - z||. \tag{4.50}$$

In particular for $i = k-1$, we have

$$\begin{aligned}
||x_{n+1} - z|| &= ||(1-\alpha_{k,n})(x_n - z) + \alpha_{k,n}(T_k^n y_{k-1,n} - z)|| \\
&\leq (1-\alpha_{k,n})||x_n - z|| + \alpha_{k,n}(1+u_{k,n})||y_{k-1,n} - z|| \\
&\leq (1-\alpha_{k,n})||x_n - z|| + \\
&\quad \alpha_{k,n}(1+u_{k,n})(1+v_n)^{k-1}||x_n - z|| \\
&\leq (1-\alpha_{k,n})||x_n - z|| + \alpha_{k,n}(1+v_n)^k ||x_n - z|| \\
&= [1 - \alpha_{k,n} + \alpha_{k,n}(1 + \sum_{j=1}^{k} \binom{k}{j} v_n^j \;)]||x_n - z|| \\
&\leq (1 + \sum_{j=1}^{k} \binom{k}{j} v_n^j \;)||x_n - z||.
\end{aligned}$$

Therefore, for each $n = 1, 2, 3, \cdots$, we have

$$||x_{n+1} - z|| \;\leq\; (1 + \sum_{j=1}^{k} \binom{k}{j} v_n^j \;)||x_n - z||. \tag{4.51}$$

If we set $b_n = \sum_{j=1}^{k} \binom{k}{j} v_n^j$, we have

$$b_n = \sum_{j=1}^{n} \binom{k}{j} v_n^j \leq \sum_{j=1}^{n} \binom{k}{j} v_n = v_n(2^k - 1). \tag{4.52}$$

Using (4.47) and (4.52), we obtain that

$$\sum_{n=1}^{\infty} b_n \leq (2^k - 1) \sum_{n=1}^{\infty} v_n < \infty. \tag{4.53}$$

Using (4.51), (4.53) to apply (ii) of Lemma 3.1 with $a_n = ||x_n - z||$, we conclude that

$$\lim_{n\to\infty} ||x_n - z|| \text{ exists.} \tag{4.54}$$

Next, we prove (ii). From (i), we have $\lim_{n\to\infty} ||x_n - z||$ exists and hence $\{x_n\}$ is a bounded sequence. Let

$$\lim_{n\to\infty} ||x_n - z|| = c \text{ for some } c \geq 0. \tag{4.55}$$

From (4.50), for each $m \in \{1, 2, 3, \cdots, k-1\}$, we have

$$||y_{m,n} - z|| \leq (1 + v_n)^m ||x_n - z|| \tag{4.56}$$

and using (4.55), we get

$$\limsup_{n\to\infty} ||y_{m,n} - z|| \leq c. \tag{4.57}$$

On the other hand, from (2.6) we have

$$\begin{aligned}
||x_{n+1} - z|| &\leq (1 - \alpha_{k,n})||x_n - z|| + \alpha_{k,n}(1 + v_n)||y_{k-1,n} - z)|| \\
&\leq (1 - \alpha_{k,n})||x_n - z|| + \alpha_{k,n}(1 + v_n)[(1 - \alpha_{k-1,n}) \\
&\quad ||x_n - z|| + \alpha_{k-1,n}(1 + v_n)||y_{k-2,n} - z||] \\
&= [1 - \alpha_{k,n} + \alpha_{k,n}(1 + v_n)(1 - \alpha_{k-1,n})]||x_n - z|| + \\
&\quad \alpha_{k,n}\alpha_{k-1,n}(1 + v_n)^2||y_{k-2,n} - z|| \\
&= (1 - \alpha_{k,n}\alpha_{k-1,n} + \alpha_{k,n}v_n - \alpha_{k,n}\alpha_{k-1,n}v_n)||x_n - z|| + \\
&\quad \alpha_{k,n}\alpha_{k-1,n}(1 + v_n)^2||y_{k-2,n} - z|| \\
&\leq (1 + v_n)[(1 - \alpha_{k,n}\alpha_{k-1,n}||x_n - z|| + \\
&\quad \alpha_{k,n}\alpha_{k-1,n})(1 + v_n)(1 - \alpha_{k-2,n})||x_n - z|| + \\
&\quad \alpha_{k-2,n}(1 + v_n)||y_{k-3,n} - z||]
\end{aligned}$$

$$\leq (1 - \alpha_{k,n}\alpha_{k-1,n}\alpha_{k-2,n})(1 + v_n)^2||x_n - z||^2 + \alpha_{k,n}\alpha_{k-1,n}\alpha_{k-2,n}(1 + v_n)^3||y_{k-3,n} - z||.$$

Continuing the process, we obtain

$$||x_{n+1} - z|| \leq (1 - \alpha_{k,n}\alpha_{k-1,n} \cdots \alpha_{j+1,n})(1 + v_n)^{k-j-1}||x_n - z|| + \alpha_{k,n}\alpha_{k-1,n} \cdots \alpha_{j+1,n}(1 + v_n)^{k-j}||y_{j,n} - z||$$

for each $j = 1, 2, 3, \cdots, k - 1$.

By rearranging, we get

$$\frac{||x_{n+1} - z||}{(1 + v_n)^{k-j-1}} \leq (1 - \alpha_{k,n}\alpha_{k-1,n} \cdots \alpha_{j+1,n})||x_n - z|| + \alpha_{k,n}\alpha_{k-1,n} \cdots \alpha_{j+1,n}(1 + v_n)||y_{j,n} - z||.$$

Which was simplified to

$$(\frac{||x_{n+1} - z||}{(1 + v_n)^{k-j-1}} - ||x_n - z||)\frac{1}{\alpha_{k,n}\alpha_{k-1,n} \cdots \alpha_{j+1,n}} + ||x_n - z|| \leq (1 + v_n)||y_{j,n} - z||.$$

Since $\alpha_{i,n} \in [\delta, 1 - \delta]$, the above inequality was further simplified to

$$(\frac{||x_{n+1} - z||}{(1 + v_n)^{k-j-1}} - ||x_n - z||)\frac{1}{(1 - \delta)^{k-j}} + ||x_n - z|| \leq (1 + v_n)||y_{j,n} - z||. \tag{4.58}$$

Taking limit inferior of (4.58) and using (4.55), we get

$$c \leq \liminf_{n\to\infty} ||y_{j,n} - z||. \tag{4.59}$$

Thus, from (4.57) and (4.59), we conclude that

$$\lim_{n\to\infty} ||y_{j,n} - z|| = c \tag{4.60}$$

for each $j = 1, 2, 3, \cdots, k-1$.

Thus, for $j = 2, 3, 4, \cdots, k-1$, we have

$$\lim_{n\to\infty} ||(1-\alpha_{j,n})(x_n - z) + \alpha_{j,n}(T_j^n y_{j-1,n} - z)|| \;=\; c. \quad (4.61)$$

For $j = 2, 3, 4, \cdots, k-1$, we have

$$||T_j^n y_{j-1,n} - z|| \;\leq\; (1 + u_{j,n})||y_{j-1,n} - z|| \quad (4.62)$$

and hence

$$\limsup_{n\to\infty} ||T_j^n y_{j-1,n} - z|| \;\leq\; c. \quad (4.63)$$

Using (4.55), (4.61), (4.63) and apply Lemma 4.3, we have

$$\lim_{n\to\infty} ||x_n - T_j^n y_{j-1,n}|| \;=\; 0 \quad (4.64)$$

for $j = 2, 3, 4, \cdots, k-1$.

For the case $j = k$, using (4.50) and G-asymptotically nonexpansiveness of T_k, we have

$$\begin{aligned} ||T_k^n y_{k-1,n} - z|| &\leq (1 + u_{k,n})||y_{k-1,n} - z|| \\ &\leq (1 + v_n)^k ||x_n - z||. \quad (4.65) \end{aligned}$$

Taking limit superior of (4.65) and using (4.55), we obtain

$$\limsup_{n\to\infty} ||T_k^n y_{k-1,n} - z|| \;\leq\; c. \quad (4.66)$$

Since

$$\begin{aligned} \lim_{n\to\infty} ||x_{n+1} - z|| &= \lim_{n\to\infty} ||(1-\alpha_{k,n})(x_n - z) - \\ &\qquad \alpha_{k,n}(T_k^n y_{k-1,n} - z)|| = c. \end{aligned}$$

Applying Lemma 4.3 once again, we get

$$\lim_{n\to\infty} ||x_n - T_k^n y_{k-1,n}|| \ = \ 0. \tag{4.67}$$

Therefore, from (4.64) and (4.67), for each

$j = 2, 3, 4, \cdots, k$, we obtain that

$$\lim_{n\to\infty} ||x_n - T_j^n y_{j-1,n}|| \ = \ 0. \tag{4.68}$$

Next, we show (iii). Since

$$\lim_{n\to\infty} ||x_n - z|| = \lim_{n\to\infty} ||y_{i,n} - z|| \ = \ c \tag{4.69}$$

for each $i = 1, 2, \cdots, k-1$, we have $\{x_n\}$ and $\{y_{i,n} - z\}$

are bounded sequences and hence $\{T_i^n y_{i,n} - z\}$ is also bounded.

Therefore, there is $R > 0$ such that

$$\bigcup_{i=1}^{k-1} \{x_n\} \bigcup \{y_{i,n}\} \bigcup \{T_i^n y_{i,n}\} \subset \overline{B(z, R)}. \tag{4.70}$$

By Lemma 3.2, there is continuous and strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ such that

$$\begin{aligned}
||y_{1,n} - z||^2 \ &\leq \ (1 - \alpha_{1,n})||x_n - z||^2 + \alpha_{1,n}||T_1^n x_n - z||^2 - \\
&\quad\ \alpha_{1,n}(1 - \alpha_{1,n}) g(||x_n - T_1^n x_n||) \\
&\leq \ (1 - \alpha_{1,n})||x_n - z||^2 + \alpha_{1,n}(1 + u_{1,n})^2||x_n - z||^2 - \\
&\quad\ \delta^2 g(||x_n - T_1^n x_n||) \\
&\leq \ (1 + u_{1,n})^2||x_n - z||^2 - \delta^2 g(||x_n - T_1^n x_n||)
\end{aligned}$$

On rearranging, we obtain

$$\delta^2 g(||x_n - T_1^n x_n||) \ \leq \ (1 + u_{1,n})^2||x_n - z||^2 -$$

$$||y_{1,n} - z||^2. \tag{4.71}$$

Taking the superior limit of (4.71) and using (4.55),(4.60) to get

$$\delta^2 \limsup_{n\to\infty} g(||x_n - T_1^n x_n) \leq 0. \tag{4.72}$$

Which gives that

$$\lim_{n\to\infty} g(||x_n - T_1^n x_n||) = 0. \tag{4.73}$$

Since g is continuous and monotonically increasing we conclude that

$$\lim_{n\to\infty} ||x_n - T_1^n x_n|| = 0. \tag{4.74}$$

Again,

$$\begin{aligned} ||x_n - T_2^n x_n|| &\leq ||x_n - T_2^n y_{1,n}|| + ||T_2^n y_{1,n} - T_2^n x_n|| \\ &\leq ||x_n - T_2^n y_{1,n}|| + (1 + u_{2,n})||y_{1,n} - x_n|| \\ &= ||x_n - T_2^n y_{1,n}|| + \alpha_{2,n}(1 + u_{2,n})||x_n - T_1^n x_n||. \end{aligned}$$

Which implies that

$$||x_n - T_2^n x_n|| \leq ||x_n - T_2^n y_{1,n}|| + \alpha_{2,n}(1 + u_{2,n})||x_n - T_1^n x_n||. \tag{4.75}$$

Applying (ii) of Lemma 4.3, using (4.74) and the fact that the sequence $\{\alpha_{2,n}(1 + u_{2,n})\}$ is bounded, we have

$$\lim_{n\to\infty} ||x_n - T_2^n x_n|| = 0. \tag{4.76}$$

Repeatedly, we apply Lemma 3.2, for $i = 3, 4, 5, \cdots, k-1$ and get

$$||y_{i,n} - z||^2 \leq (1 - \alpha_{i,n})||x_n - z||^2 + \alpha_{i,n}||T_i^n y_{i-1,n} - z||^2 -$$

$$\begin{aligned}
& \alpha_{i,n}(1-\alpha_{i,n})g(||x_n - T_i^n y_{i-1,n}||) \\
\leq\ & (1-\alpha_{i,n})||x_n - z||^2 + \alpha_{i,n}||T_i^n y_{i-1,n} - z||^2 - \\
& \delta^2 g(||x_n - T_i^n y_{i-1,n}||) \\
\leq\ & (1-\alpha_{i,n})||x_n - z||^2 + \alpha_{i,n}(1+u_{i,n})^2||y_{i-1,n} - z||^2 - \\
& \delta^2 g(||x_n - T_i^n y_{i-1,n}||).
\end{aligned}$$

On rearranging, we obtain

$$\begin{aligned}
\delta^2 g(||x_n - T_i^n y_{i-1,n}||) &\leq (1-\alpha_{i,n})||x_n - z||^2 - ||y_{i-1,n} - z||^2 \\
&\quad -\alpha_{i,n}(1+u_{i,n})^2||y_{i-1,n} - z||^2 \\
&= (||x_n - z||^2 - ||y_{i,n} - z||^2) + \\
&\quad \alpha_{i,n}[(1+u_{i,n})^2||y_{i-1,n} - z||^2 - ||x_n - z||^2 \\
&\leq 2R(||x_n - z|| - ||y_{i,n} - z||) \\
&\quad [1-\alpha_{i,n}(1+u_{i,n})^2]. \qquad (4.77)
\end{aligned}$$

Taking limit superior of (4.77) and using (4.55) and (4.60), we get

$$\delta^2 \limsup_{n\to\infty} g(||x_n - T_i^n y_{i-1,n}||) \leq 0. \qquad (4.78)$$

Using property of g we conclude that

$$\lim_{n\to\infty} ||x_n - T_i^n y_{i-1,n}|| = 0 \qquad (4.79)$$

for each $i = 3, 4, 5, \cdots, k-1$.

On the other hand, for $i = 3, 4, 5, \cdots, k$, we have

$$\begin{aligned}
||x_n - T_i^n x_n|| &\leq ||x_n - T_i^n y_{i-1,n}|| + ||T_i^n y_{i-1,n} - T_i^n x_n|| \\
&\leq ||x_n - T_i^n y_{i-1,n}|| + (1+u_{i,n})||y_{i-1,n} - x_n|| \\
&= ||x_n - T_i^n y_{i-1,n}|| +
\end{aligned}$$

$$(1+u_{i,n})\alpha_{i-1,n}||T^n_{i-1}y_{i-2,n}-x_n||.$$

This implies that

$$\begin{aligned} ||x_n - T^n_i x_n|| &\leq ||x_n - T^n_i y_{i-1,n}|| + \\ & (1+u_{i,n})\alpha_{i-1,n}||T^n_{i-1}y_{i-2,n}-x_n|| \end{aligned} \tag{4.80}$$

for each $i = 3, 4, 5, \cdots, k$.

Taking limit superior of (4.80) and using (4.79), we get that

$$\limsup_{n\to\infty} ||x_n - T^n_i x_n|| \leq 0. \tag{4.81}$$

Which implies that, for each $i = 3, 4, 5, \cdots, k$, we have

$$\lim_{n\to\infty} ||x_n - T^n_i x_n|| = 0. \tag{4.82}$$

Thus, from (4.74), (4.76) and (4.82), we conclude that

$$\lim_{n\to\infty} ||x_n - T^n_i x_n|| = 0 \tag{4.83}$$

for each $i = 1, 2, 3, \cdots, k$.

Finally, we prove (iv). Fix $m \in \{1, 2, 3, \cdots, k\}$ but arbitrary. By the G-asymptotically nonexpansiveness of T_m we have that

$$\begin{aligned} ||x_n - T_m x_n|| &\leq ||x_n - x_{n+1}|| + ||x_{n+1} - T^{n+1}_m x_{n+1}|| + \\ & ||T^{n+1}_m x_{n+1} - T^{n+1}_m x_n|| + ||T^{n+1}_m x_n - T_m x_n|| \\ &\leq ||x_n - x_{n+1}|| + ||x_{n+1} - T^{n+1}_m x_{n+1}|| + \\ & k_{n+1}||x_n - x_{n+1}|| + k_1||x_n - T^n_m x_n|| \\ &= (1+k_{n+1})||x_n - x_{n+1}|| + k_1||x_n - T^n_m x_n|| + \\ & ||x_{n+1} - T^{n+1}_m x_{n+1}||. \end{aligned}$$

where $k_n = \max_{1 \leq i \leq k}(1 + u_{i,n})$. This implies that

$$\begin{aligned} ||x_n - T_m x_n|| \;&\leq\; (1 + k_{n+1})||x_n - x_{n+1}|| + k_1||x_n - T_m^n x_n|| + \\ &\quad ||x_{n+1} - T_m^{n+1} x_{n+1}||. \end{aligned} \tag{4.84}$$

From (2.6), we have that

$$||x_{n+1} - x_n|| = \alpha_{k,n}||x_n - T_k^n y_{k-1,n}||. \tag{4.85}$$

Since $\alpha_{k,n} \in [\delta, 1 - \delta]$, using (iii) and (4.85), we obtain

$$\lim_{n\to\infty} ||x_n - x_{n+1}|| \;=\; 0. \tag{4.86}$$

Taking the superior limit of (4.84) and using (4.83) and (4.86), we conclude that

$$\lim_{n\to\infty} ||x_n - T_m x_n|| \;=\; 0. \tag{4.87}$$

Since m was arbitrary, we obtain the required result. $\square$

This completes the proof.

In the next lemma, the G-demiclosed principle is proved.

Lemma 4.4. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and suppose that C has property A. Let T be a G-asymptotically nonexpansive mapping on C with asymptotic coefficient $\{k_n\}$ such that*

$$\sum_{n=1}^{\infty}(k_n - 1) \;<\; \infty. \tag{4.88}$$

Then $I - T$ is G-demi-closed at 0.

Proof. Let $\{x_n\}$ be a sequence in C with (x_n, x_{n+1}) and (x_n, Tx_n) are in $E(G)$ such that $x_n \rightharpoonup q \in C$ as $n \to \infty$ and $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$. By Property A, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $(x_{n_j}, q) \in E(G)$ for all $j \in \mathbb{N}$. By Remark 1.3, $(x_n, q) \in E(G)$ for all $n \in \mathbb{N}$.

We claim that, as $n \to \infty$

$$T^n q \;\; \to \;\; q. \tag{4.89}$$

Note that, for each $n \in \mathbb{N}$, we have

$$\begin{aligned} ||x_k - T^n x_k|| &\leq (1 + \sum_{r=1}^{n-1} k_r)||x_k - Tx_k|| \\ &= (1 + \sum_{r=1}^{n-1}(1 + u_r))||x_k - Tx_k|| \\ &= (n + \sum_{r=1}^{n-1} u_r)||x_k - Tx_k|| \\ &\leq (n + \sum_{r=1}^{\infty} u_r)||x_k - Tx_k|| \\ &\leq (n + M)||x_k - Tx_k|| \end{aligned}$$

where, $M = \sum_{r=1}^{\infty} u_r = \sum_{r=1}^{\infty}(k_r - 1) < \infty$.

Thus, we have

$$||x_k - T^n x_k|| \;\; \leq \;\; (n + M)||x_k - Tx_k||. \tag{4.90}$$

Since $\lim_{k\to\infty} ||x_k - Tx_k|| = 0$, we have that, for fixed $n \in \mathbb{N}$, there is a positive integer $N = N(n)$, such that

$$k \geq N \Rightarrow \;\; ||x_k - Tx_k|| < \tfrac{1}{(n+M)^2}. \tag{4.91}$$

Hence, from (4.90) and (4.91), we obtain

$$k \geq N \Rightarrow ||x_k - T^n x_k|| < \frac{1}{n + M}. \tag{4.92}$$

Therefore,

$$\overline{\lim_{k}}\ ||x_k - T^n x_k|| \leq \frac{1}{n+M}. \tag{4.93}$$

This implies that

$$\overline{\lim_{n}\lim_{k}}||x_k - T^n x_k|| \ = \ 0. \tag{4.94}$$

Therefore, for an arbitrary $\epsilon > 0$, we can choose n_0 such that

$$\limsup_{k\to\infty} ||x_k - T^n x_k|| \ < \ \epsilon, \ \forall n \geq n_0. \tag{4.95}$$

Since $x_n \rightharpoonup q$, by Mazur's theorem (Cf. [78]), for each positive integer k, there exists a convex combination $y_k = \sum_{i=1}^{p(k)} t_i^{(k)} x_{i+k}$ with $t_i^{(k)} \geq 0$ and $\sum_{i=1}^{p(k)} t_i^{(k)} = 1$ such that

$$||y_k - q|| \ < \ \frac{1}{k}. \tag{4.96}$$

Since $\{x_n\}$ weakly converges in a uniformly convex Banach space X, it is bounded and hence there exists $r > 0$ such that $\{x_n\} \subset D =: C \cap \overline{B(q,r)}$. Then D is nonempty closed convex subset of C. Thus, $T : D \to C$ is G-asymptotically nonexpansive mapping. Therefore, $T^n : D \to C$ is a Lipschitzian mapping with Lipschitz constant $k_n \geq 1$.

Again, we have

$$\begin{aligned} ||T^n y_k - y_k|| \ &= \ ||T^n y_k - \sum_{i=1}^{p(k)} t_i^{(k)} T^n x_{i+k} + \sum_{i=1}^{p(k)} t_i^{(k)} T^n x_{i+k} - \\ &\quad \sum_{i=1}^{p(k)} t_i^{(k)} x_{i+k}|| \\ &\leq \ ||T^n y_k - \sum_{i=1}^{p(k)} t_i T^n x_{i+k}|| + ||\sum_{i=1}^{p(k)} t_i^{(k)} T^n x_{i+k} - \end{aligned}$$

$$\begin{aligned} & \sum_{i=1}^{p(k)} t_i^{(k)} x_{i+k}|| \\ \leq \ & ||T^n y_k - \sum_{i=1}^{p(k)} t_i^{(k)} T^n x_{i+k}|| + \\ & \sum_{i=1}^{p(k)} t_i^{(k)} ||T^n x_{i+k} - x_{i+k}||. \end{aligned} \quad (4.97)$$

From (4.95), we get

$$\sum_{i=1}^{p(k)} t_i^{(k)} ||T^n x_{i+k} - x_{i+k}|| \ < \ \epsilon, \ \forall \ n \geq n_0. \quad (4.98)$$

Using Lemma 3.4, we obtain

$$\begin{aligned} ||T^n y_k - \sum_{i=1}^{p(k)} t_i^{(k)} T^n x_{i+k}|| \ & \leq \ k_n \gamma^{-1} \{ \max(||x_{i+k} - x_{i+p}|| - \\ & \quad k_n^{-1} ||T^n x_{i+k} - T^n x_{i+p}||) \} \\ & \leq \ k_n \gamma^{-1} \{ \max(2\epsilon + \\ & \quad (1 - k_n^{-1}) k_n ||x_{i+k} - x_{i+p}||) \}. \end{aligned}$$

Since $\{x_k\} \subset D$, we must have

$$||x_{i+k} - x_{i+p}|| \ \leq \ 2r, \text{ so that}$$

$$||T^n y_k - \sum_{i=1}^{p(k)} t_i^{(k)} T^n x_{i+k}|| \ \leq \ k_n \gamma^{-1} (2\epsilon + 2r(k_n - 1)). \quad (4.99)$$

Substituting (4.98) and (4.99) in (4.97), we obtain

$$||T^n y_k - y_k|| \ \leq \ k_n \gamma^{-1} (2\epsilon + 2r(k_n - 1)) + \epsilon \quad (4.100)$$

for each $k \in \mathbb{N}$ and $n \geq n_0$. Thus we have

$$\limsup_{k \to \infty} ||T^n y_k - y_k|| \ \leq \ k_n \gamma^{-1} (2\epsilon + 2r(k_n - 1)) + \epsilon. \quad (4.101)$$

On the other hand, for each $n \in \mathbb{N}$, we have

$$||q - T^n q|| \leq ||q - y_k|| + ||y_k - T^n y_k|| + ||T^n y_k - T^n q||. \tag{4.102}$$

Since $y_k \to q$ as $k \to \infty$, which gives $y_k \rightharpoonup q$ and thus, by property A and Remark 1.3, we have $(y_k, q) \in E(G)$, for all $k \in \mathbb{N}$. This in turn gives $(T^n y_k, T^n q) \in E(G)$, So that

$$||T^n y_k - T^n q|| \leq k_n ||y_k - q||. \tag{4.103}$$

Substituting (4.103) in (4.102), we obtain

$$||q - T^n q|| \leq (1 + k_n)||y_k - q|| + ||y_k - T^n y_k||. \tag{4.104}$$

From (4.96) and (4.104), we get

$$||q - T^n q|| \leq \frac{1 + k_n}{k} + ||y_k - T^n y_k||. \tag{4.105}$$

Taking limit superior of (4.105) as $k \to \infty$, we have

$$||T^n q - q|| \leq \limsup_{k\to\infty} ||y_k - T^n y_k||. \tag{4.106}$$

Combining (4.101) and (4.106), we infer that

$$||T^n q - q|| \leq k_n \gamma^{-1}(2\epsilon + 2r(k_n - 1)) + \epsilon \tag{4.107}$$

for all $n \geq n_0$.

Taking limit superior of (4.107) as $n \to \infty$ and using the arbitrariness of ϵ, we get

$$\limsup_{n\to\infty} ||T^n q - q|| \leq \gamma^{-1}(0) = 0. \tag{4.108}$$

Therefore,

$$\lim_{n\to\infty} ||q - T^n q|| \;=\; 0. \tag{4.109}$$

But,

$$||q - Tq|| \leq ||q - T^{n+1}q|| + ||T^{n+1}q - Tq||. \tag{4.110}$$

Since $T^n q \to q$ as $n \to \infty$ and the fact that strong convergence implies weak convergence, we can obtain that $(T^n q, q) \in E(G)$, $\forall n \in \mathbb{N}$. As T is edge-preserving, we have $(T^{n+1}q, Tq) \in E(G)$, so that

$$||T^{n+1}q - Tq|| \leq k_1||T^n q - q||. \tag{4.111}$$

From (4.110) and (4.111), we get that

$$||q - Tq|| \leq ||q - T^{n+1}q|| + k_1||q - T^n q||. \tag{4.112}$$

Taking limit superior of (4.112) and using (4.109), we obtain that

$$\begin{aligned} ||q - Tq|| \;&\leq\; (1 + k_1) \limsup_{n\to\infty} ||q - T^n q|| \\ &=\; 0. \end{aligned}$$

This shows that

$$q \;=\; Tq. \tag{4.113}$$

□

Theorem 4.3. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and suppose that C has property A. Let $\{T_i\}_{i=1}^{k}$ be a finite family of G-asymptotically nonexpansive mappings on C with the nonempty common fixed points set F =*

$\bigcap_{i=1}^{k} F(T_i)$. Let $x_1 \in C$ be fixed so that (x_1, z) and (z, x_1) are in $E(G)$. If $\{x_n\}$ is a sequence generated by x_1 with iterative scheme (2.6) such that $\{\alpha_{i,n}\} \subset [\delta, 1-\delta]$ for some $\delta \in (0, \frac{1}{2})$ and $\sum_{n=1}^{\infty} u_{i,n} < \infty$ for each $i = 1, 2, 3, \cdots, k$ then $\{x_n\}$ converges weakly to a common fixed point of the family $\{T_i\}_{i=1}^{k}$.

Proof. Let $z \in F$ and $x_1 \in C$ be such that $(x_1, z), (z, x_1)$ are in $E(G)$. By Lemma 4.3, we have

$$\lim_{n\to\infty} ||x_n - z|| \text{ exists .} \tag{4.114}$$

$$\lim_{n\to\infty} ||x_n - T_i^n x_n|| = 0 \text{ for } i = 1, 2, 3, \cdots, k. \tag{4.115}$$

$$\lim_{n\to\infty} ||x_n - T_i^n y_{i-1,n}|| = 0 \text{ for } i = 2, 3, 4, \cdots, k. \tag{4.116}$$

$$\lim_{n\to\infty} ||x_n - T_i x_n|| = 0 \text{ for } i = 1, 2, 3, \cdots, k. \tag{4.117}$$

From (4.114), we see that $\{x_n\}$ is a bounded sequence in C. Since C is nonempty closed convex subset of a uniformly convex Banach space X, it is weakly compact and hence there exists a subsequence $\{x_{n_h}\}$ of the sequence $\{x_n\}$ such that $\{x_{n_h}\}$ converges weakly to some point $p \in C$. It follows from (4.115) that

$$\lim_{h\to\infty} ||T_i^{n_h} x_{n_h} - x_{n_h}|| \;=\; 0 \tag{4.118}$$

for each $i = 1, 2, 3, \cdots, k$. By proposition 4.2, we have (x_n, x_{n+1}) and $(x_n, T_i x_n)$ are in $E(G)$ for all $n \in \mathbb{N}$ and hence (x_{n_h}, x_{n_h+1}) and $(x_{n_h}, T_i x_{n_h})$ are also in $E(G)$ for each $i = 1, 2, 3, \cdots, k$. Thus, by Lemma 4.4 we conclude that $p \in F$.

To complete the proof it suffices to show that $\{x_n\}$ converges weakly to p.

To this end we need to show that $\{x_n\}$ satisfies the hypothesis of Lemma 3.5.

Let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ which converges weakly to some point $q \in C$.

By similar arguments as above q is in F.

Now for each $j \geq 1$, we have

$$x_{n_j+1} \;=\; x_{n_j} + \alpha_{k,n_j}(T_k^{n_j} y_{k-1,n_j} - x_{n_j}). \tag{4.119}$$

It follows from (4.116) that

$$\lim_{j\to\infty} ||T_k^{n_j} y_{k-1,n_j} - x_{n_j}|| \;=\; 0. \tag{4.120}$$

Since $\alpha_{k,n_j} \in [\delta, 1-\delta]$ for each $j \in \mathbb{N}$ and $\delta \in (0, \frac{1}{2})$, we have

$$\lim_{j\to\infty} \alpha_{k,n_j} ||T_k^{n_j} y_{k-1,n_j} - x_{n_j}|| \;=\; 0. \tag{4.121}$$

Thus, from (4.119) and (4.121), we conclude that

$$weak - \lim_{j\to\infty} x_{n_j+1} \;=\; q. \tag{4.122}$$

Therefore, the sequence $\{x_n\}$ satisfies the hypothesis of Lemma 3.5 which in turn implies that $\{x_n\}$ weakly converges to q so that $p = q$.

This completes the proof. $\square$

Theorem 4.4. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and suppose that C has property P. Let $\{T_i\}_{i=1}^k$ be a family of G-asymptotically nonexpansive mappings on C with the nonempty common fixed points set $F = \bigcap_{i=1}^k F(T_i)$ and $\sum_{n=1}^\infty u_{i,n} < \infty$ for each $i = 1, 2, 3, \cdots, k$. Let $x_1 \in C$ be fixed so that F dominates x_1 and F is dominated by x_1. If for some*

$l \in \{1,2,3,\cdots,k\}$, T_l^m is semi-compact for some positive integer m, then the iteration $\{x_n\}$ generated by x_1 with iterative scheme (2.6) such that $\{\alpha_{i,n}\} \subset [\delta, 1-\delta]$ for some $\delta \in (0, \frac{1}{2})$ converges strongly to a common fixed point of the family $\{T_i\}_{i=1}^k$.

Proof. Fix $m \in \{1,2,3,\cdots,k\}$ and assume that T_m^s is semi-compact for some $s \in \mathbb{N}$. For each $z \in F$, by dominance assumption, we have $(x_1, z), (z, x_1)$ are in $E(G)$. It follows from (4.115) and (4.116) and that $\{x_n\}$ is a bounded sequence in C and

$$\lim_{n\to\infty} ||x_n - T_m x_n|| \;=\; 0. \tag{4.123}$$

As $\{x_n\}$ is bounded, by the definition of semi-compactness, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that for some $z \in C$,

$$\lim_{j\to\infty} ||x_{n_j} - z|| \;=\; 0. \tag{4.124}$$

Since strong convergence implies weak convergence and using Remark 1.3, we have $(x_{n_j}, z) \in E(G)$. Now it is obvious that z is a fixed point of T_m. By the G-asymptotically nonexpansiveness of T_i for each $i \in \{1,2,3,\cdots,k\}$, and using (4.119) and (4.124), we have

$$\begin{aligned} ||T_i z - z|| \;&\leq\; ||T_i z - T_i x_{n_j}|| + ||T_i x_{n_j} - x_{n_j}|| + ||x_{n_j} - z|| \\ &\leq\; (1+k_1)||z - x_{n_j}|| + ||T_i x_{n_j} - x_{n_j}|| \to 0 \text{ as } j \to \infty. \end{aligned}$$

Thus, z is a common fixed point of the family $\{T_i\}_{i=1}^k$ so that $\lim_{n\to\infty} ||x_n - z||$ exists. Hence it must be the case that

$$\lim_{n\to\infty} ||x_n - z|| \;=\; 0. \tag{4.125}$$

This completes the proof. □

4.3 On Monotone Generalized Quasi Contraction Mappings in Modular Metric Spaces with a Graph

We first prove the following lemma.

Lemma 4.5. *Let (X, d) be a metric space and G be a reflexive and transitive digraph on X. Let C be a nonempty subset of X and $T : C \to C$ be a G-monotone generalized quasi contraction mapping. Let $x \in C$ be such that $(x, T(x)) \in E(G)$ and $\delta(x) < \infty$, then for any $n \in \mathbb{N}_0$, we have*

$$\delta(T^n(x)) \leq \varphi^n(\delta(x))$$

where $\varphi \in \Phi$ is the comparison function associated with the G-monotone generalized quasi contraction definition of T. Moreover, we have

$$d(T^n(x), T^{n+m}(x)) \leq \varphi^n(\delta(x)) \text{ for each } n, m \in \mathbb{N}_0.$$

Proof. Since T is G-monotone and $(x, T(x)) \in E(G)$, we have

$$(T^n(x), T^{n+1}(x)) \in E(G) \text{ for each } n \in \mathbb{N}_0.$$

By the transitivity of G, for each $n \in \mathbb{N}_0$, we also have

$$(T^n(x), T^{n+m}(x)) \in E(G) \text{ for any } m \in \mathbb{N}. \tag{4.126}$$

As G is reflexive, $(x, x) \in E(G)$ and using the G-monotonicity of T, we get

$$(T^n(x), T^n(x)) \in E(G) \text{ for any } n \in \mathbb{N}_0 \tag{4.127}$$

and hence (4.126) holds true for $m = 0$. Thus, we obtain

$$(T^n(x), T^{n+m}(x)) \in E(G) \text{ for any } n, m \in \mathbb{N}_0. \tag{4.128}$$

First, we show that

$$\delta(T^n(x)) \leq \varphi^n(\delta(x)) \tag{4.129}$$

for each $n \in \mathbb{N}_0$.

For $n = 0$, we have $\delta(x) = \varphi^0(\delta(x))$ and equality holds in this case.

For $n = 1$, from (4.128) and using the monotonicity of φ, we have

$$\begin{aligned} d(T(x), T^{1+m}(x)) &\leq max\{\varphi(d(x, T^m(x))), \varphi(d(x, T(x))), \\ &\quad \varphi(d(T^m(x), T^{1+m}(x))), \varphi(d(x, T^{1+m}(x))), \\ &\quad \varphi(d(T(x), T^m(x)))\} \\ &\leq \varphi(\delta(x)), \end{aligned} \tag{4.130}$$

for each $m \in \mathbb{N}_0$. This shows that

$$\delta(T(x)) \leq \varphi(\delta(x)), \tag{4.131}$$

and from (4.131) and the monotonicity of φ, we get

$$\varphi(\delta(T(x))) \leq \varphi^2(\delta(x)). \tag{4.132}$$

In addition, inequalities (4.131), (4.132) and the monotonicity of φ, gives

$$\delta(T^2(x)) = \delta(T(T(x))) \leq \varphi(\delta(T(x))) \leq \varphi^2(\delta(x)).$$

Whence, by induction we conclude that

$$\delta(T^n(x)) = \delta(T(T^{n-1}(x)))$$

$$\leq \varphi(\delta(T^{n-1}(x)))$$

$$\vdots$$

$$\leq \varphi^n(\delta(x))$$

for each $n \in \mathbb{N}_0$.

Thus, for each $n \in \mathbb{N}_0$,

$$\delta(T^n(x)) \leq \varphi^n(\delta(x)). \tag{4.133}$$

On the other hand, by using (4.128) and the definition of δ, we obtain

$$\begin{aligned} d(T^n(x), T^{n+m}(x)) &= d(T^n(x), T^n(T^m)(x)) \\ &\leq \delta(T^n(x)) \end{aligned} \tag{4.134}$$

for each $n, m \in \mathbb{N}_0$.

From (4.133) and (4.134), we conclude that

$$d(T^n(x), T^{n+m}(x)) \leq \varphi^n(\delta(x))$$

for each $n, m \in \mathbb{N}_0$. □

Theorem 4.5. *Let (X, d) be a complete metric space and G be a reflexive, transitive digraph defined on X such that the triple (X, d, G) has Property P : for any sequence $\{x_n\}_{n\in\mathbb{N}} \in X$, if $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}_0$, then $(x_n, x) \in E(G)$ for each $n \in \mathbb{N}_0$. Let C be a nonempty closed subset of X and $T : C \to C$ be a G-monotone generalized quasi contraction mapping with $\varphi \in \Phi$, the associated comparison function. For $x \in C$ with $(x, T(x)) \in E(G)$ and $\delta(x) < \infty$, we have the following:*

(a) There exists $p \in F(T)$ such that $\{T^n(x)\}$ converges to p. Moreover, we have

$$(x, p) \in E(G) \text{ and } d(T^n(x), p) \leq \varphi^n(\delta(x)) \text{ for each } n \in \mathbb{N}_0.$$

(b) If u is any fixed point of T such that $(x, u) \in E(G)$ then $u = p$.

Proof. First we show (a). By Lemma 4.5, we see that $\{T^n(x)\}$ is a Cauchy sequence in C. Since X is a complete metric space and C is closed subset of X, there exists $p \in C$ such that $\{T^n(x)\}$ converges to p. From (4.134), we have

$$d(T^n(x), T^{n+m}(x)) \leq \varphi^n(\delta(x)) \tag{4.135}$$

for any $n, m \in \mathbb{N}_0$. Thus, from (4.135) (by letting $m \to \infty$), we get that

$$d(T^n(x), p) \leq \varphi^n(\delta(x)), \text{ for } n \in \mathbb{N}_0.$$

Moreover, since T is G-monotone and $(x, T(x)) \in E(G)$, we have $(T^n(x), T^{n+1}(x)) \in E(G)$ for each $n \in \mathbb{N}_0$ and using Property P, we conclude that $(T^n(x), p) \in E(G)$ for each $n \in \mathbb{N}_0$. In particular, $(x, p) \in E(G)$.

It remains to show that p is a fixed point of T. From $(T^n(x), p) \in E(G)$ the G-monotonicity of T, we have $(T^{n+1}(x), T(p)) \in E(G)$ for each $n \in \mathbb{N}_0$. As $(T^n(x), T^{n+1}(x)) \in E(G)$ and G is transitive, we obtain that

$$(T^n(x), T(p)) \in E(G) \text{ for each } n \in \mathbb{N}_0. \tag{4.136}$$

Thus, using (4.136) and the hypothesis that T is a G-monotone generalized quasi contraction, we have a comparison function φ satisfying

$$\begin{aligned} d(T^n(x), T(p)) \;\leq\; & \max\{\varphi(d(T^{n-1}(x), p)), \varphi(d(T^{n-1}(x), T^n(x))), \\ & \varphi(d(T^{n-1}(x), T(p))), \varphi(d(T^n(x), p)), \\ & \varphi(d(p, T(p)))\} \end{aligned} \tag{4.137}$$

for each $n \in \mathbb{N}_0$. Letting $n \to \infty$ in (4.137) and using the upper semi-continuity of φ, we get

$$d(p, T(p)) \;\leq\; \varphi(d(p, T(p))$$

which implies that

$$d(p, T(p)) \;=\; 0 \;\text{ and hence p = T(p).}$$

Next we show (b). Let $u \in C$ be any fixed point of T such that $(x, u) \in E(G)$. Then for each $n \in \mathbb{N}_0$, as T is G-monotone, we have $(T^n(x), u) \in E(G)$.

Therefore,

$$\begin{aligned} d(T^n(x), u) \;\leq\; & max\{\varphi(d(T^{n-1}(x), u)), \varphi(d(T^{n-1}(x), T^n(x))), \\ & \varphi(d(T^{n-1}(x), u)), \varphi(d(T^n(x), u))\} \end{aligned} \tag{4.138}$$

for each $n \in \mathbb{N}$. If

$$\begin{aligned} & max\{\varphi(d(T^{n-1}(x), u)), \varphi(d(T^{n-1}(x), T^n(x))), \\ & \qquad \varphi(d(T^n(x), u))\} = \varphi(d(T^n(x), u)) \end{aligned} \tag{4.139}$$

for some $n \in \mathbb{N}$, then from (4.138), we have

$$d(T^n(x), u)) \;\leq\; \varphi(d(T^n(x), u)).$$

Thus, by property (ii) of φ, we get

$$d(T^n(x), u) \;=\; 0 \text{ which implies } T^n(x) = u.$$

Hence,

$$T^{n+m}(x) \;=\; T^m(T^n(x)) = T^m(u) = u \text{ for each } m \in \mathbb{N}_0,$$

which shows that the sequence $T^n(x) \to u$ as $n \to \infty$. By the uniqueness of the limit we conclude that $u = p$.

Otherwise,

$$\begin{aligned} &max\{\varphi(d(T^{n-1}(x), u)), \varphi(d(T^{n-1}(x), T^n(x))), \\ &\qquad \varphi(d(T^n(x), u))\} \neq \varphi(d(T^n(x), u)), \end{aligned}$$

for all $n \in \mathbb{N}$. Again, from (4.138) we must have

$$\begin{aligned} d(T^n(x), u) &\leq \max\{\varphi(d(T^{n-1}(x), u)), \varphi(d(T^{n-1}(x), T^n(x)))\} \\ &\leq \varphi(d(T^{n-1}(x), u)) + \varphi(d(T^{n-1}(x), T^n(x))) \quad (4.140) \end{aligned}$$

for all $n \in \mathbb{N}_0$.

If we take limit superior of (4.140), and use the upper-semi continuity of φ, we obtain

$$\begin{aligned} d(p, u) &\leq \limsup_{n\to\infty} \varphi(d(T^{n-1}(x), u)) + \limsup_{n\to\infty} \varphi(d(T^n(x), T^{n-1}(x))) \\ &\leq \varphi(d(p, u)). \end{aligned}$$

Which gives that

$$d(p, u) \leq \varphi(d(p, u)). \tag{4.141}$$

Using property (ii) of φ and (4.141), we conclude that

$$d(u, p) = 0 \text{ and hence } u = p.$$

$\square$

Next, we prove the following technical lemmas that will be used in the proof of the subsequent theorem.

Lemma 4.6. *Let (X, ω) be a modular metric space and G be a reflexive and transitive digraph on X . Let C be a nonempty subset of X and $T : C \to C$ be a G-monotone generalized ω-quasi contraction mapping. For $x \in C$ with $(x, T(x)) \in E(G)$ and $\delta_\omega(x) < \infty$, we have*

$$\delta_\omega(T^n(x)) \leq \varphi^n(\delta_\omega(x))$$

for each $n \in \mathbb{N}_0$, where φ is the comparison function associated with the G-monotone generalized ω-quasi contraction definition of T. Moreover, we have

$$\omega_1(T^n(x), T^{n+m}(x)) \leq \varphi^n(\delta_\omega(x)) \text{ for any } n, m \in \mathbb{N}_0.$$

Proof. Since T is G-monotone and $(x, T(x)) \in E(G)$, we have

$$(T^n(x), T^{n+1}(x)) \in E(G) \text{ for any } n \in \mathbb{N}_0.$$

By the transitivity of the graph G, for each $n \in \mathbb{N}_0$, we have

$$(T^n(x), T^{n+m}(x)) \in E(G) \text{ for any } m \in \mathbb{N}_0. \tag{4.142}$$

Again as G is reflexive, $(x, x) \in E(G)$ and using the monotonocity of G, we obtain

$$(T^n(x), T^n(x)) \in E(G) \text{ for each } n \in \mathbb{N}_0. \tag{4.143}$$

Thus, from (4.142) and (4.143), we infer that

$$(T^n(x), T^{n+m}(x)) \in E(G) \text{ for each } n, m \in \mathbb{N}_0. \tag{4.144}$$

Since T is a G-monotone generalized ω-quasi contraction, and using (4.144), there is a comparison function $\varphi \in \Phi$ such that

$$\begin{aligned}\omega_1(T^n(x), T^{n+m}(x)) \;\leq\; & max\{\varphi(\omega_1(T^{n-1}(x), T^{n+m-1}(x))),\\ & \varphi(\omega_1(T^{n-1}(x), T^n(x))),\\ & \varphi(\omega_1(T^{n+m-1}(x), T^{n+m}(x))),\\ & \varphi(\omega_1(T^{n-1}(x), T^{n+m}(x))),\\ & \varphi(\omega_1(T^n(x), T^{n+m-1}(x)))\} \qquad (4.145)\end{aligned}$$

for any $n \in \mathbb{N}$ and any $m \in \mathbb{N}_0$. We need to show that

$$\delta_\omega(T^n(x)) \;\leq\; \varphi^n(\delta_\omega(x)) \text{ for each } n \in \mathbb{N}_0. \qquad (4.146)$$

To see this, if $n = 0$, the result is obvious; in this case equality holds. Now, if we take $n = 1$, in (4.145), we get

$$\begin{aligned}\omega_1(T(x), T^{1+m}(x)) \;\leq\; & max\{\varphi(\omega_1(x, T^m(x))),\\ & \varphi(\omega_1(x, T(x))), \varphi(\omega_1(T^m(x), T^{1+m}(x))),\\ & \varphi(\omega_1(x, T^{1+m}(x))), \varphi(\omega_1(T(x), T^m(x)))\}\\ \leq\; & \varphi(\delta_\omega(x)) \qquad (4.147)\end{aligned}$$

for any $m \in \mathbb{N}_0$. Thus

$$\omega_1(T(x), T^{1+m}(x)) \;\leq\; \varphi(\delta_\omega(x)) \qquad (4.148)$$

for each $m \in \mathbb{N}_0$.

By the definition of δ and (4.148), we obtain

$$\delta_\omega(T(x)) \leq \varphi(\delta_\omega(x)). \qquad (4.149)$$

Using the monotonicity of φ and (4.149), we obtain

$$\varphi(\delta_\omega(T(x))) \leq \varphi^2(\delta_\omega(x)). \tag{4.150}$$

Using (4.149) and (4.150), we get

$$\begin{aligned} \delta_\omega(T^2(x)) &= \delta_\omega(T(T(x)) \\ &\leq \varphi(\delta_\omega(T(x))) \\ &\leq \varphi^2(\delta_\omega(x)). \end{aligned} \tag{4.151}$$

Repeated use of the monotonicity of φ and (4.149), gives

$$\begin{aligned} \delta_\omega(T^n(x)) &= \delta_\omega(T(T^{n-1}(x))) \\ &\leq \varphi(\delta_\omega(T^{n-1}(x))) \\ &\ \vdots \\ &\leq \varphi^n(\delta_\omega(x)). \end{aligned} \tag{4.152}$$

On the other hand, from (4.145) and using the definition of δ_ω, we obtain

$$\begin{aligned} \omega_1(T^n(x), T^{n+m}(x)) &= \omega_1(T^n(x), T^m(T^n(x))) \\ &\leq \delta_\omega(T^n(x)) \end{aligned} \tag{4.153}$$

for each $n, m \in \mathbb{N}_0$. Hence, from (4.152) and (4.153), we get that

$$\omega_1(T^n(x), T^{n+m}(x)) \leq \varphi^n(\delta_\omega(x)) \tag{4.154}$$

for each $n, m \in \mathbb{N}_0$. □

Lemma 4.7. *Let (X, ω) be a modular metric space and let C be a nonempty ω-complete subset of X. Let $T : C \to C$ be a G-monotone* *generalized ω-quasi contraction mapping and $x \in C$ be such that $\delta_\omega(x) < \infty$. Then $\{T^n(x)\}\omega$-converges to a point $u \in C$. Moreover, one has*

$$\omega_1(T^n(x), u) \leq \varphi^n(\delta_\omega(x))$$

for all $n \in \mathbb{N}_0$.

Proof. By Lemma 4.6, $\{T^n(x)\}$ is ω-Cauchy in C. Since C is ω-complete, there exists $u \in C$ such that $\{T^n(x)\}\omega$-converges to u. Again, from (4.152), we have

$$\omega_1(T^n(x), T^{n+m}(x)) \;\leq\; \varphi^n(\delta_\omega(x)) \tag{4.155}$$

for $n, m \in \mathbb{N}_0$. As ω satisfies the Fatou property, letting $m \to \infty$ in
(4.155)
to obtain

$$\omega_1(T^n(x), u) \leq \varphi^n(\delta_\omega(x), \text{ for each } n \in \mathbb{N}_0. \tag{4.156}$$

□

Theorem 4.6. *Let (X, ω) be a modular metric space and let G be a reflexive transitive digraph defined on X such that the triple (X, ω, G) have Property (B):*
for any sequence $\{x_n\}_{n\in\mathbb{N}_0}$ in X, if $x_n \overset{\omega}{\to} x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}_0$, then $(x_n, x) \in E(G)$ for each $n \in \mathbb{N}_0$. Let C be a nonempty ω-complete subset of X and $T : C \to C$ be a G-monotone generalized ω-quasi contraction mapping with the associated comparison function $\varphi \in \Phi$, the strong comparison function. For $x \in C$ with $(x, Tx) \in E(G)$ and $\delta_\omega(x) < \infty$, we have the following:

(a) The sequence $\{T^n(x)\}\omega$-converges to some point u in C.

Moreover, we have $(x,u) \in E(G)$ and $\omega_1(T^n(x),u) \leq \varphi^n(\delta_\omega(x))$ for each $n \in \mathbb{N}_0$. If in addition, $\omega_1(u,T(u)) < \infty$ and $\omega_1(x,T(u)) < \infty$,

then u is a fixed point of T.

(b) If u^* is any fixed point of T in C such that $(x,u^*) \in E(G)$ and $\omega_1(T^n(x),u^*) < \infty$ for any $n \in \mathbb{N}_0$ then $u = u^*$.

Proof. First we prove (a). By Lemma 4.7, the sequence $\{T^n(x)\}$ is ω-Cauchy in C. Since C is ω-complete, there is a $u \in C$ such that $T^n(x) \xrightarrow{\omega} u$ as $n \to \infty$. From (4.154), we have

$$\omega_1(T^n(x), T^{n+m}(x)) \ \leq \ \varphi^n(\delta_\omega(x)) \tag{4.157}$$

for each $m \in \mathbb{N}_0$ and by assumption ω satisfies the Fatou property, we get (by letting $m \to \infty$) in (4.157)

$$\begin{aligned}\omega_1(T^n(x), u) \ &\leq \ \liminf_{m\to\infty} \omega_1(T^n(x), T^{n+m}(x)) \\ &\leq \ \varphi^n(\delta_\omega(x)) \text{ for any } n \in \mathbb{N}_0.\end{aligned}$$

From property (B), we have $(T^n(x), u) \in E(G)$, for $n \in \mathbb{N}_0$. In particular $(x,u) \in E(G)$. Since T is G-monotone, we have $(T(x),T(u)) \in E(G)$ and hence

$$\begin{aligned}\omega_1(T(x),T(u)) \ \leq \ & max\{\varphi(\omega_1(x,u)), \varphi(\omega_1(x,T(x))), \varphi(\omega_1(u,T(u)), \\ & \varphi(\omega_1(x,T(u)), \varphi(\omega_1(T(x),u))\}.\end{aligned} \tag{4.158}$$

Since ω satisfies the Fatou property, and using Lemma 4.6, we have

$$\omega_1(x,u) \ \leq \ \liminf_{m\to\infty} \omega_1(x, T^m(x)) \leq \delta_\omega(x),$$

$$\omega_1(x, T(x)) \ \leq \ \delta_\omega(x),$$

and

$$\omega_1(T(x), u) \ \leq \ \liminf_{m\to\infty} \omega_1(T(x), T^m(x)) \leq \delta_\omega(T(x)) \leq \varphi(\delta_\omega(x)).$$

Substituting these values in (4.158), we obtain

$$\begin{aligned} \omega_1(T(x), T(u)) \ &\leq \ max\{\varphi(\delta_\omega(x)), \varphi(\omega_1(u, T(u))), \\ &\quad \varphi(\omega_1(x, T(u))), \varphi^2(\delta_\omega(x))\}. \end{aligned} \tag{4.159}$$

Since T is a G-monotone and $(T^n(x), u) \in E(G)$, for each $n \in \mathbb{N}_0$, we have that

$$(T^{n+1}(x), T(u)) \in E(G). \tag{4.160}$$

As $(x, T(x)) \in E(G)$ and T is a G-monotone, we also have

$$(T^n(x), T^{n+1}(x)) \in E(G)$$

for each $n \in \mathbb{N}_0$, and using the transitivity property of G, we get

$$(T^n(x), T(u)) \in E(G).$$

Now, we assume that

$$\begin{aligned} \omega_1(T^n(x), T(u)) \ &\leq \ max\{\varphi^n(\delta_\omega(x)), \varphi(\omega_1(u, (T(u)))), \\ &\quad \varphi^2(\omega_1(u, T(u))), \cdots, \varphi^n(\omega_1(u, T(u))), \\ &\quad \varphi^n(\omega_1(x, T(u)), \varphi^{n+1}(\delta_\omega(x))\}. \end{aligned} \tag{4.161}$$

Then, for each $n \in \mathbb{N}_0$, we have

$$\omega_1(T^{n+1}(x), T(u)) \ = \ \omega_1(T^n(T(x), T(u))$$

$$
\begin{aligned}
&\leq max\{\varphi^n(\delta_\omega(T(x))), \varphi(\omega_1(u, T(u))),\\
&\quad \varphi^2(\omega_1(u, (T(u)))), \cdots, \varphi^n(\omega_1(u, T(u))),\\
&\quad \varphi^n(\omega_1(T(x), T(u)), \varphi^{n+1}(\delta_\omega(T(x)))\}. \qquad (4.162)
\end{aligned}
$$

From (4.159) and the monotonicity of φ, we get

$$
\begin{aligned}
\varphi^n(\omega_1(T(x), T(u))) \leq max\{&\varphi^{n+1}(\delta_\omega(x)), \varphi^{n+1}(\omega_1(u, T(u)),\\
&\varphi^{n+1}(\omega_1(x, T(u))), \varphi^{n+2}(\delta_\omega(x))\}. \qquad (4.163)
\end{aligned}
$$

Combining (4.162) and (4.163), we obtain

$$
\begin{aligned}
\omega_1(T^{n+1}(x), T(u)) \;\leq\; & max\{\varphi^{n+1}(\delta_\omega(x)), \varphi^{n+2}(\delta_\omega(x)), \varphi(\omega_1(u, T(u))),\\
&\varphi^2(\omega_1(u, (T(u)))), \cdots, \varphi^n(\omega_1(u, T(u))),\\
&\varphi^{n+1}(\omega_1(x, T(u))\}.
\end{aligned}
$$

Hence, by induction we conclude that

$$
\begin{aligned}
\omega_1(T^n(x), T(u)) \;\leq\; & max\{\varphi^n(\delta_\omega(x)), \varphi^{n+1}(\delta_\omega(x)), \varphi(\omega_1(u, T(u))),\\
&\varphi^2(\omega_1(u, T(u))), \cdots, \varphi^n(\omega_1(u, T(u))), \varphi^n(\omega_1(x, T(u))\}
\end{aligned}
$$

for each $n \in \mathbb{N}_0$.

Since $\varphi^n(t)$ is a nonincreasing sequence for any $t > 0$ and $\omega_1(u, T(u)) < \infty$, we have that

$$
\varphi^n(\omega_1(u, T(u))) \leq \varphi(\omega_1(u, T(u))) \qquad (4.164)
$$

for each $n \in \mathbb{N}_0$.

Thus, from (4.162) and (4.164), we have

$$
\omega_1(T^n(x), T(u)) \;\leq\; max\{\varphi^n(\delta_\omega(x)), \varphi^{n+1}(\delta_\omega(x)), \varphi(\omega_1(u, T(u))),
$$

$$\begin{aligned}
& \varphi^n(\omega_1(x,T(u))\} \\
\leq\ & \varphi^n(\delta_\omega(x)) + \varphi^{n+1}(\delta_\omega(x)) + \varphi^n(\omega_1(x,T(u)) \\
& +\varphi(\omega_1(u,T(u))). \qquad (4.165)
\end{aligned}$$

Thus, taking limit superior of (4.165), gives

$$\begin{aligned}
\limsup_{n\to\infty} \omega_1(T^n(x),T(u)) &\leq \limsup_{n\to\infty}(\varphi^n(\delta_\omega(x)) + \varphi^{n+1}(\delta_\omega(x)) \\
&+ \varphi^n(\omega_1(x,T(u)) + \varphi(\omega_1(u,T(u))) \\
&\leq \limsup_{n\to\infty}(\varphi^n(\delta_\omega(x)) + \limsup_{n\to\infty}(\varphi^{n+1}(\delta_\omega(x)) \\
&+ \limsup_{n\to\infty} \varphi^n(\omega_1(x,T(u))) + \varphi(\omega_1(u,T(u)) \\
&= \varphi(\omega_1(u,T(u)).
\end{aligned}$$

Whence, using the Fatou property, we obtain

$$\begin{aligned}
\omega_1(u,T(u)) &\leq \liminf_{n\to\infty} \omega_1(T^n(x),T(u)) \\
&\leq \limsup_{n\to\infty} \omega_1(T^n(x),T(u)) \\
&\leq \varphi(\omega_1(u,T(u)).
\end{aligned}$$

Since $\omega_1(u,T(u)) < \infty$, and using property (ii) of φ, we must have $\omega_1(u,T(u)) = 0$. Since ω is regular, we conclude that $u = T(u)$.

Next we prove (b). Let u^* be any fixed point of T. Suppose that $(x,u^*) \in E(G)$, and $\omega_1(T^n(x),u^*) < \infty$ for any $n \in \mathbb{N}_0$. By induction, and using the G-monotonicty of T, we have

$$(T^n(x),u^*) \in E(G) \text{ for each } n \in \mathbb{N}_0.$$

By hypothesis, there is a $\varphi \in \Phi$, strong comparison function, such that

$$\begin{aligned}\omega_1(T^n(x),u^*) &\le max\{\varphi(\omega_1(T^{n-1}(x),u^*)),\varphi(\omega_1(T^{n-1}(x),T^n(x))),\\ &\quad \varphi(\omega_1(T^n(x),u^*))\}\end{aligned}$$

for each $n \in \mathbb{N}$. If

$$\begin{aligned}&max\{\varphi(\omega_1(T^{n-1}(x),u^*),\varphi(\omega_1(T^{n-1}(x),T^n(x))),\\ &\qquad \varphi(\omega_1(T^n(x),u^*))\} = \varphi(\omega_1(T^n(x),u^*))\end{aligned}$$

for some $n \in \mathbb{N}$, then

$$\omega_1(T^n(x),u^*) \;\le\; \varphi(\omega_1(T^n(x),u^*)).$$

Using property of φ, once again, we get

$$\omega_1(T^n(x),u^*) \;=\; 0.$$

Thus, as ω is regular, we have $T^n(x) = u^*$ which implies that $T^n(x) \xrightarrow{\omega} u^*$ as $n \to \infty$. Thus,

$$\omega_2(u,u^*) \;\le\; \omega_1(u,T^n(x)) + \omega_1(T^n(x),u^*) \xrightarrow{\omega} 0$$

as $n \to \infty$ and the regularity of of ω provides $u = u^*$.

Now, suppose that

$$\begin{aligned}&max\{\varphi(\omega_1(T^{n-1}(x),u^*),\varphi(\omega_1(T^{n-1}(x),T^n(x)),\\ &\qquad \varphi(\omega_1(T^n(x),u))\} \ne \varphi(\omega_1(T^n(x),u)),\end{aligned}$$

for each $n \in \mathbb{N}$. Then,

$$\omega_1(T^n(x),u^*) \;\le\; max\{\varphi(\omega_1(T^{n-1}(x),u^*)),\varphi(\omega_1(T^{n-1}(x),T^n(x)))\}$$

$$
\begin{aligned}
&\leq \; max\{\varphi(\omega_1(T^{n-1}(x), u^*)), \varphi^n(\delta_\omega(x))\} \qquad (4.166)\\
&\leq \; max\{\omega_1(T^{n-1}(x), u^*)), \varphi^n(\delta_\omega(x))\}\\
&\leq \; \omega_1(T^{n-1}(x), u^*)) + \varphi^n(\delta_\omega(x))
\end{aligned}
$$

for each $n \in \mathbb{N}$. Similarly, we have

$$\omega_1(T^{n-1}(x), u^*) \; \leq \; \omega_1(T^{n-2}(x), u^*)) + \varphi^{n-1}(\delta_\omega(x)).$$

Following the same procedure, we obtain

$$\omega_1(T^n(x), u^*) \; \leq \; \sum_{j=2}^{n} \varphi^j(\delta_\omega(x)) + \omega_1(T(x), u^*).$$

and

$$
\begin{aligned}
\omega_1(T(x), u^*) \; &\leq \; max\{\varphi(\omega_1(x, u^*)), \varphi(\omega_1(x, T(x)))\}\\
&\leq \; max\{\omega_1(x, u^*)), \varphi(\omega_1(\delta_\omega(x)))\}\\
&\leq \; \omega_1(x, u^*)) + \varphi(\omega_1(\delta_\omega)).
\end{aligned}
$$

So, we obtain

$$\omega_1(T^n(x), u^*) \; \leq \; \sum_{j=1}^{n} \varphi^j(\delta_\omega(x)) + \omega_1(x, u^*). \qquad (4.167)$$

Thus,

$$\limsup_{n\to\infty} \omega_1(T^n(x), u^*) \; \leq \; \sum_{j=1}^{\infty} \varphi^j(\delta_\omega(x)) + \omega_1(x, u^*) < \infty$$

as φ was assumed to be a strong comparison function.

Now, if we set

$$\gamma(u^*) : \; = \; \limsup_{n\to\infty} \omega_1(T^n(x), u^*),$$

then we have $\gamma(u^*) < \infty$. Then, from (4.166), we get

$$\omega_1(T^n(x), u^*) \; \leq \; max\{\varphi^n(\delta_\omega(x)), \varphi(\omega_1(T^{n-1}(x), u^*))\} (4.168)$$

and taking limit superior as $n \to \infty$ in (4.168), we obtain, $\gamma(u^*) \leq \varphi(\gamma(u^*))$ which implies that $\gamma(u^*) = 0$, i.e., $\limsup_{n\to\infty} \omega_1(T^n(x), u^*) = 0$. Therefore, applying Fatou property once again, we get

$$\begin{aligned} \omega_1(u, u^*) &\leq \liminf_{n\to\infty} \omega_1(T^n(x), u^*) \\ &\leq \limsup_{n\to\infty} \omega_1(T^n(x), u^*) = 0, \end{aligned}$$

and hence since ω is regular, we obtain $u = u^*$. $\square$

Remark 4.1. *If we assume that $(u, u^*) \in E(G)$ for any fixed point u^* in C and $\omega_1(u, u^*) < \infty$, then $(T(u), T(u^*)) \in E(G)$ and $\omega_1(u, u^*) = \omega_1(T(u), T(u^*)) \leq \varphi(\omega_1(u, u^*))$. This clearly shows that $u = u^*$.*

Remark 4.2. *If we take $\varphi(t) = kt$, where $k \in [0, 1)$, then Theorem 4.6 is reduced to the result of Alfuraidan [[6], Theorem 4.1]*

From Remark 1.4, we observe that the class of G-monotone generalized quasi contraction mappings contains the class of G-monotone quasi contraction mappings. The following example illustrates that the inclusion is proper.

Example 4.1. *Let $X = \mathbb{R}$ with metric d, the usual absolute value metric, i.e., $d(x, y) = |x - y|$. We see that (X, d) is a complete metric space. Let $C = [0, \infty) \subset \mathbb{R}$ which is closed.*

Define a map

$$T : C \to C \quad \text{by } T(x) = \frac{x}{1 + x}.$$

Consider a graph G on X with $E(G) = X \times X$, then G is connected, reflexive and transitive digraph.

We are going to show that T is a G-monotone generalized quasi con-

traction mapping, but not a G-monotone quasi contraction mapping.

Now, consider a function

$$\varphi : [0, \infty) \to [0, \infty) \quad \text{given by } \varphi(t) = \frac{t}{1+t}.$$

We observe that φ is a comparison function.

Let $x, y \in C$. Without loss of generality we assume that $x < y$ (as d is symmetric, the same result follows for $x > y$). Then we have

$$\begin{aligned} d(T(x), T(y)) &= \frac{y-x}{(1+x)(1+y)} = \frac{y-x}{1+x+y+xy} \\ &\leq \frac{y-x}{1+y+x} \leq \frac{y-x}{1+y-x} \\ &= \varphi(y-x) = \varphi(d(x,y)) \leq \varphi(M(x,y)), \end{aligned}$$

where,

$$M(x,y) \;=\; max\{d(x,y), d(x,T(x)), d(y,T(y)), d(x,T(y)), d(T(x),y)\}.$$

Therefore,

$$d(T(x), T(y)) \;\leq\; \varphi(M(x,y)),$$

which shows that T is a G-monotone generalized quasi contraction mapping.

Next we show that T is not a G-monotone quasi contraction mapping. Indeed,

$$d(x, T(x)) \;=\; x - \frac{x}{1+x} = \frac{x^2}{1+x},$$

$$d(y, T(y)) \;=\; y - \frac{y}{1+y} = \frac{y^2}{1+y},$$

$$d(x, T(y)) \;=\; |x - \frac{y}{1+y}| = \frac{|x-y+xy|}{1+y}$$

$$\leq \frac{y-x+xy}{1+y} \leq \frac{y-x+xy}{1+x}, \tag{4.169}$$

and

$$d(T(x),y) \;=\; |y-\frac{x}{1+x}| = \frac{y-x+xy}{1+x}. \tag{4.170}$$

Now, we claim that

$$M(x,y) \;=\; \frac{y-x+yx}{1+x}. \tag{4.171}$$

Using the fact that the function

$$f(x) = \frac{x}{1+x}$$

is strictly increasing function and the assumption that $x < y$, *we have*

$$\begin{aligned}\frac{x^2}{1+x} \;&=\; x\frac{x}{1+x} \leq x\frac{y}{1+y} \\ &\leq\; \frac{y^2}{1+y}.\end{aligned} \tag{4.172}$$

In addition, if $y-x>0$, *then we have*

$$y-x+yx-yx+y^2+y^2x > yx-yx+y^2+y^2x,$$

which implies that

$$(y-x)(1+y)+yx(1+y) \geq y^2(1+x)$$

and hence

$$\frac{y-x+yx}{1+x} \geq \frac{y^2}{1+y}. \tag{4.173}$$

On the other hand

$$d(x,T(y)) \;\leq\; \frac{y-x+yx}{1+y}$$

$$\leq \frac{y-x+yx}{1+x}. \tag{4.174}$$

Again, as $-x^2 \leq 0$*, we have*

$$y-x+yx-x^2 \leq y-x+xy$$

which implies that

$$(y-x)(1+x) \ \leq \ y-x+yx$$

and hence

$$y-x \ \leq \ \frac{y-x+yx}{1+x}.$$

From (4.169)-(4.174), we conclude that

$$M(x,y) \ = \ \frac{y-x+yx}{1+x} = d(x,Ty).$$

Now, if there is a constant number $k \in [0,1)$ *such that for each* $x, y \in C$

$$d(T(x),T(y)) \ = \ \frac{y-x}{(1+x)(1+y)} \leq kM(x,y) = k\frac{(y-x+yx)}{1+x}$$

then we must have

$$\frac{1}{1+y} \ \leq \ k(1+\frac{yx}{y-x}). \tag{4.175}$$

Letting $y \to 0$ *(and hence* $\frac{yx}{y-x} \to 0$*) in (4.175), we get* $1 \leq k$*, which is a contradiction. Hence,* T *is not a* G*-monotone quasi contraction mapping. Actually,* T *has a unique fixed point and for each* $x \in C$*,*

$$T^n(x) \ = \ \frac{x}{1+nx} \leq \frac{x}{nx} = \frac{1}{n} \to 0$$

as $n \to \infty$*. Here* 0 *is a unique fixed pint of* T*, where the* G*-monotone quasi contraction mapping could not be applied to guarantee the existence of such fixed points.*

Chapter 5

Discussion, Conclusion and Recommendations

5.1 Discussion and Conclusion

In this thesis, results that improve and generalize several recent results in the contemporary literature is investigated. In particular, studying fixed points of G-nonexpansive self mappings in spaces without Opial's condition and with graph structure is appeared for the first time in our two successive results. Furthermore, rather than working on one or two operators, we dealt with finitely many of them. In addition to this, a more general iterative methods are employed. Thus, the obtained results are far beyond extending the existing results in this direction. Specifically, the results of G-nonexpasive self mappings in uniformly convex Banach spaces with graph structure generalizes the results due to Tripak [77](and references therein), in the following ways:

(i) Finite Noor iterative scheme which is more general than Ishikawa scheme were used.

(ii) The Opial's condition assumed on the space under consideration is dispensed.

(iii) The number of mappings considered is extended arbitrary finite family.

(iv) The dominance assumption is modified.

The results for the class of G-nonexpansive self mappings in uniformly convex Banach spaces with graph structure was published in *International Journal of Advances in Mathematics,* Vol. 2017, no 6, 137-152, 2017.

The results for the class of G-asymptotically nonexpansive self mappings in uniformly convex Banach spaces is also under review in *Journal of Fixed Point Theory and Applications.*

Moreover, the results for nonlinear quasi contraction mappings in the setting of modular metric spaces with graph structure Theorem 4.5 and Theorem 4.6 generalizes Theorem 2.14 and Theorem 2.15, respectively of the results due to Alfuraidan [6], from quasi contraction to nonlinear quasi contraction mappings. Moreover, the results in this line was published in *Journal of Abstract and Computational Mathematics,* NTMSCI 2, no. 2, 20-38, 2017

5.2 Recommendations

Our recommendations for the future work in this direction are as follows:

- In all of our results, only single valued self mappings were considered. So, the following problems need further investigations.

(a) Can we extend these results to multivalued case?

(b) Can we obtain similar results, if non-self mappings were considered?

- Our Results on common fixed point approximations are obtained on uniformly convex Banach spaces. So, can we extend these results on the setting of modular function spaces?

Bibliography

[1] A.N. Abdou, M.A. Khamsi: Fixed points of multivalued contraction mappings in modular metric spaces. *Fixed Point Theory Appl.,* 2014 (2014).

[2] R.P. Agarwal, D. O'Regan, D.R. Sahu: Fixed Point Theory for Lipschitzian-type Mappings with Applications. *Springer, New York* (2009).

[3] S. Aleomraninejad, S. Rezapour, N. Shahzad: Some fixed point result on a metric space with a graph. *Topol. Appl.,* 159, 659-663 (2012).

[4] M.R. Alfuraidan: Fixed points of monotone nonexpansive mappings with a graph. *Fixed Point Theory Appl.*, 2015 (2015).

[5] M.R. Alfuriadan: The contraction principle for mappings on a modular metric space with a graph. *Fixed Point Theory Appl.,* 2015 (2015).

[6] M.R. Alfuriadan: On monotone Ciric quasi-contraction. *Fixed Point Theory Appl.,* 2015 (2015).

[7] M.R. Alfuraidan, M.A. Khamsi: Fixed points of monotone nonexpansive mappings on a hyperbolic metric space with a graph. *Fixed Point Theory Appl.,* 2015 (2015).

[8] S. Banach: Sur les opérations dans les ensembles abstraits et leurs applications. *Fundam. Math.,* 3, 133-181 (1922).

[9] I. Beg, R. Butt: The contraction principle for set valued mappings on a metric space with a graph, *Comput. Math. Appl.,* 60, 1214-1219 (2010).

[10] A. Bnouhachem, M.A. Noor, Th.M. Rassian: Three-step iterative algorithms for mixed variational inequalities. *Appl. Math. Comput.,* 183, 436-446 (2006).

[11] S.C. Bose: Weak convergence to the fixed point of an asymptotically nonexpansive maps. *Proc. Amer. Math. Soc.,* 68, 305-308 (1978).

[12] D.W. Boyd, J.S. Wong: Nonlinear contractions. *Proc. Amer. Math. Soc.,* 20, 458-464 (1969).

[13] F. Browder: Nonexpansive nonlinear operators in a Banach space. *Proc. Nat. Acad. Sci. U.S.A.,* 54, 1041-1044 (1965).

[14] S.S. Chang, Y.J. Cho, H. Zahu: Demiclosed principle and weak convergence problems for asymptotically nonexpansive mappings. *J. Korean Math. Soc.,* 38, 1245-1260 (2001).

[15] C.E. Chidume, N. Shahzad: Strong convergence of an implicit iteration process for a finite family of nonexpansive mappings. *Nonlinear Anal.,* 62, no. 6, 1149-1156 (2005).

[16] C.E. Chidume: Iterative algorithms for nonexpansive mappings and some of their generalizations. *Nonlinear Anal. Appl.,* 1, no 2, 383-429 (2003).

[17] C.E. Chidume, E.U. Ofoedu, H. Zegeye: Strong and weak convergence theorems for asymptotically nonexpansive mappings. *J. Math. Anal. Appl.,* 280, 364-374 (2003).

[18] C.E. Chidume, S.A. Mutangadura: An Example on the Mann Iteration Method for Lipschitz Pseudocontractions. *Proc. Amer. Math. Soc.,* 129, no 8, 2359-2363 (2001).

[19] C. Chifu, G. Petrusel: Generalized contractions in metric spaces endowed with a graph. *Fixed Point Theory Appl.,* 2012 (2012).

[20] V.V. Chistyakov: Modular metric spaces, I: basic concepts. *Nonlinear Anal.,* 72, no 1, 1-14 (2010).

[21] V.V. Chistyakov: Modular metric spaces, II: application to superposition operators. *Nonlinear Anal.,* 72, no 1, 15-30 (2010).

[22] V.V. Chistyakov: A fixed point theorem for contractions in modular metric spaces. *Nonlinear Anal.,* 72, no 1, 1–14 (2011).

[23] L.B. Ciric: A generalization of Banach's contraction principle. *Proc. Amer. Math. Soc.,* 45, 267-273 (1974).

[24] E.H. Connell: Properties of fixed point spaces. *Proc. Amer. Math. Soc.,* 10, 974-979 (1959).

[25] R. Diestel: Graph Theory. *Springer, New York* (2000).

[26] T. Dominguez-Benavides, M.A. Khamsi, S. Samadi: Asymptotically nonexpansive mappings in modular function spaces. *J. Math. Anal. Appl.,* 265, no 2, 249-263 (2002).

[27] T. Fujimoto: An extension of Tarski's fixed point theorem and its application to isotone complementarity problems. *Math. Program.,* 28, 116-118 (1984).

[28] H. Fukhar-ud-din, A.R. Khan: Approximating common fixed points of asymptotically nonexpansive maps in uniformly convex Banach spaces. *Comput. Math. Appl.,* 53, 1349-1360 (2007).

[29] K. Goebel, W.A. Kirk: A fixed point theorem for asymptotically nonexpansive mappings. *Proc. Amer. Math. Soc.,* 35, 171-174 (1972).

[30] S. Ishikawa: Fixed points by a new iteration method. *Proc. Amer. Math. Soc.,* 44, 147-150 (1974).

[31] I.D. Arandelovic: On Boyd-Wong-Type Fixed Point Results. *Appl. Math. Inform. Mech.,* 7, no 1, 41-46 (2015).

[32] J.J. Jachymski: The contraction principle for mappings on a metric space with a graph. *Proc. Amer. Math. Soc.,* 1, no 136, 1359-1373 (2008).

[33] R. Kannan: Some results on fixed points. *Bull. Calcutta Math. Soc.,* 60, 71-76 (1968).

[34] R. Kannan: Some results on fixed points. American Mathematical Monthly, vol. 76, 405-408 (1969).

[35] R.P. Kelisky, T.J. Rivlin: Iterates of Bernstein polynomials. *Pac. J. Math.,* 21, 511-520 (1967).

[36] M.A. Khamsi, W.A. Kirk: An Introduction to Metric Spaces and Fixed Point Theory. *John Wiley and Sons, Inc., New York* (2001).

[37] M.A. Khamsi, W.M. Kozlowski: On asymptotic pointwise nonexpansive mappings in modular function spaces. *J. Math. Anal. Appl.,* 380(2), 697-708 (2011).

[38] A.R. Khan, A.A. Domlo, H. Fukhar-ud-din: Common fixed points Noor iteration for a finite family of asymptotically quasi-nonexpansive mappings in Banach spaces. *J. Math. Anal. Appl.,* 341, 1-11 (2008).

[39] S.H. Khan, W. Takahashi: Approximating common fixed points of two asymptotically nonexpansive mappings. *Sci. Math. Jpn.,* 53, 133-138 (2001).

[40] G.E. Kim, H. Kiuchi, W. Takahashi: Weak and strong convergence theorems for nonexpansive mappings. *Sci. Math. Jpn.,* 56, no. 1, 133-141 (2002).

[41] W.M. Kozlowski: Modular Function Spaces. Series of Monographs and Textbooks in Pure and Applied Mathematics, vol. 122. *Marcel Dekker, New York,* (1988).

[42] W.M. Kozlowski: Advancements in fixed point theory in modular function spaces. *Arab. J. Math.,* 2012 (2012).

[43] J. Matkowski: Integrable solutions of functional equations. *Diss. Math. 127, Warsaw* (1975).

[44] J. Musielak: Orlicz Spaces and Modular Spaces. Lecture Notes in Mathematics, vol. 1034. *Springer, Berlin* (1983).

[45] J. Musielak, W. Orlicz: On modular spaces. *Studia Math.,* 18, 59-65 (1959).

[46] H. Nakanno: Modulared Semi-Ordered Linear Spaces. *Maruzen, Tokyo* (1950).

[47] J.J. Nieto, R.L. Pouso, R. Rodriguez-Lopez: Fixed point theorems in ordered abstract spaces. *Proc. Amer. Math. Soc.,* 135, 2505-2517 (2007).

[48] M.A. Noor: New approximation schemes for general variational inequalities. *J. Math. Anal. Appl.,* 251, 217-229 (2000).

[49] M.A. Noor, Th.M. Rassias, Z. Huang: Three-step iterations for nonlinear accretive operator equations. *J. Math. Anal. Appl.,* 274, 59-68, (2002).

[50] W.R. Mann: Mean value methods in iteration. *Proc. Amer. Math. Soc.,* 4, 506-510 (1953).

[51] Z. Opial: Weak convergence of the sequence of successive approximations for nonexpansive mappings. *Bull. Amer. Math. Soc.,* 73, 591-597 (1967).

[52] M.O. Osilike, Aniagbosor: Weak and strong convergence theorems for fixed points of asymptotically nonexpansive mappings. *Math. Comput. Modeling,* 32, 1181-1191 (2000).

[53] J.A. Park: Mann-iteration Process for the Fixed Point of Strictly Pseudocontractive Mapping in Some Banach Spaces, *J. Korean Math. Soc.,* 31(3), 333-337 (1994).

[54] G.B. Passty: Construction of fixed points for asymptotically nonexpansive mappings. *Proc. Amer. Math. Soc.,* 384, 213-216 (1982).

[55] W. Phuengrattana, S. Suantai: Comparison of the rate of convergence of various iterative methods for the class of weak contractions in Banach spaces. *Thai. J. Math.,* 11, 217-226 (2013).

[56] H. Rahimpoor, A. Ebadian, M. Ehaghi, A. Zohri: Fixed Point Theory for Generalized Quasi-contraction Maps in Modular Metric Spaces. *J. Math. comp.,* 10, 54-60 (2014).

[57] A. Ran, M. Reurings: A Fixed point theorem in partially ordered sets and some application to matrix equations. *Proc. Amer. Math. Soc.,* 132, 1435-1443 (2004).

[58] S. Reich, A.J. Zaslavski: Convergence of Krasnoselskii-Mann iterations of nonexpansive operators. *Math. Comp. Modeling,* 32(11-13), 1423-1431 (2000).

[59] B.E. Rhoades: Fixed point iterations for certain nonlinear mappings. *J. Math. Anal. Appl.,* 183, 118-120 (1994).

[60] B.E. Rhoades: Some theorems on weakly contractive maps. *Fixed Point Theory Appl.,* 2015 (2015).

[61] M.G. Sangago: Convergence of Iterative Schemes for Nonexpansive Mappings. *Asian-European J. Math.,* 4, no. 4, 671-682 (2011).

[62] S. Sharma, B. Deshpande: Common Fixed Point Theorems for Mann Type Iterations. *East Asian J. Math.,* 17(1):19-32 (2001).

[63] S. Suantai: Weak and strong convergence criteria of Noor iterations for asymptotically nonexpansive mappings. *J. Math. Anal. Appl.,* 311, no. 2, 506-517 (2005).

[64] J. Schu: Weak and strong convergence to fixed points of asymptotically nonexpansive mappings. *Bull. Austral. Math. Soc.,* 43, no. 1, 153-159 (1991).

[65] S.P. Singh, B.A. Meade: On common fixed point theorems. *Bull. Austral. Math. Soc.,* 16, 49-53 (1977).

[66] N. Shahzad, R. Al-Dubiban: Approximating common fixed points of nonexpansive mappings in Banach spaces. *Georgian J. Math.,* 13(3), 529-537 (2006).

[67] R. Suparatulatorn, W. Cholamjiak, S. Suantai: A modified S-iteration process for G-nonexpansive mappings in Banach spaces with graphs. *Numer. Algor.,* 2017 (2017).

[68] W. Takahashi, T. Tamura: Convergence theorems for a pair of nonexpansive mappings. *J. Convex Anal.,* 5, no. 1, 45-56 (1998).

[69] W. Takahashi: Nonlinear Functional Analysis. Fixed Point Theory and Its Applications. *Yokohama publisher, Yokohama, Japan* (2000).

[70] K.K. Tan, H.K. Xu: The nonlinear ergodic theorem for asymptotically nonexpansive mappings in Banach spaces. *Proc. Amer. Math. Soc.,* 114, 399-404 (1992).

[71] K.K. Tan, H.K. Xu: Fixed point iteration process for asymptotically nonexpansive mappings. *Proc. Amer. Math. Soc.,* 122, 733-739 (1994).

[72] K.K. Tan, H.K. Xu: Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process. *J. Math. Anal. Appl.,* 178, no. 2, 301-308 (1993).

[73] H.K. Xu: A Note on the lshikawa Iteration Scheme. *J. Math. Anal. Appl.,* 167, 582-587 (1992).

[74] A. Tarski: A lattice theoretical fixed point and its application. *Pac. J. Math.,* 5, 285-309 (1955).

[75] J. Tiammee, A. Kaekhao, S. Suantai: On Browder's convergence theorem and Halpern iteration process for G-nonexpansive mappings in Hilbert spaces endowed with graph. *Fixed Point Theory Appl.,* 2015 (2015).

[76] J. Tiammee, S. Suantai: Coincidence point theorems for graph-preserving multi-valued mappings. *Fixed Point Theory Appl.,* 2014 (2014).

[77] O. Tripak: Common fixed points of G-nonexpansive mappings on Banach spaces with a graph. *Fixed Point Theory Appl.,* 2016 (2016).

[78] P. Wajtaszczyk: Banach Spaces for Analysts. *Cambridge Univ. Press* (1991).

[79] B. Xu, M.A. Noor: Fixed point iteration for asymptotically nonexpansive mappings in Banach spaces. *J. Math. Anal. Appl.,* 267, 444-453 (2002).

[80] H.K. Xu: Inequalities in Banach Spaces with Applications. *Nonlinear Anal.,* 16, No. 12, 1127-1138 (1991).

[81] Y.J. Cho, R. Saadati, G. Sadeghi: Quasi-Contractive Mappings in Modular Metric Spaces. *Journal of Applied Mathematics,* 2012 (2012).