ON EXISTENCE OF POSITIVE SOLUTIONS OF A NON-LINEAR THREE-POINT BOUNDARY-VALUE PROBLEM

ADDIS ABABA UNIVERSITY

DEPARTMENT OF MATHEMATICS

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The undersigned hereby certify that they have read and recommend to the school of graduate studies for acceptance of a project entitled **On Existence of Positive Solutions of a Nonlinear Three-point Boundary-value Problem** by Wubale Demis Alamirew in partial fulfillment of the requirements for the degree of master of Science.

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Abstract

This paper is highly dependent on the work of Ruyun Ma and it stabilizes some conditions for the existence of positive solutions to the boundary value problem

$$u'' + a(t)f(u) = 0, \quad t \in (0, 1)$$

$$u(0) = 0, \quad \alpha u(\eta) = u(1),$$

where $0 < \eta < 1$ and $0 < \alpha < \frac{1}{\eta}$. We show the existence of at least one positive solution if f is either superlinear or sublinear by applying the fixed point theorem in cones.

Contents

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Notations

$\mathbb{R}$	The set of all real numbers.
$\mathbb{C}$	The set of all complex numbers.
$\|x\|$	Euclidean norm of $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$, i.e. $\|x\| = (\sum_{i=1}^{n} x_i^2)^{1/2}$.
$C[0,1]$	The set of continuous function on $[0,1]$.
$C([0,\infty),[0,\infty))$	The set of continuous function $f : [0,\infty) \to [0,\infty)$, whenever f is the function of u.
$C([0,1],[0,\infty))$	The set of continuous function $a : [0,1] \to [0,\infty)$, whenever a is the function of t.
$\partial\Omega$	The boundary of Ω
$\overline{\Omega}$	Closure of the set Ω

Chapter 1

Introduction

We recall that explicit ODE of n^{th} order can be given by

$$u^{(n)} \ + \ f(x, u, ..., u^{(n-1)}) \ = 0.$$

Here u is the unknown function.

1.1 Boundary value problem

In a boundary value problem for an n^{th} order differential equation

$$u^{(n)} \ + \ f(x, u, ..., u^{(n-1)}) \ = 0;$$

the n additional conditions that (we expect to) define a solution uniquely are not prescribed at a single point a, as in the case of the initial value problem, but at two points a and b that are the end points of the interval $a \leq x \leq b$ where the solution is considered. Boundary value problems for (real) linear second order equations

$$u'' + a_1(x)u' \ + \ a_0(x)u \ = \ g(x) \quad for \quad a \leq x \leq b \tag{1.1}$$

are particularly important because of numerous applications in science and technology.

1.2 Boundary conditions

The three most common types of boundary conditions for (1.1) are the *boundary conditions of the*

first kind:	$u(a) = \eta_1,$	$u(b) = \eta_2,$
second kind:	$u'(a) = \eta_1,$	$u'(b) = \eta_2,$
third kind :	$\alpha_1 u(a) + \alpha_2 u'(a) = \eta_1,$	$\beta_1 u(b) + \beta_2 u'(b) = \eta_2.$

Obviously, the first two conditions are special cases of the third, which is also called a *Sturmian boundary condition*. There are also other boundary conditions such as

$$u(a) - u(b) = \eta_1, \qquad u'(a) - u'(b) = \eta_2.$$

When $\eta_1 = \eta_2 = 0$, these are called *periodic boundary conditions* for the following reason: If the coefficients are continuous, periodic functions with period $l = b-a$ and if $u(x)$ is a solution to (1.1), then $v(x) := u(x+l)$ is also a solution of the differential equation. If u satisfies the periodic boundary conditions described above, then $v(a) = u(a)$ and $v'(a) = u'(a)$. This implies $u \equiv v$ by the uniqueness theorem for the initial value problem. In other words, u is a periodic function with period l.

1.3 Nonexistence and Non uniqueness

In contrast to the initial value problem, where general existence and uniqueness theorems are available, cases of non uniqueness or nonexistence arise in very simple boundary value problems. Consider the simplest example: $u'' = 0$. The solutions are linear functions $u(x) = a + bx$. A boundary value problem of first kind is always uniquely solvable, while one of the second kind has no solution if $\eta_1 \neq \eta_2$ and infinitely many solutions if $\eta_1 = \eta_2$.

The paper is divided in five chapters including this introduction.The rest of the paper is organized as follows. The second chapter is about three point boundary value problem and the main theorem is stated in this chapter. Definitions, some important concepts like the Fixed point theorem are discussed in chapter three. In chapter four , we provide some property of solutions and various lemmas, which play key roles in this paper. The proof of the main theorem and application example will be given in chapter five. And in the last part we get reference materials that I have used to write the paper.

Chapter 2

Three Point Boundary Value Problems

In this paper, we consider the existence of positive solutions to the equation

$$u'' + a(t)f(u) = 0, \quad t \in (0,1) \tag{2.1}$$

with the boundary condition

$$u(0) = 0, \quad \alpha u(\eta) = u(1) \tag{2.2}$$

where $0 < \eta < 1$. Our purpose here is to give some existence results for positive solutions to (2.1)-(2.2), assuming that $\alpha\eta < 1$ and f is either superlinear or sublinear. Our proof is based up on the Krasnoselskii's Fixed Point Theorem in a cone. From now on, we adopt assumptions,

(As1) the function f is a continuous function of u,

(As2) $a : [0,1] \to [0,\infty)$ is continuous and there exists $x_0 \in [\eta, 1]$ such that $a(x_0) > 0$.

Set $f_0 = \lim_{u\to 0^+} \frac{f(u)}{u}$, $f_\infty = \lim_{u\to\infty} \frac{f(u)}{u}$. Then $f_0 = 0$ and $f_\infty = \infty$ corresponds to the superlinear case, and $f_0 = \infty$ and $f_\infty = 0$ correspond to the sublinear case.

2.1 Asymptotic Order of Nonlinear Term

Given the second order nonlinear ODE

$$u'' + a(t)f(u) = 0; \quad 0 < t < 1,$$

we say that the nonlinear term, $a(t)f(u)$ is

i, superlinear, if $\lim_{u\to 0^+} \frac{f(u)}{u} = 0$ and/or $\lim_{u\to\infty} \frac{f(u)}{u} = \infty$.

ii, sublinear, if $\lim_{u\to 0^+} \frac{f(u)}{u} = \infty$ and/or $\lim_{u\to\infty} \frac{f(u)}{u} = 0$.

Evidently, if the nonlinearity term is superlinear then $f(u)$ outpaces(dominates) the linear function $g(u) = u$, whereas in case of sublinearity it decays(grows slower) than $g(u) = u$. And hence, one can interpret the limits

$$\lim_{u\to 0^+} \frac{f(u)}{u} = f_0 \ \ and \ \ \lim_{u\to\infty} \frac{f(u)}{u} = f_\infty$$

as a measure of the growth of the function $f(u)$ in view of the growth of $g(u) = u$.
Let $f,\ g\ : U \to Y$ be two functions. Suppose that $g(u) \neq 0$, for all $u \in U$ and $\lim_{u\to\infty} \frac{f(u)}{g(u)} = f_\infty$.

Case 1: g grows faster than f
$\Rightarrow$ g outpaces f
$\Rightarrow$ $\lim_{u\to\infty} \frac{f(u)}{g(u)} = 0$
i.e. g has greater asymptotic order than f.

Case 2: f grows faster than g
$\Rightarrow$ $\lim_{u\to\infty} \frac{f(u)}{g(u)} = \infty$
i.e. f has greater asymptotic order than g.

Case 3: f and g have the same growth rate.
$\Rightarrow$ $\lim_{u\to\infty} \frac{f(u)}{g(u)}$ is finite and nonzero
i.e.$0 < f_\infty < \infty$. In this case, we say that f and g have the same asymptotic order.

2.2 Local Asymptotic Order

Here in this subsection, we consider asymptotic relation between functions in a neighbourhood of a given point $x_0 \in U$.
Let $f,\ g\ :\ U \subset \mathbb{R} \to \mathbb{R}$ be given $x_0 \in U$ be fixed.

If there exist $k > 0$(constant) and neighbourhood $B(x_0)$ of x_0 such that $x \in B(x_0) \cap U$ implies $|f(x)| \leq k|g(x)|$ then, we say that f and g have the same asymptotic order as $x \to x_0$.
Notation: $f(x) \in O(g(x)) \ \ (as \ \ x \to x_0)$
Define a relation $\sim$ on $O(g(x))$ as follows:

$$f \ \sim \ h \ \ iff \ \ f - h \ \in \ O(g(x)) \ \ (as \ \ x \to x_0)$$

Claim! " $\sim$ " is an equivalence relation in $O(g(x))$.
Justification:

i, since $f - f \ = \ 0$
we have $|f(x) - f(x)| = 0 \ \leq \ |g(x)| \ \ for \ \ all \ \ x \in U$
thus,$x \in B(x_0) \cap U \ \Rightarrow \ |f(x) - f(x)| \ \leq \ |g(x)|$
$\Rightarrow f - f \ \in \ O(g(x)) \ \ (as \ \ x \to x_0)$
$\Rightarrow$ " $\sim$ " is reflexive.

ii, Let $f, h \ \in \ O(g(x)) \ (as \ \ x \to x_0) \ \ such \ \ that \ \ f \sim h$
$\Rightarrow there \ \ exist \ \ k > 0 \ \ and \ \ B(x_0) \ \ such \ \ that$
$x \in B(x_0) \cap U \Rightarrow |f(x) - h(x)| \ \leq \ k|g(x)|$
$\Rightarrow |h(x) - f(x)| \ \leq \ k|g(x)|$
$\Rightarrow h \ \sim \ f$
$\therefore$ " $\sim$ " is symmetric.

iii, Let $f_1, f_2, f_3 \ \in \ O(g(x)) \ (as \ \ x \to x_0)$ such that $f_1 \sim f_2$ and $f_2 \sim f_3$
$\Rightarrow$ there exist $k_1, k_2 > 0$ and $B_1(x_0), B_2(x_0)$ such that
$x \in B_1(x_0) \cap U \Rightarrow |f_1(x) - f_2(x)| \ \leq \ k_1|g(x)|$ and
$x \in B_2(x_0) \cap U \Rightarrow |f_2(x) - f_3(x)| \ \leq \ k_2|g(x)|$
Thus, for $B(x_0) = B_1(x_0) \cap B_2(x_0)$ and $k = k_1 + k_2$,

$$\begin{aligned} x \in B(x_0) \cap U \Rightarrow |f_1(x) - f_3(x)| \ &\leq \ |f_1(x) - f_2(x)| + |f_2(x) - f_3(x)| \\ &\leq \ k_1|g(x)| + k_2|g(x)| \\ &= \ (k_1 + k_2)|g(x)| \\ &= \ k|g(x)| \end{aligned}$$

$\Rightarrow f_1 \ \sim \ f_3$
$\therefore$ " $\sim$ " is transitive.
Consequently, " $\sim$ " is an equivalence relation.

2.3 Positive Solution

By the positive solution of (2.1)-(2.2) we understand a function $u(t)$ which is positive on $0 < t < 1$ and satisfies the differential equation (2.1) and the boundary conditions (2.2). The main point of this paper is to prove the following theorem , which is the main theorem for this paper.

Theorem 2.1. *Assume* (As_1) *and* (As_2)*hold. Then the problem* $(2.1)-(2.2)$ *has at least one positive solution in the case*

i, $f_0 = 0$ *and* $f_\infty = \infty$*(superlinear) or*

ii, $f_0 = \infty$ *and* $f_\infty = 0$*(sublinear)*

Chapter 3

Krasnoselskii's Fixed Point and related theorems

3.1 Definitions

Definition 3.1.1. *Boundary condition is the set of conditions specified for the behavior of the solution to a set of differential equations at the boundary of its domain. Boundary conditions are important in determining the mathematical solutions to many physical problems.*

Definition 3.1.2. *A Banach space is a vector space x over the field* $\mathbb{R}$ *of real numbers, or over the field* $\mathbb{C}$ *of complex numbers, which is equipped with a norm and which is complete with respect to that norm, that is to say, for every cauchy sequence* $\{x_n\}_{n=1}^{\infty}$, *in X their exists an element x in X such that* $\lim_{n\to\infty} x_n = x$, *i.e.* $\lim_{n\to\infty} \parallel x_n - x \parallel = 0$ *simply a Banach space is a complete normed vector space.*

Definition 3.1.3. *An operator is called completely continuous if it is continuous and maps bounded sets in to precompact sets.*

Definition 3.1.4. *Let f(x) be a differentiable function on an interval I. Assume that f'(x) is also differentiable on I.*

i, f(x) is concave up on I if and only if $f''(x) > 0$ *on I*

ii, f(x) is concave down on I if and only if $f''(x) < 0$ *on I*

Definition 3.1.5. *A function f defined on some set X with real or complex values is called bounded, if the set of its values is bounded. In other words, there exists a real number M such that* $\mid f(x) \mid \leq M$, *for all x in X.*

Definition 3.1.6. *Let S be a set.*
Let $(X, \| . \|)$ *be a normed vector space.*
Let B be the set of bounded mappings $S \rightarrow X$*. This is a vector space.*
For $f \in B$ *the supremum norm (sup norm) of f on S is*
$\| f \|_\infty = sup \{\| f(x) \|: \quad x \in S\}$.

Definition 3.1.7. *Let E be a real Banach space. A non empty closed convex set* $K \subset E$ *is called cone of E if it satisfies the following conditions;*

i, $x \in K$, $\lambda \geq 0$ *implies* $\lambda x \in K$;

ii, $x \in K$, $-x \in K$ *implies* $x=0$.

3.2 Krasnoselskii's Fixed Point and related theorems

Theorem 3.1. *[4, 9] (Krasnoselskii's Fixed Point Theorem).*
Let E be a Banach space, and let $K \subset E$ *be a cone. Assume* Ω_1, Ω_2 *are open subsets of E with* $0 \in \Omega_1, \Omega_1 \subset \Omega_2$*, and let*
$A: K \cap (\overline{\Omega_2} \setminus \Omega_1) \rightarrow K$
be completely continuous operator such that

i, $\| Au \| \leq \| u \|, \quad u \in K \cap \partial\Omega_1$, *and* $\| Au \| \geq \| u \|, \quad u \in K \cap \partial\Omega_2$, *or*

ii, $\| Au \| \geq \| u \|, \quad u \in K \cap \partial\Omega_1$, *and* $\| Au \| \leq \| u \|, \quad u \in K \cap \partial\Omega_2$.

Then A has a fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Theorem 3.2. *(Ascoli-Arzela Theorem)*
If a sequence $\{f_n\}_{n=1}^{\infty}$ *in C(x) is bounded and equicontinuous then it has a uniformly convergent subsequence.*

In this statement

i, "$F \subset C(x)$ is bounded" means that there exists a positive constant $M < \infty$ such that $| f(x) | \leq M$ for each $x \in X$ and each $f \in F$, and

ii, "$F \subset C(x)$ is equicontinuous" means that for every $\epsilon > 0$ there exists $\delta > 0$(which depends only on ϵ) such that for $x, y \in X$,

$$d(x,y) < \delta \ implies \ \ | f(x) - f(y) | < \epsilon \ \ for \ all f \in F, where \ d \ is \ the \ metric \ on \ X.$$

Proof. We have three steps to prove the Ascoli-Arzela Theorem.
Step I. We show that the compact metric space X is separable, i.e., has a countable dense subset S.

Given a positive integer n and a point $x \in X$. Let

$$B(r, \frac{1}{n}) = \{y \in X: \ d(x, y) < \frac{1}{n}\}.$$

the open ball of radius $\frac{1}{n}$, centered at x. For a given n, the collection of all these balls as x runs through X is an open cover of x, so (because X is compact) there is a finite sub collection that also covers X.
Let S_n denote the collection of centers of the balls in this finite sub collection. Thus S_n is a finite subset of X that is "$\frac{1}{n}$-dense" in the sense that every point of X lies within $\frac{1}{n}$ of a point of S_n. We know that the union S of all the sets S_n is countable, and dense in X.
Step II. We find a subsequence of $\{f_n\}$ that converges pointwise on S. This is a standard diagonal argument. Let's list the (countably many) elements of S as $\{x_1, x_2, ...\}$. Then the numerical sequence $\{sf_n(x_1)\}_{n=1}^{\infty}$ is bounded, so by Bolzano-Weierstrass it has a convergent subsequence, which we will write using double subscripts: $\{f_{1,n}(x_1)\}_{n=1}^{\infty}$. Now the numerical sequence $\{f_{1,n}(x_2)\}_{n=1}^{\infty}$ is bounded, so it has a convergent subsequence $\{f_{2,n}(x_2)\}_{n=1}^{\infty}$. Note that the sequence of functions $f_{2,n}{}_{n=1}^{\infty}$, since it is a subsequence of $\{f_{1,n}\}_{n=1}^{\infty}$, converges at both $x_1 and x_2$. Proceeding in this fashion we obtain a countable collection of subsequences of our original sequence:

$$\begin{array}{cccc} f_{1,1} & f_{1,2} & f_{1,3} & \cdots \\ f_{2,1} & f_{2,2} & f_{2,3} & \cdots \\ f_{3,1} & f_{3,2} & f_{3,3} & \cdots \\ . & . & . & ... \\ . & . & . & ... \\ . & . & . & ... \end{array}$$

where the sequence in the n-th row converges at the points $x_1, ..., x_n$, and each row is a subsequence of the one above it. Thus the diagonal sequence $\{f_{n,n}\}$ is a subsequence of the original sequence $\{f_n\}$ that converges at each point of S.
Step III. Completion of the proof. Let $\{g_n\}$ be the diagonal subsequence produced in the previous step, convergent at each point of the dense set S. Let $\epsilon > 0$ be given, and choose $\delta > 0$ by equicontinuity of the original sequence, so that d(x,y)$< \delta$ implies $\mid g_n(x) - g_n(y) \mid < \frac{\epsilon}{3}$ for each $x, y \ \in$ X and each positive integer n. Fix M $> \frac{1}{\delta}$ so that the finite subset $S_M \subset S$ that

we produced in Step I is δ-dense in X. Since $\{g_n\}$ converges at each point of S_M, there exists $N > 0$ such that

$$n; m \ > \ N \ \ implies \ \ \mid g_n(s) - g_m(s) \mid < \frac{\epsilon}{3} \ \ for \ all \ \ s \ \in \ S_M. \qquad (3.1)$$

Fix x $\in$ X. Then x lies within δ of some $s \in S_M$, so if n,m > M:

$$\mid g_n(x) - g_m(x) \mid \leq \mid g_n(x) - g_n(s) \mid + \mid g_n(s) - g_m(s) \mid + \mid g_m(s) - g_m(x) \mid$$

The first and last terms on the right are $\leq \frac{\epsilon}{3}$ by our choice of δ (which was possible because of the equicontinuity of the original sequence), and the same estimate holds for the middle term by our choice of N in (2.1). In summary: given $\epsilon > 0$ we have produced N so that for each x $\in$ X,

$$m, n \ > \ N \ \ implies \ \ \mid g_n(x) - g_m(x) \mid < \ \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

Thus on X the subsequence $\{g_n\}$ of $\{f_n\}$ is uniformly Cauchy, and therefore uniformly convergent. This completes the proof of the Arzela-Ascoli Theorem. □

Theorem 3.3. *(First mean value theorem for integration)*
If G: [a,b] $\rightarrow$ R is a continuous function and φ is an integrable function that does not change sign on the interval (a,b), then there exists a number x in (a,b) such that

$$\int_a^b G(t)\varphi(t)dt = G(x) \int_a^b \varphi(t)dt. \qquad (3.2)$$

In particular, if $\varphi(t) = 1$ for all t in [a,b], then there exists x in (a,b) such that

$$\int_a^b G(t)dt = G(x)(b - a). \qquad (3.3)$$

More commonly written as:

$$\frac{1}{b-a} \int_a^b G(t)dt = G(x) \qquad (3.4)$$

The value G(x) is called the *mean value* of G(t) on [a,b].

Proof. With out loss of generality assume the one-signed function $\varphi(t) \geq 0$ for all t(the negative case just changes direction of some inequalities). It follows from the extreme value theorem that the continuous function G has a finite infimum m and a finite supremum M on the interval [a,b].

From the monotonicity of the integral and the fact that $m \leq G(t) \leq M$, it follows from the non-negativity of $\varphi(t)$ that

$$mI = \int_a^b m\varphi(t)dt \leq \int_a^b G(t)\varphi(t)dt \leq \int_a^b M\varphi(t)dt = MI,$$

where

$$I := \int_a^b \varphi(t)dt$$

denotes the integral of $\varphi(t)$. Hence, if $I = 0$, then the claimed equality holds for every x in [a,b]. Therefore, we may assume $I \geq 0$ in the following. Dividing through by I we have that

$$m \leq \frac{1}{I}\int_a^b G(t)\varphi(t)dt \leq M.$$

The extreme value theorem tells us more than just that the infimum and supremum of G on [a,b] are finite; it tells us that both are actually attained. Thus we can apply the intermediate value theorem, and conclude that the continuous function G attains every value of the interval $[mM]$, in particular there exists x in [a,b] such that

$$G(x) = \frac{1}{I}\int_a^b G(t)\varphi(t)dt.$$

this completes the proof. □

Chapter 4

Property of Solutions

4.1 Uniqueness

Lemma 4.1. *Let* $\alpha\eta \neq 1$ *then for* $y \in C[0,1]$, *the problem*

$$u'' + y(t) = 0, \quad t \in (0,1) \tag{4.1}$$

$$u(0) = 0, \quad \alpha u(\eta) = u(1) \tag{4.2}$$

has a unique solution

$$u(t) = -\int_0^t (t-s)y(s)ds - \frac{\alpha t}{1-\alpha\eta}\int_0^\eta (\eta - s)y(s)ds + \frac{t}{1-\alpha\eta}\int_0^1 (1-s)y(s)ds$$

Proof. Since we have $u'' + y(t) = 0$, rewriting this differential equation as $u'' = -y(t)$, and integrating twice by using Theorem 3.3 , we obtain

$$u(t) = -\int_0^t (t-s)y(s)ds + At,$$

where

$$A(1-\alpha\eta) = -\alpha\int_0^\eta (\eta - s)y(s)ds + \int_0^1 (1-s)y(s)ds.$$

Then,

$$\begin{aligned} u(t) &= -\int_0^t (t-s)y(s)ds + t(-\alpha\int_0^\eta \frac{(\eta-s)}{(1-\alpha\eta)}y(s)ds + \int_0^1 \frac{(1-s)}{(1-\alpha\eta)}y(s)ds) \\ &= -\int_0^t (t-s)y(s)ds - \frac{\alpha t}{1-\alpha\eta}\int_0^\eta (\eta-s)y(s)ds + \frac{t}{1-\alpha\eta}\int_0^1 (1-s)y(s)ds \end{aligned}$$

□

Lemma 4.2. *Let $0 < \frac{1}{\eta} < 1$. If $y \in C[0,1]$ and $y \geq 0$, then the unique solution u of the problem (4.1)-(4.2) satisfies*

$$u \geq 0, \quad t \in [0,1].$$

Proof. From the fact that $u''(x) = -y(x) \leq 0$, we know that the graph of $u(t)$ is concave down on $(0,1)$ (by definition 3.1.4). So, if $u(1) \geq 0$, then the concavity of u and the boundary condition $u(0) = 0$ imply that $u \geq 0$ for $t \in [0,1]$.
If $u(1) < 0$, then we have that

$$u(\eta) < 0 \tag{4.3}$$

and

$$u(1) = \alpha u(\eta) > \frac{1}{\eta} u(\eta) \tag{4.4}$$

This contradicts the concavity of u.
This completes the proof. □

4.2 Existence of positive solutions

Lemma 4.3. *Let $\alpha\eta > 1$. If $y \in C[0,1]$ and $y(t) \geq 0$ for $t \in (0,1)$, then (4.1)-(4.2) has no positive solution.*

Proof. Assume that (4.1)-(4.2) has a positive solution u. If $u(1) > 0$, then $u(\eta) > 0$ and

$$\frac{u(1)}{1} = \frac{\alpha u(\eta)}{1} > \frac{u(\eta)}{\eta} \tag{4.5}$$

this contradicts the concavity of u. If $u(1) = 0$ and $u(\tau) > 0$ for some $\tau \in (0,1)$, then

$$u(\eta) = u(1) = 0, \tau \neq \eta \tag{4.6}$$

If $\tau \in (0,\eta)$, then $u(\tau) > u(\eta) = u(1)$, which contradicts the concavity of u. If $\tau \in (\eta, 1)$, then $u(0) = u(\eta) < u(\tau)$ which contradicts the concavity of u again. □

In the rest of the paper, we assume that $\alpha\eta < 1$. Moreover, we will work in the Banach space $C[0,1]$, and only the sup norm is used.

Lemma 4.4. *Let $0 < \frac{1}{\eta} < 1$. If $y \in C[0,1]$ and $y \geq 0$, then the unique solution u of the problem (4.1)-(4.2) satisfies*

$$\inf_{t \in [\eta,1]} u(t) \geq \gamma \|u\|$$

where $\gamma = \min\{\alpha\eta, \frac{\alpha(1-\eta)}{1-\alpha\eta}, \eta\}$.

Proof. We divide the proof into two steps.
Step 1. We deal with the case $0 < \alpha < 1$. In this case, by Lemma 4.2, we know that

$$u(\eta) \geq u(1) \tag{4.7}$$

Set

$$u(\bar{t}) = \|u\| \tag{4.8}$$

If $\bar{t} < \eta < 1$, then

$$\min_{t\in[\eta,1]} u(t) = u(1) \tag{4.9}$$

and

$$\begin{aligned} u(\bar{t}) &\leq u(1) + \frac{u(1) - u(\eta)}{1-\eta}(0-1) \\ &= u(1)[1 - \frac{1-\frac{1}{\alpha}}{1-\eta}] \\ &= u(1)\frac{1-\alpha\eta}{\alpha(1-\eta)} \end{aligned}$$

□

This together with (4.9) implies that

$$\min_{t\in[\eta,1]} u(t) \geq \frac{\alpha(1-\eta)}{1-\alpha\eta}\|u\| \tag{4.10}$$

If $\eta < \bar{t} < 1$, then

$$\min_{t\in[\eta,1]} u(t) = u(1) \tag{4.11}$$

From the concavity of u, we know that

$$\frac{u(\eta)}{\eta} \geq \frac{u(\bar{t})}{\bar{t}} \tag{4.12}$$

Combining (4.12) and boundary condition $\alpha u(\eta) = u(1)$, we conclude that

$$\frac{u(1)}{\alpha\eta} \geq \frac{u(\bar{t})}{\bar{t}} \geq u(\bar{t}) = \|u\|.$$

This is

$$\min_{t\in[\eta,1]} u(t) \geq \alpha\eta\|u\|. \tag{4.13}$$

Step 2. We deal with the case $1 \leq \alpha < \frac{1}{\eta}$. In this case, we have

$$u(\eta) \leq u(1) \tag{4.14}$$

Set

$$u(\bar{t}) = \|u\| \tag{4.15}$$

then we can choose $\bar{t}$ such that

$$\eta \leq \bar{t} \leq 1 \tag{4.16}$$

(we note that if $\bar{t} \in [0,1] \setminus [\eta, 1]$, then the point $(\eta, u(\eta))$ is below the straight line determined by $(1, u(1))$ and $(\bar{t}, u(\bar{t})$ This contradicts the concavity of u. From (4.14) and the concavity of u, we know that

$$\min_{t \in [\eta,1]} u(t) = u(\eta) \tag{4.17}$$

Using the concavity of u and Lemma 4.2, we have that

$$\frac{u(\eta)}{\eta} \geq \frac{u(\bar{t})}{\bar{t}} \tag{4.18}$$

This implies

$$\min_{t \in [\eta,1]} u(t) \geq \eta \|u\|. \tag{4.19}$$

This completes the proof.

Corollary 4.1. *Assume(As1) and (As2) hold. If A is given by*

$$\begin{aligned} Ay(t) = &- \int_0^t (t-s)a(s)f(y(s))ds - \frac{\alpha t}{1-\alpha\eta} \int_0^\eta (\eta - s)a(s)f(y(s))ds \\ &+ \frac{t}{1-\alpha\eta} \int_0^1 (1-s)a(s)f(y(s))ds, \qquad (*) \end{aligned}$$

and let a cone K in C[0,1] be

$$K := \{y/y \in C[0,1],\ y \geq 0,\ \min_{\eta \leq t \leq 1} y(t) \geq \gamma \|y\|\}$$

Then A: K $\rightarrow$ K is completely continuous.

Proof. Let $\phi, \psi \in C[0,1]$. In view of (As1), given an $\epsilon > 0$ there exists a $\delta > 0$ such that for $\|\phi - \psi\| < \delta$ we have

$$\sup_{t \in [0,1]} \Big| f(\phi) - f(\psi) \Big| < \frac{\epsilon}{P[2 + \alpha(1-\eta)]} \ , \quad where \ \ P = \int_0^1 \frac{(1-s)a(s)}{1-\alpha\eta} ds$$

Using (*) we have for t ∈ (0,1),

$$\begin{aligned}\Big|(A\phi)(t)-(A\psi)(t)\Big| &\le \int_0^1 (1-s)a(s)\Big|f(\phi(s))-f(\psi(s))\Big|ds \\ &+ \frac{\alpha}{1-\alpha\eta}\int_0^1 (1-s)a(s)\Big|f(\phi(s))-f(\psi(s))\Big|ds \\ &+ \frac{1}{1-\alpha\eta}\int_0^1 (1-s)a(s)\Big|f(\phi(s))-f(\psi(s))\Big|ds \\ &\le [(1-\alpha\eta)P+\alpha P+P]\Big|f(\phi(s))-f(\psi(s))\Big|ds \\ &\le P[\alpha+\alpha(1-\eta)]\sup_{t\in[0,1]}\Big|f(\phi)-f(\psi)\Big| \\ &< \epsilon\end{aligned}$$

Thus, A is continuous. Also, by Lemma 4.4 AK ⊂ K. Thus, we have shown that A: K → K. Next, we show that f maps bonded sets into bounded sets. Let D be a positive constant and define the set

$$S = \{x \in C[0,1] : \|x\| \le D\}.$$

Since (As1) holds, for any x,y ∈ S, there exists a $\delta > 0$ such that if $\|x-y\| < \delta$, implies $\Big|f(x)-f(y)\Big| < 1$. We choose a positive integer N so that $\delta > \frac{D}{N}$.

For y(t)∈ C[0,1], define $y_j(t) = \frac{jy(t)}{N}$, for j = 0,1,2,...,N. For y∈ S,

$$\begin{aligned}\|y_j - y_{j-1}\| &= \sup_{t\in[0,1]}\Big|\frac{jy(t)}{N} - \frac{(j-1)y(t)}{N}\Big| \\ &\le \frac{\|y\|}{N} \le \frac{D}{N} < \delta.\end{aligned}$$

Thus, $\Big|f(y_j)-f(y_j-1)\Big| < 1$. As a consequence, we have

$$f(y)-f(0) = \sum_{j=1}^{N}(f(y_j)-f(y_j-1)),$$

which implies that

$$\begin{aligned}\Big|f(y)\Big| &\le \sum_{j=1}^{N}\Big|f(y_j)-f(y_j-1)\Big| + \Big|f(0)\Big| \\ &< N + \Big|f(0)\Big|.\end{aligned}$$

Thus, f maps bounded sets into bounded sets. It follows from the above inequality and (*),that

$$\begin{aligned} \|(Ay)(t)\| &\le \frac{t}{1-\alpha\eta}\int_0^1 (1-s)a(s)\Big|f(y(s))\Big|ds \\ &\le \frac{1}{1-\alpha\eta}\int_0^1 (1-s)a(s)(N+\Big|f(0)\Big|) \\ &\le P(N+\Big|f(0)\Big|). \end{aligned}$$

Next, for t $\in (0,1)$, we have

$$\begin{aligned} (Ay)'(t) = &-\int_0^t a(s)f(y(s))ds - \frac{\alpha}{1-\alpha\eta}\int_0^\eta (\eta-s)a(s)f(y(s))ds \\ &+ \frac{1}{1-\alpha\eta}\int_0^1 (1-s)a(s)f(y(s))ds. \end{aligned}$$

Hence,

$$\begin{aligned} \Big|(Ay)'(t)\Big| &\le \frac{1}{1-\alpha\eta}\int_0^1 (1-s)a(s)\Big|f(y(s))\Big|ds \\ &\le P(N+\Big|f(0)\Big|). \end{aligned}$$

Thus, the set

$$\{(Ay) \;:\; y \in K, \|y\| \le D\}$$

is a family of uniformly bounded and equicontinuous functions on the set t $\in [0,1]$. By Ascoli-Arzela Theorem, the map A is completely continuous. This completes the proof. □

Chapter 5

Proof of Main Theorem

5.1 Proof of Theorem 2.1

In this section the proof of theorem 2.1 will be given, as stated earlier. The proof totally depends on lemmas given in previous section.
To prove the theorem we have to consider two cases. The first case is superlinear case and the second case is sublinear case.
i. Superlinear case. Suppose that $f_0 = 0$ and $f_\infty = \infty$.
We want to show the existence of positive solution of (2.1)-(2.2).
Now (2.1)-(2.2) has a solution $y = y(t)$ if and only if y solves the operator equation

$$\begin{aligned} y(t) &= -\int_0^t (t-s)a(s)f(y(s))ds - \frac{\alpha t}{1-\alpha\eta}\int_0^\eta (\eta - s)a(s)f(y(s))ds \\ &\quad + \frac{t}{1-\alpha\eta}\int_0^1 (1-s)a(s)f(y(s))ds \qquad (5.1) \\ &=: Ay(t) \quad \text{where, } A(1-\alpha\eta) = -\alpha\int_0^\eta (\eta - s)a(s)ds + \int_0^1 (1-s)a(s)ds. \end{aligned}$$

Denote

$$K = \{y/y \in C[0,1],\ y \geq 0,\ \min_{\eta \leq t \leq 1} y(t) \geq \gamma \|y\|\}. \tag{5.2}$$

We know that k is a cone in $C[0,1]$. Moreover, by lemma 4.4, $AK \subset K$. From corollary 4.1 we have also that $A : K \to K$ is completely continuous. Now since f_0 =0, we may choose $H_1 > 0$ so that $f(y) \leq \epsilon y$, for $0 < y < H_1$, where $\epsilon > 0$ satisfies

$$\frac{\epsilon}{1-\alpha\eta}\int_0^1 (1-s)a(s)ds \leq 1. \tag{5.3}$$

Thus, if $y \in K$ and $\|y\| = H_1$, then from (5.1) and (5.3) we get

$$\begin{aligned} Ay(t) &\leq \frac{t}{1-\alpha\eta}\int_0^1 (1-s)a(s)f(y(s))ds \\ &\leq \frac{t}{1-\alpha\eta}\int_0^1 (1-s)a(s)\epsilon y(s)ds \\ &\leq \frac{\epsilon}{1-\alpha\eta}\int_0^1 (1-s)a(s)ds\|y\| \\ &\leq \frac{\epsilon}{1-\alpha\eta}\int_0^1 (1-s)a(s)dsH_1. \end{aligned} \tag{5.4}$$

Now if we get

$$\Omega_1 = \{y \in C[0,1] \;\mid\; \|y\| < H_1\}, \tag{5.5}$$

then (5.4) shows that $\|Ay\| \leq \|y\|$, for $y \in K \cap \partial\Omega_1$.
Further, since $f_\infty = \infty$, there exists $\hat{H}_2 > 0$ such that $f(u) \geq \rho u$, for $u \geq \hat{H}_2$, where $\rho > 0$ is chosen so that

$$\rho\frac{\eta\gamma}{1-\alpha\eta}\int_\eta^1 (1-s)a(s)ds \geq 1. \tag{5.6}$$

Let $H_2 = \max\{2H_1, \frac{\hat{H}_2}{\gamma}\}$ and $\Omega_2 = \{y \in C[0,1] \;\mid \|y\| < H_2\}$, then $y \in K$ and $\|y\| = H_2$ implies

$$\min_{\eta \leq t \leq 1} y(t) \geq \gamma\|y\| \geq \hat{H}_2,$$

and so

$$
\begin{aligned}
Ay(\eta) &= -\int_0^\eta (\eta - s)a(s)f(y(s))ds - \frac{\alpha\eta}{1-\alpha\eta}\int_0^\eta (\eta - s)a(s)f(y(s))ds \\
&\quad + \frac{\eta}{1-\alpha\eta}\int_0^1 (1-s)a(s)f(y(s))ds \\
&= -\frac{1}{1-\alpha\eta}\int_0^\eta (\eta - s)a(s)f(y(s))ds + \frac{\eta}{1-\alpha\eta}\int_0^1 (1-s)a(s)f(y(s))ds \\
&= -\frac{1}{1-\alpha\eta}\int_0^\eta \eta a(s)f(y(s))ds + \frac{1}{1-\alpha\eta}\int_0^\eta sa(s)f(y(s))ds \\
&\quad + \frac{\eta}{1-\alpha\eta}\int_0^1 a(s)f(y(s))ds - \frac{\eta}{1-\alpha\eta}\int_0^1 sa(s)f(y(s))ds \qquad (5.7) \\
&= \frac{\eta}{1-\alpha\eta}\int_\eta^1 a(s)f(y(s))ds + \frac{1}{1-\alpha\eta}\int_0^\eta sa(s)f(y(s))ds \\
&\quad - \frac{\eta}{1-\alpha\eta}\int_0^1 sa(s)f(y(s))ds \\
&\geq \frac{\eta}{1-\alpha\eta}\int_\eta^1 a(s)f(y(s))ds - \frac{\eta}{1-\alpha\eta}\int_\eta^1 sa(s)f(y(s))ds \\
&= \frac{\eta}{1-\alpha\eta}\int_\eta^1 (1-s)a(s)f(y(s))ds.
\end{aligned}
$$

Hence, for $y \in K \cap \partial\Omega_2$,

$$\|Ay\| \geq \rho\frac{\eta\gamma}{1-\alpha\eta}\int_\eta^1 (1-s)a(s)ds\|y\| \geq \|y\|.$$

Therefore, by the first part of the Krasnoselskii's Fixed point Theorem, it follows that A has a fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$, such that $H_1 \leq \|u\| \leq H_2$. This completes the superlinear part of the theorem.

ii. Sublinear case. Suppose that $f_0 = \infty$ and$f_\infty = 0$.

We first choose $H_3 > 0$ such that $f(y) \geq My$ for $0 < y < H_3$,
where

$$M\gamma\big(\frac{\eta}{1-\alpha\eta}\big)\int_\eta^1 (1-s)a(s)ds \geq 1. \qquad (5.8)$$

By using the method together (5.7), we can get that

$$\begin{aligned} Ay(\eta) &= -\int_0^\eta (\eta - s)a(s)f(y(s))ds - \frac{\alpha\eta}{1-\alpha\eta}\int_0^\eta (\eta - s)a(s)f(y(s))ds \\ &\quad + \frac{\eta}{1-\alpha\eta}\int_0^1 (1-s)a(s)f(y(s))ds \\ &\geq \frac{\eta}{1-\alpha\eta}\int_\eta^1 (1-s)a(s)f(y(s))ds \qquad (5.9) \\ &\geq \frac{\eta}{1-\alpha\eta}\int_\eta^1 (1-s)a(s)My(s)ds \\ &\geq \frac{\eta}{1-\alpha\eta}\int_\eta^1 (1-s)a(s)M\gamma ds\|y\| \\ &\geq H_3. \end{aligned}$$

Thus, we let $\Omega_3 = \{y \in C[0,1] \mid \|y\| < H_3\}$ so that

$$\|Ay\| \geq \|y\|, \quad y \in K \cap \partial\Omega_3.$$

Now, since $f_\infty = 0$, there exists $\hat{H}_4 > 0$ so that $f(y) \leq \lambda y$ for $y \geq \hat{H}_4$, where $\lambda > 0$ satisfies

$$\frac{\lambda}{1-\alpha\eta}\Big[\int_0^1 (1-s)a(s)ds\Big] \leq 1 \qquad (5.10)$$

we consider two cases:

Case(a). Suppose f is bounded, say $f(y) \leq N$ for all $y \in [0,\infty)$.
In this case choose $H_4 = \max\{2H_3, \frac{N}{1-\alpha\eta}\int_0^1 (1-s)a(s)ds\}$
so that for $y \in K$ with $\|y\| = H_4$ we have

$$\begin{aligned} Ay(t) &= -\int_0^t (t-s)a(s)f(y(s))ds - \frac{\alpha t}{1-\alpha\eta}\int_0^\eta (\eta - s)a(s)f(y(s))ds \\ &\quad + \frac{t}{1-\alpha\eta}\int_0^1 (1-s)a(s)f(y(s))ds \\ &\leq \frac{t}{1-\alpha\eta}\int_0^1 (1-s)a(s)f(y(s))ds \\ &\leq \frac{1}{1-\alpha\eta}\int_0^1 (1-s)a(s)Nds \\ &\leq H_4. \end{aligned}$$

and therefore $\|Ay\| \leq \|y\|$.

Case(b). If f is unbounded, then we can understand from (A1) that there is $H_4 : H_4 > \max\{2H_3, \frac{1}{\gamma}\hat{H}_4\}$ such that

$$f(y) \le f(H_4) \quad \text{for} \quad 0 < y \le H_4.$$

(we are able to do this since f is unbounded). Then for $y \in K$ and $\|y\| = H_4$ we have

$$\begin{aligned} Ay(t) &= -\int_0^t (t-s)a(s)f(y(s))ds - \frac{\alpha t}{1-\alpha\eta}\int_0^\eta (\eta - s)a(s)f(y(s))ds \\ &\quad + \frac{t}{1-\alpha\eta}\int_0^1 (1-s)a(s)f(y(s))ds \\ &\le \frac{t}{1-\alpha\eta}\int_0^1 (1-s)a(s)f(H_4)ds \\ &\le \frac{1}{1-\alpha\eta}\int_0^1 (1-s)a(s)\lambda H_4 ds \\ &\le H_4. \end{aligned}$$

Therefore, in either case we put
$\Omega_4 := \{y \in C[0,1] \mid \|y\| < H_4\}$,
and for $y \in K \cap \partial\Omega_4$ we have $\|Ay\| \le \|y\|$. By the second part of the Krasnoselskii's Fixed Point Theorem, it follows that BVP (2.1)-(2.2) has a positive solution. Therefore, we have completed the proof of Theorem 2.1.

5.2 Prototypical Example

Example 5.2.1. *Consider the following boundary value problem*

$$u'' + (1-t)u^2 = 0, \quad t \in (0,1) \tag{5.11}$$

$$u(0) = 0, \quad \alpha u(\eta) = u(1),$$

where $0 < \eta < 1$ *and* $0 < \alpha < \frac{1}{\eta}$*. Now we need to check that weather the conditions are satisfied or not. To start checking,*

1. *let* $f(u) = u^2$, *then we know that* $f \in C([0,\infty),[0,\infty))$
2. *let* a = *1-t, then* $a \in C([0,1],[0,\infty))$*, there exist* $x_0 \in [\eta, 1]$ *such that* $ax_0 > 0$.
3. *since* $f(u) = u^2$, *we have* $f_0 = 0$ *and* $f_\infty = \infty$ *(superlinear case) , because* $\lim_{u\to 0} \frac{f(u)}{u} = \lim_{u\to 0} \frac{u^2}{u}$ *=0 and* $\lim_{u\to\infty} \frac{f(u)}{u} = \lim_{u\to\infty} \frac{u^2}{u} = \infty$

Hence, a function u(t) is positive on $0 < t < 1$.

Example 5.2.2. *Consider the following boundary value problem*

$$u'' + (1-t^2)e^{-u} = 0, \quad t \in (0,1) \tag{5.12}$$

$$u(0) = 0, \quad \alpha u(\eta) = u(1),$$

where $0 < \eta < 1$ *and* $0 < \alpha < \frac{1}{\eta}$*. Now we need to check that weather the conditions are satisfied or not, just like that of what we have done in the above example. To start checking,*

1. *let* $f(u) = e^{-u}$, *then we have* $f \in C([0,\infty),[0,\infty))$
2. *let* $a = 1-t^2$, *then* $a \in C([0,1],[0,\infty))$*, there exist* $x_0 \in [\eta, 1]$ *such that* $ax_0 > 0$.
3. *since* $f(u) = e^{-u}$, *we have* $f_0 = \infty$ *and* $f_\infty = 0$ *(sublinear case) , because*

$$\begin{aligned}\lim_{u\to 0} \frac{f(u)}{u} &= \lim_{u\to 0} \frac{e^{-u}}{u} \\ &= \lim_{u\to 0} \frac{1}{ue^u} \\ &= \infty\end{aligned}$$

and

$$\lim_{u\to\infty}\frac{f(u)}{u} = \lim_{u\to\infty}\frac{e^{-u}}{u}$$
$$= \lim_{u\to\infty}\frac{1}{ue^{u}} = 0$$

Thus, a function u(t) is positive on $0 < t < 1$.

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