COLLECTIVE PHENOMENA
IN
HIGHLY EXCITED BOSE GAS

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ABSTRACT

The theme of this thesis is the investigation of collective and thermodynamical properties of highly-excited Bose-gas. By excited gas is meant a gas in which there exist a significant number of excited particles. The number of excited particles depends on external sources (laser illumination, electron beams, etc.) and can be arbitrary. In typical cases the number of excited atoms N_1 is smaller than the number of particles in the ground state. As is well known the energy of interaction between particles in the same quantum state is proportional to $1/R^6$ where R is the distance between particles. This short-range interaction is responsible for the collective properties of the conventional gas such as condensation and phase transitions.

The interaction between differently-excited particles is proportional to $1/R^3$ and can be attractive as well as repulsive. Thus, the availability of even a comparatively small number of excited particles can and does lead to a significant change of the properties of the gas compared with the non-excited gas.

The properties of such excited gas are investigated on the basis of the second-quantized Hamiltonian and the method of equations of motion of Green functions. This method yields, in a most natural and direct way, the energy-spectrum of elementary excitations and the thermodynamical functions of excited gas.

CONTENTS

INTRODUCTION

On the macroscopic level the properties of gases, liquids, and solids are determined, ultimately, by the interactions between the particles that constitute them. As far as we deal with atoms and molecules, in this case, these are electrical interactions.

The main purpose of this paper is the investigation of the collective and thermodynamic properties of an excited Bose gas. By excited gas we shall understand a gas in which a definite amount of atoms are excited to the first level of electron excitation. It will be shown below that the interaction between differently excited atoms is much larger than the interaction between similarly excited atoms. This means that even a small concentration of excited atoms can give significant contribution to the thermodynamical functions of the excited gas. In particular an excited gas with comparatively small number of excited atoms will be non-ideal.

Due to strong resonance interaction between excited and non-excited atoms which is connected with the interchanging of excitation between differently excited atoms and which is proportional to $1/R^3$, (R = distance between atoms) one can expect collective properties of such a gas. The first investigation of collective properties of non-excited gas at low temperatures had been done by Bogolubov. In this paper we also considered the Bogolubov case using the thermal Green's function method which is the more natural one in the treatment of the many-body problem.

The second and main problem we have considered is the energy of spectrum of elementary excitations in an excited Bose-gas with a comparatively small number of excited atoms- where the interaction between differently excited atoms gives rise to a significant change of the energy spectrum of elementary excitations. Apart from the appearance of another branch, the availability of excited atoms leads to the appearance of an energy-gap for Bogolubov's phonons and a changed velocity of sound.

The third and main problem considered here is the energy spectrum of elementary excitations in an excited Bose-gas when the number of excited atoms is still small but we can neglect the interactions among non-excited atoms and also the interactions among excited ones.

The last problem considered in this paper is the excited Bose-gas at high temperatures. We have obtained the energy spectrum of elementary excitations and analysed its thermodynamic properties.

CHAPTER 1

QUANTUM CONSIDERATIONS OF THE INTERACTION BETWEEN ATOMS.

(VIRTUAL DIPOLES)

VAN DER WAALS FORCES

The prototype of our discussion is the problem of determining the energy of interaction of two atoms which are situated at a great distance from each other.

In the solution of this problem we apply perturbation theory, regarding the two isolated atoms as the unperturbed system, and the potential energy of their electrical interaction as the perturbation. As is known the 1st order correction to the required energy of interaction of the atoms is determined as the diagonal matrix element of the perturbing Hamiltonian, calculated with respect to the unperturbed wave functions of the system which are expressed in terms of products of the unperturbed wave functions for the isolated atoms.

The total Hamiltonian of the system of two interacting atoms is

$$H = {}^{1}H_{o} + {}^{2}H_{o} + U(R) \qquad (1.1)$$

where U(R) is the perturbing term given by the operator of dipole-dipole interaction [1] , [2]

$$U(R) = \frac{\vec{d}_1 . \vec{d}_2 - 3(\vec{d}_1 . \hat{n})(\vec{d}_2 . \hat{n})}{R^3} , \qquad (1.2)$$

${}^{1}H_{o}$ and ${}^{2}H_{o}$ are the Hamiltonians of the isolated atoms.

Let us consider two atoms in the ground states. In s-states the diagonal elements of the perturbing Hamiltonian are zero since in s states the Ψ-function is a function only of r implying, on the average, spherical symmetry in the distribution of charges in the atom. This amounts to evaluating the mean values of the dipole moments of the atoms and they vanish in s-states. So the 1st order correction term to the energy of the system of two atoms vanishes and we go over to the next correction term which is the second order.

In the special case of two atoms in their ground states, the 2nd order correction to the ground energy of the two atoms depends on the distance and can be considered as the interaction energy [2]

$$U(R) = - \frac{A}{R^6} \tag{1.3}$$

where A is a positive constant which depends on the type of atoms.

Thus, two atoms in normal s-states, at a great distance apart, attract each other with a force $(- \frac{dU}{dR})$ which is inversely proportional to the seventh power of the distance. The attractive forces between atoms (dipoles) are the Van der Waal's forces. These are short range forces effective over distances of the order of atomic diameters and are a minor effect in the present problem. The constant

A in (1.3) depends, not only on the states of the two atoms, but also on their mutual orientation, i.e., on the value of the projection of the angular momentum on the axis joining the atoms.

INTERACTION OF DIFFERENTLY EXCITED ATOMS

RESONANCE INTERACTION

As a special case consider the interaction of two similar atoms in different states. The unperturbed system (the two isolated atoms) has here an additional degeneracy due to the possibility of interchanging the states of the atoms (excitation jumps from one atom to the other). Accordingly, the correction in the first approximation will be given by the secular equation with the interaction term (1.2), in which the non-diagonal matrix elements of the perturbation appear as well as the diagonal ones.

Consider the case where the 1st atom is in the ground state and the 2nd atom in the 1st excited state. The wave function of the 1st atom is given by

$$\psi_{100}(1) \sim R_{10}(r) \sim e^{-r/a}$$

and that of the 2nd atom by

$$\psi_{21-1}(2) \sim R_{21}(r)\ \sin\ \theta\ e^{-i\phi}$$

$$\psi_{210}(2) \sim R_{21}(r)\ \cos\ \theta$$

$$\psi_{211}(2) \sim R_{21}(r)\ \sin\ \theta\ e^{i\phi}$$

The set of wave functions corresponding to the system of two differently excited atoms in S and P states are then

$$\begin{aligned} \Psi_1 &= \psi_{100}(1)\ \psi_{21-1}(2) \quad , & \Psi_4 &= \psi_{100}(2)\ \psi_{21-1}(1) \\ \Psi_2 &= \psi_{100}(1)\ \psi_{210}(2) \quad , & \Psi_5 &= \psi_{100}(2)\ \psi_{210}(1) \\ \Psi_3 &= \psi_{100}(1)\ \psi_{211}(2) \quad , & \Psi_6 &= \psi_{100}(2)\ \psi_{211}(1) \end{aligned} \tag{1.4}$$

Each of the wavefunctions corresponds to the same energy $E = (\varepsilon_o + \varepsilon_1)$ where ε_o is the energy of the ground state of one atom and ε_1 is the energy of the 1st excited state of the other atom.

The correction, $E^{(1)}$, to the energy E in the first approximation of perturbation theory is given by the secular equation, [3]

$$\begin{vmatrix} U_{11}-E^{(1)} & U_{12} & \cdots & U_{16} \\ U_{21} & U_{22}-E^{(1)} & \cdots & U_{26} \\ \vdots & & & \\ U_{61} & U_{62} & \cdots & U_{66}-E^{(1)} \end{vmatrix} = 0 \ ; \tag{1.5}$$

where

$U_{ij} = \langle \Psi_i | U | \Psi_j \rangle = \int \Psi_i^* U \Psi_j d\tau$ is the matrix element of the perturbing Hamiltonian given by (1.2) on the wave functions (1.4). Due to symmetry conditions all diagonal matrix elements vanish and only the matrix elements U_{14}, U_{41} ; U_{25}, U_{52} ; U_{36}, U_{63} ; are non-zero.

Taking the linear combinations of the wave functions of the excited atom, we can choose them in such a way that they will transform like components of the vector $\vec{r}$:

$$\phi_x=\psi_{21-1}+\psi_{211} \quad ; \quad \phi_y=\psi_{21-1}-\psi_{211} \quad ; \quad \phi_z=\psi_{210} .$$

Then, the set of wave functions given by (1.4) corresponding to the system of two differently excited atoms, can be written as

$$|\Psi_1\rangle=|\phi_o(1)\phi_x(2)\rangle=|0\rangle|x\rangle \quad ; \quad |\Psi_4\rangle=|\phi_o(2)\phi_x(1)\rangle=|x\rangle|0\rangle$$

$$|\Psi_2\rangle=|\phi_o(1)\phi_y(2)\rangle=|0\rangle|y\rangle \quad ; \quad |\Psi_5\rangle=|\phi_o(2)\phi_y(1)\rangle=|y\rangle|0\rangle \qquad (1.6)$$

$$|\Psi_3\rangle=|\phi_o(1)\phi_z(2)\rangle=|0\rangle|z\rangle \quad ; \quad |\Psi_6\rangle=|\phi_o(2)\phi_z(1)\rangle=|z\rangle|0\rangle$$

This, then, is a case of 6-fold degeneracy and

$$\sum_{j=1}^{6} \{\langle\psi_i|H_o+U|\psi_j\rangle - E\delta_{ij}\} C_j = 0 \quad . \qquad (1.7)$$

With the z-axis chosen in the direction of the unit vector $\hat{n}$, the perturbing Hamiltonian U takes the form

$$U = \frac{d_{1x}d_{2x} + d_{1y}d_{2y} - 2d_{1z}d_{2z}}{R^3} \quad .$$

The solution of (1.7) gives the following values for the 1st correction to the degenerate energy level $(\varepsilon_o+\varepsilon_1)$. [4]

$$E^{(1)}_{1,2} = \pm \frac{K}{R^3}, \quad E^{(1)}_{3,4} = \pm \frac{K}{R^3} ; \quad E^{(1)}_{5,6} = \pm \frac{2K}{R^3} \quad . ,$$

where

$$K = \langle 0|d_x|x\rangle^2 = \langle 0|d_y|y\rangle^2 = \langle 0|d_z|z\rangle^2$$

So the 6-fold degenerate level $(\varepsilon_o+\varepsilon_1)$ splits into four levels among which the levels $(\varepsilon_o+\varepsilon_1 \pm \frac{K}{R^3})$ still have 2-fold degeneracy. The linear combinations which correspond to these levels are

$$\left.\begin{aligned} |\Psi^x_s\rangle &= \frac{1}{\sqrt{2}}\{|\phi_o(1)\phi_x(2)\rangle + |\phi_o(2)\ \phi_x(1)\rangle\} \\ |\Psi^y_s\rangle &= \frac{1}{\sqrt{2}}\{|\phi_o(1)\phi_y(2)\rangle + |\phi_o(2)\ \phi_y(1)\rangle\} \end{aligned}\right\} \text{two fold degenerate}$$

$$|\Psi^z_s\rangle = \frac{1}{\sqrt{2}}\{|\phi_o(1)\phi_z(2)\rangle + |\phi_o(2)\ \phi_z(1)\rangle\} \qquad (1.8)$$

$$\left.\begin{aligned} |\Psi^x_A\rangle &= \frac{1}{\sqrt{2}}\{|\phi_o(1)\phi_x(2)\rangle - |\phi_o(2)\ \phi_x(1)\rangle\} \\ |\Psi^y_A\rangle &= \frac{1}{\sqrt{2}}\{|\phi_o(1)\phi_y(2)\rangle - |\phi_o(2)\ \phi_y(1)\rangle\} \end{aligned}\right\} \text{two fold degenerate}$$

$$|\Psi^z_A\rangle = \frac{1}{\sqrt{2}}\{|\phi_o(1)\phi_z(2)\rangle - |\phi_o(2)\ \phi_z(1)\rangle\}$$

It can be shown that the expectation value of the dipole moment in all antisymmetrical states in (1.8) equals to zero and has non-zero meaning for symmetric states. Considering the pair of differently excited atoms, the probability of dipole transition from one of the states in equation (1.8) to a state where both of them are in the ground state (probability is proportional to dipole matrix element), it is clear that the antisymmetric states in (1.8) are long-lived states with respect to dipole transitions. So among the interacting pairs of differently excited atoms we have short-lived symmetrical states (life-time $\sim 10^{-8}$ seconds which is the time of dipole transition [5]) and long-lived antisymmetrical states with the forbidden dipole transitions.

CHAPTER 2

THE CONCEPTION OF ELEMENTARY EXCITATIONS

The ideal quantum Bose gas at $T = 0$ creates the so-called "condensate" when all particles occupy the level with zero energy and zero momentum. This can be, most easily, seen from the Bose distribution

$$n_k = \frac{1}{e^{\frac{\varepsilon(k)-\mu}{T}} - 1} \, . \qquad (2.1)$$

At $T = 0$ $\mu = 0$ and

$$n_k = \frac{1}{e^{\frac{\varepsilon(k)}{T}} - 1}$$

has singularity $n_{k=0} \longrightarrow \infty$ $\left[\varepsilon(k) = \frac{k^2}{2M}\right]$. This means that all particles have energy $\varepsilon(k) = 0$.

In the case of non-ideal gas a macroscopic part of the particles are still in the condensate state; but there are some (small amount of) particles with non-zero momentum even at $T = 0$. The more removed from ideality the gas is, the more the number of particles with

$$\varepsilon(k) \neq 0 \, .$$

The theory of such weakly non-ideal degenerate Bose gas was developed by Bogolubov [6] and demonstrates that particles with $\varepsilon(k) \neq 0$ can be described in terms of phonons. These elementary excitations have the following dispersion law,

$$E(k) = \sqrt{[\varepsilon(k)]^2 + U_S^2 K^2} \, . \qquad (2.2)$$

In the long-wavelength limit $(k \to 0)$ eqn.(2.2) reduces to

$$E(k) = U_S K \quad . \qquad (2.3)$$

which corresponds to the phonon part of the excitation spectrum of a Bose liquid with U_S identified as the velocity of sound. In the case of large momenta, the excitation spectrum is given by

$$E(k) \approx \varepsilon(k) = \frac{K^2}{2M} \qquad (2.4)$$

and the quasi-particles turn out to be the (real) free particles. This agrees with the general conclusions of Landau's theory about the energy-spectrum of condensed systems at low temepratures [6] .

This result for the phonon case turns out to be extremely general. It appears now that, in most many-body systems, it is possible to perform a transformation τ from the system of strongly interacting particles to a set of approximately independent elementary excitations above the ground state. Thus, the transformation may be written in the form |10|:

$$\tau : H = \sum_i H(\vec{p}_i, \vec{r}_i) + \tfrac{1}{2}\sum_{ij} V(\vec{r}_i, \vec{r}_j ; \vec{p}_i \vec{p}_j) \rightarrow$$

$$\rightarrow H' = E_o + \sum_k E(k) A_k^\dagger A_k + \overbrace{f(\ldots A_k \ldots A_k^\dagger \ldots)}^{\text{small}} \quad . \qquad (2.5)$$

Here H' is the second quantized Hamiltonian of the system of interacting particles; the transformation τ is understood to contain the necessary transformation to the second-quantized form. E_o is the ground state energy of the interacting system; E(k) is the energy of the elementary excitation ('dispersion law' or 'excitation spectrum'); $A_k^\dagger, A_k$, $(A_k^\dagger A_k)$ are creation, destruction and number operators for the elementary excitations. The small term in eqn.(2.5) describes the interactions between elementary excitations.

Thus, at low temperatures the energy spectrum of a many-body system can be built as a sum of the energies of elementary excitations which can be considered like non-interacting (ideal) particles. This means that at low temperatures, where excitations with small energies are important and the number of them is small, the interaction of elementary excitations leading to decomposition of excitations (which occurs only for fairly large energies) and the scattering of excitations by one another (which becomes important only when the number of excitations is fairly large) are no longer important. The weakness of the interactions between excitations at low temperatures enables them to be regarded as an ideal gas of elementary excitations (quasi-particles) with specific dispersion law, $E(k)$ [8]. As temperature increases the number of elementary excitations increases and the interaction between them becomes significant.

II. THE GREEN FUNCTION METHOD. [7] , [8].

One can ask how one can find the dispersion law of these quasiparticles in the particular cases. The most simple way is to start with the second-quantized Hamiltonian of a system.

$$H=\sum_{k}\varepsilon(k)a_{k}^{\dagger}a_{k}+\frac{1}{2V}\sum_{\substack{k_1'k_2'\\k_1k_2}}\langle k_1'k_2'|U|k_2k_1\rangle a_{k_1'}^{\dagger}a_{k_2'}^{\dagger}a_{k_2}a_{k_1}. \qquad (2.6)$$

$$k_1'+k_2' = k_1+k_2 \quad \text{(conservation of momentum).}$$

where

$a_k^{\dagger}$, a_k - are creation and annihilation operators;

$\langle k_1'k_2'|U|k_2k_1\rangle$ = is the matrix element of interaction with $\langle k_1'k_2'|U|k_2k_1\rangle = \int\psi_{k_1'}\phi_{k_2'}\, U(r_1,r_2)\phi_{k_2}\phi_{k_1}d^3_{r_1}d^3_{r_2}$

U being the potential energy of pair interaction; and k is the set of quantum numbers which describes the single particle state ϕ_k.

At the first steps of theory the only way for finding the dispersion law of elementary excitations was looking for a suitable transformation from old operators $a_k^\dagger, a_k$ to new operators $b_k^\dagger, b_k$ so as to reduce the Hamiltonian (2.6) to the diagonal form

$$H=\sum_k E(k) b_k^\dagger b_k + E_o \, , \tag{2.7}$$

E_o is the energy of the ground state and $E(k)$ the energy of the quasiparticles or elementary excitations. $b_k^\dagger$ and b_k are creation and annihilation operators of excitations with energy $E(k)$. Unfortunately this method works only in the case of special Hamiltonians which can be obtained from the general one using some physical approximations.

At the end of the 50's the powerful methods of quantum field theory have been introduced into statistical physics and the physics of condensed matter. These methods are now very widely used and called the method of Green functions. The most important featueres of this method are the following:

We introduce the thermal Green function [7]

$$\mathfrak{G}_{a_k^\dagger a_{k'}}(t) = -i\theta(t) \langle [a_k^\dagger(t) \, , \, a_{k'}(0)]_{\mp} \rangle \tag{2,8}$$

where

$a_k^\dagger(t)$ is the creation operator in the Heisenberg representation

$$a_k^\dagger(t) = e^{iHt}\, a_k^\dagger(0)\, e^{-iHt} \;;$$

(H is the Hamiltonian eqn.(2,6)); $\theta(t)$ is Heaveside's step function; $[\;]_{\mp}$ stands for the commutator in case of Bose particles or anticommutator in case of Fermi particles; $\langle\;\rangle$ means thermodynamical average

$$\langle [a_k^\dagger(t)\;,\; a_k(0)]\rangle = \frac{1}{Z(\beta)}\, Tr\{e^{-\beta H}[a_k^\dagger(t)\;,\; a_{k'}(0)]\} ,$$

$Z = Tr\{e^{-\beta H}\}$, β is the reciprocal of the absolute temperature in units of reciprocal of energy. Here we have put Boltzmann's constant K_B, and Plancks constant $\hbar$ equal to unity.

The Green function (2.8) permits one to find the energy spectrum of elementary excitations. Taking the Fourier transform of (2.8) and demanding

$$G^{-1}_{a_k^\dagger a_{k'}}(\omega) = 0 \qquad (2.9)$$

one can find the dispersion law, ie., the dependence of the energy, $\omega(k)$, of elementary excitations on momentum K as the roots of equation (2.9).

The distribution of particles in the momentum space is

$$n_k = \langle a_k^{\dagger}(t) \; , \; a_k(t) \rangle = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\omega}{\exp[\beta\,\omega(k)]+1} \mathrm{Im}\; G_{a_k^{\dagger} a_k}(\omega). \qquad (2.10)$$

The Green function (2.8) has a structure of the type

$$G_{a_k^{\dagger} a_k}(\omega) \sim \frac{1}{\omega - \omega(k)} \; ,$$

and using the Dirac identity

$$\frac{1}{\omega - \omega(k) \pm i\varepsilon} = \frac{P}{\omega - \omega(k)} \mp i\pi\delta\,(\omega - \omega(k))$$

One can select the imaginary part of the Green function (2.8).

If the energy of elementary excitations and their distribution n_k in momentum space are known, then one can work out the thermodynamical properties of the system. The internal energy, for instance, is given by

$$\langle E \rangle = \sum_k \omega(k)\, n_k \; .$$

On the basis of E(V,T) we can find all thermodynamical properties.

The problem now is how to find the Green function. Below we shall use the method of equations of motion for the Green function proposed by Tyablikov [9]. The equation of motion for the Green function (2.8) has the form

$$i\dot{G}_{a_k^\dagger a_{k'}}(t) = -\delta(t)\,\delta_{kk'} + \theta(t)\langle[\dot{a}_k^\dagger(t), a_{k'}(0)]_{\mp}\rangle \qquad (2.11)$$

and can be obtained from eqn. (2.8) directly by differentiation with respect to time. The time derivative of $a_k^\dagger(t)$ is given by the standard formula

$$\dot{a}_k^\dagger(t) = -i\,[a_k^\dagger(t), H] \qquad (2.12)$$

In the real cases the interaction term in the Hamiltonian (2.6) generates the Green function of higher order of the type

$$G_{a_{k_1}^\dagger a_{k_2}^\dagger a_{k'_2}, a_{k'}}(t) = -i\theta(t)\langle[a_{k_1}^\dagger(t)\, a_{k_2}^\dagger(t)\, a_{k'_2}(t)\;,\; a_{k'}(0)]\rangle .$$

So we also need the equation for $\dot{G}_{a_{k_1}^\dagger a_{k_2}^\dagger a_{k'_2} a_{k'}}$. But such an equation will involve the Green function of still higher order and so on. Eventually we arrive at the infinite chain of coupled equations for the Green functions of different orders. On some physical grounds we can cut the system of equations for the Green functions and obtain the closed finite system. Examples of such procedures are demonstrated below.

I. SECOND-QUANTIZED HAMILTONIAN OF EXCITED GAS.

The main goal of this work is the investigation of the properties of an excited Bose gas. This is a gas in which the number of excited atoms is not determined by Maxwellian distribution

$$N_1 \sim e^{-\beta E_1} \quad (E_1 = \text{the energy of excited atom}),$$

but depends on external conditions [laser illumination, electron beams, etc.]. Using the laser one can create and maintain the constant number of excited atoms [4].

It can be shown [4] that, for example, in excited gas the lifetime of excited atom is $\sim 10^{-8}$ seconds, but the characteristic time of Maxwellization of gas under normal conditions is $\sim 10^{-11}$ seconds. So we can assume that there is equilibrium in the kinetical degrees of freedom, but the electron degrees of freedom are not in thermodynamic equilibrium.

Let us consider an excited gas in which N_1 atoms are excited to the 1st excited level and N_0 atoms in the ground state with the usual situation

$$N_1 \ll N_0 .$$

The Hamiltonian of our system in which interactions can be taken into account in the dipole-dipole approximation is

$$H=\sum_{j=1}^{N}\left(\frac{p_j}{2M}+H_j^{o}\right)+\sum_{\substack{j=1\\ j'>j}}^{N}\frac{(\vec{d}_j\cdot\vec{d}_{j'})-3(\hat{n}_{jj'}\cdot\vec{d}_j)(\hat{n}_{jj'}\cdot\vec{d}_{j'})}{|\vec{R}_j-\vec{R}_{j'}|^3}\,, \qquad (2.13)$$

where

H_j^{o} = Hamiltonian of the j^th atom,

$\frac{p_j^2}{2M}$ = operator of kinetic energy of the j^th atom,

and the 3^rd term is the energy of dipole-dipole interaction. We have also

$$H_j^{o}\phi_s(r_{j_1},r_{j_2},\ldots,r_{jn})=\varepsilon_s\phi_s(r_{j_1},r_{j_2},\ldots,r_{jn})\,.$$

We find the second-quantized Hamiltonian for excited gas using the following field operators

$$\hat{\Psi}(r,R)=\frac{1}{\sqrt{V}}\sum_{K}\left[a_k\phi_o(r)+b_k\phi_1(r)\right]e^{i\vec{K}\cdot\vec{R}}$$

$$\hat{\Psi}^{\dagger}(r,R)=\frac{1}{\sqrt{V}}\sum_{K}\left[a_k^{\dagger}\phi_o^{*}(r)+b_k^{\dagger}\phi_1^{*}(r)\right]e^{-i\vec{K}\cdot\vec{R}}\,. \qquad (2.14)$$

Here and below we shall assume, for the sake of simplicity, that the excited state is non-degenerate and $\phi_1(r)$ may correspond to one of $\phi_x(r)$, $\phi_y(r)$, or $\phi_z(r)$. Such a situation can be organized by experimentally applying to an excited gas a strong magnetic field leading to the removal of degeneracy with the condition that the Zeeman splitting

$$\Delta E >> K_B T.$$

In (2.14) $\vec{R}$ is the coordinate of the center of mass of the atom; $\vec{r}=\{\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_n\}$ is the set of electrons' coordinates; $a_k^{\dagger}/a_k$ are the creation/annihilation operators for particles in the ground state $[\phi_0(r)]$; b_k^{+}/b_k are creation/annihilation operators for particles in the excited state $(\phi_1(r))$.

Following the standard procedure [2]

$$H=\int\Psi^{+}(1)H_0(1)\Psi(1)d\tau_1 + \tfrac{1}{2}\int\Psi^{+}(1)\Psi^{\dagger}(2)U(1,2)\Psi(2)\Psi(1)d\tau_1 d\tau_2$$

we get the second-quantized Hamiltonian for excited gas,

$$H=N_0\varepsilon_0+N_1\varepsilon_1+\sum_k\left(\varepsilon(k)-\frac{U_0^*}{2V}\right)(a_k^{\dagger}a_k+b_k^{+}b_k)+\frac{1}{2V}\sum_{\substack{k'_1 k'_2\\ k_1 k_2}}U_0(q)a^{+}_{k'_1}a^{+}_{k'_2}a_{k_2}a_{k_1}+$$

$$+\frac{1}{2V}\sum_{\substack{k'_1 k'_2\\ k_1 k_2}}U_0^*(q)\left(a^{\dagger}_{k'_1}a^{+}_{k'_2}b_{k_2}b_{k_1}+b^{+}_{k'_1}b^{+}_{k'_2}a_{k_2}a_{k_1}+2a^{+}_{k'_1}b^{+}_{k'_2}a_{k_2}b_{k_1}\right\}$$

(2.15)

Noting that resonance interaction means the interaction between differently excited atoms moving with the same momenta (interchanging of excitation is possible only in this case), the Hamiltonian (2.15) takes the form (in the limit $V \to \infty$; q = transferred momentum $\to 0$)

$$H = N_0\varepsilon_0 + N_1\varepsilon_1 + \sum_k \varepsilon(k)(a_k^+ a_k + b_k^+ b_k) + \frac{U_0}{2V} \sum_{\substack{k'_1 k'_2 \\ k_1 k_2}} a_{k'_1}^+ a_{k'_2}^+ a_{k_2} a_{k_1} + \\ + \frac{U_0^*}{2V}\left(\sum_k (a_k^+ b_k + b_k^+ a_k)\right)^2 . \qquad (2,16)$$

The term $(N_0\varepsilon_0 + N_1\varepsilon_1)$ is the energy of the electrons' degrees of freedom for the non-interacting particles and we shall drop it hereafter, it being understood that we measure energy from $(N_0\varepsilon_0 + N_1\varepsilon_1)$; $\varepsilon(k) = \frac{K^2}{2M}$ is the kinetic energy of a particle of mass M; U_0 is the Fourier transform of the interaction between non-excited atoms, and U_0^* is the Fourier transform of the interaction between differently excited atoms.

It can be shown (see Chapter 1) that, in the dipole-dipole approximation

$$U_0 << U_0^* \qquad (2.17)$$

So, even in the conditions when $N_1 << N_0$ the last term in (2.16) describing the interaction of differently excited atoms can significantly change the energy spectrum of such a gas. We have neglected the interaction between excited atoms on account of their relatively small number. We shall consider three cases.

CHAPTER 3

WEAKLY NON-IDEAL BOSE-GAS AT LOW TEMPERATURES T→O.

In the absence of excited atoms, the Hamiltonian (2.16) reduces to Bogolubov's Hamiltonian

$$H_B = \sum_k \varepsilon(k) a_k^{\dagger} a_k + \frac{U_0}{2V} \sum_{\substack{k'_1 k'_2 \\ k_1 k_2}} a^{\dagger}_{k'_1} a^{\dagger}_{k'_2} a_{k_2} a_{k_1} \qquad (3.1)$$

$$k'_1 + k'_2 = k_1 + k_2$$

For simplification of the Hamiltonian (3.1) we use the following considerations. First of all we consider the case of low temperatures, T→O. At T ≈ O most of the particles should be in the state with zero momentum

$$N_o^{k=o} \approx N_o \approx N \; .$$

So we can neglect the non-commutativeness of

$$a_{k=o} a^{\dagger}_{k=o} - a_{k=o} a_{k=o} = 1 \qquad (3.2a)$$

and treat $a_{k=o}$, $a^{\dagger}_{k=o}$ as c-numbers.

$$a_{k=o} \approx a^{\dagger}_{k=o} = \sqrt{N_o^{k=o}} = \sqrt{N_o^o} \approx \sqrt{N_o} = \sqrt{N}$$

But $\qquad a_k a_k^{\dagger} - a_k^{\dagger} a_k = 1 \quad , k \neq 0 \qquad (3.2b)$

and $\qquad N_o = N_o^{k=o} + \sum_{k \neq o} a_k^{\dagger} a_k = N$

The application of perturbation theory now means the expansion of the interaction term in powers of small values $a_{k\neq o}$, $a_{k\neq o}$. Doing this one can get up to second-order terms. There are no 1st order terms since these would not conserve momenta.

$$H_B = \frac{U_o N}{2V} + \frac{1}{2}\sum_{k\neq o}\left\{\left(\varepsilon(k) + \frac{U_o N}{V}\right)(a_k^\dagger a_k + a_{-k}^\dagger a_{-k}) + \right.$$

$$\left. + \frac{U_o N}{V}(a_k a_{-k} + a_{-k}^\dagger a_k^\dagger)\right\} \qquad (3.3)$$

For the determination of the energy levels it is necessary to reduce the Hamiltonian (3.3) to the diagonal form. It can be done by appropriate linear transformation as in the case of N.N. Bogolubov [6]. We shall arrive at the well-known solutions by the Green function method as described in the previous chapter.

We shall 1st find the equation(s) of motion for the $a_k^\dagger(a_k)$ operator(s).

$$\dot{a}_k^\dagger = -i\left[a_k^\dagger, H_B\right] = i\left\{\left(\varepsilon(k) + 2A\right)a_k^\dagger(t) + 2Aa_{-k}\right\} \qquad (3.4)$$

where we have put $\hbar = 1$ and

$$A = \frac{U_o N}{2V}$$

The equation of motion for the Green function $G_{a_k^{+}a_{k'}} = G_1$

$$i\dot{G}_1(t) = -\delta(t)\,\delta_{kk'} - \Big(\varepsilon(k)+2A\Big)G_1(t) - 2AG_2(t) \qquad (3.5a)$$

involves the Green function $G_{a_{-k}a_{k'}} = G_2$. The equation of motion for G_2 is

$$i\dot{G}_2(t) = 2AG_1(t) + \Big(\varepsilon(k)+2A\Big)G_2(t) \quad . \qquad (3.5b)$$

The system of equations (3.5) represents the closed chain of equations for determining G_1 & G_2. Passing to the Fourier transform of these equations we get

$$\begin{aligned} \delta_{kk'} &= \Big\{\omega - \Big[\varepsilon(k)+2A\Big]\Big\}G_1(\omega) - 2A\;G_2(\omega) \\ 0 &= 2A\;G_1(\omega) + \Big\{\omega + \Big(\varepsilon(k)+2A\Big)\Big\}G_2(\omega) \quad . \end{aligned} \qquad (3.6)$$

The solution to (3.6) is

$$\begin{aligned} G_1(\omega) &= \frac{E(k) + \varepsilon(k)+2A}{2E(k)\Big(\omega(k)-E(k)\Big)}\,\delta_{kk'} - \frac{\varepsilon(k)+2A-E(k)}{2E(k)\Big(\omega(k)+E(k)\Big)}\,\delta_{kk'} \\ G_2(\omega) &= -\frac{A}{E(k)\Big(\omega(k)-E(k)\Big)}\,\delta_{kk'} + \frac{A}{E(k)\Big(\omega(k)+E(k)\Big)}\,\delta_{kk'} \end{aligned} \qquad (3.7)$$

and the energy spectrum of elementary excitations

$$E(k) = \sqrt{\varepsilon^2(k) + U_S^2K^2} \qquad (3.8)$$

where $U_S^2 = \dfrac{U_0N}{mV}$ is the velocity of sound. This result coincides with Bogolubov's [8].

The transformation required to put the Hamiltonian (3.3) in the form

$$H_B = \underbrace{E_o}_{\text{ground state}} + \sum_{k\neq o} E(k) b_k^\dagger b_k \qquad (3.9)$$

is suggested by eqn.(3.4) and can right-away be written in the form

$$a_k^\dagger = U_k b_k + V_k b_{-k}^\dagger \qquad (3.10)$$

$b_k/b_k^\dagger$ being annihilation/creation operators for quasi-particles that obey Bose commutation rules; U_k, V_k are real numbers whose values are still to be determined, and

$$\begin{aligned} b_k^\dagger(t) &= b_k^\dagger(0)\ \exp\left[iE(k)t\right] \\ b_k(t) &= b_k(0)\ \exp\left[-iE(k)t\right]. \end{aligned} \qquad (3.11)$$

Using equations (3.10) and (3,11) the Green functions $G_1(t)$ and $G_2(t)$ may be put in the form

$$\begin{aligned} G_1(t) &= i\theta(t)U_k^2\exp(iEt)\delta_{kk'} - i\theta(t)V_k^2\exp(-iEt)\delta_{kk'} \\ G_2(t) &= i\theta(t)U_kV_k\exp(iEt)\delta_{kk'} - i\theta(t)U_kV_k\exp(-iEt)\delta_{kk'} \end{aligned} \qquad (3.12)$$

Taking the Fourier transform of (3,12) we obtain

$$G_1(\omega) = \frac{U_k^2}{\omega-E}\ \delta_{kk'} - \frac{V_k^2}{\omega+E}\ \delta_{kk'} \qquad (3.13a)$$

$$G_2(\omega) = \frac{U_kV_k}{\omega-E}\ \delta_{kk'} - \frac{U_kV_k}{\omega+E}\ \delta_{kk'} \qquad (3.13b)$$

On comparing equations (3.13) and (3.7) we see that

$$U_k^2 = \tfrac{1}{2}\left[1/\sqrt{1-\Gamma_k^2} + 1\right] \; ; \quad V_k^2 = \tfrac{1}{2}\left[1/\sqrt{1-\Gamma_k^2} - 1\right]$$

$$U_k V_k = -\tfrac{1}{2}\left(\Gamma_k/\sqrt{1-\Gamma_k^2}\right) . \qquad (3.14)$$

with

$$\Gamma_k = 2A/(\varepsilon+2A) \; ; \quad E = \sqrt{\varepsilon^2+4\varepsilon A} = 2A\left(\sqrt{1/\Gamma_k^2 - 1}\right).$$

which coincide with [8].

It will be useful to note that Bogolubov used for the diagonalization of the Hamiltonian (3.3) the algebraic method introducing the operators of quasiparticles b_k

$$b_k = U_k a_k + V_k a_{-k}^{+} .$$

The necessary coefficients of this U-V transformation were obtained from the condition of the vanishing of non-diagonal terms. The Green function method, as we have seen, systematically and directly yields not only the dispersion law for elementary excitations but also the sought-for transformation is indicated in the equations of motion of the operators involved.

CHAPTER 4

ELEMENTARY EXCITATIONS IN AN EXCITED BOSE-GAS AT LOW TEMPERATURES.

I. Let us consier an excited Bose-gas with a comparatively small number of excited particles. We shall take into account the interactions among non-excited particles and between differently excited particles and shall show that even in the case when the number of excited particles is small their contribution to the energy spectrum of elementary excitations is significant.

We shall start by assuming that the ground state of an excited Bose-gas is as follows:

In the Ground State there exist the macroscopic number of particles in the state with $k = 0$. We denote them as

$$N_0^{k=0} = N_0 \quad ; \quad N_1^{k=0} = N_1 \tag{4.1}$$

A comparatively small number of particles have non-zero momenta ; i.e.,

$$\sum_{k\neq 0} a_k^\dagger a_k \ll N_0^{k=0} \quad ; \quad \sum_{k\neq 0} b_k^\dagger b_k \ll N_1^{k=0} \tag{4.2}$$

Expanding the Hamiltonian (2.16) in powers of small operators

$a_{k\neq 0}$, $a^{\dagger}_{k\neq 0}$, $b_{k\neq 0}$, $b^{\dagger}_{k\neq 0}$ up to second order terms we obtain

$$H' = E_0 + \sum_{k\neq 0} E(k) a^{\dagger}_k a_k + \sum_{k\neq 0} \varepsilon(k) b^{\dagger}_k b_k + 2A_0 \sum_{k\neq 0} (a_k a_{-k} + a^{\dagger}_{-k} a_k) +$$

$$+ 2A_1 \xi \sum_{k\neq 0} (a^{\dagger}_k b_k + b^{\dagger}_k a_k) \quad . \qquad (4.3)$$

Where

$$E_0 = A_0 N_0 + 2A_1 N_1 \quad ; \quad A_0 = \frac{U_0 N_0}{2V} \quad ; \quad A_1 = \frac{U^*_0 N_0}{V}$$

$$(4.4)$$

$$E(k) = \varepsilon(k) + 2A_0 - 2A_1 \xi^2 \quad ; \qquad \xi^2 = \frac{N_1}{N_0} \ll 1$$

Equation (4.3) is obtained on the assumptions that

$$N_1 << N_0$$

so that we have improved N_0 in accordance with

$$N_0^{k\neq 0} + \sum_{k\neq 0} a^{\dagger}_k a_k = N_0$$

leaving N_1 as it is.

The equations of motion of our operators have the form

$$\dot{a}^{\dagger}_k = i\{E(k) a^{\dagger}_k + 2A_1 \xi b^{\dagger}_k + 2A_0 a_{-k}\}$$

$$\dot{b}^{\dagger}_k = i\{\varepsilon(k) b^{\dagger}_k + 2A_1 \xi a^{\dagger}_k\} \qquad (4.5)$$

The closed system of equations of motion of the Green's functions needs also $\dot{a}_{-k}$ and $\dot{b}_{-k}$ which can be obtained from (4.5) by taking adjoints and substituting k by -k.

Introducing the following Green's functions

$$
\begin{aligned}
G_1(t) &= -i\,\theta(t)\langle[a_k^\dagger(t),\ a_{k'}(0)]\rangle\\
G_2(t) &= -i\,\theta(t)\langle[b_k^\dagger(t),\ a_{k'}(0)]\rangle\\
G_3(t) &= -i\,\theta(t)\langle[a_{-k}(t),\ a_{k'}(0)]\rangle\\
G_4(t) &= -i\,\theta(t)\langle[b_{-k}(t),\ a_{k'}(0)]\rangle
\end{aligned}
\qquad (4.6)
$$

we can obtain the closed system of equations of motion of the Green's functions (4,6)

$$
\begin{aligned}
i\dot{G}_1(t) &= -\delta(t)\delta_{kk'} - E(k)G_1(t) - 2A_1\xi G_2(t) - 2A_0G_3(t)\\
i\dot{G}_2(t) &= -2A_1\xi G_1(t) - \varepsilon(k)G_2(t)\\
i\dot{G}_3(t) &= 2A_0G_1(t) + E(k)G_3(t) + 2A_1\xi G_4(t)\\
i\dot{G}_4(t) &= 2A_1\xi G_3(t) + \varepsilon(k)G_4(t)
\end{aligned}
\qquad (4.7)
$$

Passing to the Fourier transform of (4,7) one can arrive at the system of linear algebraic equations from which one can obtain the equation for the energy spectrum of elementary excitations $\omega(k)$.

$$
\begin{aligned}
&\left[\omega^2(k)-\varepsilon^2(k)\right]^2-\left[\omega^2(k)-\varepsilon^2(k)\right]\Big\{4\varepsilon(k)A_0-2\left[\varepsilon(k)+2A_0-2A_1\right](2A_1)\xi^2+\\
&\quad+4(2A_1)^2\xi^4\Big\}-\Big\{4(2A_1)^2\left[\varepsilon^2(k)+\varepsilon(k)A_0\right]\xi^2-\\
&\quad-2(2A_1)^3\left[\varepsilon(k)+A_1\right]\xi^4\Big\} = 0
\end{aligned}
\qquad (4.8)
$$

---

we shall seek the solution to (4.8) in the form

$$\Omega = \Omega_0 + \Omega_1 \xi^2 \quad , \quad \Omega = \omega^2 - \varepsilon^2 \qquad (4.9)$$

which corresponds to keeping terms $\sim \xi^2$ and neglecting $\sim \xi^4$ in (4.8). Then step-by-step approximation yields the two branches which we shall designate by $\omega_B^2(k)$, and $\omega_e^2(k)$

i) $$\omega_B^2 \simeq \varepsilon^2 + 4\varepsilon A_0 \left[1 + \left(\frac{A_1}{A_0}\xi\right)^2 \right] + 12 A_1^2 \xi^2 \qquad (4.10a)$$

ii) $$\omega_e^2 \simeq \left[\varepsilon - \frac{1}{2A_0}(2A_1\xi)^2\right]^2 - (2A_1\xi)^2 \left[1 + \left(\frac{A_1}{A_0}\xi\right)^2\right] . \qquad (4.10b)$$

The condition

$$\varepsilon(k) > 2\left(\frac{A}{A_0}\xi\right)^2 + 2A_1\xi\sqrt{1 + \left(\frac{A_1}{A_0}\xi\right)^2}$$

is imposed on (4.10b) in order for the second branch of excitations to possess non-imaginary energies.

Thus we see that there exist two sets of elementary excitations for the excited Bose-gas with the dispersion laws.

i) $$\omega_B(k) = \sqrt{\varepsilon^2 + 4\varepsilon A_0 \left(1 + \left(2\frac{U_0^*}{U_0}\xi\right)^2\right) + 12(A_1\xi)^2} \qquad (4.11a)$$

ii) $$\omega_e(k) = \sqrt{\left(\varepsilon - 4\left(\frac{U_0^*}{U_0}\right) \frac{U_0^* N_1}{V}\right)^2 - \left(2\frac{U_0^*\sqrt{N_1 N_0}}{V}\right)^2 \left(1 + \left(2\frac{U_0^*}{U_0}\xi\right)^2\right)} \qquad (4.11b)$$

The 1st set of excitations (4,11a) exists at all momenta but with an energy gap Δ given by

$$\Delta = \sqrt{(3-\frac{U_0}{U_0^*})(2A_1\xi)^2} + \frac{2\sqrt{3}\; U_0^* \sqrt{N_0 N_1}}{V} \tag{4.12}$$

Thus the energy gap is connected with the presence of excited particles since in the limit $N_1 \to 0$ $\Delta \to 0$. If we calculate the energy of elementary excitations from $\epsilon(k_0) = \Delta$, then for $k > k_0$ we have the conventional phonon branch

$$\omega_B(k) \xrightarrow[\text{limit } \frac{k-k_0}{k} \to 0]{} U_s^*(k-k_0) \tag{4.13}$$

where U_s^* is the velocity of sound given by

$$U_s^* = \sqrt{\frac{2A_0}{M}\kappa} = \sqrt{\kappa}\; U_s^B , \tag{4.14}$$

and

$$\kappa = (2\frac{U_0^*}{U_0})^2 \xi^2 + 1 ,$$

Whence the aforementioned appearance of an energy-gap (eqn.4.12) for the Bogolubov phonons and an increase in the velocity of sound by a factor of $\sqrt{\kappa} > 1$ (eqn.4.14).

The second set of excitations (4.11b) exists only at momenta greater than the critical value k_0

$$k_0 = \{4m\left((\frac{A_1}{A_0}\xi)^2 + A_1\xi\sqrt{1+(\frac{A_1}{A_0}\xi)^2}\right)\}^{\frac{1}{2}} . \tag{4.15}$$

II. Now, we consider the excited Bose gas in which we can neglect the interaction among non-excited particles. In this case the Hamiltonian has the form

$$H' = E_0 + \sum_{k \neq 0} (E_1 a_k^\dagger a_k + E_2 b_k^\dagger b_k) + 2A_1 \xi \sum_{k \neq 0} (a_k^\dagger b_k + b_k^\dagger a_k) . \qquad (4.16)$$

Where

$$E_0 = 2A_1 N \qquad ; \quad \xi^2 = \frac{N_1}{N_0} < 1$$

$$E_1 = E_1(k) = \varepsilon(k) - 2A_1 \xi^2 ; \qquad E_2(k) = \varepsilon(k) - 2A_1 \quad ; \qquad A_1 = \frac{U^* N_0 0}{V}$$

The expression (4.16) is obtained by improving both N_0^0 and N_1^0 according to

$$N_0^0 = N_0 - \sum_{k \neq 0} a_k^\dagger a_k$$

$$N_1^0 = N_1 - \sum_{k \neq 0} b_k^\dagger b_k \quad .$$

The equations of motion for our operators have the form

$$\begin{aligned} \dot{a}_k^\dagger &= i\{ E_1(k) a_k^\dagger + 2A_1 \xi b_k^\dagger \} \\ \dot{b}_k^\dagger &= i\{ E_2(k) b_k^\dagger + 2A_1 \xi a_k^\dagger \} . \end{aligned} \qquad (4.17)$$

Using the Green function method as employed earlier in this paper, one can arrive at

$$G_{a_k a^+_{k'}}(\omega) = \left(\frac{1}{1+\xi^2}\right)\frac{\delta_{kk'}}{\omega-\omega_1} + \left(\frac{\xi^2}{1+\xi^2}\right)\frac{\delta_{kk'}}{\omega-\omega_2}$$

$$G_{a_k b^+_{k'}}(\omega) = G_{b_k a^+_{k'}}(\omega) = \left(\frac{\xi}{1+\xi^2}\right)\left(\frac{1}{\omega-\omega_1} - \frac{1}{\omega-\omega_2}\right)\delta_{kk'} \tag{4.18}$$

$$G_{b_k b^+_{k'}}(\omega) = \left(\frac{\xi^2}{1+\xi^2}\right)\frac{\delta_{kk'}}{\omega-\omega_1} + \left(\frac{1}{1+\xi^2}\right)\frac{\delta_{kk'}}{\omega-\omega_2} \quad .$$

The spectrum of elementary excitations in this case also has two branches and may be written as

$$\omega_1(k) = \varepsilon(k)$$

$$\omega_2(k) = \varepsilon(k) - 2\,\frac{U^*_0 N}{V} \tag{4.19}$$

where $N = N_0 + N_1$ = total number of particles.

In this particular case the spectrum of elementary excitations $\omega_2(k)$ depends on the sign of U^*_0 , the Fourier transform of the interaction potential between differently excited particles. In the case of attractive potential the elementary excitations have the character of dressed particles defined at all momenta

$$\omega_2(k) = \varepsilon(k) + 2\,\frac{|U^*_0|}{V}\,N \quad ,$$

In the case of repulsive potential the elementary excitations with the +ve energy exist above a critical value k_0 of momentum

$$k_o = \{4 \frac{U_0^* N}{V} m\}^{\frac{1}{2}} \quad ,$$

It is interesting to investigate the linear combination of operators of real particles $a_k/a_k^\dagger$, $b_k/b_k^\dagger$ which correspond to the above mentioned quasi-particles. As usual, we may right away write the sought-for transformation [see eqn.(4.17)] as

$$a_k = U_k \alpha_k + V_k \beta_k \tag{4.20a}$$

$$b_k = V_k \alpha_k - U_k \beta_k \tag{4.20b}$$

where $U_k^2 + V_k^2 = 1$, and the new operators α_k/β_k satisfy the Bose commutation relations. We then find the coefficients U_k , V_k

$$U_k = \sqrt{\frac{N_0}{N}} \; ; \quad V_k = \sqrt{\frac{N_1}{N}} \quad . \tag{4.21.}$$

The dispersion law $\omega_1(k)$ corresponds to the symmetrical combination (4.20a) and $\omega_2(k)$ to the antisymmetrical combination (4.20b).

CHAPTER 5

EXCITED BOSE GAS AT HIGH TEMPERATURES

Here we shall consider the excited gas in which the number, N_1, of excited particles is comparatively less than the number, N_0, of non-excited particles but sufficiently big enough to allow us to neglect the interaction between the non-excited ones.

The Hamiltonian in this case has the form

$$H=E_0 + \sum_k (\varepsilon_k - \frac{U_0^*}{2V})(a_k^\dagger a_k + b_k^\dagger b_k) + \frac{U_0^*}{2V}\left(\sum_k (a_k^\dagger b_k + b_k^\dagger a_k)\right)^2 \qquad (5.1)$$

This Hamiltonian should be diagonalized under the following conditions:

$$\begin{aligned} &1. \quad \sum_k (a_k^\dagger a_k + b_k^\dagger b_k) = N \\ &2. \quad \sum_k b_k^\dagger b_k = N_1 \end{aligned} \qquad (5.2)$$

where $N = N_0 + N_1$ is the total number of particles;

$E_0 = N_0 \varepsilon_0 + N_1 \varepsilon_1$.

It is convenient to pass to a new Hamiltonian

$$H' = H - \mu N + \lambda N_1 - E_0 \quad \text{or}$$

$$H' = \sum_k \left(\varepsilon_k - \frac{U_0^*}{2V} - \mu\right) a_k^\dagger a_k + \sum_k \left(\varepsilon_k - \frac{U_0^*}{2V} - \mu + \lambda\right) b_k^\dagger b_k + \frac{U_0^*}{2V}\left(\sum_k (a_k^\dagger b_k + b_k^\dagger a_k)\right)^2 \qquad (5.3)$$

The interaction term in this Hamiltonian is a complex one. In the general case it generates the higher order Green functions. So we decided to simplify the Hamiltonian reducing it to the quadratic form

$$H = E_0 + \sum_k \left(\varepsilon(k) - \mu\right) a_k^\dagger a_k + \sum_k \left(\varepsilon(k) - \mu + \lambda\right) b_k^\dagger b_k + \Lambda \sum_k (a_k^\dagger b_k + b_k^\dagger a_k) \qquad (5.4)$$

where

$$\Lambda = \frac{U_0^*}{V} \sum_k \langle a_k^\dagger b_k + b_k^\dagger a_k \rangle$$

Introducing such a Λ is equivalent to introducing a self-consistent field in the following sense. On the basis of the Hamiltonian (5.4) we can find the Green functions $G_{a_k^\dagger b_k}(\Lambda)$ and the energy spectrum of elementary excitations which will depend on Λ and after this we should determine Λ. Physically this means that pairs of differently excited particles interact through some average field Λ.

Using the previously described Green function method we can diagonalize Hamiltonian (5.4),

$$H' = \sum_k \{\omega_1(k) n_A(k) + \omega_2(k) n_B(k)\} , \qquad (5.5)$$

where

$$\omega_{1,2}(k) = \varepsilon(k) - \mu + \frac{\lambda}{2}(1 \mp Q) \qquad (5.6)$$

with

$$Q^2 = 1 + 4\,\frac{\Delta^2}{\lambda^2}$$

$$n_A = \left\{\exp\left(\frac{\omega_1(k)}{T}\right) - 1\right\}^{-1} ; \quad n_B = \left\{\exp\left(\frac{\omega_2(k)}{T}\right) - 1\right\}^{-1}$$

with still undetermined μ , λ and Δ . For determination of these quantities we should use the constraints (5.2) and the defining equation for Δ ,

$$N = \sum_k \left(n_A(k) + n_B(k)\right)$$

$$Q(N_0 - N_1) = \sum_k \{ n_A(k) - n_B(k)\} \qquad (5.7)$$

$$Q = -\lambda \frac{U^*_0}{V} \sum_k \{n_A(k) - n_B(k)\} .$$

From the defining expressions for the energy of quasiparticles, $\omega_{1,2}(k)$, one can see that $n_A > n_B$. From (5.7) one can see that the last equation can be solved only for attractive potentials ($U^* \leqslant 0$). In the general case the solution of the system (5.7)

REFERENCES

[1] Hirschfelder, J.O.; Curtis, C.F.; Bird, R.B., 'Molecular Theory of Gases and Liquids', John Wiley & Sons, Fourth Printing (1967).

[2] Landau, L.D. & Lifshitz, E.M. 'Quantum Mechanics-Non-Relativistic Theory'. (Vol.3) Pergamon Press Ltd.(1958).

[3] Harris, E.G., 'Introduction to Modern Theoretical Physics, (Vols. 1&2) John Wiley & Sons (1975).

[4] Mal'nev, V.N., Pekar S.I., Sov.Phys. JETP, (Vol.51), 1964. pp.1811-1820.

[5] Blokintsev, D.I., 'Quantum Mechanics'. D. Reidel Publishing Company, Dordrecht, Holland 1964.

[6] Landau & Lifshitz, 'Statistical Physics', Pergamon Press Ltd. (Vol.5) 1958.

[7] Mahanty, J. 'The Green Function Method in Solid State Physics'. Affiliated East-West Press Pvt.Ltd., New Delhi 1974.

[8] Abrikosov, A.A., Gor'kov, L.P., Dzyaloshinskii, I.Ye. 'Quantum Field Theoretical Methods in Statistical Physics'. Pergamon Press. 2nd ed. 1965.

[9] Bonch-Bruevich, V.L. and Tyablikov, S.V. 'The Green Function in Statistical Mechanics',North-Holland, Amsterdam 1962.

[10] Mattuck, R.D. 'A Guide to Feynman Diagrams in the Many-Body Problem'.McGraw-Hill Publishing Company, England 1967.