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GRADUATE PROJECT REPORT

On

Duality in Multiobjective Programming

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Abstract

Nowadays, human beings have been confronted with multiple criteria decision making problems. We want to have a good life, which may mean more wealth, more power, more respect and more time for our selves, together with a good health and a good second generation, e.t.c. Unlike single objective optimization in solving multi-objective optimization problem, we have solution set that is called efficient set. It is from this set decision is made by taking elements of efficient set as alternatives, which is given by analysts. This graduate project report contains the mathematical theories in multi-objective optimization, necessary and sufficient condition for existence of efficient solutions and their properties in partial ordered vector space. Furthermore, the dual problem has (under additional conditions) the same optimal value as the given "primal" optimization problem, but solving the dual problem could be done with other methods of analysis or numerical mathematics. An approximate solution of the given minimization problem gives an estimation of the minimal value p^* from above, whereas an approximate solution of the dual problem is an estimation of p^* from below, so that one gets intervals which contain p^* . Lagrange method, saddle points, equilibrium points of two person games, shadow prices in economics, perturbation methods or dual variational principles, it becomes clear that optimal dual variables often have a special meaning for the given problem. Thus, in this report different duality approach for multiobjective optimization problem is discussed using point-to-set map.

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Table of contents

Contents Page

CHAPTER ONE

1. Introduction and Problem Formulation.

Every day we encounter varies kinds of decision making problems as manager, resource planner, designers, a dministrative of ficers, mere i ndividuals, a nd so on. I n t hese p roblems, t he f inal decision is usually made through several steps; the structural model, the impact model, and the evaluation model even though they sometimes not be perceived explicitly.

[6] By structural modeling, we mean constructing a model in order to know the structure of the problem, what the problem is, which factors comprise the problem, how they interrelate, and so on. Though the process, the objective of the problem and alternatives to perform it are specified. Hereafter, we shall use the notation Y for the objective and X for the set of alternatives, which is supposed to be a subset of an n-dimensional vector space.

In order to solve our decision making problem by some systems analytical methods, we usually require that degrees of objectives be represented in numerical terms, which may be of multiple kinds even for one objective. In order to exclude subjective value judgment at this stage, we restrict these numerical terms to physical measures (for example money, weight, length, and time). As a performance index, for the objective Y_i an objective function $f_i: X \to \mathbb{R}^1$ is introduced. Where R^1 denotes one dimensional Euclidean space. The value $f_i(x)$ indicates how much impact is given on objective Y_i by performing an alternative x. In this report we assume that a smaller value for each objective function is preferred to large one.

Now we can formulate our decision making problems as a Multiobjective optimization problem:

(P) Minimize $f(x) = (f_1(x), f_2(x), \ldots, f_p(x))^T$ over $x \in X$.

In some cases, some of the objective functions are required to be maintained under given levels prior to minimizing other objective functions. Denoting these objective functions by $g_j(x)$, we require that

$g_j(x) \leq 0, \quad j = 1,2, \ldots, m,$

Such a function $g_j(x)$ is generally called a constraint function. According to the situation, we consider either the problem (P) itself or (P) accompanied by the constraint conditions

$g_j(x) \leq 0, \quad j = 1,2, \ldots, m.$

Of course, an equality constraint $h_k(x) = 0$ can be embedded within two inequalities $h_k(x) \leq 0$ $and - h_k(x) \leq 0$, and hence, it does not appear explicitly in this paper.

[8] We call the set of alternatives X the **feasible set** of the problem. The space containing the feasible set is said to be a **decision space**, whereas the space that contain the image of the feasible set $Y = f(X)$ is referred to as **criterion space**.

Unlike the traditional mathematical programming with a single objective function, an optimal solution in the sense of one that minimizes all the objective function simultaneously does not necessarily exist in multiobjective optimization problem, and hence, we are in trouble of conflicts among objectives in decision making problems with multiple objectives, the final decision should be made by taking the total balance of objectives into account. Here we assume a decision maker who is responsible for the final decision. [6] The decision maker's value is usually represented by saying whether or not an alternative x is preferred to another alternative x' or equivalently whether or not $f(x)$ is preferred to $f(x')$. In other words, the decision maker's value is represented by some binary relation over X or $f(X)$. Since such a binary relation representing the decision maker's preference usually become an order, it is called *a preference order* and it is supposed to be defined on the so called criteria space Y, which includes the set $f(X)$. Several kind of preference orders could be possible, sometimes the decision maker cannot judge whether or not $f(x)$ is preferred to $f(x')$. Such an order that admits incomparability for a pair of objects is called *partial order*, whereas the order requiring the comparability for every pair of objects is called *a weak order or total order*. In practice, we often observe a partial order for the decision maker's preference. Unfortunately, however, an optimal solution in the sense of one that is more preferred with respect to the order, hence the notion of optimality does not necessarily exist for partial orders. Instead of strict optimality, we introduce in multiobjective optimization the notion of efficiency. A vector $f(x')$ is said to be efficient if there is no $x \in X$ such that $f(x)$ is preferred to $f(x')$ with respect to the preference order. The final decision is made among the set of efficient solutions. This report is mainly concerned with some of the theoretical aspects in vector optimization problem; in particular we will focus on existence, necessary and sufficient conditions for efficient solutions.

Chapter two is devoted to mathematical notions and preliminaries. The first section gives a review of convex sets, cones, convex functions, and other properties related to convexity, which have important role in multiobjective optimization. The second section introduce point-to-set map that play a very important role in the analysis, since efficient solutions usually constitutes a set. The third section is concerned with a brief explanation of preference order. These concepts are fundamental for existence and necessary /sufficient condition for efficient solutions.

Chapter three begins with the introduction of several possible concepts for solutions in multiobjective optimization. Above all, efficient solutions will be the subject of primary consideration in subsequent theories. Next, some properties of efficient solutions, such as existence, properly efficient, connectedness, and external stability will be discussed.

Chapter four will be devoted to the duality theory in multiobjective optimization. Duality is a fruitful result in traditional mathematical programming and is very useful both theoretically and practically. Consequently, it is quite interesting to extend the duality theory to the case of multi-objective optimization. In first section the duality theory in nonlinear cases will be discussed in parallel with the case of ordinary convex programming. Given a convex multiobjective programming problem, some new concepts such as the primal map, the dual map, and the vector valued Lagrangian will be defined. The Lagrangian multiplier theorem, the saddle point theorem, and the duality theorem will be obtained. The second section will develop the conjugate duality theory in multiobjective optimization in this section we introduce the concepts of dual map, which are the extension of conjugate function, and develop duality for multiobjective optimization. A first order necessary and sufficient condition for unconstrained cone d.c . programming problems is also discussed. This condition is given in terms of directional derivative and subdfferential of component functions. Moreover, a weak duality theorem is proved in a more general partially ordered linear topological vector space.

CHAPTER TWO

2. Mathematical preliminaries

In th is c hapter w e in vestigate o ptimization p roblems w ith f eature m ore th an o ne o bjective function. I n t he be ginning f undamental c oncepts t hat a re e ssential f or multiobjective optimization ar e est ablished. T hat is, o rdering co nes a re i ntroduced a nd key properties of efficient s ets a re e xamined i n de tail. S ubsequently, the e xistence of e fficient poi nts a nd techniques to compute these are analyzed. One method to determine solutions to multiobjective problems is to c onsider re lated s calarized pr oblems, which l eads t o the de finition of pr operly efficient points. Finally the major result, that the properly efficient points are dense within the set of efficient solutions, is presented at the end of this chapter.

2.1. Elements of convex analysis

2.1.1. Convex set

Definition 2.1.1 (algebraic sum of two sets)

1. The algebraic sum of two sets is defined as

   $S^1 + S^2 := \{z^1 + z^2 \mid z^1 \in S^1, z^2 \in S^2\}$.

   In case $S^1 = \{z^1\}$ is a singleton we use the form $z^1 + S^2$ instead of $\{z^1\} + S^2$.

2. Let $s, s^1, s^2 \subset \mathbb{R}^p$ and $\alpha \in \mathbb{R}$. The multiplication of a scalar with a set is given by

   $\alpha S := \{\alpha z \mid z \in S\}$, in particular $-S = \{-z \mid z \in S\}$

Definition 2.1.2

i. The point $X \in \mathbb{R}^n$ is said to be a convex combination of two points $x^1, x^2 \in \mathbb{R}^n$ if $X = \alpha x^1 + (1 - \alpha) x^2$, for some $\alpha \in \mathbb{R}, O \leq \alpha \leq 1$.

ii. The point $X \in \mathbb{R}^n$ is said to be a convex combination of m points $x^1, x^2, \dots, x^m \in \mathbb{R}^n$ if

$$X = \sum_{i=1}^{m} \alpha_i x^i , for\ \alpha_i \geq 0, \sum_{i=1}^{m} \alpha_i = 1 .$$

Definition 2.1.3 A set $C \subset \mathbb{R}^n$ is said to be convex if for any $x^1, x^2 \in C$ and every real number $\alpha \in \mathbb{R}, O \leq \alpha \leq 1$, the point $\alpha x^1 + (1 - \alpha) x^2 \in C$

In other words, C is convex if the convex combination of every pair of points in C lies in C.

The intersection of all convex sets containing a given subset C of $\mathbb{R}^n$ is called the convex hull of C and denoted by conv(C).

Definition 2.1.4 (cone)

$C \subset \mathbb{R}^p$ is called a cone if $\alpha c \in C$ for all $c \in C$ *and for all* $\alpha \in \mathbb{R}, \alpha > 0$.

A cone C is referred to as:

• Nontrivial or proper, if $C \neq \emptyset$ *and* $C \neq \mathbb{R}^p$.

• Convex, if $\alpha d^1 + (1-\alpha)\, d^2 \in C$ for all d^1 *and* $d^2 \in C$ for all $0 < \alpha < 1$

• Pointed, if for $d \in C, d \neq 0, -d \notin C$, *i e.* $C \cap -C \subseteq \{0\}$

Theorem 2.1.1

A cone C is convex if and only if it is closed under addition. In other words,

$$\alpha c^1 + (1-\alpha)c^2 \in C \Leftrightarrow c^1 + c^2 \in C \;\; \forall\, c^1, c^2 \in C, \forall \alpha \in [0,1]$$

Proof First, a ssume t hat t he c one C is convex. T hen w e c an c onclude f or c^1, $c^2 \in C$ and $\alpha = 1/2$ in combination with the cone property of C that

$$\left(\frac{1}{2}\right)c^1 + \left(1 - \frac{1}{2}\right)c^2 \in C$$

$$2(\left(\frac{1}{2}\right)c^1 + \left(\frac{1}{2}\right)c^2) \in C$$

$$c^1 + c^2 \in C.$$

Secondly, suppose that the cone C is closed under addition. Exploiting the fact that C is a cone we deduce for c^1,$c^2 \in C$ and $\alpha \in [0, 1]$ that $\alpha c^1 \in C$ and $(1-\alpha)\, c^2 \in C$. Furthermore, since the cone is closed under addition we derive for these elements that $\alpha c^1 + (1-\alpha)c^2 \in C$.

Remark 2.1.1 Let $C \subset \mathbb{R}^n$ is a convex cone, $C + C \subseteq C$.

A cone $C \subset \mathbb{R}^p$ is said to be acute if there is an open half space

°$H^+ = \{x \in \mathbb{R}^n : x^T x^* > 0\}, x^* \neq 0$ such that $\mathrm{cl}C \subset H^+ \cup \{0\}$.

The pos itive pol ar a nd s trict pos itive pol ar of C, denoted by C^0 and C^{s0}, respectively are defined by

$C^0 = \{x^* \in \mathbb{R}^n : \langle x, x^* \rangle \geq 0$ for any $x \in C\}$

$C^{s0} = \{x^* \in \mathbb{R}^n : \langle x, x^* \rangle > 0$ for any non zero $x \in C\}$,

A set C in $\mathbb{R}^n$ is said to be a polyhedral convex set if it can be expressed as the intersection of a finite collection of closed half spaces, that is if

$C = \{x : \langle b^i, x \rangle \leq \beta_i \; for \; i = 1, \dots, m\}$, where $b^i \in \mathbb{R}^n, \beta_i \in \mathbb{R}$

Furthermore, if $\beta_i = 0$ for all $i = 1, \dots, m$, in the above expression, C is said to be a polyhedral convex cone.

Note that given a set X and a convex cone C in $\mathbb{R}^n$, X is said to be C-convex if $X + C$ is a convex set.

Note also that a set X is convex if and only if X is $\{0\}$ −convex. Moreover, if X is a convex set it is C-convex for arbitrary nonempty convex cone C

2.1.2 Convex functions

Definition 2.1.5 The epigraph of a function $f: \mathbb{R}^n \to [-\infty, \infty]$ is the set

$$epi(f) = \{(x, y) \in \mathbb{R}^{n+1} \mid x \in \mathbb{R}^n, y \in \mathbb{R}, f(x) \leq y\}.$$

Definition 2.1.6 A function $f: \mathbb{R}^n \to [-\infty, \infty]$ is said to be convex if its epigraph epi(f) is a convex subset of $\mathbb{R}^{n+1}$. Furthermore, if $\text{dom}(f) \neq \emptyset$ and $f(X) > -\infty\ \forall x \in \mathbb{R}^n$, and $f(X) < \infty$ for at least one $x \in X$, then f is a proper convex function.

f is said to be concave if $-f$ is convex.

Alternatively, a more conventional definition is as follows. Let $X \subset \mathbb{R}^n$ be a nonempty convex set.

A function $f: X \to \mathbb{R}$ is said to be convex if $f(\lambda z^1 + (1-\lambda)z^2) \leq \lambda f(z^1) + (1-\lambda)f(z^2)$ is satisfied for all $z^1, z^2 \in X$ and for all $\lambda \in [0, 1]$

Moreover, a function f is said to be strictly convex if for any

$$z^1, z^2 \in X, z^1 \neq z^2 \text{ and for all } \lambda \in (0, 1),$$

$$f(\lambda z^1 + (1-\lambda)z^2) < \lambda f(z^1) + (1-\lambda)f(z^2)$$

Furthermore, $f(x) = (f_1(x), \dots, f_p(x)) : \mathbb{R}^n \to \mathbb{R}^p$ is convex if the functions f_i are convex for all $i = 1, \dots, p$.

Remark 2.1.2 If $f: X \to \mathbb{R}$ is convex, then f is continuous in ri(X).

2.1.3 Conjugate function

Definition 2.1.7 [1], [5] Let f be a convex function $f: \mathbb{R}^n \to [-\infty, \infty]$, then the conjugate (or polar) function of f is the function $f^*: R^n \to [-\infty, \infty]$ defined as:

$$f^*(y) = \sup_{x \in R^n} \{y^T x - f(x) : x \in \mathbb{R}^n\}, y \in \mathbb{R}^n$$

The *biconjugate (or bipolar)* f^{**}: $\mathbb{R}^n \to [-\infty, \infty]$ of f is the conjugate of f^* is defined as:

$$f^{**}(x) = \sup_{x \in R^n} \{y^T x - f^*(y) : y \in \mathbb{R}^n\}, x \in \mathbb{R}^n$$

Proposition 2.1.1 (Fenchel's or Young's Inequality)

$$f^*(y) + f(x) \geq y^T x \;\; for\ any\ x\ and\ y$$

Proof: Follows directly from Definition 2.1.3.1.

2.1.4 Subgradients of convex functions

Definition 2.1.8

Let X be a convex subset in $\mathbb{R}^n$ and $f : X \to \mathbb{R}^p$ be a vector valued function.

i. The directional derivative of the function f at $x \in X$ in the direction $d \in \mathbb{R}^n$ is defined, if it exists, by: $f'(x; d) = \lim_{t \downarrow 0} \frac{f(x+td) - f(x)}{t}$

ii. f is said to be Gateaux differentiable at x if there exists a $p\ x\ n$ matrix $\nabla f(x)$ such that for any $d \in \mathbb{R}^n$, $f'(x; d)$ exist and $f'(x; d) = \nabla f(x)d.$

iii. If f is Gateaux differentiable at every x of X, then f is said to be Gateaux differentiable on X.

Definition 2.1.9 Let $f : \mathbb{R}^n \to [-\infty, \infty]$ be a convex function, and let $x \in dom(f)$. The vector $z \in \mathbb{R}^n$ is said to be a *subgradient* of f at X if

$z^T(y - x) \leq f(y) - f(x) \;\; \forall y \in \mathbb{R}^n,$

the set $\partial f(x) = \{z \in \mathbb{R}^n | z^T(y - x) \leq f(y) - f(x) \;\; \forall y \in \mathbb{R}^n\}$ is called the *subdfferential* of f at x.

Proposition 2.1.2: Let f be a convex function from $\mathbb{R}^n$ to $[-\infty, \infty]$. Then

$$f(x^*) = Min\{f(x): x \in R^n\} \iff 0 \in \partial f(x^*)$$

The result is a generalization for the condition $\nabla f(x^*) = 0$

Proof x^* is optimal if and only if $f(x) \geq f(x^*)$ for all x, or equivalently

$f(x) \geq f(x^*) + 0^T(x - x^*)$ for all $x \in \mathbb{R}^n$

Thus, $x^*$is optimal if and only if $0 \in \partial f(x^*)$

Remark 2.1.3

If $x \notin dom(f)$, then $\partial f(x) = \emptyset$ by convention

It can be easily established that for convex functions, $\partial f(x)$ is closed and convex;

$\partial f(x)$ is a singleton if and only if f is differentiable at x. In this case $\partial f(x) = \{\nabla f(x)\}$

Proposition 2.1.3

Let X be a convex subset in $\mathbb{R}^n$ and $f: X \to \mathbb{R}^p$ be a vector-valued function. Assume that f is Gateaux differentiable on X. Then f is convex on X if and only if for every $x^1, x^2 \in X$,

$$f(x^2) \geq f(x^1) + \nabla f(x^1)(x^2 - x^1)$$

Proof: By definition, if f is convex, then for all $t \in [0, 1]$

$$f\left(x^1 + t(x^2 - x^1)\right) \leq tf(x^2) + (1-t)f(x^1) = f\left(x^1\right) + t(f(x^2) - f(x^1))$$

$\frac{f\left(x^1+t\left(x^2-x^1\right)\right)-f(x^1)}{t} \leq f(x^2) - f(x^1)$ The **Proposition** holds by taking limit $as \ t \downarrow 0$

2.2. Point-To-Set Maps

[6] A point to set map F from a set X to a set Y is a map that associates a subset of Y with each point of X. Equivalently, F can be viewed as a function from the set X into the power set of Y (2^Y). In multiobjective optimization problem, it is difficult to obtain a unique solution.

Solving the problem often leads to a set solutions. Thus, if the problem has a parameter, the solution defines a point –to- set map from parameter space into the objective space.

Definition 2.2.1: If F is a point to set map from a set X to a set Y, then F is said to be

(i) lower semicontinuous (l.s.c) at a point $x \in X$ if $\{x^k\} \subset X$, $\{x^k\} \to x$, and $y \in \mathrm{F}(x)$ all implies the existence of an integer m and a sequence $\{y^k\} \subset Y$ such that $y^k \in \mathrm{F}(x^k)$ for $k \geq m$ and $y^k \to y$.

(ii) upper semicontinuous (u.s.c) at a point $x \in X$ if $\{x^k\} \subset X$, $x^k \to x$, $y^k \in \mathrm{F}(x^k)$ and $y^k \to y$ implies $y \in F(x)$.

(iii) continuous at point $x \in X$ if it is both l.s.c and u.s.c at x; and

(iv) l.s.c (resp. u.s.c, continuous) on $X' \subset X$ if it is l.s.c (resp.u.s.c, continuous) at every $x \in X'$.

Definition 2.2.2 let F be a point-to- set map from $\mathbb{R}^n$ into $\mathbb{R}^p$ and C be a convex cone in $\mathbb{R}^p$. The set $\{(x, y): x \in \mathbb{R}^n, y \in \mathbb{R}^p, y \in \mathrm{F}(x) + \mathrm{C}\}$ is called the C-epigraph of F and denoted by C-epi F.

Note that for a convex cone C in $\mathbb{R}^p$, F is said to be C-convex if C-epi F is a convex set.

2.3 Preference Orders and Domination structures

2.3.1 Preference Orders

A preference order represents the preference attitude of the decision maker in the objective space. It is a binary relation on a set $Y = f(X) \subset \mathbb{R}^p$ where f is a vector valued objective function, and X is a feasible decision set.

The basic binary relation $>$ means strict preference i.e $y > z$ for $y, z \in Y$ means objective value y is preferred to z.

Definition 2.3.1 (Binary relation)

Let S be any set. A binary relation R on S is a subset of $S \times S$.

It is called

1. reflexive, if $(s, s) \in R$ *for all* $s \in S$,
2. irreflexive if $(s, s) \notin R$ for all $s \in S$,
3. symmetric if $(s^1, s^2) \in R \Rightarrow (s^2, s^1) \in R$ for all $s^1, s^2 \in S$,
4. asymmetric if $(s^1, s^2) \in R \Rightarrow (s^2, s^1) \notin R$ for all $s^1, s^2 \in S$,
5. antisymmetric, if $(s^1, s^2) \in R$ and $(s^2, s^1) \in R \Rightarrow s^1 = s^2$ for all $s^1, s^2 \in S$,
6. transitive, if $(s^1, s^2) \in R$ and $(z^2, z^3) \in R \Rightarrow (s^1, s^3) \in R$ for all $s^1, s^2, s^3 \in S$,
7. negatively transitive if $(s^1, s^2) \notin R$ and $(s^2, s^3) \notin R \Rightarrow (s^1, s^3) \notin R$ for all $s^1, s^2, s^3 \in S$,
8. connected if $(s^1, s^2) \in R$ or $(s^2, s^1) \in R$ for all $s^1, s^2 \in S$ with $s^1 \neq s^2$,
9. strongly connected (or total) if $(s^1, s^2) \in R$ or $(s^2, s^1) \in R$ for all $s^1, s^2 \in S$.

Definition 2.3.2 A binary relation R on a set S is a preorder (quasi-order) if it is *reflexive* and *transitive*.

Instead of $(s^1, s^2) \in R$ we shall also write $s^1 R\, s^2$. In the case of R being a preorder the pair (S, R) is called a preordered set. In the context of (pre)orders yet another notation for the relation R is convenient. We shall write $s^1 \leq s^2$ as shorthand for $(s^1, s^2) \in R$ and $s^1 \nleq s^2$ for $(s^1, s^2) \notin R$ and indiscriminately refer to the relation R or the relation $\leq$. This notation can be read as "preferred to''.

Given any preorder $\leq$, two other relations are closely associated with $\leq$.

We define them as follows:

$s^1 < s^2 \Leftrightarrow s^1 \leq s^2$ and $s^2 \nleq s^1$,

$s^1 \sim s^2 \Leftrightarrow s^1 \leq s^2$ and $s^2 \leq s^1$.

Actually, $<$ and $\sim$ can be seen as the strict preference and equivalence (or indifference) relation, respectively, associated with the preference defined by preorder $\leq$.

Definition 2.3.3

A relation is named a partial order *if it is* ***reflexive****,* ***transitive*** *and* ***antisymmetric****.*

If R solely satisfies the first two properties it is denoted as a preorder *(**or quasi-order**).*

2.3.2 Domination structure

[6] Preference order (and more generally, binary relationship) on a set Y can be represented by a point-to-set map from Y in to Y. in fact, a binary relationship may be considered to be a subset of the product set YxY, and so it can be regarded as a graph of a point-to-set map from Y in to Y. That means we identify the preference order $\succ$ with the graph of set valued map P:

$$P(y) := \{y' \in Y \mid y \succ y'\}$$

[P(y) is the set of elements in Y less preferred to y]

[4], [6] Another approach for specifying preferences of the decision maker are the so called domination structures (ordering cone) where the preference order is represented by a set -valued map. Therefore for all $y \in \mathbb{R}^p$, we define the set of domination factors

$$C(y) := \{d \in \mathbb{R}^p \mid y \succ y + d\} \cup \{0\}$$

is defined where $y \succ y'$ means that the decision maker prefers y more than y'.

A deviation of $d \in C(y)$ from y is hence less preferred than the original y.

The most important and interesting special case of a domination structure is when $C(.)$ is a constant set-valued map, especially if $C(y)$ equals a pointed convex cone for all $y \in \mathbb{R}^p$ i.e. $C(.) = C$.

A cone C is called pointed if $C \cap (-C) = \{0p\}$.

Given an order relation R on $\mathbb{R}^p$, we can define a set

$C_R := \{y^2 - y^1 : y^1 R\, y^2\}$, (**)

Which we would like to interpret as the set of nonnegative elements of $\mathbb{R}^p$ according to R

It is interesting to consider the definition (**) with $y^1 \in \mathbb{R}^p$, fixed, i.e.,

$C_R(y^1) = \{y^2 - y^1 : y^1 R\, y^2\}$. If R is an order relation, $y^1 + C_R(y^1)$ is the set of elements of $\mathbb{R}^p$, that y^1 is preferred to (or that are dominated by y^1).

A natural question to ask is: Under what conditions is C_R (y) the same for all y ∈ $\mathbb{R}^p$? In order to answer that question, we need another assumption on order relation R.

[2] Definitions 2.3.4

1. A binary relation R is said to be compatible with addition if (y^1+z, y^2+z) ∈ R for all z ∈ $\mathbb{R}^p$, and all (y^1, y^2) ∈ R.
2. A binary relation R is said to be compatible with scalar multiplication if (αs^1, αs^2) ∈ R holds for all(s^1,s^2) ∈ R and for all α ∈ ℝ, and $\alpha > 0$

Theorem *2.3.1* *Let R be a relation which is compatible with scalar multiplication, then C_R is a cone.*

Proof Let d ∈ C_R then $d = y^2 - y^1$ for some $y^1, y^2 \in \mathbb{R}^p$ with (y^1, y^2) ∈ R.

Thus, (αy^1, αy^2) ∈ R for all α > 0, Hence $\alpha d = \alpha (y^2 - y^1) = \alpha y^2 - \alpha y^1 \in C_R$,

for all $\alpha > 0$.

The f ollowing re sult c onstitutes h ow c ertain properties o f a b inary relation R affects the characteristics of the cone C_R .

Lemma *2.3.1* If R is compatible with addition and $d \in C_R$then $0Rd$.

Proof

Let $d \in C_R$. Then there are $y^1, y^2 \in \mathbb{R}^p$ with y^1R y^2 such that $d = y^2 - y^1$.

Using z = $-y^1$, compatibility with addition implies $(y^1 + z) R (y^2 + z)$ or $0Rd$.

The above Lemma indicted that if R is compatible with addition, the sets $C_R(y)$, $y \in \mathbb{R}^P$, do not depend on y. In this report, we will be mainly concerned with this case.

Theorem *2.3.2* Let R be a binary relation on $\mathbb{R}^p$ which is compatible with scalar multiplication and addition. Then the following statements hold.

1. $0 \in C_R$ if and only if R is reflexive.
2. C_R is pointed if and only if R is antisymmetric.
3. C_R is convex if and only if R is transitive.

Proof

1. Let R be reflexive and let $y \in \mathbb{R}^p$. Then yRy and $y - y = 0 \in C_R$. Let $0 \in C_R$. Then there is some $y \in \mathbb{R}$ with yRy. Now let $y' \in \mathbb{R}^p$. Then

y' = y +z for some $z \in \mathbb{R}^p$. Since yRy and R is compatible with addition we get $y'Ry'$

2. Let R be antisymmetric and let $d \in C_R$ such that $-d \in C_R$, too. Then there are $y^1, y^2 \in \mathbb{R}^p$ such that $y^1 R y^2$ and $d = y^2 - y^1$ as well as $y^3, y^4 \in \mathbb{R}^p$ such that $y^3 \; R y^4$ and $-d = y^4 - y^3$. Thus, $y^2 - y^1 = y^3 - y^4$ and there must be y $\in \mathbb{R}^p$ such that $y^2 = y^3 + y$ and $y^1 = y^4 + y$. Therefore compatibility with addition implies $y^2 \; R y^1$. Antisymmetry of R now yields $y^2 = y^1$ and therefore,

$d = 0$, i.e. C_R is pointed.

Let $y^1, y^2 \in \mathbb{R}^p$ with $y^1 \; R y^2$ and $y^2 \; R y^1$. Thus, $d = y^2 - y^1 \in C_R$ and $-d = y^1 - y^2 \in C_R$. If C_R is pointed we know that $\{d, -d\} \subset C$ implies $d = 0$ and therefore $y^1 = y^2$, i.e., R is antisymmetric.

3. Let R be transitive and let $d^1, d^2 \in C_R$. Since R is compatible with scalar multiplication, C_R is a cone and we only need to show $d^1 + d^2 \in C_R$. By Lemma *2.3.1* we have $0Rd^1$ and $0Rd^2$. Compatibility with addition implies $d^1 R(d^1 + d^2)$, transitivity yields $0R(d^1 + d^2)$, from which $d^1 + d^2 \in C_R$.

Let C_R be convex and let $y^1, y^2, y^3 \in R^p$ be such that $y^1 \; R y^2$ and $y^2 \; R y^3$. Then $d^1 = y^2 - y^1 \in C_R$ and $d^2 = y^3 - y^2 \in C_R$. Because C_R is convex, $d^1 + d^2 = y^3 - y^1 \in C_R$. By Lemma 2.3.1 we get $0R(y^3 - y^1)$ and by compatibility with addition $y^1 \; R y^3$.

Example The weak componentwise order $\leq$ is compatible with addition and scalar multiplication. $C_{\leq} = \mathbb{R}^P_+$ contains 0, is pointed, and convex.

We have defined cone C_R given a relation R. We can also use a cone to define an order relation.

The concept of a constant set-valued map defining a domination structure has a direct connection to partial orderings. We recall the definition of partial orderings [3] as

Definition *2.3.2*

(a) A nonempty subset R $\subset \mathbb{R}^P \times \mathbb{R}^P$ is called a binary relation on $\mathbb{R}^P$.
We write xRy for $(x, y) \in$ R.

(b) A binary relation $\leq$ on $\mathbb{R}^P$ is called a partial ordering on $\mathbb{R}^P$ if for arbitrary $w, x, y, z \in \mathbb{R}^P$:

(i) $x \leq x$ (Reflexivity),

(ii) $x \leq y, y \leq z \Rightarrow x \leq z$ (Transitivity),

(iii) $x \leq y, w \leq z \Rightarrow x + w \leq y + z$ (Compatibility with addition),

(iv) $x \leq y, \alpha \in \mathbb{R}_+ \Rightarrow \alpha x \leq \alpha y$ (Compatibility with the scalar multiplication).

(c) A partial ordering $\leq$ on $\mathbb{R}^P$ is called antisymmetric if for arbitrary $x, y \in \mathbb{R}^P$

$x \leq y, y \leq x \Rightarrow x = y$.

A linear space $\mathbb{R}^P$ equipped with a partial ordering is called a partially ordered linear space.

An example for a partial ordering on $\mathbb{R}^P$ is the natural (or componentwise) ordering $\leq_p$ defined by

$\leq_p := \{(x, y) \in \mathbb{R}^P \times \mathbb{R}^P \mid x_i \leq y_i \text{ for all } i = 1, \ldots, p\}$.

Partial orderings can be characterized by convex cones.

Any partial ordering $\leq$ in $\mathbb{R}^P$ defines a convex cone by

$C := \{x \in \mathbb{R}^P \mid 0p \leq x\}$ and any convex cone, then also called ordering cone, defines a partial ordering on $\mathbb{R}^P$ by

$$\leq_C := \{(x, y) \in \mathbb{R}^P \times \mathbb{R}^P \mid y - x \in C\}.$$

For example the ordering cone representing the natural ordering in $\mathbb{R}^P$ is the positive orthant $\mathbb{R}^p_+$

A partial ordering $\leq_C$ is antisymmetric if and only if C is pointed.

Thus, preference orders which are partial orderings correspond to domination structures with a constant set-valued map being equal to a convex cone.

The point-to- set map $C: Y \to \mathbb{R}^p$ reprsents preference order and we call C domination structure (ordering cone)

CHAPTER THREE

3. PROPERTIES AND EXISTENCE OF THE EFFICIENT SOLUTION

3.1 Efficiency Solutions

The concept of optimal solutions to multiobjective optimization is not trivial. It is closely related to the preference attitudes of the decision makers.

The most fundamental solution concept is that of efficient solution (non dominated, noninferior) solution with respect to the domination structure of the decision maker.

[2] Let $Y = f(X)$ be the image of the feasible set X and $y = f(x)$ for $x \in X$. Hence, a multiobjective optimization problem is given by

(MOP) Minimize $f(x) = (f_1(x), \dots, f_p(x))$

subject to $x \in X \subset \mathbb{R}^n$.

A domination structure representing a preference attitude of the decision maker is supposed to be given as a point-to-set map C from Y to $\mathbb{R}^n$.

Hence, the optimization problem (MOP) can be restated as:

(P) $''Min''\ y = f(x)$

$$s.t. \quad y \in Y = f(X).$$

Definition 3.1.1 (Efficiency via order relations) An element $y^* \in Y$ is denoted as efficient, if there exists no other element $y \in Y$, that differs from y^* and is less than it with respect to the employed ordering relation $\leq_C$. That is,

$$\nexists y \in Y : y \neq y^*, y \leq_C y^*.$$

Definition 3.1.2 [1] ,[4], [6](Efficiency via ordering cones) Let Y be a set and C a related ordering cone. Then $y^* \in Y$ is called efficient, if there exists no $y \in Y$ such that $y^* - y \in C\backslash\{0\}$.

Moreover, all efficient elements for problem (P) are combined into the efficient set

$$E(Y, C) := \{y \in Y \,|\, Y \cap (y - C) = \{y\}\}.$$

In addition, if Y is the image of the feasible set X, $x^* \in X$ is labelled as Pareto optimal if $y^* \in Y$ satisfying $y^* = f(x^*)$ is efficient.

Definition 3.1.3 A feasible solution $x' \in X$ is called efficient or Pareto optimal, if there is no other $x \in X$ such that $f(x) \leq f(x')$. If x' is efficient, $f(x')$ is called nondominated point. If $x^1, x^2 \in X$ and $f(x^1) \leq f(x^2)$ we say x^1 dominates x^2 and $f(x^1)$ dominates $f(x^2)$.

A feasible vector $x^{'} \in X$ is called efficient or Pareto optimal $(C = \mathbb{R}^p_+)$, if there is no other decision vector $x \in X$ such that $f_i(x) \leq f_i(x^{'})$ for all $i = 1,2,\ldots,p$, and $f_i(x) \leq f_i(x^{'})$ for at least one objective function.

In this case, $\left(f(x^{'}) - \mathbb{R}^p_+\right) \cap (Y) = \{f(x^{'})\}$ or equivalently

$$(Y - \left(f(x^{'})\right) \cap \left(-\mathbb{R}^p_+\right) = \{0\}.$$

$x^{'} \in X$ is called weak Pareto optimal solution to the problem if there does not exist another decision vector $x \in X$ such that $f(x) < f(x^{'})$. This means,

$$(Y - (f(x^{'})) \cap (-intC) = \{0\}.$$

$$(Y - (f(x^{'})) \cap (-C \setminus \{0\}) = \{0\}.$$

$$(Y - \left(f(x^{'})\right) \cap \left(-int\mathbb{R}^p_+\right) = \emptyset$$

(If there is no other decision vector $x \in X$ such that $f_i(x) < f_i(x^{'})$ for all $i = 1,2,\ldots,p$.

Remark *3.1.1*

We can introduce a domination structure $C^{'}$ on a given domination structure C on Y as follows:

$$C^{'}(x) = \left\{d^{'} \in R^n : f(x + d^{'}) \in f(x) + C\left(f(x)\right) \setminus \{0\}\right\} \cap \{0\}.$$

If we denote $C^{'} = f^{-1}(C)$ and the set of efficient solution in the decision space is

$$\{x : f(x) \in E(y, C)\} = E(x, f^{-1}(C))$$

Proposition3.1.1 Let Y and Z be two sets in $\mathbb{R}^p$, and let C be constant ordering cone on $\mathbb{R}^p$ then $\mathrm{E(Y + Z, C)} \subset \mathrm{E(Y, C)} + \mathrm{E(Z, C)}$.

Proof

Let $y^* \in \mathrm{E(Y + Z, C)}$, then $y^* = y + z$ for some $y \in Y, z \in Z$.

We want to show $y \in \mathrm{E(Y, C)}$ and $z \in \mathrm{E(Z, C)}$.

Suppose not, then there exist $y^{'} \in Y$ and $0 \neq d \in C$ such that $y = y^{'} + d$.

Then $y^* = y^{'} + z + d$ and $y^{'} + z \in Y + Z$ which contradict the supposition $y^* \in \mathrm{E(Y + Z, C)}$.

Similarly we can show for Z.

Theorem 3.1.1 Let C be a pointed convex cone, then $\mathrm{E(Y,C)} = \mathrm{E(Y + C,C)}$.

Proof

The result is trivial if $Y = \emptyset$, because $\mathrm{Y + C} = \emptyset$ and the nondominated subsets of both are empty, too.

So let $Y \neq \emptyset$. First, assume $y \in \mathrm{E(Y + C,C)}$, but $y \notin \mathrm{E(Y,C)}$. There are two possibilities. If $y \notin Y$ there is $y' \in Y$ and $0 \neq d \in C$ such that $y = y' + d$.

Since, $y' = y' + 0 \in \mathrm{Y + C}$,we get $y \notin \mathrm{E(Y + C,C)}$ wich is a contradiction.

If $y \in Y$ there is $y' \in Y$ such that

$y' \leq y.$

Let $d = y - y' \in C.$ Therefore $\mathrm{y} = y' + \mathrm{d}$ and $y \notin \mathrm{E(Y + C,C)}$, a gain contradicting the assumption.

Hence in either cases $y \in \mathrm{E(Y,C)}$

Second, assume $y \in \mathrm{E(Y,C)}$ but $y \notin \mathrm{E(Z + C,C)}$. Then there is some $y' \in Y + C$ with $y - y' = d' \in C$. That is $y' = y'' + d''$ with $y'' \in Y, d'' \in C$ and therefore $y = y' + d' = y'' + (d' + d'') = y'' + d$ with $d = d' + d'' \in C$.

This implies $y \notin \mathrm{E(Y,C)}$, contradicting the assumption.

Hence, $y \in \mathrm{E(Y + C,C)}$.

Theorem 3.1.2 Let C be a nonempty ordering cone with $\mathrm{C} \neq \{0\}$.

Then $\mathrm{E(Y,C)} \subseteq \mathrm{bd(Y)}$.

Proof Let ($C = \mathbb{R}^p_+$), and let $y \in \mathrm{E(Y,C)}$ and suppose $y \notin bd(Y)$.

Therefore $y \in intY$ and there exists an ε-neighborhood $B(y,\varepsilon)$ of y (with $B(y,\varepsilon) := y + B(0,\varepsilon) \subset Y$, $B(0,\varepsilon)$ is an open ball with radius ε centered at the origin). Let $d \neq 0, d \in C.$

Then we can choose some $\alpha \in \mathbb{R}, 0 < \alpha < \varepsilon$ such that $\alpha d \in B(0,\varepsilon)$. Now,

$y - \alpha d \in Y$ with $\alpha d \in C \backslash \{0\}, i.e. y \notin \mathrm{E(Y,C)}$.

Proposition 3.1.2 Let $C_1$and C_2 be domination structures. Then C_1 is said to be included by C_2 if $C_1(y) \subset C_2(y)$ for all $y \in Y$ in this case, $\mathrm{E}(\mathrm{Y}, C_2) \subset \mathrm{E}(\mathrm{Y}, C_1)$.

Proposition 3.1.3 $\boldsymbol{E}(\alpha \mathrm{Y,C}) = \alpha E(\mathrm{Y,C})$, for $\alpha \in \mathbb{R}, \alpha > 0$.

Proof The easy proof is left to the reader,

Definition 3.1.4 (Connectedness) A set $Y \subset \mathbb{R}^p$ is said to be connected if there do not exist two open sets $O_1, O_2 \subset \mathbb{R}^p$ such that $Y \subseteq O_1 \cup O_2$ and $Y \cap O_1 \neq \emptyset$, $Y \cap O_2 \neq \emptyset$,

$Y \cap O_1 \cap O_2 = \emptyset$

Otherwise, Y is called connected.

Theorem 3.1.3 If Y is a closed and convex set and the section $(y - C) \cap Y$ is compact for all $y \in Y$, then $E(Y, C)$ is connected.

3.1.2 Properly Efficient Solutions

We know that $E(Y, C)$ is a set from which the decision maker has to choose an alternative among infinitely many optimal solutions. So we need to have relatively smaller size optimal solution set. Therefore it needs a more restricted concept then efficient (Pareto optimal) solution which is properly efficient solution. Under certain prerequisites these properly efficient points are dense within the efficient set and consequently, these sets are identical from a numerical point of view.

In the given context scalarization is considered to be a technique that transforms a multiobjective optimization problem into a family of single-criterion ones. This method employs scalarising functionals and we show in the following discussion that certain elements induce such functionals if they belong to a distinct set, namely the dual cone.

Definition3.1.5 (Domination in the sense of Geoffrion) [2]

A point x' is a properly efficient solution of the multiobjective optimization problem if it is efficient and there exist some real number $M > 0$ such that for each $i \in \{1, \ldots, p\}$ and each $x \in X$ satisfying

$f_i(x) < f_i(x')$, there exists at least one $j \in \{1, \ldots, p\}$ such that $f_j(x') < f_j(x)$ and

$$\frac{f_i(x') - f_i(x)}{f_j(x) - f_j(x')} \leq M$$

The corresponding point $y' = f(x')$ is called properly nondominated point.

Remark: To be Properly Pareto optimal unbounded trade-offs are not allowed.

Definition 3.1.6 (Dual cone) Let $C \subset \mathbb{R}^p$ be a cone. The dual cone C^0 is defined by

$C^0 = \{c^0 \in \mathbb{R}^p \mid (c^0)^T c \geq 0 \ \forall c \in C\}$.

The set $C^+ = \{\lambda \in \mathbb{R}^p \mid (\lambda)^T c > 0 \ \forall c \in C\backslash\{0\}\}$ is named the quasi-interior of the dual cone. Furthermore, we can easily derive that $C^+ = \text{int}(C^0)$ if $\text{int}(C^0) \neq \emptyset$.

Remark 3.1.2

1. Regarding the pr operties of t hese s ets w e note t hat t he c ase $C^+ = \emptyset$ can oc cur. For example, consider $C = \{(x,y) \in \mathbb{R}^2 \mid x \geq 0\}$. We deduce that

   $C^0 = \{(x,0) \in \mathbb{R}^2 \mid x \geq 0\}$ and moreover, $C^+ = \emptyset$.
2. The dual cone C^0 is a closed and convex cone since the scalar product is linear in c^0 and the limit of every convergent sequence in C^0 is also an element of C^0
3. If the non-negative orthant $C = \mathbb{R}^p{}_+$ is used as the ordering cone then the dual cone is $C^0 = \mathbb{R}^p{}_+$. Hence, we notice that both cones are pointed.

   For example the nonnegative orthant cone $\mathbb{R}^p{}_+$ is its own dual: because $y^T x \geq 0$, for all $x \geq 0$ and $y \geq 0$. We call such a cone self-dual.

Definition 3.1.7 (**C-monotonicity**) Let $\leq_c$ and $\leq$ be relations on $\mathbb{R}^p$ and $\mathbb{R}$, respectively. Furthermore, let φ be a function $\varphi : \mathbb{R}^p \to \mathbb{R}$. Then φ is named C-monotone with respect to $\leq_c$ if

$x \leq_c y, x \neq y \Rightarrow \varphi(x) \leq \varphi(y)$ for all $x, y \in \mathbb{R}^p$.

Moreover, the function φ is called strictly C-monotone with respect to $\leq_c$ if

$x \leq_c y, \ x \neq y \Rightarrow \varphi(x) < \varphi(y)$ for all $x, y \in \mathbb{R}^p$.

Remark 3.1.3 The elements $\lambda \in C^+$ generate strictly C-monotone functionals $\varphi(y) = (\lambda)^T y$. That is, the functionals created by these elements preserve the order established by the ordering cone.

Definition 3.1.8 (Proper Efficiency) Let $C \subseteq \mathbb{R}^p$ be a cone and $Y \subseteq \mathbb{R}^p, Y \neq \emptyset$ be a set. Then $y' \in Y$ is d enoted a s p roperly ef ficient w ith respect to p roblem (P) if th ere e xists a $\lambda \in C^+$ such that

$(\lambda)^T y' \leq (\lambda)^T y$, for all $y \in Y$.

The set of all properly efficient elements with respect to (P) is denoted by $P(Y, C)$.

Theorem 3.1.4 Let $Y, C \in \mathbb{R}^p$ be sets. Then the following statements are true.

1. If C is a convex cone, then $P(Y, C) \subseteq E(Y, C)$.
2. If C is a closed, convex and pointed cone and the set Y is closed and convex then

$$E(Y, C) \subseteq cl(P(Y, C)).$$

Proof

1. Concerning the first statement, suppose $y' \notin E(Y, C)$. If $y' \notin Y$ then it is apparent that $y' \notin P(Y, C)$. Therefore, assume that $y' \notin E(Y, C)$ and thus, deduce that there exists a $y \in Y$ such that $y' - y \in C\backslash\{0\}$.

Consequently, we obtain by the definition of the dual cone that for all $\lambda \in C^+$

$$(\lambda)^T (y' - y) > 0$$

$$(\lambda)^T y' - (\lambda)^T y > 0$$

$$(\lambda)^T y' > (\lambda)^T y.$$

Hence, $y' \notin P(Y, C)$.

2. Since $E(Y + C, C) = E(Y, C)$ and $P(Y + C, C) = P(Y, C)$ when C is a pointed closed convex cone. It is suffices to consider the case Y is closed and convex.

   If $E(Y, C) = \emptyset$, then the result is obvious. On the other hand, if $E(Y, C) \neq \emptyset$, then Y is C-closed and C-bounded.

   The weaker condition for the above theorem and the connectedness of the efficient solution is that Y is C-convex

Definition 3.1.9(cone convex function)

Let X be a convex set in $\mathbb{R}^n$, f be a function from X into $\mathbb{R}^p$, and C be a convex cone in $\mathbb{R}^p$. Then, f is said to be C-convex if for any $x^1, x^2 \in X$ and $\lambda \in [0, 1]$

$$\lambda f(x^1) + (1 - \lambda)f(x^2) - f(\lambda x^1 + (1 - \lambda)x^2) \in C$$

Proposition 3.1.4: Let $X \subset \mathbb{R}^n$ be a nonempty convex set. A function $f : X \to \mathbb{R}^p$ and C be a convex cone in $\mathbb{R}^p$. If the function f is C-convex, then the set $f(X)$ is C-convex.

Proof For $y^1, y^2 \in f(X) + C$ there exist $x^1, x^2 \in X$ and $c^1, c^2 \in C$ such that

$$y^i = f(x^i) + c^i$$

for $i = 1, 2$. Hence, we can derive for $\alpha \in [0, 1]$, $x^3 \in X$ and $c^3 \in C$ that

$$\alpha y^1 + (1 - \alpha)y^2 = \alpha(f(x^1) + c^1) + (1 - \alpha)(f(x^2) + c^2)$$

$= \alpha f(x^1) + (1 - \alpha)f(x^2) + \alpha c^1 + (1 - \alpha)c^2$

$\leq f(\alpha x^1 + (1 - \alpha)x^2) + c^3$. Since C is convex and f is c-convex

$= f(x^3) + c^3$. Since X is convex

$\subset f(X) + C.$

Proposition 3.1.5: *Let* $X \subset \mathbb{R}^n$ *be a nonempty convex set. A function* $f(x) = \left(f_1(x),\dots,f_p(x)\right)$ *be a function from* $\mathbb{R}^n$ *in to* $\mathbb{R}^p$. The function f is $\mathbb{R}^p{}_+$-convex, then if and only if f_i is convex and in this case $f(X)$ is $\mathbb{R}^p{}_+$-convex.

Proof Let $\mathbb{R}^p{}_+ = C$

($\Rightarrow$) Let f be $\mathrm{C}-$ convex function.

If $x, y \in R^n$, then $\lambda f(x) + (1-\lambda)f(y) - f(\lambda x + (1-\lambda)y) \in C$, for $\lambda \in [0, 1]$ which implies

$$\lambda\left(f_1(x),\dots,f_p(x)\right) + (1-\lambda)\left(f_1(y),\dots,f_p(y)\right) - \left(f_1(\lambda x + (1-\lambda)y),\dots,f_p(\lambda x + (1-\lambda)y)\right) \in C$$

$$\lambda\left(f_1(x),\dots,f_p(x)\right) + (1-\lambda)\left(f_1(y),\dots,f_p(y)\right) - \left(f_1(\lambda x + (1-\lambda)y),\dots,f_p(\lambda x + (1-\lambda)y)\right) \in C$$

$$(\lambda f_1(x) + (1-\lambda)f_1(y),\dots,\lambda f_p(x) + (1-\lambda)f_p(y)) - \left(f_1(\lambda x + (1-\lambda)y),\dots,f_p(\lambda x + (1-\lambda)y)\right) \in C$$

$\lambda f_i(x) + (1-\lambda)f_i(y) - f_i(\lambda x + (1-\lambda)y) \in C$

Hence f_i is convex for each i.

($\Leftarrow$) Suppose f_i is convex for each i.

$\lambda f(x) + (1-\lambda)f(y) - f(\lambda x + (1-\lambda)y) =(\lambda f_1(x) + (1-\lambda)f_1(y),\dots,\lambda f_p(x) + (1-\lambda)f_p(y)) - \Big(f_1(\lambda x +$

1 − λy, . . . , fpλx + 1 − λy,And since fi is convex for each i

$\lambda f_i(x) + (1-\lambda)f_i(y) - f_i(\lambda x + (1-\lambda)y) \in C$,

. For each i

Therefore$\lambda f(x) + (1-\lambda)f(y) - f(\lambda x + (1-\lambda)y) \in C$

Hence, altogether we deduce that the set of properly efficient points is dense within the efficient set.

3.2 Existence and External stability of Efficient Solution

Recall that in ordinary optimization problem,

Minimize $f(x)$

Subject to $x \in X \subset \mathbb{R}^n$,

the existence of optimal solution x^* is guaranteed if X is compact and the objective functions f is lower semicontinuous.

This idea can be extended to multiobjective optimization problem.

3.2.1 Existence of efficient solution

Definition 3.2.1 A domination structure C is said to be acyclic if it has no cycle.

That means for $n = 1,2, ...$ it never occurs that

$$y^1 \in y^2 + C(y^2) \setminus \{0\}, y^2 \in y^3 + C(y^3) \setminus \{0\}, \dots, y^n \in y^1 + C(y^1) \setminus \{0\}.$$

(i.e. $y^1 < y^2 < \cdots < y^n < y^1$).

Remark 3.2.1 if a domination structure C on Y is acyclic, it is also asymmetric. Conversely every transitive and asymmetric domination structure is acyclic.

Theorem 3.2.1

If the domination structure C on Y is acyclic, the set $C(y) \setminus \{0\}$ is open and Y is nonempty and compact, then $\mathrm{E(Y, C)} \neq \emptyset$

Proof suppose the contrary, that is $\mathrm{E(Y, C)} = \emptyset$, then for any $y \in Y$ there exist $y' \in Y$ such that $y \in y' + C(y') \setminus \{0\}$. thus $Y \subset \bigcup_{y \in Y}(y + C(y) \setminus \{0\})$

Thus, the families of the sets $\{y + C(y) \setminus \{0\}\}$, form an open cover of Y. Since Y is compact, then there is finite subcover $\left\{y^i + C\left(y^i\right) \setminus \{0\}\right\}$, $(i = 1,2, \dots n)$. then for $i \in \{1, \dots, n\}$,

$$y^i \in y^j + C\left(y^j\right) \setminus \{0\} \text{ for some } j \in \{1, \dots, n\}.$$

However, this contradicts the assumption that C is acyclic.

Hence, $\mathrm{E(Y, C)} \neq \emptyset$

Definition 3.2.2 Let $(S, \leq_C)$ be a preordered set, i.e. $\leq_C$ is reflexive and transitive. $(S, \leq_C)$ is inductively ordered, if every totally ordered subset of $(S, \leq_C)$ has a lower bound. A totally ordered subset of $(S, \leq_C)$ is also called a chain.

Definition 3.2.3 A set $Y \subset \mathbb{R}^p$ is called C-semicompact if every open cover of Y of the form$\{\left(y^i - clC\right)^c : y^i \in Y, i \in I\}$ has a finite subcover. For some indexed set I

This means that whenever $Y \subset \bigcup_{i \in I}\left(y^i - clC\right)^c$ there exist $m \in N$ and $\{i_1, \dots, i_m\} \subset I$ such that $Y \subset \bigcup_{k=1}^{m}\left(y^{i_k} - clC\right)^c$ here $\left(y^i - clC\right)^c$ denotes the complement $C \setminus (y^i - clC)$ of $(y^i - clC)$.

Note that these sets are always open

Theorem 3.2.2 If C is a n a cute c onvex c one and Y is none mpty C-semicompact set in $\mathbb{R}^p$, then $\mathrm{E(Y,C)} \neq \emptyset$.

Proof: As $C \subset clC \Rightarrow \mathrm{E(Y,clC)} \subset \mathrm{E(Y,C)}$ (by Proposition 3.1.2**)**

It is enough to show the case in which C is a pointed closed convex cone. In this case, C define a partial order $\leq_c$ on Y as $y^1 \leq_c y^2$ iff $y -^2 y^1 \in C$.

An element in $\mathrm{E(Y,C)}$ is a minimal element with respect to $\leq_c$. therefore we can show that Y is inductively ordered and applying Zorn's lemma to establish the existence of minimal element.

Now, suppose the contrary that Y is not inductively ordered, then there exist a totally ordered set $\bar{Y} = \{y^\Upsilon : \Upsilon \in \tau\}$ in Y which has no lower bound in Y. Thus

$\bigcap_{\Upsilon \in \tau}[y^\Upsilon - C \cap Y] = \emptyset$.

Otherwise any elements of this intersection is a lower bound of $\bar{Y}$in Y. Now it follows that for any $y \in Y$ there exist $y^\Upsilon \in \bar{Y}$ such that $y^\Upsilon \notin \bar{Y} - C$. since $y^\Upsilon - C$ is closed, the family of $\{(y^\Upsilon - C)^c : \Upsilon \in \tau\}$ forms an open cover of Y. Moreover, $y^\Upsilon - C \subset y^{\Upsilon'} - C$ iff $y^\Upsilon \leq_c y^{\Upsilon'}$, and so they are totally ordered by inclusion.

Since Y is C -semicompact, t he co ver ad mits f inite su bcover, an d h ence t here ex ist a si ngle $y^\Upsilon \in \bar{Y}$ such that $Y \subset (y^{\bar{\Upsilon}} - C)^c$.

However, this contradicts the fact that $y^\Upsilon \in Y$.

Therefore Y is inductively ordered by $\leq_c$ and $\mathrm{E(Y,C)} \neq \emptyset$ by Zorn's lemma.

Definition 3.2.4 (cone compactness)

[2] Let C be a cone in $\mathbb{R}^p$. A set $Y \subset \mathbb{R}^p$ is said to be C-compact if, for any $y \in Y$, the set $(y - clC) \cap Y$ is compact.

Proposition 3.2.1 If Y is C-compact, then Y is C- semicompact.

Proof Let a family of sets $\{(y^\Upsilon - clC)^c : y^\Upsilon \in Y, \Upsilon \in \tau\}$ be an open cover of Y.

For an arbitrary $y^{\bar{\Upsilon}} \in Y$, the subfamily

$\{(y^\Upsilon - clC)^c : y^\Upsilon \in Y, \Upsilon \in \tau, \Upsilon \neq \bar{\Upsilon}\}$ forms an open cover of $(y^{\bar{\Upsilon}} - clC) \cap Y$.

Since f rom th e d efinition o f C -compactness, $(y^{\bar{\Upsilon}} - clC) \cap Y$ is c ompact, t his s ubfamily ha s a finite subcover, which together with $(y^{\bar{\Upsilon}} - clC)^c$ constitutes a finite subcover of Y.

Remark 3.2.2 A compact se t i s C -compact and C -semicompact. H owever, a C -compact set i s not necessarily compact.

For example let $C = \mathbb{R}^2{}_+$ and $Y = \{y_1 + y_2 \geq 0\}$

Here Y is not bounded, which implies not compact, but Y is C-compact.

Theorem 3.2.3 Let C is an acute convex cone in $\mathbb{R}^p$. If $Y \subset \mathbb{R}^p$, is nonempty and C-compact, then $\mathrm{E(Y, C)} \neq \emptyset$.

Proof This result is immediate from theorem 3.2.2

Definition 3.2.5 (cone semicontinuous) A function $f : \mathbb{R}^n \rightarrow \mathbb{R}^p$ is C-semicontinuous iff $f^{-1}(y - \mathrm{clC}) = \{x \in \mathbb{R}^n : y - f(x) \in \mathrm{cl}C\}$ is closed for all $y \in \mathbb{R}^p$ i.e. the preimage of the translated negative orthant is always closed.

Lemma 3.2.1 A function $f : \mathbb{R}^n \rightarrow \mathbb{R}^p$ is C-semicontinuous if and only if the component functions $f_k : \mathbb{R}^n \rightarrow \mathbb{R}$ are lower semicontinuous for all $k = 1, \ldots, p$.

Proposition 3.2.2 Let $X \subset \mathbb{R}^n$ be nonempty and compact, $f : \mathbb{R}^n \rightarrow \mathbb{R}^p$ be C -semicontinuous. Then $Y = f(X)$ is C -semicompact.

Proof Let $\{(y^i - clC)^c : y^i \in Y, i \in I\}$ be an open cover of Y. By C -semicontinuity of f, $\{f^{-1}((y^i - clC)^c) : y^i \in Y, i \in I\}$ is an open cover of X.

Because X is compact there is a finite subcover in this open cover. The image of this subcover is a finite subcover of Y whence Y is C- semicompact.

Theorem 3.2.4 Let $X \subset \mathbb{R}^n$ be a nonempty and compact set. Let f be C -semicontinuous. Then there exist efficient solution.

Proof The result follows directly from Theorem **3.2.3** and Proposition **3.2.2**

3.2.2 External stability of efficient solution

In this subsection we introduce a new concept, external stability of the efficient set. We have defined the efficient set, which is the set of all nondominated points in objective space. Each point outside the efficient set is therefore, dominated by some other points in the feasible set. However, is it also dominated by a point in the efficient set? If this is the case, the efficient set is said to be externally stable.

Definition 3.2.6 Let Y be a set of feasible points in $\mathbb{R}^p$, S is a subset of Y, and C be a domination structure on Y. S is said to be externally stable if, for each $y \in Y \setminus S$ there exist some $y' \in S$ such that $y \in y' + C(y')$.

Remark Since each C(y) is assumed to contain the zero vectors the external stability condition can be rewritten as follows: for each y$\in Y$, there exist $y' \in S$ such that $y \in y' + C(y')$. hence, if $C(y) = C$ (constant) for all y, this can be rewritten as $Y \in S + C$.

Definition 3.2.7 A set S$\subset Y$ is said to be internally stable if $y \in y' + C(y') \setminus \{0\}$.

(i.e. $y \not< y'$ whenever $y, y' \in S$.

Which implies that $E(Y, C)$ is internally stable.

Definition 3.2.8 Let Y be set in $\mathbb{R}^p$ and C be domination structure on Y. A subset of Y is called a kernel of Y with respect to C, denoted as $K(Y, C)$ if it is both externally and internally stable.

Proposition3.2.3 If C is transitive, asymmetric, and a kernel exists, then it is unique.

Proof Assume to the contrary that there are two different kernels $K(Y, C)$ and $K'(Y, C)$, then we have $y \in K(Y, C)$ but $y \notin K'(Y, C)$ for some $y' \in Y$. Form $y \notin K'(Y, C)$ there exist $y' \in K'(Y, C)$ such t hat $y \in y' + C(y') \setminus \{0\}$. S ince $K(Y, C)$ is in ternally stable a nd $y \in K(Y, C)$, then $y' \notin K(Y, C)$ and there exist $y'' \in K(Y, C)$ such that $y' \in y'' + C(y'') \setminus \{0\}$. Since C is transitive and asymmetric, $y \in y'' + C(y'') \setminus \{0\}$, which contradict the internal stability of $K(Y, C)$.

Proposition 3.2.4 Suppose that C is transitive, asymmetric. If $K(Y, C)$ exist, then $K(Y, C) = E(Y, C)$.

Proof Let $y' \in K(Y, C) \setminus E(Y, C)$, then there exist $y \in Y$ such that $y' \in y + C(y) \setminus \{0\}$.

From the external stability of $K(Y, C)$, there exist $y^* \in K(Y, C)$ such that $y \in y^* + C(y^*) \setminus \{0\}$, since C is transitive and asymmetric $y' \in y^* + C(y^*) \setminus \{0\}$ which contradict the internal stability of $K(Y, C)$, there exist

$y \in K(Y, C) \subset Y$ such that $y^* \in y + C(y) \setminus \{0\}$, however this is a contradiction.

Therefore, $K(Y, C) = E(Y, C)$.

Proposition3.2.5 Let Y be nonempty set and C be transitive, asymmetric domination structure on Y. then $K(Y, C)$ exist if $E(Y, C)$ is externally stable.

Proof $K(Y, C) = E(Y, C)$ if $K(Y, C)$ exist. Hence $E(Y, C)$ is externally stable.

Conversely if $E(Y, C)$ is externally stable, then it becomes kernel because it is internally stable.

Theorem 3.2.5 Let Y be a none mpty c ompact s et. S uppose t hat a dom ination s tructure C i s transitive a nd uppe r semicontinuous (as a poi nt-to-set m ap) on Y. m oreover, for e ach compact subset $Y' of\ Y$, $E(Y', C)$ is assumed to be nonempty. Then $E(Y, C)$ is internally stable and the so is the kernel of Y.

Proof Let y be an arbitrary point in Y, and define a set

$$Y^{'} = \{y^{'} \in Y: y \in y^{'} + C(y^{'})\}.$$

This implies, the set Y' consists of y and all points in Y that dominate y. We must show that $Y^{'} \cap E(Y,C) \neq \emptyset$. it is suffices to show that

i. $E(Y',C) \neq \emptyset$.

ii. $E(Y^{'},C) \subset E(Y,C)$.

We prove the compactness of Y' since it implies that $E(Y',C) \neq \emptyset$.

Since from the assumption $Y' \subset Y$ and Y is bounded then Y' is bounded. To show the closeness let $\{y^k\} \subset Y'$ and $y^k \to y$. then $y \in y^k + C(y^k)$,

i.e. $y - y^k \in C(y^k)$ and $y - y^k \to y - y'$.

Since C is upper semiconscious,$y - y' \in C(y^{'})$, hence $y' \in Y'$.

A vector $y^* \in R^p$ is supposed not to be contained in $E(Y,C)$. We suppose $y^* \in Y'$ since otherwise it is clear that $y^* \notin E(Y',C)$. Then $y^* \in Y$, and there exist $y'' \in Y$ such that $y^* \in y^{''} + C(y^{'}{}') \setminus \{0\}$. Since C is transitive and $y^* \in Y'$,$y \in y^{''} + C(y^{'}{}')$, which implies that $y'' \in Y'$. Hence, $y^* \notin E(Y',C)$.

Theorem 3.2.6 Let C (y) = C be *a pointed closed convex cone* and Y be a nonempty C-compact set. Then $E(Y,C)$ is externally stable;

That is $Y \subset E(Y,C) + C$.

Proof Let $y \in Y$. Define $\mathrm{Y}^{'} := (\mathrm{y} - \mathrm{C}) \cap \mathrm{Y}$, i.e. all points in Y dominating y. We need to show that $\mathrm{Y}' \cap \mathrm{E}(\mathrm{Y},\mathrm{C}) \neq \emptyset$.

To do so it is enough to show that $\mathrm{E}(\mathrm{Y}',\mathrm{C}) \neq \emptyset$ and that $\mathrm{E}(\mathrm{Y}',\mathrm{C}) \subset \mathrm{E}(\mathrm{Y},\mathrm{C})$.

$\mathrm{Y}^{'}$ is C-compact since Y is. Therefore, $\mathrm{E}(\mathrm{Y}',\mathrm{C}) \neq \emptyset$

Assume that $\mathrm{y}'$is not in $\mathrm{E}(\mathrm{Y},\mathrm{C})$, but $\mathrm{y}' \in \mathrm{Y}'$ (otherwise y' is certainly not contained in $\mathrm{E}(\mathrm{Y}',\mathrm{C})$).

Thus $\mathrm{y}' \in \mathrm{Y}$ and there is some $\mathrm{y}'' \in \mathrm{Y}$ such that $\mathrm{y}'' \leq \mathrm{y}'$.

Therefore $\mathrm{y}'' \leq \mathrm{y}' \leq \mathrm{y}$ and $\mathrm{y}'' \notin \mathrm{Y}'$.

This implies $\mathrm{y}' \in \mathrm{E}(\mathrm{Y}',\mathrm{C})$

CHAPTER FOUR

4. DUALITY IN MULTIOBJECTIVE PROGRAMMING

4.1 Lagrangian Duality in Nonlinear Multiobjective Optimization

In this section, we consider a Lagrangian duality theory for efficient solution in multiobjective optimization. In the most general case, a complete duality theory can be established for multiobjective optimization problems based on an arbitrary cone ordering. For our present purpose, however, the domination structure is supposed to be a pointed closed convex cone. Therefore, for convenience in this chapter we define:

C-Minimizing to be finding efficient solution w.r.t. cone $\leq_C$. In particular, we define minimizing for the cases with the cone ordering $\mathbb{R}^p_+$. Similarly we use the notation Min $_C$Y for representing $E(Y, C)$.

In particular, we define Min Y as $E\left(Y, \leq_{\mathbb{R}^p_+}\right)$. C-maximization and Max_C are used in similar fashion for the ordering cone $\leq_{-C}$.

Consider a nonlinear multiobjective problem formulated as follows:

(P) C-minimize $\{f(x): x \in X\}$

Where $X = \{x \in X': g(x) \leq_Q 0, X' \subset \mathbb{R}^n\}$

Throughout this section we impose the following assumption:

i. X' is a nonempty compact set.

ii. C and Q are pointed convex cones with nonempty interiors of $\mathbb{R}^p$ and $\mathbb{R}^m$ respectively.3

iii. f is continuous and C-convex.

iv. g is continuous and Q-convex.

Under these assumptions it can be readily shown that for every $u \in \mathbb{R}^m$ both sets

$X(u) = \{x \in X': g(x) \leq_Q u\}$ and

$Y(u) = f[X(u)] = \{y \in \mathbb{R}^p: y = f(x), x \in X', g(x) \leq_Q u\}$ (4*)

are compact, $X(u)$ is convex $Y(u)$ is C-convex.

4.1.1. Perturbation (or Primal) Map.

Let us consider the primal problem (P) by embedding it in a family of perturbed problems with $Y(u)$ given in (4*):

(P_u) C-Min $Y(u)$

Clearly primal problem (P) is identical to problem (P_u) with $u = 0$.

Now define the set Γ as Γ = $\{u \in \mathbb{R}^m : X(u) \neq \emptyset\}$.

It is easy to show that Γ is convex.

The point – to –set map W: $\Gamma \to \mathbb{R}^p$ defined by

$$W\ (u) = Min_C\, Y(u) = \{y^* \in Y(u) : \nexists y \in Y(Y), y^* \in y + C\backslash\{0\}\}$$

is called perturbation (or primal) map.

Observe that the perturbation map corresponds to the perturbation (or primal) function $W\ (u) = min\{f(x) : x \in X^{'}, g(x) \leq u\}$ in ordinary mathematical programming.

Obviously the original problem (P) can be regarded as determining the set W(0) and $f^{-1}[W(0)] \cap X$. In the following we shall investigate the properties of W.

Proposition 4.1.1: For any u ∈ Γ, $W(u) + C = Y(u) + C$

Proof Note that $W(u) = Min_C Y(u) \subset Y(u)$.

This implies that $W(u) + C \subset Y(u) + C$.

On the other hand since $Y(u)$ is C-compact, then by theorem3.2.6 it is externally stable.

$Y(u) \subset W(u)$ + C.

From which $Y(u) + C \subset W(u) + C + C = W(u) + C$, because *C is a convex cone*

Hence, $W(u) + C = Y(u) + C$.

Proposition 4.1.2: The map W is C-monotone on Γ, namely $W(u^1) \subset W(u^2) + C$ for some $u^1, u^2 \in \Gamma$ such that $u^1 \leq_Q u^2$.

Proof $Y(u^1) \subset Y(u^2)$ whenever $u^1 \leq_Q u^2$. Hence,

$W(u^1) \subset Y(u^1) \subset Y(u^2) \subset W(u^2) + C$.

Proposition 4.1.3 For each $u \in \Gamma$, W (u) is a C-convex point-to-set map on Γ

Proof It suffices to show that

$\mu Y(u^1) + (1-\mu)Y(u^2) \subset Y(\mu u^1 + (1-\mu)u^2)$ + C for some

$u^1, u^2 \in \Gamma$ and $\mu \in [0,1]$ if w e s uppose t hat y ∈ $\mu Y(u^1) + (1-\mu)Y(u^2)$, then t here ex ist $x^1, x^2 \in X'$ such that

$g(x^1) \leq_Q u^1, g(x^2) \leq_Q u^2$, and y= $\mu f(x^1) + (1-\mu)f(x^2)$.

Since $X^{'}$ is a convex set $\mu u^2 + (1-\mu)x^2 \in X^{'}$.

Furthermore, from the Q-convexity of g,

$g(\mu x^1 + (1-\mu)x^2) \leq_Q \mu g(x^1) + (1-\mu)g(x^2) \leq_Q \mu u^2 + (1-\mu)u^2$

which implies that

$\mu x^1 + (1-\mu)x^2 \in X(\mu(u^1) + (1-\mu)u^2)$ and thus,

$$f(\mu(x^1) + (1-\mu)x^2) \in \ Y(\mu u^1 + (1-\mu)u^2).$$

On the other hand, from the C-convexity of f we get

$\mu f(x^1) + (1-\mu)f(x^2) \in f(\mu x^1 + (1-\mu)x^2) + \mathrm{C}$

Finally we have

$\mathrm{y} \in Y(\ \mu u^2 + (1-\mu)u^2) + C$. This completes the proof of the proposition.

Remark 4.1.1: The C-cconvexity of a point-to-set map W ensures that if an appropriate linear vector valued functional Λu is added to $W(u)$ there will exist no point of $W(u) + \Lambda u$ that dominates a given point of $W(0)$. This vector valued functional leads to a supporting canonical variety (i.e. a translation of cone) of $W(0)$ at the given point.

Theorem 4.1.1 (Lagrange Multiplier Theorem)

[6] If x' is properly efficient solution to (P) and if Slater's constraint qualification condition holds (i.e. there exist $x' \in X$ such that $g(x^{'}) <_Q 0$), then there exist a $p\mathrm{x}m$ matrix Λ such that $\Lambda\mathrm{Q} \subset \mathrm{C}$ and $f(x^{'}) \in Min_C\{f(x) + \Lambda \mathrm{g}(x) \colon x \in \mathrm{X}^{'}\}$ and $\Lambda \mathrm{g}(x^{'}) = 0$

Proof

Let $X = \{x \in \mathbb{R}^n \colon g(x) \leq_Q 0\} \cap X^{'}$. Since x' is properly efficient solution of $f(X)$ with respect to $\leq_C$, there exist a vector $\mu' \in intC^0$ such that

$\langle \mu^{'}, f(x^{'}) \rangle \leq \langle \mu^{'}, f(x) \rangle$ for any $x \ \in X$.

Note that $\langle \mu^{'}, f(x) \rangle$ is a convex function on $X^{'}$. In fact, due to the C-convexity of f, since

$\alpha f(x^1) + (1-\alpha)f(x^1) - f(\alpha x^1 + (1-\alpha)x^2) \in C$ for some $x^1, x^2 \in X'$ and $\alpha \in [0,1]$.

We have:

$$\alpha\langle \mu^{'}, f(x^1) \rangle + (1-\alpha)\langle \mu^{'}, f(x^2) \rangle - \langle \mu^{'}, f(\alpha x^1 + (1-\alpha)x^2) \rangle \geq 0 .$$

Therefore, the well known Lagrange multiplier theorem in scalar convex optimization leads to the existence of a vector $\lambda' \ \in Q^0$ such that:

$\langle \mu^{'}, f(x^{'}) \rangle + \langle \lambda^{'}, g(x^{'}) \rangle \leq \langle \mu^{'}, f(x) \rangle + \langle \lambda^{'}, g(x) \rangle$ for any $x \in X'$ and (*)

$$\langle \lambda^{'}, g(x) \rangle = 0$$

Now for such $\mu^{'} and\ \lambda^{'}$ take Λ' with $\Lambda'^T \mu^{'} = \lambda'$ in such a way that

Λ' =$(\lambda_1^{'} e, \lambda_2^{'} e, \dots, \lambda_m^{'} e)$,

Where e is a vector of C with $\langle \mu^{'}, e \rangle$=1.

Then clearly $\Lambda' Q \subset C$ and $\Lambda' g(x') = 0$.

If we suppose that for this Λ'

$$f(x') \notin Min_C\{f(x) + \Lambda' g(x) : x \in X'\}.$$

There exist $\bar{x} \in X'$ such that

$$f(x') - f(\bar{x}) - \Lambda' g(\bar{x}) \in C \setminus \{0\}.$$

Hence,

$$\langle \mu', f(x') \rangle > \langle \mu', f(\bar{x}) \rangle + \langle \mu', \Lambda' g(\bar{x}) \rangle$$

$$= \langle \mu', f(\bar{x}) \rangle + \langle \lambda', g(\bar{x}) \rangle,$$

which contradict the relation in (*).

Therefore, $f(x') \in Min_C\{f(x) + \Lambda g(x) : x \in X'\}$

Note that in the above case where $(1,1,\ldots,1) \in C$, by normalizing μ' such that a particular way that $\sum_{i=1}^{p} \mu' = 1$, we can take $e = (1,1,\ldots,1)^T$ in the proof of the theorem.

We then have $\wedge g(x) = (\langle \lambda, g(x) \rangle, \langle \lambda, g(x) \rangle, \ldots, \langle \lambda, g(x) \rangle)^T$.

4.1.2 Vector Valued Lagrangian Function and Its Saddle Point

Let $\mathcal{L} \subset R^{pxm}$ denote the family of all positive matrices: $\mathcal{L} = \{\Lambda \in \mathbb{R}^{pxm} \mid \Lambda Q \subseteq C\}$ such matrices are said to be positive in some literature.

Note: for given $\mu \in C^0 \setminus \{0\}$ and $\lambda \in Q^0$ there exist $\Lambda \in \mathcal{L}$ such that

$$\Lambda^T \mu = \lambda$$

For some $e \in C$ with $\langle \mu, e \rangle = 1$, $\Lambda = (\lambda_1 e, \lambda_2 e, \ldots, \lambda_m e)$ is the desired pxm matrix.

Definition 4.1.1 A vector valued Lagrangian function for the problem (P) is defined as:

$L : X' x \mathcal{L} \to \mathbb{R}^p$ and given by

$L(x, \Lambda) = f(x) + \Lambda g(x)$.

where Λ is pxm matrix, the dual variable is an m-dimensional vector λ as in the ordinary duality.

Definition 4.1.2 A pair $(x', \wedge') \in X' x \mathcal{L}$ is said to be a saddle point for the vector valued Lagrangian function $L(x, \wedge)$ if

$L(x', \wedge') \in min_C\{L(x, \wedge') : x \in X'\} \cap max_C \{L(x', \wedge) : \wedge \in \mathcal{L}\}$.

Theorem 4.1.2 The following conditions are necessary and sufficient for a pair $(x^{'},\wedge^{'}) \in X'\text{x}\mathcal{L}$ to be a saddle point for a vector valued Lagrangian function $\text{L}(x,\Lambda)$

i. $\text{L}(x^{'},\wedge^{'}) \in min_C\{\text{L}(x,\ \wedge^{'}): x \in X^{'}\}$

ii. $g(x^{'}) \leq_Q 0$

iii. $\wedge^{'}\ g(x^{'}) = 0$

Proof

(⟹)Suppose that $(x^{'},\wedge^{'})$ is saddle point of $\text{L}(x,\Lambda)$.

Then

i. is the same as part of the condition in the definition of saddle point.

ii. $\text{L}(x^{'},\wedge^{'}) \in max_C\{f(x^{'}) + \wedge\ g(x^{'}), \wedge \in \mathcal{L}\}$

This implies $f(x^{'}) + \wedge{'}g(x^{'}) \nleq_C f(x^{'}) + \wedge\ g(x^{'})$ for any $\wedge \in \mathcal{L}$ (4.1.3)

From which we have $\langle \mu^{'}, \wedge\ g(x^{'}) - \wedge^{'}\ g(x^{'})\rangle \leq 0$ for some (4.1.4)

$\mu' \in C^0 \setminus \{0\}$ and for any $\wedge \in \mathcal{L}$

Suppose that $g(x^{'}) \nleq_Q 0$, then there exist $\lambda' \in Q^0$ such that $\langle \lambda^{'}, g(x^{'})\rangle > 0$

Making $||\lambda'||$ sufficiently large and taking $\wedge \in \mathcal{L}$ such that $\mu^T\ \wedge = \lambda'^T$, we obtain the relation

$\langle \mu^{'}, \wedge\ g(x^{'})\rangle - \langle \mu^{'}, \wedge^{'}\ g(x^{'})\rangle > 0$ this contradicts (4.1.4). Thus $g(x^{'}) \leq_Q 0$

iii. Using the result $\wedge{'}g(x^{'}) \nleq_Q 0$ for $\wedge{'} \in \mathcal{L}$.

On the other hand substituting $\wedge = 0$ in to eq (4.1.3) yields $\wedge{'}g(x^{'}) \nleq_Q 0$

Finally we have $\wedge^{'}\ g(x) = 0$

(⟸) Since $\wedge\ g(x^{'}) \in -C$ for any $\wedge \in \mathcal{L}$ as long as $g(x^{'}) \leq_Q 0$, it follows that $Max_C\{\wedge\ g(x^{'}): \wedge \in \mathcal{L}\} = \{0\}$. Thus from $\wedge^{'}\ g(x') = 0$, we have $\text{L}(x^{'},\wedge^{'}) \in max_C\{f(x^{'}) + \wedge\ g(x^{'}), \wedge \in \mathcal{L}\}$ this result and condition (i) implies that pair $(x^{'},\wedge^{'})$ is a saddle point.

Corollary 4.1.1 Suppose that x' is a properly efficient solution to the problem (P) and let Slater's constraint qualification is satisfied. Then there exist a pxm matrix $\wedge{'} \in \mathcal{L}$ such that $(x^{'},\wedge^{'})$ is a saddle point for the vector valued Lagrangian function $\text{L}(x,\wedge)$.

Proof: immediate from Theorem 4.1.1 and Theorem 4.1.2

Thus, we have verified that properly efficient solutions to the problem (P) together with a matrix give a saddle point for the vector valued Lagrangian function under convexity assumptions an appropriate regularity conditions. Conversely, the saddle point provides a sufficient condition for optimality of problem (P).

Theorem 4.1.3 If $(x',\wedge') \in X'x\mathcal{L}$ is a saddle point for one vector valued Lagrangian function $L(x,\wedge)$ then x' is an efficient solution to the problem (P).

Proof

Suppose that x' is not a solution to the problem (P), this implies there exists $x^* \in X'$ such that $f(x^*) \leq_C f(x')$. Since $g(x') \leq_Q 0$ and $\wedge' \in \mathcal{L}$ yield $\wedge' g(x') \in -C$, we finally have $f(x^*)+ \wedge' g(x^*) \leq_C f(x')$, which contradicts $L(x',\wedge') \in min_C\{L(x, \wedge'): x \in X'\}$.

Thus, x' is an efficient solution to the problem (P).

4.1.3 Dual map and Duality theory

Recall that the dual function for ordinary optimization is defined by

$$\Phi(\lambda) = \inf\{L(x,\lambda): x \in X'\} = \inf\{f(x) + \langle\lambda, g(x)\rangle: x \in X'\},$$

Definition 4.1.2 (dual map)

For any,$\Lambda \in \mathcal{L}$ the dual set-valued map Φ is defined by

$$\Omega(\Lambda) = \{L(x,\Lambda): x \in X'\} = \{f(x) + \Lambda g(x): x \in X'\},$$

and

$$\Phi(\Lambda) = Min_C\{\Omega(\Lambda)\}$$

The point-to-set map $\Phi: \mathcal{L} \to \mathbb{R}^p$ is called the dual map.

The dual problem is formulated as:

(D) $\quad C$-Maximize $\bigcup_{\Lambda \in \mathcal{L}} \Phi(\Lambda)$

Though Φ is not a function, but it is a set-valued map.

[6] **Proposition4.1.5** For $\wedge \in \mathcal{L}$, $\Phi(\Lambda)$ is a C-convex set in $\mathbb{R}^p$.

Proof Since the map f and g are C-convex and Q-convex respectively, then map $L(.,\Lambda)$ is C-convex over X' for each fixed $\wedge \in \mathcal{L}$.

Hence, $\Omega(\Lambda)$ is a compact set in $\mathbb{R}^p$ for X' a compact convex set.

Therefore, $\Phi(\Lambda) + C = \Omega(\Lambda) + C$.

Thus, $\Phi(\Lambda)$ is also C-convex.

Proposition 4.1.6 The dual function Φ is a C-concave point-to-set map on Γ.

Namely, for $t \in [0,1]$, $\Lambda^1, \Lambda^2 \in \mathcal{L}$, we have

$\Phi(t\Lambda^1 + (1-t)\Lambda^2) \subset t\,\Phi(\Lambda^1) + (1-t)\Phi(\Lambda^2) + C$

Proof:

Since X' *is compact*,

$Min_C\{t\,(f(x) + \Lambda^1 g(x)) + (1-t)(f(x) + \Lambda^2 g(x)) : x \in X'\}$ is externally stable.

For $t \in [0,1]$, $\Lambda^1, \Lambda^2 \in \mathcal{L}$, we have

$$\Phi(t\Lambda^1 + (1-t)\Lambda^2) = min_C\{f(x) + (t\Lambda^1 + (1-t)\Lambda^2)g(x)\} : x \in X')$$

$$= min_C\{t\,(f(x) + \Lambda^1 g(x)) + (1-t)(f(x) + \Lambda^2 g(x)) : x \in X'\}$$

$$\subset min_C\{t\{f(x) + \Lambda^1 g(x)\} + (1-t)\{f(x) + \Lambda^2 g(x))\ (x \in X') + C$$

$$\subset min_C\{t\{f(x) + \Lambda^1 g(x)\} : x \in X') + min_C\{(1-t)\{f(x) + \Lambda^2 g(x)) : x \in X'\} + C$$

$$= t\,min_C\{f(x) + \Lambda^1 g(x) : x \in X') + (1-t)min_C\{f(x)) + \Lambda^2 g(x) : x \in X') + C$$

$$= t\,\Phi(\Lambda^1) + (1-t)\Phi(\Lambda^2) + C.$$

Note that proposition 4.1.6 is an extension of the fact that the dual function $\Phi(\lambda)$ is concave .We can now establish the following relationship between the dual map $\Phi(\Lambda)$ and the primal map $W(u)$, which is an extension of the following relationship between the dual function $\Phi(\lambda)$ and the primal function $W(u)$:

$$\Phi(\lambda) = \inf\{W(u) + \langle\lambda, u\rangle : u \in \Gamma\}$$

Proposition 4.1.7 The following relation holds:

$\Phi(\Lambda) = \mathrm{Min}_C \bigcup_{u\in\Gamma}\{W(u) + \Lambda u\}$.

Proof Let $y^1 = f(x^1) + \Lambda g(x^1)$ for any $x^1 \in X'$.

Then, let $u^1 = g(x^1)$ which yields $y^1 = f(x^1) + \Lambda u^1$.

Note here that $f(x^1) \in W(u^1) + C$, because $f(x^1) \in Y(u^1)$.

Hence, $y^1 \in W(u^1) + \Lambda u^1 + C$. From which $\Omega(\Lambda) \subset \bigcup_{u\in\Gamma}\{W(u) + \Lambda u\} + C$ which is equivalent to $\Omega(\Lambda) + C \subset \bigcup_{u\in\Gamma}\{W(u) + \Lambda u\} + C$.

Conversely suppose that

$y^1 \in W(u^1) + \wedge\, u^1$ for some $u^1 \in \Gamma$, this implies that

$$y^1 - \wedge\, u^1 \in Min_C Y(u^1).$$

Thus, $y^1 - \wedge\, u^1 = f(x^1)$ for some $x^1 \in X'$ such that $g(x^1) \leq_Q u^1$

Then for $\wedge \in \mathcal{L}$, $y^1 = f(x^1) + \wedge\, u^1 \geq_C f(x^1) + \wedge\, g(x^1)$, and $y^1 \in L(x^1, \wedge) + C \subset \Omega(\wedge) + C$.

Therefore, $\bigcup_{u \in \Gamma}\{W(u) + \wedge\, u\} \subset \Omega(\wedge) + C$, and thus

$\bigcup_{u \in \Gamma}\{W(u) + \wedge\, u\} + C \subset \Omega(\wedge) + C$.

Finally, we have

$\Omega(\wedge) + C = \bigcup_{u \in \Gamma}\{W(u) + \wedge\, u\} + C$ and hence

$\mathrm{Min}_C(\Omega(\wedge) + C) = Min_C(\bigcup_{u \in \Gamma}\{W(u) + \wedge\, u\} + C)$.

Which establishes the proposition, because in general $\mathrm{Min}_C A = Min_C(A + C)$ whenever C is a pointed convex cone.

Recall that (Theorem 4.1.4) implies that, given a properly efficient solution x', then

$f(x') \in \Omega(\wedge')$ for some $\wedge' \in \mathcal{L}$. Hence we may see from (proposition 4.1.7) that

$f(x') \in Min_C(\bigcup_{u \in \Gamma}\{W(u) + \wedge' u\}$ for some $\wedge' \in \mathcal{L}$.

The following theorem represents some properties of efficient solutions to primal problem (P) in connection with the dual map Φ and might be considered as duality theorem for multiobjective optimization problem.

Theorem 4.1.4 (Weak duality) [6] (duality theorem for multiobjective optimization)

If $x \in X$ is feasible for the primal problem (P), and $\wedge \in \mathcal{L}$ is feasible for the dual problem (D), then

$$y \ngeq_C f(x) \qquad \forall y \in \Phi(\wedge).$$

Proof For any feasible $x \in X$, and feasible $\wedge \in \mathcal{L}$,

$$\wedge\, g(x) \leq_C 0.$$

For any $y \in \Phi(\wedge)$

$y \ngeq_C f(x) + \wedge\, g(x)$ for all $x \in X'$. Hence the theorem is proved.

Theorem4.1.4 (Strong Duality) [4] If $x' \in X$, $\wedge' \in \mathcal{L}$ and $f(x') \in \Phi(\wedge')$, then x' is simultaneously a minimal point to the primal problem (P) and a maximal point to the dual problem (D).

Proof: If $f(x')$ is not minimal for the primal problem (P), i.e.

$f(x') \notin Min_C \{f(x): x \in X\}$, then there exists $x \in X$ such that

$f(x') \geq_C f(x)$,

$g(x') \leq_Q 0$ and $\wedge' \in \mathcal{L}$ together imply that

$\wedge' g(x') \leq_C 0$.

This implies that $L(x,\wedge') = f(x) + \wedge' g(x) \leq_C f(x) \leq_C f(x')$. But t his c ontradicts t he premise that

$f(x') \in \Phi(\wedge') = \min_C\{L(x,\wedge'): x \in X\}$.

Hence $f(x')$ is minimal for the primal problem (P).

If $f(x')$ is not maximal for the dual problem (D), i.e.,

$f(x') \notin Max_C\{\Phi(\wedge):\wedge \in \mathcal{L}\}$, then there exists $y \in \bigcup_{\wedge\in\mathcal{L}} \Phi(\wedge)$ such that $y \geq_C f(x')$. Let $\wedge^0 \in \mathcal{L}$ be such that $y \in \Phi(\wedge^0)$). Since $\wedge^0 g(x') \leq_C 0$ this implies that

$y \ngeq_C L(x',\wedge^0) = f(x') + \wedge^0 g(x') = L(x',\wedge^0)$.

This is contradicts the fact that $y \in \Phi(\wedge^0) = \min_C\{L(x,\wedge): x \in X\}$.

Hence $f(x')$ is maximal for the dual problem (D).

Corollary 4.1.2

If $f(x^*)$ is a p roperly m inimal p oint to the p rimal p roblem (P), and the S later's c onstraint qualification is satisfied, then $f(x^*)$ is a maximal point to the dual problem (D).

Proof: By Theorem4.1.4 there exists $\wedge^* \in \mathcal{L}$, such that $f(x^*) \in \Phi(\wedge^*)$.

The conclusion thus follows from Theorem4.1.4.

4.2 Conjugate Duality in Convex Multiobjective Optimization

In this section, we present a generalization of the conjugate duality theory for scalar optimization. Although the theory is established for an unconstrained multicriteria optimization problem, it is only easy to allow for constraints implicitly by defining the underlying objective function to assume an abstract infinity whenever the explicit constraints are violated.

As conjugate duality theory for multicriteria optimization involves set-valued function, we begin this section by further defining the tools required for these set-valued functions.

The conjugate map, which is also a set -valued map, takes efficient values instead of the minimum value. Through conjugate maps, they define the conjugate dual problem and obtain the duality result. The concepts of Subgradients and subdfferential are also introduced.

Notations

$\mathbb{R}$ is the set of real numbers;

$\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$;

$\mathbb{R}^n = \mathbb{R} x\, \mathbb{R}\, x \ldots x\, \mathbb{R}$ is the Euclidean vector space of dimension n;

$\mathbb{R}^n_+ = \{ y \in \mathbb{R}^n : y_i \geq 0 , i = 1, 2, \ldots , n\}$ is the closed positive orthant of $\mathbb{R}^n$; and

$\text{int } \mathbb{R}^n_+ = \{y \in \mathbb{R}^n : y_i > 0 , i = 1, 2, \ldots , n\}$ is the interior of $\mathbb{R}^n_+$.

$\overline{\mathbb{R}}^p = \mathbb{R}^p \cup \{\infty\}$ where ∞ is the imaginary point which has ∞ for every component.

For the following section we use $C = \mathbb{R}^n_+$.

[7] Let $\bar{Y}$ denote the extended space of Y (i.e. $\bar{Y} := Y \cup \{\pm\infty\}$). Given a set $Z \subset \bar{Y}$, we define the set $A(Z)$ of $\bar{Y}$ by $A(Z) = \{y \in \bar{Y} | y' <_C y \text{ for some } y' \in Z\}$ which is the set of all points above Z and the set $B(Z) = \{y \in \bar{Y} | y' >_C y \text{ for some } y' \in Z\}$ which is the set of all points below Z.

Clearly $A(Z) \subseteq Y \cup \{+\infty\}$ and $B(Z) \subseteq Y \cup \{-\infty\}$.

Definition 4.2.1 [7] Let Z be a nonempty subset of $\bar{Y}$ such that $Z \neq \{+\infty\}$. A point $p \in \bar{Y}$ is said to be a C-infimal point of a set Z, if there is no $y \in Z$ such that $y <_C p$ and if the relation $y' <_C p$ implies the existence of some $z \in Z$ such that $y' >_C z$

The set of all C-infimal point of Z is called a C-infimum of Z and is denoted by C-inf Z

C-Sup Z is defined similarly.

As an easy consequence from the definition

- **i.** $-C$-Max(-Z)=C-MinZ and $-C$-inf(-Z)= C-SupZ.
- **ii.** C-Max$\emptyset = \emptyset$ and C-Sup$\emptyset$=$\{-\infty\}$.

Proposition4.2.1 $\bar{Y}$=(C-SupZ)∪ A(C-SupZ)∪ B(C-SupZ) and these three sets in the right-hand side are disjoint.

Definition 4.2.2 Let $F: \mathbb{R}^n \rightarrow 2^{\overline{\mathbb{R}}^p}$ be a set-valued function, and $y \in F(x)$.

i. The conjugate dual of F, is a set-valued function denoted by $F^*: \mathbb{R}^{pxn} \rightarrow 2^{\overline{\mathbb{R}}^p}$ and is defined as follows:

$$F^*(T) = \mathrm{Max}_C \cup_{x\in R^n}[Tx - F(x)],\ T\in \mathbb{R}^{pxn}$$

ii. The biconjugate of F, or the conjugate of F^*, is a set-valued function denoted by $F^{**}: \mathbb{R}^n \rightarrow 2^{\overline{\mathbb{R}}^p}$ and is defined as follows:

$$F^{**}(x) = \mathrm{Max}_C \cup_{T\in R^{pxn}} [Tx - F^*(T)],\ x \in \mathbb{R}^n$$

iii. T is said to be a subgradient of the set-valued function F at $(x_0; y)$ if

$$y - Tx_0 \in \mathrm{Min}_C \cup_{x\in R^n}[F(x) - Tx],$$

Or equivalently

$$F(x) \geq_C F(x_0) + T(x - x_0) for\ all\ x \in \mathbb{R}^n$$

iv. The set of all Subgradients of F at $(x; y)$ is denoted by $\partial F(x; y)$, the subdfferential of F at $(x; y)$. Moreover, we let $\partial F(x_0) = \cup_{y\in F(x_0)} \partial F(x_0;\ y)$

Note Unlike to the scalar case, the subdfferential of vector- valued function may not closed convex set even when f is a finite C-convex function.

v. F is said to be subdifferentiable at x if $\partial F(x; y) \neq \emptyset, \forall y \in F(x)$.

Let $f: \mathbb{R}^n \rightarrow 2^{\overline{\mathbb{R}}^p}$ be an extended vector-valued function. A primal multiobjective optimization problem is defined as follows:

(P) $\mathrm{Min}_C\{f(x) \in \mathbb{R}^p:\ x \in \mathbb{R}^n\}$.

In other words (P) is the problem to find $x^* \in \mathbb{R}^n$ such that

$$f(x^*) \in \mathrm{Min}_C\{f(x) \in \mathbb{R}^p: x \in \mathbb{R}^n\}.$$

To construct a conjugate duality theory for the above problem, we embed f into a family of perturbed functions.

Let $\psi: \mathbb{R}^n x \mathbb{R}^m \rightarrow \overline{\mathbb{R}}^p$ be another vector-valued function such that

$$f(x) = \psi(x, 0), \forall x \in \mathbb{R}^n.$$

The perturbation function is a set-valued function $W: \mathbb{R}^{\boldsymbol{m}} \rightarrow \overline{\mathbb{R}}^p$ defined as:

(Perturbation Function) $W(u) = \mathrm{Min}_C \{\psi(x, u): x \in \mathbb{R}^n\}$. Clearly

$W(0) = \text{Min}_C\, f(\mathbb{R}^n)$, the minimal frontier for the problem (P).

The problem (P) now is stated as the primal of a pair of dual optimization problems.

(Primal Problem P) Minimize$_x$ $\{\psi(x, 0): x \in \mathbb{R}^n\}$

The conjugate dual of ψ, denoted as $\psi^*: \mathbb{R}^n x \mathbb{R}^m \to \overline{\mathbb{R}}^p$ is a set-valued function defined in the usual manner:

$\psi^*(T,\wedge) = \text{Max}_C \cup_{u \in R^m, x \in R^n} [Tx + \wedge u - \psi(x, u) \in \mathbb{R}^p]$,

The problem (P) may now be stated as the primal of a pair of dual optimization problems.

(Dual problem D) $\text{Max}_C \cup_{\wedge \in R^{pxn}} (-\psi^*(0,\wedge))$,

(D) is not an ordinary multiobjective optimization problem. To avoid any possible confusion, a more accurate statement of the dual problem is given as follows:

(D) Find: $\wedge^* \in \mathbb{R}^{pxm}$ such that [4]

$$-\psi^*(0,\wedge^*) \cap \text{Max}_C \cup_\wedge (-\psi^*(0,\wedge)) \neq \emptyset$$

Proposition 4.2.2 (**Weak Duality**) for any $x \in \mathbb{R}^n$ and $\wedge \in \mathbb{R}^{pxm}$.

$$\psi(0,\wedge) \notin -\psi^*(0,\wedge) - C\backslash\{0\}$$

Proof: Let $y = \psi(0,\wedge)$ and $y' \in \psi^*(0,\wedge)$. By definition of $\psi^*(0,\wedge)$,

$y' \nleq_C \wedge u - \psi(x, u), \forall x, u$

In particular when u =0,

$y' \nleq_C \wedge 0 - \psi(x, 0)$,

$y' \nleq_C -\psi(x, 0) = -y$ Or $y + y' \nleq_C 0$.

Corollary 4.2.2 $\forall y \in \text{Min}_C\, \psi(x, 0)$ and $\forall y' \in \text{Max}_C \cup_\wedge \{-\psi^*(0,\wedge)\}$

$$y \nleq_C y'$$

Proof immediate from the proposition

Definition 4.2.3 (i) The (set-valued) perturbation function

$W(u) = \text{Min}_C \{\psi(x, u) : x \in \mathbb{R}^n\}$, is said to be externally stable if

$\{\psi(x, u) \in \mathbb{R}^p : x \in \mathbb{R}^n\} \subset W(u) + C$, for each $u \in \mathbb{R}^m$

(ii) Let $h: \mathbb{R}^n \to 2^{\overline{\mathbb{R}}^p}$ be a set-valued function. We say that the set-valued function

$\text{Min}_C\, h(x)$ is externally stable if

$h(x) \subset \text{Min}_C h(x) + C$

(iii) Given a set $B \subset R^P$, Min_C (B) is said to be externally stable if

$$B \subset \text{Min}_C (B) + C$$

Lemma4.2.1 If ψ is convex on $\mathbb{R}^n x \mathbb{R}^m$ and the perturbation function $W(u)$ is externally stable, then the perturbation function is convex.

Lemma 4.2.2 Let $F: \mathbb{R}^n \to 2^{\overline{\mathbb{R}}^p}$ be a set-valued function, and $y \in F(x)$. Then $\partial F(x;\ y) \neq \emptyset$ if and only if $y \in F^{**}(x)$.

In other words, F is subdifferentiable at x if and only if $F(x) \subset F^{**}(x)$.

Proof: By definition of F^*, $\partial F(x;\ y) \neq \emptyset$ if and only if

$\exists \wedge \in R^{pxn}$ such that $\wedge x - y \in F^*(\wedge)$.

Hence if $y \in F^{**}(x) = \max_C\{\cup_\wedge[\wedge x - F^*(\wedge)]\}$, then

$y \in F^{**}(x) = \wedge x - F^*(\wedge)$ for some $\wedge \in \mathbb{R}^{pxn}$

$\wedge x - y \in F^*(\wedge)$ for some $\wedge \in \mathbb{R}^{pxn}$

Thus, we have $\partial F(x;\ y) \neq \emptyset$.

Conversely, if $\partial F(x;\ y) \neq \emptyset$, then $y \in \wedge x - F^*(\wedge)$ for some$\wedge$.

Pick any $y' \in \wedge x - F^*(\wedge)$ or $\wedge x - y' \in F^*(\wedge)$.

From Definition 4.2.1 (i) and the fact that $y \in F(x)$,

we have $y + \wedge x - y' \nleq_C \wedge x$, or $y \nleq_C y'$.

Since $y' \in \wedge x - F^*(\wedge)$ with any $\wedge$.

Therefore, $y \in \max_C \cup_\wedge[\wedge x - F^*(\wedge)] = F^{**}(x)$.

Definition 4.2.4 The primal problem (P) is said to be stable if the perturbation function W is subdifferentiable at $u = 0$.

Theorem 4.2.2 (Strong Duality) [4], [6]

(i) The primal problem (P) is stable if and only if for each solution x^* of the primal problem (P), there exists a solution $\wedge^*$ for the dual problem (D) such that

$$\psi(x^*, 0) \in -\psi^*(0,\wedge^*) \qquad (**)$$

(ii) Conversely, if x^* and $\wedge^*$ satisfy $(**)$ then x^* is a solution of (P) and $\wedge^*$ is a solution of (D).

Proof: (i) The primal problem (P) is stable if and only if

$$\partial W(0; z) \neq \emptyset \quad \forall z \in W(0) = \mathrm{Min}_C\{\psi(x, 0): x \in \mathbb{R}^n\},$$

if and only if

$$z \in W^{**}(0) = \mathrm{Max}_C \cup_\wedge (\wedge 0 - W^*(\wedge))$$
$$= \mathrm{Max}_C \cup_\wedge \left(-\psi^*(0,\wedge)\right),$$

if and only if,

$$\mathrm{Min}_C\{\psi(x,0): x \in \mathbb{R}^n\} \subset \mathrm{Max}_C \bigcup_{\Lambda} (-\psi^*(0,\Lambda))$$

Thus for each solution x^* of the primal problem (P) and for each solution Λ^* of the dual problem (D), we have

$$\psi(x^*, 0) \in (-\psi^*(0, \Lambda^*).$$

(ii) Follows from the definitions.

Notes that given a perturbation map $\mathrm{W}(u)$, Min (P) $= \mathrm{W}(0) \subset W^{**}(0) = Max\ (D)$.

Moreover, if W is subdifferentiable at 0, then Min (P) = Max (D).

Convexity assumptions essentially guarantee the subdifferentiability of W.

4.2.2 Optimality Conditions

[7] Let X be a real topological space and assume $Y = \mathbb{R}^p$, which is partially ordered by a pointed closed convex cone C with nonempty interior in Y.

Now, consider the following unconstrained minimization problem.

(P1) Min $f(x)$ such that $x \in X$

Definition 4.2.5 A function $f: X \to Y$ is said to be locally C-Lipschitz if for every $x \in X$ there exist a neighborhood $B(x,\delta)$ for x with radius $\delta > 0$ and $L \in C$ such that

$$-L\|x-y\| \leq_C f(x) - f(y) \leq_C L\|x-y\| \quad \forall x \in B(x,\delta)$$

Here, L is called Lipschitz Constant.

For a vector valued function $f: X \to \bar{Y}$ the directional derivative of f at a point $x_0 \in X$ in the direction $u \in X$ is given by

$$f'(x_0,u) = \lim_{t\to 0^+} \frac{f(x_0+tu) - f(x_0)}{t}$$, if the limit exist.

Note that every C-convex function is both directionally differentiable and C-Lipschitz.

Lemma4.2.3 Suppose that f is a directionally differentiable vector-valued function.

Then $T \in \partial f(x_0)$ if and only if $f'(x_0,u) \geq_C Tu$ for all $u \in X$.

Proof Let $T \in \partial f(x_0)$, then we have $tf'(x_0,u) + o(t) = f(x_0+tu) - f(x_0) \geq_C Tu$ for all $t > 0$, and for all $u \in X$. when $\frac{o(t)}{t} \to 0$ as $t \to 0$, or

$f'(x_0,u) + \frac{o(t)}{t} \geq_C Tu$ for all $t > 0$, which in turn implies that

$f'(x_0,u) \geq_C Tu$ for all $u \in X$

The proof of other the side is obvious.

Lemma4.2.4 For a C-convex function f, we have a stronger condition

i. $f^{'}(x_0, u) = \lim_{t\to 0^+} \frac{f(x_0+tu)-f(x_0)}{t} \leq_C f(x_0+u) - f(x_0)$for all $u \in X$

ii. $f^{'}(x_0, u) \in$ C-Max{$Tu: T \in \partial f(x_0)$}

Proof Let f be a C-convex function. Then for all $x_0, u \in X$ and $t \in [0,1]$

$$f(x_0+tu) - f(x_0) = f(x_0 - tx_0 + tx_0 + tu) - f(x_0)$$

$$\Rightarrow f((1-t)x_0 + t(x_0+u)) - f(x_0) \leq_C (1-t)f(x_0) + tf(x_0+u) - f(x_0)$$

$$= t[f(x_0+u) - f(x_0)]$$

Dividing by t and taking a limit as $t \to 0^+$ we get

$f^{'}(x_0, u) = \lim_{t\to 0^+} \frac{f(x_0+tu)-f(x_0)}{t} \leq_C f(x_0+u) - f(x_0)$ for all $u \in X$. Hence (i) is proved.

C-convexity of f implies that f is directionally differentiable hence using Lemma 4.2.3 we have $Tu \leq_C f^{'}(x_0, u)$ if and only if $T \in \partial f(x_0)$ for all $u \in X$. This implies that $f^{'}(x_0, u) \in$ C-Max $\{Tu: T \in \partial f(x_0)\}$ which is (ii).

A point $x_0 \in \text{dom} f$ is said to be a local efficient point for (P1) if there exist a neighborhood of x_0 such that $f(x) \geq_C f(x_0)$ $\forall x$ in the neighborhood.

Proposition4.2.3

Let f be directionally differentiable at $x_0 \in X$. If x_0 is a local efficient point for (P 1), then $f^{'}(x_0, u) \geq_C 0$ for all $u \in X$.

Proof Since f is directionally differentiable at x_0, we have

$f(x_0+tu) = f(x_0) + tf^{'}(x_0+u) + o(t), \forall t \geq 0, \forall u \in X$, where $o(t) = o(t, x_{0,}u)$ and

$$\lim_{t\to 0^+} t^{-1}\left[o(t, x_{0,}u)\right] = 0 \;\forall u \in X.$$

Then $f^{'}(x_0, u) = t^{-1}[f(x_0+tu) - f(x_0)] - \frac{o(t)}{t}$. Since x_0 is a local minimum point of f, the assertion of the proof follows.

Proposition4.2.4 Let f be a function which is directionally differentiable at $x_0 \in X$.

If f is locally C-Lipschitz in a neighborhood of x_0 and if $f^{'}(x_0, u) \geq_C 0$ $\forall u \in$ X and $u \neq 0$, then x_0 is local efficient point of f.

Proof: Suppose to the contrary. Then there exist a net of vectors $\{u_i\}$ with $\|u_i\| = 1$, $\forall i$ in a neighborhood of x_0 and a sequence $\{t_i\}$ with $t_i \geq 0, t_i \to 0$ and $u_i \to u$ $as\ i \to \infty$ such that

$$f(x_0 + t_i u_i) <_C f(x_0).$$

But $f(x_0 + t_i u_i) - f(x_0) = f(x_0 + t_i u) - f(x_0) + f(x_0 + t_i u_i) - f(x_0 + t_i u) <_C 0$.

Since f is locally C-Lipschitz, there exists $L \in C$ such that for sufficiently large i,

$$-Lt_i\|u_i - u\| \leq_C f(x_0 + t_i u_i) - x_0 + t_i u \leq_C Lt_i\|u_i - u\|$$

Then $f'(x_0, u) = \lim_{t\to\infty}[\frac{f(x_0+t_i u)-f(x_0)}{t_i}] <_C \lim_{i\to\infty} L\|u_i - u\| = 0$

i.e. $f'(x_0, u) <_C 0$ contradicting the assumption. Therefore the conclusion of the proposition is true.

A C-convex function f is said to be proper if $f(x) >_C -\infty, \forall x \in X$.

Definition4.2.6 A vector valued function $f: X \to \overline{Y}$ is said to be a C-d.c. function iff it can be written as a difference of two proper cone-convex functions i.e. $f(x) = g(x) - h(x)$ where g and h are C- convex and proper vector valued functions.

Note: Let $g, h: X \to \overline{Y}$ be C-convex proper vector valued function on X and it is easy to verify that f is locally Lipschitz at each point of X and is directionally differentiable on X with

$$f'(x_0, \mathrm{u}) = g'(x_0, u) - h'(x_0, \mathrm{u}) \text{ for all } u, x_0 \in \mathrm{X}.$$

Consider the following optimization problem.

(P) Min $f(x)$ s.t $x \in \mathrm{X}$

Where $f = g - h$ and g and h as above. We conventionally assume that

$+\infty - (+\infty) = +\infty$.

To state the necessary condition for minimality we first define the strong subdfferential of a C-convex vector valued function f at a point x_0, denoted by $\partial_s f(x_0)$

$$\partial_s f(x_0) = \{\mathrm{T} \in \mathrm{L}(x, \mathrm{y}): \mathrm{T}(x - x_0) \leq_\mathrm{C} f(x) - f(x_0) \quad \forall x \in \mathrm{X}\}$$

Note that: $\partial_s f(x_0) \subseteq \partial f(x_0)$ for any x_0 at which f is subdifferentiable.

Lemma 4.2.5 For C-convex vector valued function f, we have

$\mathrm{T} \in \partial_s f(x_0)$ if and only if $\mathrm{T}u \leq_\mathrm{c} f'(x_0, u)$ for all $u \in \mathrm{X}$

For a C-convex f, $\partial_s f(x_0)$ is nonempty as it at least contains the directional derivative of f at the point x_0.

Proof (i) suppose $\mathrm{T} \in \partial_s f(x_0)$ and f is C-convex implies that it is directionally differentiable. Thus,

$$\mathrm{t}f'(x_0, u) + \circ(\mathrm{t}) = f(x_0 + \mathrm{t}u) - f(x_0) \geq_\mathrm{C} \mathrm{tT}u$$

$\Rightarrow f'(x_0, u) + \lim_{\mathrm{t}\to 0^+} \frac{\mathrm{o(t)}}{\mathrm{t}} \geq_\mathrm{C} \mathrm{T}u$ where $\lim_{\mathrm{t}\to 0^+} \frac{\mathrm{o(t)}}{\mathrm{t}} = 0$

$f'(x_0,u) \geq_C Tu$ for all $u \in X$.

The converse follows easily

Since f is C-convex which implies directional differentiable form (ii) from lemma 4.2.4 we have

$f'(x_0,u) \leq_C f(x_0+u) - f(x_0)$ for all $u \in X$.

This implies $f'(x_0,u) \in \partial_s f(x_0)$ for all $u \in X$.

Hence $\partial_s f(x_0) \neq \emptyset$

Theorem 4.2.1 (Necessary condition) For $f = g - h$ to attain its local C-minimal value at a point $x_0 \in X$ it is necessary that $\partial_s h(x_0) \subseteq \partial_s g(x_0)$.

Proof If x_0 is local minimum point for f, then there exists a neighborhood θ of x_0 such that $f(x_0) \leq_C f(x)$ for all $x \in \theta$. which implies $g(x_0) - h(x_0) \leq_C g(x) - h(x)$, or

$h(x) - h(x_0) \leq_C g(x) - g(x_0)$ for all $x \in \theta$.

But for $T \in \partial_s h(x_0)$ we have $T(x - x_0) \leq_C h(x) - h(x_0)$ for all $x \in X$. then one can conclude that $g(x) - g(x_0) \nless_C T(x - x_0)$ or $T \in \partial g(x_0)$.

For otherwise, $g(x) - g(x_0) <_C T(x - x_0)$ together with the relation

$T(x - x_0) \leq_C h(x) - h(x_0)$ we will have $g(x) - g(x_0) <_C h(x) - h(x_0)$,

which is contradiction. Hence the theorem is proved.

Theorem4.2.2 (Sufficient condition)

If $\partial_s h(x_0) \subseteq \operatorname{int} \partial g(x_0)$, then the criterion Vector x_0 is a local efficient point for (P).

Proof

Let $f(x) = g(x) - h(x)$ for all $x \in X$, then clearly f is directionally differentiable on X and it is locally C-Lipschitz. From the assumption of the theorem, we have

$\partial_s h(x_0) \subseteq \partial g(x_0)$, and from the relation (ii) of the Lemma (4.2.3) it follows immediately that $g'(x_0,u) \geq_C h'(x_0,u)$ for all $u \in X, u \neq 0$.

This implies $f'(x_0,u) = g'(x_0,u) - h'(x_0,u) \geq_C 0$.

Hence using proposition (4.2.3) we have the conclusion.

4.2.3 Conjugate Duality in Cone d.c Optimization

[7] Let X be a real topological vector space and Y be a locally convex linear topological vector space. Assume that Y is partially ordered by a pointed, closed, convex cone C which has a nonempty interior in Y.

Let g and h be proper vector valued C-convex function from X in to $\bar{Y}$.

Now consider the C-d.c optimization

(P) Min $f(x)$ s.t $x \in$ X

Solving this problem means to find the set

C-Inf(P) =C-Inf $\{g(x) - h(x): x \in X\}$

Let U $\subseteq$ X be another locally convex linear topological vector space. We introduce the special perturbation function [7]

ψ: XxU $\to \bar{Y}$ such that $\psi(x, u) = h(x + u) - g(x)$ for all $(x, u) \in$ XxU

Then clearly $\psi(x, 0) = -f(x)$ for all $x \in$ X.

For $\Lambda \in \mathrm{M} = \mathrm{L}(\mathrm{U}, \mathrm{Y})$, the space of all linear continuous operators from U to Y, let the Lagrangian of problem (P) be given by

$$-\mathrm{L}(x, \Lambda) = C\text{-Sup}\{\Lambda u - \psi(x, u): u \in \mathrm{U}\}$$

$$= C\text{-Sup}\{\Lambda(x + u) + g(x) - h(x + u) - \Lambda x: u \in \mathrm{U}\}$$

$$= h^*(\Lambda) + g(x) - \Lambda x,$$

Where $h^*(\Lambda)$ denotes the conjugate map of h.

Now we put $-\mathrm{J}(\Lambda) = C\text{-Sup}\bigcup_{x\in \mathrm{X}} \mathrm{L}(x,\Lambda) = g^*(x) - h^*(x)$.

Then the dual optimization problem for (P) is written as

(Dc) $C\text{-Inf}\bigcup_{\Lambda\in \mathrm{M}} h^*(\Lambda) - g^*(\Lambda)$

We can also observe the symmetry between (P) and (Dc). But since both $h^*(\cdot)$ and $g^*(\cdot)$ are set valued maps the (Dc) is not a vector optimization problem. However it can be understood as determining the set $C\text{-Inf}\bigcup_{\Lambda\in \mathrm{M}} h^*(\Lambda) - g^*(\Lambda)$ on the other hand,

$$C\text{-Sup}\bigcup_{x\in \mathrm{X}}[-\mathrm{L}(x,\Lambda)] = C\text{-Sup}\bigcup_{x\in \mathrm{X}} \mathrm{C} - \mathrm{Sup}\{\Lambda u - \psi(x, u): u \in \mathrm{U}\}$$

$$= C\text{-Sup}\bigcup_{(x,u)\in \mathrm{XxU}}\{\Lambda u + 0x - \psi(x, u): u \in \mathrm{U}\}$$

$$= \psi^*(0, \Lambda),$$

Therefore, $\psi^*(0, \Lambda) = h^*(\Lambda) - g^*(\Lambda)$

Theorem4.2.3 Suppose a ny $x \in X$ and $\wedge \in M$, $f(x) \notin A(h^*(\wedge) - g^*(\wedge))$ and thus C-Inf(P) $\cup$ $A(C - \mathrm{Inf}(Dc)) = \emptyset$.

Proof Suppose the contrary. Then there exist $y \in h^*(\wedge) - g^*(\wedge)$ such that $y <_C f(x)$. But since $h^*(\wedge) - g^*(\wedge) = C\text{-Sup}\cup_{(x,u)\in XxU}\{\Lambda u + 0x - \psi(x,u)\}$

$y \geq_C (\Lambda u - \psi(x,u))$ for all $u \in U$. In particular, if we put $u = 0$ and noting that $f(x) = -\psi(x, 0)$.

It follows that $y \geq_C -\psi(x,0) = f(x)$ $\forall y \in h^*(\wedge) - g^*(\wedge)$ which contradiction our assumption. Hence the theorem is proved.

The above theorem assures us that for any x, $f(x) \in \mathrm{Inf}\cup_{\wedge\in M}[h^*(\wedge) - g^*(\wedge)]$.

If we can find $\Lambda_0 \in M$ such that $f(x_0) \in h^*(\wedge_0) - g^*(\wedge_0)$ for some x_0, then it means that $\wedge_0$ solves (Dc), then next theorem reflects this fact.

Theorem 4.2.4 If x_0 solves (P), then there exist some $\wedge_0 \in M$ which solves (Dc).

Proof If x_0 solves (P), then $f(x) = -\psi(x,0)$ the same If x_0 solves

(P$'$) Min$-\psi(x,0)$ $x \in X$.

There exists some $\wedge_0 \in M$ such that $(0,\wedge_0) \in \partial\psi(x,0)$.

But this in turn implies that

$(0,\wedge_0)(x_0,0)^T - \psi(x,0) \in \psi^*(0,\wedge_0)$ which means that

$$f(x_0) = -\psi(x,0) \in \psi^*(0,\wedge_0) = h^*(\wedge) - g^*(\wedge).$$

Now assume that $\wedge_0$ does not solve (Dc). Then there exist $\wedge \in M$ such that

$$h^*(\wedge) - g^*(\wedge) \cup B(h^*(\wedge_0) - g^*(\wedge_0)) \neq \emptyset$$

Since $f(x_0) \in h^*(\wedge_0) - g^*(\wedge_0)$ there exist $y \in h^*(\wedge) - g^*(\wedge)$ such that $y <_C f(x_0)$.

This implies $f(x_0) \in A(h^*(\wedge) - g^*(\wedge))$ but this contradicts the statement in theorem 4.2.3

Hence $\wedge_0$ solves (Dc).

Corollary 4.2.1 If x_0 solves (P) and $\wedge_0 \in \partial h(x_0)$, then $\wedge_0$ solves (Dc).

Proof from the assumption we have $\partial_s h(x_0) \subseteq \partial g(x_0)$, and the relation

$T \in \partial f(x_0)$ if and only if $Tx_0 - f(x_0) \in f^*(T)$ gives

$$g(x_0) - h(x_0) = (\wedge_0\, x_0 - h(x_0)) - (\wedge_0\, x_0 - g(x_0)) \in h^*(\wedge_0) - g^*(\wedge_0)$$

That is $f(x_0) \in h^*(\wedge_0) - g^*(\wedge_0)$, hence $\wedge_0$ solves (Dc).

Proposition 4.2.3 Let $x_0 \in$ domf. If $f(x_0) \in h^*(\wedge_0) - g*(\wedge_0)$ for $\wedge_0 \in$ M, then x_0 solves (P) at least locally.

Proof By assumption we have

$$-\psi(x_0, 0) = f(x_0) \in h^*(\wedge_0) - g*(\wedge_0) = \psi^*(0, \wedge_0)$$

Which is equivalent to $(0, \wedge_0)(x_0, 0)^{\mathrm{T}} - \psi(x_0, 0) \in \psi^*(0, \wedge_0)$ and by Lemma 4.2.1 we have

$$(0, \wedge_0) \in \partial\psi(x_0, 0) \qquad (**)$$

Then the relation $-\psi(x_0, 0) = f(x_0)$ and (**), we can see that

$0 \in \partial \mathrm{f}(x_0)$, or

x_0 is a local minimum of $f(x)$

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