ADDIS ABABA UNIVERSITY
COLLEGE OF NATURAL AND COMPUTATIONAL SCIENCES
DEPARTMENT OF MATHEMATICS

ON FOURIER TRANSFORM AND SOME OF ITS APPLICATIONS

A thesis submitted to the college of Natural and Computational Science of the Addis Ababa University in partial fulfillment of the requirements for degree of Master of Science.

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Approval

This thesis has been examined and approved as meeting the requirements for the partial fulfillment of Master of Science in Mathematics.

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DECLARATION

I, Demeku Tegegn Tadesse, a student ID GSK/6055/05, hereby declare that this thesis entitled "**On Fourier Transform And Some Of Its Applications**" has been written and organized by myself under the supervision of Dr. Tadesse Abdi and that it has never been submitted for completion of graduate qualification at any higher learning institution. Any work done by others has been acknowledged and referenced accordingly.

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ABSTRACT

The focus of this thesis is on the property and the use of Fourier analysis in different fields containing differential equations. The Fourier transform of an integrable function *f*, is

$$F(\xi)=\int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx$$

We present properties of Fourier transform and visit Fourier transform in connection with heat transfer in a conducting wire and wave propagation in an infinite bar.

1. INTRODUCTION

The Fourier transform is a valuable theoretical technique, used widely in fields such as applied mathematics, statistics, physics, and engineering. However, the relationship between a function and its transform is given by an integral, and a certain amount of tedious integration may be required to obtain the transform in a given application. In general, the user of this mathematical tool is interested in the functions and their transforms and not in the process of obtaining one from the other, which, can be complicated and require care to avoid any small slip that might lead to an error in the result. If the transform function could be obtained without integration, this would be welcomed by most users. In fact, anyone performing many transforms in a particular field, such as radar, where the spectra corresponding to various, perhaps rather similar, waveforms are required, would notice that certain waveforms have certain transforms and that certain relationships between waveforms lead to corresponding relationships between spectra. With the knowledge of a relatively small number of waveform-transform pairs and the rules for combining and scaling transforms, a very substantial amount of Fourier transform analysis can be carried out without any explicit integration at all the integrations are prepackaged within the set of rules and pairs.

Definition: Let Ω be an open subset of R^n and $1 < p < \infty$. We define $L^p(\Omega)$ to be the space of all measurable functions $f: \Omega \to$ R for which $\|f\|_{L^p(\Omega)} < \infty$, where

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f|^p dx\right)^{1/p}$$

Definition: A function $f: \mathbb{R} \to \mathbb{C}$ is Lebesgue integrable, written $f \in \mathcal{L}^1(\mathbb{R})$, if there exists a series $w_n = \sum_{j=1}^n f_j, f_j \in C(\mathbb{R})$ which is absolutely summable, $\sum_j \int |f_j| < \infty$ and such that

$$\sum_j \int |f_j| < \infty \Rightarrow \lim_{n\to\infty} w_n(x) = \sum_{j=1}^n f_j = f(x)$$

1.1 Fourier series

Definition 1.1.1 A function f is said to satisfy Dirichlet's conditions in the interval

$-a < x < a$, if

(i). f has only a finite number of finite discontinuities in $-a < x < a$ and has no infinite discontinuities

(ii). f has only a finite number of maxima and minima in $-a < x < a$.

Definition 1.1.2 Let f be a function with period 2π that is absolutely Riemann-integrable over a period T. Define the numbers $c_n, n \in Z$, by

$$c_n = \frac{1}{2\pi} \int_T f(t) e^{-int}\, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int}\, dt.$$

These numbers are called the Fourier coefficients of f, and the Fourier series of f is the series $\sum_{n \in Z} c_n\, e^{int}$.

Notice that the definition does not state anything about the convergence of the series, even less what its sum might be if it happens to converge.

When dealing simultaneously with several functions and their Fourier coefficients it is convenient to indicate to what function the coefficients belong by writing things like $c_n(f)$. Another commonly used way of denoting the Fourier coefficients of f is $\hat{f}(n)$.

When we want to state, as a formula, that f has a certain Fourier series, we write

$$f(t) \sim \sum_{n \in Z} c_n\, e^{int}$$

1.2 Dirichlet's kernels

It is a regrettable fact that a Fourier series need not be convergent. For example, it is possible to construct a continuous function such that its Fourier series diverges at a specified point. We shall see, in due time, that if we impose somewhat harder requirements on the function, such as differentiability, the results are more positive.

It is however true that the Fourier series of a continuous function is Ces`aro summable to the values of the function, and this is the main result of this section.

We start by establishing a formula for the partial sums of a Fourier series. To this end we shall use the following formula:

Lemma 1.2.1 $D_N(u) := \frac{1}{\pi}\left(\frac{1}{2} + \sum_{n=1}^{N} cosnu\right) = \frac{1}{2\pi}\sum_{n=-N}^{N} e^{inu} = \frac{\sin\,(N+\frac{1}{2})u}{2\pi sin\frac{1}{2}u}$

Proof : The equality of the two sums follows from Euler's formulae. Let us then start from the "complex" version of the sum and compute it as a finite geometric sum:

$$2\pi D_N(u) = \sum_{n=-N}^{N} e^{inu} = e^{-iNu}\sum_{n=0}^{2N} e^{inu} = e^{-iNu}.\frac{1-e^{i(2N+1)u}}{1-e^{iu}}$$

$$= e^{-iNu}\frac{e^{i(N+\frac{1}{2})u}\left(e^{-i(N+\frac{1}{2})u} - e^{i(N+\frac{1}{2})u}\right)}{e^{\frac{iu}{2}}\left(e^{-\frac{iu}{2}} - e^{\frac{iu}{2}}\right)}$$

$$= \frac{e^{-iNu+i(N+\frac{1}{2})u}}{e^{\frac{iu}{2}}}.\frac{-2i\sin\,(N+\frac{1}{2})u}{-2isin\frac{1}{2}u} = \frac{\sin\,(N+\frac{1}{2})u}{sin\frac{1}{2}u}$$

The function D_N is called the ***DIRICHLET*** kernel.

When discussing the convergence of Fourier series, the natural partial sums are those containing all frequencies up to a certain value. Thus we define the partial sum $s_N(t)$ to be :

$$s_N(t) := \frac{1}{2}a_0 + \sum_{n=1}^{N}(a_n cosnt + b_n sinnt) = \sum_{n=-N}^{N} c_n e^{int}$$

Using the Dirichlet kernel we can obtain an integral formula for this sum, assuming the c_n to be the Fourier coefficients of a function f.

$$s_N(t) = \sum_{n=-N}^{N} c_n e^{int} = \sum_{n=-N}^{N}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(u)\, e^{-inu}du.\, e^{int}$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(u)\,.\frac{1}{2}\sum_{n=-N}^{N} e^{in(t-u)}\, du = \int_{-\pi}^{\pi} f(u) D_N(t-u)\, du$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t-u)\frac{\sin\,(N+\frac{1}{2})u}{sin\frac{1}{2}u}du$$

In the last step we change the variable ($t - u$ is replaced by u) and make use of the periodicity of the integrand. We shall presently take another step and form the arithmetic means of the $N + 1$ first partial sums. To achieve this we need a formula for the mean of the corresponding Dirichlet kernels.

Lemma 1.2.2 $F_N(u) = \frac{1}{N+1}\sum_{n=0}^{N} D_n(u) = \frac{1}{2\pi(N+1)}\left(\frac{\sin\frac{1}{2}(N+1)u}{sin\frac{1}{2}u}\right)^2$

2. THE FOURIER TRANSFORM

The concept of the Fourier series seems intuitively very reasonable: that any periodic function can be represented by a sum of elementary periodic functions, either sine and cosine functions or, equivalently, complex exponentials. The frequencies of the elementary functions are integer multiples (including zero, giving a constant function) of the repetition frequency of the periodic function. The sum may turn out to be infinite, but users of this mathematical tool are generally content to let mathematicians justify such a sum, determining the conditions under which it converges; however, for problems arising in practice, in physics or engineering, for example, it is ‘‘obvious’’ that such a sum does converge. Thus, we can put for f a real or complex function of a real variable, with period X [so that $f\ (x\ +\ X\ )\ =\ f\ (x\ )$],

$$f(x) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{2\pi nx}{X}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nx}{X}\right) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x/X}$$

(2.1)

(By expressing the trigonometric functions as complex exponentials, we can relate c_n to a_n and b_n. From now on we restrict our attention exclusively to the complex exponential series.) The coefficients of the series are found by integration over one cycle of the function, so that, for example,

$$c_n = \frac{1}{X}\int_{x_0-X/2}^{x_0+X/2} f(x)\, e^{-2\pi i n x/X} dx \tag{2.2}$$

The Fourier transform can be obtained as the limiting case of the Fourier series when the period is increased towards infinity and the fundamental frequency falls to zero. In this case, as $X \to \infty$ we put $n/X \to y, 1/X \to dy$, $c_n \to g(y)\, dy$, where g is a continuous function replacing the discrete series c_n and the summations in (2.1) become integrals.

Thus (2.1) and (2.2) become, respectively,

$$f(x) = \int_{-\infty}^{\infty} g(y)\, e^{2\pi i x y} dy \tag{2.3}$$

and

$$g(y) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x y} dx$$

(2. 4)

Suppose $f : R \to C$ is an $\mathcal{L}^2$ function. Its restriction to $(-l, l)$ clearly lies in $\mathcal{L}^2\ (-l, l)$ for any $l > 0$. On the interval $(-l, l)$ we can always represent f by the Fourier series:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l} \tag{2.5}$$

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x)\, e^{-in\pi x/l} dx, \quad n \in \mathbb{Z} \tag{2.6}$$

Let $\Delta\xi = \frac{\pi}{l}$ and $\xi_n = n\Delta\xi = \frac{n\pi}{l}$. The pair of equations (2.5) and (2.6) then take the form

$$f(x) = \frac{1}{2\pi}\sum_{-\infty}^{\infty} C(\xi_n) e^{i\xi_n x}\Delta\xi \tag{2.7}$$

$$C(\xi_n) = 2lc_n = \int_{-l}^{l} f(x)\, e^{-i\xi_n x} dx \tag{2.8}$$

If we now let $l \to \infty$, that is, if we allow the period (−l, l) to increase to ℝ so that f loses its periodicity, then the discrete variable ξ_n, will behave more as a real variable ξ, and the formula (2.8) will tend to the form

$$c(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-i\xi x} dx \tag{2.9}$$

The right hand of (2.7), on the other hand, looks very much like Riemann sum which in the limit as $l \to \infty$ approaches the integral

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} C(\xi) e^{ix\xi} d\xi \tag{2.10}$$

Thus, the Fourier coefficients C_n are transferred to the function $C(\xi)$, the Fourier transform of f, and the Fourier series (2.5) which represents f on $(-l, l)$ is replaced by the Fourier integral (2.10) which presumably represent the function f on $(-\infty, \infty)$.

Definition(2.1): For any $f \in \mathcal{L}^1(\mathbb{R})$ we define the Fourier transform of f as the function $\hat{f}: \mathbb{R} \to \mathbb{C}$ by

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-i\xi x} dx \tag{2.11}$$

A sufficient condition for $f(x)$ to have a fourier transform is that $f(x)$ is absolutely integrable on $(-\infty, \infty)$. The convergence of the integral (2.11) follows from the fact that $f(x)$ is absolutely integrable.

We also use the symbol $\mathcal{F}(f)$ instead of $\hat{f}$ to denote the Fourier transform of f.

Since $\left|e^{i\xi x}\right| = 1$, we have $\left|\hat{f}(\xi)\right| \le \int_{-\infty}^{\infty} |f(x)|\, dx < \infty$

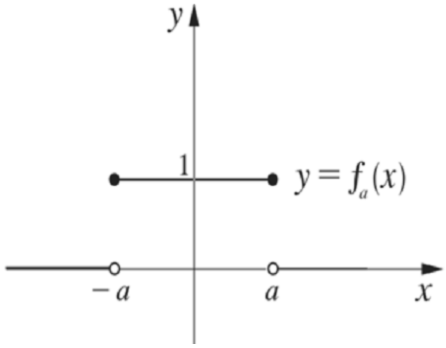
That is $\hat{f}$ is a bounded function on ℝ by the linearity of the integral

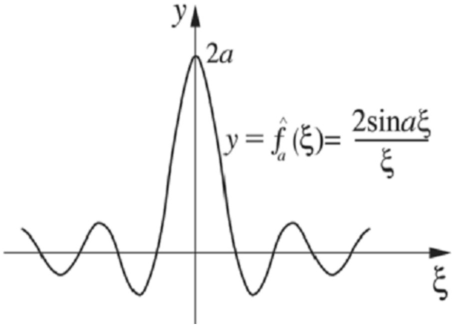
$\mathcal{F}(c_1f_1 + c_1f_2) = c_1\mathcal{F}(f_1) + c_2\mathcal{F}(f_2)$ for all $c_1, c_2 \in \mathbb{C}$ and all $f_1,\ f_2 \in \mathcal{L}^1(\mathbb{R})$,which means that the Fourier transformation $\mathcal{F}: f \to \hat{f}$ is linear.

Example2.1: For any positive constant a, let $f(x) = \begin{cases} 1, |x| \le a \\ 0,\ |x| > a \end{cases}$

Then $\hat{f}_a(\xi) = \int_{-a}^{a} e^{-i\xi x}\, dx = \frac{1}{-i\xi}\left(e^{-i\xi x}\right)\Big|_{-a}^{a} = \frac{1}{-i\xi}\left(e^{-i\xi a} - e^{i\xi a}\right) = \frac{2}{\xi}\sin\,(a\xi)$

Note that $\lim_{a\to\infty} \hat{f}_a(\xi)$ does not exist and that $f(x) = 1$ does not lie in $\mathcal{L}^1(\mathbb{R})$.





Example 2.2: Let $f(x) = e^{-|x|}$

$$\hat{f}(x) = \int_{-\infty}^{0} e^{x}\, e^{-i\xi x} dx + \int_{0}^{\infty} e^{-x}\, e^{-i\xi x} dx$$

$$= \lim_{b\to-\infty}\left(\int_{b}^{0} e^{(1-i\xi)x}\, dx\right) + \lim_{b\to\infty}\left(\int_{0}^{b} e^{(-1-i\xi)x}\, dx\right)$$

$$= \frac{1}{1-i\xi} + \frac{1}{1+i\xi}$$

$$= \frac{2}{1+\xi^2}$$

When $|f|$ is integrable over $\mathbb{R}$ (i.e when $f \in \mathcal{L}^1(\mathbb{R})$, we have seen that its Fourier transform is bounded. But we can also prove that $\hat{f}$ is bounded.

This result relies on a well-known theorem on real analysis, the Lebesgue dominated convergence theorem, which states the following.

Theorm 2.*1*: *Let $(f_n: n \in \mathbb{Z})$ be a sequence of measurable functions in $\mathcal{L}^1(I)$ where I is a real interval and suppose $f_n \to f$ point wise on I. If there is a positive function $g \in \mathcal{L}^1(I)$ such that $|f_n| \le g(x)$ for all $x \in I, n \in N$ then $f \in \mathcal{L}^1(I)$ and*

$$lim_{n\to\infty} \int_I \ f_n\, dx = \int_I \ f(x)\, dx$$

Proof :

Suppose $|f_n| \leq g(x)$ and g is integrable, $\int_I |f_n|\, d\mu \leq \int_I g\, d\mu < \infty$. So f_n is integrable. f is measurable (as a point wise limit of measurable functions) and then, similarly, $|f(x)| \leq \lim_{n\to\infty}|f_n(x)| \leq g(x)$ implies that f is integrable too.

This proof does not work properly if $g = \infty$ for some x. We know that $g(x) < \infty$ almost everywhere. So we can take $E = \{x \in \mathbb{R}: g(x) = \infty\}$ and multiply g and each of the functions f_n and f by $1 - \chi_E$ to make sure all the functions have finite values. As we are changing them all only on the set E of measure 0, this change does not affect the integrals or the conclusions. We assume then all have finite values.

Let $h_n = g - f_n$, so that $h_n \geq 0$. By Fatou's lemma

$$\text{liminf}_{n\to\infty} \int_I (g - f_n) d\mu \geq \int_I \liminf_{n\to\infty}(g - f_n) d\mu = \int_I (g - f) d\mu$$

and that gives

$$\liminf_{n\to\infty}\left(\int_I g d\mu - \int_I f_n d\mu\right) = \int_I g d\mu - \limsup_{n\to\infty} \int_I f_n d\mu \geq \int_I g d\mu - \int_I f d\mu$$

or

$$\limsup_{n\to\infty} \int_I f_n d\mu \leq \int_I f d\mu \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (*)$$

Repeat this Fatou's lemma argument with $g + f_n$ rather than $g - f_n$.

We get

$$\liminf_{n\to\infty} \int_I (g + f_n) d\mu \geq \int_I \liminf_{n\to\infty}(g + f_n) d\mu = \int_I (g + f) d\mu$$

and that gives

$$\liminf_{n\to\infty}\left(\int_I g d\mu + \int_I f_n d\mu\right) = \int_I g d\mu + \liminf_{n\to\infty} \int_I f_n d\mu \geq \int_I g d\mu + \int_I f d\mu$$

$$\liminf_{n\to\infty} \int_I f_n d\mu \geq \int_I f d\mu \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (**)$$

Combining (*) and (**) we get

$$\int_I f d\mu \leq \liminf_{n\to\infty} \int_I f_n d\mu \leq \limsup_{n\to\infty} \int_I f_n d\mu \leq \int_I f d\mu$$

Which implies

$$\int_I f d\mu = \liminf_{n\to\infty} \int_I f_n d\mu = \limsup_{n\to\infty} \int_I f_n d\mu = \int_I f d\mu$$

And that gives the result because if $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n$ (for a sequence $\{a_n\}$, it implies that $\lim_{n\to\infty} a_n$ exists and $\lim_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n$. ■

In order to study the behavior of $\hat{f}$ as $\xi \to \infty$, we need the following result, often referred to as Riemann Lebesgue Lemma.

Lemma 2.1 Let f be a piece wise smooth function on $\mathbb{R}$

i. If $[a, b]$ is a bounded interval, then $\lim_{|\xi|\to\infty} \int_a^b f(x)e^{i\xi x}dx = 0$

ii. If $f \in \mathcal{L}^1(I)$, then $\lim_{|\xi|\to\infty} \int_{-\infty}^{\infty} f(x)e^{i\xi x}dx = 0$

Proof:

(i). Let $x_1, x_2, x_3, \ldots, x_n$ be the points of discontinuity of f and f' in (a, b), arranged in increasing order, and $a = x_0$ and $b = x_{n+1}$. We then have $\int_a^b f(x)e^{i\xi x}dx = \sum_{k=0}^{n} \int_{x_k}^{x_{k+1}} f(x)e^{i\xi x}dx$ and it suffices to prove that $\lim_{|\xi|\to\infty} \int_{x_k}^{x_{k+1}} f(x)e^{i\xi x}dx = 0$ for all k.

Integrating by parts,

$$\int_{x_k}^{x_{k+1}} f(x)e^{i\xi x}dx = \frac{1}{i\xi} f(x)e^{i\xi x} \Big|_{x_k}^{x_{k+1}} - \frac{1}{i\xi}\int_{x_k}^{x_{k+1}} f'(x)e^{i\xi x}dx$$

And the right hand side of this equation tends to 0 as $|\xi| \to \infty$.

(ii). Let ξ be any positive number. Since $|f|$ is integrable on $(-\infty, \infty)$, we know that there is a positive number L such that

$$\left|\int_{-\infty}^{\infty} f(x)e^{i\xi x}dx - \int_{-L}^{L} f(x)e^{i\xi x}dx\right| \leq \int_{|x|>L} |f(x)|dx < \frac{\varepsilon}{2}$$

But from (i), we also know that there is a positive number K, such that

$$\left|\int_{-L}^{L} f(x)e^{i\xi x}dx\right| < \frac{\varepsilon}{2} \text{ for all } |\xi| > K$$

Therefore, if $|\xi| > K$, then $\left|\int_{-\infty}^{\infty} f(x)e^{i\xi x}dx\right| < \varepsilon$

We have therefore proved the following theorem

Theorem 2.2 For any $\in \mathcal{L}^1(\mathbb{R})$, the Fourier transform $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx$ is a bounded continuous function on $\mathbb{R}$. If, furthermore, f is piecewise smooth, then

$$\lim_{|\xi|\to\infty} \hat{f}(\xi) = 0 \tag{2.12}$$

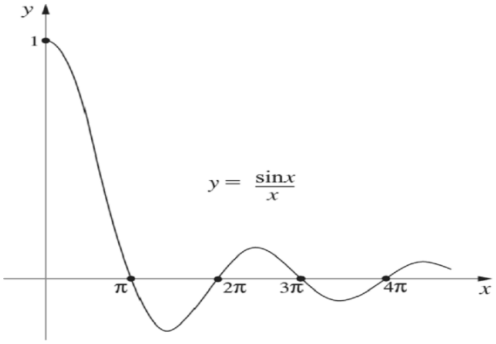
2.1 The Fourier Integral

The main result of this section is Theorem 2.3.1, which establishes the inversion formula for the Fourier transform. The proof of the theorem relies on evaluating the improper integral $\int_0^\infty \frac{sinx}{x} dx$. Which is known as Dirichlet's integral. To show that this integral exist, we write

$$\int_0^\infty \frac{sinx}{x} dx = \int_0^1 \frac{sinx}{x} dx + \lim_{b\to\infty} \int_1^b \frac{sinx}{x} dx \tag{2.13}$$

Because the function $\frac{sinx}{x}$ is continuous and bounded on $(0, 1]$, where it satisfies $0 \le \frac{sinx}{x} \le 1$, the first integral on the right hand side of (2.13) exists. Using integration by parts in the second integral $\int_1^b \frac{sinx}{x} dx = cos1 - \frac{cosb}{b} - \int_1^b \frac{cosx}{x^2} dx$, and noting that

$\left|\int_1^b \frac{cosx}{x^2} dx\right| \le \int_1^b \left|\frac{cosx}{x^2}\right| dx \le \int_1^b \frac{1}{x^2} dx \le 1 - \frac{1}{b}$



We see that $\lim_{b\to\infty} \int_1^b \frac{cosx}{x^2} dx$ exists. Hence the integral $\int_1^\infty \frac{sinx}{x} dx$ is convergent. Now we know Dirichlet's integral exist, it remains to determine its value.

Lemma 2.1.1

$$\int_0^\infty \frac{sinx}{x} dx = \frac{\pi}{2} \tag{2.14}$$

Proof:

Define the function $f(x) = \begin{cases} \frac{1}{x} - \frac{1}{2sin\frac{1}{2}x}, 0 < x \leq \pi \\ 0, \quad x = 0 \end{cases}$

Then f and f' are both continuous on $[0, \pi]$.

By lemma (2.1(ii)) then

$$\lim_{|\xi|\to\infty} \int_{-\infty}^{\infty} f(x) sin\xi x dx = 0 \tag{2.15}$$

Therefore, $\int_0^\infty \frac{sinx}{x} dx = \lim_{\xi\to\infty} \int_0^{\xi\pi} \frac{sinx}{x} dx$

$$= \lim_{\xi\to\infty} \int_0^{\pi} \frac{sin\xi x}{x} dx$$

$$= \lim_{\xi\to\infty} \frac{1}{2} \int_0^{\pi} \frac{sin\xi x}{sin\frac{1}{2}x} dx$$

$$= \lim_{\xi\to\infty} \frac{1}{2} \int_0^{\pi} \frac{\sin{(n+\frac{1}{2})x}}{sin\frac{1}{2}x} dx$$

Going back to the Dirichlet's Kernel

$D_n = \frac{1}{2\pi} \sum_{k=-n}^{n} e^{ikx}$ then we write $D_n = \frac{1}{2\pi} \frac{\sin{(n+\frac{1}{2})x}}{sin\frac{1}{2}x}$,

we conclude that $\int_0^\infty \frac{sinx}{x} dx = \lim_{n\to\infty} \int_0^{\pi} D_n(x) dx = \frac{\pi}{2}$. ■

Theorem 2.1.1: Let f be a piece wise smooth function in $\mathcal{L}^1(\mathbb{R})$ if

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx, \xi \in \mathbb{R} \tag{2.16}$$

Then

$$\lim_{L\to\infty} \frac{1}{2\pi} \int_{-L}^{L} \hat{f}(\xi) e^{i\xi x} dx = \frac{1}{2}[f(x^+) + f(x^-)] \tag{2.17}$$

2.2. Properties of Fourier Transform

The following theorem gives the fundamental properties of the Fourier transformation under differentiation. The formula for the transform of the derivative is particularly important, not only as a tool for solving linear partial differential equations, but also as a fundamental result on which the existence of solutions to such equations is based.

Theorem 2.2.1 Let $f \in \mathcal{L}^1(\mathbb{R})$

(i) If $f' \in \mathcal{L}^1(\mathbb{R})$ and f is continuous on $\mathbb{R}$, then

$$\mathcal{F}(f')(\xi) = i\xi\mathcal{F}(f)(\xi), \xi \in \mathbb{R}$$

(ii) If $xf(x) \in \mathcal{L}^1(\mathbb{R})$, then $\mathcal{F}(f)$ is differentiable and its derivative

$\frac{d}{d\xi}\mathcal{F}(f)(\xi) = \mathcal{F}(-ixf)(\xi), \xi \in \mathbb{R}$ is continuous on $\mathbb{R}$.

Proof:

(i) $|f'|$ being integrable, its Fourier transform $\mathcal{F}(f')$ exists and

$\mathcal{F}(f')(\xi) = \int_{-\infty}^{\infty} f'(x)e^{-i\xi x}dx.$

The continuity of f' allows us to write $\int_0^x f'(t)dt = f(x) - f(0)$.

Hence the two limits

$\lim_{x\to\pm\infty} f(x) = f(0) + \lim\limits_{x\to\pm\infty} \int_0^x f'(t)dt$ exist.

But because f is continuous and integrable on $\mathbb{R}$, $\lim\limits_{x\to\pm\infty} f(x) = 0$.

Now integration by parts yields

$$\begin{aligned}\mathcal{F}(f')(\xi) &= f(x)e^{-i\xi x}\Big|_{-\infty}^{\infty} + i\xi\int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx \\ &= i\xi\,\mathcal{F}(f)(\xi)\end{aligned}$$

(ii) $\frac{\hat{f}(\xi+\Delta\xi)-\hat{f}(\xi)}{\Delta\xi} = \int_{-\infty}^{\infty} f(x)\frac{e^{-i(\xi+\Delta\xi)x}-e^{-i\xi x}}{\Delta\xi}dx$

In the limit as $\Delta\xi \to 0$, we obtain

$\frac{d}{d\xi}\mathcal{F}(f)(\xi) = \lim\limits_{\Delta\xi\to 0}\int_{-\infty}^{\infty} f(x)\frac{e^{-i(\xi+\Delta\xi)x}-e^{-i\xi x}}{\Delta\xi}dx$

By hypothesis, the limit

$\lim\limits_{\Delta\xi\to 0} f(x)\frac{e^{-i(\xi+\Delta\xi)x}-e^{-i\xi x}}{\Delta\xi} = -ixf(x)\,e^{-i\xi x}$ is dominated by

$|xf(x)| \in \mathcal{L}^1(\mathbb{R})$ for every $\xi \in \mathbb{R}$.

Therefore, we can use Theorem 2.1 to conclude that

$\frac{d}{d\xi}\mathcal{F}(f)(\xi) = \int_{-\infty}^{\infty}[-ixf(x)]e^{-i\xi x}dx = \mathcal{F}(-ixf)(\xi).$

Using induction, this result can be generalized.

Corollary 2.2.1: Suppose $f \in \mathcal{L}^1(\mathbb{R})$ and n is any positive integer.

(i) If $f^{(k)} \in \mathcal{L}^1(\mathbb{R})$ $for\ all\ 1 \le k \le n$ and if $f^{(n-1)}$ is continuous on $\mathbb{R}$, then $\mathcal{F}(f^n)(\xi) = (i\xi)^n\mathcal{F}(f)(\xi)$.

(ii) If $x^n f(x) \in \mathcal{L}^1(\mathbb{R})$ then

$$\frac{d^n}{d\xi^n}\mathcal{F}(f)(\xi) = \mathcal{F}((-ix)^n f)(\xi)$$

The integrability of $|x^n f(x)|$ on $\mathbb{R}$ may be viewed as a measure of how fast the function $f(x)$ tends to zero as $x \to \pm\infty$, in the sense that $f(x)$ tends to 0 faster when n is larger. As the order of differentiability (or smoothness) of f increases, so does the rate of decay of $\hat{f}$.

A good function, $g(x)$ is a function in $C^\infty(\mathbb{R})$ that decays sufficiently rapidly that $g(x)$ and all of its derivatives decay to zero faster than $|x|^{-N}$ as $|x| \to \infty$ for all $N > 0$.

Definition 2.2.1: Suppose a real or complex valued function $g(x)$ is defined for all $x \in \mathbb{R}$ and is infinitely differentiable everywhere, and suppose that each derivative tends to zero as $|x| \to \infty$ faster that any positive power of (x^{-1}), or in other words, suppose that for each positive integer N and n,

$$\lim_{|x|\to\infty} x^N g^{(n)}(x) = 0$$

Then $g(x)$ is called a *good function*

Usually, the class of good functions is represented by $\mathbb{S}$. The good functions play an important role in Fourier analysis because the inversion, convolution, and differentiation theorems as well as many others take simple forms with no problem of convergence. The rapid decay and infinite differentiability proper-ties of good functions lead to the fact that the Fourier transform of a good function is also a good function.

Good functions also play an important role in the theory of generalized functions. A good function of bounded support is a special type of good function that also plays an important part in the theory of generalized functions. Good functions also have the following important properties. The sum (or difference) of two good functions is also a good function. The product and convolution of two good functions are good functions. The derivative of a good function is a good function; $x^n g(x)$ is a good function for all non-negative integers n whenever $g(x)$ is good function. A good function belongs to $\mathcal{L}^p$(a class of p^{th} power Lebesgue integrable functions) for every p in $1 \le p \le \infty$. The integral of good function is not necessarily good.

Theorem 2.2.2 The Fourier transform of a good function is a good function. The Fourier transform of a good function $f(x)$ exists and is given by

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx$$

Differentiating $\hat{f}(\xi)$ n −times and integrating N times by parts, we get

$$\left|\hat{f}^{(n)}(\xi)\right| \leq \left|\frac{(-1)^N}{(-ik)^N}\int_{-\infty}^{\infty} e^{-i\xi x}\frac{d^N}{dx^N}[(-ix)^n f(x)]dx\right|$$

$$\leq \frac{1}{k^N}\int_{-\infty}^{\infty}\left|\frac{d^N}{dx^N}x^n f(x)\right|dx$$

Evidently, all derivatives tend to zero as fast as $|k|^{-N}$ as $|k| \to \infty$ for any $N > 0$ and hence, $\hat{f}(\xi)$ is a good function. ■

2.3 The Inverse Fourier Transform

Theorem 2.3.1 (Inversion theorem) suppose that $f \in L^1(R)$. That f is continuous except for a finite number of finite jumps in any finite interval, and that $f(t) = \frac{1}{2}\left(f(t^+) + f(t^-)\right)$ for all t. Then

$$f(t_0) = \lim_{A\to\infty}\frac{1}{2\pi}\int_{-A}^{A}\hat{f}(\omega)e^{i\omega t_0}d\omega \qquad (2.18)$$

for every t_0 where f has left and right derivatives. In particular, if f is piecewise smooth(i.e continuous and with a piecewise continuous derivative), then the formula holds for all $t_0 \in R$.

Proof: put $s(t_0, A) = \lim_{A\to\infty}\frac{1}{2\pi}\int_{-A}^{A}\hat{f}(\omega)e^{i\omega t_0}d\omega$

and rewrite this expression by inserting the definition of $\hat{f}(\omega)$:

$$s(t_0, A) = \frac{1}{2\pi}\int_{-A}^{A}\left(\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt\right)e^{i\omega t_0}d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-A}^{A} f(t)e^{i\omega t_0 - i\omega t}d\omega\right)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\left[\frac{e^{i\omega(t_0-t)}}{i(t_0-t)}\right]_{\omega=-A}^{\omega=A}dt$$

$$\frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\frac{\sin A(t_0-t)}{t_0-t}dt = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t_0-u)\frac{\sin Au}{u}du.$$

Switching the order of integration is permitted, because the improper double integral is absolutely convergent over the strip $(t,\omega) \in R \times [-A,A]$, and in the last step we have put $t_0 - t = u$. We are now in a situation very much the same as in the proof of Fourier series; but there is a complication in as much as the interval of integration is unbounded. Then we can write

$$\frac{2}{\pi}\int_0^{\infty} f(t_0-u)\frac{sinAu}{u}du - f(t_0-) = \frac{2}{\pi}\int_0^{\infty}(f(t_0-u)-f(t_0-))\frac{sinAu}{u}du \qquad (2.19)$$

Now let $\epsilon > 0$ be given. Since we have assumed that $f \in L^1(R)$, there exists a number X such that

$$\frac{2}{\pi}\int_X^{\infty}|f(t_0-u)|du < \epsilon.$$

changing the variable, we find that

$$\int_X^{\infty}\frac{sinAu}{u}du = \int_{AX}^{\infty}\frac{sint}{t}dt \to 0 \text{ as } A \to \infty. \qquad (2.20)$$

The last integral in (2.19) can be split in to three terms:

$$\frac{2}{\pi}\int_0^{X}\frac{(f(t_0-u)-f(t_0-))}{u}sinAudu + \frac{2}{\pi}\int_X^{\infty}(f(t_0-u))\frac{sinAu}{u}du - \frac{2}{\pi}f(t_0-)\int_X^{\infty}\left(\frac{sinAu}{u}\right)du$$

$$= I_1 + I_2 + I_3$$

The term I_3 tends to zero as $A \to \infty$ because of (2.20). the term I_2 can be estimated:

$$|I_2| = \left|\frac{2}{\pi}\int_X^{\infty}(f(t_0-u))\frac{sinAu}{u}du\right| \le \frac{2}{\pi}\int_X^{\infty}|f(t_0-u)|du \le \epsilon.$$

In the term I_1 we have the function $u \longmapsto g(u) = \frac{f(t_0-u)-f(t_0)}{-u}$. This is continuous except for jumps in the interval $(0, X)$, and it has the finite limit $g(0^+) = f_L'(t_0)$ as $u \searrow 0$; this means that g is bounded and thus integrable on the interval. By the Riemann-Lebesgue lemma, we conclude that $I_1 \to 0 \; as \; A \to \infty$. All this together gives, since ϵ can be taken as small as we wish, $\frac{2}{\pi}\int_0^{\infty} f(t_0-u)\frac{sinAu}{u}du \to f(t_0-) \; as \; A \to \infty$.

A parallel argument implies that the corresponding integral over $(-\infty, 0)$ tends to $f(t_0+)$. Taking the mean value of these two results, we have completed the proof of the theorem ■

3. APPLICATIONS OF FOURIER TRANSFORM

3.1 Heat Transfer in an Infinite Bar

As Fourier series served us in the construction of solutions to boundary-value problems in bounded space domains, we now show how such solutions can be represented by Fourier integrals when the space domain becomes un-bounded. The question of the uniqueness of a solution obtained in this manner, in general, is not addressed here, as it properly belongs to the theory of partial differential equations. But the equations that we have already introduced (Laplace's equation, the heat equation, and the wave equation) all have unique solutions under the boundary conditions imposed, whether the space variable is bounded or not. This follows from the fact that, due to the linearity of these equations and that of their boundary conditions, the difference between any two solutions of a problem satisfies a homogeneous differential equation under homogeneous boundary conditions, which can only have a trivial solution.

Example 3.1.1

Suppose that an infinite thin bar has an initial temperature distribution along its length given by $f(x)$. We wish to determine the temperature $u(x,t)$ along the bar for all $t > 0$. To solve this problem by the Fourier transform, we assume that f is piecewise smooth and that $|f|$ is integrable on $(-\infty, \infty)$.

The temperature $u(x,t)$ satisfies the heat equation

$$u_t = ku_{xx} \quad -\infty < x < \infty \,, t > 0 \qquad (3.1)$$

And the initial condition

$$u(x,0) = f(x), -\infty < x < \infty \qquad (3.2)$$

We resort to separation of variables by assuming that

$$u(x,t) = v(x)w(t)$$

Substituting into Equation (3.1), we obtain the equation

$$\frac{v''(x)}{v(x)} = \frac{1}{k}\frac{w'(t)}{w(t)} \qquad -\infty < x < \infty\,, t > 0$$

Which implies that each side must be a constant, say $-\lambda^2$. The resulting pair of equations leads to the solutions

$$v(x) = A(\lambda)cos\lambda x + B(\lambda)sin\lambda x$$

$$w(t) = c(\lambda)e^{-k\lambda^2 t}$$

Where A, B, and C are constants of integration which depend on the parameter λ. Since there are no boundary conditions on the solution, λ will be a real (rather than a discrete) variable, and $A = A(\lambda)$ and $B = B(\lambda)$ will therefore be functions of λ, and we can set $C(\lambda) = 1$ in the product $v(x)w(t)$.

Corresponding to each $\lambda \in R$, the function

$$u_\lambda(x,t) = [A(\lambda)cos\lambda x + B(\lambda)sin\lambda x]e^{-k\lambda^2 t}$$

therefore satisfies equation (3.1) $A(\lambda)$ and $B(\lambda)$ being arbitrary functions of $\lambda \in \mathbb{R}$.

We do not lose any generality by assuming that $\lambda \geq 0$ because the negative values of λ do not generate additional solutions. $u_\lambda(x,t)$ cannot be expected to satisfy the initial condition (3.2) for a general function f, so we assume that the desired solution has the form

$$u(x,t) = \frac{1}{\pi}\int_0^\infty u_\lambda(x,t)d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty [A(\lambda)cos\lambda x + B(\lambda)sin\lambda x]e^{-k\lambda^2 t}\,d\lambda \qquad (3.3)$$

At $t = 0$, we have

$$u(x,0) = \frac{1}{\pi}\int_0^\infty [A(\lambda)cos\lambda x + B(\lambda)sin\lambda x]\;d\lambda \qquad (3.4)$$

$$= f(x),\; x \in \mathbb{R}$$

By Theorem 2.3.1, we see that Equation (3.4) uniquely determines A and B as the cosine and sine transforms, respectively, of f:

$$A(\lambda) = \int_{-\infty}^{\infty} f(y)\cos(\lambda y)\,dy$$

$$B(\lambda) = \int_{-\infty}^{\infty} f(y)\sin(\lambda y)\,dy$$

Substituting back into (3.3), and using the even and odd properties of A and B, we arrive at the solution of the heat equation (3.1) which satisfies the initial
condition (3.2)

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y)[\cos(\lambda y)\cos(\lambda x) + \sin(\lambda y) sin(\lambda x)]\, e^{-k\lambda^2 t} dy d\lambda$$

$$= \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(y)[\cos(x-y)\,\lambda]\, e^{-k\lambda^2 t} dy d\lambda \qquad (3.5)$$

The solution (3.5) can also be obtained by first taking the Fourier transform of both sides of the heat equation, as functions of x, and using Corollary 2.1 to obtain

$$\hat{u}_t = k(i\xi)^2\hat{u}(\xi,t) = -k\xi^2\hat{u}(\xi,t)$$

The solution of this equation is

$$\hat{u}(\xi,t) = ce^{-k\xi^2 t}$$

Where, by the initial condition, $c = \hat{u}(\xi,0) = \hat{f}(\xi)$

Thus, as a function in ξ,

$$\hat{u}(\xi,t) = \hat{f}(\xi)e^{-k\xi^2 t}$$

is the Fourier transform of $u(x,t)$ for every $t > 0$, and u can therefore be represented, according to Theorem 2.3.1, by the integral

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)e^{-k\xi^2 t}\, e^{i\xi x} d\xi$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(y)\, e^{-i\xi y} dy\right] e^{-k\xi^2 t}\, e^{i\xi x} d\xi$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y) e^{i(x-y)\xi} e^{-k\xi^2 t} d\xi\, dy$$

Here the fact that $|\hat{f}(\xi)|e^{-k\xi^2 t}$ is integrable on $-\infty < \xi < \infty$ allows us to replace the Cauchy principal value in (2.17) by the corresponding improper integral, and the change of the order of integration in the last step is justified by the assumption that $|f|$ is integrable. Now, because $e^{-k\xi^2 t}$ is an even function of ξ,we have

$$u(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(y) \int_{0}^{\infty} cos(x-y)\xi e^{-k\xi^2 t} d\xi\, dy$$

which coincides with (3.5).

Using the integration formula

$$\int_{0}^{\infty} cos z\xi e^{-b\xi^2} d\xi = \frac{1}{2}\sqrt{\frac{\pi}{b}}\, e^{\frac{-z^2}{4b}} \text{ for all } z \in \mathbb{R}, b > 0 \qquad (3.6)$$

We therefore have

$$u(x,t) = \frac{1}{\sqrt{\pi kt}}\int_{-\infty}^{\infty} f(y)\, e^{\frac{-(y-x)^2}{4kt}} dy$$

$$= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} f(x + 2\sqrt{\pi kp}) e^{-p^2}\, dp$$

Integrating by parts, verify that this last expression for $u(x,t)$ satisfies the heat equation. It also satisfies the initial condition

$$u(x,0) \; = f(x)$$

Example3.1.2

The corresponding boundary-value problem for a semi-infinite bar, which is insulated at one end, is defined by the system of equations

$$u_t = ku_{xx}, 0 < x < \infty \tag{3.7}$$

$$u_x(0,t) = 0,\; t > 0 \tag{3.8}$$

$$u(x,0) = f(x),\; 0 < x < \infty \tag{3.9}$$

The solution of equations (3.7) by separation of variables leads to the solutions obtained by example 3.1,

$$u_\lambda(x,t) = [A(\lambda)cos\lambda x + B(\lambda)sin\lambda x]e^{-k\lambda^2 t},\; 0 \le \lambda < \infty$$

For each solution in this set to satisfy the boundary condition at $x = 0$, we must have

$$\left.\frac{\partial u_\lambda}{\partial x}\right|_{x\,=\,0} = \lambda B(\lambda)e^{-k\lambda^2 t} = 0\, for\ all\ t > 0$$

If $\lambda = 0$ we obtain the constant solution $u_0 = A(0)$, and if $B(\lambda) = 0$ the solution is given by

$$u_\lambda(x,t) = A(\lambda)cos\lambda x e^{-k\lambda^2 t} \tag{3.10}$$

That means (3.8), where $0 \le \lambda < \infty$, gives all the solutions of the heat equation which satisfy the boundary condition (3.8).

In order to satisfy the initial condition (3.9), we form the integral

$$u(x,t) = \frac{1}{\pi}\int_0^{\infty} u_\lambda(x,t)d\lambda = \frac{1}{\pi}\int_0^{\infty} A(\lambda)cos\lambda x e^{-k\lambda^2 t}d\lambda$$

When $t = 0$

$$u(x,0) = \frac{1}{\pi}\int_0^\infty A(\lambda) cos\lambda x d\lambda \tag{3.11}$$

By extending f as even function in to $(-\infty,\infty)$, we see from the representation (3.11) that $A(\lambda)$ is the cosine transform of f. Hence, with $f \in \mathcal{L}^1(\mathbb{R})$,

$$A(\lambda) = \int_{-\infty}^\infty f(y)\cos\lambda y dy = 2\int_0^\infty f(y)\cos\lambda y dy$$

and the solution of the boundary-value problem is given by

$$u(x,t) = \frac{2}{\pi}\int_0^\infty\int_0^\infty f(y)\cos\lambda y \, cos\lambda x e^{-k\lambda^2 t} dy d\lambda \tag{3.12}$$

Using the identity $2\cos\lambda y\cos\lambda x = \cos\lambda(y-x) + \cos\lambda(y+x)$ and the formula (3.6), this double integral may be reduced to the single integral representation

$$u(x,t) = \frac{1}{2\sqrt{\pi k t}}\int_0^\infty f(y)\left[e^{-(y-x)^2/4kt} + e^{-(y+x)^2/4kt}\right]dy$$

To obtain explicit expression for the solution when

$$f(y) = \begin{cases} 1, & 0 < y < a \\ 0, & y > a \end{cases},$$

We can use the definition of error function given by

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-p^2}\,dp,$$

to write

$$\begin{aligned} u(x,t) &= \frac{1}{2\sqrt{\pi k t}}\int_0^a\left[e^{-(y-x)^2/4kt} + e^{-(y+x)^2/4kt}\right]dy \\ &= \frac{1}{\sqrt{\pi}}\int_{-x/2\sqrt{kt}}^{(a-x)/2\sqrt{kt}} e^{-p^2}\,dp + \frac{1}{\sqrt{\pi}}\int_{x/2\sqrt{kt}}^{(a+x)/2\sqrt{kt}} e^{-p^2}\,dp \\ &= \frac{1}{\sqrt{\pi}}\int_0^{(a-x)/2\sqrt{kt}} e^{-p^2}\,dp + \frac{1}{\sqrt{\pi}}\int_0^{(a+x)/2\sqrt{kt}} e^{-p^2}\,dp \\ &= \frac{1}{2} erf\left(\frac{a-x}{2\sqrt{kt}}\right) + \frac{1}{2} erf\left(\frac{a+x}{2\sqrt{kt}}\right) \end{aligned}$$

From the properties of the error function for all $t > 0$,

$$u(x,t) \to 0 \; as \; x \to \infty$$

$$u(x,t) \to erf\left(\frac{a}{2\sqrt{kt}}\right) \quad as \; x \to 0$$

and that, for all $x > 0$, $u(x,t) \to 0 \;\; as \; t \to \infty$

as $t \to \infty$,

$$u(x,t) \to \begin{cases} 1 & when\ 0 \le x < a \\ 0 & when\ x > \quad a \end{cases}$$

Thus, at any point on the bar, the temperature approaches 0 as $t \to \infty$; and at any instant, the temperature approaches 0 as $x \to \infty$. This is to be expected, because the initial heat in the bar eventually seeps out to ∞. It is also worth noting that $u(a,t) \to \frac{1}{2} = \frac{[f(a^{+})+f(a^{-})]}{2}$ as $t \to 0$, as would be expected in the following figure.

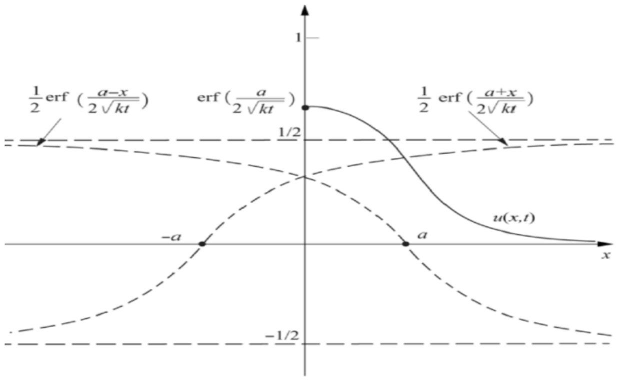


Figure *Temperature distribution on a semi-infinite rod*.

3.2 Non-Homogeneous Equations

The nonhomogeneous differential equation $y - y'' = f$ can be solved directly by well-known methods when f is a polynomial or an exponential function, for example, but such methods do not work for more general classes of functions.

If f has a Fourier transform, we can use Corollary 2.1 to write

$$\hat{y} + \xi^2 \hat{y} = \hat{f}$$

Thus the solution of the differential equation is given, formally, by

$$y(x) = \mathcal{F}^{-1}\left(\hat{f}(\xi)\frac{1}{\xi^2+1}\right)(x) \tag{3.13}$$

We know the inverse transforms of both $\hat{f}$ and $(\xi^2+1)^{-1}$, but we have no obvious method for inverting the product of these two transforms, or even deciding whether such a product is invertible. Now we show how we can express $\mathcal{F}^{-1}(\hat{f}.\hat{g})$ in terms of f and g under relatively mild restrictions on the functions f and g.

Definition 3.2.1 Let I be a real interval. A function $f: I \to C$ is said to be locally integrable on I if $|f|$ is integrable on any finite subinterval of I.

Thus all piecewise continuous functions on I and all functions in $\mathcal{L}^1(I)$ are locally integrable.

Definition 3.2.2: If the functions $f, g: \mathbb{R} \to \mathbb{C}$ are locally integrable, their convolution is the function defined by the integral

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-t)g(t)dt$$

for all $x \in \mathbb{R}$ where the integral converges.

By setting $x - t = s$ in the above integral we obtain the commutative relation

$$(f * g)(x) = \int_{-\infty}^{\infty} f(s)g(x-s)ds = g * f(x) .$$

The convolution $f * g$ exists as a function on $\mathbb{R}$ under various conditions on f and g. Here are some examples.

1. If either of the function is absolutely integrable and the other is bounded.

   Suppose $f \in \mathcal{L}^1(\mathbb{R})$ $|g| \leq M$ then

$$|f * g(x)| \leq \int_{-\infty}^{\infty} |f(x-t)||g(t)|dt$$
$$\leq M \int_{-\infty}^{\infty} |f(t)|dt < \infty.$$

2. If both f and g vanish on $(-\infty, 0)$, in which case

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-t)g(t)dt = \int_{0}^{x} f(x-t)g(t)dt$$

3. If either f or g is bounded and vanishes outside a finite interval, then $f * g$ is bounded.

***Theorem* 3.2.1** If both f and g belong to $\mathcal{L}^1(\mathbb{R})$ and either function is bounded, then $f * g$ also lies in $\mathcal{L}^1(\mathbb{R})$ and

$$\mathcal{F}(f * g) = \hat{f}\hat{g}. \tag{3.14}$$

Proof: For $f * g$ to have a Fourier transform we first have to prove that $|f * g|$ is integrable on R. Suppose, without loss of generality, that $|g|$ is bounded on R by the positive constant M. Because

$$|f(x-t)g(t)| \leq M|f(x-t)|$$

And $M|f(x-t)|$ is integrable (as a function of x) on $(-\infty, \infty)$ for all $t \in R$, it follows that the integral

$$f * g(x) = \int_{-\infty}^{\infty} f(x-t)g(t)dt$$

is uniformly convergent. Consequently, for any $a > 0$, we can write

$$\int_0^a |f * g(x)|dx = \int_0^a \left|\int_{-\infty}^{\infty} f(x-t)g(t)dt\right|dx$$

$$\leq \int_0^a \left|\int_{-\infty}^{\infty} f(x-t)g(t)\right|dtdx$$

$$= \int_{-\infty}^{\infty} \int_0^a |f(x-t)g(t)|\, dtdx$$

$$= \int_{-\infty}^{\infty} |g(t)| \int_0^a |f(x-t)|\, dxdt$$

$$\leq \int_{-\infty}^{\infty} |g(t)| \int_{-\infty}^{\infty} |f(x-t)|\, dxdt$$

$$= \int_{-\infty}^{\infty} |g(t)dt| \int_{-\infty}^{\infty} |f(x-t)|\, dx$$

This inequality holds for any $a > 0$, therefore $|f * g|$ is integrable on $(0, \infty)$.

Similarly, $|f * g|$ is integrable on $(-\infty, 0)$ and hence on $\mathbb{R}$.

To prove the equality (3.8) we write

$$\mathcal{F}(f * g)(\xi) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x-t)g(t)e^{-i\xi x} dt\, dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x-t)e^{-i\xi(x-t)} g(t)e^{-i\xi t}\, dxdt$$

$$= \hat{f}(\xi)\hat{g}(\xi)$$

Where the change in the order of integration is justified by the uniform convergence of the convolution integral. ■

Using Equation (3.8) and the result of Example(2.1) we therefore conclude that a particular solution of $y - y'' = f$ is given by

$$y(x) = \mathcal{F}^{-1}\left(\hat{f}(\xi)\frac{1}{\xi^2+1}\right)(x)$$

$$= f * \left(\frac{1}{2}e^{-|.|}\right)(x) \tag{3.15}$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} f(x-t)e^{-|t|}dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} f(t)e^{-|x-t|}dt \tag{3.16}$$

This integral expression satisfies the equation $y - y'' = f$. provided f is continuous.

In this connection, it is worth noting that the kernel function $e^{-|x-t|}$ in the integral representation of y in equation (3.16) is no other than Green's function $G(x,t)$ for the SL operator

$$L = -\frac{d^2}{dx^2} + 1 \tag{3.17}$$

On the interval $(-\infty, \infty)$, subject to the boundary conditions $\lim_{x\to\pm\infty} G(x,t) = 0$; for the nonhomogeneous equation $Ly = f$ is solved, formally by

$$y(x) = L^{-1}f = \int_{-\infty}^{\infty} G(x,t)f(t)dt$$

This solution tends to 0 as $x \to \pm\infty$,as it should, being continuous and integrable on $(-\infty, \infty)$. Under such boundary conditions, the solution of the corresponding homogeneous equation $y - y'' = 0$ namely $c_1e^x + c_2e^{-x}$, can only be the trial solution.

To solve the same equation $y - y'' = f$ on the semi-infinite interval $[0, \infty)$ we need to impose a boundary condition at $x = 0$, say $y(0) = y_0$.

In this case the same procedure above leads to the particular solution

$$y_p(x) = \frac{1}{2}\int_0^{\infty} f(t)e^{-|x-t|}dt$$

But here the homogeneous solution

$$y_h(x) = ce^{-x}, \ \ x \geq 0$$

Is admissible for any constant c. Applying the boundary condition at $x = 0$ to the sum $y = y_p + y_h$ yields

$$y_0 = \frac{1}{2}\int_0^{\infty} f(t)e^{-t}dt + c$$

From which c can be determined. The desired solution is therefore

$$y(x) = \frac{1}{2}\int_0^{\infty} f(t)e^{-|x-t|}dt + \left(y_0 - \frac{1}{2}\int_0^{\infty} f(t)e^{-t}dt\right)e^{-x}$$

If $y_0 = 0$, that is, if the boundary condition at $x = 0$ is homogeneous, then

$$y(x) = \frac{1}{2}\int_0^{\infty} f(t)\left(e^{-|x-t|} - e^{-|x+t|}\right)dt$$

And, once again, we conclude the green's function for the same operator(3.17)) on the interval $[0, \infty)$ under the homogeneous boundary condition $y(0) = 0$ is now

$$G(x,t) = \frac{1}{2}\int_0^{\infty} f(t)\left(e^{-|x-t|} - e^{-|x+t|}\right)dt.$$

3.3 Application to PDEs

The usual difficulty with PDEs is that the solution involves more than one independent variable. The transform method allows us to reduce one independent variable. We commonly try to transform the $x - dependance$ through Fourier transform, provided that the domain is infinite, i.e $-\infty < x < \infty$.

Consider the function $u(x,t)$, with $-\infty < x < \infty,\ t > 0$. Let

$$U(\omega,t) = \mathcal{F}[u(x,t)] = \int_{-\infty}^{\infty} u(x,t)\, e^{i\omega x} dx \tag{3.18}$$

Be the fourier transform of $u(x,t)$ with respect to x. The original function $u(x,t)$ can then be recovered from the fourier inverse transform:

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U(\omega,t)\, e^{-i\omega x} d\omega \tag{3.19}$$

[note that in (3.18) and (3.19) t plays no role; it may be regarded as arbitrary.] this is very similar to our previous method of writing the solution in the form of an eigenfunction expansion when the domain is finite.

With (3.19), taking derivatives with respect to x is now:

$$u_x(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} (-i\omega) U(\omega,t)\, e^{-i\omega x} d\omega$$

$$u_{xx}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} (-i\omega)^2 U(\omega,t)\, e^{-i\omega x} d\omega \tag{3.20}$$

$$u_t(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U_t(\omega,t)\, e^{-i\omega x} d\omega$$

$$u_{tt}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U_{tt}(\omega,t)\, e^{-i\omega x} d\omega \tag{3.21}$$

Provided of course that these integral exists. At this point, there is no need to worry about these mathematical issues of integrability because we don't even know what $U(\omega,t)$ is yet.

3.3.1. The Wave Equation in an Infinite Domain

$$\begin{cases} u_{tt} = c^2 u_{xx}, -\infty < x < \infty, t > 0 \\ u(x,t) \to 0 \ as\ x \to \pm\infty \\ u(x,0) = f(x) \\ u_t(x,0) = 0, -\infty < x < \infty \end{cases} \tag{3.22}$$

We assume the solution to be of the form of an integral (3.19) which we substitute in to PDE(3.21). This yields, using (3.20)

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} (U_{tt}(\omega,t) + c^2\omega^2 U(\omega,t)) e^{-i\omega x} d\omega = 0$$

Which is the same as

$$\mathcal{F}^{-1}(U_{tt} + c^2\omega^2 U) = 0 \tag{3.23}$$

So by taking $\mathcal{F}$ of the above equation

$$U_{tt} + c^2\omega^2 U = 0 \tag{3.24}$$

This is an ODE; the partial derivatives $\frac{\partial^2}{\partial x^2}$ have been converted to $(-i\omega)^2$, an algebraic multiplication. The ODE in t is to be solved subject to the folloing initial conditions:

$$u(x,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U(\omega,0)e^{-i\omega x}d\omega = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{-i\omega x}d\omega.$$

These imply:

$$U_t(\omega,0) = 0 \tag{3.25}$$

and

$$U(\omega,0) = F(\omega) \tag{3.26}$$

Where the Fourier transform $F(\omega)$ of $f(x)$ is known if $f(x)$ is known.

The general solution to the ODE (3.24) is

$U(\omega,t) = A(\omega)\sin(c\omega t) + B(\omega)\cos(c\omega t)$.

The ICs(3.24) and (3.25) can be used to determine the constants A and B to be

$B(\omega) = F(\omega)$ and $A(\omega) = 0$. Thus

$$U(\omega,t) = F(\omega)\cos(c\omega t).$$

(3.27)

We recover $u(x,t)$ by substuting (3.27) back into (3.19).

$$u(x,t) = \mathcal{F}^{-1}[U(\omega,t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} U(\omega,t)e^{-i\omega x}d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\cos(c\omega t)e^{-i\omega x}d\omega$$

(3.28)

Typically one cannot perform the integral explicitly unless $F(\omega)$ is known. In the particular case of the wave equation however, progress can be made by noting that

$$\cos(c\omega t) = \frac{1}{2}\left(e^{ic\omega t} + e^{-ic\omega t}\right)$$

and so (3.28) can be written as

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{2}F(\omega)e^{-i\omega(x-ct)}d\omega + \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{2}F(\omega)e^{-i\omega(x+ct)}d\omega$$

$$= \frac{1}{2}f(x-ct) + \frac{1}{2}f(x+ct) \tag{3.29}$$

Since $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{-i\omega x}d\omega$

So

$$f(x-ct)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)e^{-i\omega(x-ct)}d\omega \text{ and } f(x+ct)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)e^{-i\omega(x+ct)}d\omega$$

The physical interpretation of the solution (3.29) to the wave equation (3.21) is that an initial displacement of $f(x)$ will split in to two shapes for $t>0$, each with half the amplitude of the original shape, one propagates to the left and one propagates to the right, both with speed c. The quantity c is therefore called the wave speed.

3.3.2 Diffusion Equation in an Infinite Domain

$$\begin{aligned}&u_t=\alpha^2u_{xx},-\infty<x<\infty,t>0\\&u(x,t)\to 0\ as\ x\to\pm\infty\\&u(x,0)=f(x)\ -\infty<x<\infty\end{aligned}\tag{3.30}$$

We assume a solution of the form of an integral (3.19) and substitute it in to the PDE (3.30). this yields, using (3.20)

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}(U_t+\alpha^2\omega^2U)\,e^{-i\omega x}d\omega=0$$

Which implies

$$U_t+\alpha^2\omega^2U=0\tag{3.31}$$

The ODE (3.30) is solved subject to

$$U(\omega,0)=F(\omega)\tag{3.32}$$

Which is obtained by taking the fourier transform of (3.30)

The solution is

$$U(x,t)=A(\omega)e^{-\alpha^2\omega^2t}=F(\omega)e^{-\alpha^2\omega^2t}\tag{3.33}$$

The final solution is obtained by substituting (3.33) into (3.19)

$$u(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)e^{-\alpha^2\omega^2t-i\omega x)}d\omega\tag{3.34}$$

For the special case of

$$f(x)=ae^{-(x/L)^2}$$

Since $F(\omega)=\mathcal{F}(f(x))=aL\sqrt{\pi}e^{-(L\omega)^2/4}$

Then

$$u(x,t)=\frac{aL}{2\pi}\int_{-\infty}^{\infty}\sqrt{\pi}e^{-(L\omega)^2/4-\alpha^2\omega^2t-i\omega x)}d\omega$$

Can be evaluated by completing squares

$$u(x,t)=\frac{aL}{2\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-(\alpha^2t+L^2/4)\omega^2-i\omega x}d\omega$$

$$= \frac{aL}{\sqrt{4\alpha^2 t + L^2}} e^{-x^2/(4\alpha^2 t + L^2)} \tag{3.35}$$

The physical interpretation of the solution (3.35) is that an initial concentration near $x = 0$ an initial with width of approximately $2L$ spreads out in to a wider and wider region while its amplitude at $x = 0$ decreases monotonically to zero. This is a typical behavior of solutions to the diffusion equation. The underlying physical process reduces gradients and spreads any initial concentration/heat to wider regions.

CONCLUSION

The last section on application demonstrates the use of Fourier transform in solving problems involving differential equations. These transforms are very useful for solving differential or integral equations for the following reasons. First, these equations are replaced by simple algebraic equations, which enable us to find the solution of the transform function. The solution of the given equation is then obtained in the original variables by inverting the transform solution. Second, the Fourier transform of the elementary source term is used for determination of the fundamental solution that illustrates the basic ideas behind the construction and implementation of Green's functions. Third, the transform solution combined with the convolution theorem provides an elegant representation of the solution for the boundary value and initial value problems.

Generally from this thesis we see that Fourier transform is very useful when dealing with differential and integral equations that could be difficult to solve. So it is beneficial to have a firm understanding of how to use Fourier transform and about Fourier transformation.

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