ADDIS ABABA UNIVERSITY

DEPARTMENT OF MATHEMATICS

GLOBAL SOLUTION THEOREMS FOR ODEs

A Thesis Submitted to the Department of Mathematics of Addis Ababa University in Partial Fulfillment of the requirements of the Master of Science Degree in Mathematics

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We, the under signed, hereby certify that we have read and examined this thesis, a thesis Global Solution Theorems for ODEs which is done by TUBA NEGESSO in partial fulfillment of requirement for the degree of master of science in Mathematics and recommend to the school of graduate studies for acceptance of thesis.

Advisor: Dr Tadesa Abdi Signature------------------------ Date---------------------

Examiners:

1. Dr.______________________ Signature------------------------- Date-------------------
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Abstract

This thesis explores the globalization of the implicit function theorem (IFT) within the context of global solution theorems for ordinary differential equations (ODEs). Traditionally, the IFT provides powerful local results, but its global applicability has been less thoroughly examined. By extending the IFT to a global setting, this work develops new theoretical frameworks for understanding and solving ODEs on a broader scale. The study introduces generalized formulations of the IFT, derives significant global bifurcation results, and applies topological methods such as the Leray-Schauder degree to substantiate these results. Emphasis is placed on deriving conditions under which global solutions can be effectively analyzed and obtained.

Table of Contents

Chapter 1

Introduction

1. Background and Motivation

1.1 Importance of Global Solutions to ODEs

Global solutions to ordinary differential equations (ODEs) are crucial for understanding the comprehensive behavior of dynamical systems over extended domains, beyond local neighborhoods. While local solutions provide insights into the behavior of solutions near specific points, global solutions offer a complete picture, revealing the overall dynamics, stability, and bifurcation phenomena that may emerge over larger intervals. The globalization of the implicit function theorem (IFT) enhances this by extending its applicability from local to global contexts, thus facilitating the analysis of global properties such as persistence, bifurcations, and qualitative behavior of solutions across the entire domain. This extension is vital for addressing complex systems where interactions and global constraints significantly impact the system's dynamics, leading to a more robust understanding and potentially new insights into system behavior and control.

The implicit function theorem (IFT) is a fundamental result in differential calculus that provides conditions under which a relation defines a function implicitly. Classically, the IFT guarantees that if a system of equations is sufficiently smooth and certain conditions are met (such as the Jacobian being invertible at a point), then locally around that point, one can solve for some variables as functions of others. This local solvability is crucial for analyzing the behavior of systems near specific points. In the context of ordinary differential equations (ODEs), the IFT is used to determine local existence and uniqueness of solutions and to analyze the stability and behavior of equilibria. For example, in studying dynamical systems, the IFT helps in understanding how small perturbations affect solutions and in characterizing local bifurcations. Globalizing the IFT extends these local results to a broader setting, allowing for the examination of solutions and bifurcations across entire domains rather than just local neighborhoods. This expansion is particularly useful in analyzing complex systems where global behavior, such as global bifurcations and stability, plays a significant role. By applying the IFT globally, one can derive comprehensive results about the existence and behavior of solutions over larger intervals, providing a more complete understanding of the syst

1.2 Objective

1.2.1 General Objective

The goal is to derive comprehensive global bifurcation results, understand the persistence of solutions across entire domains, and apply topological methods such as the Leray-Schauder degree to substantiate these findings.

1.2.2 Specific objectives

To globalize the Implicit Function Theorem for ODEs

To derive Global Bifurcation Results

To apply Topological Methods Such as the

Leray-Schauder.

Chapter 2

Preliminary

2.1 Implicit Function Theorem (IFT).

In the preliminary section of my thesis on global solution theorems for ODEs, it is important to outline the classical local form of the implicit function theorem (IFT) to establish a foundation for understanding its extension to global contexts:

Define $A = \frac{\partial F}{\partial y}$.

Since A is an mxm matrix and is invertible, A^{-1}exists.

By the Inverse Function Theorem, since $A = \frac{\partial F}{\partial y}$ is invertible, we can find a neighborhood U of x_0 and a neighborhood V of y_0 such that F (x, y) can be locally expressed in terms of x and y in these neighborhoods.

Define F (x, y) in the neighborhood around $(x_0, y_{0)}$.

Let's consider the Taylor expansion of F around $(x_0, y_0\ )$For $(x\ , y\ )$close to(x_0, y_0), we have:

$$F(x\ , y)=F(x_0, y_{0)}+\frac{\partial F}{\partial x}(x_0, y_0)(x - x_0)+\frac{\partial F}{\partial y}(x_0, y_0)(y - y_0)+\text{higher-order terms.}$$

To solve for y in terms of x, we isolate $y - y_0$ and focus on the linear approximation and the invertibility of $\frac{\partial F}{\partial x}(x_0, y_0)$.

We can write:

$$F((x\ , y\ ) \approx F(x_0, y_{0)}+\frac{\partial F}{\partial x}(x_0, y_0)(x - x_0)+\frac{\partial F}{\partial y}(x_0, y_0)(y - y_0).$$

Then by rearranging for y, we get:

$$\frac{\partial F}{\partial y}(x_0, y_0)(y - y_0) \approx \frac{\partial F}{\partial x}(x_0, y_0)(x - x_0).$$

Therefore, locally, we can define:

$$g(x) = y_0 - \frac{\partial F}{\partial y}(x_0, y_0)^{\ -1}\frac{\partial F}{\partial x}(x_0, y_0)(x - x_0).$$

Substitute g(x) back into $F(x, y)$

$$F\big(x, g(x)\big) = F(x, y_0) - \frac{\partial F}{\partial y}(x_0, y_0)^{\ -1}\frac{\partial F}{\partial x}(x_0, y_0)x - x_0).$$

By the construction and approximation, this should satisfy F(x, g(x)) ≈ 0, verifying that g(x) implicitly defines y as a function of x locally around (x_0, y_0).

4Example 2.1: Consider the implicit equation:

$$F(x, y) = x^2 + y^2 - 1 = 0.$$

Solution: Let's choose the

$$(x_0, y_0) = (0,1).$$

At this point:

$$F(0,1) = 0^2 + 1^2 - 1 = 0$$

So, $(0,1)$ satisfies the equation

$$\frac{\partial F}{\partial y} = 2y$$

$\frac{\partial F}{\partial y}$ At (0,1):

$$\frac{\partial F}{\partial y}(0,1) = 2 \times 1 = 2.$$

which is non-zero. So, by Theorem 3.1 in chapter 3, when we express y as a function of x, we have:

$$x^2 + y^2 - 1 = 0 \Rightarrow y^2 = 1 - x^2$$

$$\Rightarrow y = \pm\sqrt{1 - x^2}$$

Since, near (0,1) we select the positive branch because $y = 1$ is positive.

Hence, near the point (0,1) the function y can be implicitly defined as:

$$y = \sqrt{1 - x^2}$$

Example 2.2: Consider the implicit equation:

$$F(x, y, z) = x^2 + y^2 + z^2 - 1 = 0.$$

The equation $x^2 + y^2 + z^2 - 1 = 0$, represents a sphere of radius 1 centered at the origin in R^3

The partial derivatives of the given implicit equation is:

- $F_x(x, y, z) = \frac{\partial F}{\partial x} = 2x$
- $F_y(x, y, z) = \frac{\partial F}{\partial y} = 2y$
- $F_z(x, y, z) = \frac{\partial F}{\partial z} = 2z$

Next, we solve for z in terms of x and y.

- For example, let's consider the point $(x_o, y_0, z_0) = (0,0,1)$

At this point:

- $F_x(0,0,1) = 2.0 = 0$
- $F_y(0,0,1) = 2.0 = 0$

- $F_z(0,0,1) = 2.1 = 2$

Here, $F_z \neq 0$.

This means that we can solve for z as a function of x and y in a neighborhood around this point using the Implicit Function Theorem.

Near the point (0,0,1),

we can solve:

$$x^2 + y^2 + z^2 - 1 = 0 \; for \; z$$

$$z^2 = 1 - x^2 - y^2$$

Thus, $z = \pm 1 - x^2 - y^2$

So locally around (0,0,1) we can write:

$$z = 1 - x^2 - y^2$$

This function represents the upper part of the sphere. The other branch

$$z = 1 - x^2 - y^2$$

corresponds to the lower part of the sphere. The Implicit Function Theorem confirms that locally around (0,0,1), z can be expressed as a function of x and y because $F_Z \neq 0$.

Specifically, the function $z = 1 - x^2 - y^2$

describes the upper hemisphere of the sphere in a local neighborhood around (0,0,1).

2.2. *Bifurcation theory*

Bifurcation theory is the mathematical study of changes in the qualitative or topological structure of a given family of curves, such as the integral curves of a family of vector fields, and the solutions of a family of differential equations. Most commonly applied to the mathematical study of dynamical systems, a bifurcation occurs when a small smooth change made to the parameter values (the bifurcation parameters) of a system causes a sudden 'qualitative' or topological change in its behavior. [1] Bifurcations occur in both continuous systems (described by ordinary, delay or partial differential equations) and discrete systems (described by maps)

Definition 2.1. If the phase portrait of Eq. (A[μ]) undergoes a structural change at $\mu = \mu_0$, we say that a bifurcation occurs at $\mu = \mu_0$, and $\mu = \mu_0$ is called a bifurcation value of Eq. (A[μ]). Such structural changes include

(i) a change of the number of equilibria,

(ii) a change of the type or stability of any equilibrium,

(iii) a change of the number of closed orbits,

(iv) a change of the orbital stability of any closed orbit;

and more.

To explain the bifurcation phenomena, we look at the examples below.

Example 2.3. Consider the equation

(2.2.1) ………………… $x = f(x, \mu) := \mu - x^k, k \in N$.

.

(a) *k* is an odd number.

In this case, Eq. (2.2.1) has exactly one equilibrium $x_0(\mu) = \sqrt[k]{\mu}$ for each $\mu \in R$. Moreover, it is easy to see that $f(x, \mu) > 0$ for $x < x_0(\mu)$ and $f(x, \mu) < 0$ for $x > x_0(\mu)$. Therefore, $x_0(\mu)$ is asymptotically stable for every$\mu \in$ R. This means that no bifurcation.

See Fig.2.2.1

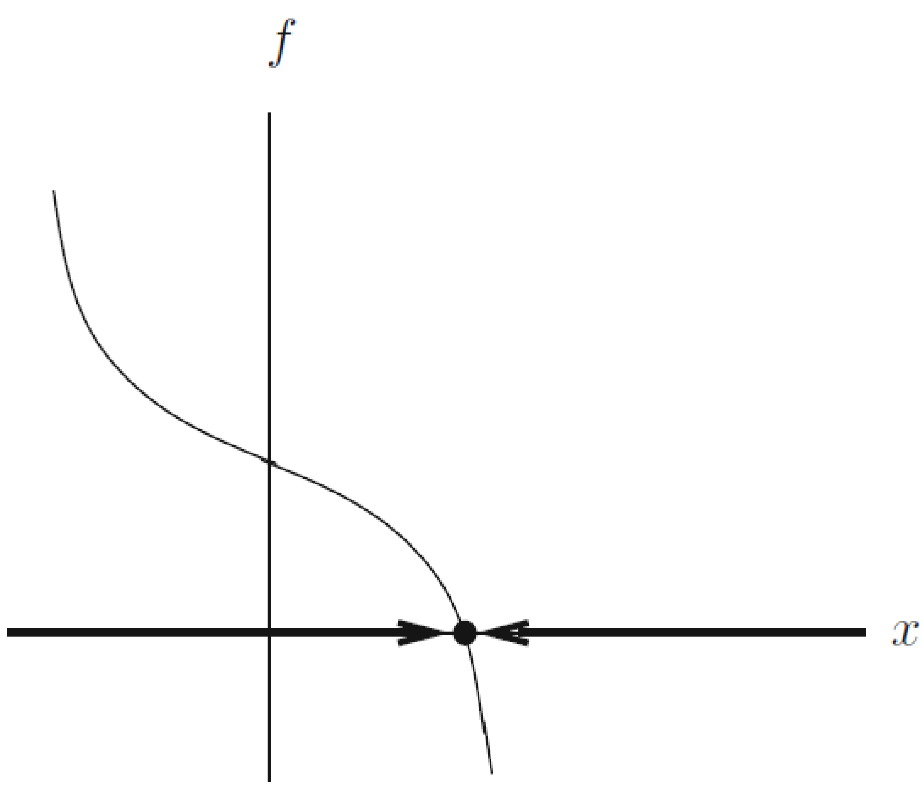

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Figure 2.2.1 Example 2.2 a (1 equilibrium for all μ)

(b) k is an even number.

In this case, Eq. (2.2.1) has no equilibrium when $\mu < 0$, exactly one equilibrium $x_0(0) = 0$ when $\mu = 0$, and exactly two equilibria $x\pm(\mu) = \pm\sqrt[k]{\mu}$ when $\mu > 0$.

Moreover, it is easy to see that

(i) when $\mu = 0$, $f(x, 0) < 0$ for both $x < x_0(0)$ and $x > x_0(0)$, so $x_0(0)$ is unstable.

We may say that $x_0(0)$ is unstable from the left and asymptotically stable from the right.

(ii) when $\mu > 0$,

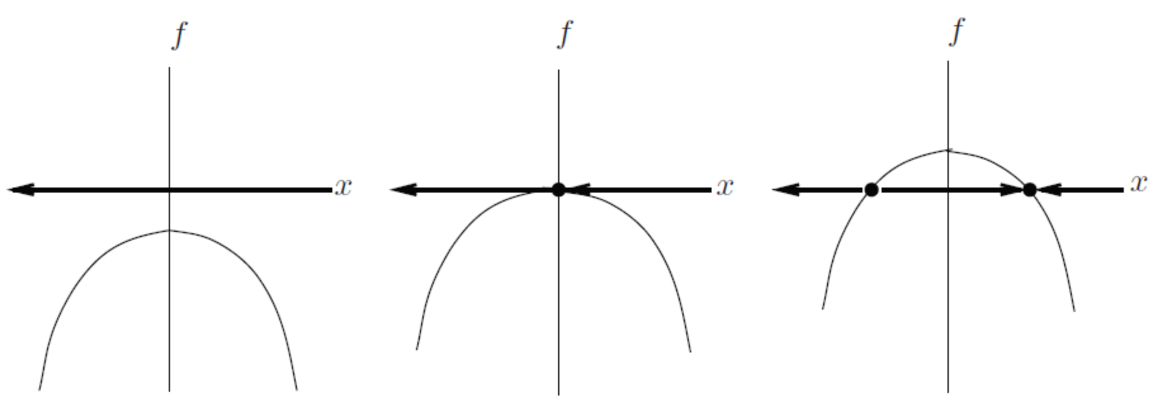
$$f(t, \mu) < 0 \text{ for } x < x-(\mu) \text{ and } x > x+(\mu),$$

and

$$f(t, \mu) > 0 \text{ for } x-(\mu) < x < x+(\mu), \text{ so } x-(\mu)$$

is unstable and $x+(\mu)$is asymptotically stable.

We note that both the number and stability of the equilibria change as μ passes through 0. Therefore, $\mu = 0$ is a bifurcation value. See Fig. 2.3



$\mu < 0$: no equilibrium $\mu = 0$: 1 equilibrium $\mu > 0$: 2 equilibria

Figure 2.2. Example 2.2. b

2.2.1 Bifurcation Types

It is useful to divide bifurcations into two principal classes:

Definitions 2.1:

Local Bifurcation: Involves changes in the solution structure in a small neighborhood around a bifurcation point. Local bifurcation theory typically uses tools like the local implicit function theorem and linear stability analysis

More technically, consider the continuous dynamical system described by the ordinary differential
equation (ODE):

$$\dot{x} = f(x, \lambda), f: R^n \times R \to R^n$$

A local bifurcation occurs at $(x_{0,}\lambda_0)$if the Jacobian matrix $d_{f_{[x_0,\lambda_0]}}$ has an eigenvalue with zero real part. If the eigenvalue is equal to zero, the bifurcation is a steady state bifurcation, but if the eigenvalue is non-zero but purely imaginary, this is a Hopf bifurcation.

For discrete dynamical systems, consider the system: x_0, λ_0

$$x_{n+1} = f(x_n, \lambda)$$

Then a local bifurcation occurs at $(x_{0,}\lambda_0)$ if the matrix $d_{f_{[x_0,\lambda_0]}}$ has an eigenvalue with modulus equal to one. If the eigenvalue is equal to one, the bifurcation is either a saddle-node (often called fold bifurcation in maps), transcritical or pitchfork bifurcation. If the eigenvalue is equal to -1, it is a period-doubling (or flip) bifurcation, and otherwise, it is a Hopf bifurcation.

Examples of local bifurcations include:

-Saddle-node (fold) bifurcation: Occurs when two solutions (one stable, one unstable) merge and annihilate each other as a parameter changes

-**Transcritical bifurcation**: Involves the exchange of stability between two branches of solutions as a parameter crosses the bifurcation point.

-**Pitchfork bifurcation**: Results in the splitting of a single solution branch into multiple branches, typically involving symmetry considerations

-**Period-doubling (flip) bifurcation**

-**Hopf bifurcation:**

-**Neimark–Sacker (secondary Hopf) bifurcation:**

Global bifurcation refers to the study of bifurcation phenomena that occur over the entire parameter space, rather than just locally. It encompasses scenarios where the structure of the solution set changes significantly as parameters vary, often involving complex interactions or constraints.

Examples of global bifurcations include:

-Homoclinic bifurcation: in which a limit cycle collides with a saddle point.[3] Homoclinic bifurcations can occur supercritically or subcritically

-Heteroclinic bifurcation: in which a limit cycle collides with two or more saddle points; they involve a heteroclinic cycle.

-Infinite-period bifurcation: in which a stable node and saddle point simultaneously occur on a limit cycle.

-Blue sky catastrophe: in which a limit cycle collides with a nonhyperbolic cycle.

A **global solution** to an ODE system is a solution that exists over a broad range of parameters or initial conditions, rather than being limited to a local neighborhood. The existence of global solutions often requires extending local results to account for boundary conditions, global constraints, and the overall structure of the solution space.

In the context of extending the implicit function theorem (IFT) to global solution theorems for ordinary differential equations (ODEs), understanding global bifurcation theory is crucial. This section will define key concepts and outline their significance in the analysis of global solution behaviors.

Global Bifurcation Considers the entire parameter space and solution domain, focusing on how solutions behave globally rather than just near bifurcation points. Global bifurcation theory often involves topological methods and continuity arguments to understand how solutions evolve over large regions.

A **bifurcation point** is a parameter value at which the qualitative nature of the solutions to an ODE system changes. At a bifurcation point, new solutions can emerge, or existing solutions can change stability or structure. Formally, for a parameter-dependent system of ODEs $\frac{\partial F}{\partial y} = f(x, \lambda)$, where λ is a parameter, a bifurcation point λ^* is a value where the number or stability of equilibrium points or periodic orbits changes.

A **bifurcation branch** is a continuous curve in the parameter space along which solutions change in a specific manner as the parameter varies. For example, in a one-parameter family of ODEs, a bifurcation branch represents the trajectory of solutions as the parameter evolves past a bifurcation point. This can include branches of equilibria, periodic orbits, or other types of solution structures.

The **continuation method** is a technique used to trace the paths of solutions as parameters vary, often starting from known solutions at specific parameter values and following them as parameters change. This method is crucial in studying global bifurcations as it helps in identifying and analyzing the global structure of solution branches.

Critical points in the context of bifurcation theory are points in parameter space where the Jacobian of the system with respect to the bifurcation parameter vanishes or becomes singular, indicating potential bifurcation points.

Techniques in Global Bifurcation Theory

a) Continuation Methods

b) Topological Methods

2.2.2 Applications

1. **Engineering Systems**

Global bifurcation theory is used to study stability and performance of engineering systems, such as mechanical structures or electrical circuits, where bifurcations can lead to significant changes in behavior.

2. **Biological Systems**

In biological models, such as population dynamics or neural networks, global bifurcations help understand how system behaviors change with varying environmental conditions or internal parameters.

3. **Ecological Models**

Global bifurcation analysis helps in studying ecological systems where bifurcations can lead to changes in species distribution or ecosystem stability.

2.3 The Leray-Schauder Degree

The Leray-Schauder degree is used to prove the existence of solutions to nonlinear ODEs. For instance, if you have a nonlinear operator $F(x)$ and can show that $\deg(F, U, O) \neq 0$ for some bounded set U, then there exists at least one solution in U.

Example 2.4: Nonlinear Elliptic Equation

Find:

$$u \in H_0^1(\Omega)$$

such that:

$$-\Delta u = \lambda u + u^3 in\ \Omega, \mathrm{u} = 0 \text{ on } \partial\Omega,$$

where Ω is a bounded domain in R^n with a smooth boundary, and λ is a parameter define the operator

$$F: H_0^1(\Omega) \rightarrow H^1(\Omega) \text{byF(u)} = -\Delta \text{u} - \lambda u - u^3$$

We want to find:

$$u \in H_0^1(\Omega)$$

such that:

$$F(u) = 0.$$

The operator F can be written as:

$$F(u) = L(u) + N(u),$$

where

$$L(u) = -\Delta \text{u} - \lambda u$$

is the linear part, and $N(u) = -u^3$ is the nonlinear part.

The linear operator L is:

$$L(u) = -\Delta u - \lambda \text{u}.$$

This operator is a compact perturbation of the identity because the Laplacian $-\Delta$ is a well-known compact operator in

$H_0^1(\Omega)$ and λu is linear.

The nonlinear term $N(u) = -u^3$ is compact in the sense that it maps bounded sets to relatively compact sets in $H^1(\Omega)$ under appropriate growth conditions.

2.3.1 Topological Methods

In the context of global solution theorems for ordinary differential equations (ODEs), the Leray-Schauder degree is a critical tool in topological methods used to analyze and understand global solution behaviors. This section provides an overview of the Leray-Schauder degree, its definition, properties, and its application in global bifurcation theory.

Definition 2.2: Consider a continuous map $F: X \times Y$ between two topological spaces, where X and Y are Banach spaces. The Leray-Schauder degree is a mapping:

$$\deg(F, U, O)$$

that assigns an integer (the degree) to the map F on a bounded set $\text{U} \subset X$, which represents the number of pre images of the point 0 in Y under F. The degree is defined for maps that satisfy certain conditions, typically involving compactness and continuity.

The Leray-Schauder degree is invariant under homotopy. If F and G are homotopic maps (i.e., there exists a continuous deformation between them), then:

$$\deg(F, U, O) = \deg(G, U, O).$$

The degree is additive with respect to the decomposition of the domain. If U can be decomposed into disjoint subsets U_1 and U_2, then:

1. **Additivity**:

$$\deg(F, U, O) = \deg(F, U_1, O) + \deg(F, U_2, O).$$

2. **Solution existence:**

$$\text{If } \deg(F, U, O) \neq 0,$$

then there exists at least one solution to $F(x) = 0$ within the set U.

3. **Identity map:** For the identity map

$$id: X \to X$$

on a bounded set U, the degree is always 1 if 0 is in the interior of U.

The Leray-Schauder degree is used to prove the existence of solutions to nonlinear ODEs. For instance, if you have a nonlinear operator $F(x)$ and can show that $\deg(F, U, O) \neq 0$ for some bounded set U, then there exists at least one solution in U.

The Leray-Schauder degree is instrumental in proving global bifurcation theorems, which describe how solution branches evolve and how new solutions emerge as parameters change. It helps in understanding the global impact of bifurcations on the solution space.

Definitions 2.3:

A metric space (X, d) is compact if every sequence in X has a convergent subsequence whose - limit is in X. Compact metric spaces are essential in global bifurcation theory as they ensure that certain topological properties, such as connectedness and continuity, hold over the entire space.

A continuum is a compact, connected, and non-empty metric space. In the study of solutions to ODEs, continua often represent solution sets or bifurcation branches that are connected and persist across parameter changes.

A space is connected if it cannot be divided into two disjoint non-empty open subsets. Connectedness is crucial for understanding how solution sets can be continuously transformed and how they behave under perturbations.

Lemma 2.1: Let X and Y be compact metric spaces, and let $f: X \to Y$, be a continuous function. If $C \subset X$ is a continuum (i.e., a compact connected subset of X), then $f(C) \subset Y$ is also a continuum (i.e., a compact connected subset of Y).

Proof: By assumption, C is a continuum, which means C is compact in X.

Since f is continuous and C is compact, the image $f(C)$ is compact in Y (this follows from the fact that the continuous image of a compact set is compact). Thus, $f(C)$ is compact in. By assumption, C is connected (in addition to being compact). Since f is continuous and C is connected, the image $f(C)$ is connected. This is a standard result in topology: the continuous image of a connected space is connected.

Chapter 3

3. Globalization of the Implicit Function Theorem

In the globalization of the Implicit Function Theorem (IFT) within the thesis on global solution theorems for ordinary differential equations (ODEs), understanding the conditions required to extend local results to global contexts is crucial. This involves identifying and ensuring that certain conditions hold to allow local results to be applied on a global scale.

3.1. Overview of the Implicit Function Theorem

The classical Implicit Function Theorem (IFT) provides conditions under which a system of equations $F(x, y) = 0$ can be locally solved for y as a function of x, given that F is continuously differentiable and the Jacobian matrix has full rank. The theorem guarantees the existence of a local differentiable function near a point where certain conditions are met.

To extend local results to global contexts,

The domain where the IFT is applied should be compact. Compactness ensures that the solution sets are bounded and the problem remains manageable globally. For ODEs, this often involves working within a compact region of the phase space or parameter space.

If working in metric spaces, completeness is necessary to ensure that limits of Cauchy sequences exist within the space. This property is crucial for extending local results globally, as it guarantees that solution branches do not escape to infinity.

The map:

$$x \to y(x),$$

which is implicitly defined by the solution to $F(x, y) = 0$, should be continuous over the global domain. This ensures that small changes in x lead to small changes in y, maintaining the solution structure globally.

Solution sets should be connected to ensure that solution branches are continuous and do not break into separate pieces. Connectedness is essential for analyzing global bifurcations and understanding how solution branches interact.

Theorem 3.1: Suppose F is continuously differentiable and the Jacobian matrix:

$$\frac{\partial F}{\partial y} \text{(denoted as } J_F\text{)}$$

is invertible at a point:

$$(x_0, y_0) \text{ where } F(x_0, y_0) = 0.$$

Then, there exists a neighborhood around (x_0, y_0) in which there is a unique continuously differentiable function

$$y = g(x)$$

such that:

$$\left(x_0, g(x)\right) = 0$$

for all x in this neighborhood.

<u>Proof :</u>- Let $F: R^n \times R^m \to R^m$ be continuously differentiable, and suppose at the point (x_0, y_0) where $F(x_0, y_0) = 0$,

the Jacobian matrix:

$$J_F = \frac{\partial F}{\partial x}(x_0, y_0)$$

is invertible. This means J_F is an m×m matrix with full rank m(a rank equal to its dimension m).

First we, have to find neighborhood around (x_0, y_0) and a function such that for x in this neighborhood, $F\left(x_0, g(x)\right) = 0$ and g is continuously differentiable.

Since $J_F$is invertible at (x_0, y_0), by the definition of differentiability, F can be approximated locally by its linear part:

$$F((x\,, y) \approx F(x_0, y_{0)} + \frac{\partial F}{\partial x}(x_0, y_0)x - x_0) + \frac{\partial F}{\partial y}(x_0, y_0)(y - y_0).$$

Given:

$$F(x_0, y_0) = 0),$$

this simplifies to:

$$F((x\,, y) \approx F(x_0, y_{0)} + \frac{\partial F}{\partial x}(x_0, y_0)x - x_0) + \frac{\partial F}{\partial y}(x_0, y_0)(y - y_0).$$

In a sufficiently small neighborhood around (x_0, y_0), the approximation is good, and the function F can be viewed so:

$$F((x\,, y) \approx F(x_0, y_{0)} + \frac{\partial F}{\partial x}(x_0, y_0)x - x_0) + J_F(x_0, y_0)(y - y_0).$$

To find y such that $F((x\,, y) = 0$, set:

$$F((x\,, y) \approx \frac{\partial F}{\partial x}(x_0, y_0)x - x_0) + J_F(x_0, y_0)(y - y_0).$$

Rearranging gives:

$$J_F(y - y_0) = -\frac{\partial F}{\partial x}(x_0, y_0)(x - x_0).$$

By solving for y yields:

$$(y - y_0) = -J_F^{-1}.\left(\frac{\partial F}{\partial x}(x_0, y_0).(x - x_0)\right).$$

Therefore:

$$y = y_0 - J_F^{-1}.\left(\frac{\partial F}{\partial x}(x_0, y_0).(x - x_0)\right)$$

Define as:

$$g(x) = y_0 - J_F^{-1}.\left(\frac{\partial F}{\partial x}(x_0, y_0).(x - x_0)\right)$$

This function $g(x)$ s continuously differentiable (since it is defined by continuous operations on x).

Verify that $F(x, g(x) = 0$:

$$F(x, g(x)) = F(x, y_0 - J_F^{-1}.\left(\frac{\partial F}{\partial x}(x_0, y_0).(x - x_0))\right).$$

This satisfies $F(x, g(x)) = 0$

Within the neighborhood where the linear approximation is valid.

The solution $g(x)$ and the condition $F(x, g(x)) = 0$:

are valid in a sufficiently small neighborhood around (x_0, y_0) where the approximation holds true.

Theorem 3.2: Let X be a Banach space, and let $F: X \times R \to X$, be a continuous map, where R represents the parameter space.

Suppose:

1. There exists $(u_0, \lambda_0) \in X \times R$ such that $F(u_0, \lambda_0) = 0$.
2. The linearized operator $DFu(u_0, \lambda_0)$, has a non-trivial kernel, i.e., there is a non-zero solution v to the linearized problem.
3. The map F is continuous and X is a Banach space.

Then, there exists a global bifurcation branch of solutions $(u(s), \lambda(s))$ such that $u)(s) \in X$ and $\lambda(S) \in R$, which emanates from (u_0,λ_0) and extends continuously for s in some interval, possibly unbounded, and this branch is not limited to a local neighborhood of (u_0,λ_0).

Proof: Define the linearization operator L at (u_0, λ_0) by:

$$L_v = DuF(u_0, \lambda_0)v$$

Since $L_v = 0$ has non-trivial solutions, there exists $v \neq 0$ such that:

$$DuF(u_0, \lambda_0)v = 0$$

This indicates that (u_0, λ_0) is a bifurcation point.

By the local bifurcation theorem proved in chapter 2: there exists a local bifurcation branch

$$(u(s), \lambda(s))$$

for s in a neighborhood $s = 0$, such that:

$$F\big(u(s), \lambda(s)\big) = 0, \text{ with } \big(u(s), \lambda(s)\big) = (u_0, \lambda_0)$$

To show that this local branch extends globally, we use the following arguments:

Consider the map:

$$\Phi: X \times R \to X, \Phi(\mathrm{u}, \lambda) = \mathrm{F}(\mathrm{u}, \lambda).$$

The fact that

$$D_u F(u_0, \lambda_0)$$

has a non-trivial kernel implies that Φ has direction in which the solution can branch out from

(u_0, λ_0).

This non-trivial kernel indicates that the solution set to:

$$\Phi(u_0, \lambda_0) = 0$$

is not confined to a local neighborhood of (u_0, λ_0).

By applying the global bifurcation results (such as those found in the works of Crandall-Rabinowitz or other bifurcation theory sources), it follows that if the linearized operator

$$D_u F(u_0, \lambda_0)$$

has a non-trivial kernel, there exists a global bifurcation branch. This means that the branch

$$\big(u(s), \lambda(s)\big)$$

extends beyond any local neighborhood of

$$(u_0, \lambda_0)$$

and can be extended to s in an interval that may be unbounded.

The continuity of F and the fact that the local bifurcation branch $\big(u(s), \lambda(s)\big)$ can be extended globally follow from standard results in bifurcation theory. Specifically, the branch $\big(u(s), \lambda(s)\big)$ is continuous in s and can be shown to extend beyond the local neighborhood of (u_0, λ_0) by leveraging the properties of the bifurcation set and the continuous dependence on the parameters.

Example 3.1: Let $f: R \times R \to R$ be given by:

$$f(u, \lambda) = u^2 + \lambda^2 - 1.$$

Solution: We need to solve:

$$f(u, \lambda) = 0,$$

which means:

$$u(u^2 + \lambda^2 - 1) = 0.$$

This equation is satisfied if either:

$$u = 0, \text{ or } u^2 + \lambda^2 - 1 = 0.$$

Thus, the equilibrium points are:

For $u = 0$, we have $\lambda \in R$. So, $(0, \lambda)$ is an equilibrium for any λ.

For $u^2 = 1 - \lambda^2$,

we need:

$$1 - \lambda^2 \geq 0, \text{ i.e., } |\lambda| \leq 1.$$

If

$$|\lambda| \leq 1, \text{ then } u = 1 - \lambda^2,$$

.

The linearized operator around a solution $(u_0, \lambda_0$ is given by:

$$\frac{\partial f}{\partial u}(\text{u},\lambda) = u^2 + \lambda^2 - 1 + 2u^2 = 3u^2 + \lambda^2 - 1.$$

Thus, the linearized operator at (u_0, λ_0) is:

$$\frac{\partial f}{\partial u}(u_0, \lambda_0) = 3{u_0}^2 + {\lambda_0}^2 - 1.$$

By apply the Global Bifurcation Theorem, we need to find conditions where bifurcation occurs:

Choose a specific:

$$(u_0, \lambda_0)$$

For simplicity, let's choose:

$$(u_0, \lambda_0) = (0,0).$$

Here, $f(0,0) = 0$.

The linearization at $(0,0)$ is:

$$\frac{\partial f}{\partial u}(0,0) = -1.$$

This has a non-trivial kernel if we consider the existence of a solution to the linearized problem

$\frac{\partial f}{\partial u}(u_0, \lambda_0)v = 0$ with $v \neq 0$. For $(0,0)$, the linearized operator is simply -1, which does not have a non-trivial kernel. However, we are interested in local bifurcation from points where the linearized operator has a non-trivial kernel or in non-linear settings where solutions might bifurcate.

Let's find bifurcation points. For u≠0, we have:

$$u = 1 - \lambda^2.$$

For $\lambda = 0, u = \pm 1$.

For $|\lambda| \leq 1, = \pm\sqrt{1 - \lambda^2}$.

So, we have equilibrium points:

$$(\pm 1 - \lambda^2, \lambda) for\ |\lambda\ | \leq 1.$$

Around $(u_0, \lambda_0) = (0,0)$ there is a local bifurcation because the solutions exist when $|\lambda\ | \leq 1$.

To show that this bifurcation is global, note that as λ varies within the interval $[-1,1]$ the solutions $u = \pm 1 - \lambda^2$ sweep out a continuous family of solutions in $R \times [-1,1]$.

Therefore, there is a global bifurcation branch extending for all λ in the interval $[-1,1]$ and for corresponding u values, showing that the branch is not restricted to a local neighborhood around $(0,0)$ but extends globally within the specified parameter range.

Example 3.2: Consider the equation

$$x' = f(x, \mu) := \mu x - x^{2m}, m \in \text{N}.$$

By letting $f(x, \mu) = 0$ we find two equilibria $x_1(\mu) = 0$ and $x_2(\mu) = \mu^{1/(2m-1)}$.

Since $f_x(x, \mu) = 2\text{m}x^{2m-1}$, we have the following:

(a) For $x_1(\mu) = 0,\ \lambda_1(\mu) = \mu$. It follows that $x_1(\mu)$ is asymptotically stable for $\mu < 0$ and unstable for $\mu > 0$.

(b) For $x_2(\mu) = \mu^{1/(2m-1)}\ \lambda_2(\mu) = -(2m-1)\mu$. Thus, $x_2(\mu)$ is unstable for $\mu < 0$ and asymptotically stable for $\mu > 0$.

As a result, $\mu = 0$ is a bifurcation value, and the bifurcation diagram and the phase portraits are given in

Fig. 2.3.

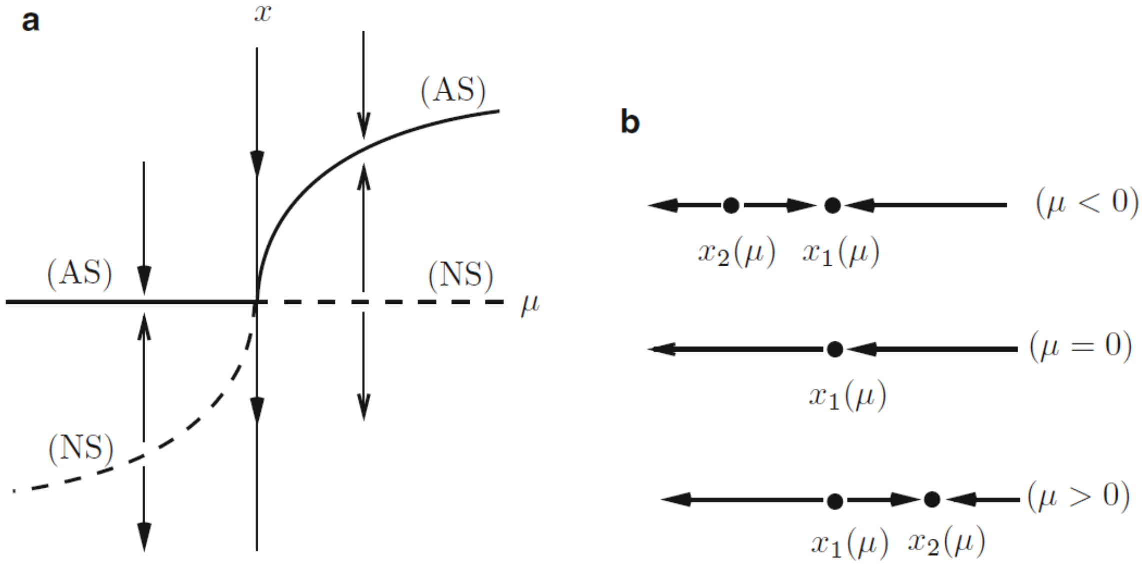


Figure 2.3. Transcritical bifurcation. (a) Bifurcation diagram. (b) Phase portraits

In this diagram, two $C^1 - branches\ of\ equilibria$ with opposite stabilities inter change their stabilities at the bifurcation value. Such a bifurcation is called a transcritical bifurcation.

Topological Methods: Employ topological methods such as the Leray-Schauder degree to study the global structure of solutions. These methods help in understanding the number and behavior of solutions over large domains.

Lemma 3.1: Let X be a topological space, and let $F: X \times X \to X$ be a continuous map. Suppose U is a compact subset of X and K is a closed subset of X such that:

1. $U \cap K = \emptyset$.
2. There exists a continuous function $\Phi: X[0,1]$ such that:
    - $\Phi(\mathrm{x}) = 0$ for all $\mathrm{x} \in \mathrm{k}$.
    - $\Phi(\mathrm{x}) = 1$ for all $\mathrm{x} \in \mathrm{U}$.

Then if F is homotopic to the identity map on X through a homotopy that does not intersect K during the homotopy, then there exists at least one point $\mathrm{x} \in \mathrm{U}$ such that $F(x) = x$.

Proof: By assumption,

$$H(x,t) \notin K \text{ for all } \mathrm{x} \in \mathrm{k} \text{ and } t \in [0,1].$$

This implies that the map F does not map any point in X to K.

Define:

$$G(x) = \Phi\text{F(x)}).$$

Since

$$F(x,) \notin K, \Phi\text{F(x)}) \geq 0.$$

The function Φ maps U to 1 and K to 0.

Therefore, $G(x) = \Phi\text{F(x)}$ must be less than 1 if $F(x)$ is not in .

U Since Φ is continuous, G is continuous and achieves its maximum on the compact set X.

The maximum value of G is 1 (because $\Phi\text{(x)} = 1$ on U). Thus, there must be some $x \in X$ such that $G(x) = 1$. This means $\Phi\text{F(x)} = 1$, which implies $F(x) \in U$ and hence $\Phi\text{F(x)} = 1$ for

$F(x) \in U$.

Since $\Phi\text{F(x)} = 1$

implies $F(x) \in U$.

only if ($x \in U$.)

$x \in U$, it follows that $F(x) = x$ for some $x \in U$. Thus, there exists at least one point $x \in U$ such that $F(x) = x$.

In extending the Implicit Function Theorem (IFT) from a local to a global context, it's essential to understand how classical results compare with and contribute to this extension. Here's a detailed comparison that highlights how classical local results relate to and inform the global results:

Theorem (Local Version)) 3.3: Let $F(x, y) = 0$ can be locally solved for y as a function of x. Then if F is continuously differentiable and the Jacobian matrix J_F has full rank at a point (x_0, y_0), then there exists a neighborhood around (x_0, y_0) where y can be expressed as a differentiable function of x.

Proof: Given:

$$F(x, y) = 0,$$

with

$$F: R^m \times R^n \to R^{P\ \mathrm{p}}.$$

Suppose F is continuously differentiable and $J_F(x_0, y_0)$ the Jacobian matrix of F with respect to y, has full rank.

$$\text{Let } F(x, y) = F_1(x, y), \dots, F_p(x, y)$$

The Jacobian matrix of F with respect to y at (x_0, y_0), is:

$$J_F(x_0, y_0) = \frac{\partial \mathrm{Fi}}{\partial \mathrm{yj}}(x_0, y_0).$$

Assume this matrix has full rank n (the number of variables y) at (x_0, y_0)

By the Implicit Function Theorem, if $J_F(x_0, y_0)$ has full rank (i.e., rank n), then there exists a neighborhood around (x_0, y_0) where we can locally solve the equation $F(x, y) = 0$,

,for y as a differentiable function of x.

Explicitly, there exist neighborhoods $U \subset R^m$ around x_0 and $V \subset R^m$ around y_0, and a differentiable function $g: U \to V$ such that for all $x \in U$, $F\big(x, g(x)\big) = 0$.

Here, $g(x)$ is the function expressing y in terms of x in the neighborhood.

Thus, we have shown that in a neighborhood of (x_0, y_0), the equation $F(x, y) = 0$,

,can be locally solved for y as a differentiable function of x, as required.

Example 3.2: Consider the equation:

$$F(x, y, z) = e^2 + y^2 + z^2,$$

We want to solve :

$$F(x, y, z) = 0,$$

for z as a function of x and y near a point where $F(x, y, z) = 0$

Specifically, let’s choose the point:

$$(x_0, y_0, z_0) = (0,1,1)$$

Plug (0,1,1) into $F(x, y, z)$.

$$F(0,1,1) = e^0 + 1^2 - 1^3 = 1 + 1 - 1 = 1$$

Since F(0,1,1)≠0, this point does not satisfy the equation:

$$F(x, y, z) = 0,$$

Therefore, this choice of point does not work, so let's pick a point where the equation is satisfied.

Let’s choose:

$$(x_0, y_0, z_0) = (0,1,2).$$

$$F(0,1,2) = e^0 + 1^2 - 2^3 = 1 + 1 - 8 = -6.$$

This also does not satisfy the equation. Correcting the choice:

Let’s choose:

$$(x_0, y_0, z_0) = (0,0,1).$$

$$F(0,0,1) = e^0 + 0^2 - 1^3 = 1 + 0 - 1 = 0.$$

So

$$(0,0,1)$$

satisfies

$$F(x,y,z) = 0.$$

The Jacobian matrix of F with respect to (x,y) is:

$$J_F(x,y,z) = \left[\frac{\partial F}{\partial x}\frac{\partial x}{\partial y}\right] = [e^x 2y]$$

At

$$(0,0,1)\colon J_F(0,0,1) = [e^0 2.0] = [1, 0]$$

We also need the Jacobian of F with respect to zz:

$$\frac{\partial \mathrm{F}}{\partial \mathrm{z}} = -3z^2.$$

At:

$$\frac{\partial \mathrm{F}}{\partial \mathrm{z}}\ (0,0,1)\colon \frac{\partial \mathrm{F}}{\partial \mathrm{z}}\ (0,0,1) = -3.\,1^2 = -3.$$

This is non-zero, so the Jacobian matrix has full rank. Since $\frac{\partial \mathrm{F}}{\partial \mathrm{z}} \neq 0$ at (0,0,1), the Implicit Function Theorem guarantees that z can be locally expressed as a differentiable function of x and y in a neighborhood around (0,0,1)

Solve $F(x,y,z) = 0$ for z:

$$e^x + y^2 + z^3 = 0 \Longrightarrow z^3 = e^x + y^2 \Longrightarrow z = \sqrt[3]{e^x + y^2}\,.$$

Thus, z can be expressed as a differentiable function of x and y in a neighborhood around (0,0,1) as:

$$z = g(x) = z = \sqrt[3]{e^x + y^2}.$$

3.2 Leray-Schauder Degree and Topological Lemmas

3.2.1.Leray-Schauder Degree Theory

The Leray-Schauder degree is a topological invariant used to study the existence of solutions to nonlinear equations in Banach spaces. It extends the concept of the topological degree to infinite-dimensional spaces, providing valuable insights into the existence and multiplicity of solutions.

Let X be a Banach space and $F\colon X \to X$ be a continuous map**.** The Leray-Schauder degree of F with respect to a set $U \subset C$ is a tool to determine whether $F(x) = 0$ has solutions in U, and if so, how many solutions there are.

Definition 3.1: Given a compact set $U \subset C$ and a continuous map $F: X \rightarrow X$, the Leray-Schauder degree $\deg(F, U, O)$ is defined as follows:

The Leray-Schauder degree $\deg(F, U, O)$ measures the number of solutions to $F(x) = 0$ inside U, considering the behavior of F at the boundary of U and its interaction with the zero element.

3.2.2. Properties of the Leray-Schauder Degree

- **Homotopy Invariance**

If F and G are homotopic maps (i.e., there exists a continuous map $H: X \times [0,1] \rightarrow X$ such that $H(x, 0) = F(x) and\ H(x, 1) = G(x)$) and if F and G map into the same compact set, then:

$\deg(F, U, O)$= $\deg(G, U, O)$

- **Additivity**

If F is a continuous map on the union of two disjoint compact subsets U_1 and U_2 of X, and if F is well-defined and non-zero on their boundary, then:

$$\deg(F, U_1 \cup U_2, O) = \deg(F, U_1 \cup U_1, O) + \deg(F, U_1 \cup U_2, O).$$

- **Invariance under Isomorphisms**

If $T: X \rightarrow X$:

is an isomorphism, then:

$$\deg(T \circ F \circ T^{-1} T(U), O) = \deg(F, U, O)\ where\ T(U)$$

is the image of U under T.

- **Non-zero Degree Implies Existence of Solutions**

If $\deg(F, U, O) \neq 0$, then there exists at least one solution to $F(x) = 0$.

3.2.4. Topological Lemmas Involving the Leray-Schauder Degree

Lemma 3.2: Let X be a Banach space, and let $F: X \rightarrow X$ be a continuous map such that $F(x) - x$ is compact. If 0 is an interior point of $F(X)$, then F has at least one fixed point.

This lemma uses the Leray-Schauder degree to ensure the existence of fixed points by showing that the degree of the map F^{-1} is non-zero, indicating that F indeed has fixed points.

Lemma 3.3: Given a parameter-dependent family of maps $F(x, \lambda): X \times \Lambda \rightarrow X$, if there exists a point $(\bar{x}, \bar{\lambda}) \in X \times \Lambda$ such that $\deg(F(.\lambda), U, O) \neq 0)$, then there exists a bifurcation branch of solutions that continues from $(\bar{x}, \bar{\lambda})$

This lemma establishes the existence of global branches of solutions by leveraging the non-zero degree to guarantee that solutions persist as parameters vary.

Lemma 3.4: Let $F: X \to X$ be a continuous map and let U be a compact subset of X. If $\deg(F, U, O) \neq 0$) and F is continuous, then $F^{-1}(0)$ is non-empty.

This lemma ensures that the set of solutions is not empty if the degree is non-zero, providing a topological guarantee of the existence of solutions within the specified set.

3.3. Application to ODEs and Global Solution Theorems

In the context of global solution theorems for ODEs, the Leray-Schauder degree is used to analyze the global existence and bifurcation of solutions:

By applying the degree theory, one can show that under certain conditions (e.g., compactness and boundedness), solutions to nonlinear ODEs exist over global parameter ranges.

The degree theory helps in identifying global bifurcation points where the structure of solutions changes globally, not just locally. For instance, it can show the existence of global branches of periodic solutions or the emergence of new solution branches as parameters vary.

By understanding the global degree, researchers can predict how solutions will behave over a wide range of parameters, which is crucial for analyzing complex dynamical systems.

The Leray-Schauder degree provides a tool for ensuring that solutions are robust and persist across different parameter values, which is essential for practical applications in engineering, physics, and other fields.

In the context of global solution theorems for Ordinary Differential Equations (ODEs), the Leray-Schauder degree theory and associated topological lemmas provide powerful tools for understanding and establishing the existence, uniqueness, and global behavior of solutions. Below, we explore how these concepts are applied to global solution theorems for ODEs.

The Leray-Schauder degree can be used to prove the global existence of solutions to nonlinear ODEs. Suppose we have a nonlinear ODE of the form:

$$\frac{dx}{dt} = f(x, t)$$

where f is a continuous function. To apply Leray-Schauder degree theory, one often reformulates this problem into a fixed-point problem:

$$x = g(x, t)$$

where $g(x,t)$ is derived from the integral form of the ODE or other equivalent formulations. If g satisfies the conditions required for the Leray-Schauder degree, such as compactness and the presence of a fixed point, one can apply the degree theory to establish the existence of a solution.

Example 3.3: Consider the ODE:

$$\frac{dx}{dt} = f(x),$$

with f a nonlinear function. If f is continuous and maps bounded sets to bounded sets, and if the associated fixed-point problem $x = \Phi(x)$ has a non-zero Leray-Schauder degree, it implies that there is at least one solution to the ODE.

Implication: The degree theory helps in proving that solutions exist globally, not just locally or in a limited range of parameters. This is crucial for understanding the overall behavior of the system.

3.5. Global Bifurcation Analysis

Global bifurcation results can be derived using the Leray-Schauder degree by analyzing how the number and structure of solutions change as a parameter varies.

For a parameter-dependent family of ODEs:

$$\frac{dx}{dt} = f(x, \lambda),$$

where λ is a parameter, one can study the map:

$$\frac{dx}{dt} - f(x, \lambda),$$

By analyzing the Leray-Schauder degree of F with respect to λ , one can determine how the solution set changes with λ, identifying bifurcation points where the number of solutions changes.

Example 3.4: For a family of ODEs where $f(x, \lambda)$ changes with λ, applying the Leray-Schauder degree theory helps in proving that solutions branch out as λ crosses certain values, leading to the appearance of new solution branches.

Implication: This approach helps in identifying and analyzing global bifurcations, which is essential for understanding how solutions evolve over a wide range of parameters.

3.6. Continuity Lemma

The Continuity Lemma provides guarantees about the continuity and existence of solutions based on the non-zero degree of a map. It ensures that if the Leray-Schauder degree of a map is non-zero, then solutions exist and vary continuously with parameters.

3.6.1 **Application of Topological Lemma for Continua:**

To find limit cycles (periodic solutions), reformulate the ODE into a fixed-point problem using:

$$x = \Phi(x, \lambda)$$

where Φ is derived from the integral formulation of the ODE. By analyzing the continua formed by the solutions, apply the Topological Lemma for Continua to ensure that the periodic orbits exist if certain conditions are met.

Example 3.5: In the context of parameter-dependent ODEs:

$$\frac{dt}{dx} = f(x, \lambda),$$

applying the Continuity Lemma helps in proving that solutions exist continuously as parameters λ vary, ensuring that solution branches do not abruptly disappear or change discontinuously.

Implication: This lemma supports the robustness of solutions across varying parameters, contributing to the global understanding of the solution set.

Lemma 3.5: Let X be a compact metric space, and let $F: X \to X$ be a continuous map. Suppose C is a continuum (a compact connected subset) of X such that $F(C)$ is also a continuum in X. If F is a continuous map and $F(C)$ intersects C, then C contains a point x such that $F(x) = x$.

In other words, C contains at least one fixed point of F.

Definition 3.2: Let X be a compact metric space, and let $F: X \to X$ be a continuous map. Assume $C \subset X$ is a continuum and $F(C)$ is also a continuum. We need to show that if $F(C) \neq \emptyset$, then C contains a fixed point of F.

Since F is continuous and C is compact, $F(C)$ is compact and connected. Also, g is continuous on C because it is the difference of two continuous functions.

If $F(C)$ intersects C, then there exist points $x_1, x_2 \in C$ such that $F(x_1) \in C$ and $F(x_2) \in C$.

By the intermediate value theorem for continua, if C is connected, the image under F will also connect these points, implying $g(x) = F(x) - x$ takes values of both signs if

$$F(x_1) \neq x_1 and\ F(x_2) \neq x_2.$$

3.6.2 Application of Leray-Schauder Degree Theory:

To analyze the existence of equilibrium populations (fixed points), rewrite the ODE in the form:

$$N = \frac{\mathrm{rN}(1-\frac{N}{K}}{1-\frac{N^2}{1+N^2}}.$$

Define:

$$\Phi(\mathrm{N}) = \frac{\mathrm{rN}(1-\frac{N}{K}}{1-\frac{N^2}{1+N^2}}.$$

The existence of equilibria can be studied by analyzing the fixed points of Φ using the Leray-Schauder degree theory.

3.7 Global Formulation

To extend the local results of the IFT to global contexts, we need to address how local solutions behave over larger domains and under varying parameters. This involves ensuring that local properties extend to global properties and understanding the global structure of solution sets.

To simplify the discussion, throughout this chapter we make the following
assumption:

$$(H)\ \lambda_i(\mu),\ i = 1, \ldots, n,$$

are distinct for all μ in the domain.

We note that assumption ($\mathcal{H}$) implies that bifurcations occur only when
Re $\lambda_i(\mu)$ changes sign for some $i \in \{1, \ldots, n\}$, and hence eliminates the possibility
of an occurrence of a bifurcation caused by a multiplicity change of
eigenvalues. We comment that in some references, the phase portraits for nodes and spiral-points are regarded as topologically equivalent as long as they have the same stability. Consequently, the structural change. Since the matrix $A(\mu)$ is real-valued, all non-real eigenvalues are in complex-conjugate pairs. This means that under assumption ($\mathcal{H}$), a bifurcation
may occur at $\mu = \mu_0$ when the two conditions below are satisfied:

(a) Either one of the following holds:

(i) there exists an $i \in \{1 \ldots n\}$ such that $\lambda_i(\mu) \in R$ and $\lambda_i(\mu)$
changes sign at μ_0;

(ii) there exist j pairs of complex-conjugate eigenvalues $\lambda_i \pm (\mu)$
$i = 1 \ldots j$, such that Re $\lambda_i \pm (\mu)$ changes sign at μ_0 and Im

$$\lambda_i \pm (\mu_0) \neq 0;$$

(iii) the combination of (i) and (ii).

(b) Re $\lambda_k(\mu_0) \neq 0$ for all other eigenvalues λ_k.

In particular, a bifurcation where only one real eigenvalue passes through
zero is called a one-dimensional bifurcation, and a bifurcation where the real
parts of one pair of complex-conjugate eigenvalues pass through zero is called
a Poincar´e–Andronov–Hopf bifurcation or simply a Hopf bifurcation.
Clearly, this bifurcation is a one-dimensional bifurcation. The eigenvalues of $A(0)$ are
Hopf bifurcations.

Definition 3.3: A subset K of a topological space X is called **compact** if every open cover of K has a finite sub-cover.

The classical IFT ensures local solvability, but the global extension requires analyzing the rank of the Jacobian over a larger domain. Compactness and regularity ensure that local solutions remain valid and well-behaved globally.

Example 3.6: Consider the following system of equations:

$$x^2 + y^2 - 1$$

Solution: Compute the Jacobian matrix J_F of F:

$$JF(x,y) = \begin{pmatrix} \frac{\partial}{\partial x}(x^2 + y^2 - 1 & \frac{\partial}{\partial y}(x^2 + y^2 - 1 \\ \frac{\partial}{\partial x}(x - y) & \frac{\partial}{\partial y}(x - y) \end{pmatrix} = \begin{pmatrix} 2x & 2y \\ 1 & -1 \end{pmatrix}$$

So, use the Implicit Function Theorem, we need to find points where the Jacobian matrix has full rank. For a 2×2 matrix, the Jacobian has full rank if its determinant is non-zero.

Calculate the determinant of :

$$\det(JF(x,y) = det\begin{pmatrix} 2x & 2y \\ 1 & -1 \end{pmatrix} = 2x)(-1) - (2y)(1) = -2x - 2y$$

The Jacobian matrix has full rank if:

$$-2x - 2y \neq 0 \; or \; x + y \neq 0$$

Choose the point $(1,1)$:

$$J_F(1,1) = \begin{pmatrix} 2(1) & 2(1) \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 1 & -1 \end{pmatrix}$$

Compute the determinant:

$$\deg(J_F(1,1) = (2.-1) - (2.1) = -2 - 2 = -4 \neq 04.$$

The determinant is non-zero, so $J_F(1,1)$ has full rank. By the Implicit Function Theorem, there are local solutions around $(1,1)$ where $F(x,y) = 0$.

$$\begin{cases} x^2 + y^2 = 1 \\ x = y \end{cases}$$

Thus:

$$x = \pm\frac{1}{\sqrt{2}} \text{ and } y = \pm\frac{1}{\sqrt{2}}$$

The solutions are:

$$(x,y) = \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \quad \text{and} \quad (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$$

The solution set $\left\{(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})\right\}$is finite and hence compact in R^2

Definition 3.4: A topological space X is said to be **connected** if there do not exist two non-empty disjoint open subsets U and V in X such that:

$$X = U \cup V.$$

Example 3.7: Let's consider the interval $[0,1]$ in the real line R with the standard topology.

Claim: The interval $[0,1]$ is connected.

Proof: Suppose $[0,1]$ is not connected. Then there exist two non-empty disjoint open subsets U and V in $[0,1]$ such that:

$$[0,1] = U \cup V$$

Here, U and V are open in the subspace topology of $[0,1]$. This means there are open sets U' and V' in R such that:

$$U = U' \cap [0,1]$$

$$V = V' \cap [0,1]$$

Since U and V are disjoint and cover $[0,1]$ the complements U^c and V^c in $[0,1]$are non-empty. Consider:

$$(U^c) = [0,1]\backslash U$$

$$(V^c) = [0,1]\backslash V$$

Both U^c and V^c are closed in $[0,1]$and non-empty. Therefore:

$$U^c = V \text{ and } V^c = U$$

Since U and V are non-empty and disjoint, there must be a point in $[0,1]$ that is not in U or V, but this contradicts our assumption that:

$$[0,1] = U \cup V.$$

In more detail, if you split [0,1] into disjoint non-empty open subsets, you would get an open interval (or union of intervals) that isn't open in R, which is a contradiction.

Therefore, [0,1] cannot be divided into two disjoint non-empty open subsets in R. Thus, [0,1] is connected.

Consider the function:

$$F(x, y) = \binom{y^2 - x^2}{e^y - z}.$$

We want to compute the Jacobian matrix $DF(x, y)[0,1]$ at the point (0,0)

For the function components:

$$F_1(x, y) = x^2 - y^2$$

$$F_2(x, y) = e^y - x$$

- **Partial Derivatives of F_1:**

$$\frac{\partial F_1}{\partial x} = -2x$$

$$\frac{\partial F_1}{\partial y} = 2y$$

- **Partial Derivatives of F_2:**

$$\frac{\partial F_2}{\partial x} = -1$$

$$\frac{\partial F_2}{\partial y} = e^y$$

Substitute $x = 0 \; and \; y = 0$ into the partial derivatives:

For $\frac{\partial F_1}{\partial x} : \frac{\partial F_1}{\partial x}(0,0) = -2.0 = 0$

For $\frac{\partial F_1}{\partial y} : \frac{\partial F_1}{\partial y}(0,0) = 2.0 = 0$

For $\frac{\partial F_2}{\partial x} : \frac{\partial F_2}{\partial x}(0,0) = -1$

For $\frac{\partial F_2}{\partial y} : \frac{\partial F_2}{\partial y}(0,0) = e^0 = 1$

By combining these results, the Jacobian matrix at (0,0) is:

$$DF(0,0) = \begin{pmatrix} \frac{\partial F_1}{\partial x}(0,0) & \frac{\partial F_1}{\partial y}(0,0) \\ \frac{\partial F_2}{\partial x}(0,0) & \frac{\partial F_2}{\partial y}(0,0) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}$$

To verify, let's double-check:

The matrix provided matches our computed Jacobian matrix:

$$DF(0,0) = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}$$

The Jacobian matrix of $F(x,y) = \begin{pmatrix} y^2-x^2 \\ e^y-z \end{pmatrix}$ at the point (0,0) is indeed:

$$\begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}$$

This matrix shows that the function F does not have a full rank at $(0,0)$ which implies that at this point, the Jacobian matrix is singular. For analysis and application in global contexts, further methods would be required to understand the behavior of solutions, particularly if extending local results globally.

3.8 . Global Bifurcation Results

3.8.1. Apply the Global Implicit Function Theorem

To apply the global IFT:

a) **Define a Suitable Function Space:** Consider the space of all possible solutions x and parameters λ.

b) **Verify Conditions:**

- ✓ **Existence of a solution:** Ensure that the equation $G(x, \lambda) = 0$ has a solution for each λ in the parameter space.
- ✓ **Continuity and Boundedness:** Verify that G is continuous and satisfies the necessary boundedness conditions.

c) **Use Topological Degree:**

This step involves calculating the topological degree of the operator G over the compact parameter space.

Determine if the topological degree is non-zero, indicating that solutions exist globally.

Let's illustrate the derivation with a concrete example: a nonlinear eigenvalue problem.

Example Problem 3.8: Consider the nonlinear eigenvalue problem:

$$-\Delta u + \lambda u = f(u)$$

where Δ denotes the Laplacian operator, and $f(u)$ is a nonlinear function.

Steps: Define:

$$G(x,\lambda)$$

such that:

$$G(x,\lambda) = -\Delta u + \lambda u - f(u)$$

Verify that G is continuous in u and λ.

Ensure f(u) satisfies global boundedness conditions.

Use the Leray-Schauder degree to analyze G.

For example, determine where λ changes leading to a change in the number of solutions.

Plot how solutions u change with λ.

Ensure that the bifurcation results align with the expected global behavior.

Check for continuity and stability of solution branches.

In deriving global bifurcation results using a global version of the Implicit Function Theorem (IFT), several key results and their implications emerge. These results provide a comprehensive understanding of how solutions to differential equations change globally with varying parameters. Below, we summarize the major results and their implications.

3.8.2. Existence of Global Solution Branches

Using global IFT and topological methods (like the Leray-Schauder degree), one can demonstrate the existence of global branches of solutions to a given differential equation as parameters vary.

The global IFT allows us to extend local bifurcation results to show that there exist continuous branches of solutions for a range of parameters, not just in a local neighborhood. For instance, if a parameter crosses a bifurcation point, global IFT helps in confirming that the solution branch persists globally and is not confined to local regions.

3.8.3. Global Bifurcation Theorems

Global bifurcation theorems, such as the Crandall-Rabinowitz theorem, can be extended to analyze more complex scenarios where multiple parameters are involved. These theorems often use the concept of the topological degree to establish global results.

These theorems provide conditions under which global bifurcations occur, meaning new solution branches emerge from an equilibrium or periodic orbit. The results show how bifurcations are not limited to local changes but lead to global changes in the solution structure.

In the context of global solution theorems for ordinary differential equations (ODEs), globalization techniques can be highly effective in analyzing and understanding the behavior of solutions across entire parameter spaces, not just locally. Below are several case studies and examples where globalization techniques have been successfully applied to ODEs.

Problem: Consider the nonlinear oscillator described by:

$$\frac{d^2x}{dt^2} + \alpha x + \beta x^3 = 0$$

where α and β are parameters. This equation models oscillatory systems with nonlinear restoring forces.

For example, in the case of a double-well potential:

$$V(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2,$$

global bifurcation results can reveal multiple stable and unstable periodic orbits depending on energy levels.

In the study of the Duffing oscillator (a Hamiltonian system with a cubic potential), global bifurcation analysis shows how varying the amplitude of forcing can lead to a variety of dynamic behaviors, including chaotic motion and periodic orbits.

3.8.4. Global Solutions to Boundary Value Problems

Problem: Consider boundary value problems (BVPs) of the form:

$$u''(x) + f\big(u(x)\big) = 0$$

with boundary conditions

$$u(0) = u(L) = 0.$$

Solution: The BVP can be rewritten as:

$$u''(x) = f(u(x))$$

with the boundary conditions: $u(0) = 0, u(L) = 0$.

Existence and uniqueness of solutions depend on the nature of the nonlinearity $f(u)$ and additional conditions. For many problems, the following conditions can be useful. If $f(u)$ is a linear function, say $f(u) = \lambda$, then the problem simplifies to a linear ordinary differential equation (ODE) with boundary conditions. The theory of linear BVPs provides conditions for existence and uniqueness. For nonlinear functions $f(u)$, existence and uniqueness might be more complex. Techniques such as the Leray-Schauder Fixed Point Theorem**,** Schauder's Fixed Point Theorem, or the Galerkin Method can be used. The specific methods depend on the properties of f (e.g., if f is Lipschitz continuous or satisfies certain growth conditions). If $f(u)$ is simple, direct integration might work. For example, if $f(u) = constant$, then: $u''(x) = constant$. Integrating twice and applying boundary conditions will give a solution. For more complex $f(u)$, consider a variational approach. Reformulate the problem as finding a critical point of an energy functional:

$$J(u) = \frac{1}{2}\int_0^L (u'(x))^2\, dx - \int_0^L (u\,(x))\; dx$$

Where $F'(u) = f(u)$.

Solve for u such that $J'(u) = 0$, which corresponds to solving the Euler-Lagrange equation. Numerical methods can approximate solutions. Finite difference methods or finite element methods are commonly used for solving BVPs, especially when f(u) is nonlinear and difficult to handle analytically. If $f(u) = \lambda$, then:

$$-u''(x) + \lambda \mathrm{u(x)} = 0.$$

The general solution is:

$$u(x) = \left(A sin\left(\sqrt{\lambda \mathrm{x}}\right) + \mathrm{Bcos}\left(\sqrt{\lambda \mathrm{x}}\,\right)\right).$$

Applying boundary conditions $u(0) = 0$, and $u(L) = 0$, we find:

$$B = 0, A = \sqrt{\lambda} = \frac{n\pi}{L} u(x) = \sqrt{\lambda} = A sin(\frac{n\pi}{L}).$$

where n is a positive integer, and $\lambda_n = (\frac{n\pi}{L})$.

For nonlinear $f(u)$, if $f(u) = u^3$, the equation becomes:

$$u''(x) = u(x)^3$$

This is a more challenging problem and may require numerical or approximate methods to solve. Such problems often arise in physical contexts, such as in beam deflection theory or in certain models of reaction-diffusion systems. Understanding the physical context can often provide insights into appropriate methods and solution behavior. Global solutions describe how boundary conditions and nonlinearities interact to produce a range of possible solution profiles.

Example 3.9: *prove the assertion of the system scalar equations*:

$$\begin{cases} x = \lambda x + y^3 \\ y = \lambda y - x^3 \end{cases}$$

Proof: To analyze the system of scalar equations:

$$\begin{cases} x = \lambda x + y^3 \\ y = \lambda y - x^3 \end{cases}$$

we need to find the conditions for which this system has solutions and to understand the behavior of these solutions. Rewrite the system as:

$$\begin{cases} x = \lambda x + y^3 \\ y = \lambda y - x^3 \end{cases}$$

Let's solve for x and y in terms of λ.

From the first equation:

$$x = \lambda x + y^3$$

Rearrange it to:

$$x - \lambda x = y^3$$

$$x(1 - x) = y^3$$

So:

$$x = \frac{y^3}{1 - \lambda}$$

(assuming λ≠1)

From the second equation:

$$y = \lambda y - x^3$$

Rearrange it to:

$$y - \lambda y = x^3$$

So:

$$y = \frac{-x^3}{1 - \lambda}$$

(assuming λ≠1).

Substitute $x = \frac{y^3}{1-\lambda}$ into $= \frac{-x^3}{1-\lambda}$:

$$\frac{(\frac{y^3}{1 - \lambda})^3}{1 - \lambda} = \frac{\frac{y^9}{(1 - \lambda)^3}}{1 - \lambda}$$

Rearrange to get:

$$y(1-\lambda)^{4} = y^9.$$

Rearrange and solve for y:

$$y^9 + y(1-\lambda)^{4} = 0.$$

Factor out y:

$$y(y^8 + (1-\lambda)^{4}) = 0.$$

So the solutions are:

$$y = 0$$

or

$$y^8 + (1-\lambda)^{4} = 0$$

For $y \neq 0$:

$$y^8 = -(1-\lambda)^{4}$$

Since $y^8 \geq 0$, this equation has solutions if and only if $(1-\lambda)^{4} \leq 0$, which implies:

$$1-\lambda = 0 \text{ or } \lambda = 1$$

However, substituting λ=1 into the original system leads to:

$$\begin{cases} x = y3 \\ y = -x3 \end{cases}$$

Substitute $x = y^3\ into\ y = -x^3 : y = -(y^3)^{3}$

$$y = -y^9$$

This simplifies to:

$$y + y^9 = 0$$

$$y(1 + y^8) = 0$$

So $y = 0\ or\ y^8 = -1$. Since $y^8 \geq 0$, the only solution is $y = 0$. Substituting $y = 0$, gives $x = 0$

Conclusion

Recap of global IFT, bifurcation results, and topological methods.

In concluding a thesis on global solution theorems for Ordinary Differential Equations (ODEs), it's essential to recap the significant findings related to the Global Implicit Function Theorem (IFT), bifurcation results, and topological methods. These concepts form the backbone of understandisng and analyzing global behaviors of ODEs and their solutions.

Global IFT extends the classical Implicit Function Theorem by providing conditions under which a solution to a system of ODEs can be globally parameterized as a function of one or more variables. It ensures the existence and smoothness of solutions to differential equations in a global context, beyond local neighborhoods.

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