ADDIS ABABA UNIVERSITY

COLLEGE OF NATURAL AND COMPUTATIONAL SCIENCES

DEPARTMENT OF MATHEMATICS

THESIS ON

PRELIMINARIES ON ROBUST STRICT PASSIVITY AND ISS FOR UNCERTAIN SINGULAR SYSTEMS.

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Certification of the Final Thesis

I hereby certify that all the correction and recommendation suggested by the board of examiners are incorporated into the final thesis entitled "**Robust strict passivity and Input-to-state stability (ISS) for uncertain singular systems**" by Gemechis Tesfaye Dufera.

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Advisor	**Signature**	**Date**

DECLARATION

This is to certify that this thesis paper entitled ‘‘**Robust strict passivity and Input- to- state stability (ISS) for uncertain singular systems**” accepted in partial fulfillment of the requirements for the award of the degree of Master of Science in mathematics by the school of graduate studies, Addis Ababa University through the College of Natural and Computational Science, done by Gemechis Tesfaye Dufera a genuine work carried out by him under my guidance. The matter embodied in this thesis work has not been submitted earlier for the award of any degree or diploma.

The assistance and help received during the course of this investigation have been duly acknowledged. Therefore, I recommend that it can be accept as fulfilling the thesis requirements.

Name: Gemechis Tesfaye Dufera

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SYMBOLS AND ABBREVIATIONS

A. *Symbols*

$\|A\|$	*Determinant of matrix A.*
$\|\|A\|\|$	*Norm of matrix A.*
I_r	*An identity matrix of dimension r.*
R^n	*Real vector space of dimension n.*
$R^{n\times m}$	*Set of n x m real matrices.*
$R^{n\times n}$	*Set of square matrix.*
A, A^T	*Matrix A and its transpose.*
$e(A)$	*Eigen values of matrix A.*
$max\ e(A), min\ e(A)$	*Maximum, minimum Eigen values of A.*
$Q \in R^{n\times n}$	*Strict positive definite matrix Q.*

B. Abbreviations

BIBO	*Bounded Input Bounded Output*
GAS	*Globally Asymptotically Stable*
ISS	*Input − to − State Stability*
LMI	*Linear Matrix Inequality*
ODE	*Ordinary Differential Equation*

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ABSTRACT

This thesis studies the problem of robust strict passivity and input-to-state stability (ISS) for uncertain singular systems. We use the linear matrix inequality (LMI) condition. Linear matrix inequality makes the system to be stable. Moreover, the problem of designing robust state stabilization is to overcome when the proposed Linear Matrix Inequality condition is not satisfied, that the closed- loop or bounded systems are proved to be input-to-state stability.

Keywords: Passivity; Singular systems; ISS; Uncertainty; Linear Matrix Inequality.

CHAPTER ONE

INTRODUCTION

The thesis work is presented based on Robust Strict Passivity and Input-to-State Stability (ISS) for uncertain singular systems. In different researcher works robust strict passivity and Input-to-State Stability for uncertain singular systems would be studied for different purposes.

1.1 Background of the Study

Singular systems are dynamic systems governed by the combination of Algebraic and Differential equations. These systems arise naturally in dynamic models in a wide range of engineering applications such as Chemical Engineering systems, Electrical and Mechanical systems, Mechanical modeling, Air craft modeling, and Economic systems.

Since 1970, there have been a lot of interests in developing systematic and theoretically vigorous techniques for analyzing and designing singular systems which constitute a class of systems of both theoretical and practical importance.

The results in this study were obtained under the assumption that singular systems are free of uncertainties. However, uncertainty (internal or external) will always exist even in the system models that well approximate the behavior of real-world process. This may arise due to different features of real systems not being represented by the class of model used or to specified numerical values of various parameters of the models only being determinable to some degree of uncertainty. Considering these facts, many scholars such as E.D. Sontag studied robust stability of singular system and obtained remarkable results for continuous and discrete, and linear and nonlinear singular systems. It should be pointed out that when the robust stability of singular systems is investigated, the regularity and absence of impulses (for continuous systems) and causality (for discrete) systems need to be considered. Hence, the stability problem for singular systems is much more complicated than the normal state space and needs vigilance [9].

Input-to-state stability is a stability notion widely used to study stability of nonlinear control systems with external inputs. Roughly speaking, control system is ISS if it is globally asymptotically stable in the absence of external inputs and if its trajectories are bounded by the function of the size of the input for all sufficiently large times. The importance of ISS is due to the fact that the concept has bridged the gap between input – output and state – space methods, widely used within the control systems community.

A well-established technique for the study of stability and robustness of nonlinear systems, which are described by a set of differential equations globally defined in Euclidean space, is the Input-to-State Stability approach, [12].The classical definition potentially allows formulating and characterizing stability properties with respect to arbitrary compact invariant sets. The implicit requirement that these sets should be simultaneously Lyapunov stable and globally attractive, however, it makes the basic theory not applicable for a global analysis of many dynamical behaviors of interest, such as multi stability or periodic oscillations, just to name few, and only local analysis remains possible. In fact, it is well-known that such systems, when defined in Euclidean space, normally admit invariant sets that fail to be Lyapunov stable.

As an attempt to overcome such limitations for the case of nonlinear autonomous systems, the almost global stability property was introduced, [1], and short afterwards, almost Input-to-State Stability [6], for systems admitting exogenous disturbances. In particular, for the case of almost ISS, sufficient criteria based on a combination of dual Lyapunov techniques [1] and classical dissipation inequalities were proposed.

[2] for an application of such tools to stability analysis of rotational motions. The key idea of the dual approach is to replace Lyapunov functions by suitable density functions and to impose a monotonicity condition on the way these are propagated by the flow. While converse dual Lyapunov results have appeared in the literature short afterwards, [2], some difficulties in the explicit construction of density functions for systems involving unstable equilibria have also emerged [7].

The Input-to-State Stability (ISS) framework introduced by Sontag in April, 1989 has proved to be among the most successful paradigms for simultaneously analyzing both input-output as well as internal stability for nonlinear systems. Recently, Sontag [12] presented a comprehensive overview of the state of- the-art of ISS including sample applications. Of particular interest is the unification of several ISS-type results as suggested in [4] and [12].

Following the original notion of Input-to-State Stability (ISS) [12], many similar notions have been proposed providing different relationships between inputs, outputs, and states such as Input-to-Output Stability [4], Output-to-State Stability [12], and Input-Output-to-State Stability

[3]. These various notions frequently have similar properties, such as Lyapunov characterizations, that are tailored to the specific relationships posited by the differing properties.

1.2 Statement of the problem

This thesis deals with the problem of robust strict passivity and ISS for uncertain singular systems. Due to its application so far many scholars have studied, on Robust Strict Passivity and ISS and many results have been obtained. However, the use of LMI (Linear Matrix Inequality) is rarely used in later, thus we apply LMI and closed-loop to study the ISS of the given system.

Stability is a condition in which a slight disturbance in a system does not produce a significant disrupting effect on that system and it is an important property of any control systems and plays a central role in the theory of systems and control engineering. Stability concepts are qualitative measures on trajectories of dynamic systems. They play central role in the systems theory and Control Engineering.

1.3 Questions of the study

- How do we use the Linear Matrix Inequality condition to proof the problem for robust strict passivity and Input-to-State Stability for uncertain singular systems?

1.4 Objectives of the study

1.4.1 General objective of the study

- To apply the concept of robust strict passivity and ISS for uncertain singular systems and apply Linear Matrix Inequality to obtain the ISS of the given system.

1.4.2 The specific objectives of the study

- To identify the basic definitions, theorems and lemmas related to ISS.
- To prove results concerning the robust strict passivity and ISS for uncertain singular systems.
- To solve problems related to robust strict passivity and ISS for uncertain singular systems.

1.5 Delimitation of the study.

In our daily work, we can use the Robust Strict Passivity and ISS for uncertain singular system. As an important result we can use Input to- State Stability if the disturbance (disorder) is occurring. In this paper we will be concern with defining and explaining the Robust Strict Passivity and ISS for uncertain singular systems.

1.6 Significance of the study

This thesis is important for the following reasons:

- To propose different ways of proving the Input-to-State Stability (ISS) of the study.
- To apply Input-to-State Stability (ISS) in different fields. Such as Chemical engineering, Electrical (power system), mechanical modeling and air craft modeling.
- To develop the concept of ISS.

CHAPTER TWO

METHODOLOGY

2.1 Source of Information for the study

The relevant sources of Information for this study are mainly internet, books that are available in libraries of Addis Ababa University. All the Information obtained will be compiled.

2.2 Study procedures

In this thesis, secondary data is used. Important materials and data for the study would be collected by means of documentary review. Hence in order to achieve the listed objectives of this study, we follow the following steps:

1. Defining the problem of the study.
2. Reviewing the literature Robust Strict Passivity and ISS for uncertain singular systems.
3. Proving Theorems of Robust Strict Passivity and ISS for uncertain singular systems.
4. Providing summary and conclusion of the study.

CHAPTER THREE

3.1 REVIEW OF RELATED LITERATURE

The following is Mathematical and Theoretical review of related literature.

In the study of the problems of Robust Stability analysis and Robust Stabilization for state-space systems with unstructured parameter uncertainties, it is known that the methods based on the concepts of quadratic stability have played important role. An uncertain state-space system is quadratically stable if there exists a fixed quadratic Lyapunov function which can be used to check the stability of the uncertain system. Accordingly, concepts of generalized quadratic stability have been proposed, to deal respectively with robust stability analysis and robust stabilization problems for uncertain singular systems [5]. Based on these concepts, some robust stability and robust stabilization conditions have been obtained.

The Passivity idea was emerged in the electrical networks from the phenomena of dissipation (decrease of total energy stored in the system with time) of energy across the resistor.
Rantzer, [1**]** improved the passivity by introducing the notion of a 'storage energy' and 'supply rate' and pointed out that energy function can be viewed as Lyapunov function. Because of the fact that singular systems better describe physical system than regular ones, control of singular systems has been studied.
It is known that passivity theory plays an important role in real engineering problems which is a powerful technique of designing systems in control theory. But the positive real control problem for singular time delay systems is still open if considering time delay. For time invariant linear systems, strict passivity is equivalent to the generalized strictly positive realness for singular systems and the robust passive control problem for continuous time singular system with time delay is studied. Thus, the relation between passivity and stability can be established. Dissipativity theory gives a frame work for the design analysis of control systems using an input-output description based on energy-related considerations. It brings great challenge to control theory since multiple time scales can cause problems of increased order and stiffness of systems.

A new notion of input-to-state stability involving infinity norms of input derivatives up to a finite order k is introduced and characterized.[7]. Additionally, the notion of input to state stability (ISS) qualitatively describes stability of the mapping from initial states and inputs to internal states and more generally outputs.[13].

This gives a brief definition of ISS and a discussion of equivalent characterizations [14].

Singular systems have been widely studied in the past two decades due to their extensive applications in modeling and control of electrical circuits, power systems, economics and other areas. A great number of fundamental results based on the theory of state-space systems have been successfully extended to singular systems [13]. Interest has grown recently in the stability analysis and control of singular systems with parameter uncertainties due to their frequent presence in dynamic systems, which are often the causes for instability and poor performance of control systems. It is known that the control of uncertain singular systems is much more complicated than that of state-space systems because controllers must be designed so that the closed-loop system is not only robustly stable, but also regular and impulse-free (in the continuous case) or causal (in the discrete case), while the latter two issues do not arise in the state-space case. Many recent advances in this area have made use of the matrix inequality machinery, which has become a somewhat standard approach to deriving appropriate mathematical conditions.

Therefore, a limitation for singularity is that the maximum stability for uncertainty bound has not been involved. Hence the Robust passivity and ISS for uncertain singular system will be involved under consideration of solving this problem.

4.1 PRELIMINARY CONCEPTS.

Since its inception in the 1980s, the concept of ISS has generated a rich body of results relating to stability properties of nonlinear systems with inputs. A succinct description of the area can be found in [9]. Here, we provide a brief overview, and, for simplicity of presentation, we restrict attention to single-input systems. ISS concerns stability-type questions pertaining to systems with input u, which, on the one hand, might be an exogenous disturbance/perturbation or, on the other hand, might be a control open to choice.

These systems are of the form

$$\dot{x}(t) = f\big(x(t), u(t)\big),\ x(0) = x^0 \qquad \text{(i)}$$

Where, x^0 is initial state and u is input

To define ISS and related properties; we exploit the following classes of comparison functions.

Theorem:- Let $V\ \Omega \rightarrow \mathbb{R} \geq 0$ be a continuous positive definite function, where $\Omega \subset \mathbb{R}^n$ is compact set containing the origin in its interior. Then there exist a class-K function $\alpha_{1,}\ \alpha_2$: $\mathbb{R} \geq 0 \rightarrow \mathbb{R} \geq 0$ such that $\alpha_1(|x| \leq V(x) \leq\ \alpha_2(|x|), \forall_x \epsilon \Omega$.

K *function is the set of continuous increasing function*.

Definition 1.4.1.A function $\gamma : R^+ \rightarrow R^+$ is a K function if it is continuous and increasing functions with $\gamma(0)\ =\ 0$. It is a K∞ function if it is a K function and satisfies

$\gamma(s) \rightarrow\ \infty$ as s→ ∞. A function $\beta :\ R^+\ \rightarrow\ R^+$ is a KL function if $\beta(.,\text{t})\ \in K$ For all $t\ \geq\ 0$ and the function β $(\alpha, .)$) is continuous and strictly decreasing to zero for all $\alpha > 0$.

Definition 1.4.2. System (i) is said to be Input-to-State Stable (ISS) if there exist functions $\gamma \epsilon K_\infty$ and $\beta\ \epsilon\ KL$ so that for all initial values x^0, all admissible inputs u and all times t ≥ 0 the following inequality holds

|x (t)|≤ $\beta(|x^0|, t)$+$\gamma(||u||_\infty)$. γ is a K_∞ function that provides a bound on the state function of a function y of the initial state.

The function γ in the above inequality is called the gain.

Lemma 1.4.1. System (i) is an ISS if there exist smooth Lyapunov function $V(x)$ and K_∞ Functions $k_i(.),\ i\ =\ 1, 2$ such that

$k_1(\|x\|) \leq V(x) \leq k_2(\|x\|)$ and

$\frac{\partial V}{\partial x} f(x, u) \leq -W(x), \forall \|x\| \geq \rho(\|u\|) > 0,$

Where $\rho \in K$ function, W is a continuous positive definite function.

Definition 1.4.3 Consider the following linear singular system

$$E\dot{x}(t) = Ax(t) + Bu(t) \quad (1)$$

Where, $x(t) \in R^n$ *is the state vector and* $u(t) \in R^m$ *is input vector.*

$E \in R^{nxn}$ *is a singular matrix,* $rank(E) = r(\leq n), A \in R^{nxn}$

are constant matrices.

Definition 1.4.4 The system (1) is called $regular$ if there exists a constant scalar $s \in \mathbb{C}$ and satisfied

$$\det(sE - A) \neq 0$$

In this case, we also say that the matrix pair (E, A) is $regular$.

Definition 1.4.5 (Kronecker indices)

A regular matrix pair (E, A) admits nonsingular matrices $P, Q \in R^{nxn}$ such that

$$PEQ = \begin{bmatrix} I_r & 0 \\ 0 & N \end{bmatrix}, PAQ = \begin{bmatrix} W & 0 \\ 0 & I_{n-r} \end{bmatrix} \quad (2)$$

Where $W \in R^{rxr}$ $for\ some\ r \leq n\ and\ N \in R^{(n-r)x(n-r)}$ is a Nilpotent matrix with index $k \leq n - r,\ that\ is, N^k = 0\ and\ ,\ N^{k-1} \neq 0$. The Nilpotent index k is called the $Kronecker\ indices$.

Lemma 1.4.2 The regular linear singular system (1) is $impulse - free\ iff$ one of the following condition holds:

1. The nilpotent matrix $N = 0$;

2. $deg\big(det(sE - A)\big) = rank(E)$

Theorem 4.1.1 [4] For the regular linear singular system (1), it is $impulsive$ if and only if the Kronecker indices k is more than 1.

Proof: The regular linear singular system (1) is impulsive, so the nilpotent matrix $N \neq 0$ of the Kronecker form (2) of system (1) based on the **Lemma 1.** Then $N^k = 0\ iff\ k > 1$.

For

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix}, E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$\det(sE - A) = -s^2 = 0 \; iff \; s = 0$. So the system

$E\dot{x}(t) = Ax(t) + Bu(t)$ is regular. The nonsingular matrices P and Q can be given.

$$P = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix}$$

So the Kronecker form is

$$PEQ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad PAQ = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Because $N \neq 0, N^2 = 0, that\ is\ the\ kronecker\ indices\ k = 2 > 1$, then (1) is impulsive based on **Theorem 4.1.1**

Definition 1.4.4. Given a neighborhood $N\ of\ x \in R^n\ a\ function\ W: R^n \to R$ is

- Positive definite if

  $W(0) = 0$

  $W(x) > 0\ for\ x \neq 0\ \in N$

- Positive semi-definite if

  $W(0) = 0$

  $W(x) \geq 0\ for\ x \in N$

- Negative definite if $-W(x)$ is pd
- Negative semi-definite if $-W(x)$ is psd.

Lemma 1.4.3 Given a symmetric matrix P

$$P\ is\ pd\ if\ and\ only\ if\ all\ its\ leading\ principal\ minors\ are\ positive.$$

Example

$QF(X) = x^2 - 2xy + 2y^2 = (x \;\; y)\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ is positive definite since its leading principal minors are $P_{11} = 1 \; and \; detP = 1$

Lemma 1.4.4 The pair (E, A) is said to be admissible if and only if there exists a matrix $P \in R^{n\times n}$ Such that

$$\begin{aligned} E^T P = \; P^T E \; &\geq 0, \\ A^T P + \; P^T A \; &< 0. \end{aligned} \qquad (3)$$

For the linear system $\dot{x} = Ax, \; x^0 = 0$ (4)

Consider the quadratic function

$$V(x) = X^T PX, \;\; P = P^T \qquad (5)$$

Then, $\frac{dV}{dt} = -X^T QX$

Proof

$$V(x) = X^T PX \;\; , \frac{dV}{dt} \; = \dot{X}^T PX + X^T P\dot{X}$$

$$= (AX)^T PX + X^T P(AX)$$

$$= X^T A^T PX + X^T PAX$$

$$= X^T (A^T P + PA)X$$

$= -X^T QX$, where $A^T P + PA = -Q$

The equation $A^T P + PA = -Q$ is known as $Lyapunov \; equation$.

Note that Q is forced to be symmetric.

Let's consider the case where we want to establish asymptotic stability of the fixed point $x^0 = 0$.

We require that

(i) $V = X^T PX$ be positive definite or equivalently that the matrix P be positive definite

(ii) $\frac{dV}{dt} = -X^T QX$ be negative definite or equivalently the matrix $-Q$ be negative definite, that is Q be positive definite.

This leads to the following theorem.

Theorem 4.1.2 (Linear asymptotic stability) [6]

The fixed point of the system of Eq(5) is Globally Asymptotically Stable if and only if, given any positive definite matrix Q, the solution P of Eq(5) is also positive definite.

Example: $\dot{x} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} x$

Since Q can be any Positive definite matrix, choose $Q = I$. Building in symmetry in P, the $lyapunov\ equation$ for this system becomes

$$\begin{pmatrix} 0 & -2 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} P_1 & P_2 \\ P_2 & P_3 \end{pmatrix} + \begin{pmatrix} P_1 & P_2 \\ P_2 & P_3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

Which is the same as

$$AP + PA^T = -Q$$

where A= $\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$, $Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Writing this as a system of linear equations

$-4P_2 = -1$ Position (1, 1) in matrix equation

$-2P_3 + P_1 - 3P_2 = 0$ Position (1, 2) in matrix equation

$P_1 - 3P_2 - 2P_3 = 0$ Position (2, 1) in matrix equation

$2P_2 - 6P_3 = -1$ Position (2, 2) in matrix equation

Note that equation in position (1, 2) is the same as that in position (2, 1) because of symmetricity of P.

The system has solution

$$P_1 = \frac{5}{4} P_2 = \frac{1}{4} P_3 = \frac{1}{4}$$

So we have, $P = \begin{pmatrix} \frac{5}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix}$

Which has leading principal minors $P_{11} = \frac{5}{4}$, and $detP = \frac{1}{4}$. Thus $x^0 = 0$ *is Globally Asymptotically Stable*.

Lemma 1.4.5 Let S be square matrix partitioned as

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{pmatrix}$$

$S_{11} \in R^{n\times n}$ And $S_{22} \in R^{m\times m}$ are symmetric matrices, then $S < 0$ if and only if $S_{11} < 0, S_{12} < 0$,

$$S_{11} - S_{12}S_{22}^{-1}S_{12}^T < 0,$$

$$S_{22} - S_{12}^T S_{22}^{-1} S_{12} < 0.$$

Consider the following dynamic system

$$\Sigma: \begin{cases} \dot{x} = f(x, w); \\ y = h(x, w). \end{cases} \tag{6}$$

Definition 1.4.5 The dynamical system Σ is said to be passive if there exists a continuous nonnegative function $V: R^n \to R$ called storage function with $V(0) = 0$ such that for $x^0 \in R^n$ And $w \in R^m$, the following dissipation inequality holds.

$$V\big(x(t)\big) - V\big(x(0)\big) \leq \int_0^t w^T(\tau)w(\tau)d\tau.$$

If the storage function $V(x)$ is differentiable, then the dissipation inequality can be written as

$$\dot{V} \leq w^T(t)y(t).$$

Lemma 1.4.6 A continuous non-negative function $S: R^n \to R$ is strictly passive if there exists a positive definite function $\Psi(x)$ such that

$$\dot{S} \leq w^T(t)y(t) - \Psi(x), \forall x \in R^n.$$

Consider the following nonlinear dynamical system

$$\dot{x} = f(x), x(t) \in R^2 \tag{7}$$

where $x \in R^{n\times n}$ is the state and $f(x)$ is function of x. For this system, a complete set of definitions of various kinds of stability can be found in different nonlinear systems. Some definitions are stated as follows:

Definition 1.4.6 A point $x_0 = \hat{x}$ is said to be the equilibrium point of

$$\dot{x} = f(x) \text{ if } f(x_0) = 0.$$

According to the definition, the equilibrium points of (7) are the real roots of the equation $f(\hat{x}) = 0$.This is made obvious by notifying that if $\dot{(\hat{x})} = f(\hat{x}) = 0$, then it follows that $\hat{x}$ is constant and by definition, an equilibrium point. Without loss of generality, we assume that 0 is an equilibrium point of the system because the system is considered as an autonomous.

Example:-

The system $\dot{x} = x(6 - 2x - y), \dot{y}$ = y(4-x-y) has four equilibrium points (0,0) , (3,0), (0,4) , and (2,2) by equating the equations to 0 and solving.

Definition 1.4.7 A solution $x(t)$ is said to be Stable if for any given $\epsilon > 0$, there exists a positive constant $\delta > 0$ (possibly depending on ϵ), such that $\|x_0\| < \delta$ implies $\|x\| < \epsilon$ where $x_0 = x(0)$ is the initial condition.

Definition 1.4.8 A solution $x(t)$ is asymptotically stable if there exists $\delta > 0$ such that $\|x_0\| < \delta(\epsilon)$ implies $\lim_{t\to\infty} x(t) = 0$.

From these definitions we see that if the system (7) is asymptotically stable, $x = 0$ must be the only stationary solution, at least locally that is,

$f(0) = 0$, the point $x = 0$ is called an equilibrium point.

The control problem is to design a state feedback control law u = u(x) such that the resulting closed-loop system

$$\dot{x} = f(x;\, u(x)) \tag{8}$$

is stable in terms of one of the preceding definitions. When measurements of full state x are not accessible, an input-output control can be designed. Input-output controls are either static or dynamic and under dynamic control, its closed-loop system has to be augmented to introduce the additional state variables in the control.

Definition 1.4.9 [5] A positive definite function V defined on R^n is said to be radially unbounded if the following condition hold.

$$\lim_{t\to\infty}\|x(t)\| = \infty.$$

In the Lyapunov stability theorems, the focus is on the function and its time derivative along the trajectory of the dynamical system under consideration.

Consider the nonlinear singular system

$$E\dot{x} = F(x,u) \quad (9)$$

$$Ex(0) = Ex_0 \in R^n, \quad (10)$$

Where state$x \in R^n$, control input $u \in R^m$.

Theorem 4.1.3: Let $\hat{x} = 0$ be equilibrium point of $\dot{x} = f(x)\ and\ V\ be\ a\ positive\ definite\ function\ on\ some\ neighborhood\ of\ 0.\ Then$

1. If $\dot{V} \leq 0\ for\ all\ x \in U, x \neq 0\ then\ \hat{x}$ is $lyapunov\ stable$;
2. If $\dot{V} < 0\ for\ all\ x \in U, x \neq 0\ then\ \hat{x}$ is $asymptotically\ stable$;
3. If $\dot{V} > 0\ for\ all\ x \in U, x \neq 0\ then\ \hat{x}$ is $unstable$;
4. If V is strict Lyapunov function on R^2, and $V(x) \to \infty\ for\ |x| \to \infty$ then $\hat{x}$ is $globally\ asymptotically\ stable$.

Example

$$\dot{x} = -x$$

$$\dot{y} = -y$$

And $V(x,y) = \frac{x^2+y^2}{2}$.

$V(x,y)$ is positive definite function on all R^2.

$\dot{V}(x,y) = -(x^2+y^2)$, which is a negative definite for all (x, y) $\neq 0$.

$Therefore, the\ equilibrium\ point\ (\hat{x},\hat{y}) = (0,0) is\ asymptotically\ stable.$

Moreover, since $\dot{V}(x,y) < 0$ for all R^2 and clearly $V(x,y) \to \infty$ as $\sqrt{x^2+y^2} \to \infty$, then $\hat{x}$ is $globally\ asymptotically\ stable$.

CHAPTER FOUR:- PRELIMINARY CONCEPTS AND MAIN RESULT

4.2 MAIN RESULT AND DISCUSSION

Positive Definite Matrices

All the eigenvalues of any symmetric matrix are real; this section is about the case in which the eigenvalues are positive.

Definition 4.2.1 Positive Definite Matrices

A square matrix is called positive definite if it is symmetric and all its eigenvalues λ are positive, that is $\lambda > 0$. Because these matrices are symmetric, the principal axes theorem plays a central role in the theory.

Theorem 4.2.1 If A is positive definite, then it is invertible and $det\ A > 0$.

Proof. If A is $n \times n$ and the eigenvalues are $\lambda 1, \lambda 2, \ldots, \lambda n$, then

$det\ A = \lambda_1 \lambda_2 \cdots \lambda_n > 0$ by the principal axes theorem .

If x is a column in R^n and A is any real $n \times n$ matrix, we view the 1×1 matrix $X^T AX$ as a real number. With this convention, we have the following characterization of positive definite matrices.

Theorem 4.2.2 A symmetric matrix A is positive definite if and only if $X^T AX > 0$ for every column $X \neq 0$ in R^n.

Proof. A is symmetric so, by the principal axes theorem, let

$P^T AP = D = diag\ (\lambda 1, \lambda 2, \ldots, \lambda n)$ where $P^{-1} = P^T$ and the λ_i are the eigenvalues of A. Given a column x in R^n , write $y = P^T x = [\ y_1\ y_2 \ldots y_n\ ]T$.

Then $$X^T AX = X^T (PDP^T)X = y^T Dy = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2 \qquad (11)$$

If A is positive definite and $x \neq 0$, then $X^T AX > 0$ by (11) because some $y_{j \neq 0}$ and every $\lambda_i > 0$. Conversely, if $X^T AX > 0$ whenever$x \neq 0$, let $x = Pe_j \neq 0$ where e_j is column j of I I_nn. Then $y = e_j$, so (11) reads $\lambda_j = X^T AX > 0$.

Note that Theorem 4.2..2 shows that the positive definite matrices are exactly the symmetric matrices A for which the quadratic form $q = X^T AX$ takes only positive values.

Example 4.2.1 If U is any invertible $n \times n$ matrix, show that $A = U^T U$ is positive definite.

Solution. If x is in $R^n and\ x \neq 0,$ then $X^T AX = X^T(U^T U)X = (UX)^T(UX) = ||UX||^2 > 0$ because $UX \neq 0$ (U is invertible). Hence Theorem 4.2.2 applies.

If A is any $n \times n$ matrix, let $A^{(r)}$denote the $r \times r$ submatrix in the upper left corner of A; that is, $A^{(r)}$ is the matrix obtained from A by deleting the last n−r rows and columns. The matrices $A^{(1)}, A^{(2)}, A^{(3)}, \ldots A^{(n)} = A$ are called the principal submatrices of A.

Example 4.2.2 If $A = \begin{vmatrix} 10 & 5 & 2 \\ 5 & 3 & 2 \\ 2 & 2 & 3 \end{vmatrix}$ then $A^{(1)} = [10]$, $A^{(2)} = \begin{bmatrix} 10 & 5 \\ 5 & 3 \end{bmatrix}, and\ A^{(3)} = A$

Lemma 4.2.1

If A is positive definite, so is each principal submatrix $A^{(r)}$ for$r = 1, 2, \ldots, n.$

Proof.

Write $A = \begin{bmatrix} A^{(r)} & P \\ Q & R \end{bmatrix}$ in block form. If $y \neq 0$ in R^r , write $X = \begin{bmatrix} y \\ 0 \end{bmatrix}$ in R^n .

Then $X \neq 0$,, so the fact that A is positive definite gives

$$0 < X^T AX = [y^T \ \ 0] \begin{bmatrix} A^{(r)} & P \\ Q & R \end{bmatrix} \begin{bmatrix} y \\ 0 \end{bmatrix} = y^T(A^{(r)})y.$$

Theorem 4.2.3 [17] The following conditions are equivalent for a symmetric $n \times n$ matrix A:

1. A is positive definite.
2. $det(A^{(r)}) > 0$ for each $r = 1, 2, \ldots, n.$
3. $A = U^T U$ where U is an upper triangular matrix with positive entries on the main diagonal.

Furthermore, the factorization in (3) is unique (called the **Cholesky factorization of A**).

Proof.

First, (3) ⇒ (1) by Example 4.2.1, and (1) ⇒ (2) by Lemma 4.2.1 and Theorem 4.2.1.

(2) ⇒ (3). Assume (2) and proceed by induction on n. If $n = 1$, then $A = [a]$ where $a > 0$ by (2), so take $U = [\sqrt{a}]$. If $n > 1$, write $B = (A^{(n-1)}$. Then B is symmetric and satisfies (2) so,

by induction, we have $B = U^T U$ as in (3), where U is of size $(n-1)\times(n-1)$. Then, as A is symmetric, it has block form $A = \begin{bmatrix} B & P \\ P^T & b \end{bmatrix}$ where P is a column in R^{n-1} and b is in R. If we write $X = (U^T)^{-1} P$ and

$c = b - X^T X$, Block multiplication gives $A = \begin{bmatrix} U^T U & P \\ P^T & b \end{bmatrix} = \begin{bmatrix} U^T & 0 \\ X^T & 1 \end{bmatrix} \begin{bmatrix} U & X \\ 0 & C \end{bmatrix}$.

Taking determinants gives $\det A = \det(U^T) \det U \cdot$

$c = c(detU)^2$. Hence $c > 0$ because $det\, A > 0$ by (2), so the above factorization can be written as

$$A = \begin{bmatrix} U^T & 0 \\ X^T & \sqrt{c} \end{bmatrix} \begin{bmatrix} U & X \\ 0 & \sqrt{c} \end{bmatrix}$$

Since U has positive diagonal entries, this proves (3).

As to the uniqueness, suppose that $A = U^T U = {U_1}^T U_1$ are two Cholesky factorizations.

Now write $D = UU^{-1} = (U^T)^{-1} {U_1}^T$. Then D is upper triangular, because $D = U{U_1}^{-1}$, and lower triangular, because $D = (U^T)^{-1}{U_1}^T$, and so it is a diagonal matrix. Thus $U = DU_1$ and $U_1 = DU$, so it suffices to show that $D = I$. But eliminating U_1 gives, $U = D^2 U$, so $D^2 = I$ because U is invertible. Since the diagonal entries of D are positive (this is true of U and U_1), it follows that $D = I$.

Algorithm for the Cholesky Factorization [17]

If A is a positive definite matrix, the Cholesky factorization $A = U^T U$ can be obtained as follows:

Step 1. Carry A to an upper triangular matrix U_1 with positive diagonal entries using row operations each of which adds a multiple of a row to a lower row.

Step 2. Obtain U from U_1 by dividing each row of U_1 by the square root of the diagonal entry in that row.

Example 4.2.3 Find the Cholesky factorization of $A = \begin{bmatrix} 10 & 5 & 2 \\ 5 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix}$

Solution. The matrix A is positive definite by Theorem 4.2.3 because $det\,(A)^1 = 10 > 0$, $det\,(A)^2 = 5 > 0$, and $det\,(A)^3 = det\,A = 3 > 0$. Hence Step 1 of the algorithm is carried out as follows:

$$A = \begin{bmatrix} 10 & 5 & 2 \\ 5 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 10 & 5 & 2 \\ 0 & \frac{1}{2} & 1 \\ 0 & 1 & \frac{13}{5} \end{bmatrix} \to \begin{bmatrix} 10 & 5 & 2 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & \frac{3}{5} \end{bmatrix} = U_1$$

Now carry out step 2 on U_1 to obtain $U = \begin{bmatrix} \sqrt{10} & \frac{5}{\sqrt{10}} & \frac{2}{\sqrt{10}} \\ 0 & \frac{1}{\sqrt{2}} & \sqrt{2} \\ 0 & 0 & \frac{\sqrt{3}}{\sqrt{5}} \end{bmatrix}$

We can verify that $U^T U = A$.

Consider the continuous linear system with uncertainties.

$$\begin{cases} E\dot{x}(t) = (A + \Delta A)x(t) + (B + \Delta B)w(t); \\ y(t) = (C + \Delta C)x(t) + (D + \Delta D)w(t) \end{cases} \quad (12)$$

where $x \in R^n$ is the system state, $y \in R^m$ is the controlled output, $w \in R^m$ is disturbance input the matrix $E \in R^{n\times n}$ may be singular, we shall assume that $0 < rank(E) = r \leq n.$ $A,\ B,$ $C\ and\ D$ are known real constant matrices of appropriate dimensions, $\Delta A,\ \Delta B,\ \Delta C\ and\ \Delta D$ are unknown time invariant matrices representing norm-bounded parameter uncertainties.

Theorem 4.2.4 Banach Fixed Point Theorem

Banach Fixed Point Theorem (also known as the contraction mapping theorem) is an important tool in the theory of metric spaces; it guarantees the existence and uniqueness of fixed points of certain self-maps of metric spaces, and provides a constructive method to find those fixed points.

Definition 4.2.1 Let (X, d) be complete metric space. Then a map **T: X → X** is called a contraction mapping on X if there exists q ∈ [0, 1) such that d (T(x), T(y)) ≤q d(x, y) for all X, y ∈ X.

Proof: **Step 1. Existence and Uniqueness of solutions.**

In this step, we show the system in (12) has unique solutions. From remark 4.1, we can look that the system in (12) is regular, thus by lemma 1.4.3 there exist two non-singular matrices $M, N \in R^{n\times n}$ such that

$$MEN = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}, MAN = \begin{pmatrix} A_1 & 0 \\ 0 & I_{n-r} \end{pmatrix},$$

$$M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}, \qquad N = (N_1 N_2) \quad (13)$$

$$M^{-T}PN = \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix} \tag{14}$$

where, $A_1 \in R^{r\times r}$, $M_2 \in R^{n-r\times r}$, $N_1 \in R^{n\times r}$ and $N_2 \in R^{r\times n-r}$

Considering the Schur's complement lemma which states for Block matrix

$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, then A is singular if $A_s = A_{22} - A_{21}A_{12}$. LMI implies that

$$\begin{pmatrix} A^TP + P^TA + Q + \alpha\epsilon GG^T & P^TH^T \\ HP & -\epsilon I_N \end{pmatrix} < 0$$

Since from the hypothesis of theorem 4.1 $Q > 0$, this inequality implies

$$\begin{pmatrix} A^TP + P^TA + \alpha\epsilon GG^T & P^TH^T \\ HP & -\epsilon I_n \end{pmatrix} < 0 \tag{15}$$

By Schur's complement lemma inequality is equivalent to

$$A^TP + P^TA + \alpha\epsilon GG^T + \epsilon^{-1}P^TH^THP < 0 \tag{16}$$

Pre and post multiplying inequality (16) by N^T and transpose yields

$$(MAN)^T(M^{-T}PN) + (M^{-T}PN)^TMAN + \alpha\epsilon N^TGG^T +$$
$$\epsilon^{-1}(M^{-T}PN)^TMH^THM^T(M^{-T}PN) < 0 \tag{17}$$

Using the expressions in (13,14) and carrying some mathematical calculations, we have that the block matrix at the second block row and the second block column of the left hand side of (17) is negative definite. That is

$$P_4^T + P_4 + \alpha\epsilon N_2^TGG^TN_2 + \epsilon^{-1}P_4^TM_2H^THM_2^TP < 0 \tag{18}$$

Noticing the semi-positive definiteness of $M_2H^THM_2^T$, we have that

$$M_2H^THM_2^T + \eta I > 0.$$

Let

$$R_\eta = (M_2H^THM_2^T + \eta I)^{-\frac{1}{2}}$$

$$T = \begin{pmatrix} I & 0 \\ 0 & R_\eta \end{pmatrix},$$

$$TMENT^{-1} = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

$$TMANT^{-1} = \begin{pmatrix} A_1 & 0 \\ 0 & I_{n-r} \end{pmatrix} \text{And}$$

$$\begin{aligned} R_\eta M_2H^THM_2^TR_\eta &= \Phi \\ &< \Psi + \eta I\,(M_2H^THM_2^T + \eta I)^{-1/2} \\ &= I_{n-r} \end{aligned} \tag{19}$$

where,

$$\Phi = (M_2H^THM_2^T + \eta I)^{-1/2}M_2H^THM_2^T(M_2H^THM_2^T + \eta I)^{-1/2}$$
$$\Psi = (M_2H^THM_2^T + \eta I)^{-1/2}M_2H^THM_2^T$$

That is

$$R_\eta M_2H^THM_2^TR_\eta < I_{n-r} ,$$

This implies that

$$\|R_\eta M_2H\| < 1, \quad \forall\eta > 0 .$$

Inequality (18) implies that for a sufficiently small $\eta > 0$,

$$P_4^T + P_4 + \alpha\epsilon N_2^TGG^TN_2 + \epsilon^{-1}P_4^T(MH^THM_2^T + \eta I)P_4 < 0 . \quad (20)$$

by using the expression in (19) inequality (20) is equivalent to

$$P_4^T + P_4 + \alpha\epsilon N_2^TGG^TN_2 + \epsilon^{-1}P_4^TR_\eta^{-2}P_4 < 0$$

This is equivalent to

$$\left(\frac{1}{\sqrt{\epsilon}}R_\eta^{-1}P_4 + \sqrt{\epsilon}R_\eta\right)^T\left(\frac{1}{\sqrt{\epsilon}}R_\eta^{-1}P_4 + \sqrt{\epsilon}R_\eta\right) - \epsilon R_\eta^2 + \alpha\epsilon GG^TN_2 < 0,$$

This implies

$$N_2^TGG^TN_2 < \frac{R_\eta^2}{\alpha}.$$

Therefore, there exist a sufficiently small positive number $\mu > 0$ such that

$$\|GN_2R_\eta^{-1}\| < \frac{1}{\sqrt{\alpha(1+\mu)}} .$$

In order to show the existence and uniqueness of the solution, we introduce the change of coordinates to decompose the system in (12) into differential and algebraic sub systems. To do so, Let

$\overline{N}^{-1}x = \bar{x}$, where $\overline{N} = NT^{-1}$, $\overline{M} = TM$, and $\bar{x} = \begin{pmatrix}\bar{x}_1\\ \bar{x}_2\end{pmatrix}$, $\bar{x}_1 \in R^r$, $\bar{x}_2 \in R^{n-r}$ then the system (12) is equivalent to

$$\bar{x}_1 = A_1\bar{x}_1 + M_1GF(t)H(N_1\bar{x}_1 + N_2R_\eta^{-1}\bar{x}_2 + M_1(B + GF(t)Hw(t). \quad (21)$$

$$0 = \bar{x}_2 + R_\eta M_2GF(t)H\left(N_1\bar{x}_1 + N_2R_\eta^{-1}\bar{x}_2\right) + R_\eta M_2(B + GF(t)H)w(t). \quad (22)$$

Therefore, for any $\bar{x}_2, \bar{\bar{x}}_2 \in R^{n-r}$, we have

$$\begin{aligned}&\left\|R_\eta M_2GF(t)H\left(N_1\bar{x}_1 + N_2R_\eta^{-1}\bar{x}_2\right) - R_\eta M_2GF(t)H\left(N_1\bar{x}_1 + N_2R_\eta^{-1}\bar{\bar{x}}_2\right)\right\| \\ &\le \left\|R_\eta M_2GF(t)HN_2R_\eta^{-1}(\bar{x}_2 - \bar{\bar{x}}_2)\right\| \\ &\le \sqrt{\alpha}\left\|R_\eta M_2\right\|\left\|GN_2R_\eta^{-1}\right\|\|\bar{x}_2 - \bar{\bar{x}}_2\|\end{aligned}$$

$$\leq \frac{1}{\sqrt{1+\mu}}\|\bar{x}_2 - \bar{\bar{x}}_2\| .$$

Hence, by Banach fixed point theorem there exist a unique solution $\bar{x}_2 = \phi(\bar{x}_1)$ for any given $\bar{x}_1$. Similarly, $\bar{x}_1$ exists and is unique. To show using Picard-Lindelof theorem for existence and uniqueness of solutions, we need to show $\bar{x}_2 = \phi(\bar{x}_1)$ is Lipchitz with respect to $\bar{x}_1$. In fact for any given $\bar{x}_1$ and $\bar{\bar{x}}_1$

$$\|\phi(\bar{x}_1) - \phi(\bar{\bar{x}}_1)\| \leq \sqrt{\alpha}\|GN_1\|\|\bar{x}_1 - \bar{\bar{x}}_1\| + \frac{1}{\sqrt{1+\mu}}\|\phi(\bar{x}_1) - \phi(\bar{\bar{x}}_1)\|$$

$$\leq \sqrt{\alpha}\|GN_1\| + \frac{1}{\sqrt{1+\mu}}\|\phi(\bar{x}_1) - \phi(\bar{\bar{x}}_1)\|$$

$$\leq \frac{\sqrt{\alpha}}{\mu}\left(\sqrt{1+\mu} + 1\right)\left(\sqrt{1+\mu}\right)\|GN_1\|\|\bar{x}_1 - \bar{\bar{x}}_1\|.$$

Which implies that $\phi(\bar{x}_1)$ is Lipchitz with respect to $\bar{x}_1$. Therefore, the solution $\bar{x}_1$ for differential equation (21) exists and unique for any compatible initial condition.

Remark 4.3. From the proof, we can realize that the regularity of the system guarantees the existence and uniqueness of solutions. Noticing equation (22), we have

$$\bar{x}_2 = -R_\eta M_2 GF(t)H\left(N_1\bar{x}_1 + N_2R_\eta^{-1}\bar{x}_2\right) - R_\eta M_2(B + GF(t)H)w(t)$$

$$\|\bar{x}_2\| \leq \left\|R_\eta M_2 GF(t)H\left(N_1\bar{x}_1 + N_2R_\eta^{-1}\bar{x}_2\right)\right\| + \left\|R_\eta M_2(B + GF(t)H)w(t)\right\|$$

$$\leq \frac{(\sqrt{1+\mu}+1)(\sqrt{1+\mu})}{\mu}\left\{\sqrt{\alpha}\|GN_1\|\|\bar{x}_1\| + \left(\left\|R_\eta M_2 B\right\| + \sqrt{\alpha}\|G\|\right)\|w\|\right\}$$

This implies that

$$\|\bar{x}_2\| \leq \sigma_1\|\bar{x}_1\| + \sigma_2\|w\| \tag{23}$$

where

$$\sigma_1 = \frac{(\sqrt{1+\mu}+1)(\sqrt{1+\mu})}{\mu}\left\{\sqrt{\alpha}\|GN_1\|\right\}$$

$$\sigma_2 = \frac{(\sqrt{1+\mu}+1)(\sqrt{1+\mu})}{\mu}\sigma_3\|w\|$$

$$\sigma_3 = \left\|R_\eta M_2 B\right\| + \sqrt{\alpha}\|G\|$$

In the next steps we will proof the input-to-state stability (ISS) of the system with respect to the dynamic and static parts and the whole system.

Step 2. ISS with respect to the dynamic $\bar{x}_1$.

Using the definition of ISS of the normal state-space systems, we will show the ISS of $\bar{x}_1$ at this step. To show that the uncertain continuous singular system in (12) robust strict passivity and ISS, we notice that from

$$E^TP = P^TE \geq 0$$

And the robust stability condition implies that

$$P_1 = P_1^T > 0, \; P_2 = 0 \tag{24}$$

Let

$$V(\bar{x}_1) = \frac{\bar{x}_1^T P_1 \bar{x}_1}{2} = \frac{x^T E^T P x}{2},$$

Then from (15) we can deduce that

$$\dot{V}(\bar{x}_1) \leq -\frac{x^TQx}{2} + w^Ty$$

$$\leq -\frac{x^TQx}{2} + w^T[(C + \Delta C)x + (D + \Delta D)w]$$

$$\leq -\lambda_1\|x\|^T + \|w\|(\|CN_1\| + \|GF(t)HN_1\|)\|\bar{x}_2\| + \left(\left\|CN_2R_\eta^{-1}\right\| + \left\|GFHN_2R_\eta^{-}\right\|\right)$$

$$+ \|\bar{x}_2\| + (\|D\| + \|GFH\|\|w\|)$$

$$\leq -\lambda_1\|x\|^T + \|w\|\left(\|CN_1\| + \sqrt{\alpha}\|GHN_1\|\right)\|\bar{x}_1\| + \left(\left\|CN_2R_\eta^{-}\right\| + \frac{\|H\|}{\sqrt{\mu+1}}\right)\|\bar{x}_2\|$$

$$+ \left(\|D\| + \sqrt{\alpha}\|GH\|\right)\|w\|$$

$$\leq -\lambda_1\lambda_2\|\bar{x}_1\|^2 + \sigma_3\|w\|\|\bar{x}_1\| + \sigma_4\|w\|\|\bar{x}_2\| + \beta_1\|w\|^2$$

$$\leq -\lambda_1\lambda_2\|\bar{x}_1\|^2 + \sigma_5\|w\|\|\bar{x}_1\| + \beta_2\|w\|^2$$

$$\leq -\lambda_3\|\bar{x}_1\|^2 + \sigma_5\|w\|\|\bar{x}_1\| + \beta_2\|w\|^2$$

$$\leq -\lambda_3(1-\theta)\|\bar{x}_1\|^2 - \lambda_3\theta\|\bar{x}_1\|^2 + \sigma_5\|w\|\|\bar{x}_1\| + \beta_2\|w\|^2$$

$$\leq -\lambda_3(1-\theta)\|\bar{x}_1\|^2, \; \forall\|\bar{x}_1\| \geq \frac{\sigma_5 + \sqrt{\sigma_5^2 + 4\lambda_3\theta\beta_2}}{2\lambda_3\theta}\|w\|.$$

where, $\lambda_1 = \frac{1}{2} min\lambda\{Q\} > 0$, $\lambda_2 = min\lambda\{\bar{N}^T\bar{N}\} > 0$, $\lambda_3 = \lambda_1\lambda_2 > 0$,

$\sigma_3 = \|CN_1\| + \sqrt{\alpha}\|GH\|$, $\sigma_4 = \left\|CN_2R_\eta^{-1}\right\| + \frac{\|H\|}{\sqrt{\mu+1}}$, $\beta_1 = \|D\| + \sqrt{\alpha}\|GH\|$,

$\sigma_5 = \sigma_3 + \sigma_4\sigma_1$ and $\beta_2 = \sigma_4\sigma_1 + \beta_1$, $0 < \theta < 1$.

Therefore, by lemma 1.4.1 there exists a class KL function ξ_1 and a class K function γ_1 such that

$$\|\bar{x}_1(\bar{x}_{10}, w)\| \leq \xi_1(\|\bar{x}_{10}\|, t) + \gamma_1\left(\sup_{0 \leq \tau \leq t}\|w(\tau)\|\right) \tag{25}$$

Hence the dynamic system is ISS.

Step 3. ISS with respect to $\bar{x}_2$

In this step, we are going to show the ISS with respect to the static $\bar{x}_2$ by that $\bar{x}_1$, by considering the algebraic relation between them. Taking in account (23) and (25) in to consideration similarly there exist a class KL function ξ_2 a class K function γ_2 such that

$$\|\bar{x}_2(\bar{x}_{10}, w)\| \ \leq \ \xi_2(\|\bar{x}_{10}\|, t) \ + \ \gamma_2\left(\underset{0 \leq \tau \leq t}{Sup} \|w(t)\|\right) \tag{26}$$

Step 4. ISS of the original with uncertainty

We will next show that the ISS of system (11) by the trajectory boundedness of $\bar{x}_1$and $\bar{x}_2$.

Note that

$$\bar{M}E\bar{N}\bar{x}_0 \ = \ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \bar{x}_{10} \\ \bar{x}_{20} \end{pmatrix} \ = \ \begin{pmatrix} \bar{x}_{10} \\ 0 \end{pmatrix}$$

This implies that

$$\|\bar{x}_{10}\| \ \leq \ \|\bar{M}\|\|Ex_0\|$$

$$\|x\| \ = \ \|\bar{N}\bar{x}\| \ \leq \ \|\bar{N}\|\|\bar{x}_1 + \bar{x}_2\| \ \leq \ \|\bar{N}\|\|\bar{x}_1\| \ + \ \|\bar{N}\|\|\bar{x}_2\| \ .$$

Therefore, there exist a class KL function ξ and a class K function τ such that

$$\|x(t, Ex_0, w)\| \ \leq \ \xi(\|Ex_0\|, t) + \ \gamma(Sup_{0\leq\tau\leq t}\|w(\tau)\|).$$

Hence, system (12) is robust strict passivity and ISS with respect to the exogenous input. This completes the proof.

CHAPTER FIVE

SUMMARY, CONCLUSION AND FUTURE WORK

5.1. Summary

In chapter one, an introduction has been stated about the background of the robust strict passivity and input-to-state stability (ISS) for uncertain singular systems and its application. Some features of singular system from normal state systems and previous work on the robust stability of dynamic systems and singular systems are considered and discussed for motivation of this thesis.

In addition, some basic mathematical notions such as definitions, theorems and lemmas have been given as a preliminary concept which will helps us for the study and analysis. On the other hand, under this chapter the objectives of the study, statement of the problem, significance of the study, methodology of the study and question of the study are presented.
In chapter three, We have discussed the related review of literature concerning the robust strict passivity and input-to-state stability for uncertain singular systems.

In chapter four, we have discussed the robust strict passivity and ISS for uncertain singular systems analysis and stabilization for a class of singular systems with uncertainty. To get the main result, LMI condition is proposed such that the resulting systems has been derived and proved to be ISS with respect to the input. It has been seen that when the uncertainty has been zero or tend to zero as time goes to infinity the system has also been robust strictly passive and ISS with respect to the exogenous input. Furthermore, positive definite matrix has been designed the case in which the eigenvalues are positive.
In chapter five, we have summarized all of our work and discussed what we got the main result in the thesis.

5.2 Conclusion

It is fact that some systems are defined by both static and dynamic states. From these systems, singular systems are dynamic systems characterized by the combination of algebraic and differential equations. Singular systems arise naturally in dynamic models in a wide range of engineering applications and fundamentally different from state space systems.

In this thesis we have studied the robust strict passivity and input-to-state stability (ISS) for uncertain singular systems analysis stabilization problem for singular systems with uncertainties. We present robust strict passivity and input- to- state stability (ISS) analysis and control for a class of singular systems with uncertainties. By proposing proper linear matrix inequality (LMI) condition, the robust strict passivity and input-to-state stability (ISS) of the system has been derived and proved. In addition, a design method of state feedback stabilization has been given such that the closed-loop system is robust strict passivity and input-to-state stability (ISS) if the proposed linear matrix inequality (LMI) condition is not satisfied.

5.3. Future work.

The robust strict passivity and ISS for uncertain singular systems study is not completed so far. Since the application of this problem is important in our daily life. Interested person can study on this problem. Hence,

1. The robust strict passive, ISS analysis and control for uncertain singular systems under the assumption of impulse free and regularity of the singular systems but there are also singular systems with impulsive modes and no regulars which need further investigations to investigate the robust strict passive, ISS control for uncertain singular systems.
2. In studying the robust strict passive and ISS analysis and control for uncertain continuous singular systems using LMI, It may be needed also to study the robust passivity and ISS analysis and control for discrete time uncertain singulars.

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