OBJECTS COUNTED BY CENTRAL DELANNOY NUMBERS

By

Frether Getachew

Addis Ababa University

School of Graduate Studies

Department of Mathematics

SUBMITTED IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE
AT
ADDIS ABABA UNIVERSITY
ADDIS ABABA, ETHIOPIA
FEBRUARY 2013

ADDIS ABABA UNIVERSITY
DEPARTMENT OF
MATHEMATICS

The undersigned hereby certify that they have read this manuscript and recommends to the School of Graduate Studies its acceptance. The title of the project is, "**OBJECTS COUNTED BY CENTRAL DELANNOY NUMBERS**" by **Frether Getachew** in partial fulfillment of the requirements for the degree of **Master of Science**.

Dated: February 2013

Advisor: ______________________________
Dr. Seyoum Getu

Table of Contents

Summary

This work deals with *Central Delannoy numbers*, enumerated as $D_{n,n} = (D_n)_{n\geq 0}$ $= 1, 3, 13, 63, 321, 1683, 8989, 48639, \ldots$, which counts the number of lattice paths running from $(0,0)$ to (n,n) by using the steps $(1,0)$, $(0,1)$ and $(1,1)$. We will see some collection of examples that these numbers count. Also we will try to solve a problem on the Delannoy numbers, that is, proving that the central Delannoy numbers also exist on the diagonal, in the rectangular format of another array of numbers, namely Sulanke numbers.

Introduction

The Delannoy numbers were introduced and studied by Henri-Auguste Delannoy (1833-1915)[1]. He investigated the possible moves on a chessboard. The numbers under consideration here appear when one studies "la marche de la Reine," i.e., how the queen moves.[2]

For integers n and k, we define these numbers to satisfy

$$D_{n,k} = D_{n-1,k} + D_{n,k-1} + D_{n-1,k-1}$$

with the conditions $D_{0,0} = 1$ and $D_{i,j} = 0$ if $n < 0$ or $k < 0$. The members of the sequence $(D_n)_{i \geq 0} := (D_{n,n})_{i \geq 0} = 1, 3, 13, 63, 321, 1683, 8989, 48639, \ldots$(A001850 of sloane [5]), are known as the central Delannoy numbers.

In the following section we give a catalogue of some of the configurations counted by the central Delannoy numbers. Each example is accompanied by an illustration of a set of configurations corresponding to $D_1 = 3$, $D_2 = 13$ and $D_3 = 63$.

Chapter 1

Preliminaries

In this chapter we recall some notions and results which are useful in the latter chapters.

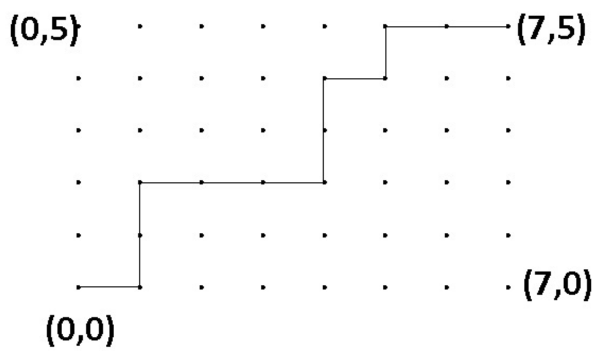
1.1 Lattice Paths (Rectangular Lattice Paths)

Definition 1.1.1. Given two points (p, q) and (r, s) with $p \geq r$ and $q \geq s$, a **rectangular lattice path** from (r, s) to (p, q) is a path from (r, s) to (p, q) that is made up of **horizontal steps** $H = (1, 0)$ and **vertical steps** $V = (0, 1)$. Thus, a rectangular lattice path from (r, s) to (p, q) starts at (r, s) and gets to (p, q) using unit horizontal and vertical segments [8].

Example 1.1.1. *In the figure below we show a rectangular lattice path from* $(0, 0)$ *to* $(7, 5)$*, consisting of* 7 *horizontal steps and* 5 *vertical steps. Given that the path starts at* $(0, 0)$*, it is uniquely determined by the sequence*

$$H, V, V, H, H, H, V, V, H, V, H, H$$

of 7 $H's$ and 5 $V's$.



Remark 1.1.1. In the integer plane, we will take lattice paths to be represented as concatenations of the directed steps belonging to various specified sets.

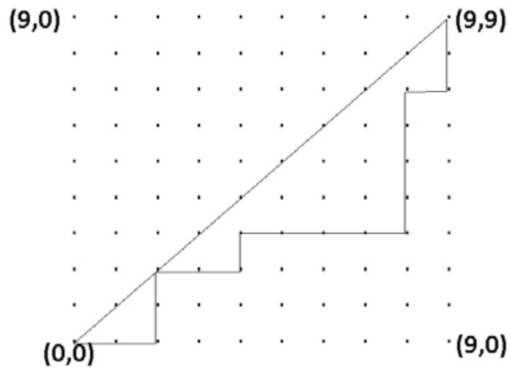
Theorem 1.1.2. *The number of rectangular lattice paths from (r, s) to (p, q) equals the binomial coefficient*

$$\binom{p-r+q-s}{p-r} = \binom{p-r+q-s}{q-s}$$

Example 1.1.3. *Consider a rectangular lattice path from (r, s) to (p, q), where $p \geq r$ and $q \geq s$. Such a path uses exactly $(p-r)+(q-s)$steps, and there is no loss of generality in assuming that $(r, s) = (0, 0)$. This is because we may simply translate (r, s) back to $(0, 0)$ and (p, q) back to $(p-r, q-s)$ and obtain a one to one correspondence between rectangular lattice path from (r, s) to (p, q) and those from $(0, 0)$ to $(p-r, q-s)$. By the above theorem ,if $p \geq 0$ and $q \geq 0$, the number of rectangular lattice paths from $(0, 0)$ to (p, q) equals*

$$\binom{p+q}{p} = \binom{p+q}{q}$$

we now consider a rectangular lattice paths from $(0, 0)$ to (p, q) that are restricted to lie on or below the line $y = x$ in the coordinate plane. We call such paths *sub diagonal rectangular lattice paths*. A sub-diagonal path from $(0, 0)$ to $(9, 9)$ is shown in the following figure.



1.2 Delannoy Numbers

Definition 1.2.1. The Delannoy numbers $D_{n,k}$ are the number of lattice paths from $(0, 0)$ to (n, k) in which only east $(1, 0)$, north $(0, 1)$, and north east $(1, 1)$ steps are allowed (i,e., $\longrightarrow, \uparrow, and \nearrow$). They are given by the recurrence relation

$D_{n,k} = D_{n-1,k} + D_{n,k-1} + D_{n-1,k-1}$

together with the initial conditions $D_{0,0} = 1$ and $D_{n,k} = 0$ if $n < 0$ or $k < 0$ [4].Two immediate consequences of this definition are the following:

1. $D_{n,0} = D_{0,n} = 1$,
2. $D_{n,1} = D_{1,n} = 2n + 1$.

   Verification:

1. $D_{n,0} = D_{n-1,0} + D_{n,-1} + D_{n-1,-1} = D_{n-1,0} = D_{n-2,0} = \cdots = D_{0,0} = 1$

similarly $D_{0,n} = D_{-1,k} + D_{0,k-1} + D_{-1,k-1} = D_{0,k-1} = D_{0,k-2} = \cdots = D_{0,0} = 1$

$$\begin{aligned} 2.D_{n,1} &= D_{n-1,1} + D_{n,0} + D_{n-1,0} \\ &= D_{n-1,1} + 2 \\ &= D_{n-2,1} + D_{n-1,0} + D_{n-2,0} + 2 \\ &= D_{n-2,1} + 4 = D_{n-3,1} + 6 = \cdots = D_{n-n,1} + 2n = 2n + 1 \end{aligned}$$

similarly $D_{1,n} = D_{1,n-1} + 2 = \cdots = D_{1,n-n} + 2n = 2n + 1$

One can also see that $D_{n,k} = D_{k,n}$, so the matrix of Delannoy numbers is symmetric[7].

They are also given by the sums

$$D_{n,k} = \sum_{d=0}^{min(n,k)} \binom{n}{d}\binom{k}{d} 2^d$$

A table of values for the Delannoy numbers is given by

n/k	0	1	2	3	4	...
0	1	1	1	1	1	...
1	1	3	5	7	9	...
2	1	5	13	25	41	...
3	1	7	25	63	129	...
4	1	9	41	129	321	...
⋮	⋮	⋮	⋮	⋮	⋮	...

The members of the sequence $(D_n)_{n\geq 0} := (D_{n,n})_{n\geq 0}$ =1,3,13,63,321,1683,8989,48639,. . . , are known as the *central delannoy numbers*[7]. This can be expressed as

$$D_n = \sum_{d=0}^{n} \binom{n}{d}^2 2^d$$

Generating Functions

In counting problems, we are often interested in counting the number of objects of "size n", which we denote by a_n. By varying n, we get different values of a_n. In this way we get a sequence of real numbers

$$a_0, a_1, a_2, \ldots$$

from which we can define a power series (which in some sense can be regarded as an 'infinite degree polynomial')

$$G(x) := a_0 + a_1 x + a_2 x^2 + \cdots .$$

The above power series $G(x)$ is called the *generating function* of the sequence $a_0, a_1, a_2, \ldots$.

The generating function of the sequence D_n is

$$\frac{1}{\sqrt{1 - 6x + x^2}}.$$

Riordan Array

$$\textit{Let} \quad g(x) := 1 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$\textit{and}$$

$$f(x) := 0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots$$

then the array $(g(x), f(x))$, given as follows,

$$\begin{pmatrix} \vdots & \vdots & \vdots & \cdots & \vdots & \\ \vdots & \vdots & \vdots & \cdots & \vdots & \\ g & gf & gf^2 & \cdots & gf^k & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \\ \vdots & \vdots & \vdots & \cdots & \vdots & \end{pmatrix}$$

is called a Riordan array.

Riordan array multiplication is defined as $(g(x), f(x)) * (h(x), l(x)) = (g(x)h(f(x)), l(f(x)))$. If the second Riordan array is simply a function, then the Riordan array multiplication is $(g(x), f(x)) * h(x) = g(x)h(f(x))$.

The row sum of a Riordan array is the Riordan array multiplication of the Riordan array and the function $\frac{1}{1-x}$.

Chapter 2

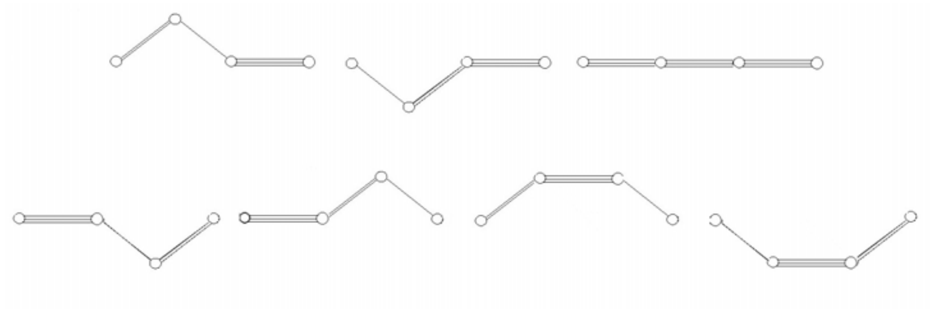
Objects Counted By Central Delanoy Numbers

When the the directed steps of the lattice path are weighted, the weight of a path is the product of the weights of its steps, and the weight of a path set is the sum of the weights of its paths. Throughout, we will denote the diagonal up and down steps as $U := (1,1)$ and $D := (1,-1)$[6].

Example 2.0.1. *A classic example is the set of paths from* $(0,0)$ *to* $(2n,0)$ *using the steps* U*,* D*, and* $E_2 = (2,0)$*.*
For $n = 1$ *there are* $D_1 = 3$ *unrestricted paths from* $(0,0)$ *and* $(2,0)$ *using the steps* U*,* D*, and* $E_2 = (2,0)$*.*

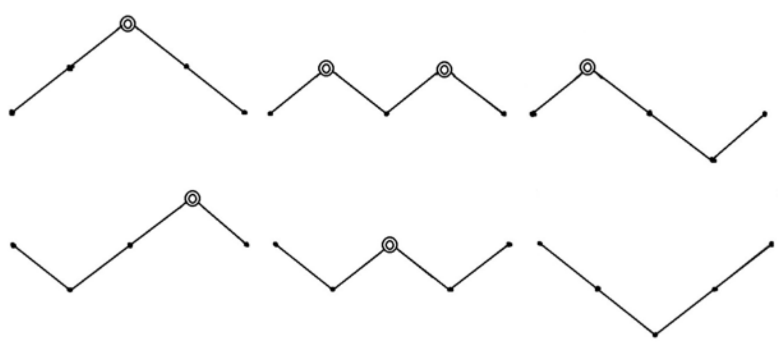
Example 2.0.2. *The Delannoy number* D_n *is the weight of the set of paths from* $(0,0)$ *to* $(n,0)$ *using the steps* U_2, D*, and* $E_1 = (1,0)_3$*, where the up step* U_2 *and the*

horizontal step $E_1 = (1,0)_3$ *have weights* 2 *and* 3*, respectively. Take* $n = 3$*, then* $D_3 = 63$ *counts the set of paths from* $(0,0)$ *to* $(3,0)$ *using the given steps.*



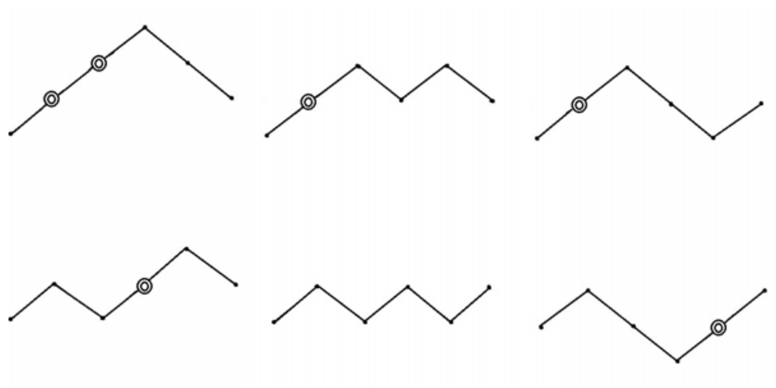
$$2.3 + 2.3 + 3.3.3 + 3.2 + 3.2 + 2.3 + 3.2 = 63 = D_3.$$

Example 2.0.3. *Using the steps* U *and* D*, we find* D_n *to be the weighted sum of the paths from* $(0,0)$ *to* $(2n,0)$ *where within each path the right-hand turns, or* ***peaks****, have weight* 2*. Thus, for* $n = 2$*,*



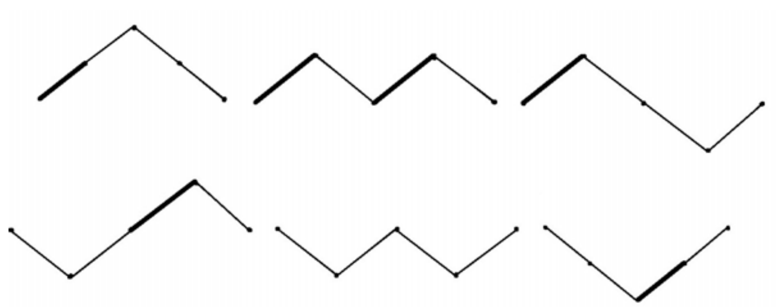
the sum of the weights of the paths from $(0,0)$ to $(4,0)$ is $2+4+2+2+2+1 = 13 = D_2$.

Example 2.0.4. *Using the steps* U *and* D*, we find that* D_n *is the sum of the weights of the paths from* $(0,0)$ *to* $(2n+1,1)$ *that begin with an up step and where the intermediate vertices of double ascents have weight* 2*. If we let* $n = 2$*, then*



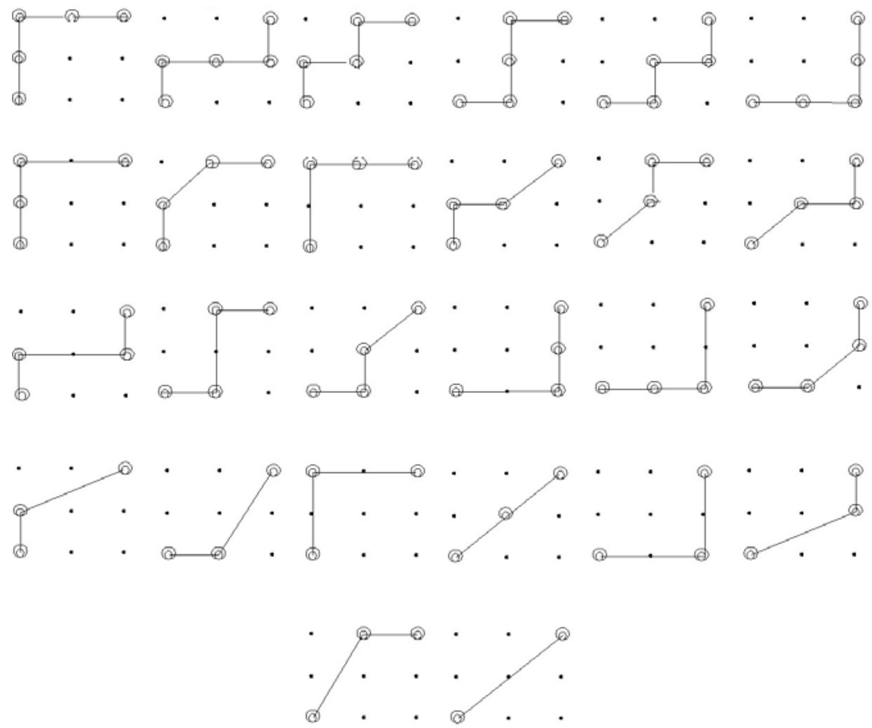
the sum of the weights of the paths from $(0,0)$ to $(5,1)$ is $4+2+2+2+1+2 = 13 = D_2$.

Example 2.0.5. *Using the steps U and D, we find that D_n weighted sum over the paths from $(0,0)$ to $(2n,0)$ where each U step which is oddly positioned along its path has weight 2.*
Consider for $n = 2$, then

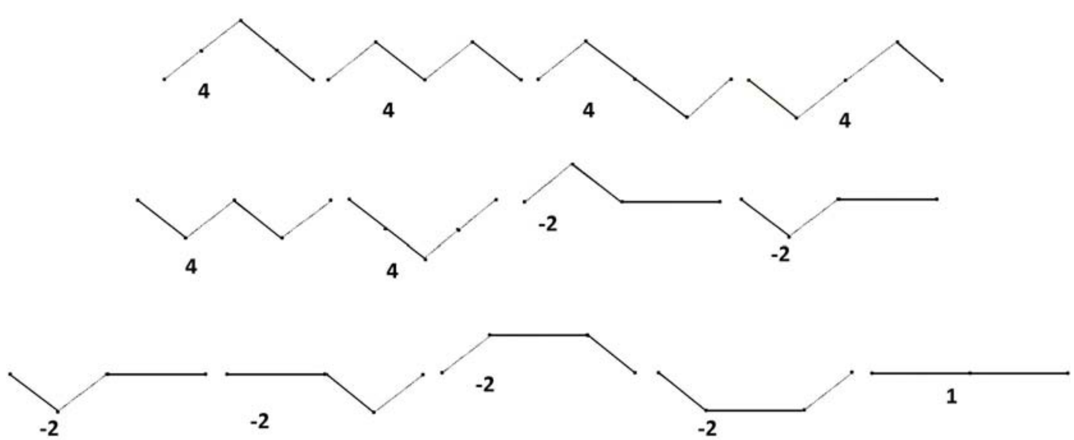


the sum over the weights of the paths from $(0,0)$ to $(4,0)$ is $2+4+2+2+1+2 = 13 = D_2$.

Example 2.0.6. *The product $2^{n-1}D_n$ counts the set of all paths from $(0,0)$ to (n,n) with steps of the form (x,y) where x and y are non negative integers, not both 0. Thus the product $2^{2-1}D_2 = 2.13$, for $n = 2$, gives us the set of paths $(0,0)$ to $(2,2)$ with the steps $(0,1), (1,0)$, $(1,1)$, $(2,0)$, $(2,1)$, $(0,2)$, $(1,2)$ and $(2,2)$.*

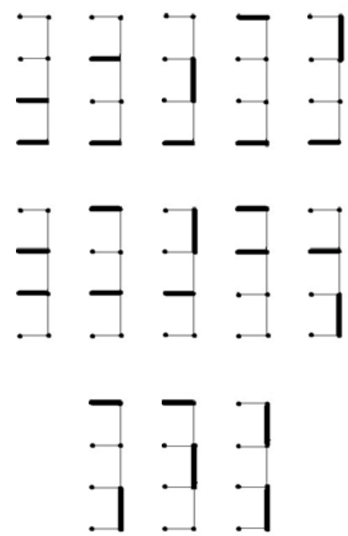


Example 2.0.7. *Using the steps* U_2, D*, and* $E_2 = (2,0)_{-1}$ *where the up step and the horizontal step have weights of* 2 *and* -1*, respectively,* D_n *is the sum of the weights of the paths running from* $(0,0)$ *to* $(2n,0)$*.*

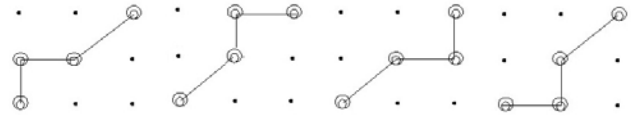


On the above picture, the sum of the weight of the paths running from $(0,0)$ to $(4,0)$ by using the steps U_2, D, and $E_2 = (2,0)_{-1}$ is 13, which is D_2.

Example 2.0.8. *Consider counting matchings in the comb graph (which is a tree, all vertices are of degree at most* 3 *and all the vertices of degree* 3 *lie on a single simple path). For a comb with* $2n$ *teeth, there are* D_n *ways to have an* n*-set of non-adjacent edges. Thus, for* $n = 2$*, there are* $D_2 = 13$ $2-$*matchings in the comb with* 2.2 *teeth.*



Example 2.0.9. *Define a triangle* $T(n,k)(0 \leq k \leq n)$ *by the number of Delannoy paths of length* n*, having* k $D' = (1,1)$*-steps on the line y=x(a Delannoy path of length* n *is a path from* $(0,0)$ *to* (n,n)*, consisting of steps* $E = (1,0), N = (0,1)$ *and* $D' = (1,1)$*). For example* $T(2,1) = 4$ *because we have the paths*



First few rows of the triangle are

1					
2	1				
8	4	1			
36	20	6	1		
172	104	36	8	1	
852	552	212	56	10	1

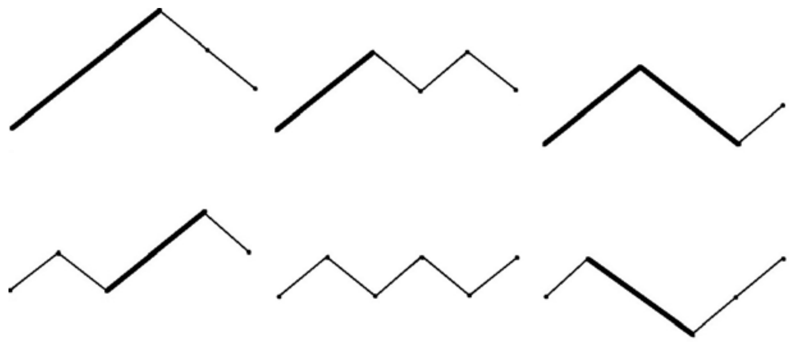
Riordan array of this triangle is $\left(\frac{1}{x+\sqrt{1-6x+x^2}}, \frac{x}{x+\sqrt{1-6x+x^2}}\right)$ [5]. *Row sums of this triangle are the central Delannoy numbers. This can be shown as follows:*

The row sum of the triangle is

$$\begin{aligned}
\left(\frac{1}{x+\sqrt{1-6x+x^2}}, \frac{x}{x+\sqrt{1-6x+x^2}}\right) * \frac{1}{1-x} &= \frac{1}{x+\sqrt{1-6x+x^2}} \frac{1}{1-\frac{x}{x+\sqrt{1-6x+x^2}}} \\
&= \frac{1}{x+\sqrt{1-6x+x^2}} \frac{x+\sqrt{1-6x+x^2}}{x+\sqrt{1-6x+x^2}-x} \\
&= \frac{1}{x+\sqrt{1-6x+x^2}-x} \\
&= \frac{1}{\sqrt{1-6x+x^2}}
\end{aligned}$$

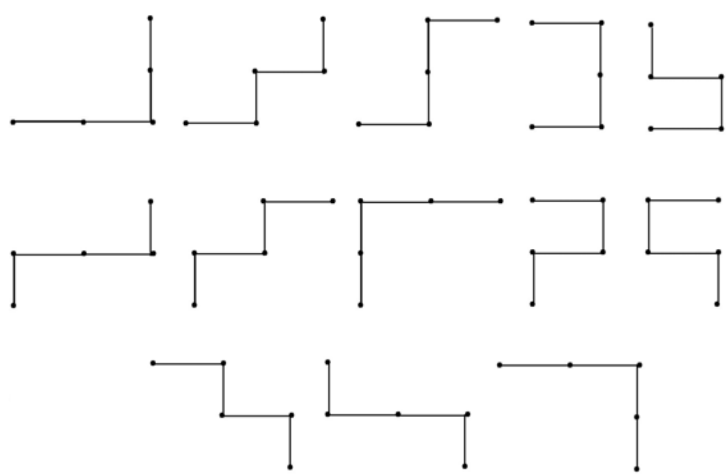
which is the generating function of the central Delannoy numbers.

Example 2.0.10. *In a lattice path using the steps U and D,* ***a long****, is a maximal subpath having at least two steps, all of the same type. The number D_n is the weighted sum over the paths running from $(0,0)$ to $(2n+1,1)$ which begin with a U step and whose non final longs have the weight 2. Hence the sum of the weight of the paths from $(0,0)$ to $(5,1)$, for $n=2$, is*



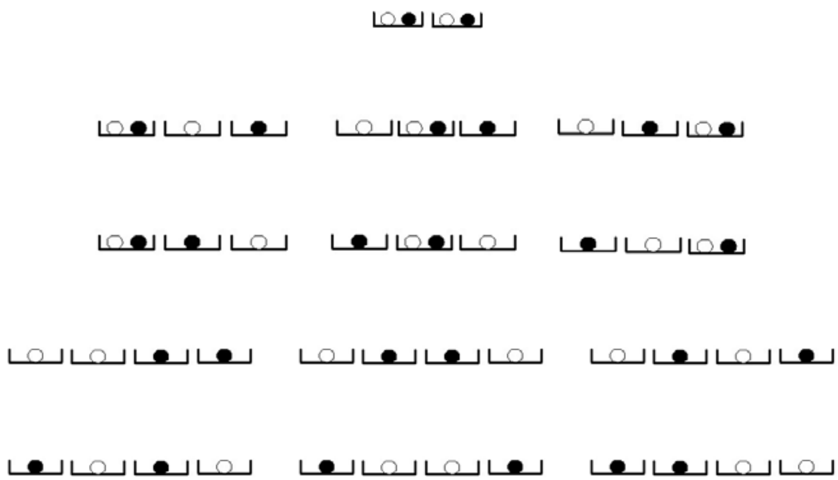
$2 + 2 + 4 + 2 + 1 + 2 = 13 = D_2$.

Example 2.0.11. *Consider the walks that begin at the origin and use the unit steps: east* (E)*, west* (W)*, and north* (N)*. If these walks never start with* W *and are self-avoiding, that is,* E *and* W *are non-adjacent, then* D_n *counts the walks with* $2n$ *steps and final height* n*. Taking* $n = 2$*, then there are* $D_2 = 13$ *walks with* 4 *steps and final height* 2*.*



Example 2.0.12. *The number* D_n *counts the ways to distribute* n *white and* n *black balls into* r *labelled urns where* r *takes on the values from* n *to* $2n$ *and where each urn is non empty and does not contain more than one ball of each color. (The balls*

are unlabelled and are ordered so that white precedes black when two are present in an urn.) If $n = 2$*, the urns takes on the values from* 2 *to* 4 *and* D_2 *counts the ways to distribute* 2 *white and* 2 *black balls in to the urns.*

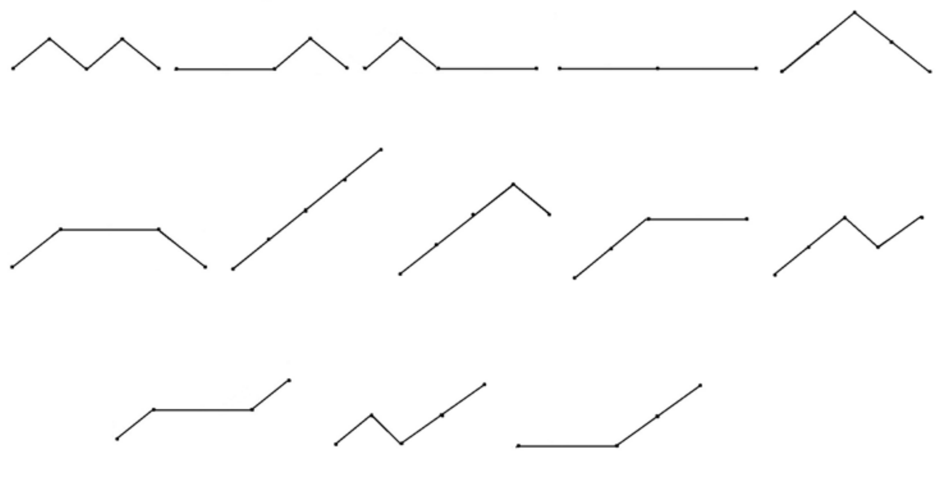


$D_2 = 13$ balls-in-urns distributions.

Example 2.0.13. *The number* D_n *counts the words from the alphabet* $\{a, b, \{a, b\}\}$ *where the total occurrences of* a *and* b *in each word is* n*. If we take* $n = 2$*, then we will get 13 words from the alphabet* $\{a, b, \{a, b\}\}$*.*

$$\{a, b\}\{a, b\}, \{a, b\}ab, \{a, b\}ba, a\{a, b\}b, b\{a, b\}a, ab\{a, b\}, ba\{a, b\},$$
$$aabb, abab, abba, baab, baba, bbaa$$
$$D_2 = 13 \textit{ words.}$$

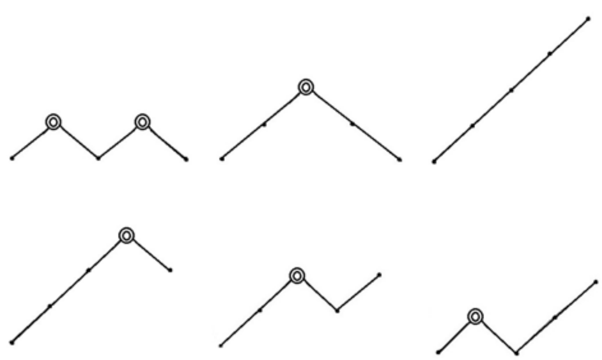
Example 2.0.14. *The number* D_n *counts the set of paths using the three steps,* U*,* D*, and* $E_2 = (2, 0)$*, running from* $(0, 0)$ *to the line* $x = 2n$*, and remaining weakly above the* x*-axis.*



13 paths running from $(0,0)$ to the line $x = 4$, for $n = 2$, and remaining weakly above the x-axis.

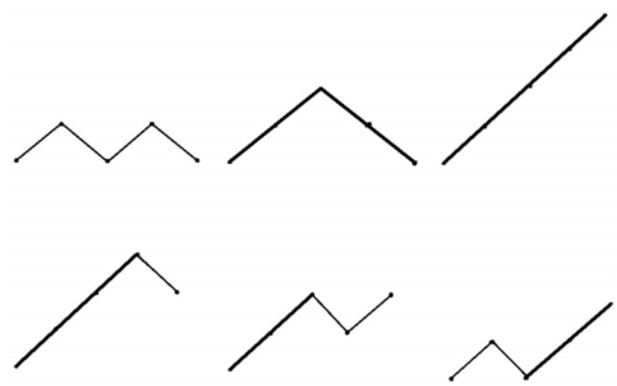
Example 2.0.15. *For the steps U and D, D_n is the weighted sum of the paths running from $(0,0)$ to the line $x = 2n$ and remaining weakly above the x-axis, where within each path the right-hand turns have weight 2.*

The sum of the weights of the paths from $(0,0)$ to the $x = 4$, for $n = 2$, and remaining weakly above the $x-$axis is



$4 + 2 + 1 + 2 + 2 + 2 = 13 = D_2.$

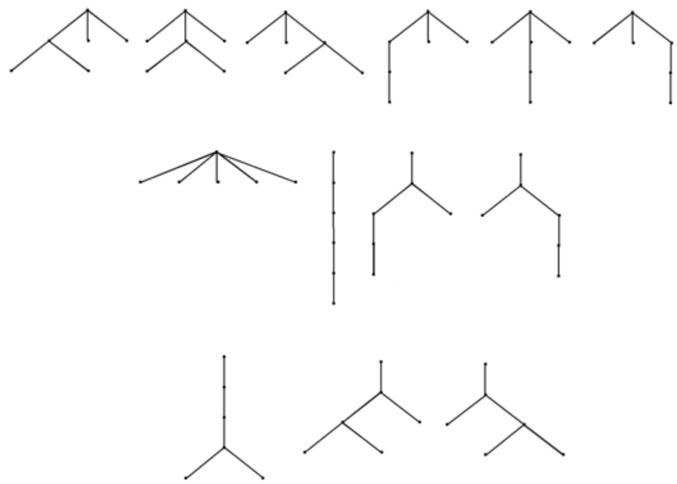
Example 2.0.16. *For the steps U and D, D_n is the weighted sum of the paths running from $(0,0)$ to the line $x = 2n$ and remaining weakly above the x-axis, where within each path each **long** has weight 2. Here a **long** is a maximal subpath of the same step type of length exceeding one. Thus, for $n = 2$*



the sum of the weights of the paths from $(0,0)$ to the $x = 4$ is

$$1 + 4 + 2 + 2 + 2 + 2 = D_2.$$

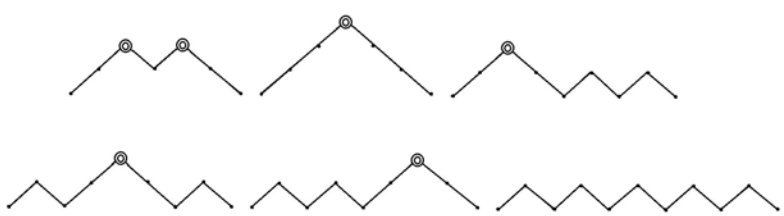
Example 2.0.17. *Let $T(n)$ denote the set of plane trees with $2n+1$ edges, with roots of odd degree, with the non-root vertices having degree 1 (for the leaves), 2, or 3, and with an even number of vertices of degree two between any two vertices of odd degree. Thus $D_2 = 13$ counts the set of plane trees $T(2)$ with 5 edges, with roots of odd degree, with the non-root vertices having degree 1, 2, or 3, and with an even number of vertices of degree two between any two vertices of odd degree as follows:*



The specified trees counted by $D_2 = 13$.

Example 2.0.18. *A **high peak** is the intermediate vertex of a UD pair with ordinate exceeding 1. Let $P(n,k)$ denote the set of paths using the steps U and D, running from $(0,0)$ to $(n,0)$, remaining weakly above the x-axis, intersecting the x-axis k times, and having high peaks of weight 2. Then the Delannoy number counts a union of sets:*

$$D_n = |\bigcup_{i=1}^{n+1} p(2n+2i, 2i)|.$$

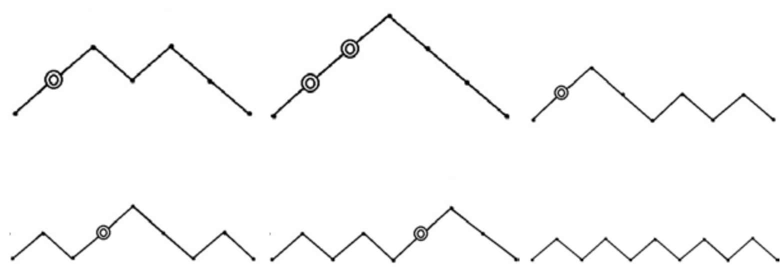


$$4+2+2+2+2+1 = 13 = D_2.$$

Example 2.0.19. *A **double ascent (or double rise)** is just a consecutive UU pair. Let $P(n,k)$ denote the set of paths using the steps U and D, running from*

$(0,0)$ *to* $(n,0)$*, remaining weakly above the* x*-axis, intersecting the* x*-axis* k *times, and having double ascents of weight* 2*. Then the Delannoy number counts a union of sets:*

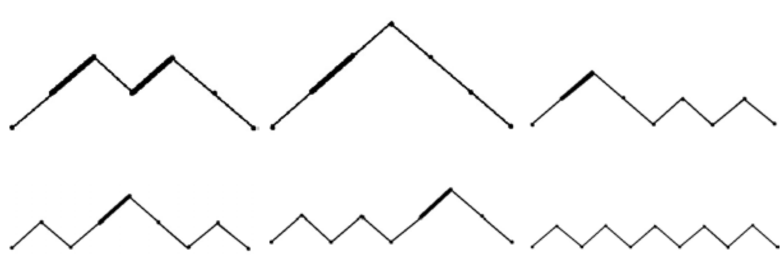
$$D_n = |\bigcup_{i=1}^{n+1} p(2n+2i, 2i)|.$$



$$2+4+2+2+2+1 = 13 = D_2.$$

Example 2.0.20. *Let* $P(n,k)$ *denote the set of paths using the steps* U *and* D*, running from* $(0,0)$ *to* $(n,0)$*, remaining weakly above the* x*-axis, intersecting the* x*-axis* k *times, and evenly positioned ascents of weight* 2*. Then the Delannoy number counts a union of sets:*

$$D_n = |\bigcup_{i=1}^{n+1} p(2n+2i, 2i)|$$



$$4+2+2+2+2+1 = 13 = D_2.$$

Example 2.0.21. *On a path using the steps U and D, a* ***restricted long*** *is a maximal subpath of a single step type having length exceeding* 1*, except when the subpath ends at the x-axis, in which case the length of the subpath must exceed* 2*. Let $P(n,k)$ denote the set of paths using the steps U and D, running from $(0,0)$ to $(n,0)$, remaining weakly above the x-axis, intersecting the x-axis k times, and having restricted longs of weight* 2*. Then the Delannoy number counts a union of sets:*

$$D_n = |\bigcup_{i=1}^{n+1} p(2n+2i, 2i)|$$

$$2+4+2+2+2+1 = 13 = D_2.$$

Example 2.0.22. *The central Delannoy number D_n counts the matrices with* 2 *rows and entries* 0 *or* 1 *such that there are exactly n* 1*'s in each row and at least one* 1 *in each column.*

There are $D_2 = 13$ such matrices for $n = 2$.

$$\begin{bmatrix}1&1\\1&1\end{bmatrix}, \begin{bmatrix}1&1&0\\1&0&1\end{bmatrix}, \begin{bmatrix}1&0&1\\1&1&0\end{bmatrix}, \begin{bmatrix}1&1&0\\0&1&1\end{bmatrix}, \begin{bmatrix}0&1&1\\1&1&0\end{bmatrix},$$

$$\begin{bmatrix}1&0&1\\0&1&1\end{bmatrix}, \begin{bmatrix}0&1&1\\1&0&1\end{bmatrix}, \begin{bmatrix}1&1&0&0\\0&0&1&1\end{bmatrix}, \begin{bmatrix}1&0&1&0\\0&1&0&1\end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

Example 2.0.23. *The product $2^{n-1}D_n$ counts the matrices having two rows and non negative integer entries where each row sum is n and each column has at least one positive entry.*

Thus, there are $2^{2-1}D_2 = 2.13$ such matrices for $n = 2$.

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 2 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

Chapter 3

A Problem on the Delannoy Numbers

Consider two recurrences for arrays of numbers which appear to have the same central, or diagonal, sequences, namely the central Delannoy numbers.
For $(n, k) \in \mathbb{Z} \times \mathbb{Z}$,

$$D_{n,k} = D_{n-1,k} + D_{n,k-1} + D_{n-1,k-1}$$

with $D_{0,0} = 1$ and $D_{n,k} = 0$ if $n < 0$ or $k < 0$, which are Delannoy numbers.

For $(n, k) \in \mathbb{Z} \times \mathbb{Z}$, let $s_{n,k}$ satisfies

$$s_{n,k} = \begin{cases} s_{n,k-1} + s_{n-1,k}, & if \quad n+k \quad is \quad even \\ s_{n,k-1} + 2s_{n-1,k}, & if \quad n+k \quad is \quad odd \end{cases}$$

with $s_{n,k} = 0$ if $n < 0$ or $k < 0$ and $s_{0,0} = 1$, which are called Sulanke numbers.

The problem is, giving a proof that the central sequences are the same. Before solving this problem, first we need to know more about Sulanke numbers, Z transform and Convolution of sequences.

3.1 Sulanke numbers

The Sulanke numbers, defined above, were introduced by R.Sulanke, in relation with a problem that involves the central Delannoy numbers.
In a rectangular format Sulanke numbers can be displayed as follows:

$$\begin{array}{cccccc} s_{0,0}(1) & s_{0,1}(1) & s_{0,2}(1) & s_{0,3}(1) & s_{0,4}(1) & ... \\ s_{1,0}(2) & s_{1,1}(3) & s_{1,2}(5) & s_{1,3}(6) & s_{1,4}(8) & ... \\ s_{2,0}(2) & s_{2,1}(8) & s_{2,2}(13) & s_{2,3}(25) & s_{2,4}(33) & ... \\ s_{3,0}(4) & s_{3,1}(12) & s_{3,2}(38) & s_{3,3}(63) & s_{3,4}(129) & ... \\ s_{4,0}(4) & s_{4,1}(28) & s_{4,2}(66) & s_{4,3}(192) & s_{4,4}(321) & ... \end{array}$$

Let us use the pascal parametrization of the indices i, j in the Sulanke numbers, so that $t_{i,j}$ stands for the number in the line i (beginning with the 0 line) and in the position j of that line (beginning with $j = 0$). We also set $t_{i,j} = 0$ if $j > i$ and $t_{i,j} = s_{j,r}$ where $j + r = i$. Thus we get a family of sequences labeled by the row number namely,

$t_{0,j} = (1, 0, 0, ...)$,

$t_{1,j} = (1, 2, 0, 0, ...)$,

$t_{2,j} = (1, 3, 2, 0, 0, ...)$,

$t_{3,j} = (1, 5, 8, 4, 0, 0, ...)$,

$t_{4,j} = (1, 6, 13, 12, 4, 0, 0, ...)$,

$t_{5,j} = (1, 8, 25, 38, 28, 8, 0, 0, ...)$,

$t_{6,j} = (1, 9, 33, 63, 66, 36, 8, 0, 0, ...)$, and so on. We will see later that this sequences have some interesting properties. It turns out that this numbers are in fact coefficients of certain polynomials.

Z transform

The Z transform maps complex sequences x_n in to complex (holomorphic) functions $X : U \subset \mathbb{C} \longrightarrow \mathbb{C}$ given by the Laurent series

$$\begin{aligned} X(z) &= \sum_{n=0}^{\infty} \frac{x_n}{z^n} \\ &= \sum_{n=0}^{\infty} x_n z^{-n} \end{aligned}$$

All the values of z that makes the summation to exist is *a region of convergence.* We can also denote the Z transform of the sequence $(x_n)_{n=0}^{\infty}$ by $Z(x_n)$ [7].

Example 3.1.1. *Consider the sequence,* $x_n = (1, 1, 1, 1, ...)$, *then the Z transform of* x_n *is given by*

$$\begin{aligned} X(z) &= \sum_{n=0}^{\infty} x_n z^{-n} \\ &= 1 + z^{-1} + z^{-2} + z^{-3} + \cdots \\ &= 1 + (z^{-1}) + (z^{-1})^2 + (z^{-1})^3 + \cdots \end{aligned}$$

let $r = z^{-1}$

$$\Longrightarrow \quad X(z) = 1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r} \qquad when \quad |r| < 1$$
$$\Longrightarrow \quad X(z) = \frac{1}{1-z^{-1}} \qquad when \quad |z^{-1}| < 1$$
$$\Longrightarrow \quad X(z) = \frac{1}{1-\frac{1}{z}} = \frac{z}{z-1} \qquad when \quad |z| > 1,$$

which is the region of convergence.

Example 3.1.2. *The Z transform of the sequence* $x_n = n$ *is*

$$
\begin{aligned}
X(z) &= \sum_{n=0}^{\infty} n z^{-n} \\
&= 0 + z^{-1} + 2z^{-2} + 3z^{-3} + \cdots \\
&= -z\frac{d}{dz}\left[\frac{1}{1-z^{-1}}\right] \qquad \textit{when} \quad |z| > 1 \\
&= -z\frac{d}{dz}\left[\frac{z}{z-1}\right] \\
&= -z\left[\frac{(z-1)-z}{(z-1)^2}\right] \\
&= -z\left[\frac{-1}{(z-1)^2}\right] \\
&= \frac{z}{(z-1)^2} \qquad \textit{when} \quad |z| > 1
\end{aligned}
$$

Similarly for $x_n = n^2$ *and* $x_n = n^3$*,*

$$
\begin{aligned}
X(z) &= -z\frac{d}{dz}\left[\frac{z}{(z-1)^2}\right] \\
&= \frac{z(z+1)}{(z-1)^3}
\end{aligned}
$$

and

$$
\begin{aligned}
X(z) &= -z\frac{d}{dz}\left[\frac{z(z+1)}{(z-1)^3}\right] \\
&= \frac{z(z^2+4z+1)}{(z-1)^4}
\end{aligned}
$$

respectively. All are valid for $|z| > 1$.

Example 3.1.3. *The Z transform of sequence* $x_n = \binom{n}{p}$*, where* p *is non negative integer, is given by*

$$\begin{aligned} X(z) &= \sum_{n=0}^{\infty} \binom{n}{p} z^{-n} \\ &= 0 + \binom{1}{p} z^{-1} + \binom{2}{p} z^{-2} + \binom{3}{p} z^{-3} + \cdots \\ &= \frac{z}{(z-1)^{p+1}} \end{aligned}$$

this can be proved by using mathematical induction on $p \geq 0$ *as follows,*

For $p = 0$*,*

$$\begin{aligned} X(z) &= \sum_{n=0}^{\infty} \binom{n}{0} z^{-n} \\ &= \sum_{n=0}^{\infty} z^{-n} \\ &= \frac{z}{z-1} \end{aligned}$$

For $p = 1$*,*

$$\begin{aligned} X(z) &= \sum_{n=0}^{\infty} \binom{n}{1} z^{-n} \\ &= \sum_{n=0}^{\infty} n z^{-n} \\ &= \frac{z}{(z-1)^2} \end{aligned}$$

Assume for $p = k$ *the statement is true,*

$$i.e. \quad X(z) = \sum_{n=0}^{\infty} \binom{n}{k} z^{-n} = \frac{z}{(z-1)^{k+1}}.$$

we need to show that

$$\begin{aligned} X(z) &= \sum_{n=0}^{\infty} \binom{n}{k+1} z^{-n} \\ &= \frac{z}{(z-1)^{(k+1)+1}}. \end{aligned}$$

$$Consider \quad X(z) = \sum_{n=0}^{\infty} \binom{n}{k+1} z^{-n}$$

$$\begin{aligned}
\implies X(z) &= \sum_{n=0}^{\infty} \left(\binom{n-1}{k} + \binom{n-1}{k+1} \right) z^{-n} \\
&= \sum_{n=0}^{\infty} \binom{n-1}{k} z^{-n} + \sum_{n=0}^{\infty} \binom{n-1}{k+1} z^{-n} \\
&= z^{-1} \sum_{n=0}^{\infty} \binom{n-1}{k} z^{-n+1} + \sum_{n=0}^{\infty} \binom{n-1}{k+1} z^{-n} \\
&= z^{-1} \sum_{n=1}^{\infty} \binom{n-1}{k} z^{-n+1} + \sum_{n=0}^{\infty} \binom{n-1}{k+1} z^{-n} \\
&= z^{-1} \sum_{n=0}^{\infty} \binom{n}{k} z^{-n} + \sum_{n=0}^{\infty} \left(\binom{n-2}{k} + \binom{n-2}{k+1} \right) z^{-n} \\
&= z^{-1} \frac{z}{(z-1)^{k+1}} + z^{-2} \sum_{n=0}^{\infty} \binom{n-2}{k} z^{-n+2} + \sum_{n=0}^{\infty} \binom{n-2}{k+1} z^{-n} \\
&= z^{-1} \frac{z}{(z-1)^{k+1}} + z^{-2} \sum_{n=2}^{\infty} \binom{n-2}{k} z^{-n+2} + \sum_{n=0}^{\infty} \binom{n-2}{k+1} z^{-n} \\
&= z^{-1} \frac{z}{(z-1)^{k+1}} + z^{-2} \frac{z}{(z-1)^{k+1}} + z^{-3} \frac{z}{(z-1)^{k+1}} + z^{-4} \frac{z}{(z-1)^{k+1}} + \cdots \\
&= \frac{1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + \cdots}{(z-1)^{k+1}} \\
&= \frac{\frac{1}{1-z^{-1}}}{(z-1)^{k+1}} \\
&= \frac{z}{(z-1)^{(k+1)+1}}
\end{aligned}$$

done!

Properties of Z transform [3]

1. Linearity:

   $Z(a(x_1)_n + b(x_2)_n) = aZ((x_1)_n) + bZ((x_1)_n)$, a and b are arbitrary constants.

2. Delay-shifting property:

   $Z(x_{n-m}) = z^{-m}Z(x_n) = z^{-m}X(z)$, for $m \in \mathbb{N}$.

   Verification:

$$\begin{aligned} Z(x_{n-m}) &= \sum_{n=0}^{\infty} x_{n-m}z^{-n} \\ &= x_{-m}z^{-0} + x_{1-m}z^{-1} + \cdots + x_{-1}z^{-1(m-1)} + x_0 z^{-m} + x_1 z^{-(m+1)} + \cdots \end{aligned}$$

$$\left[x_{n-m} = (\underbrace{0, 0, 0, \cdots, 0}_{m-times}, x_0, x_1, x_2, \cdots) \right]$$

   Since x_n is assumed to be causal (i.e. $x_n = 0, for \quad n < 0$): $x_{-m} = x_{-m+1} = \cdots = x_{-1} = 0$.

   Then we achieve,

$$Z(x_{n-m}) = x_0 z^{-m} + x_1 z^{-m-1} + x_2 z^{-m-2} + \cdots$$

$$\begin{aligned} \Longrightarrow Z(x_{n-m}) &= z^{-m}\left[x_0 + x_1 z^{-1} + x_2 z^{-2} + \cdots\right] \\ &= z^{-m}Z(x_n). \end{aligned}$$

3. Advance-shifting property:

   For $m \in \mathbb{N}$, $Z(x_{n+m}) = z^m \left(Z(x_n) - x_0 - \frac{x_1}{z} - \cdots - \frac{x_{m-1}}{z^{m-1}} \right)$

   Here x_{n+m} is the sequence $(x_m, x_{m+1}, x_{m+2}, \cdots)$.

$Verification:$

$$
\begin{aligned}
Z(x_{n+m}) &= \sum_{n=0}^{\infty} x_{n+m} z^{-n} \\
&= x_m + x_{m+1} z^{-1} + x_{m+2} z^{-2} + \cdots \\
&= x_m z^{m-m} + x_{m+1} z^{m-(m+1)} + x_{m+2} z^{m-(m+2)} + \cdots \\
&= x_0 z^m + x_1 z^{m-1} + x_2 z^{m-2} + \cdots + x_{m-1} z^{m-(m-1)} + x_m z^{m-m} \\
&\quad + x_{m+1} z^{m-(m+1)} + x_{m+2} z^{m-(m+2)} + \cdots - x_0 z^m - x_1 z^{m-1} \\
&\quad - x_2 z^{m-2} - \cdots - x_{m-1} z^{m-(m-1)} \\
&= z^m (x_0 + x_1 z^{-1} + x_2 z^{-2} + \cdots + x_{m-1} z^{-(m-1)} + x_m z^{-m} + x_{m+1} z^{-(m+1)} \\
&\quad + x_{m+2} z^{-(m+2)} + \cdots - x_0 - x_1 z^{-1} - x_2 z^{-2} - \cdots - x_{m-1} z^{-(m-1)}) \\
&= z^m \left(\sum_{n=0}^{\infty} x_n z^{-n} - x_0 - \frac{x_1}{z} - \frac{x_2}{z^2} - \cdots - \frac{x_{m-1}}{z^{m-1}} \right) \\
&= z^m \left(Z(x_n) - x_0 - \frac{x_1}{z} - \frac{x_2}{z^2} - \cdots - \frac{x_{m-1}}{z^{m-1}} \right)
\end{aligned}
$$

4. Multiplication by the sequence $n = (0, 1, 2, 3, \cdots)$:
   $Z(nx_n) = -z\dfrac{d}{dz}Z(x_n) = -z\dfrac{d}{dz}X(z).$

Convolution of sequences

The convolution of the sequence $a = (a_0, a_1, a_2, \cdots)$ with the sequence $b = (b_0, b_1, b_2, \cdots)$, denoted by $a * b$, is the sequence

$a * b = (a_n * b_n)_{n=0}^{\infty}$ where

$$(a_n * b_n)_{n=0}^{\infty} = \sum_{i=0}^{n} a_i b_{n-i}. \quad [7]$$

Example 3.1.4. *Consider the sequence* $u_n = (1, 1, 1, \cdots)$*, then*

$$\begin{aligned} u_n * u_n &= \sum_{i=0}^{n} u_i u_{n-i} \\ &= u_0 u_n + u_1 u_{n-1} + \cdots + u_{n-1} u_1 + u_n u_0 \\ &= 1 + 1 + 1 + \cdots + 1 + 1 \\ &= n + 1 \end{aligned}$$

Example 3.1.5. *The convolution of the sequence* $x_n = n$ *with the sequence* $u_n = (1, 1, 1, \cdots)$*, is*

$$\begin{aligned} x_n * u_n &= \sum_{i=0}^{n} x_i u_{n-i} \\ &= x_0 u_n + x_1 u_{n-1} + \cdots + x_{n-1} u_1 + x_n u_0 \\ &= 0 + 1 + 2 + 3 + \cdots + (n-1) + n \\ &= \frac{n(n-1)}{2} = \binom{n}{2} \end{aligned}$$

Remark 3.1.1.

⋇ Convolution is always commutative and associative and that also distributive over the sum.

⋇ The convolution of the sequence a_n with itself k times is given by $a_n * a_n * a_n * \cdots * a_n$ and denoted as $*^k a_n$.

⋇ Given sequences a_n and $\delta = (1, 0, 0, 0, \cdots)$, then $a_n * \delta = a_n$

⋇ The convolution of two eventually zero sequences $a = (a_0, a_1, a_2, \cdots, a_m, 0, 0, \cdots)$

and $b = (b_0, b_1, b_2 \cdots, b_l, 0, 0, \cdots)$ is given by

$$a_i * b_i = \sum_{j=0}^{m} a_j b_{i-j},$$

for $i = 0, 1, 2, \cdots, m+l$ and $a_i * b_i = 0$ for $i > m+l$.

Theorem 3.1.6 (Convolution Theorem)**.** [7]

$$Z\left((a_1)_n * (a_2)_n * \cdots * (a_k)_n\right) = Z((a_1)_n)Z((a_2)_n)\cdots Z((a_k)_n)$$

where $(a_1)_n, (a_2)_n, \cdots, (a_k)_n$ *are given sequences.*

It is not difficult to see that, the Z transform of the sequence $x_n = (x_1)_n * (x_2)_n$ is $X(z) = X_1(z)X_2(z)$, where $X(z) = Z(x_n), X_1(z) = Z((x_1)_n)$ and $X_2(z) = Z((x_2)_n)$.

Theorem 3.1.7. *1. If* $0 \le p_0 \le p$*, then we have*

$$Z\left(\binom{n+p_0}{p}\right) = \frac{z^{p_0+1}}{(z-1)^{p+1}}$$

2. For given $p_1, p_2, \cdots, p_k$ *non negative integers we have that*

$$Z\left(\binom{n}{p_1} * \binom{n}{p_2} * \cdots * \binom{n}{p_k}\right) = z^{k-1}\frac{z}{(z-1)^{p_1+p_2+\cdots+p_k+k}}.$$

Thus

$$\binom{n}{p_1} * \binom{n}{p_2} * \cdots * \binom{n}{p_k} = \binom{n+k-1}{p_1+p_2+\cdots+p_k+k-1}. \quad [7]$$

Proof. 1. Using the advance-shifting property,

$$\begin{aligned} Z\left(\binom{n+p_0}{p}\right) &= z^{p_0}\left(Z\left(\binom{n}{p}\right) - \binom{0}{p} - \binom{1}{p}z^{-1} - \cdots - \binom{p_0-1}{p}z^{p_0-1}\right) \\ &= z^{p_0}\left(Z\left(\binom{n}{p}\right) - 0\right) \quad \textit{since} \quad 0 \le p_0 \le p \\ &= z^{p_0}\frac{z}{(z-1)^{p+1}} \quad \textit{since} \quad Z\left(\binom{n}{p}\right) = \frac{z}{(z-1)^{p+1}} \\ &= \frac{z^{p_0+1}}{(z-1)^{p+1}} \end{aligned}$$

2.

$$\begin{aligned} Z\left(\binom{n}{p_1} * \binom{n}{p_2} * ... * \binom{n}{p_k}\right) &= Z\left(\binom{n}{p_1}\right) Z\left(\binom{n}{p_2}\right) ... Z\left(\binom{n}{p_k}\right) \\ &= \frac{z}{(z-1)^{p_1+1}} \frac{z}{(z-1)^{p_2+1}} \cdots \frac{z}{(z-1)^{p_k+1}} \\ &= z^{k-1} \frac{z}{(z-1)^{p_1+p_2+...+p_k+k}} \end{aligned}$$

and

$$\begin{aligned} Z\left(\binom{n}{p_1} * \binom{n}{p_2} * \cdots * \binom{n}{p_k}\right) &= z^{k-1} \frac{z}{(z-1)^{p_1+p_2+\cdots+p_k+k}} \\ &= Z\left(\binom{n+k-1}{p_1+p_2+\cdots+p_k+k-1}\right) \end{aligned}$$

$$\Longrightarrow \binom{n}{p_1} * \binom{n}{p_2} * ... * \binom{n}{p_k} = \binom{n+k-1}{p_1+p_2+\cdots+p_k+k-1}$$

$\square$

We begin by considering the Z-transform of the sequence n^2, which can be written as follows:

$$Z(n^2) = \frac{z(z+1)}{(z-1)^3} = \frac{z}{(z-1)^2} + 2\frac{z}{(z-1)^3}, \tag{3.1.1}$$

$$\Longrightarrow n^2 = \binom{n}{1} + 2\binom{n}{2} \tag{3.1.2}$$

And let $k \in \mathbb{N}$ and consider the convolution $*^k n^2$. We have

$$Z(*^k n^2) = \frac{z^k(z+1)^k}{(z-1)^{3k}} = z\frac{z^{k-1}(z+1)^k}{(z-1)^{3k}} \tag{3.1.3}$$

Let $a_{k,i}, i = 0, 1, 2, \cdots, 2k-1$, be the coefficients in the expansion of the polynomial $z^{k-1}(z+1)^k$ in powers of $z-1$. From [7],

$$z^{k-1}(z+1)^k = \sum_{i=0}^{2k-1} a_{k,i}(z-1)^{2k-1-i}. \tag{3.1.4}$$

Thus we have

$$\begin{aligned} Z(*^k n^2) &= z\frac{z^{k-1}(z+1)^k}{(z-1)^{3k}} \\ &= z\frac{\sum_{i=0}^{2k-1} a_{k,i}(z-1)^{2k-1-i}}{(z-1)^{3k}} \\ &= \sum_{i=0}^{2k-1} a_{k,i}\frac{z}{(z-1)^{k+i+1}} \end{aligned}$$

$$\Longrightarrow *^k n^2 = \sum_{i=0}^{2k-1} a_{k,i}\binom{n}{k+i} \tag{3.1.5}$$

Thus for each $k \in \mathbb{N}$ we have the sequence $a_{k,i} = (a_{k,0}, a_{k,1}, \cdots, a_{k,2k-1}, 0, 0, \cdots)$. Now, let us consider the convolution $*^k n^2 * n$ whose Z transform is

$$Z(*^k n^2 * n) = \frac{z^k(z+1)^k}{(z-1)^{3k}}\frac{z}{(z-1)^2} = z\frac{z^k(z+1)^k}{(z-1)^{3k+2}}. \tag{3.1.6}$$

Let $b_{k,i}, i = 0, 1, 2, \cdots, 2k$, be the coefficients in the polynomial $z^k(z+1)^k$ in powers of $z-1$. From [7]

$$z^k(z+1)^k = \sum_{i=0}^{2k} b_{k,i}(z-1)^{2k-i} \tag{3.1.7}$$

Thus we have that

$$\begin{aligned} Z(*^k n^2 * n) &= z\frac{\sum_{i=0}^{2k} b_{k,i}(z-1)^{2k-i}}{(z-1)^{3k+2}} \\ &= \sum_{i=0}^{2k} b_{k,i}\frac{z}{(z-1)^{k+i+2}}. \end{aligned}$$

from which we get

$$*^k n^2 * n = \sum_{i=0}^{2k} b_{k,i}\binom{n}{k+i+1}. \tag{3.1.8}$$

Then we have sequences $b_{k,i} = (b_{k,0}, b_{k,1}, \cdots, b_{k,2k}, 0, 0, \cdots)$ for each $k \in \mathbb{N}$. Observe that (3.1.7) makes sense for $k = 0$, if we simply set $b_{0,0} = 1$ and then we understand (3.1.8) in this case $*^k n^2 * n$ is simply taken as n.

So the sequence $b_{k,i}$ are defined for k nonnegative integers, where $b_{0,i} = (1, 0, 0, \cdots)$ is the δ sequence.

Proposition 3.1.8. *The sequences $a_{k,i}$ and $b_{k,i}$ are related by*

$$b_{k,i} = a_{k,i} + a_{k,i-1} \tag{3.1.9}$$

$$a_{k+1,i} = b_{k,i} + 2b_{k,i-1} \tag{3.1.10}$$

Proof. By using (3.1.7) and (3.1.4) we get

$$\begin{aligned}
\sum_{i=0}^{2k} b_{k,i}(z-1)^{2k-i} &= z^k(z+1)^k \\
&= z(z^{k-1}(z+1)^k) \\
&= (z-1+1)\sum_{i=0}^{2k-1} a_{k,i}(z-1)^{2k-1-i} \\
&= \sum_{i=0}^{2k-1} a_{k,i}(z-1)^{2k-i} + \sum_{i=1}^{2k} a_{k,i-1}(z-1)^{2k-i} \\
&= \sum_{i=0}^{2k}(a_{k,i}+a_{k,i-1})(z-1)^{2k-i},
\end{aligned}$$

$$\Longrightarrow b_{k,i} = a_{k,i} + a_{k,i-1}$$

And also from (3.1.4) with k replaced by $k+1$, we get

$$\begin{aligned}
\sum_{i=0}^{2(k+1)-1} a_{k+1,i}(z-1)^{2(k+1)-1-i} &= \sum_{i=0}^{2k+1} a_{k+1,i}(z-1)^{2k+1-i} \\
&= (z+1)z^k(z+1)^k \\
&= (z-1+2)\sum_{i=0}^{2k} b_{k,i}(z-1)^{2k-i} \\
&= \sum_{i=0}^{2k} b_{k,i}(z-1)^{2k+1-i} + 2\sum_{i=1}^{2k+1} b_{k,i-1}(z-1)^{2k+1-i} \\
&= \sum_{i=0}^{2k+1}(b_{k,i}+2b_{k,i-1})(z-1)^{2k+1-i},
\end{aligned}$$

$$\Longrightarrow a_{k+1,i} = b_{k,i} + 2b_{k,i-1}.$$

□

Thus, we have two kinds of sequences, namely $a_{k,i}$ and $b_{k,i}$, satisfying relations (3.1.9) and (3.1.10). We claim that these are precisely Sulanke numbers $s_{n,m}$. In fact, the elements $a_{k,i}$, $i = 0, 1, 2, \cdots, 2k-1$, and $k \geq 1$ correspond to the Sulanke numbers $s_{i,2k-1-i}$, and the elements $b_{k,i}$, $i = 0, 1, 2, \cdots, 2k$, and $k \geq 0$ correspond to the Sulanke numbers $s_{i,2k-i}$. Conversely, the Sulanke numbers $s_{n,m}$ with $n + m$ even, correspond to the elements $b_{\frac{n+m}{2},n}$, and the Sulanke numbers $s_{n,m}$ with $n + m$ odd, correspond to the elements $a_{\frac{n+m+1}{2},n}$.

We will refer to the first index of the sequences $a_{k,i}$ and $b_{k,i}$ as *level* of the sequence. Beginning with the sequence $b_{0,i} = (1, 0, 0, \cdots)$ in the level $k = 0$, and use (3.1.10) to move to level 1. We get

$a_{1,i} = b_{0,i} + 2b_{0,i-1} = (1, 0, 0 \cdots) + 2(0, 1, 0, 0, \cdots) = (1, 2, 0, 0, \cdots)$.

Now, from (3.1.9) we get the sequence $b_{1,i}$

$b_{1,i} = a_{1,i} + a_{1,i-1} = (1, 2, 0, 0, \cdots) + (0, 1, 2, 0, 0, \cdots) = (1, 3, 2, 0, 0, \cdots)$.

And again with (3.1.10) we can move from level 1 to level 2 using $b_{1,i}$. We have that

$a_{2,i} = b_{1,i} + 2b_{1,i-1} = (1, 3, 2, 0, 0, \cdots) + 2(0, 1, 3, 2, 0, 0, \cdots) = (1, 5, 8, 4, 0, 0, \cdots)$,

and we complete level 2 with (3.1.9)

$b_{2,i} = a_{2,i}+a_{2,i-1} = (1, 5, 8, 4, 0, 0, \cdots)+(0, 1, 5, 8, 4, 0, 0, \cdots) = (1, 6, 13, 12, 4, 0, 0, \cdots)$,

and so on.

Summarizing, the steps we follow for obtaining all the sequences $a_{k,i}$ and $b_{k,i}$ are

$$b_{0,i} \longrightarrow^{(3.1.10)} \underbrace{a_{1,i} \longrightarrow^{(3.1.9)} b_{1,i}}_{Level 1} \longrightarrow^{(3.1.10)} \underbrace{a_{2,i} \longrightarrow^{(3.1.9)} b_{2,i}}_{Level 2} \longrightarrow^{(3.1.10)} a_{3,i} \longrightarrow \cdots,$$

some of the first sequences are

$b_{0,i} = (1, 0, 0, ...)$

$a_{1,i} = (1, 2, 0, 0, ...)$

$b_{1,i} = (1, 3, 2, 0, 0, ...)$

$a_{2,i} = (1, 5, 8, 4, 0, 0, ...)$

$b_{2,i} = (1, 6, 13, 12, 4, 0, 0, ...)$

$a_{3,i} = (1, 8, 25, 38, 28, 8, 0, 0, ...)$

$b_{3,i} = (1, 9, 33, 63, 66, 36, 8, 0, 0, ...)$

$a_{4,i} = (1, 11, 51, 129, 192, 168, 80, 16, 0, 0...)$

$b_{4,i} = (1, 12, 62, 180, 321, 360, 248, 96, 16, 0, 0...)$

Corollary 3.1.9. *The sequence* $a_{k+1,i}$ *can be obtained using only the sequence of the same type* $a_{k,i}$ *of the previous level (and similarly for* $b_{k,i}$*). That is,*

$$a_{k+1,i} = a_{k,i} + 3a_{k,i-1} + 2a_{k,i-2} \tag{3.1.11}$$

$$b_{k+1,i} = b_{k,i} + 3b_{k,i-1} + 2b_{k,i-2} \tag{3.1.12}$$

Proof. Combine (3.1.9) and (3.1.10) to get

$a_{k+1,i} = b_{k,i} + 2b_{k,i-1} = a_{k,i} + a_{k,i-1} + 2(a_{k,i-1} + a_{k,i-2}) = a_{k,i} + 3a_{k,i-1} + 2a_{k,i-2}$

Similarly for (3.1.12). □

Thus, beginning with $a_{1,i} = (1, 2, 0, 0, ...)$ we obtain

$$a_{2,i} = a_{1,i} + 3a_{1,i-1} + 2a_{1,i-2} = (1, 5, 8, 4, 0, 0, ...),$$

and then

$$a_{3,i} = a_{2,i} + 3a_{2,i-1} + 2a_{2,i-2} = (1, 8, 25, 38, 28, 8, 0, 0, ...),$$

and so on. Similarly, from with $b_{0,i} = (1, 0, 0, ...)$

$$b_{1,i} = b_{0,i} + 3b_{0,i-1} + 2b_{0,i-2} = (1, 3, 2, 0, 0, ...),$$

and then

$$b_{2,i} = b_{1,i} + 3b_{1,i-1} + 2b_{1,i-2} = (1, 6, 13, 12, 4, 0, 0, ...),$$

and so on. Now let us consider the numbers $b_{k,k}$ which are the numbers corresponding to the diagonal in the rectangular format of the sulanke numbers i.e. $s_{k,k}$.

From (3.1.7)

$$\begin{aligned} z^k(z+1)^k &= \sum_{i=0}^{2k} b_{k,i}(z-1)^{2k-i} \\ \Longrightarrow b_{k,k} = \frac{1}{k!}\frac{d^k}{dz^k} z^k(z+1)^k|_{z=1}. \end{aligned}$$

Since

$$\begin{aligned} \frac{d^k}{dz^k}\left(z^k(z+1)^k\right) &= \sum_{j=0}^{k}\binom{k}{j}\left(z^k\right)^{(j)}\left((z+1)^k\right)^{(k-j)} \\ &= \sum_{j=0}^{k}\binom{k}{j}\frac{k!}{(k-j)!}\frac{k!}{j!}z^{k-j}(z+1)^j, \end{aligned}$$

we have that

$$\begin{aligned} b_{k,k} &= \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}\frac{k!}{(k-j)!}\frac{k!}{j!}z^{k-j}(z+1)^j|_{z=1} \\ &= \sum_{j=0}^{k}\binom{k}{j}^2 2^j, \end{aligned}$$

which are the central Delannoy numbers.

References

[1] H. Delannoy, Emploi de l'echiquier pour la resolution de certain problems de probabilities. Assoc. Franc.Bordeaux, 24 (1895) 70-90.

[2] Christer O.Kiselman, Asymptotic properties of the Delannoy numbers and similar arrays, 2012

[3] Dr.Ghulam Muhammad, Z-Transform, CE352, King Saud University, (power point).

[4] $Mathworld.wolfram.comDelannoyNumbers.html$

[5] On-Line Encyclopaedia of Integer Sequences, http://www.research.att.com/ njas/sequences/

[6] ROBERT A. SULANKE, OBJECTS COUNTED BY THE CENTRAL DELANNOY NUMBERS, Journal of Integer Sequences, Vol.6 (2003), Article 03.1.5

[7] Claudio de Jesús Pita Ruiz Velasco, Convolution and Sulanke numbers, Universided Panamericana, Mexico City, Mexico, Journal of Integer Sequences, Vol.13(2010), Article 10.1.8.

[8] $www.cord.edufacultyLenarzMath380handouts8_5pdf, LatticepathsandSchrederNumbers$