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CHROMATIC POLYNOMIAL

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A Project submitted in partial fulfillment of the requirement of the degree of master of science in mathematics

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May, 2018

Addis Ababa, Ethiopia

ADDIS ABABA UNIVERSITY
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The undersigned hereby certify that they have read and recommend to the department of mathematics for acceptance of this project entitled **"Chromatic Polynomials"** by **Abebu Tamiru** in partial fulfillment of the requirements for the degree of Master of Science in Combinatorics.

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Degree: M.Sc.

Convocation: May

Year: 2018

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Acknowledgement

First of all I would like to thank my God for his blessing and wisdom for without his grace this work could have not been accomplished.

Nextly I would like to thank my advisor Dr.Yirgalem Tsegaye for guiding, giving me supporting materials and directing the success of this project. And also I want to thank my families, specially my husband Ayele Regasa for his support in every part of my life.

Also I want to thanks Addis Ababa university and all staff members of Mathematics Department at Addis Ababa university for helping me during my study. And also I want to thanks to Debre Tabor University for its financial support in attending the degree of masters of Science. Lastly but not least, I would like to thank Mesfin Masre for his support by giving me necessary materials and to all my friends that I haven't mentioned here, for their encouragement and support.

Abstract

In this project, based on articles published by(Coudy Fouts), we see how Incidence Algebra in particular, Möbius Inversion Theorem is used to compute the Chromatic Polynomials of graphs. This is one of the various applications of Möbius Inversion Theorem.

Introduction

A graph can be colored by assigning a different color to each of its vertices. However, for most graphs a coloring can be found that uses fewer colors than the number of vertices in the graph. A vertex coloring of a graph is proper if no two adjacent vertices have the same color. This is called a **proper coloring** of a graph. Furthermore, we want to count the possible number of different proper coloring on a graph with a given number of colors. We can compute each of these values by using special function that is associated with each graph called **Chromatic Polynomial**.

For simple graph Chromatic polynomial can be determined by examining the structure of the graph. For other graphs, it is very difficult to determine Chromatic polynomial by examining the structure of the graph.

Utilizing bond lattices and a special theorem called the Möbius Inversion Theorem, we determine an algorithm for calculating the Chromatic Polynomial for some graph.

The content of this work is given below.

In **chapter 1** the basics of graph theory and partially ordered set will be introduced . In**chapter 2** chromatic polynomial will be introduced and the chromatic polynomials of some classes of graphs will be computed. We will also see some techniques, deletion-contraction that allow us compute the chromatic polynomials of the graphs.

All graphs considered in this project are finite, undirected, and simple graphs a coloring always refers to vertex coloring.

Chapter 1

Preliminaries

1.1 Basic concepts in graph theory

Definition 1.1.1. *A graph G is a pair of sets (V, E), where V is a nonempty set of elements called vertices, and E is called an edge set, which is a set of two element subsets of V .*

Definition 1.1.2. *Let $G = (V, E)$ be a graph. Two vertices v_1 and v_2 are said to be adjacent if there exists $e = v_1v_2$ in G. An isolated vertex is a vertex that has no other vertices adjacent to it.*

Definition 1.1.3. *The complement of a graph G denoted by $\bar{G}$ is a graph with $V(\bar{G}) = V(G)$ but $uv \in E(\bar{G})$ if and only if $uv \notin E(G)$.*

Definition 1.1.4. *Let $G = (V, E)$ be a graph and let $v \in V$. The neighbor of v denoted by $N(v)$ is the set of vertices that are adjacent to v.*
That is, $N(v) = \{u \in V : \exists\ e \in E \text{ where } e \text{ is edge between } u \text{ and } v\}$.

Definition 1.1.5. *The order of a graph G is the number of vertices, denoted by $|G|$ and the size of a graph G is the number of edges, denoted by $||G||$.*

Definition 1.1.6. *Let $G = (V, E)$ be a graph and let $v \in V$. The degree of v in G, written $deg_G(v)$ is the number of vertices adjacent to v.*
That is, $deg_G(v) = |N(v)|$.

Definition 1.1.7. *A loop is an edge whose end points are equal i.e., an edge joining a vertex to itself is called a loop. We say that the graph has multiple edges if in the graph two or more edges joining the same pair of vertices.*

Definition 1.1.8. *A graph with no loops or multiple edges is called a simple graph.*

Definition 1.1.9. *A graph G is connected if there is a path in G between any given pair of vertices, otherwise it is disconnected. Every disconnected graph can be split up into a number of connected subgraphs called components.*

Definition 1.1.10. *Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A subgraph of G is a graph all of whose vertices belongs to $V(G)$ and all of whose edges belong to $E(G)$.*

Definition 1.1.11. *A graph is regular if all the vertices of G have the same degree. In particular, if the degree of each vertex is r,then G is regular of degree r or an r-regular graph.*

1.2 Some example of graphs

- **Null Graphs**

  Definition 1.2.1. *A null graph is one in which the edge set is empty. A null graph of n vertices is denoted by N_n.*
  NOTE:N_n is a 0- regular graph.

- **Complete Graphs**

  Definition 1.2.2. *A complete graph is a simple graph in which each pair of distinct vertices are adjacent. Complete graphs on n vertices are denoted by K_n.*

- **Path Graphs**

  Definition 1.2.3. *A path graph is a simple graph whose vertices can be ordered on a line so that adjacency means consecutiveness in the line. A path graph of n vertices is denoted P_n.*

- **Cycle Graphs**

  Definition 1.2.4. *A cycle graph is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle. A cycle graph of order n is denoted by C_n.*

- **Bipartite Graph**

  Definition 1.2.5. *A bipartite graph is a graph whose vertex-set can be split into two sets in such a way that each edge of the graph joins a vertex in the first set to a vertex in the second set.*

- **Complete Bipartite Graphs**

  Definition 1.2.6. *A complete bipartite graph is a simple bipartite graph in which each vertex in the first set is adjacent to each vertex in the second set. A complete bipartite graph G with a bipartition (x, y) ,where $|x| = r$, $|y| = s$ is denoted by $K_{r,s}$.*

  NOTE:A complete bipartite graph of the form $K_{1,s}$ is called a star graph.

Definition 1.2.7. *A tree is a connected graph which has no cycles.*

1.3 Graph coloring and Chromatic Number

A graph can be colored by assigning a different color to each of its vertices. However, for most graphs a coloring can be found that uses fewer colors than the number of vertices in the graph.

Definition 1.3.1. *Graph coloring, or more specifically vertex coloring means the assignment of colors to the vertices of a graph in such a way that no two adjacent vertices share the same color.*

A coloring of a graph is the result of giving to each node(vertex) of the graph one of a specified
set of colors.

Definition 1.3.2. *Given a simple graph $G = (V, E)$, a k-coloring of the vertices of G is an assignment of one of k given colors to each of the vertices of G.*
A vertex coloring of a graph is proper if no two adjacent vertices have the same color.

Definition 1.3.3. *A graph G is called k-colourable if it has a proper k- coloring.*

Definition 1.3.4. *The chromatic number of a graph is the least number of colors needed for a coloring of the graph. and denoted by $\chi(G)$.*

There is no general formula for the chromatic number of a graph.
Remark:

1. $\chi(N_n) = 1$
2. $\chi(P_n) = 2$
3. $\chi(C_n)$ is

$$\begin{cases} 2, & \text{if } n \textit{ is even} \\ 3, & \text{if } n \textit{ is odd} \end{cases}$$

4 . The chromatic number of complete graph of order n is n.

1.4 Partially Ordered Set

Definition 1.4.1. *A partially ordered set S(or poset, for short) is a set, together with a binary relation denoted $\preceq$ satisfying the following three axioms:*

1. *For all $x \in S$, $x \preceq x$ (reflexivity)*
2. *If $x \preceq y$ and $y \preceq x$, then $x = y$ (antisymmetry)*

3. If $x \preceq y$ and $y \preceq z$, then $x \preceq z$ (transitivity)

Definition 1.4.2. *Let S be partially ordered set. Then S is locally finite if every interval $[a, b] := \{x \in S : a \leq x \leq b\}$ contains a finite number of elements.*

Definition 1.4.3. *Let S be partially ordered set. Then S is locally infinite if every interval $[a, b] := \{x \in S : a \leq x \leq b\}$ contains an infinite number of elements.*

Example 1.1. *An open interval, $(1, 2)$ over a set $\mathbb{R}$ is locally infinite poset.*

Example 1.2. *The usual "greater than or equal" relation $(\geq)$ is a partial ordering on the set of integers. Because $a \geq a$ for every integer a, " $\geq$ " is reflexive. If $a \geq b$ and $b \geq a$, then $a = b$. Hence," $\geq$ " is antisymmetric.*
Finally, " $\geq$ " is transitive because $a \geq b$ and $b \geq c$ imply that $a \geq c$.
It follows that " $\geq$ " is a partial ordering on the set of integers and $(\mathbb{Z}, \geq)$ is a poset.

Example 1.3. *Inclusion relation " $\subseteq$ " is a partial ordering on the power set of a set S. Because $A \subseteq A$ whenever A is a subset of S, " $\subseteq$ " is reflexive.*
It is antisymmetric since $A \subseteq B$ and $B \subseteq A$ imply that $A = B$.
Finally, " $\subseteq$ " is transitive, because $A \subseteq B$ and $B \subseteq C$
$\Rightarrow A \subseteq C$.
Hence, " $\subseteq$ " is a partial ordering on power set of a set S is a poset.

Definition 1.4.4. *The elements a and b of a partially ordered set $(S, \leq)$ are called comparable if either $a \leq b$ or $b \leq a$. When a and b are elements of S such that neither $a \leq b$ nor $b \leq a$,a and b are called incomparable.*

Definition 1.4.5. *If for every two elements w and z in partially ordered set either $w \leq z$ is true or $z \leq w$ is true then partially ordered set is called a linearly ordered set or chain or totally ordered.*

When every two elements in the set are comparable, the relation is called a total ordering or linear ordering.

Example 1.4. *The poset $(\mathbb{Z}, \leq)$ is linearly ordered, because $a \leq b$ or $b \leq a$ whenever a and b are integers.*

An interval $[u, v]$ is the set of all elements between u and v.
Thus $[u, v] = \{t \in S : u \leq t \leq v\}$.

Definition 1.4.6. *Suppose Q is a partially ordered set and let q and r be elements of Q. If x is another element of Q such that $x \leq q$ and $x \leq r$, then x is called a* ***lower bound*** *of q and r. If v is a lower bound of q and r such that $x \leq v$ for all other lower bounds x, then v is the greatest lower bound or* ***meet*** *of q and r.*

Similarly, an element y such that $q \leq y$ and $r \leq y$ is called an upper bound of q and r. If u is an element such that $u \leq y$ for all other upper bounds y, then u is the **least upper bound** or **join** of q and r.

Remark: If $x < y$ and there is no z for which $x < z < y$ then we say y covers x. The Hasse diagram of a finite poset P is the graph whose vertices are the elements of P, whose edges are the cover relations, and such that if $x < y$ then y is drawn above x.

Definition 1.4.7. *A partially order set with the property that every pair of elements has a greatest lower bound and least upper bound is called a lattice.*

Example 1.5. *Let D_{12} be the set of all divisors of 12 under the relation divides is a lattice.This can be represented in the following figure.*

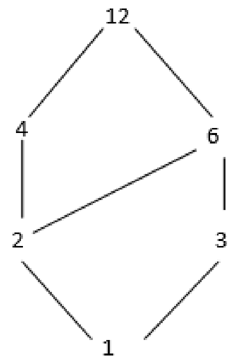


Figure 1.1: The divisor lattice D_n,for $n = 12$

.

Definition 1.4.8. *A partition of a set R is a set of subsets of R which are disjoint and whose union is R.*
Each element of a partition is known as a part.

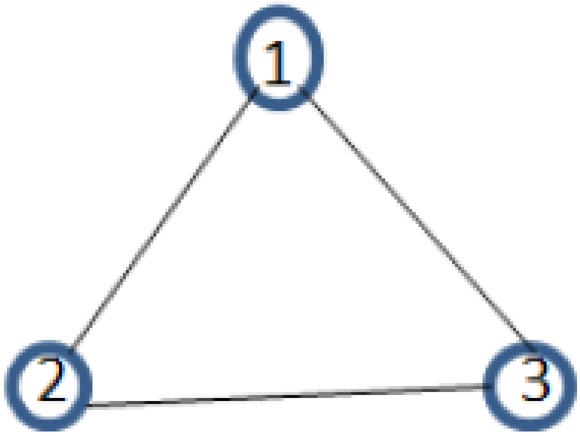
As an example, let $R = \{1, 2, 3, 4\}$.Two different partitions of R which are labeled as P and Q , let $P = \{\{1, 2\}, \{3\}, \{4\}\}$ and $Q = \{\{1, 2, 3\}, \{4\}\}$. The parts of P are $\{1, 2\}, \{4\}, \{3\}$ and $\{1, 2, 3\}, \{4\}$ are parts of Q.
Given partitions P and Q of a set R, we can define a relationship between them in which we say that P is finer than Q if every subset, or part, in P is a subset of a subset (part) in Q, where $P \neq Q$. We denote this relationship by $P \prec Q$. Also, within this relationship we say that Q is coarser than P. In our example, $P \prec Q$ because $\{1, 2\} \subseteq \{1, 2, 3\}$, $\{3\} \subseteq \{1, 2, 3\}$, and $\{4\} \subseteq \{4\}$.

This relationship is known as the Refinement ordering on the partitions of a set. For the purpose of partially ordered sets we use to compute the chromatic polynomial,we will allow the Refinement ordering to be $\preceq$ and we can have $P \preceq P$.

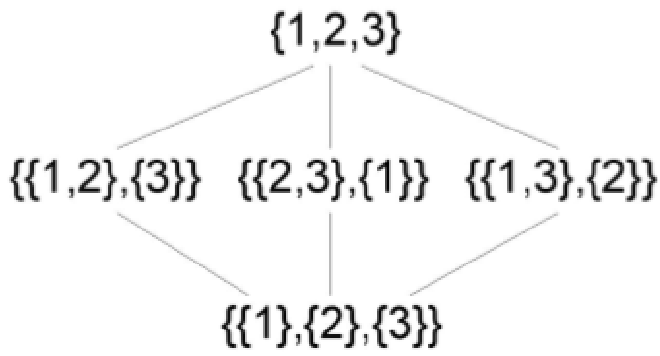
Definition 1.4.9. *A bond of a graph G is a partition of its vertices such that all vertices in the same part are connected within the graph G. The set of the bonds of a graph form the* ***bond lattice*** *of the graph.*

Given all possible partitions of a set R, we can create a partition lattice which organizes the partitions based on the relationship $\prec$,with the "finest" partitions at the bottom of the lattice and the "coarsest" at the top.

Example 1.6. *Let $V = \{1, 2, 3\}$ be vertex set of the graph below.*



Then all possible partitions of $V(G)$ are $\{\{1\}, \{2\}, \{3\}\}, \{\{1, 2\}, \{3\}\}, \{\{1, 3\}, \{2\}\}, \{\{2, 3\}, \{1\}\}$ and $\{\{1, 2, 3\}\}$
We can create bond lattice which organizes the partitions based on the relationship $\prec$,with the "finest" partitions at the bottom of the lattice and the "coarsest" at the top.Thus the finest partition is $\{\{1\}, \{2\}, \{3\}\}$ and the coarsest partition is $\{\{1, 2, 3\}\}$
Therefore bond lattice of the graph will be as the following.



1.5 Incidence Algebra

Consider a partially ordered set P. A function f defined on P maps the direct product of the elements of P to $\mathbb{R}$, the set of real numbers we can add and multiply these function forming an algebra which refers to as the Incidence Algebra. For most functions in this Incidence Algebra, $f(x, y) = 0\ if\ x \nleq y\ in\ P$.

1.5.1 Zeta Function and Möbius Function

Definition 1.5.1. *The most basic of these functions is Zeta Function some times referred to as the Indicator Function, it indicates whether or not $x \leq y$ in the set P.*
The zeta Function defined as:

$$\zeta(x, y) = \begin{cases} 1, & if\ \ x \leq y \\ 0, & if\ \ x \nleq y \end{cases} \tag{1.1}$$

Definition 1.5.2. *Let P be partially ordered set. A function defined on P called Möbius function of P for any two elements x, y of P,is defined as*

$$\mu(x, y) = \begin{cases} 1, & if\ \ x = y \\ 0, & if\ \ x \nleq y \\ -\sum_{z:x \leq z < y} \mu(x, z), & if\ \ x < y \end{cases} \tag{1.2}$$

If P is the set of integers with the usual ordering then, the Möbius function defined over p in the following manner:

$$\mu(n, k) = \begin{cases} 1, & \text{when} \ \ n = k \\ -1, & \text{when} \ \ n + 1 = k \\ 0, & \text{otherwise} \end{cases} \tag{1.3}$$

1.5.2 Kronecker delta

Definition 1.5.3. *If x and y are elements of a partially ordered set P, the Kronecker delta denoted by $\delta(x, y)$,is defined as follows.*

$$\delta(x,y) = \begin{cases} 1, & if \ \ x = y \\ 0, & if \ \ x \neq y \end{cases} \tag{1.4}$$

Note:Möbius function is the inverse of Zeta function.

The product of the Möbius function and Zeta function over the interval $[a, d]$ is

$$\delta(a,d) = \sum_{a \leq x \leq d} \mu(a,x)\zeta(x,d)$$

.

For all x such that $x \leq d$, $\zeta(x, d) = 1$ also $\delta(a, d) = 0$ because $a \neq d$.
Thus we have the following:

$$0 = \delta(a,d) = \sum_{a \leq x \leq d} \mu(a,x)1 = \sum_{a \leq x \leq d} \mu(a,x)$$

$$\Rightarrow \sum_{a \leq x \leq d} \mu(a,x) = 0$$

1.5.3 Möbius Inversion Theorem

The principle of Möbius Inversion is a significant module of the method of computing chromatic polynomials using bond lattices and the Möbius function.

Theorem 1.1. *Let $N_e(x)$ (read "N sub equal to") be a real-valued function defined for all x in a locally finite partially ordered set $(S, \leq)$ and assume that there is an element $m \in S$ such that $N_e(x) = 0$ when $x \nleq m$. Define $N_a(x)$ (read "N sub at least") by*

$$N_a(x) = \sum_{y:y \geq x} N_e(y).$$

Then

$$N_e(x) = \sum_{y:y \geq x} \mu(x,y) N_a(y).$$

Proof. We have $N_a(x) = \sum_{y:y\geq x} N_e(y) = \sum_{y:x\leq y\leq m} N_e(y)$ because $N_e(y) = 0$ for all $y > m$. Since our partially ordered set is locally finite this sum is finite. Next, we substitute this into the right side of the equation for $N_e(x)$.

$$\sum_{y:y\geq x} N_a(y)\mu(x,y) = \sum_{x\leq y}\sum_{y\leq z\leq m} N_e(z)\mu(x,y)$$

because, $N_a(y) = \sum_{z:z\geq y} N_e(z)$

$$\sum_{x\leq y}\sum_{y\leq z\leq m} N_e(z)\mu(x,y) = \sum_{x\leq y}\sum_{z} N_e(z)\zeta(y,z)\mu(x,y)$$

because, $\zeta(y,z) = 0$ if $y \nleq z$ and $N_e(z) = 0$ if $z \nleq m$. Rearranging the summands, we find

$\sum_{x\leq y}\sum_{z} N_e(z)\zeta(y,z)\mu(x,y) = \sum_{z} N_e(z)\sum_{x\leq y}\zeta(y,z)\mu(x,y)$

However, we know $\sum_{x\leq y}\zeta(y,z)\mu(x,y) = \delta(x,z)$. Thus

$\sum_{z} N_e(z)\sum_{x\leq y}\zeta(y,z)\mu(x,y) = \sum_{z} N_e(z)\delta(x,z)$
$\delta(x,z) = 1$ when $z = x$ and 0 otherwise, and we have

$$\sum_{z} N_e(z)\delta(x,z) = \sum_{x} N_e(x)$$

$$\Rightarrow N_e(x) = \sum_{y:y\geq x}\mu(x,y)N_a(y)$$

□

Chapter 2

Chromatic polynomials

In this chapter, we introduce the concept of chromatic polynomial and we calculate the chromatic polynomial for some special classes of graphs using partition lattice and Möbius Inversion Theorem and introduce some basic properties of chromatic polynomials and also we show that these chromatic polynomials are equivalent to the identities obtained from the auxiliary methods like deletion-contraction algorithm.

Definition 2.0.4. *Let k be the number of colors available for a graph G. Then chromatic polynomial of the graph G denoted by $P_G(k)$ or $P(G,k)$, is the number of ways we can color the graph with atmost k colors.*
Formally, the chromatic Polynomial of a graph G is a special function that describes the number of ways we can achieve a proper coloring on a graph G given k- colors.

2.1 Chromatic polynomials of some graphs

In this section, we shall enumerate $P(G,k)$ for some special graph G, for illustration purposes.

proposition 2.1. *The Chromatic polynomial of the empty (null) graph on n vertices is $P(N_n,k) = k^n$.*

Proof. Let $k \in N$ and N_n be empty graph. Each of the n vertices can be independently colored using any of the k colors, which gives a total of k^n possibilities, by multiplication principle.
Hence $P(N_n,k) = k^n$. □

Example 2.1. *Let N_3 be null graph of order 3.*
If we want to color this graph with k- colors, we have k color options for each vertex since no vertex is adjacent to another as shown in figure below.

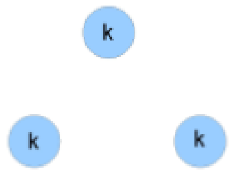


Figure 2.1: Calculating the chromatic function of N_3

Therefore, the number of ways we can achieve a proper coloring on the vertices of N_3 is k^3.
i.e $P(N_3, k) = k^3$.

proposition 2.2. *The Chromatic polynomial of tree graph on n vertices is* $P(T_n, k) = k(k-1)^{n-1}$.

Proof. Let $k \in N$ and T_n be tree graph.
We start with an end vertex and note that this vertex can be colored in k ways. As we move across the graph to the right, each successive vertex can be colored $(k-1)$ ways as it cannot be the same color as the vertex to its left which gives a total of $k(k-1)^{n-1}$ possibilities.
Hence $P(T_n, k) = k(k-1)^{n-1}$. □

Example 2.2. *Let P_3 be path graph of 3 vertices.*
we start with an end vertex and note that this vertex can be colored in k ways.
As we move across the graph to the right, each successive vertex can be colored $(k-1)$ ways as it cannot be the same color as the vertex to its left as we see in the figure below.

Figure 2.2: chromatic function of P_3

Thus, P_3 can be colored $k(k-1)^2$ ways with k colors.
$\Rightarrow P(p_3, k) = k(k-1)^2$.

proposition 2.3. *The chromatic polynomial of graph G is $k(k-1)(k-2)\cdots(k-n+1)$ if and only if G is complete graph on n vertices.*

Proof. With k colors, the first vertex G can be colored in k ways, a second vertex of G can be colored in $k-1$ ways if and only if the first vertex is adjacent to the second vertex, the third vertex of G can be colored in $k-2$ ways if and only if the third vertex is adjacent to the first two vertices ..., and the n^{th} vertex can be colored in $k-n+1$ ways, if and only if every vertex is adjacent to every other vertex, i.e this happen if and only if G is complete.
Therefore $P(K_n, k) = k(k-1)(k-2)\cdots(k-n+1)$ □

Example 2.3. *For the complete graph K_3, we begin by selecting a random vertex and note that it can be colored k ways. If we move from this vertex to any other, we notice that this second one can only be colored $k-1$ ways as it is adjacent to the first. The third and final vertex can only be colored $k-2$ ways as it is adjacent to both of the first two as we observe in Figure below . As a result, we nd that K_3 can be colored $k(k-1)(k-2)$ ways with k colors.*

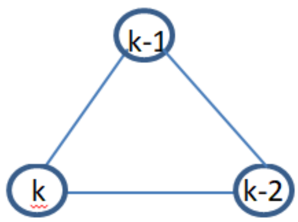


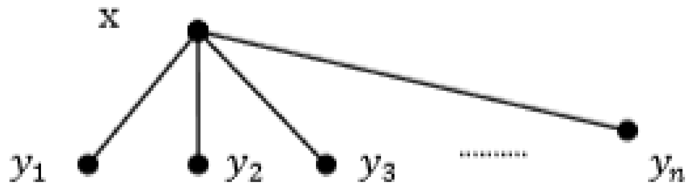
Figure 2.3: chromatic function $P(K_3, k)$

proposition 2.4. *The chromatic polynomial of $K_{2,n}$ is $k(k-1)[(k-1)^{n-1} + (k-2)^n]$*

Proof. Let v and w be the vertices in the bipartite set of size 2.
We consider two cases.
Case -1: If v and w are colored the same, then we choose a single color of the available k colors. As both v and w are adjacent to all of the remaining n vertices, we color using the remaining $(k-1)$ colors.
As these n vertices are independent, we can color freely, and so arrive at $k(k-1)^n$ colorings for this case.

Case -2: If v and w are colored differently, then there are $k(k-1)$ ways to color them. As before, we use the remaining $k-2$ colors to color the remaining n vertices freely, giving $k(k-1)(k-2)^n$ colorings for this case.
Using the addition principle for combining mutually exclusive cases, we get
$P(K_{2,n}, k) = k(k-1)^n + k(k-1)(k-2)^n)$
$\Rightarrow P(K_{2,n}, k) = k(k-1)[(k-1)^{n-1} + (k-2)^n]$ □

Corollary 2.1. *The chromatic polynomials of* $K_{1,n}$ *is* $k(k-1)^n$.



Proof.

Figure 2.4: Chromatic function $K_{1,n}$

The graph is a tree on $(n+1)$ vertices so $P(K_{1,n}, k) = k(k-1)^n$ □

For many graphs, it is very difficult to determine the chromatic polynomials by analysis of the structure of the graphs.
However, deletion-contraction method is also one of the methods to determine such polynomials.
When we contract two vertices, we identify them as a single vertex and all edges incident with either vertex become incident with the new vertex of the fused.

2.2 The deletion-contraction method

In this section we will define two important operations deletion and contraction and also we state the chromatic recurrence that would help us determine the chromatic polynomial of a graph. Let G be a graph and e an edge of G. There are two important operations (deletion and contraction).

Definition 2.2.1. *$G-e$ is the graph with the same vertices as G, and the same edges except, the removed one.*

Definition 2.2.2. *For a graph G and $e \in E$, the contracting of an edge denoted by $G.e$ is the graph obtained from G by contracting e; that is by identifying the ends of e and then delating e.*

By deleting an edge and contracting the corresponding vertices until it became a null graph which forms a chromatic polynomial known as the deletion contraction algorithm.

Theorem 2.1. *(Chromatic recurrence)*
If G is simple graph and $e \in E(G)$, then $P(G, k) = P(G - e, k) - P(G.e, k)$.

Proof. Every proper $k-$coloring of G is a proper $k-$coloring of $G - e$. A proper $k-$coloring of $G - e$ is a proper $k-$coloring of G if and only if it gives distinct color to the end points u, v of e.

Hence we can count the proper $k-$colorings of G by subtracting from $P(G - e, k)$ the number of proper $k-$colorings $G - e$ that gives u and v the same color.

Coloring of $G - e$ in which u and v have the same color corresponds directly to proper k- coloring of $G \cdot e$ in which the color of contracted vertex is the common color of u and v. □

Example 2.4. *If G is a tree of order 4.*

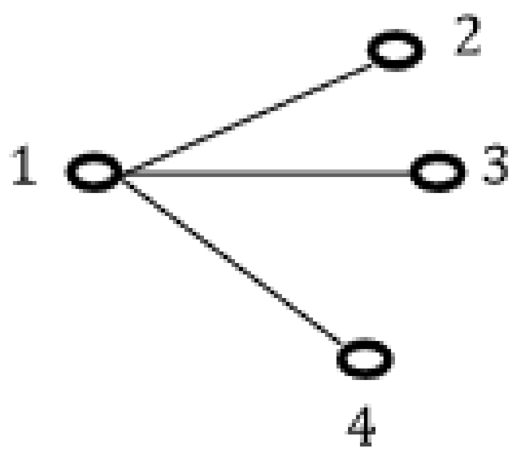


Figure 2.5: Graph of tree order 4

Then the chromatic polynomial of G using deletion-contraction formula $P(G, k) = P(G - e, k) - P(G.e, k)$ is as follows.

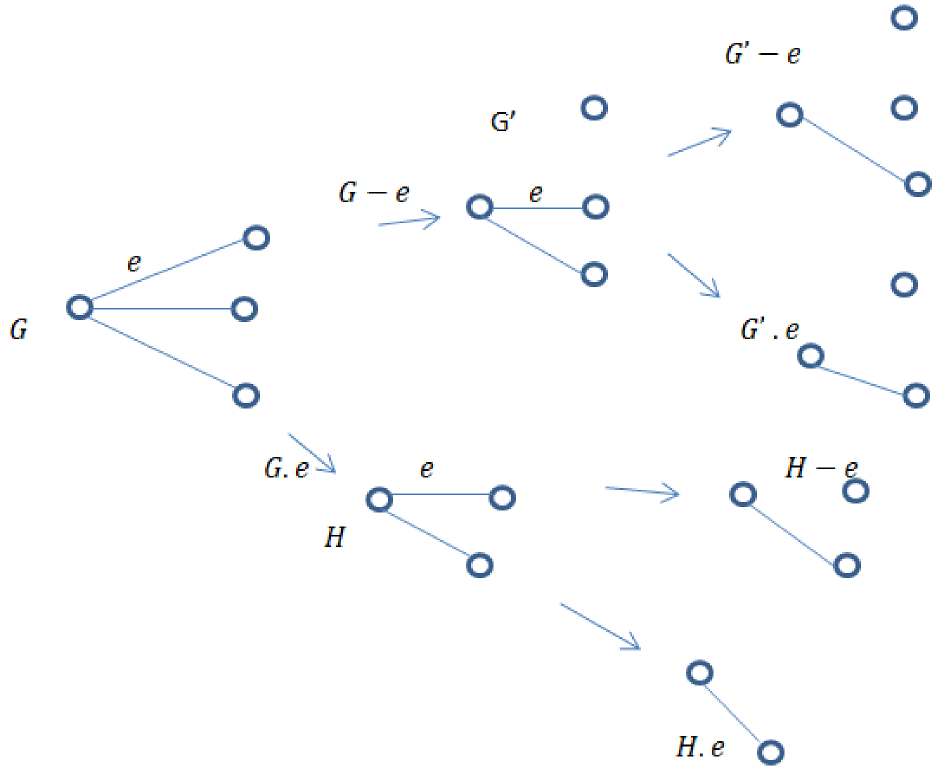


Figure 2.6: Deletion and contraction of the graph G

$$
\begin{aligned}
P(G,k) =& P(G',k) - P(H,k) \\
=& P(G'-e,k) - P(G'.e,k) - (P(H-e,k) - P(H.e,k)) \\
=& k^3(k-1) - k^2(k-1) - (k^2(k-1) - k(k-1)) \\
=& k^3(k-1) - k^2(k-1) - (k^2(k-1) - k^2 + k) \\
=& k^4 - 2k^3 - k^3 + k^2 - k^3 + k^2 + k^2 - k \\
=& k^4 - 3k^3 + 3k^2 - k \\
=& k(k-1)^3.
\end{aligned}
$$

proposition 2.5. *Let $x_{(r)} = x(x-1)...(x-r+1)$. If P_r denotes the number of partition of $V(G)$ in to r non empty independent sets, then $P(G,k) = \sum_{r=1}^{n(G)} P_r(G)k_{(r)}$ which is polynomial in k of degree $n(G)$.*

Proof. When r colors are used in a proper coloring, the color classes partition $V(G)$ in to exactly r independent sets which can happen in $P_r(G)$ ways. When k colors are available there are exactly $k_{(r)}$ ways to choose colors and assign them to the classes. All the proper colorings arise in this way, so the formula for $P(G,k) = \sum_{r=1}^{n} P_r(G)k_{(r)}$ is correct.

Since $k_{(r)}$ is polynomial in k and $P_r(G)$ is a constant for each r, this formula implies that $P(G,k)$ is polynomial function of k.

When G has n vertices, there is exactly one partition of $V(G)$ in to n independent sets and no partition using more sets, so the leading term is k^n.

□

Example 2.5. *For the graph given in the figure 2.5 we can compute the chromatic polynomial of G using partition of $V(G)$ in to r independent subset as follows.*

We have vertex set $V = \{1,2,3,4\}$.
It is not possible to partition $V(G)$ in to one independent set.
Thus $P_1(G) = 0$
There is one way to partition $V(G)$ in to two independent subsets.
.i.e $\{\{1\},\{2,3,4\}\}$
$\Rightarrow P_2(G) = 1$

There are three ways to partition $V(G)$ in to three independent subset.
i.e $\{\{2,3\},\{1\},\{4\}\},\{\{2,4\},\{1\},\{3\}\},\{\{4,3\},\{1\},\{2\}\}$

$\Rightarrow P_3(G) = 3.$

And also there are one way to partition $V(G)$ in to four independent subset.

$\Rightarrow P_4(G) = 1$
Therefore $P(G,k) = \sum_{r=1}^{4} P_r(G)k_r$

$$\begin{aligned} P(G,k) =& P_1(G)k_1 + P_2(G)k_2 + P_3(G)k_3 + P_4(G)k_4 \\ =& 0 + k(k-1) + 3k(k-1)(k-2) + k(k-1)(k-2)(k-3) \\ =& k^4 - 3k^3 - 8k^2 + 11k^2 - k \\ =& k^4 - 3k^3 + 3k^2 - k \\ =& k(k-1)^3 \end{aligned}$$

Theorem 2.2. *(Whitney)*
The Chromatic polynomial $P(G,k)$ has degree $n(G)$ with integer coefficients alternating in sign beginning $k^n - e(G)k^{n-1} + \ldots$.

Proof. We use induction on $e(G)$.

The claim hold when $e(G) = 0$, where $P(\bar{G},k) = k^n$.

For the induction step let G be an n-vertex graph with $e(G) \geq 1$.

Each of $G - e$ and $G.e$ has fewer edges than G and $G.e$ has $n-1$ vertices.

By induction hypothesis, there are nonnegative integers $\{a_i\}$ and $\{b_i\}$ such that
$P(G-e,k) = \sum_{i=0}^{n}(-1)^i a_i k^{n-i}$ and $P(G.e,k) = \sum_{i=0}^{n-1}(-1)^i b_i k^{n-i-1}$

By chromatic recurrence we have

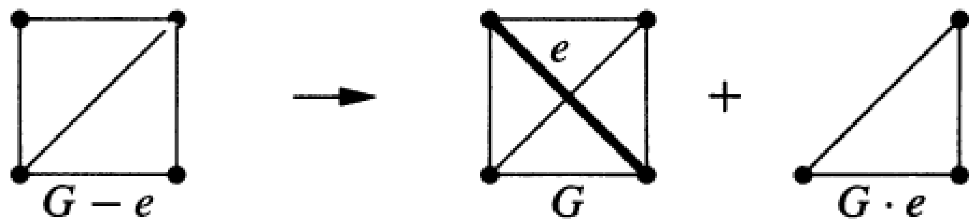
$$\begin{aligned}P(G,k) =& P(G-e,k) - P(G.e,k)\\ =& k^n - (e(G)-1)k^{n-1} + a_2k^{n-2} - ... + (-1)^i a_i k^{n-i} - (k^{n-1} - b_1 + ... + (-1)^i b_{i-1}k^{n-i})\\ =& k^n - e(G)k^{n-1} + (a_2+b_1)k^{n-2} - ... + (-1)^i(a_i+b_{i-1})k^{n-i}\end{aligned}$$

Hence $P(G,k)$ is a polynomial with leading coefficient $a_0 = 1$ with next coefficient $-(a_1+b_0) = -e(G)$ its coefficients alternating in sign. □

When adding an edge yields a graph whose chromatic polynomial is easy to compute, we can use chromatic recurrence in different ways.

Instead of $P(G,k) = P(G-e,k) - P(G.e,k)$ we can write $P(G-e,k) = P(G,k) + P(G.e,k)$. Thus we can compute $P(G-e,k)$ using $P(G,k)$.

Example 2.6. *Let G complete graph of order 4. Then we can compute $P(G-e,k)$ as follow.*



$P(G,k) = k(k-1)(k-2)(k-3)$ *and* $P(G.e,k) = k(k-1)(k-2)$

$$\begin{aligned}\Rightarrow P(G-e,k) = P(G,k) + P(G.e,k) =& k(k-1)(k-2)(k-3) + k(k-1)(k-2) \\ =& k(k-1)((k-2)(k-3)+1) \\ =& k(k-1)(k-2)^2\end{aligned}$$

In general, $P(K_{n-e},k) = P(K_n,k) + P(K_{n-1},k) = (k-n+2)^2 \prod_{i=0}^{n-3}(k-i)$

For some graphs, the method in deletion-contraction is either insufficient or too tedious to use for computing the Chromatic Polynomial.

However, we can use partition lattices and a special function called the Möbius Function to find these polynomials.

2.3 Möbius Inversion Theorem and Chromatic polynomial

In this section we apply the Möbius Inversion Theorem to the problem of determining the Chromatic Polynomial of a graph.

Example 2.7. *consider a tree of order* 4 *in the figure below.*

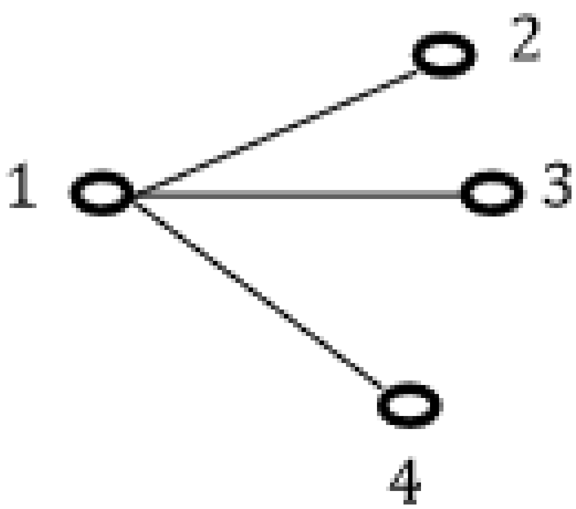


Figure 2.7: The tree graph of order 4

By using method of analysing the structure of graph the chromatic polynomial of tree order n is $k(k-1)^{n-1}$.
$\Rightarrow P(T_4, k) = k(k-1)^3$

To find the chromatic polynomial of the given graph by using Möbius Inversion Theorem, first we have to find the bond lattice of the graph, so that we have an ordering in the bonds of the vertices of the given graph. Further we find N_e and N_a for any bond.

Let the function $N_e(b)$ stands for the number of coloring on the vertices of the given graph that have exactly b as their bonds.
Let $N_a(b)$ stands for the number of coloring on the vertices of the given graph that have atleast b as their bonds.
Let the set of vertices of T_4 be $\{1, 2, 3, 4\}$. To find the chromatic polynomial of T_4, first we have to find the bonds of vertices set and construct the bond lattice with these bonds.

Let bonds be

$$\begin{aligned} p &= \{\{1\}, \{2\}, \{3\}, \{4\}\} \\ p_2 &= \{\{\{1, 2\}, \{3\}, \{4\}\}, \{\{1, 4\}, \{2\}, \{3\}\}, \{\{1, 3\}, \{2\}, \{4\}\}\} \\ p_3 &= \{\{1, 2, 3\}, \{4\}\}, \{\{1, 4, 3\}, \{2\}\}, \{\{1, 2, 4\}, \{3\}\}\} \\ p_4 &= \{\{1, 2, 3, 4\}\} \end{aligned}$$

The bond lattice of the graph is given in Figure below.

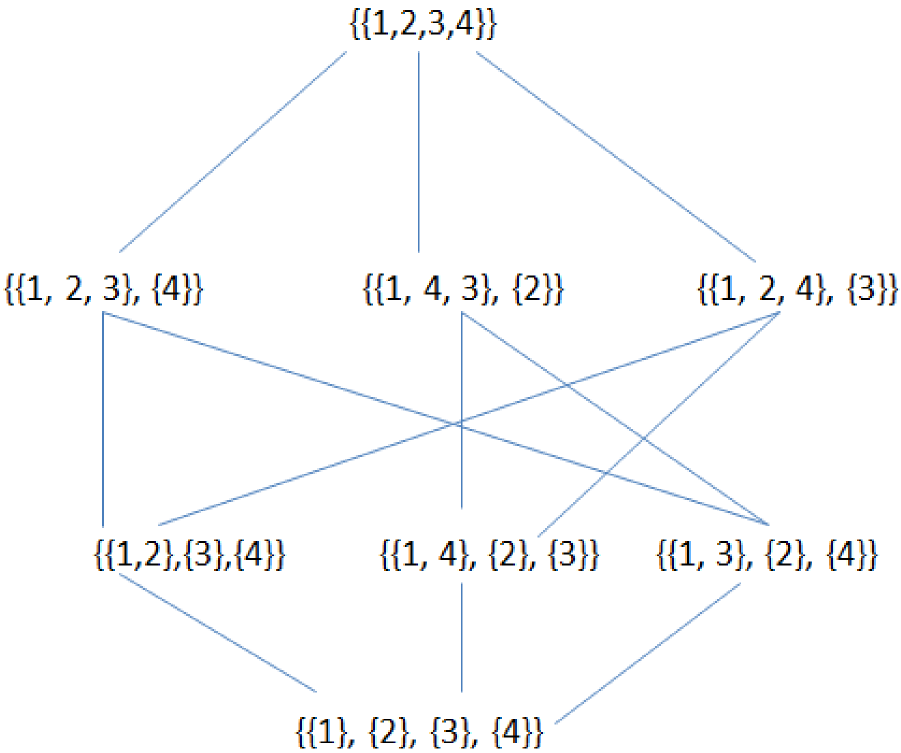


Figure 2.8: The bond lattice of T_4

.

As the bond $\{\{1\},\{2\},\{3\},\{4\}\}$ represents all coloring in which no two adjacent vertices are the same color.
Now we have to calculate $\mu(P,Q)$ for all bonds Q in the lattice.

The finest bond has value 1, while each bond in the second level has value -1.
To calculate the Möbius function value for bonds in the 3^{rd} level, we set up all finer bonds for each bond.
The following figure represents the bond $\{\{1,4,3\},\{2\}\}$ and all its finer bonds.

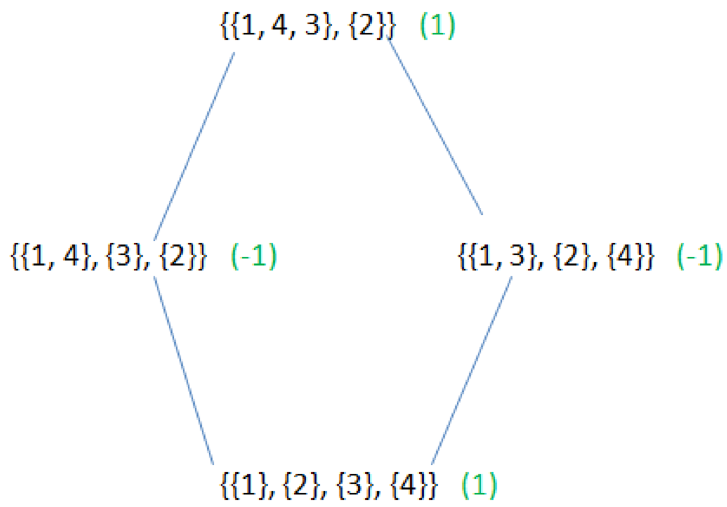


Figure 2.9: Finer bonds of the bond $\{\{1, 4, 3\}, \{2\}\}$,with the Möbius Function values

Again using Möbius function, we find that $\mu(P, \{\{1, 4, 3\}, \{2\}) = 1$ and all bonds on this level have a function value 1.
We can now fill out bond lattice with the Möbius function for each bonds as in figure below.

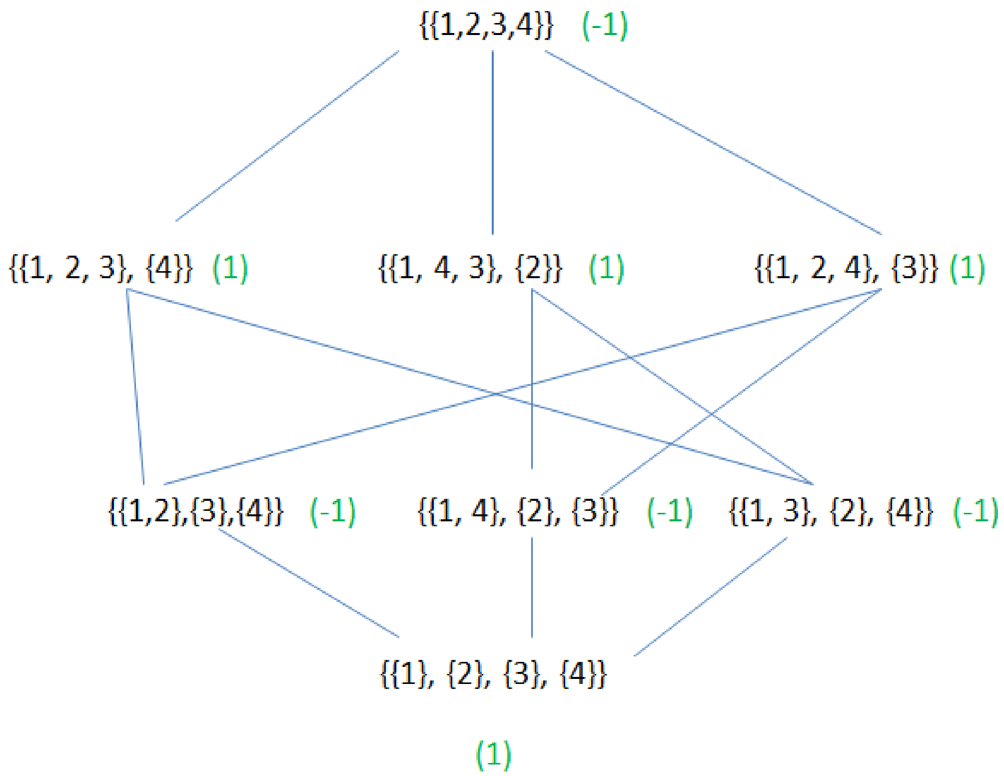


Figure 2.10: The bond lattice of T_4, with the Möbius Function values

All bonds that are founds on the same level of the lattice have the same number of parts. Thus all bonds Q that occurs on the same level, $N_a(Q) = k^i$, where i is the number of parts of the bond b.

Then, we once more sum up the Möbius function values for all the bonds on particular level and use this resulting values as the coefficient of k^i in our sum.
Accordingly $N_e(P) = \sum_{Q:P \preccurlyeq Q} \mu(P,Q)N_a(Q) = k^4 - 3k^3 + 3k^2 - k$, which is equivalent to the chromatic polynomial of tree order 4.

Therefore $P(T_4, k) = k^4 - 3k^3 + 3k^2 - k = k(k-1)^3$

Example 2.8. *Let $K_4 - e$ be a graph in the figure given below.*

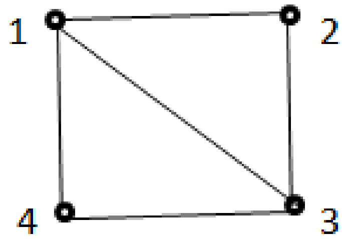


Figure 2.11: The of graph $K_4 - e$.

From example 2.6 $P(K_4 - e, k) = k(k-1)(k-2)^2 = k^4 - 5k^3 + 8k^2 4k$ *This should be the same result yielded by the Möobius Inversion Theorem. In order compute the chromatic polynomial, first we have to find the bond vertices of the graph and construct the bond lattice with these bonds.*
Let $V(G) = \{1, 2, 3, 4\}$ *and bonds of the vertices be given as follow.*

$p = \{\{1\}, \{2\}, \{3\}, \{4\}\}$
$p_1 = \{\{\{1, 4\}, \{2\}, \{3\}\}, \{\{1, 3\}, \{2\}, \{4\}\}, \{\{1, 2\}, \{3\}, \{4\}\}, \{\{2, 3\}, \{1\}, \{4\}\}, \{\{3, 4\}, \{1\}, \{2\}\}\}$
$p_2 = \{\{1, 4, 2\}, \{3\}\}, \{\{2, 4, 3\}, \{1\}\}, \{\{1, 3, 4\}, \{2\}\}, \{1, 2, 3\}, \{4\}\}, \{3, 4\}, \{1, 2\}\}$
$p_3 = \{\{1, 2, 3, 4\}\}$

The bond lattice of the graph is given in Figure below.

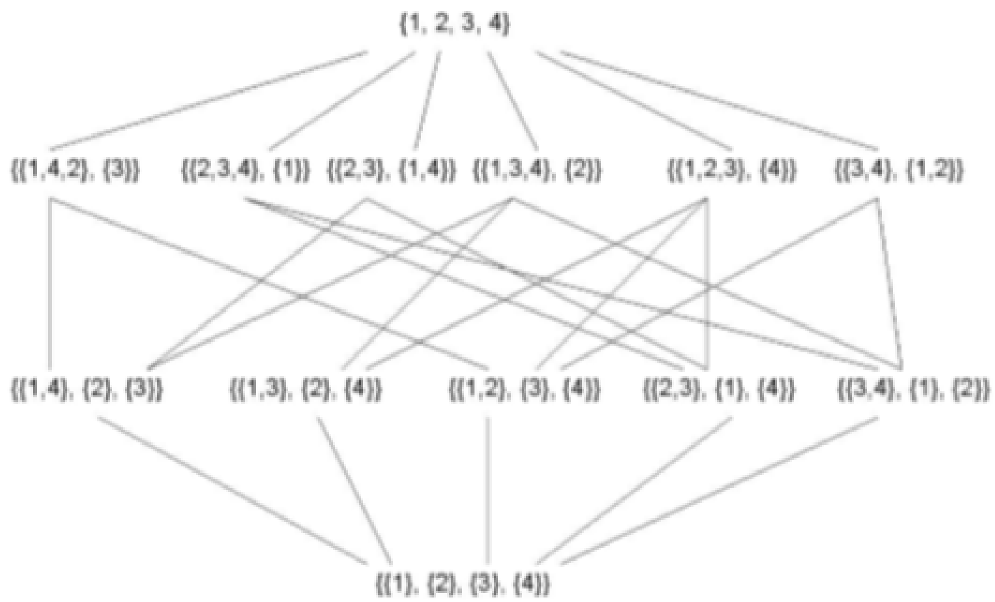


Figure 2.12: The bond lattice of $K_4 - e$

Also, we need to define the functions N_e and N_a for any bond Q.
The function $N_e(Q)$ represents the number of colorings on the vertices of G that have exactly Q as their bond representation. The function $N_a(Q)$ represents the number of colorings on the vertices of G that have at least Q as their bond; that is, all colorings whose exact bond representation is the same or coarser than Q.

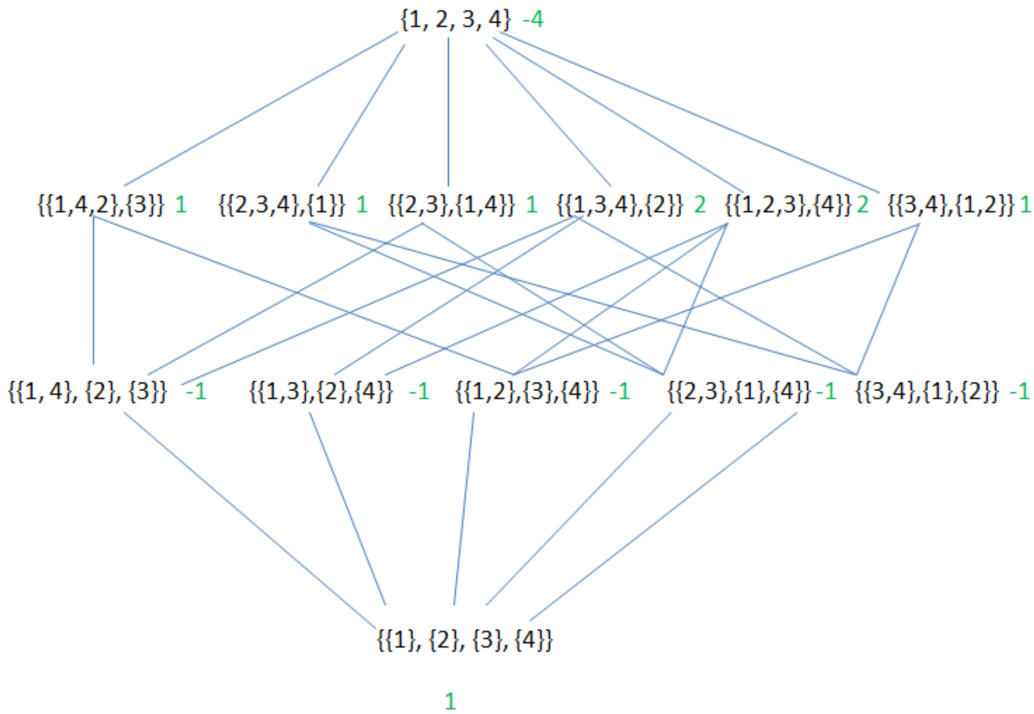


Figure 2.13: The bond lattice of $K_4 - e$,with the Möbius Function values included

*For all bonds Q on the same level and any number of colors k, $N_a(Q) = k^i$, where i is the number of parts of the bond Q. $N_a(p) = k^4$, $N_a(P_1) = k^3$, $N_a(P_2) = k^2$, $N_a(P_3) = k$
To find the chromatic polynomial of $K_4 - e$, we calculate $N_e(P_1)$.By Möbius inversion theorem, $N_e(P) = \sum_{Q:P \preccurlyeq Q} \mu(P, Q) N_a(Q) = k^4 - 5k^3 + 8k^2 - 4k$.*

2.4 Properties of chromatic polynomial

From the method of computing Chromatic Polynomials using of the bond lattices and the Möbius Inversion Theorem we can determine characteristic of the Chromatic Polynomial. Let G be graph of order n. In the Chromatic Polynomial of the graph G, the coefficient of the k^{n-1} is always the negative of the number of edges in the graph. This is because the second level of any bond lattice is composed of bonds that contain only an edge and singleton vertices. Thus, the number of bonds in the second level of a bond lattice is always the number of edges of the graph.

And also the first level is always composed of only one bond that always has a Möbius Function value of 1 and is always finer than every bond on the second level, each bond on the second level will have a function value of -1.
Because $N_a(b) = k^{n-1}$ for all bonds b on the second level of any lattice, the coefficient on the k^{n-1} term will always be the negative of the number of edges in the graph.

Next, we show why the coefficients in a Chromatic Polynomial always sum to 0. This is because the Möbius Function value for each bond in the bond lattice of a graph is calculated so that the function value for a particular bond and function values of all finer bonds sum to 0. Particularly, the function value for the coarsest bond in the bond lattice is chosen so that the sum of all function values of all bonds in the lattice is 0.
Also, we note that plugging $k = 1$ into any Chromatic Polynomial is the same as summing together all of the coefficients. For any graph with at least one edge, this sum will always be 0 as such a graph cannot be properly colored with $k = 1$ colors.

If we compare the chromatic polynomials of N_3, P_3 and K_3, we have some interesting properties.

$$P(N_3, k) = k^3$$
$$P(P_3, k) = k(k-1)^2 = k^3 - 2k^2 + k$$
$$P(K_3, k) = k(k-1)(k-2) = k^3 - 3k^2 + 2k$$

In each of the polynomials above there is no constant term. Thus, if $k = 0$, $P(k) = 0$, as we would expect. Also, except in the case of the null graph, we notice that the sum of the coefficients of each polynomial is 0, which tells us that $P(1) = 0$. This is also as expected because any graph with more than 1 vertex and at least one edge cannot be properly colored with only 1 color. Our final two observations are that the coefficients of these polynomials have alternating signs and that the absolute value of the coefficient on the term k^{n-1} is the number of edges of the graph.

Conclusion

In this project, we computed the chromatic polynomial of simple un directed graphs by using different methods such as analysing the structure of graph, deletion contraction, partition of $V(G)$ in to r independent subset, bond lattice and Möbius Inversion Theorem. We need the bond lattice of a given graph to apply the Möbius Inversion Theorem to compute the chromatic polynomial of a given graph.

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