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SEMINAR REPORT

ON

THE FUNDAMENTAL GROUP

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Symbols and notations

$\sim, \approx$, homotopic maps

$\approx$, homeomorphic spaces

$\cong$, isomorphic groups

ker (f), kernel of a homomorphism

$I = I^1 = [0,1] \subset R$, the unit interval.

$D^n = \{x \in R^n : |x| \leq 1\}$, the unit n-disk.

$S^{n-1} = \{x \in R^n : |x| = 1\}$, the unit sphere of dimension n-1.

$S^1 = \{x \in R^2 : |x| = 1\}$, the one dimensional sphere, circle group.

$S^\infty = \{x \in l^2 : |x| = 1\}$, the infinite dimensional sphere.

Introduction

One of the basic problems in topology is to determine whether two given two topological spaces are homeomorphic or not. Showing that two spaces are homeomorphic is a matter of constructing a continuous mapping from one to the other having a continuous inverse, and constructing continuous functions is a problem that we have developed techniques to handle.

But the topological properties we have studied up to now do not carry us very far in solving the problem. For instance, how does one show that the plane R^2 is not homeomorphic to three-dimensional space R^3? As one goes down the list of topological properties—*compactness, connectedness, local connectedness, metrizability, and so on* one can find no topological property that distinguishes between them. As another example, consider such surfaces as the 2-sphere S^2, the torus T (surface of a doughnut), and the double torus T#T (surface of two-holed doughnut). None of the topological properties we have studied up to now will distinguish them.

So we must introduce new properties and new techniques. One of the most natural such properties is *simple connectedness*. Roughly speaking, one says that a space X is simply connected if every closed curve in X can be shrunk to a point in X. The property of simple connectedness, it turns out, will distinguish between R^2 and R^3; deleting a point from R^3 leaves a connected space remaining, but deleting from R^2 does not. It will also distinguish between S^2 (which is simply connected) and the torus T (which is not). But it will not distinguish between T and T#T; neither of them is simply connected.

There is an idea more general than the idea of simple connectedness, an idea that includes simple connectedness as a special case. It involves a certain group that is called the fundamental group of the space. Two spaces that are homeomorphic have fundamental groups which are isomorphic. And the condition of simple connectedness is just the condition that the fundamental group of X is the trivial (one element) group.

How can we distinguish between S^2 *and* T^2?

Figure1. Torus, T^2.

Figure 2. Sphere, S^2 .

A closed loop on S^2 can always be deformed into a constant loop: but this fails on T^2 .Thus the proof that S^2 and T^2 are not homeomorphic can be rephrased by saying that the fundamental group of S^2 is trivial and the fundamental group of T is not. The fundamental group will distinguish between more spaces than the condition of simple connectedness will. It can be used, for example, to show that T and T#T are not homeomorphic; it turns out that T has an abelian fundamental group and T#T does not.

Important Definitions and Theorems

1. The pasting lemma: Let $X = AUB$, where A and B are closed in X. Let $f : A \to Y$ *and* $g : B \to Y$ be continuous. If $f(x) = g(x)$ for every $x \in A \cap B$ then f *and* g combine to give a continuous function $h(x) = \begin{cases} f(x), & \text{if } x \in A \\ g(x) & \text{if } x \in B \end{cases}$.

2. Definition: A space X is said to be arcwise connected or pathwise connected if any two points of X can be joined by an arc. The maximal arcwise-connected subsets of X, called arcwise connected components (arc components) of X, form a partition of X.

3. Definition: A space X is locally arcwise connected if each point has a basic family of arcwise connected neighborhoods.

4. An arcwise-connected space is connected, but the converse is not true. A space which is connected and locally arcwise connected is arcwise connected.

5. Theorem (Seifert-Van Kampen Theorem, special case)
Suppose $X = U \cup V$, where U *and* V simply connected open are subspaces of X with $U \cap V$ nonempty and pathwise connected then X is simply connected.

Homotopy of Paths

The homotopy of paths plays an important role not only in the course complex analysis, in evaluating complex integrals but also in the study of the fundamental group.

Before defining the fundamental group of a space X, we shall consider paths on X and an equivalence relation called path homotopy between them. And we shall define a certain operation on the collection of the equivalence classes that makes it into what is called algebra of groupoid.

Definition: If f_0 and f_1 are continuous maps of the space X in to the space Y, we say that f_0 is homotopic to f_1 if there is a continuous map F: X × I →Y such that

F(x, 0) = f_0(x) and F(x, 1) = f_1(x) for each x∈ X. Here (I= [0, 1].).

The map F is called homotopy between f_0 and f_1. If f_0 is homotopic to f_1, we write $f_0 \sim f_1$.

If $f_0 \sim f_1$ and f_1 is a constant map, we say that f_0 is *nullhomotopic*.

Now we consider the special case in which f is a path in X.

Definition: If f: [0, 1] $\rightarrow$X is a continuous map such that f (0) =x_0 and f (1) =x_1, we say that f is a path in X from x_0 to x_1. We also say x_0 is the initial point, and x_1 is the terminal point, of the path f.

Remark: From now on we use the interval I= [0, 1] as the domain for all paths.

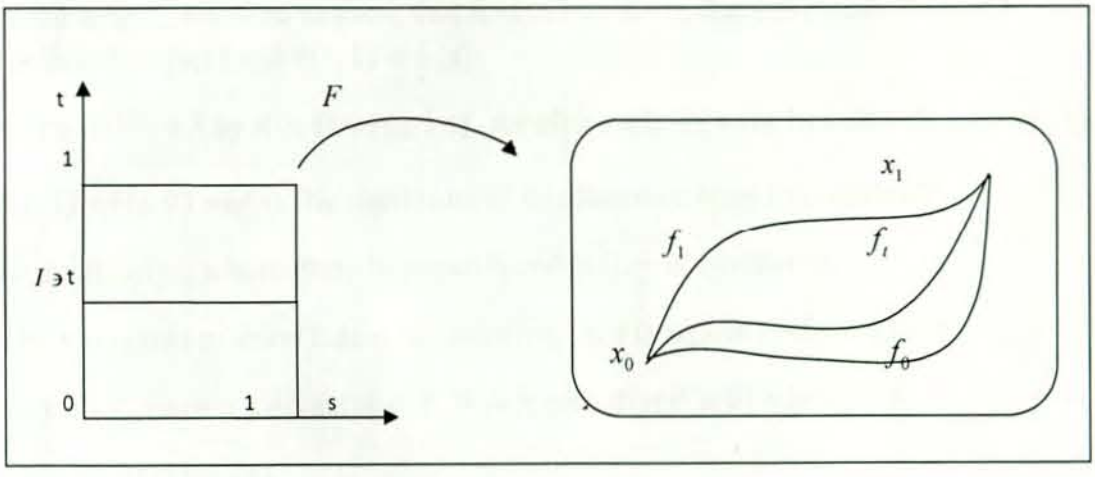
Definition: Two paths f_0 and f_1, mapping the interval I=[0,1] into X , are said to be *path homotopic* if they have the same initial point x_0 and the same terminal point x_1 , and if there is a continuous map F: I x I $\rightarrow$X such that

F(s, 0) =f_0 (s) and F(s, 1) =f_1(s), F(0,t)=x_0 and F(1,t)=x_1 for each s $\in$ I and each t $\in$ I.

We call F a path homotopy between f_0 and f_1. See Figure 1.

If f_0 is path homotopic to f_1, we write $f_0 \sim f_1$.

If we imagine the parameter t as representing time, then the homotopy f represents a continuous "deforming" of the map f_0 to the map f_1, as t goes from 0 to 1.



The first condition says simply that F is a homotopy between f_0 and f_1, and the second condition says that for each t, the path f_t defined by the equation $f_t(s) = F(s,t)$ is a path from x_0 to x_1. Said differently, the first condition says that F represents a continuous way of deforming the path f_0 to f_1, and the second condition says that the end points of the path remain fixed during the deformation.

Lemma The relation ~ is an equivalence relation.

Proof: Given paths f, f_1 and f_2.To check the properties of equivalence relation.

Now the homotopy F: $I^2 \to X$, between f and f is given by $F(s, t) = f(s)$ $\forall s,t \in I$.

We must show that F (s, t) is continuous and that $F(s, 0) = F(s, 1) = f(s)$.

Given $\varepsilon > 0$ we want to find $\delta > 0$ such that if $0 < \sqrt{(s-s_0)^2 + (t-t_0)^2} < \delta$ then $|F(s,t) - F(s_0,t_0)| < \varepsilon$.

Now $|F(s,t) - F(s_0,t_0)| = |f(s) - f(s_0)|$, and since f is continuous on I given $\varepsilon > 0$ there is a $\delta > 0$ such that if $0 < |s - s_0| < \delta'$ then $|f(s) - f(s_0)| < \varepsilon$, but this implies $|F(s,t) - F(s_0,t_0)| = |f(s) - f(s_0)| < \varepsilon$

Thus F is continuous on I x I. Clearly, F(s, 0) =F(s, 1) =f(s).So ~ is reflexive.

Suppose $f_0 \sim f_1$, we need to show that $f_1 \sim f_0$. Since $f_0 \sim f_1$, there is a homotopy F: $I^2 \to X$ such that $F(s, 0) = f_0(s)$ and $F(s, 1) = f_1(s)$.

Define H: $I^2 \to X$ by H(s, t) =F(s, 1-t) for all s, t ∈ I. We have $H(s, 0) = F(s, 1) = f_1(s)$, and

$H(s, 1) = F(s, 0) = f_0(s)$.The continuity of H is followed from the continuity of F.

Hence, H(s, t) is a homotopy between f_1 and f_0.So ~ is symmetric.

Now it remains to show that ~ is transitive, i.e. if $f_0 \sim f_1$ and $f_1 \sim f_2$then $f_0 \sim f_2$.

Since $f_0 \sim f_1$ there is a homotopy F: $I^2 \to X$ such that $F(s, 0) = f_0(s)$ and

$F(s, 1) = f_1(s)$ for s∈ I.

Similarly, since $f_1 \sim f_2$, there is a homotopy G: $I^2 \to X$ such that

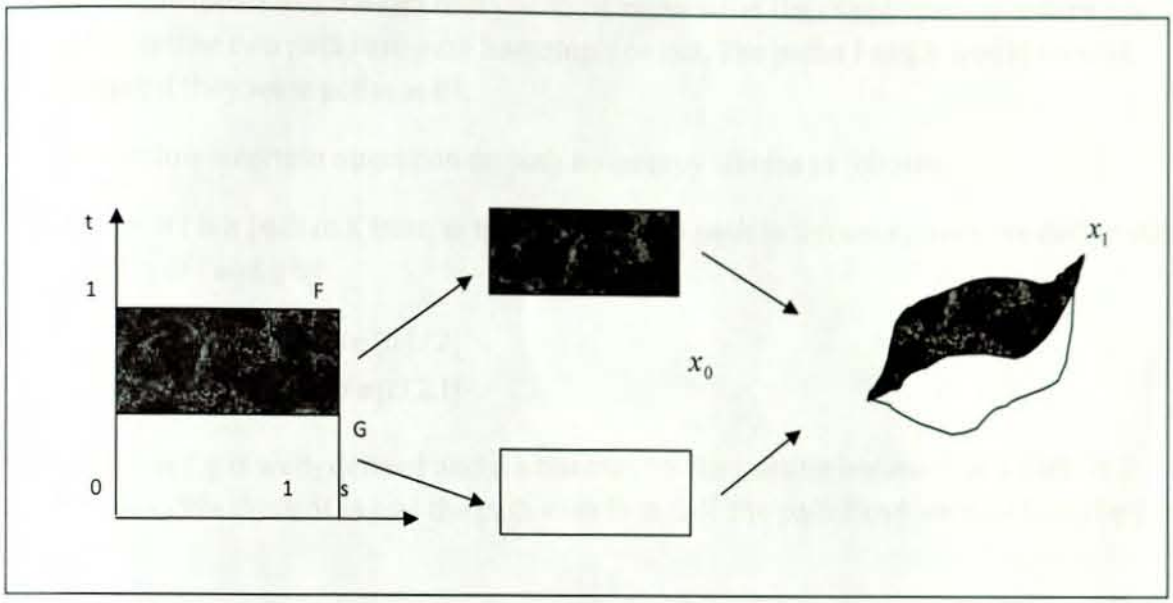
$G(s, 0) = f_1(s)$ and $G(s, 1) = f_2(s)$ for all s∈ I. Let us define a mapping h: $I^2 \to X$ by

$$h(s, t) = \begin{cases} F(s,2t) & for \quad 0 \le t \le 1/2 \\ G(s,2t-1) & for \ 1/2 \le t \le 1 \end{cases}$$ for all s∈ I.

Note that $h(s, 0) = F(s, 0) = f_0(s)$ and $h(s, 1) = G(s, 1) = f_2(s)$ $h(s, 1/2) = F(s, 0) = G(s, 1) = f_1(s)$

Mapping h(s, t) is well defined, since if t=1/2, we have F(s,2t) = $f_1(s)$ =G(s, 2t-1) and since h is continuous on the two closed subsets I x [0, 1/2] and I x [1/2, 1] of I^2, it is continuous on all of I^2, by the pasting lemma.

Thus h(s, t) is a homotopy between f_0 and f_2. So ~ is transitive. See figure 2.



Therefore, the relation ~ is an equivalence relation.

Definition: The family of all continuous maps from I into X, which are homotopic to a continuous map f, is called the *homotopy class* of f, denoted by [f].

Example1. Let f and g be any two maps of a space X in to R^2. It is easy to see that f and g are homotopic; the map

F(x, t) = (1- t)f(x) + t g(x) is a homotopy between them . It is called a *straight-line homotopy* because it moves the point f(x) to the point g(x) along the straight line segment joining them.

If f and g are paths from x_0 to x_1, then F will be a path homotopy. This situation is pictured in Figure 3.More generally, let A be any convex subspace of R^n. (This means that for any two points a, b of A, the straight line segment joining a and b is contained in A.).Then any two paths f , g in A from x_0 to x_1 are path homotopic in A, for the straight-line homotopy F between them has image in set A.

Example 2 Let X denotes the punctured plane, R^2-{0}. The following paths in X,

$f(s) = (\cos \pi s , \sin \pi s)$, $g(s) = (\cos \pi s , 2\sin \pi s)$ are path homotopic ; the straight line homotopy between them is an acceptable path homotopy. But the straight line homotopy between f and the path $h(s) = (\cos \pi s , -\sin \pi s)$ is not acceptable , for its image does not lie in the space X= R^2-{0}.See figure below.

This example illustrates the fact that you must know what the range space is before you can tell whether two paths are path homotopic or not. The paths f and h would be path homotopic if they were paths in R^2.

Now we define a certain operation on path homotopy classes as follows:

Definition: If f is a path in X from x_0 to x_1, and if g is a path in X from x_1 to x_2, we define the product f .g of f and g by

$$(f.g)(t) = \begin{cases} f(2t) & for\ t \in [0,1/2] \\ g(2t-1) & for\ t \in [1/2,1] \end{cases}$$

The function f .g is well- defined and continuous, by the pasting lemma; it is a path in X from x_0 to x_2.We think of f • g as the path with first half the path f and second half the path g.

The product operation on paths induces a well-defined operation on path-homotopy classes, defined by the equation

[f] . [g] = [f .g]

Step 1 let h_1, h_2, h_3 be paths such that $h_1 \sim h_3$, then we will show $h_1 \bullet h_2 \sim h_3 \bullet h_2$

Since $h_1 \sim h_3$,a homotopy $h : I^2 \to X$ exists: h(s, 0) = h_1 (s) and h(s, 1)= h_3(s)

We define H: I^2 ⟶X by $H(s,t) = \begin{cases} h(2s,t) & s \in [0,1/2] \\ h_2(2s-1) & s \in [1/2,1] \end{cases}$

Thus, $H(s,0) = \begin{cases} h(2s,0) & s \in [0,1/2] \\ h_2(2s-1) & s \in [1/2,1] \end{cases} = \begin{cases} h_1(2s) & s \in [0,1/2] \\ h_2(2s-1) & s \in [1/2,1] \end{cases} = (h_1 \bullet h_2)(s) \quad s \in [0,1]$

Also $H(s,1) = \begin{cases} h(2s,1) & s \in [0,1/2] \\ h_2(2s-1) & s \in [1/2,1] \end{cases} = \begin{cases} h_3(2s) & s \in [0,1/2] \\ h_2(2s-1) & s \in [1/2,1] \end{cases} = (h_3 \bullet h_2)(s) \quad s \in [0,1]$

The continuity of H follows from the continuity of h and h_2. Hence, $h_1 \bullet h_2 \sim h_3 \bullet h_2$

Step 2 Suppose $h_1 \sim h_3$ and $h_2 \sim h_4$ .By step 1, we have that $h_1 \bullet h_2 \sim h_3 \bullet h_2$

Similarly, we can show that $h_1 \bullet h_2 \sim h_1 \bullet h_4$

The homotopy will be

$$G : I^2 \to X \text{ by } G(s,t) = \begin{cases} h_1(2s) & s \in [0,1/2] \\ h'(2s-1,\, t) & s \in [1/2,1] \end{cases}$$

Where h' is the homotopy between h_2 and h_4.

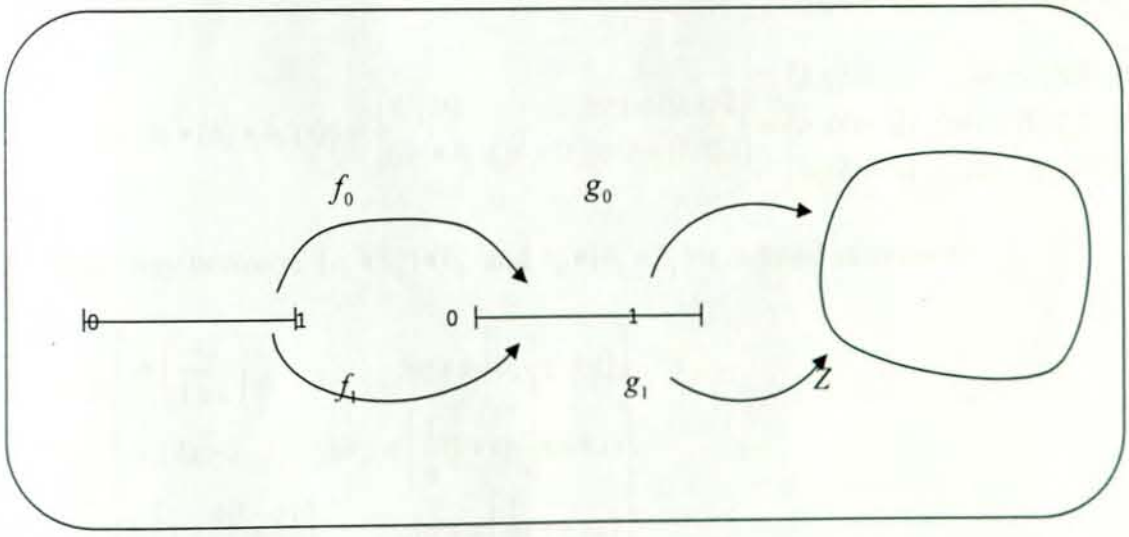
Thus, $[h_1 \bullet h_2] = [h_1] \bullet [h_2], [h_3 \bullet h_2] = [h_3] \bullet [h_2]$ and $[h_1] \bullet [h_2] = [h_1] \bullet [h_4]$

We have $h_1 \bullet h_2 \sim h_3 \bullet h_2 \sim h_3 \bullet h_4$.

Hence, if $[h_1] = [h_3]$ *and* $[h_2] = [h_4]$, *then* $[h_1] \bullet [h_2] = [h_3] \bullet [h_4]$

Lemma The equivalence relation and the product we have defined are compatible in the following sense: If $f_0 \sim f_1$ and $g_0 \sim g_1$, then $f_0 \bullet g_0 \sim f_1 \bullet g_1$ (it is assumed, off course, that the terminal point of f_0 is the initial point of g_0)

Proof: First let's draw diagram for the above statement



Since $f_0 \sim f_1$, a continuous map $h_1(s, t) \in I$ exists: $h_1(s, 0) = f_0(s)$, $h_1(s, 1) = f_1(s)$

Since $g_0 \sim g_1$, a continuous map $h_2(s, t) \in I$ exists: $h_2(s, 0) = g_0(s)$, $h_2(s, 1) = g_1(s)$

To show $g_0 f_0 \sim g_1 f_1$. It suffices to show that a continuous function $H(s, t) \in Z$ exists:

$H(s, 0) = (g_0 f_0)(s)$ and $H(s, 1) = (g_1 f_1)(s)$ Define $H(s, t) = h_2(h_1(s, t), t)$.

Then, $H(s, 0) = h_2(h_1(s, 0), 0) = h_2(f_0(s), 0) = g_0(f_0(s)) = (g_0 f_0)(s)$

$H(s, 1) = h_2(h_1(s, 1), 1) = h_2(f_1(s), 1) = g_1(f_1(s)) = (g_1 f_1)(s)$

And since h_1 and h_2 are continuous so is their composition, and H is also continuous.

Thus, H is a homotopy between $g_0 f_0$ and $g_1 f_1$.

Lemma The multiplication of equivalence classes of paths is associative.

Proof: Let $[h_1]$, $[h_2]$, $[h_3]$ represent equivalence class of paths h_1, h_2 and h_3 respectively.

To show $([h_1] \bullet [h_2]) \bullet [h_3] = [h_1] \bullet ([h_2] \bullet [h_3])$. It suffices to show that

$(h_1 \bullet h_2) \bullet h_3 \sim h_1 \bullet (h_2 \bullet h_3)$

By definition $((h_1 \bullet h_2) \bullet h_3)(t) = \begin{cases} (h_1 \bullet h_2)(2t) & for\ t \in [0,1/2] \\ h_3(2t-1) & for\ t \in [1/2,1] \end{cases} = \begin{cases} h_1(4t) & for\ t \in [0,1/4] \\ h_2(4t-1) & for\ t \in [1/4,1/2] \\ h_3(2t-1) & for\ t \in [1/2,1] \end{cases}$

and $(h_1 \bullet (h_2 \bullet h_3))(t) = \begin{cases} h_1(2t) & for\ t \in [0,1/2] \\ (h_2 \bullet h_3)(2t-1) & for\ t \in [1/2,1] \end{cases} = \begin{cases} h_1(2t) & for\ t \in [0,1/2] \\ h_2(4t-2) & for\ t \in [1/2,3/4] \\ h_3(4t-3) & for\ t \in [3/4,1] \end{cases}$

The homotopy between $(h_1 \bullet h_2) \bullet h_3$ and $h_1 \bullet (h_2 \bullet h_3)$ is defined as follows:

$$H(s,t) = \begin{cases} h_1\left(\dfrac{4s}{1+t}\right) & for\ s \in \left[0, \dfrac{1}{4}(1+t)\right] \\ h_2(4s-1-t) & for\ s \in \left[\dfrac{1}{4}(1+t), \dfrac{1}{4}(2+t)\right] \\ h_3\left(1 - \dfrac{4(1-s)}{2-t}\right) & for\ s \in \left[\dfrac{1}{4}(2+t), 1\right] \end{cases}$$

Then, H is continuous as h_1, h_2 and h_3 are and by the pasting lemma. And

$$H(s,0)=\begin{cases} h_1(4s) & for\ s\in\left[0,\frac{1}{4}\right] \\ h_2(4s-1) & for\ s\in\left[\frac{1}{4},\frac{1}{2}\right] \\ h_3(2s-1) & for\ s\in\left[\frac{1}{2},1\right] \end{cases}=((h_1\bullet h_2)\bullet h_3)(s) \qquad H(s,1)=\begin{cases} h_1(2s) & for\ s\in\left[0,\frac{1}{2}\right] \\ h_2(4s-2) & for\ s\in\left[\frac{1}{2},\frac{3}{4}\right] \\ h_3(4s-3) & for\ s\in\left[\frac{3}{4},1\right] \end{cases}=(h_1\bullet(h_2\bullet h_3))(s)$$

Thus H(s, t) is indeed the homotopy between $(h_1\bullet h_2)\bullet h_3$ and $h_1\bullet(h_2\bullet h_3)$.

Lemma Let $\xi_x:[0,1]\to x\in X$ denote the equivalence class of the constant map of [0, 1] onto $x\in X$. Let φ be an equivalence class of paths with initial point x and terminal point y. Then $\xi_x\bullet\varphi=\varphi$ and $\varphi\bullet\xi_y=\varphi$.

Proof: Let $e:I\to X$ be the constant map such that $e(I)=\{x\}$ *and let* $f:I\to X$ be a representative of the path class φ.

To show $\xi_x\varphi=\varphi$, it suffices to prove that $e\bullet f\sim f$.

Define $F:I^2\to X$ by $F(s,t)=\begin{cases} x, & s\in\left[0,\frac{1-t}{2}\right] \\ f\left(\frac{2s-1+t}{1+t}\right), & s\in\left[\frac{1-t}{2},1\right] \end{cases}$

Then, $F(s,0)=\begin{cases} x, & s\in\left[0,\frac{1}{2}\right] \\ f(2s-1), & s\in\left[\frac{1}{2},1\right] \end{cases}=\begin{cases} e(2s), & s\in\left[0,\frac{1}{2}\right] \\ f(2s-1), & s\in\left[\frac{1}{2},1\right] \end{cases}=(e\bullet f)(s)$ for $s\in[0,1]$

$F(s,1)=\begin{cases} x, & s=0 \\ f(s), & s\in[0,1] \end{cases}=f(s)$ *for* $s\in[0,1]$, this is true since $x\in\varphi$.

The continuity of F is followed from the continuity of e *and* f and the pasting lemma.

Thus $F(s,t)$ is the required homotopy between $e\bullet f$ and f. And hence we have $\xi_x\varphi=\varphi$.

Now we show that $\varphi\bullet\xi_y=\varphi$. We use the above procedure.

Let $e_y : I \to X$ be the constant map such that $e_y(I) = \{y\}$ *and let* $f : I \to X$ be a representative of the path class φ .We want to show $\varphi \bullet \xi_y = \varphi$ i.e. to show $f \bullet e_y \sim f$

Define $G : I^2 \to X$ by $G(s,t) = \begin{cases} f\left(\dfrac{2s}{1+t}\right), & s \in \left[0, \dfrac{1}{2}(1+t)\right] \\ y, & s \in \left[\dfrac{1}{2}(1+t), 1\right] \end{cases}$

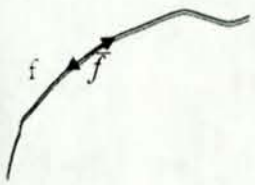
Then $G(s,0) = \begin{cases} f(2s), & s \in \left[0, \dfrac{1}{2}\right] \\ y, & s \in \left[\dfrac{1}{2}, 1\right] \end{cases} = \begin{cases} f(2s), & s \in \left[0, \dfrac{1}{2}\right] \\ e_y(2s-1), & s \in \left[\dfrac{1}{2}, 1\right] \end{cases} = (f \bullet e_y)(s)$

$$G(s,1) = \begin{cases} f(s), & s \in [0,1] \\ y, & s = 1 \end{cases} = f(s) \quad \text{for } s \in [0,1].$$

The continuity of G is followed from the continuity of e_y *and* f and the pasting lemma.

Thus $G(s,t)$ is the required homotopy between $f \bullet e_y$ and f . And hence we have $\varphi \bullet \xi_y = \varphi$.

Definition: For any path $f : I \to X$, let $\overline{f}$ denote the path defined by $\overline{f}(t) = f(1-t), t \in I$ .The path $\overline{f}$ is obtained by traversing the path f in opposite direction. That is , $\overline{f}(0) = f(1)$ *and* $\overline{f}(1) = f(0)$. See figure below.



Note that $\overline{f}$ has the same shape as of f , but it is drawn in opposite direction.

Lemma Let γ *and* $\overline{\gamma}$ denote the equivalence classes of the paths f and $\overline{f}$, respectively. Then,

$\gamma \bullet \overline{\gamma} = \xi_x$, $\overline{\gamma} \bullet \gamma = \xi_y$, where x *and* y are the initial and terminal points of the path f .

Proof: First let's show $\gamma \bullet \bar{\gamma} = \xi_x$. It is sufficient to show that $f \bullet \bar{f} \sim e$, where e is the constant path at the point x.

Define $F : I^2 \to X$ by $F(s,t) = \begin{cases} f(2s), & s \in \left[0, \frac{1-t}{2}\right] \\ f(1-t), & s \in \left[\frac{1-t}{2}, \frac{1+t}{2}\right] \\ f(2-2s), & s \in \left[\frac{1+t}{2}, 1\right] \end{cases}$

$$Then,\ F(s,0) = \begin{cases} f(2s), & s \in \left[0, \frac{1}{2}\right] \\ f(1), & s = \frac{1}{2} \\ f(2-2s), & s \in \left[\frac{1}{2}, 1\right] \end{cases} = \begin{cases} f(2s), & s \in \left[0, \frac{1}{2}\right] \\ \bar{f}(2s-1), & s \in \left[\frac{1}{2}, 1\right] \end{cases} = (f \bullet \bar{f})(s) \text{ and}$$

$$F(s,1) = \begin{cases} f(2s), & s = 0 \\ f(0), & s \in [0,1] = f(0) = x = e(s) \text{ for } s \in [0,1] \\ f(2-2s), & s = 1 \end{cases}$$

Since F is continuous by pasting lemma, $F(s,t)$ is the homotopy between $f \bullet \bar{f}$ and e.

Thus, $\gamma \bullet \bar{\gamma} = \xi_x$.

Now let's show, $\bar{\gamma} \bullet \gamma = \xi_y$. It is sufficient to show that $\bar{f} \bullet f \sim e_y$, where e_y is the constant path at the point y.

Define $G : I^2 \to X$ by $G(s,t) = \begin{cases} \bar{f}(2s), & s \in \left[0, \frac{1}{2}(1-t)\right] \\ \bar{f}(1-t), & s \in \left[\frac{1}{2}(1-t), \frac{1}{2}(t+1)\right] \\ \bar{f}(2-2s), & s \in \left[\frac{1}{2}(t+1), 1\right] \end{cases}$

$$\text{Then } G(s,0)=\begin{cases}\overline{f}(2s), & s\in\left[0,\frac{1}{2}\right]\\ \overline{f}(1), & s=\frac{1}{2}\\ \overline{f}(2-2s), & s\in\left[\frac{1}{2},1\right]\end{cases}=\begin{cases}\overline{f}(2s), & s\in\left[0,\frac{1}{2}\right]\\ \overline{f}(1-(2s-1)), & s\in\left[\frac{1}{2},1\right]\end{cases}$$

$$=\begin{cases}\overline{f}(2s), & s\in\left[0,\frac{1}{2}\right]\\ \overline{\overline{f}}(2s-1), & s\in\left[\frac{1}{2},1\right]\end{cases}=\begin{cases}\overline{f}(2s), & s\in\left[0,\frac{1}{2}\right]\\ f(2s-1), & s\in\left[\frac{1}{2},1\right]\end{cases}=(\overline{f}\bullet f)(s),\text{and}$$

$$G(s,1)=\begin{cases}\overline{f}(2s), & s=0\\ \overline{f}(0), & s\in[0,1]\\ \overline{f}(2-2s), & s=1\end{cases}=\overline{f}(0)=f(1)=y=e(s)\ \textit{for}\ s\in[0,1].$$

Since G is continuous by pasting lemma, $G(s,t)$ is the homotopy between $\overline{f}\bullet f$ and e_y

Therefore, $\overline{\gamma}\bullet\gamma=\xi_y$.

From now on we will denote the path class $\overline{\gamma}$ by γ^{-1}. The conditions of the above lemma characterize γ^{-1} uniquely.

Lemma If $f_0, f_1: I\to X$, such that $f_0\sim f_1$, then $\overline{f_0}\sim\overline{f_1}$.

Proof: Since $f_0\sim f_1$ there is a continuous map $F: I^2\to X$ Such that $F(s,0)=f_0(s)$ and $F(s,1)=f_1(s)\ \forall s\in I$.

Define $G: I^2\to X$ by $G(s,t)=F(1-s,t)$ .Then G(s, t) is continuous as F is. And

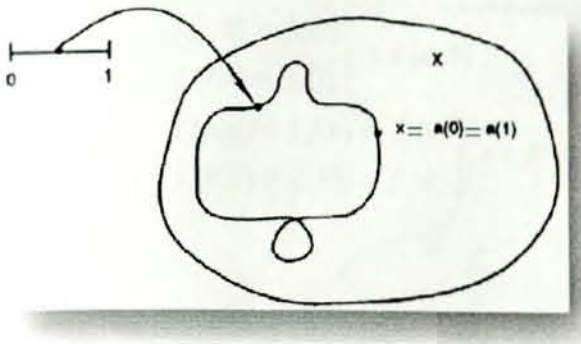
$G(t,0)=F(1-t,0)=f_0(1-t)=\overline{f_0}(t)$ *and* $G(t,1)=F(1-t,1)=f_1(1-t)=\overline{f_1}(t)\ \forall t\in[0,1]$

Thus G is the required homotopy between $\overline{f_0}$ and $\overline{f_1}$.

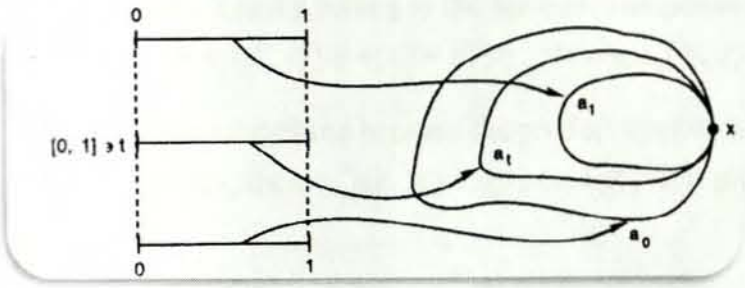
The lemmas just proved showed that all path classes in X satisfy the axioms for a group, if the product of two paths is always defined.

Definition: A path, or path class, is called closed, or a *loop*, if the initial and terminal points are the same. The loop is said to be based at the common end point.

(or a continuous function $a : I \to X$ *such that* $a(0) = a(1) = x$ is called a loop in X based at the point x. See the first figure below.)



The figure below shows homotopy of loops based at a point x.



Let x be any point of X; since the conditions in the above lemmas are satisfied, the set of all loops based at x is a group. This group is called the ***fundamental group*** or ***Poincarè group*** of X at the base point x and is denoted by $\pi_1(X, x)$.

i.e. $\pi_1(X, x) = \{[\gamma]: \gamma \text{ is a loop based at } x\}$ is a group with respect to the multiplication $[\gamma][\lambda] = [\gamma\lambda]$ and with neutral element $\xi_x = \{x\}$.

Note: The fundamental group is introduced by the great French Mathematician Henri Poincarè((29 April 1854 – 17 July 1912)) in 1895(Analysis Situs, J. Ecole Polytechn. 1(1895), 1-121).

Definition: If for any $\gamma \in \pi_1(X, x)$ such that $\gamma \sim \xi_x$, then the fundamental group of X is trivial. That is $\pi_1(X, x) = \{1\}$

Definition: Let $f_0, f_1 : [a, b] \to X$ be two paths in X such that $f_0(a) = f_1(a), f_0(b) = f_1(b)$ (i.e. the two paths have the same initial and terminal points).We say that these two paths are equivalent, denoted by $f_0 \sim f_1$,if and only if there exists a continuous map

$f : [a,b]xI \to X$, such that

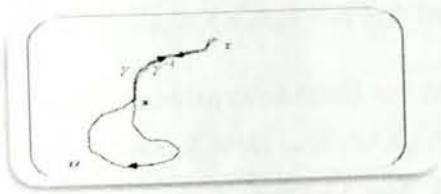
$$\left.\begin{aligned} f(t,0) &= f_0(t) \\ f(t,1) &= f_1(t) \end{aligned}\right\} \; t \in [a,b]$$

$$\left.\begin{aligned} f(a,s) &= f_0(a) = f_1(a) \\ f(b,s) &= f_0(b) = f_1(b) \end{aligned}\right\} \; s \in I$$

Theorem If X is arcwise connected, the groups $\pi_1(X, x)$ and $\pi_1(X, y)$ are isomorphic for any two points $x, y \in X$.

Proof: Let x and y two points in X, and let γ be a path class with initial point x and terminal point y (hence, x and y belong to the same arc component of X). Define a mapping $u : \pi_1(X,x) \to \pi_1(X,y)$ by $u(\alpha) = \gamma^{-1}\alpha\gamma$, for $\alpha \in \pi_1(X, x)$.

The map u is well-defined because the product operation of loops is well-defined i.e. if α is a loop based at x, then $\gamma^{-1}\alpha\gamma$ is a loop based at y. It is pictured below.



Then for $\alpha, \beta \in \pi_1(X,x)$,

$u(\alpha \bullet \beta) = \gamma^{-1}(\alpha \bullet \beta)\gamma = \gamma^{-1}(\alpha\gamma \bullet \gamma^{-1}\beta)\gamma = (\gamma^{-1}\alpha\gamma) \bullet (\gamma^{-1}\beta\gamma) = u(\alpha) \bullet u(\beta)$

Thus u is a homomorphism of $\pi_1(X,x)$ *into* $\pi_1(X,y)$.

Also $v : \pi_1(X,y) \to \pi_1(X,x)$ *by* $v(\beta) = \gamma\beta\gamma^{-1}$, $\beta \in \pi_1(X,y)$ is a homomorphism.

Since, for $\lambda, \beta \in \pi_1(X,y)$,

$v(\lambda \bullet \beta) = \gamma(\lambda \bullet \beta)\gamma^{-1} = \gamma(\lambda\gamma^{-1} \bullet \gamma\beta)\gamma^{-1} = (\gamma\lambda\gamma^{-1}) \bullet (\gamma\beta\gamma^{-1}) = v(\lambda) \bullet v(\beta)$

Thus v is a homomorphism of $\pi_1(X,y)$ *into* $\pi_1(X,x)$.

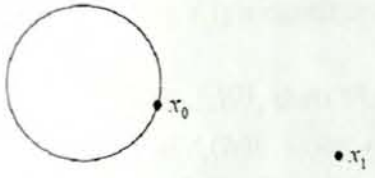
Also, $(uv)(\beta) = u(v(\beta)) = u(\gamma\beta\gamma^{-1}) = \gamma^{-1}(\gamma\beta\gamma^{-1})\gamma = \beta \ \forall \beta \in \pi_1(X,y).$

and $(vu)(\alpha) = v(u(\alpha)) = v(\gamma\alpha\gamma^{-1}) = \gamma(\gamma\alpha\gamma^{-1})\gamma^{-1} = \alpha \ \ \forall \alpha \in \pi_1(X,x).$

Thus the composed homomorphisms vu *and* uv are the identity maps of $\pi_1(X,x)$ *and* $\pi_1(X,y)$ respectively. Whence, u *and* v are isomorphisms, each of which is the inverse of the other. Therefore, $\pi_1(X,x)$ *and* $\pi_1(X,y)$ are isomorphic.

Remark: Given two base points x_0 *and* x_1 in X, we cannot in general expect any relationship between $\pi_1(X,x_0)$ and $\pi_1(X,x_1)$. For example, if x_0 *and* x_1 do not lie in a common connected subset of X, there can be no relationship.

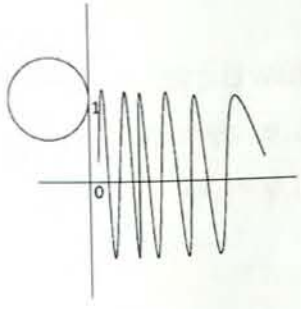
Consider the disjoint union X of a circle and a point x_1 (See fig. below)



For x_0 in the circle, $\pi_1(X,x_0)$ has non trivial elements, whereas $\pi_1(X,x_1)$ is clearly trivial.

Also, if X is not arcwise connected, we can expect no relationship between these groups. Consider the space $X = X_1 \cup X_2 \cup X_3 \subset R^2$, where X_1 is the graph of $\sin(1/x)$ *for* $0 < x < 1$, $X_2 = \{0\} \times I$ *and* X_3 *is a circle* tan*gent to the y axis at* (0, 1) with center to the left of the y axis(See fig below).

For $x_0 \in X_1$, $\pi_1(X,x_0)$ is trivial. But for $x_1 = (0,1)$, $\pi_1(X,x_1)$ has non-trivial elements.



The Effect of a Continuous Mapping on the fundamental Group

We now see how continuous maps effect π_1:

Let $\varphi : X \to Y$ be a continuous mapping, and let $f_0, f_1 : I \to X$ be paths in X.

Then if $f_0 \sim f_1$, then $\varphi f_0 \sim \varphi f_1$ represented by the composed functions.

Proof: Since $f_0 \sim f_1$, there is a homotopy $F: I^2 \to X$ such that $F(s, 0) = f_0(s)$ and $F(s, 1) = f_1(s)$

.

To show $\varphi f_0 \sim \varphi f_1$. Define $G : I^2 \to Y$ by $G(s,t) = \varphi(F(s, t))$

Then, $G(s,0) = \varphi(F(s, 0)) = \varphi(f_0(s)) = (\varphi f_0)(s), G(s,1) = \varphi(F(s, 1)) = \varphi(f_1(s)) = (\varphi f_1)(s)$

and since G is continuous as it is composition of continuous maps, G is the required homotopy between $\varphi f_0 \sim \varphi f_1$.

Note: If $\varphi : X \to Y$ is a continuous map and if $f_0, f_1 : I \to X$ be paths in X

with $f_0(1) = f_1(0)$, then $\varphi(f_0 \bullet f_1) = (\varphi f_0) \bullet (\varphi f_1)$ i.e.

$$\varphi(f_0 \bullet f_1)(t) = \begin{cases} \varphi(f_0(2t)) & for\ \ t \in [0, 1/2] \\ \varphi(f_0(2t-1)) & for\ \ t \in [1/2, 1] \end{cases} = \begin{cases} (\varphi f_0)(2t) & for\ \ t \in [0, 1/2] \\ (\varphi f_0)(2t-1) & for\ \ t \in [1/2, 1] \end{cases} = ((\varphi f_0) \bullet (\varphi f_1))(t)$$

$for\ t \in [0,1]$

Definition: (Based map) Suppose that $\varphi : X \to Y$ is a continuous map that carries the point x_0 of X to the point y_0 of Y, denote this by $\varphi : (X, x_0) \to (Y, y_0)$. If f is a loop in X based at x_0, then the composite $\varphi \circ f : I \to Y$ is a loop in Y based at y_0.

Definition: Let $\varphi : (X, x_0) \to (Y, y_0)$ be a continuous map. Define $\varphi_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ by $\varphi_*([f]) = [\varphi \circ f]$. The map φ_* is called the homomorphism induced by φ relative to the base point x_0.

The induced map is well-defined since a homotopy f_t of loops based at x_0 yields a composed homotopy φf_t of loops based at y_0, and so if $[f_0] = [f_1]$, then $\varphi_*[f_0] = [\varphi f_0] = [\varphi f_1] = \varphi_*[f_1]$.

Theorem Let α denotes the path class that contains a homotopy f_t of loops based at x_0 i.e. $\alpha = [f_t], t \in I$ and denote $\varphi_*(\alpha) = [\varphi f_t]$ the path class that contains the paths φf_t . $\varphi_*(\alpha)$ is the image of the path class α in the space Y. Then we have the following:

If α and β are path classes in X such that $\alpha \bullet \beta$ is defined, then

$\varphi_*(\alpha \bullet \beta) = (\varphi_*\alpha) \bullet (\varphi_*\beta)$.

For any point $x \in X, \varphi_*(\xi_x) = \xi_{\varphi(x)}$.

$\varphi_*(\alpha^{-1}) = (\varphi_*\alpha)^{-1}$.

If $\psi : Y \to Z$ is also a continuous map, and $id_X : X \to X$ is the identity map, then

d) $(\psi\varphi)_* = \psi_*\varphi_*$

e) $id_{X*}(\alpha) = \alpha$ for any path class α in X ; i.e. id_{X*} is the identity homomorphism.

Proof: By definition of the map φ_*, i.e. $\varphi_*([f]) = [\varphi \circ f]$, we have that

$\alpha = [f_t]$, $\beta = [g_t]$ such that $f_t g_t$ is defined for $t \in I$ .Then $\varphi_*(\alpha) = [\varphi f_t]$, $\varphi_*(\beta) = [\varphi g_t]$ and $\alpha \cdot \beta = [f_t] \cdot [g_t] = [f_t g_t]$.

$$\varphi_*(\alpha \cdot \beta) = \varphi_*[f_t \cdot g_t] = [\varphi \circ (f_t \cdot g_t)] = [(\varphi \circ f_t) \cdot (\varphi \circ g_t)] = [\varphi \circ f_t] \cdot [\varphi \circ g_t] = \varphi_*(\alpha) \cdot \varphi_*(\beta)$$

Let $\xi_x = [e]$, $e : I \to X$ be the constant map such that $e(I) = \{x\}$, denote the equivalence class of the constant map of [0, 1] onto $x \in X$. Then $\varphi \circ e = \{\varphi(x)\}$ is the constant path in Y.and, $\varphi_*(\xi_x) = \varphi_*[e] = [\varphi \circ e] = [\varphi(x)] = \xi_{\varphi(x)}$.

$$\varphi_*(\alpha^{-1}) = \varphi_*[f_t]^{-1} = \varphi_*\left[f_t^{-1}\right] = \left[\varphi \circ f_t^{-1}\right]$$

but $(\varphi \circ f_t)^{-1}(s) = (\varphi \circ f_t)(1-s) = \varphi(f_t(1-s)) = \varphi(f_t^{-1}(s)) = (\varphi \circ f_t^{-1})(s)$ i.e. $(\varphi \circ f_t)^{-1} = \varphi \circ f_t^{-1}$.
Thus, $\varphi_*(\alpha^{-1}) = (\varphi_*\alpha)^{-1}$.

d) $(\psi\varphi)_*(f) = (\psi\varphi) \circ (f) = (\psi(\varphi \circ f)) = \psi_*(\varphi \circ f) = \psi_*\varphi_*(f)$

e) $id_{X*}(\alpha) = id_{X*}[f_t] = [id_X \circ f_t] = [f_t] = \alpha$

Thus the mapping $\varphi \mapsto \varphi_*$ preserves composition and identity maps. In the language of category theory, such mapping is called a *functor.*

Theorem If $\varphi : (X, x_0) \to (Y, \varphi(x_0))$ is a homeomorphism, then $\varphi_* : \pi_1(X, x_0) \to \pi_1(Y, \varphi(x_0))$ is an isomorphism.

Proof: Let $\varphi^{-1} : Y \to X$, and $[\alpha],[\beta] \in \pi_1(X, x_0)$, then for $\varphi \circ \varphi^{-1} = id_{(Y,\varphi(x_0))}$ and $\varphi^{-1} \circ \varphi = id_{(X,x_0)}$, we have $(id_{(X,x_0)})_* = (\varphi^{-1} \circ \varphi)_* = \varphi_* \circ (\varphi^{-1})_* = id_{\pi_1(Y,\varphi(x_0))}$, similarly $(\varphi^{-1})_* \circ \varphi_* = id_{\pi_1(X,x_0)}$

φ_* is homomorphism: $\varphi_*([\alpha][\beta]) = \varphi_*([\alpha\beta]) = \varphi \circ ([\alpha\beta]) = (\varphi \circ \alpha).(\varphi \circ \beta) = \varphi_*([\alpha])\varphi_*([\beta])$.

φ_* is monomorphism: Let $\varphi_*([\alpha]) = \xi_{\varphi(x_0)}$, then $\varphi \circ \alpha = \xi_{\varphi(x_0)} \Rightarrow (\varphi \circ \alpha)(t) = \xi_{\varphi(x_0)}(t)\ \forall t \in I$

$\Rightarrow \alpha(t) = \varphi^{-1}(\xi_{\varphi(x_0)}(t)) = \varphi^{-1}(\varphi(x_0)) = x_0 = \xi_{x_0}(t)\ \forall t \in I$. Thus, $[\alpha] = \xi_{x_0}$.

φ_* is epimorphism: Let $\gamma \in \pi_1(Y, \varphi(x_0))$, then $\varphi^{-1}(\gamma(0)) = \varphi^{-1}(\gamma(1)) = x_0$ and since φ^{-1} is continuous $\varphi^{-1}\gamma \in \pi_1(X, x_0)$.

Therefore, φ_* is an isomorphism with $(\varphi^{-1})_* = (\varphi_*)^{-1}$.

Thus the fundamental group is a topological invariant in that homeomorphic spaces have isomorphic fundamental groups.

Definition: Two continuous maps $\varphi_0, \varphi_1 : X \to Y$, are homotopic if and only if there exists a continuous map $\varphi : XxI \to Y$ such that for $x \in X, \varphi(x,0) = \varphi_0(x), \varphi(x,1) = \varphi_1(x)$.

Notation: If two maps φ_0 and φ_1 are homotopic, we shall denote this by $\varphi_0 \approx \varphi_1$

Here $\varphi_t(x) = \varphi(x,t)$ *for any* $(x,t) \in X\,x\,I$. Then for any $t \in I$, $\varphi_t : X \to Y$ is a continuous map and as t varies from 0 to 1 φ_0 is continuously deformed to φ_1.

Definition: Two maps $\varphi_0, \varphi_1 : X \to Y$, are homotopic relative to the subset A of X, if and only if there exists a continuous map $\varphi : X \times I \to Y$ such that $\begin{matrix}\varphi(x,0) = \varphi_0(x), \\ \varphi(x,1) = \varphi_1(x).\end{matrix}$ for $x \in X$, and $\varphi(a,t) = \varphi_0(a) = \varphi_1(a)$, $a \in A$, $t \in I$. (i.e. $\varphi_0|_A = \varphi_1|_A$), denoted by $\varphi : \varphi_0 \approx \varphi_1 \ rel\, A$.

Theorem: Let $\varphi_0, \varphi_1 : X \to Y$ be maps that are homotopic relative to the subset $\{x\}$.Then $\varphi_{0*} = \varphi_{1*} : \pi_1(X, x) \to \pi_1(Y, \varphi_0(x))$, i.e. the induced homomorphisms are the same.

Proof. Let $\varphi_0, \varphi_1 : X \to Y$ maps that are homotopic relative to the subset{x}.

Then exists a continuous map $\varphi : XxI \to Y$ such that $\begin{matrix}\varphi(x,0) = \varphi_0(x), \\ \varphi(x,1) = \varphi_1(x).\end{matrix}$ for $x \in X$, and since A={x}, $\varphi(x,t) = \varphi_0(x) = \varphi_1(x)$, $t \in I$. Let f be a loop in X based at x, then $\varphi_0 \circ f$ *and* $\varphi_1 \circ f$ are loops in Y based at $\varphi_0(x) = \varphi_1(x)$, define $\widetilde{F}(s, t) = \varphi(f(s), t)$ then $\widetilde{F}$ is the required homotopy between $\varphi_0 \circ f$ *and* $\varphi_1 \circ f$ i.e. $[\varphi_0 \circ f] = [\varphi_1 \circ f]$. So, $\varphi_{0*}[f] = [\varphi_0 \circ f] = [\varphi_1 \circ f]$ $= \varphi_{1*}[f]$, since f is arbitrary loop in X based at x, we have $\varphi_{0*} = \varphi_{1*}$.

Definition: A subset A of a topological space X is called a retract of X if there exists a continuous map $r : X \to A$ (called a retraction) such that $r(a) = a$ *for any* $a \in A$.

i.e. a map $r : X \to X$ such that $r(X) = A$ *and* $r|_A = id_A$.

Equivalently, denoting by the inclusion, $i: A \hookrightarrow X$ a retraction is a continuous map r such that

$r \circ i = id_A$ that is, the composition of r with the inclusion map is the identity of A.

For instance, any space retracts to a point in the obvious way (the constant map yields a retraction).

Definition: A subset A of a topological space X is called a *deformation retract* of X if there exists a retraction $r : X \to A$ and a homotopy $f : X \times I \to X$ such that

$$\left.\begin{matrix} f(x,0) = x \\ f(x,1) = r(x) \end{matrix}\right\} x \in X,$$
$$f(a,t) = a, \ a \in A, t \in I.$$

i.e. a family of maps $f_t : X \to X,\ t \in I$: $f_0 = id_X$, $f_1(X) = A$,*and* $f_t|_A = id_A\ \forall t \in I$

where the associated map $f(x,t) = f_t(x)\ \ \forall x \in X$ is continuous.

In other words, a deformation retraction is a homotopy between a retraction and the identity map on X. The subspace A is called a deformation retract of X.

Remark 1. A retract need not be a deformation retract. For instance, having a single point as a deformation retract would imply a space is path connected(in fact it would imply contractibility of the space).

2. A continuous map r from X into A is a deformation retraction if it is a retraction and its composition with the inclusion map is homotopic to the identity map on X. In this formulation, a deformation retraction carries with it a homotopy between the identity map on X and itself.

Examples

1. Let $S^1 = \{x \in R^2 : |x| = 1\}$, be the circle. Then $S^1 \times S^1$ is a torus, with a center circle $S^1 \times \{0\}$.

The homotopy $f(x,y,t) = (x,(1-t)y)$ shows that the solid torus can be deformed onto the center circle , so $S^1 \times \{0\} \cong S^1$ is a deformation retract of $S^1 \times S^1$.

Proof. Let $X = S^1 \times S^1$ *and* $A = S^1 \times \{0\}$ .Then $X = \{(x,y) : x \in R^2, y \in R^2, |x| = 1, |y| = 1\}$ and $A = \{(x,0) : x \in R^2, |x| = 1\}$. Let $r : X \to A$ be such that $r(x,y) = (x,0),\ \forall x \in X$ then, r is retraction of X, then . Define a homotopy $f : X \times I \to X$ by $f(x,y,t) = (x,(1-t)y)$, $x, y \in X$ *and* $t \in I$ .Then,

$f(x,y,0) = (x,y)$ *and* $f(x,y,1) = (x,0) = r(x)\ \forall x \in X$ and $f(a,t) = (a,0) \in A$ .Thus A is the deformation retract of X .

But $A \cong S^1$, and hence S^1 is the deformation retract of $S^1 \times S^1$.

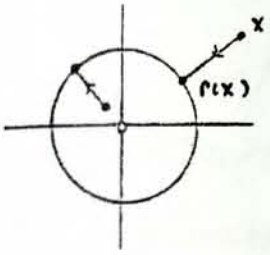
2. Let $X = [0,1]$, $A = \{0\}$. Then $f(x,t) = (1-t)x$ shows that A is a deformation retract of X .

Proof. Let $r : X \to A$ be the retraction of X, then $r(x) = 0\ \forall x \in X$. Define a homotopy $f : X \times I \to X$ by $f(x,t) = (1-t)x$, $x \in X$ *and* $t \in I$ .Then, $f(x,0) = x$, $f(x,1) = 0 = r(x)$ $\forall x \in X$ and $f(0,t) = 0$ .Thus A is the deformation retract of X .

3. Let B denote the z-axis in R^3. Consider the space R^3-B.It has the punctured x-y plane, $(R^2-0)\times 0$, as deformation retract. The map f defined by $f(x,y,z,t)=(x,y,(1-t)z)$ is a deformation retraction.

4. S^1 is a deformation retract of $R^2-\{(0,0)\}$.

Proof: Define r: $R^2-\{(0,0)\}\longrightarrow S^1$ defined by $r(x)=\frac{x}{\|x\|}$, then for every $x\in R^2-\{(0,0)\}$,$r(x)\in S^1$.Then, r a retraction.



If $i: S^1\hookrightarrow R^2-\{(0,0)\}$, then $i\circ r: R^2-\{(0,0)\}\to R^2-\{(0,0)\}$ is just regarded as a map into $R^2-\{(0,0)\}$. Since $R^2-\{(0,0)\}$ contains the line segment joining x and $r(x)$

For every $x\in R^2-\{(0,0)\}$,

$$F(x,t)=t\big((i\circ r)(x)\big)+(1-t)x \text{ is a homotopy from } id_{R^2-\{(0,0)\}} \text{ to } i\circ r.$$

(i.e. $F(x,0)=x$, $F(x,1)=(i\circ r)(x)$and satisfies $F(a,t)=t\big((i\circ r)(a)\big)+(1-t)a=ta+a-ta=a\ for\ all\ a\in S^1\ and\ all\ t\in[0,1]$.)

So, S^1 is a deformation retract of $R^2-\{(0,0)\}$.

Theorem: If A is a deformation retract of X, then the inclusion map $i: A\hookrightarrow X$ induces an isomorphism of $\pi_1(A,a)$ onto $\pi_1(X,a)$ for any $a\in A$.

Proof: Let $r: X\to A$ be a retraction and $i: A\hookrightarrow X$ the inclusion map. For any point $a\in A$, consider the induced homomorphisms

$$i_*:\pi_1(A,a)\to\pi_1(X,a), \text{ and } \quad r_*:\pi_1(X,a)\to\pi_1(A,a).$$

Then ri is the identity map of A and r_*i_* is the identity homomorphism of the group $\pi_1(A,a)$.

$i_* : \pi_1(A,a) \to \pi_1(X,a)$ is a monomorphism: Let $[\gamma] \in \pi_1(A,a)$ and $i_*([\gamma]) = \{\xi_a\}$, α *and* e be the representatives of $[\gamma]$ *and* ξ_a. Then

$i_*(\alpha)(t) = e(t)\ \forall t \in I$ which implies $(i \circ \alpha)(t) = e(t)\ \forall t \in I$ i.e. $(\alpha)(t) = e(t)\ \forall t \in I$ or $\alpha = e$.

Thus $[\gamma] = \xi_a$. So i_* is also injective.

$i_* : \pi_1(A,a) \to \pi_1(X,a)$ is an epimorphism: Let $[\beta] \in \pi_1(A,a)$. If $f_t : X \to X,\ t \in I$ is the deformation retraction, then the composition $f_t\beta : I \to X$ gives a homotopy of β to a loop in A,

i.e. $f_t\beta(0) = f_t\beta(1) = f_t(a) = f(a,t) = a$ and

$f_0\beta(s) = f(\beta(s),0) = \beta(s)$ *i.e.* $f_0\beta = \beta$ also $f_1\beta(s) = f(\beta(s),1) = r(\beta(s))$ *i.e.* $f_1\beta = r\beta$

Since r is a retraction $r\beta \in \pi_1(A, a)$. So i_* is also surjective. As required.

Example In the above examples, the corresponding spaces are homeomorphic to the corresponding deformation retracts.

Definition: A topological space X is contractible to a point if there exists a point $x_0 \in X$ such that $\{x_0\}$ is a deformation retract of X.

Definition: A topological space X is simply connected if it is arcwise connected and $\pi_1(X, x) = \{1\}$ for some (and hence any) $x \in X$.

Corollary: If X is contractible to a point, then X is simply connected.

Proof: Let X be contractible to a point then there exists a point $x_0 \in X$ such that $\{x_0\}$ is a deformation retract of X.

That is there is a retraction $r : X \to \{x_0\}$ of X i.e. $r(x) = x_0\ \forall x \in X$ and a homotopy

$f : X \times I \to X$ such that $\left.\begin{matrix} f(x,0) = x \\ f(x,1) = x_0 \end{matrix}\right\} x \in X,\ f(x_0,t) = x_0,\ t \in I.$

Now $f(x,t) \in X$ *for all* $x \in X, t \in I$. Define $F : I \to X$ by $F(t) = f(x,t)$ for $x \in X, t \in I$.

Then $F(0) = f(x,0) = x$, $F(1) = f(x,1) = x_0$ and $F(t) = f(x,t) \in X$ for $x \in X, t \in I$.

Thus F is a path in X, with initial point x and terminal point x_0, and $F(I) \subseteq X$. Thus X is arcwise connected.

Now we show $\pi(X, x) = \{1\}$. Let $\gamma : I \to X$ be a loop based at x_0 Define $H : X \times I \to X$ by

$H(s, t) = f(\gamma(s), t)$ where $f : X \times I \to X$ such that $\left.\begin{matrix} f(x,0) = x \\ f(x, 1) = x_0 \end{matrix}\right\} x \in X,\ f(x_0, t) = x_0,\ t \in I.$

Then $H(s, 0) = f(\gamma(s), 0) = \gamma(s)$, $s \in I$ and $H(s, 1) = f(\gamma(s), 1) = x_0$, and

$H(0, t) = f(\gamma(0), t) = f(x_0, t) = x_0$, $H(1, t) = f(\gamma(1), t) = f(x_0, t) = x_0$

Thus $\gamma \sim x_0$ and hence since γ is arbitrary, we have $\pi(X, x) = \{1\}$.

Example 1 A subset X of the plane or, more generally, of Euclidean n-space R^n is called convex if the line segment joining any two points of X lies entirely in X.

Any convex subset of R^n is contractible to a point. In particular the unit disk D^2 is contractible to a point.

Proof: Let $x_0 \in X$ be arbitrary, define $f : X \times I \to X$ by the formula $f(x,t) = (1-t)x + tx_0$

For any $(x, t) \in X \times I$ (i.e. $f(x,t)$ is the point on the line segment joining x and x_0 which divides it in the ratio (1-t): t).Then f is continuous, $f(x,0) = x$, and $f(x,1) = x_0$.

Example 2 The unit (n-1)-sphere S^{n-1} is a deformation retract of $D^n - \{0\}$, the closed unit n-dimensional disc minus the origin.

Proof: Define $f : X \times I \to X$, where $X = D^n - \{0\} = \{x \in R^n : 0 < |x| \le 1\}$, by the formula

$f(x,t) = (1-t)x + t\dfrac{x}{|x|}$. Then f is continuous, $f(x,0) = x$, $f(x,1) = \dfrac{x}{|x|} \in S^{n-1}$, *and*

if $x \in S^{n-1}$, *then* $f(x,t) = x$ *for all* $t \in I$.

In particular, for n=2, we see that the boundary circle is a deformation retract of a punctured disc.

The fundamental group of a circle

So far we only know $\pi_1(X)$ for $X \subseteq R^n$ convex and in that case $\pi_1(X) = \{1\}$.

We now turn to techniques for calculating π_1 of given spaces.

The circle S^1 is path connected, and thus its fundamental group is independent of the choice of the base point. The natural base point is $x_0 = (1, 0) \in S^1$.

Theorem: $\pi_1(S^1) \cong Z$ .Intuitively the homotopy class of a loop on S^1 depends only on how many times it winds around the circle and in what direction.

We will show that the homotopy class of a loop at $x_0 \in S^1$ is uniquely determined by the direction and the number of times it "wraps around" S^1, if $x_0 = (1,0)$, then the elements of $\pi_1(S^1, x_0)$ are represented by the maps

$$\alpha_n : I \to S^1$$

$$\alpha_n(s) = (\cos 2\pi ns, \sin 2\pi ns) \text{ for } n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

and $n \to [\alpha_n]$ is an isomorphism of Z onto $\pi_1(S^1, x_0)$.

Definition: Given continuous maps $f : X \to Y$, $p : E \to Y$. If there exist a continuous map $\tilde{f} : X \to E$ such that $p \circ \tilde{f} = f$ then we say that $\tilde{f}$ *lift* of f to E

The map $e : R \to S^1$ is defined by $e(t) = \exp(2\pi it)$. Then e is continuous, $e(t_1 + t_2) = e(t_1)e(t_2)$ and, $e(t_1) = e(t_2)$ if and only if $t_1 - t_2$ is an integer. It follows that $e|_{(-1/2,1/2)}$ is a homeomorphism of the open interval $(-1/2,1/2)$ onto $S^1 \setminus \{\exp(\pi i)\}$.

Let $\log : S^1 \setminus \{\exp(\pi i)\} \to (-1/2,1/2)$ be the inverse of $e|_{(-1/2,1/2)}$.

Definition: A subset $X \subseteq R^n$ is called starlike from a point x_0 if, whenever $x \in X$, the closed segment $[x_0, x]$ from x_0 to x lies in X.

Lemma (Existence) Let X be compact and starlike from $x_0 \in X$. Given any continuous map $f: X \to S^1$ and any $t_0 \in R$ such that $e(t_0) = f(x_0)$, there is a continuous map $\tilde{f}: X \to R$ such that $\tilde{f}(x_0) = t_0$ and $e \circ \tilde{f}(x) = f(x)$ for all $x \in X$.

Proof. Clearly we can translate X so that it is star like from the origin; hence there is no loss of generality in assuming $x_0 = 0$. Since X is compact, f is uniformly continuous and there exists $\varepsilon > 0$ such that if $\|x - x'\| < \varepsilon$, *then* $\|f(x) - f(x')\| < 2$. Since X is bounded there exist a positive integer n such that $\frac{\|x\|}{n} < \varepsilon \ \forall x \in X$. Then for each $0 \le j < n$ *and all* $x \in X$

$$\left\| \frac{(j+1)x}{n} - \frac{jx}{n} \right\| = \frac{\|x\|}{n} < \varepsilon \text{ and } \left\| f\left(\frac{(j+1)x}{n}\right) - f\left(\frac{jx}{n}\right) \right\| < 2.$$

It follows that the quotient $f((j+1)x/n) \Big/ f(jx/n)$ is a point of $S^1 \setminus \{\exp(\pi i)\}$.

Let $g_j : X \to S^1 \setminus \{\exp(\pi i)\}$ *for* $0 \le j < n$ be the map defined by

$$g_j(x) = \frac{f((j+1)x/n)}{f(jx/n)}$$

Then for all $x \in X$, we see that $f(x) = f(0) g_0(x) g_1(x) \ldots g_{n-1}(x)$.

We define $\tilde{f}: X \to R$ by

$$\tilde{f}(x) = t_0 + \log(g_0(x)) + \log(g_1(x)) + \ldots + \log(g_{n-1}(x)).$$

Since $\tilde{f}$ is the sum of $n+1$ continuous functions from X *to* R, it is continuous. Clearly $\tilde{f}(0) = t_0$ *and* $e \circ \tilde{f} = f$.

Lemma (Uniqueness) Let X be a connected space and let $\tilde{f}, \tilde{g}: X \to R$ be maps such that $e \circ \tilde{f} = e \circ \tilde{g}$ *and* $\tilde{f}(x_0) = \tilde{g}(x_0)$ *for some* $x_0 \in X$. *Then* $\tilde{f} = \tilde{g}$.

Proof. Let $h = \tilde{f} - \tilde{g}: X \to R$. Since $e \circ \tilde{f} = e \circ \tilde{g}$, $e \circ h$ is the constant map of X *to* $1 \in S^1$.

Thus h is a continuous map from X *to* R, taking only integral values. Because X is connected, h is constant, and since $h(x_0) = 0$, $h(x) = 0 \ \forall x \in X$.

Let $\alpha : I \to S^1$ be a closed path at 1. Because I is starlike from 0 and $\alpha(0) = 1 = e(0)$ it follows from the above lemmas there exists a unique lifting $\tilde{\alpha} : I \to R$ such that $e \circ \tilde{\alpha} = \alpha$. Because $e(\tilde{\alpha}(1)) = \alpha(1) = 1$, it follows that $\tilde{\alpha}(1)$ is an integer. We define the degree of α by

$$\deg(\alpha) = \tilde{\alpha}(1).$$

Lemma Let α *and* β be homotopic closed paths in S^1 *at* 1. Then, $\deg(\alpha) = \deg(\beta)$.

Proof. Let $F : I \times I \to S^1$ be a homotopy for α *and* β. Because $I \times I$ is starlike subset of R^2 from $(0,0)$, it follows that there is a (unique) lifting $\tilde{F} : I \times I \to R$ such that $\tilde{F}(0,0) = 0$ and $e \circ \tilde{F} = F$ and $F(0,t) = F(1,t) = 1 \ \forall t \in I$. Thus $\tilde{F}(0,t)$ *and* $\tilde{F}(1,t)$ take on only integral values for all $t \in I$. It follows that $\tilde{F}(0,t)$ must be a constant and $\tilde{F}(1,t)$ must be a constant. Because $\tilde{F}(0,0) = 0, \tilde{F}(0,t) = 0$ *for all* t. Let $\tilde{\alpha}, \tilde{\beta} : I \to R$ be the maps defined by $\tilde{\alpha}(t) = \tilde{F}(t,0)$ *and* $\tilde{\beta}(t) = \tilde{F}(t,1)$. Then $\tilde{\alpha}(0) = \tilde{\beta}(0) = 0$, $e \circ \tilde{\alpha} = \alpha$ *and* $e \circ \tilde{\beta} = \beta$. Thus

$$\deg(\alpha) = \tilde{\alpha}(1) = \tilde{F}(1,0) = \tilde{F}(1,t) = \tilde{F}(1,1) = \tilde{\beta}(1) = \deg(\beta).$$

It follows that there is a well-defined function deg from $\pi_1(S^1,1)$ *to* Z defined by

$$\deg([\alpha]) = \deg(\alpha)$$

Theorem The function deg is an isomorphism of groups $\pi_1(S^1,1)$ *to* Z.

Proof. To Prove that deg is a homomorphism, let α *and* β be two closed paths in S^1 *at* 1 and let $\alpha\beta$ be the closed path which is their pointwise product in the group multiplication of S^1. Let $\tilde{\alpha}, \tilde{\beta} : I \to R$ such that $\tilde{\alpha}(0) = \tilde{\beta}(0) = 0$, $e \circ \tilde{\alpha} = \alpha$ *and* $e \circ \tilde{\beta} = \beta$. Let $\tilde{\gamma} = \tilde{\alpha} + \tilde{\beta} : I \to R$. Then

$\tilde{\gamma}(0) = 0$ *and* $e(\tilde{\gamma}) = \alpha\beta$. Thus

$$\deg([\alpha] * [\beta]) = \deg([\alpha\beta]) = \tilde{\gamma}(1) = \tilde{\alpha}(1) + \tilde{\beta}(1) = \deg([\alpha]) + \deg([\beta]).$$

The map deg is an epimorphism: For any integer n, let $\tilde{\alpha} : I \to R$ be the path defined by $\tilde{\alpha}(t) = tn$ *and let* $\alpha = e \circ \tilde{\alpha} : I \to S^1$. Then, $\deg([\alpha]) = n$. The map deg is a monomorphism: If $\deg([\alpha]) = 0$, then there is a path $\tilde{\alpha} : I \to R$ with $\tilde{\alpha}(0) = \tilde{\alpha}(1) = 0$ *and* $e \circ \tilde{\alpha} = \alpha$. Since R is contractible, $\tilde{\alpha} \approx \xi_0$; and $\alpha = e \circ \tilde{\alpha} \approx e \circ \xi_0 = \xi_1$.

Example $\pi_1(S^n) \cong 1$ for $n \geq 2$.

Proof: Case 1. For $2 \leq n < \infty$.

Let $p = (0, \ldots, 0, 1) \in R^{n+1}$ and $q = (0, \ldots, 0, -1)$ be the "north pole" and the "south pole" of S^n respectively.

we show that if $n \geq 1$, the punctured sphere $S^n - p$ is homeomorphic to R^n.

Define $f: (S^n - p) \to R^n$ by the equation

$$f(x) = f(x_1, x_2, \ldots, x_{n+1}) = \frac{1}{1 - x_{n+1}}(x_1, x_2, \ldots, x_n)$$

The map f is called stereographic projection. (If one takes the straight line in R^{n+1} passing through the north pole p and the point x of $S^n - p$, then this line intersects the $n - plane$ $R^n \times 0 \subset R^{n+1}$ in the point $f(x) \times 0$). One checks that f is a homeomorphism by showing that the map

$g: R^n \to (S^n - p)$ given by

$g(y) = g(y_1, y_2, \ldots, y_n) = \left(\frac{2}{1+|y|^2} y_1, \ldots, \frac{2}{1+|y|^2} y_n, 1 - \frac{2}{1+|y|^2}\right)$ is the right and left inverse for f.

Also the reflection map $(x_1, \ldots, x_n, x_{n+1}) \to (x_1, \ldots, x_n, -x_{n+1})$ defines a homeomorphism of $S^n - p$ with $S^n - q$, so the latter is also homeomorphic to R^n.

Now we prove the theorem. Let $U = S^n - p$ with $V = S^n - q$ be the open sets of S^n. Note that for $n \geq 1$, *the sphere* S^n is path connected. This follows from the fact that the spaces U *and* V are path connected (being homeomorphic to R^n) and $U \cap V = (1, 0, \ldots, 0) \in S^n$.

And for $n \geq 2$, the sphere S^n is simply connected. The spaces $U and V$ are simply connected, being homeomorphic to R^n. Their intersection equals $S^n - p - q$, which is homeomorphic under stereographic projection to $R^n - 0$. The latter space is path connected, for every point of $R^n - 0$ can be joined to a point of S^{n-1} by a straight-line path, and S^{n-1} is path connected for $n \geq 2$. Then by the Seifert-Van Kampen Theorem, special case, we have $\pi_1(S^n) \cong 1$ for $n \geq 2$.

Case 2. For $n = \infty$, $S^\infty = \{\xi \in l^2 : \|\xi\| = 1\}$ ("infinite dimensional sphere"). Where l^2 denote the set of sequences $\xi = (\xi_1, \xi_2, \ldots)$ of complex numbers that are square summable

$(\sum_{n=1}^{\infty}|\xi_n|^2 < \infty)$ with its usual inner product ($<\xi,\eta>= \sum_{n=1}^{\infty}\xi_n\overline{\eta}_n$) and corresponding norm ($\|\xi\|^2 =<\xi,\xi>$). l^2 is a metric space ($d(\xi,\eta)=\|\xi-\eta\|$)and so a topological space, and $S^\infty = \{\xi \in l^2 : \|\xi\| = 1\}$ is a subspace of l^2.

To show $\pi_1(S^\infty) \cong 1$i.e. S^∞ is contractible.

Let $T : l^2 \to l^2$ be the shift operator $T(\xi) = T(\xi_1,\xi_2,\ldots) = (0,\xi_1,\xi_2,\ldots)$

We show that T is an isometry with no eigenvalues.

T is injective: As , if $x \in \ker T \Rightarrow Tx = 0 \Rightarrow (0,\xi_1,\xi_2,\ldots) = (0,0,\ldots) \Rightarrow \xi_i = 0\ \forall i = 1,2,\ldots \Rightarrow \ker T = \{0\}$.

T is surjective: For $y = (0,\eta_1,\eta_1,\ldots) \in l^2, \exists x = (\eta_1,\eta_1,\ldots) \in l^2$ such that $Tx = y$.

Thus, T is bijective.

Also, $\|Tx\|^2 = \|(0,\xi_1,\xi_2,\ldots)\|^2 = \|(\xi_1,\xi_2,\ldots)\|^2 = \|x\|^2$.Thus T is isometry .

Now we show T has no eigen values.

Let $\lambda \in K$ be the eigen value , then
$\exists x \neq 0 : Tx = \lambda x \Rightarrow (0,\xi_1,\xi_2,\ldots) = \lambda(\xi_1,\xi_2,\ldots) \ldots\ldots (*) \Rightarrow (0,\xi_1,\xi_2,\ldots) = (\lambda\xi_1,\lambda\xi_2,\ldots) \Rightarrow \lambda\xi_1 = 0\ and\ \lambda\xi_{i+1} = \xi_i, i = 1,2,\ldots \Rightarrow \lambda = 0\ or$

$$\xi_1 = 0\ and\ \lambda\xi_{i+1} = \xi_i, i = 1,2,\ldots$$

$if\ \lambda = 0, then\ (*) \Rightarrow \xi_i = 0\ \forall i$ which is a contradiction. $if\ \xi_1 = 0, from\ \lambda\xi_{i+1} = \xi_i\ we\ have\ \xi_i = 0\ \forall i$ which is again a contradiction.

Thus T has no eigen values.

Now, let $x \in S^\infty$, then $\|x\| = 1, Tx = (0,\xi_1,\xi_2,\ldots) \Rightarrow Tx \in S^\infty$and hence $x \to Tx$ is a bijection.

Define $F: S^\infty \times I \to S^\infty$by

$$F(x,t) = \frac{(1-t)x+Tx}{\|(1-t)x+Tx\|}$$

Then, since $(1-t)x + Tx \neq 0\ \forall x \in S^\infty$, F is continuous and

$F(x,0) = \frac{x}{\|x\|} = id_{S^\infty}(x)$ *i.e.* $F_0 = id_{S^\infty}$, $F(x,1) = \frac{Tx}{\|Tx\|} = T|_{S^\infty}$ which implies $id_{S^\infty} \approx T|_{S^\infty}$.

Now we show that $T|_{S^\infty} \approx x_0 = (1,0,0,\ldots)$.

Define $G : T|_{S^\infty} \times I \to S^\infty$ by $G(x,t) = \frac{(1-t)x_0 + Tx}{\|(1-t)x_0 + Tx\|}$ then $G(x,0) = \frac{x_0}{\|x_0\|} = x_0$ *and*

$$G(x,1) = \frac{Tx}{\|Tx\|} = T|_{S^\infty}$$

i.e. $T|_{S^\infty} \approx x_0$

Therefore, id_{S^∞} is null-homotopic and hence S^∞ is contractible to a point, and since $S^\infty \subseteq l^2$ is pathwise connected, we have that $\pi_1(S^\infty) = 1$.

The fundamental group of a product space

Let X, Y, *and* A be topological spaces. If $f : A \to X \times Y$ is any map, let us denote the coordinates of $f(a)$ by $(f_1(a), f_2(a))$ for any point $a \in A$. Then f_1 *and* f_2 are maps of A *into* X *and* Y, respectively, and it is well known f is continuous if and only if both f_1 *and* f_2 are continuous.

Also if we denote by $p : X \times Y \to X$ *and* $q : X \times Y \to Y$, the projection of the product space onto its factors, then $f_1 = pf$ *and* $f_2 = qf$.

Now let us apply this for $A = I$, the unit interval. We see that there is a natural one-to-one correspondence between paths $f : I \to X \times Y$ and in the product space and pairs of paths $f_1 : I \to X$, $f_2 : I \to X$ in the factors. Also it holds $f_1 = pf$ *and* $f_2 = qf$ as before.

Theorem If X *and* Y are path connected, then the fundamental group of the product space, $\pi_1(X \times Y, (x,y))$ is naturally isomorphic to the direct product of the fundamental groups, $\pi_1(X, x) \times \pi_1(Y, y)$. The isomorphism is defined by assigning to any element $\alpha \in \pi_1(X \times Y, (x,y))$, the ordered pair $(p_*\alpha, q_*\alpha)$, where $p : X \times Y \to X$ *and* $q : X \times Y \to Y$ the projection of the product space onto its factors.

Proof Let us denote the fundamental group of the product space by $\pi_1(X \times Y, (x,y))$. Choose base points $x_0 \in X$, $y_0 \in Y$, then $(x_0, y_0) \in X \times Y$ then we have continuous projections $p : X \times Y \to X$ and $q : X \times Y \to Y$ and thus the induced homomorphisms

$p_* : \pi_1(X \times Y,(x,y)) \to \pi_1(X,x)$ and $q_* : \pi_1(X \times Y,(x,y)) \to \pi_1(Y,y)$

and thus a homomorphism $\varphi : \pi_1(X \times Y,(x_0,y_0)) \to \pi_1(X,x_0) \times \pi_1(Y,y_0)$ defined by

$$[\alpha] \to (p_*[\alpha], q_*[\alpha])$$

We show φ is an isomorphism. φ is onto: Let $([\beta],[\gamma]) \in \pi_1(X) \times \pi_1(Y)$ then $\alpha = (\beta, \gamma)$ is a loop in $X \times Y$ with $p(\alpha) = \beta$ and $q(\alpha) = \gamma$ so that $\varphi([\alpha]) = ([\beta],[\gamma])$.

φ is one-to-one: Suppose $\varphi([\alpha]) = (\xi_{x_0}, \xi_{y_0}) = \xi$ i.e. $p \circ \alpha \sim e_{x_0}$ via F *and* $q \circ \alpha \sim e_{y_0}$ via g then $H(s,t) = (F(s,t), g(s,t))$ is the required homotopy for α and $e_{(x_0,y_0)}$ i.e. $[\alpha] = \xi$.

Thus φ is an isomorphism.

Example Torus: $T^2 = S^1 \times S^1, \pi_1(T^2) \cong Z \times Z$.

Proof: Since $T^2 = S^1 \times S^1$ and S^1 is path connected, we have by the above theorem that $\pi_1(S^1 \times S^1) \cong \pi_1(S^1) \times \pi_1(S^1)$ and $\pi_1(S^1) \cong Z$ implies $\pi_1(T^2) \cong Z \times Z$. Geometrically, the representatives of two generators for $\pi_1(T^2)$ are described below.

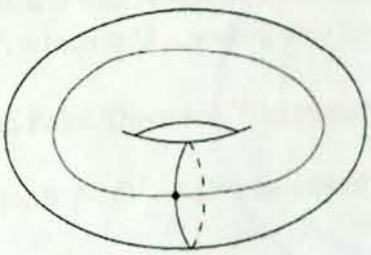


Figure Two representatives of generators of the torus.

In general, for n-torus we have $\pi_1(S^1 \times ... \times S^1) \cong Z \times ... \times Z$.

Corollary If Y is simply connected then $\pi_1(Y \times X) \cong \pi_1(X)$ for any X.

For instance, cylinder $S^1 \times [0,1] \approx$ *circle* S^1, solid torus $S^1 \times D^2 \approx$ *circle* S^1

Proof. Assume Y is contractible. Choose $y_0 \in Y$ and a homotopy $F : Y \times I \to Y$ such that $F(y,0) = y$, $F(y,1) = y_0$ from id_Y to the constant map with value y_0.

Define

$$f: X \times Y \to X \times \{y_0\} : f(x,y) = (x, y_0), \quad i: X \times \{y_0\} \to X \times Y : i(x, y_0) = (x, y_0)$$

Thus, $F \circ i = id_{X \times \{y_0\}}$. Define $G: (X \times Y) \times I \to X \times Y$ by $G((x,y),t) = (x, F(y,t))$

$G((x,y),0) = (x, F(y,0)) = (x, y) = id_{X \times Y}(x,y)$, and
$G((x,y),1) = (x, F(y,1)) = (x, y_0) = (i \circ f)(x,y)$ so $i \circ f \approx id_{X \times Y}$.

Therefore, $X \times Y \approx X \times \{y_0\} \cong X$.

Example 1. $\pi_1(S^2 \times S^1) \cong Z$.

2. Since $\pi_1(S^3) \cong 1$, $S^2 \times S^1$ and S^3 are not homeomorphic spaces.

Applications

1. The No-Retraction Theorem (Dimension 2)

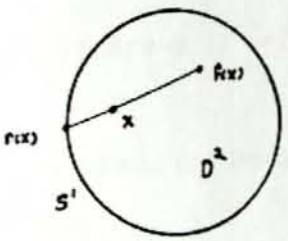
S^1 is not a retract of D^2.

Proof. Since D^2 is contractible, any map defined on D^2 is nullhomotopic, and in particular r would be so too. But this would be a contradiction, since the composition of r with the inclusion $S^1 \hookrightarrow D^2$, which is id_{S^1}, would also be null homotopic. Such r cannot exist.

2. Brouwer's Fixed Point Theorem (Dimension 2)

Any continuous map $f: D^2 \to D^2$ has a fixed point. (i.e. there is a point $x_0 \in D^2$ *such that* . $f(x_0) = x_0$)

Proof. Let $f: D^2 \to D^2 : f(x) \neq x \ \forall x \in D^2$. We use it to build a retraction $r: D^1 \to S^1$ thus contradicting the existence of this retraction.



Let $x \in D^2$, then $f(x) \neq x \ \forall x \in D^2$ implies x and $f(x)$ determine a line $tx + (1-t)f(x), t \in R$

On this line the norm is a quadratic function of t.

$$\|tx + (1-t)f(x)\|^2 = (tx + (1-t)f(x)).(tx + (1-t)f(x))$$

$$= \|x - f(x)\|^2 + 2f(x).(x - f(x))t + \|f(x)\|^2$$

This quadratic is ≤ 1 for $0 \leq t \leq 1$ (D^2 is convex). Thus, there is a unique $t_1 \geq 1$ for which it takes the value 1. The quadratic equation gives this t_1 as a continuous function of x ($f(x)$ is continuous and the denominator $2f(x).(x - f(x))$ is nonzero). Thus, the map $r : D^1 \to S^1$ defined by

$$r(x) = t_1 x + (1-t_1)f(x)$$

is continuous. By definition, $r(x) \in S^1$ and, if $x \in S^1$, then $t_1 = 1$, and we get $r(x) = x$. This is a contradiction. Thus, there is $x \in D^2$ such that $f(x) = x$.

3. Fundamental theorem of algebra

Every non constant polynomial with complex coefficients has a root.

That is, if $f(z) = z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0 \quad n > 0, a_0, a_1, ..., a_{n-1} \in C,$

then there exists $z_0 \in C$ *such that* $f(z_0) = 0.$

Proof: Assume f does not have a root, the mapping $z \mapsto f(z)$ would determine a map $f : C \to C - \{0\}$ i.e. $f(z) \neq 0 \ \forall z \in C$ and we show that this implies $n = 0$.

The plan is to construct, in two steps, a homotopy relative to {0, 1}from $e^{2\pi si}$ to the trivial loop at 1 in S^1 and then appeal to ...to conclude $n = 0$.

For each fixed $r \geq 0$ define $\alpha_r : I \to S^1$ by $\alpha_r(s) = \dfrac{f(re^{2\pi si})/f(r)}{|f(re^{2\pi si})/f(r)|}$

Then, $|\alpha_r(s)| = 1 \ \forall s \in I$ and $\alpha_r(0) = \alpha_r(1) = 1$ implies each α_r is a loop at 1 in S^1

Moreover, $\alpha_0(s) = 1\ \forall s \in I$ and $F(s,t) = \alpha_{rt}(s)$ is a homotopy relative to $\{0, 1\}$ from α_0 to α_r so $[\alpha_r] = [1]$ i.e. $\forall r \geq 0$ (all of the α_r represent the trivial element of $\pi_1(S^1)$).

Now fix some r such that $r > \max\{1, |a_{n-1}|, |a_{n-2}| + \ldots + |a_0|\}$.

Then on $|z| = r$,

$$|z^n| = r^n = r\,r^{n-1} > (|a_{n-1}| + |a_{n-2}| + \ldots + |a_0|)|z^{n-1}| = |a_{n-1}||z^{n-1}| + |a_{n-2}||z^{n-1}| + \ldots + |a_0||z^{n-1}|$$

$$\geq |a_{n-1}||z^{n-1}| + |a_{n-2}||z^{n-2}| + \ldots + |a_0| \quad (r > 1)$$

$$\geq |a_{n-1}z^{n-1} + \ldots + a_0|$$

So $|z^n| \geq |a_{n-1}z^{n-1} + \ldots + a_0|$ *on* $|z| = r$ ..... (*)

Now $\forall t \in I$, define $f_t(z) = z^n + t(a_{n-1}z^{n-1} + \ldots + a_0)$,(So that $f_0(z) = z^n$ *and* $f_1(z) = f(z)$).

Then $f_t(z)$ has no roots on $|z| = r$ because $0 = f_t(z)$ implies

$$0 = |z^n + t(a_{n-1}z^{n-1} + \ldots + a_0)| \geq |z^n| - t|a_{n-1}z^{n-1} + \ldots + a_0| \geq |z^n| - |a_{n-1}z^{n-1} + \ldots + a_0|$$

i.e. $|z^n| \leq |a_{n-1}z^{n-1} + \ldots + a_0|$ and this is impossible on $|z| = r$ by (*).

Now for this fixed r and $t \in I$ define $\quad F(s,t) = \dfrac{f_t(re^{2\pi si})/f_t(r)}{|f_t(re^{2\pi si})/f_t(r)|}$

Then $F : I^2 \to S^1$ is continuous ($|re^{2\pi si}| = |r| = r$ and $f_t(z) \neq 0$ *for* $|z| = r$), and

$$F(s,0) = \frac{(re^{2\pi si})^n / r^n}{|(re^{2\pi si})^n / r^n|} = e^{2\pi nsi} \qquad (f_0(z) = z^n)$$

$$F(s,1) = \alpha_r(s)\ (f_1 = f) \qquad F(0,t) = F(1,t) = 1$$

And therefore, $F : e^{2\pi nsi} \approx \alpha_r\ rel\ \{0,1\} \approx 1\ rel\ \{0,1\}$ This implies $n = 0$ i.e. $f(z)$ has degree 0.

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