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An Overview of Maximum Principle For Elliptic PDEs

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Abstract

In this paper, we shall study linear elliptic partial differential equations of second order,

$$Lu = \sum_{i,j} a_{ij}(x)u_{ij} + \sum_{i} b_i(x)u_i + c(x)u = 0$$

on a bounded domain $\Omega \subseteq R^n$ with regular boundary $\partial\Omega$. The coefficients a_{ij}, b_i and c are assumed to be continuous. We will see that if u is a solution of the PDE then it attains its maximum/minimum on $\partial\Omega$.

Key words: - maximum principle, elliptic PDEs, Laplace equation, Dirichlet problem, Harmonic function, existence and uniqueness, A prior bounds

1. Introduction

The maximum principle, which is employed in the study of partial differential equations, enables one to get an idea regarding the uniqueness, approximation, boundedness and symmetry of the solution. Indeed, the maximum principle enables us to obtain information about solution of differential equation and inequalities without any explicit knowledge of the solutions themselves, and hence can be regarded a valuable tool in mathematical researches.

E. Hopf extended the Maximum Principle for harmonic functions, i.e. the fact that a harmonic function cannot have an interior maximum unless it is constant is extended to more general elliptic partial differential equations. Basically, the Maximum Principle rests on the following observation, let $\Omega \subset R^2$ be a domain, if

$$u : \Omega \to R$$

a smooth such that $u \in C^2(\Omega)$ and it has a maximum value at a point $x_0 \in \Omega$, then

$$\nabla u(x_0) = 0 \quad \text{and} \quad \nabla^2 u(x_0) \leq 0$$

where ∇u and $\nabla^2 u$ are the gradient and the Hessian of u at the point $x_0 \in \Omega \subset R^2$. In particular, a harmonic function cannot have an interior maximum unless it is constant. The maximum of E. Hopf can be considered as a classical result in the theory of second orders elliptic PDE. At this point, we mention that the maximum principle for harmonic functions is believed to be known to Gauss on the basis of the mean value theorem.

The maximum principle for harmonic function is one of the basic results in the classical theory of PDEs. It guarantees that every solution of the Dirichlet problem for the Laplace equation with bounded boundary data is bounded. In his work, Poincare gave a complete proof in a rather general domain of existence and uniqueness of a solution of the Laplace equation for a continuous Dirichlet boundary data. He introduces an iterative method that relies on solving the Dirichlet problem on a ball in the domain and makes extensive use of the maximum principle and Harnack’s inequality.

For equations of higher order, the maximum principle has been established only for domains with some regularity conditions. It was shown to be true for operators with analytic coefficients in smooth domain of dimensions two and higher. In the early 1990s, it was extended to three-dimensional domains diffeomorphic to a polyhedron or having a Lipchitz boundary. However, in general domains no direct connection to the maximum principle exists. In the modern theory of partial differential equations, the seminal paper of E.Hopf on the maximum principle for linear elliptic differential equation has a central place. This paved the way for others like J.schauder, to start their treatment of partial differential equations by functional analytic methods. Elliptic partial differential equation is a typical one.

The maximum principle claims that solutions of some second order scalar elliptic partial differential equations cannot have a maximum in the interior of the domain. The basic idea is straightforward. Consider, for the sake of demonstration, the Laplace's equation

$$\Delta u = 0.$$

If u has a maximum at a point x and the second derivatives of u do not all vanish at x, then Δu is negative at x, but this is in contradiction to the equation. The only case left to be ruled out is that of degenerate maxima where all second derivatives vanish. This can be done by an approximation argument that removes the degeneracy.

The maximum principle can also be used to show that solution of elliptic PDEs must be non-negative. This is important for quantities, which have a physical interpretation as densities, concentrations, probabilities, etc. The maximum principle also leads to easy uniqueness results. The maximum principle itself can be used to construct existence proofs. As a recent application of the maximum principle, we may mention "viscosity solution" for Hamilton –Jacobi equation.

In general, maximum principles play an important role in the theory of PDEs, particularly in uniqueness theorem for boundary value problems for elliptic equation, specially the Dirichlet and Neummann problem in n-dimensional. Again, we have seen interior sphere property and existence of solution.

This paper is organized as follows, in the next section i.e. in chapter *two* we give an overview of elliptic, uniformly elliptic and degenerate elliptic partial differential equations, in chapter *three* we visit the maximum principle for second order elliptic partial differential equations in higher dimensions in chapter *four* we see the Dirichlet problem followed by summary.

2. Elliptic partial differential Equations

In order to facilitate the understanding and application of maximum principle first we give some definition and notation.

Definition 2.1:- A partial differential equation (henceforth abbreviated as PDE) in a domain $\Omega \subset \Re^n$ is a relation of independent variables$x \in \Omega$, an unknown function u defined in Ω and a finite number of its partial derivatives. Solving a PDE is to find this unknown function. The order of a PDE is the order of the highest order derivatives in the relation. Hence, for a positive integer *m*, the general form of an *m*-th order PDE in a domain $\Omega \subset \Re^n$ is given by

$$F(x, u, \nabla u(x), \nabla^2 u(x), \dots, \nabla^m u(x)) = 0 \qquad \text{for any } x \in \Omega. \tag{2.1}$$

Here F is a function, which is continuous in all its argument and u, is a C^m-function in Ω.

APDE is linear if it is linear in the unknown functions and their derivatives; with coefficients depending on independent variable *x* otherwise, it is nonlinear.

A general *m*-th order linear PDE in Ω is given by

$$\sum_{|\alpha| \le m} a_\alpha(x) \partial^\alpha u = f(x) \qquad \text{for } x \in \Omega. \tag{2.2}$$

Here a_α is called the coefficient of $\partial^\alpha u$. A PDE of order *m*, with coefficients depending on independent variables *x* and derivatives of solutions of order less than *m*. In general, an *m*-th order quasilinear PDE in Ω is given by

$$\sum_{|\alpha| \le m} a_\alpha(x, u, \dots, \nabla^{m-1} u) \partial^\alpha u = f(x, u, \dots, \nabla^{m-1} u) \text{ for } x \in \Omega. \tag{2.3}$$

Several PDEs involving one or more unknown functions and their derivatives form a differential system.

Our main points is to view second order linear PDEs.

2.1 Second order partial differential Equations

Now to see second order linear PDEs and classified it according to its leading coefficient matrix.

Definition2.2:- Let $\Omega \subset \Re^n$ be a domain with $n \in \mathbb{N}$, where the continuous coefficient functions $a_{ij}(x), b_i(x), c(x) \colon \Omega \to \Re \in C^0(\Omega)\ for\ i, j = 1, \dots, n$ are defined. Furthermore, let the matrix $\left(a_{ij}\right)_{i,j=1}^{n}$ is symmetric for all$x \in \Omega$.The linear partial differential operator of the second order

$$L \colon C^2(\Omega) \to C^0(\Omega) \text{ is defined by}$$

$$Lu \coloneqq \sum_{i,j=1}^{n} a_{ij}(x)u_{x_i x_j} + \sum_{i=1}^{n} b_i(x)u_{x_i} + c(x)u(x), x \in \Omega \tag{2.4}$$

For the operator L and a function f in Ω, we consider the equation

$$Lu = f \text{ in } \Omega \tag{2.5}$$

The function f is called the non-homogeneous term.

We now turn our attention to the classification of second order linear PDEs in$\Re^n$

Consider the coefficient matrix $A = (a_{ij})_{i,j=1}^{n}$

(a) $\sigma\ (A) \subset \Re \backslash \{0\}$
(b) $\lambda_i, \lambda_j \in \sigma(A); \sigma\ (A) = \{\lambda: det(\lambda I - A = 0)\}$

(Where $\lambda_i, \lambda_j$are Eigen values of the n x n matrix A=(a_{ij}))

We shall suppose that (2.4) is

1. Parabolic if exactly one of the eigenvalues is zero and all the others have the same sign

2. Hyperbolic if none of the eigenvalues are zero and one of them has opposite sign of the (n-1) others.

3. Elliptic if the eigenvalues of A= (a_{ij}) are all positive or all negative.

When we see second order linear differential equation in $\Re^2$, where complete classification are available.

Let Ω be a domain in $\Re^2$ consider

$$\sum_{i,j=1}^{2} a_{ij}(x)u_{x_i x_j} + \sum_{i=1}^{2} b_i(x)u_{x_i} + c(x)u = f(x) \text{ in } \Omega \tag{2.6}$$

Here we assume that $A = (a_{ij})$ is a 2x2 symmetric matrix.

Definition2.3: - Let L be a differential operator defined in $\Omega \subset \Re^n$ as in (2.6)

1. L is elliptic at $x \in \Omega$ if $\det\left(a_{ij}(x)\right) > 0$;
2. L is hyperbolic at $x \in \Omega$ if $det\left(a_{ij}(x)\right) < 0$;
3. L is parabolic at $x \in \Omega$ if $\det\left(a_{ij}(x)\right) = 0$.

For the operator L in (2.6), the symmetric matrix A always has two Eigenvalues. Then

L is elliptic if the two Eigenvalues have the same sign.

L is hyperbolic if the two Eigenvalues have different sign.

L is parabolic if one of the Eigenvalues vanishes.

The n-dimensional Laplace operator Δ is defined by

$$\Delta u = \sum_{i,j=1}^{n} u_{x_i x_j} \tag{2.7}$$

Obviously, the Laplace operator is elliptic. The Laplace operator in two dimensions can be expressed in polar form

$$\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} \tag{2.8}$$

The equation $\Delta u = 0$ is called the Laplace equation and its solutions are called harmonic functions. In this case, (a_{ij}) is the identity matrix. Note that Δ is invariant under rotations. If x = Ay for an orthogonal matrix A, then

$$\sum_{i=1}^{n} u_{x_i x_i} = \sum_{i=1}^{n} u_{y_i y_i} \tag{2.9}$$

For a non-zero function *f*, we call equation $\Delta u = f$ the Poisson equation.

2.2 Sign –definite operators

Before to define uniformly elliptic operators and degenerate elliptic operators first of all let as state about sign- definite operators.

Definition2.4 An n x n matrix A is said to be

(a). positive semi-definite if and only if $V^T A V \geq 0, \forall V \epsilon R^n \setminus \{0\}$

(b). positive definite if and only if $V^T A V > 0, \forall V \epsilon R^n \setminus \{0\}$

(C).negative semi-definite if and only if $V^T A V \leq 0, \forall V \epsilon R^n \setminus \{0\}$.

(d).negative definite if and only if $V^T A V < 0, \forall V \epsilon R^n \setminus \{0\}$.

Now using definition (2.2) or by considering sign-definite operators we can define degenerated and uniformly elliptic operators.

2.3. Degenerated elliptic operators

The linear partial differential operator of second order in (2.3) is named elliptic (or alternatively degenerated elliptic) if and only if the quadratic form $VAV^T \geq 0$ for all V eigenvectors of matrix A=$\left(a_{ij}\right)_{i,j=1}^{n}$ (or show that A is positive semi-definite.

$$i.e\ V^T AV = \sum_{i,j=1}^{n} a_{ij}(x)v_i\, v_j > 0 \left(\text{or } \sum_{i,j=1}^{n} a_{ij}\,(x)v_i v_j \geq 0\right)$$

for all $V = \left(v_{1,}v_{2,} \dots .\, v_n\right)\epsilon R^n \setminus \{0\}$ and all xϵ Ω is satisfied

2.4. Uniformly elliptic operators

When we have the ellipticity constants $0 < m \leq M < +\infty$ such that

$$m|V|^2 \leq \sum_{ij=1}^{n} a_{ij}(x)v_i v_j \leq M|V|^2$$

for all $V = (v_1 v_2, \dots, v_n) \in \Re^n \setminus \{0\}$.

and for xϵD holds true ,the operator L is called uniformly elliptic. On the other hand, the matrix A= $\left(a_{ij}\right)_{i,j=1}^{n}$ is apositive definite . $i.e\ V^T AV > 0$ for all $V\epsilon\, R^n \setminus \{0\}(V =$ eigenvectors of A).

Remark: - A uniformly elliptic differential operator is elliptic and an elliptic differential operator is degenerating elliptic. The Laplace operator appears for $a_{ij}(x) \not\equiv 0, b_i(x) = c(x) = 0$ with $i, j = 1, \dots, n$ and is consequently uniformly elliptic with $m = M = 1$. In case *c(x) =0, x*ϵ Ω,we use the notation *Mu(x) =Lu(x)*, and we call it the reduce differential operators.

3. Maximum principle for second order elliptic PDEs in $\mathfrak{R}^n$

Maximum principles provide powerful tools for linear and nonlinear elliptic equation of second order. One of the important tools in studying harmonic function is the maximum principle. In this section, we discuss the maximum principle for a class of elliptic differential equations slightly more general than the Laplace equation.

Throughout this section, we shall consider a second- order linear operator of the form (2.6) in definition (2.2).

3.1 Weak maximum principle

Theorem3.1 Assume that $Lu \geq 0($ or, respectively $Lu \leq 0)$ in a bounded domain Ω and that *c(x)* =0 in Ω. The maximum (or respectively the minimum) of *u* is achieved on $\partial\Omega$.

Proof: - If $Lu > 0$ in Ω ,then *u* cannot achieve its maximum anywhere in Ω. Suppose it did, say at the point x_0.Then all first derivatives of *u* vanish at this point, and hence

$$Lu = a_{ij}\frac{\partial^2}{\partial x_i \partial x_j}u \tag{3.1}$$

But at a maximum the matrix of second partial derivatives is negative semi-definite and we conclude that $Lu(x_0) \leq 0$ a contradiction.

For the general case, consider the function $u_\varepsilon = u + \varepsilon\exp(\gamma x_1)$. we find

$$Lu_\varepsilon = Lu + \varepsilon(\gamma^2 a_{11} + \gamma b_1)\exp(\gamma x_1) \tag{3.2}$$

We now choose γ large enough so that $\gamma^2 a_{11} + \gamma b_1 > 0$ throughout Ω. This is possible since a_{11} is positive and continuous on $\overline{\Omega}$ .Then $Lu_\varepsilon > 0$ for any positive ε.we conclude that

$$\max_{\overline{\Omega}} u_\varepsilon = \max_{\partial\Omega} u_\varepsilon \tag{3.3}$$

Then letting $\varepsilon \to 0$ we get the result $\max_{\overline{\Omega}} u_0 = \max_{\partial\Omega} u_0$ this implies that u is a Maximum at the boundary. However, u may assume a maximum at many points.

We have the following corollary of the above theorem.

Corollary3.1 Let Ω be bounded and assume $\mathrm{c} \leq 0$ in $\Omega. Let\ Lu \geq 0($or respectively $Lu \leq$ *0*. Then

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u^+ \left(\text{or, resp.}, \min_{\overline{\Omega}} u \geq \min_{\partial\Omega} u^-\right) \tag{3.4}$$

Here $u^+ = \max(u, 0)$, $u^- = \min(u, 0)$. In particular, if $Lu = 0$ in Ω , then

$$\max_{\bar{\Omega}}|u| = \max_{\partial\Omega}|u| \tag{3.5}$$

Proof:-If $u \leq 0$ throughout Ω, the corollary is trivially true. Hence, we may assume that $\Omega^+ = \Omega \cap \{u > 0\} \neq \emptyset$. On Ω^+, we have $-cu \geq 0$, and hence

$$a_{ij}\frac{\partial^2}{\partial x_i \partial x_j}u + b_i\frac{\partial}{\partial x_i}u \geq 0 \tag{3.6}$$

Hence, weak maximum principle implies that the maximum of u on the closure of $\Omega^+$is equal to its maximum on$\partial\Omega^+$. Since u = 0 on $\partial\Omega^+ \cap \Omega$, this maximum must be achieved on the boundary of Ω ($\partial\Omega$).

3.1.1 Comparison principle

Corollary 3.2 Let Ω be bounded and $c \leq 0$. If $Lu \leq Lv$ in Ω and $u \geq v$ on $\partial\Omega$, then $u \geq v$ in Ω.

Proof:-If Lw =0 in Ω, then the maximum principle and the above corollary show that

$$\max|w| \leq \max\left(\max_{\bar{\Omega}} w, -\min_{\bar{\Omega}} w\right)$$

$$\leq \max\left(\max_{\partial\Omega} w^+, -\min_{\partial\Omega} w^-\right)$$

$$\leq \max_{\partial\Omega}|w|$$

Applying this now to *w=u-v* with *w=0* on $\partial\Omega$ then it follows that $w \equiv 0$ in Ω.

$$u - v \geq w$$

$$u - v \geq 0$$

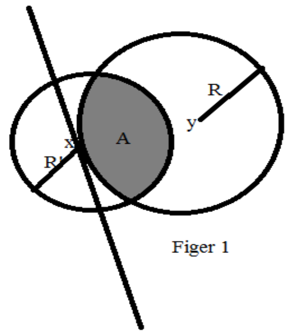
$$u \geq v$$

.

3.2 Strong maximum principle

Weak maximum principle states that u assumes its maximum at the boundary. However, u may assume its maximum at many points, and therefore the theorem does not rule out the possibility that some of these points are interior. The strong maximum principle states that this is impossible, unless u is a constant.

3.2.1 Interior sphere conditions (Hopf's Boundary Lemma)



Figer 1

Lemma 3.2.1:- Suppose that Ω lies on one side of $\partial\Omega$. Assume$Lu \geq 0$, and let x_0 be a point on the $\partial\Omega$ such that $u(x_0) > u(x)$ forevery $x \epsilon \Omega$.Also assume that, in a neighborhood of $x_{0,}$ $\partial\Omega$ is a $c^2 -$ surface and that u is differentiable at $x_{0.}$ Moreover, suppose that either

1. c=0
2. c $\leq$ 0 and $u(x_0) \geq 0$, or
3. $u(x_0) = 0$

Then $\frac{\partial u}{\partial n}(x_0) > 0,$ $where$ $\frac{\partial u}{\partial n}$ denotes the derivatives in the direction of the outer normal to ∂D.

Proof - Since $\partial\Omega$ was assumed c^2, we can choose a ball $B_R(y)$ such that $B_R(y) \subset D$ and $x_0 \epsilon\, \partial B_R(y)$. Here R and y denotes the radius and center of the ball.

For $0 \leq r = |x - y| \leq R$, define

$$v(x) = \exp(-\alpha r^2) - \exp(-\alpha R^2). \tag{3.7}$$

We find

$$Lv(x) = \exp(-\alpha r^2)\left[4\alpha^2 a_{ij}(x_i - y_i)(x_j - y_j) - 2\alpha\big(a_{ii} + b_i(x_i - y_i)\big)\right] + cv. \tag{3.8}$$

Now let $A = B_R(y) \cap B_{R'}(x_0)$, with R' Chosen small. For large enough α, we have $Lv > 0$ in A. Moreover, if we choose $\varepsilon > 0$ small enough, then $u - u(x_0) + \varepsilon v \leq 0$ on $\partial A \cap \partial B_{R'}(x_0)$, and also on $\partial A \cap \partial B_R(y)$, where $v = 0$. Thus we find $L(u - u(x_0)) + \varepsilon v \geq -cu(x_0) \geq 0$ in A and $u - u(x_0) + \varepsilon v \leq 0$ on ∂A.

If $c \leq 0$, the weak maximum principle implies that, $u - u(x_0) + \varepsilon v \leq 0$ throughout A. we take the normal derivative at x_0, and obtain

$$\frac{\partial u}{\partial n}(x_0) \;\geq\; -\varepsilon\frac{\partial v}{\partial n}(x_0) = 2\alpha R exp(-\alpha R^2) > 0, \tag{3.9}$$

This implies the lemma.

If $u(x_0) = 0$, then, by assumption, u is negative in Ω. Now let $c^+(x) = \max\big(0, c(x)\big)$. We find that $(L - c^+)u = Lu - c^+u \geq Lu \geq 0$, and hence we can apply the argument above with $L - c^+$ in place of L.

Remark: - The above lemma still holds if the matrix a_{ij} is only positive semi – definite and n is not a null space.

Because of Hopf's boundary lemma, we can also proof strong maximum principle.

Theorem3.2.2 Assume $Lu \geq 0 (Lu \leq 0)$in $\Omega(\text{not necessarily bounded})$ and assume that u is not constant. If c = 0, then u does not achieve its maximum (minimum) in the interior of Ω. If $c \leq 0$, u cannot achieve a non-negative maximum (non-positive minimum) in the interior. Regardless of the sign of c, u cannot be zero at an interior maximum (minimum).

Proof: - Assume that u is its maximum M at an interior point and let $\Omega^- = \Omega \cap \{x_i < M\}$. If $\Omega^-$is not empty, and then $\partial\Omega^- \cap \Omega$ is not empty. Let y be a point in Ω^- that is closer to $\partial\Omega^-$ than to $\partial\Omega$ and let B be the largest ball contained in $\Omega^-$and centered at y. Let $x_0$be a point on $\partial B \cap \partial\Omega^-$ .Then we can apply Hopf's boundary lemma to B .We conclude that ∇u is nonzero at x_0, this contradicting the assumption that u assumes its maximum there. Hence u does not achieve its maximum (minimum) in the interior of Ω.

4. The Dirichlet problem

Dirichlet problems are named after Lejeune Dirichlet, who proposed a solution by various methods, which become Dirichlet principle.

Dirichlet problem is a problem of finding a function, which solves a specified differential equation in the interior of a given region that takes prescribed values on the boundary of the region.

Given a function f that has values everywhere on the boundary of a region in $\Re^n$, is there a unique continuous function u twice continuously differentiable in the interior and continuous on the boundary , such that u is harmonic in the interior and u=f on the boundary This requirement is called the Dirichlet boundary condition. The main issue is to prove the uniqueness and existence of the solution of the Dirichlet problem using the maximum principles.

$$\Delta u + cu = f \text{ in } \Omega$$

$$u = g \text{ on } \partial\Omega \tag{4.1}$$

Where g is known function on the $\partial\Omega$ or Ω.

Example: - Find the solution of PDE of second order where the values of the solutions are prescribed along a closed curve. Consider the equation

$$\Delta u + cu = f \ \text{ in } \Omega \ f : \Omega \subseteq R^n \to R_0^+$$

$$u = \varphi \ \text{ on } \partial\Omega$$

Where $c \leq 0$ is a give function position in Ω and f is a positive function on $\partial\Omega$.

4.1 Subsolutions

We shall need a notion of sub solutions to the Dirichlet problem, which does not require them to of class $C^2(\Omega)$. The definition is motivated by the maximum principle.

Definition4.2 A function u in $C^0(\overline{\Omega})$is called subharmonic (superharmonic), if for every ball B with $\bar{B} \subset \Omega$ and every function $h \epsilon C(\bar{B})$ with h harmonic in B and $u \leq h(u \geq h)$ on $\partial\Omega$, we have $u \leq h(u \geq h)$ in B. A sub solution (super solution) of the Dirichlet problem is a function $u \epsilon C(\overline{\Omega})$ which is sub harmonic (super harmonic) and such that $u \leq g(u \geq g)$on $\partial\Omega$.

Clearly, if $\Delta u \geq 0$, then u is also subharmonic in the sense of the new definition. We note the following property;

1. The strong maximum principle holds, i.e. if u is sub harmonic and v is superharmonic with $v \geq u$ on $\partial\Omega$, then either $v \geq u$ in Ω or $v = u$ everywhere. we prove this by contradiction.

Assume that $u - v$ has its maximum M at some point $x_0 \epsilon\, \Omega$, where $M \geq 0$. If u-v =M throughout Ω, it follow that u=v; hence we may assume that there are points in Ω where $u - v \neq M$. In that case, we can choose x_0 in such a way that there is a ball B⊂Ω centered at x_0 such that u-v does not equal M on all of boundary of B. Let $\bar{u}$ and $\bar{v}$ denotes the harmonic functions on B which are equal to u and v, respectively, on the boundary of B. We find

$$M = (u - v)(x_0) \leq (\bar{u} - \bar{v})(x_0) \tag{4.2}$$

In addition, the right–hand side is strictly less than M by the strong maximum principle for harmonic functions. Hence we have a contradiction. An immediate consequence is that every sub solution for the Dirichlet problem is less than or equal to every super solution.

2. Let u be sub harmonic in D and let B be a ball with $\bar{B} \subset D$. Let $\bar{u}$ be the harmonic function on B satisfying $\bar{u} = u$ on ∂B. then the function

$$U(x) = \begin{cases} \bar{u}\,(x), & x\epsilon\, B \\ \quad u(x) & x\epsilon\, D \setminus B \end{cases} \tag{4.3}$$

is also sub harmonic in D. U is called the harmonic lifting of u with respect to B.

3. If $u_1, u_2, \dots, u_N$ are sub harmonic, then max $\{u_1, u_2, \dots, u_N\}$ is also sub harmonic.

4.2 Uniqueness

A consequence of comparison principle is the uniqueness of solution of the Dirichlet problem in a bounded domain. Now discusses the uniqueness of solution of a class of boundary- value problems with general boundary condition.

Theorem 4.2.1:- Let Ω be a bounded domain in $\mathfrak{R}^n$ and $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a solution of

$$\Delta u + cu = f \text{ in } \Omega$$

$$u = \varphi \text{ on } \partial\Omega \tag{4.4}$$

For some $f \in C(\overline{\Omega})$ and $\varphi \in C(\partial\Omega)$. If $c(x) \leq 0$ in Ω, then u is unique solution.

Proof: - Suppose *u* and *v* are any two solution of Dirichlet problem given above on Ω.

Let *w* =*u*-*v* .*w* satisfies $\Delta u + cu = 0$ in Ω and $w = 0$ on $\partial\Omega$, then by linearity of the problem

$$\Delta w = \Delta u - \Delta v = f - f = 0 \text{ in } \Omega$$

$$w = u - v = \varphi - \varphi = 0 \text{ on } \partial\Omega$$

Now applying maximum principle to *w*, we see that;

$0 = \min_{\partial\Omega} w(x) \leq \min_{\Omega} w(x) \leq \max_{\Omega} w(x) \leq \max_{\partial\Omega} w(x) = 0$

That is $w \equiv 0$ and so $u - v \equiv 0$. This implies that $u \equiv v$.

Another problem used to show uniqueness of solution is Neumann problem. We have seen this next.

Theorem 4.2.1:- Suppose Ω is bounded C^1-domain in $\Re^n$ c is continuous function in Ω and α is a continuous function on$\partial\Omega$. Let $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ be a solution of the boundary value problem

$$\Delta u + cu = f \text{ in } \Omega$$

$$\frac{\partial u}{\partial n} + \alpha u = \varphi \text{ on } \partial\Omega \tag{4.5}$$

For some $f \in C(\bar{\Omega})$ and $\varphi \in C(\partial\Omega)$. Assume, in addition that $c \leq 0$ in Ω and $\alpha \geq 0$ on $\partial\Omega$. Then u is the unique solution if $c \not\equiv 0$ or $\alpha \not\equiv 0$ .If $c \equiv 0$ and $\alpha \equiv 0$, u is unique up to additive constants.

Proof: - Suppose u is a solution of the homogenous problem

$$\Delta u + cu = 0 \text{ in } \Omega$$

$$\frac{\partial u}{\partial n} + \alpha u = 0 \text{ on } \Omega$$

Case 1-$c \not\equiv 0$ or $\alpha \not\equiv 0$. suppose that u has a positive maximum at $x_0 \in \bar{\Omega}$. If u is a positive constant, there leads to a contradiction to the condition $c \not\equiv 0$ in Ω or $\alpha \not\equiv 0$ on $\partial\Omega$,Otherwise $x_0 \in \partial\Omega$ and $\frac{\partial u}{\partial n}(x_0) > 0$ by Hopf lemma, which contradicts the boundary value. Therefore $u \equiv 0$.

Case 2 - $c \equiv 0$ and $\alpha \equiv 0$. Suppose u is a non-constant solution. Then its maximum in $\bar{\Omega}$ is assumed only on $\partial\Omega$ by strong maximum principle say at $x_0 \in \partial\Omega$. Again Hopf lemma implies that $\frac{\partial u}{\partial n}(x_0) > 0$; this contradiction show that u is a constant.

Remark: - The boundedness of domain D is essential, since it guarantees the existence of maximum and minimum of u in $\bar{\Omega}$. The uniqueness does not holds if domains are unbounded.

Example: - (1) u = sin x is a nontrivial solution of the problem

$$u'' + u = 0 \text{ in } (0, \pi) \qquad u(0) = u(\pi) = 0$$

This example is equally important for non-positiveness of the coefficient c.

(2) Consider the Dirichlet problem in an unbounded domain Ω

$$\Delta u = 0 \text{ in } \Omega$$

$$u = 0 \text{ on } \partial\Omega$$

First, we consider the case $\Omega = \{x \in \Re^n : |x| > 1\}$. we have a nontrivial solution u given by

$$u(x) = \begin{cases} \log|x| & \text{for } n = 2 \\ |x|^{2-n} & \text{for } n \geq 3 \end{cases}$$

Note that $u \to \infty$ as $|x| \to \infty$ for n=2 and u is bounded for $n \geq 3$. Next, we consider the upper half space $D = \{x \in \Re^n : x_n > 0\}$.Then $u(x) = x_n$ is a non-trivial solution, which is unbounded.

Theorem4.2.2:- The solution of Dirichlet problem continuously depend on the boundary value there assigned.

Proof: - Let *u* and *v* be two solution of Dirichlet problem in Ω satisfying

$$|u - v| \leq \varepsilon\, , \varepsilon \geq 0 \quad \text{on } \Omega \tag{4.6}$$

According to maximum principle their difference u-v cannot exceed ε and less than $-\,\varepsilon$ in the interior of D, otherwise it would achieve a positive maximum or a negative minimum there. It follow that the inequality $|u - v| \leq \varepsilon, \varepsilon \geq 0$ is valid inside D too, which provides us with consistently satisfactory statement about continuous dependence values.

$$\Longrightarrow |w| \leq \varepsilon \quad \text{in } \Omega$$
$$\text{But } |w| = 0 \quad \text{on } \partial\Omega$$

Hence by maximum principle *w=0* throughout Ω.

$$u - v = 0 \text{ in } \Omega$$
$$u = v \quad \text{on } \Omega$$

Hence, the solution is unique.

Now applying maximum principle to the general second order PDE.

Consider the general elliptic equation

$$\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{i=1}^{n} b_i u_{x_i} + cu = 0, c \leq 0 \tag{4.7}$$

We consider elliptic equation with $c=0$ .And assume that the remaining coefficients are continuous. It is clear that in this case any $u=\alpha$, a constant can be a solution of Dirichlet problem.

Maximum principle states that constants are the only solutions, which can assume a maximum, or a minimum in the interior of their domain Ω, no solution can possess a maximum or a minimum at an interior point of its domain of existence.

Every solution of Dirichlet problem achieves its maximum and minimum values on the boundary of Ω ($\partial\Omega$) of any region Ω where it is known to be continuous.

4.3 Existence

In this section, we will establish existence of solution for the Dirichlet problem. Specifically, we shall prove existence theorem. It will be evident from the proof that the assumption that $\partial\Omega$ is of a class C^2 can be relaxed; for example; all convex domain are permissible.

The proof will be based on the ideas of Perron, which make use of the following notations. We call v a subsolution (supersolution) if $\Delta v \geq 0 (\Delta v \leq 0)$ in Ω and $v \leq g (v \geq g)\ on\ \partial\Omega$. Obviously subsolutions exists, e.g., every sufficiently small constant is a subsolution. The maximum principle (weak form) shows that if u is a solution, then $v \leq u$ for every subsolution (this why we call them subsolutions). Thus, the point wise supermum of all solutions (which is a well-defined function) gives us an obvious candidate for a solution. If we can show that this function actually solves the problem, our existence proof will be complete. Before we prove this, we first need to develop a number of prerequisites.

4.3.1 The Dirichlet problem on a ball

One of important properties of the Laplace equation is its spherical symmetry. The equation is preserved by rotations about some point $a \in \Re^n$. Hence, it is plausible that there exists special solutions that are invariant under rotations about a. We begin this section by seeking a harmonic function $u\ in\ \Re^n$which depends only on $r = |x - a|$ for some fixed $a \in \Re^n$ by setting $v(r) = u(r)$, we have

$$\Delta u = v'' + \frac{n-1}{r} v' = 0, \tag{4.8}$$

and hence
$$v(r) = \begin{cases} c_1 + c_2 \log r \ \ for\ n = 2 \\ c_3 + c_4 r^{2-n}\ for\ n \geq 3 \end{cases} \tag{4.9}$$

where c_i are constants for i=1,2,3,4. We are interested in solution with a singularity such that

$$\int_{\partial B_r} \frac{\partial v}{\partial r} d\sigma = 1 \quad \text{for any } r > 0 \tag{4.10}$$

Hence we set for any fixed $a \in \Re^n$

$$\Gamma(a,x) = \begin{cases} \dfrac{1}{2\pi}\log|a-x| \;\; for\; n=2 \\ \dfrac{1}{\omega_n(2-n)}|a-x|^{2-n} for\; n \geq 3, \end{cases} \tag{4.11}$$

where ω_n is the surface area of the unit sphere in $\Re^n$. In summary, for any fixed $a \in \Re^n$, $\Gamma(a,.)$ is harmonic in $\Re^n\backslash\{0\}$, $i.e.$,

$$\Delta_x\Gamma(a,x) = 0 \text{ for any } x \neq 0 \tag{4.12}$$

and has a singularity at $x=a$. Moreover,

$$\int_{\partial B_r(x)} \frac{\partial\Gamma}{\partial n_x}(a,x)dS_x = 1 \quad \text{for any } r > 0 \tag{4.13}$$

The function Γ is called the fundamental solution of the Laplace operator.

Now to discuss the Dirichlet boundary-value problem

$$\Delta u = f \text{ in } \Omega$$

$$u = \varphi \text{ on } \partial\Omega$$

Finding Green's function involves solving a Dirichlet problem for the Laplace equations. Meanwhile, Green's functions are introduced to solve Dirichlet problems. We need to break this cycle. In fact, we can construct Green's functions explicitly for special domains. the next result yields an expression for Green's functions in a balls.

Theorem4.1.1:- The Green's function in a ball $B_R \subset \Re^n$ is given by

$$G(x,y) = \frac{1}{2\pi}\left(\log|x-y| - \log\left|\frac{R}{|x|}x - \frac{|x|}{R}y\right|\right) \quad \text{for } n=2, \tag{4.14}$$

$$G(x,y) = \frac{1}{(2-n)\omega_n}\left(|x-y|^{2-n} - \left|\frac{R}{|x|}x - \frac{|x|}{R}y\right|^{2-n}\right) \quad \text{for any } n \geq 3 \tag{4.15}$$

For $x=0$, we have

$$G(0,y) = \frac{1}{(2-n)\omega_n}(|y|^{2-n} - R^{2-n}) \quad \text{for any } n \geq 3$$

and $$G(0,y) = \frac{1}{2\pi}(\log|y| - \log R) \quad \text{for any } n = 2$$

Proof:-We need to adjust the fundamental solution by adding a harmonic function so that the sum vanishes on the boundary. Fix an $x \neq 0$ with $|x| < R$, and consider $X \in \Re^n/\bar{B}_R$ given by $X = \frac{R^2}{|x|^2}x$. In other words, X and x are reflexive of each other with respect to the sphere ∂B_R. Note that the map $x \to X$ is conformal, i.e., preserves angles. For $|y| = R$, we have by the similarity of triangles

$$\frac{|x|}{R} = \frac{R}{|X|} = \frac{|y-x|}{|y-X|},$$

which implies

$$|y-x| = \frac{|x|}{R}|y-X| = \left|\frac{|x|}{R}y - \frac{R}{|x|}x\right| \qquad \text{for any } y \in \partial B_R. \tag{4.16}$$

Therefore, in order to have a zero boundary value, we take for $n \geq 3$.

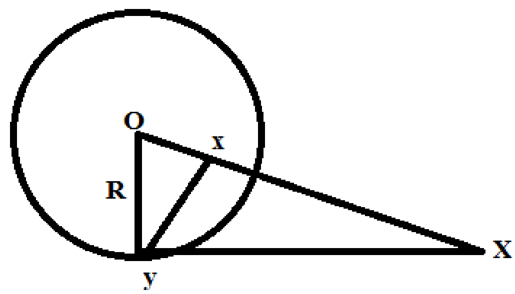


figure4.1 The reflection about the sphere

$$G(x,y) = \frac{1}{(2-n)\omega_n}\left(\frac{1}{|x-y|^{n-2}} - \left(\frac{R}{|x|}\right)^{n-2}\frac{1}{|y-X|^{n-2}}\right) \tag{4.17}$$

We point out that the second term is smooth for $y \in B_R$, since $X \notin \bar{B}_R$ .The case *n=2* is similar.

Next, we calculate normal derivatives of the Green's function on sphere.

Corollary4.1.2 Suppose G is the Green's function in B_R as in theorem 4.1.1 .Then

$$\frac{\partial G}{\partial n_y}(x,y) = \frac{R^2 - |x|^2}{\omega_n R|x-y|^n} \quad \text{for any } x \in B_R \text{ and } y \in \partial B_R. \tag{4.18}$$

Proof:- We only consider the case $n \geq 3$ with $X = \frac{R^2 x}{|x|^2}$, we have

$$G(x,y) = \frac{1}{(2-n)\omega_n}\left(|y-x|^{2-n} - \left(\frac{R}{|x|}\right)^{n-2} |y-x|^{2-n}\right),$$

for any $x \in B_R$ and $y \in \partial B_R$. Hence we get for such x and y

$$G_{y_i}(x,y) = \frac{1}{\omega_n}\left(\frac{y_i - x_i}{|y-x|^n} - \left(\frac{R}{|x|}\right)^{n-2} \frac{y_i - X_i}{|y-X|^n}\right) = \frac{y_i}{\omega_n R^2}\frac{R^2 - |x|^2}{|x-y|^n}$$

by (4.3) $\left(|y-x| = \frac{|x|}{R}|y-X| = \left|\frac{|x|}{R}y - \frac{R}{|x|}x\right| \text{ for any } y \in \partial B_R\right)$

in the proof of theorem 4.1.1 with $n_i = \frac{y_i}{R}$ for $|y| = R$, we obtain

$$\frac{\partial G}{\partial n_y}(x,y) = \sum_{i=1}^{n} n_i G_{y_i}(x,y) = \frac{1}{\omega_n R}.\frac{R^2 - |x|^2}{|x-y|^n}$$

This yields the desired results.

The function in corollary 4.1.2 denoted by *K(x,y)* , i.e.,

$$K(x,y) = \frac{R^2 - |x|^2}{\omega_n R|x-y|^n} \quad \text{for any } x \in B_R, y \in \partial B_R. \tag{4.19}$$

It is called the Poisson kernal and has the following properties.

(K1) K(x,y) is smooth for any $x \in B_R$ and $y \in \partial B_R$.

(K2) K(x,y) > 0 for any $x \in B_R$ and $y \in \partial B_R$.

(K3) For any fixed $x_0 \in \partial B_R, \lim_{x \to x_0} K(x,y) = 0$ $|x| < R$ uniformly in y for $|y - x_0| > \delta > 0$.

(K5) $\Delta_x K(x,y) = 0$ for any $x \in B_R$ and $y \in \partial B_R$.

(K5) $\int_{\partial B_R} K(x,y) dS_y = 1$ for any $x \in B_R$.

Here (K1), (K2) and (K3) follows easily from the explicit expression for K and (K4) follows easily from the definition $K(x,y) = \frac{\partial G(x,y)}{\partial n_y}$ and the fact that G(x,y) is harmonic in x. By taking a $C^2(\bar{B}_R)$ harmonic function u, we conclued

$$u(x) = \int_{\partial B_R} K(x,y)u(y)dS_y \quad \text{for any } |x| < R. \tag{4.20}$$

This is called the poisson integral formula . Then we have (K5) easily by taking $u \equiv 1$.

Now, we are ready to solve the Laplace equation in a ball with prescribed Dirichlet boundary value.The most old – fashioned way to prove existence of solutions is to give a formula for them. We now obtain an explicit solution for the Dirichlet problem on a ball.

Theorem 4.1.2 Let B be the ball of radius R centered at the origin and let g be continuous on ∂B. then the function

$$u(x) = \frac{R^2 - |x|^2}{n\omega_n R} \int_{\partial B} \frac{g(y)}{|x-y|^n} dS_y \tag{4.21}$$

is of class $C^2(B)$ and satisfies $\Delta u = 0.$ More over, for every $y \epsilon\ \partial B,$ we have

$$\lim_{x \to y} u(x) = g(y) \tag{4.22}$$

Here ω_n denotes the volume of the unit ball in R^n, and the notation ds_y in the surface integral indicates the variable of integration.

Proof: - Since we can differentiate under the integral, u is in fact of class C^∞in B and it is a simple calculation to show that it is a harmonic function. It remains to establish (4.1.2). Let

$$K(x,y) = \frac{R^2 - |x|^2}{n\omega_n R|x-y|^n}, x \in B, y \in \partial B,$$

and

$$\psi(x) = \int_{\partial B} K(x,y) dS_y. \tag{4.23}$$

The above $\psi(x)$ is the special case g=1in (4.1.1); hence ψ is harmonic. It is also obvious that ψ is a radially symmetric function. For radially symmetric functions, Laplace's equation reads

$$u_{rr} + \frac{n-1}{r} u_r = 0$$

Moreover, the only solutions of this equation, which are regular at the origin, are constants. Hence $\psi(x) = \psi(0) = 1$.

Now let $x_0 \epsilon\ \partial B$ and let $\varepsilon \geq 0$ be given. Choose $\delta\ \geq 0$ so that $|g(y) - g(x_0)| \leq \varepsilon$ for

$|y - x_0| < \delta$ and let Mbe an upper bound for g on ∂B. For $|x - x_0| < \frac{\delta}{2}$, we have

$$|u(x) - g(x_0)| = \left| \int_{\partial B} K(x,y)\big(g(y) - g(x_0)\big) dS_y \right|$$

$$\leq \int\limits_{|y-x_0|\leq\delta} K(x,y)|g(y)-g(x_0)|\,dS_y \tag{4.24}$$

$$+ \int\limits_{|y-x_0|\geq\delta} K(x,y)|g(y)-g(x_0)|\,dS_y$$

$$\leq \varepsilon + \frac{2M(R^2-|x|^2)R^{n-2}}{\left(\delta/_2\right)^n}$$

As x→ x_0, the last term on the right-hand side tends to zero and the theorem follows.

Definition4.1 The Laplace operator Δ is defined on a C^2-function u in a domain$\Re^n$ given by

$$\Delta u = \sum_{i=1}^{n} u_{x_i x_i} \tag{4.25}$$

The equation $\Delta u = 0$ is called the Laplace equation and its C^2-solutions are called harmonic functions. We have seen this in section (1).

By simple calculation that harmonic functions satisfy the mean value property.

Theorem 4.1.1:-Let Ω be a domain in $\Re^n$ and $u \in C^2(\Omega)$ be harmonic in Ω. Then

$$u(x) = \frac{1}{\omega_n r^{n-1}} \int\limits_{\partial B_r(x)} u(y)dS_y \quad \text{for any } B_r(x) \subset \Omega \tag{4.26}$$

Where ω_n is the area of the unit sphere∂B_1 in $\Re^n$.

Proof:- Take any ball $B_r(x) \subset D$. For any $\rho \in (0,r)$, we apply the Green's formula in $B_r(x)$ and get

$$\int\limits_{B_\rho(x)} \Delta u = \int\limits_{\partial B_\rho(x)} \frac{\partial u}{\partial n} ds = \rho^{n-1} \int_{\partial B_1} \frac{\partial u}{\partial \rho}(x+\rho\varphi)\, ds_\omega$$

$$= \rho^{n-1} \frac{\partial}{\partial \rho} \int\limits_{\partial B_1} u(x+\rho\omega) ds_\omega$$

Hence for the harmonic function u, we have $\rho \in (0,r)$

$$\frac{\partial}{\partial \rho} \int_{\partial B_1} u(x+\rho\omega) ds_\omega = 0$$

Integrating from 0 to r, we obtain

$$\int_{\partial B_1} u(x+r\omega) ds_\omega = \int_{\partial B_1} u(x) ds_\omega = u(x)\omega_n$$

or

$$u(x) = \frac{1}{\omega_n} \int_{\partial B_r} u(x+r\omega) ds_\omega$$

A simple change of variables yields the desired result.

The above theorem asserts that harmonic functions equal to their mean-value properties, mean-value on a sphere and mean-value on a ball.

Definition:- We assume that Ω is a domain in $\mathfrak{R}^n$. For $u \in C(\Omega)$,

i. *u* satisfies the first mean-value property if

$$u(x) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x)} u(y) dS_y \quad \text{for any } B_r(x) \subset \Omega.$$

ii. *u* satisfies the second mean-value property if

$$u(x) = \frac{n}{\omega_n r^{n-1}} \int_{B_r(x)} u(y) dy \quad \text{for any } B_r(x) \subset \Omega,$$

Where ω_n denotes the surface area of the unit sphere in $\mathfrak{R}^n$ we note that $\omega_n r^{n-1}$ is the surface area of the sphere $\partial B_r(x)$ and $\frac{\omega_n r^n}{n}$ is the volume of the ball $B_r(x)$.

The two definitions are equivalent. In fact, if we differentiate two to get one.

For function u satisfying mean-value properties, u is required only to be C^2.

Now we prove maximum principle for harmonic function.

Theorem4.1.2:- Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be harmonic in Ω. Then u assumes its maximum and minimum only on $\partial\Omega$ unless u is a constant.

proof:- we prove only for the maximum. Set

$$G = \left\{ x \in D : u(x) = M = \max_{\overline{\Omega}} u \right\} \subset \Omega \tag{4.27}$$

It is obvious that G is relatively closed; namely, for any sequence $\{x_m\} \subset G, if\, x_m \to x \in \Omega$ then $x \in G$. This follow easily from the continuity of u. Next, we show that G is open. For any $x_0 \in G$, take $\bar{B}_r(x_0) \subset \Omega$ for some $r > 0$ by mean value property, we have

$$M = u(x_0) = \int_{B_r(x_0)} u(y) dy \leq M \frac{n}{\omega_n r^n} \int dy = M$$

This implies that $u = M$ in $B_r(x_0)$. Hence G is relatively closed and open in Ω.

Therefore, either G= $\emptyset$ or G=Ω. A similar argument can be used to prove the minimum.

Definition4.3.1 Let f_m be a sequence of real valued functions defined in a sub set Ω of R^n. Let x ϵ Ω.The sequence is called ***equicontinuous*** at x if, for every $\varepsilon > 0,$ there is $\delta \geq 0$, independent of *m*, such that $|f_m(y) - f_m(x)| < \varepsilon$ for $y\epsilon$ Ω with $|y - x| < \delta$.

If a sequence f_m is equicontinuous at each point of a compact set S, it is uniformly equicontinuous, i.e., δ in the definition above can be chosen independently of x ϵ S (it is not necessary that Ω=S). We note that a sequence of function is equicontinuous at x if there exists a bound (independent of *m*) for the derivatives in some neighborhood of x.

Theorem4.3.2 (Arzela-Ascoil).Let f_m be a sequence of real- valued functions defined on a compact subset S of$\Re^n$. Assume that there is a constant M such that$|f_m(x)| \leq M$ for every $m \in \mathbb{N}$ and every $x \in$ S. Moreover, assume that the sequence f_m is equicontinuous at every point of S. Then there exists a subsequence, which converges uniformly on S.

We remark that (with rather obvious modifications) the theorem can be extended to the case where S is a compact set in an abstract topological space and the values of f_m are in complete metric space.

Proof: - Let=$x_{i,}$ i $\epsilon\mathbb{N}$ be a sequence of points that is dense in S. The sequence $f_m(x_1)$ is bounded; hence, it has a convergent subsequence. That is, we can choose a sequence m_{1j} *such that* $f_{m_{1j}}(x_1)$ converges as $j \to \infty$. similarly, we can choose a subsequence m_{2j} of a sequence $f_{m_{1j}}$ such that $f_{m_{2j}}(x)$ converges. Since m_{2j} is a subsequence of $m_{1j}, f_{m_{2j}}(x)$ converges as well. Next, we choose a subsequence m_{3j} of a sequence m_{2j} such that , $f_{m_{3j}}$ converges also at $x_{3.}$ we precede in this manner ad infinitum. Finally, consider the "diagonal" sequence $f_{m_{jj}}$. Except the first n-1 terms, m_{jj} is a subsequence of m_{ij}; hence $f_{m_{jj}}(x_i)$ converges for every i ϵ $\mathbb{N}$. To simplify notation, we shall set $g_i = f_{m_{jj}}$in the following.

To conclude the proof, we show that the sequence g_m is uniformly Cauchy. Let $\varepsilon \geq 0$ be given. The g_m , being a subsequence of the f_m, are uniformly equicontinuous on S; hence there is a $\delta \geq 0$ such that $|g_m(y) - g_m(x)| < \frac{\varepsilon}{3}$ whenever $|y - x| < \delta$. Since S is compact, there is a $K \in \mathbb{N}$ such that for every xϵ S there exists i ϵ $\{1,2, \dots, K\}$ with

$|x_i - x| < \delta$. Now choose N large enough so that $|g_m(x_i) - g_k(x_i)| < \frac{\varepsilon}{3}$ for m, kN and every $i\epsilon$ $\{1, \dots, K\}$. For $m, k > N$ and arbitrary $x\epsilon$ S,

We now have

$$|g_m(x) - g_k(x)| \leq |g_m(x) - g_m(x_i)| + |g_m(x_i) - g_k(x_i)| + |g_k(x_i) - g_k(x)| < \varepsilon$$

For some i ϵ $\{1, \dots, K\}$.

Below, we shall have to apply the Arzela – Ascoli theorem to sequence of harmonic functions. In this particular case, we have the following result.

Theorem4.3.2 Let Ω is a domain in R^n.Let f_m be a sequence of harmonic functions on Ω, which is uniformly bounded, i.e., $|f_m(x)| \leq M$ for every $x\epsilon\ D$ and $m\epsilon\ \mathbb{N}$. Then f_m has a subsequence, which converges to a harmonic function on Ω, uniformly on compact subset of Ω.

Proof: - Let $\Omega_k = \{\ x\epsilon\ \Omega: |x| \leq k, \operatorname{dist}(x, \partial\Omega) \geq {}^{1}/_{k}\ \}$.Then Ω_k is compact and $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$. If x is in$\Omega_K$, then, in a neighborhood of x, we can represent f_m by a Poisson's formula, using a ball centered at x with radius${}^{1}/_{2k}$. This yields uniform estimates for the derivatives of$f_m\ on\ \Omega_k$. In particular, the f_m are equicontinuous onΩ_k. By the Arzela-Ascoli theorem, we can extract a uniformly convergent subsequence on eachΩ_k, and using a diagonal argument similar to that used in the proof above, we obtain a subsequence converging on all of Ω, uniformly on eachΩ_k.

It remains to show that the limits of this subsequence is harmonic. To see this, we simply note that harmonic functions are characterized by the property that they obey Poisson's formula on every ball. If Poisson's formula is obeyed by all functions in a uniformly convergent sequence, it also holds for the limit.

In this section, we shall establish existence of solution for the Dirichlet problem. Specifically, we shall prove the following theorem:

Theorem4.2.3 (Existences of the solution of Dirichlet problem) Let Ω be a bounded domain in $\Re^n$ witha $C^2 -$ boundary . Then for any function $g\epsilon\ C(\partial\Omega)$, there is a unique $u\epsilon C^2(\Omega) \cap C(\overline{\Omega})$satifing $\Delta u = 0$ and $u = g$ on $\partial\Omega$.

Proof: - We now construct a solution following the Perron method. Let

$$S_g = \{v\epsilon C(\overline{\Omega}) : v \text{ is subharmonic}, v \leq g \text{ on } \partial\Omega\}.$$

Define $u(x) = \sup_{v\epsilon S_g} v(x) < \infty$. this is well defined, because every subsolution is less than or equal to every supersoltion sub/supersoltion exist by choosing appropriate;

$$v_0(x) = \min_{\partial D} g = \text{const}\ \epsilon S_g. \tag{4.28}$$

Bounded from above by

$$v(x) = \max_{\partial D} g = \text{const} \tag{4.29}$$

Step 1- u is harmonic in Ω.

Let x be an arbitrary point in Ω and v_m be a sequence in S_g such that $v_m(x) \rightarrow u(x)$.From above v_m is bounded (by any supersoltion), we may also assume it is bounded from below; if necessary, replacev_m by $\max(v_m, v_0)$, where v_0 is any subsolution. Choose R such that the closure of $B = B_R(x)$is contained in Ω and let V_m be the harmonic lifting of $v_m$with respect to B. Then $V_m(x) \to u(x)$ and, by theorem 4.2.2 above, V_m has a subsequence V_{m_k} which converges on B to a harmonic function *v;* the convergence is uniform on every compact subset of with $v \leq u \; in \; B$ and *u(x) =v(x).*We aim to show that *v=u* in B.

Assume on the contrary that $v(y) < u(y)$forsome $y\epsilon B.$ Then there exists a function $W\epsilon S_g$ such that $v(y) < W(y). Let \; w_k = \max(W, V_{m_k}) \; and \; let \; W_k$ be the harmonic lifting of w_k with respect to B. As before, a subsequence of W_k converges to a function *w*, harmonic in B. Then since $v_{m_k} \leq V_{m_k} \leq w_k \leq W_k,$ we have $v \leq w.$ Moreover, as $v_{m_k}(x) \to u(x),$ and $w_k \in S_g$,we have *v(x) =w(x).* Hence, the strong maximum principle applied to the harmonic function *v-w* on B implies *v=w* in B, thus contradicting the choice of *W.*

Step 2- *u* satisfies the boundary data.

It remains to show that *u* is actually continuous up to $\partial\Omega$ and assumes the given boundary data. Let ξ is a point on $\partial\Omega.$ Since $\partial\Omega$ was assumed of class $C^2,$ there is a ball $B = B_R(y)$ in the exterior of Ω such that $B \cap \Omega = \emptyset \; and \; \bar{B} \cap \bar{\Omega} = \{\xi\}.$

We now define

$$\omega(x) = \begin{cases} R^{2-n} - |x-y|^{2-n} & ,n \geq 3 \\ \log\dfrac{|x-y|}{R} & ,n = 2 \end{cases} \tag{4.30}$$

Then

1. ω is subharmonic (actually harmonic) in Ω.
2. $\omega > 0$ on $\bar{\Omega}\backslash\{\xi\},\ \omega(\xi) = 0.$

A continuous function with property (1) and (2) is called a ***barrier*** at ξ relative to Ω. We have

Then for n$\geq$ 3

$$\partial_i^2\omega(x) = \partial_i\left(|x-y|^{1-n}(n-2)\frac{x_i - y_i}{|x-y|}\right)$$
$$= (n-2)\left[-n|x-y|^{-(n+1)}\frac{(x_i-y_i)^2}{|x-y|} + |x-y|^{-n}\right]$$

So then summing over the index I we have

$$\sum_{i=1}^{n} \partial_i^2\,\omega(x) = (n-2)[n|x-y|^{-n} - n|x-y|^{-n-2}|x-y|^2]$$

=0

Similarly for n=2

$$\left|\partial_i^2\omega(x) = \partial_i\left(|x-y|^{-1}\frac{R(x_i-y_i)}{|x-y|}\right)\right|$$
$$= R\left[-2|x-y|^{-3}\frac{(x_i-y_i)^2}{|x-y|} + |x-y|^{-2}\right]$$

And $\partial_1^2\omega + \partial_2^2\omega = 0$

Hence ω is harmonic. Furthermore, $\omega \geq 0$ on $\overline{\Omega} \setminus \{\xi\}$ and $\omega(\xi) = 0$. Now we are in a position to prove continuity.

Let$\varepsilon \geq 0$ be arbitrary, let m=max$_{\partial D}|g|$, let δ, k be positive constants such that $|g(x) - g(\xi) < \varepsilon$ for $x \in \partial\Omega, x - \xi < \delta$ and $k\omega x \geq 2m$ for all $x \epsilon \Omega$ such that $x - \xi \geq \delta$.The existence of $k > 0$ is ensured by $\frac{\omega}{2m} > 0$ on a compact set $\partial\Omega \cap \{|x-\xi| \geq \delta\}$

Since, for $x\epsilon\ \partial\Omega$ with $|x-\xi| \leq \delta$ we have

$$\mathrm{g}(\xi) - \varepsilon - \mathrm{k}\omega(\mathrm{x}) \leq \mathrm{g}(\mathrm{x}) - \mathrm{k}\omega(\mathrm{x}) \leq \mathrm{g}(\mathrm{x}),$$

And for $x\ \epsilon\ \partial D$ with $|x-\xi| \geq \delta$ we have

$$g(\xi) - \varepsilon - k\omega(x) \leq g(\xi) - \varepsilon - 2m \leq -m - \varepsilon \leq g(x)$$

Then the function $g(\xi) - \varepsilon - k\omega(x)$ is a subsolution. Similarly, for $x\epsilon\partial D$ we have

$$g(\xi) + \varepsilon + k\omega(x) \geq g(x)$$

Hence, the function $g(\xi) + \varepsilon + k\omega(x)$ is a supersoltion. Hence we obtain that;

$$g(\xi) - \varepsilon - k\omega(x) \leq \mathrm{u}(\mathrm{x}) \leq \mathrm{g}(\xi) + \varepsilon + \mathrm{k}\omega(\mathrm{x}).$$

Since $\omega(x) \to 0$as $x \to \xi, u(x) \to g(\xi)$.

4.4 A priori estimates

Proving the uniqueness of boundary value problems is an important application of maximum principles. Equally important or more important is to derive a prior estimates, essential steps consists of construction of auxiliary functions. Next, we derive a prior estimate for solution of Dirichlet problem.

Theorem4.4.1:- Suppose $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies

$$\Delta u + cu = f \ \text{ in } \Omega$$

$$u = \varphi \ \ \text{on } \partial\Omega$$

for some $f \in C(\overline{\Omega})$ and $\varphi \in C(\partial\Omega)$. If $c(x) \leq 0$, then

$$\max_{\Omega}|u| \leq \max_{\partial\Omega}|\varphi| + C\max_{\Omega}|f| \tag{4.31}$$

Where C is a positive constant depending only on n and diam(Ω).

If $\Omega = B_R(x_0)$, then

$$\max_{B_R(x_0)}|u| \leq \max_{\partial B_R(x_0)}|\varphi| + \frac{R^2}{2n}\max_{B_R(x_0)}|f|$$

This follows from the proof below easily.

Proof: - we construct an auxiliary function w in Ω such that

(a) $(\Delta + c)(w \pm u) = (\Delta + c)w \pm f \leq 0, or\ (\Delta + c)w \leq \mp f$ in Ω;
(b) $w \pm u = w + \varphi \geq 0$, or $w \geq \mp\varphi$ on $\partial\Omega$

Set

$$F = \max_{\Omega}|f|\ , \qquad \Phi = \max_{\partial\Omega}|\varphi| \tag{4.32}$$

We need

$$\Delta w + cw \leq -F \text{ in } \Omega$$

$$w \geq \Phi \text{ on } \partial\Omega$$

Without loss of generality, we assume that Ω is contained in a ball of radius R and centered at the origin, i.e.$\Omega \subset B_R$. Set

$$w = \Phi + \frac{F}{2n}(R^2 - |x|^2) \tag{4.33}$$

Then by $c \leq 0$ and $|x| \leq R$ in Ω, we have

$$\Delta w + cw = -F + c\Phi + \frac{cF}{2n}(R^2 - |x|^2) \leq -F$$

We also have $w \leq \Phi$ on $\partial\Omega$. Hence w satisfies (a) and (b), by comparison principle, we conclude $-w \leq u \leq w$ in Ω, and in particular,

$$|u(x)| \leq \Phi + \frac{1}{2n}(R^2 - |x|^2)F \quad \text{for any } x \in \Omega$$

This yields the desired results.

Now consider a class of general boundary value problems.

Theorem 4.4.2:- Suppose $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies

$$\Delta u + cu = f \text{ in } \Omega$$

$$\frac{\partial u}{\partial n} + \alpha u = \varphi \quad \text{on } \partial\Omega$$

For some $f \in C(\overline{\Omega})$ and $\varphi \in C(\partial\Omega)$. $If\ c(x) \le 0$ in Ω and $\alpha(x) \ge \alpha_0$ on $\partial\Omega$ for a positive constant α_0, then

$$\max_{\Omega}|u| \le C\left(\max_{\partial\Omega}|\varphi| + \max_{\Omega}|f|\right) \tag{4.34}$$

Where C is a positive constant depending only on α_0 and diam(Ω).If $c \equiv 0$ in Ω and $\alpha \equiv 0$ on $\partial\Omega$, the homogenous problem in the above theorem (with $f \equiv 0$ in Ω and $\varphi \equiv 0$ on $\partial\Omega$) admits a nontrivial solution $u \equiv 1$ in Ω. Hence there does not hold a sup-norm estimates in this case.

Proof: - By assuming $\Omega \subset B_R$ for some $R > 0$, we prove

$$\sup_{\Omega}|u| \le \frac{1}{\alpha_0}\max_{\partial\Omega}|\varphi| + \frac{1}{2n}\left(\frac{1+R^2}{\alpha_0} + R^2\right)\max_{\Omega}|f|$$

Set

$$\Phi = \max_{\partial\Omega}|\varphi| \ , F = \max_{\Omega}|f|$$

and

$$v(x) = \frac{1}{\alpha_0}\Phi + \frac{1}{2n}\left(\frac{1+R^2}{\alpha_0} + R^2 - |x|^2\right)F$$

Then

$$\Delta v + cv = -F + cv \le -F \text{ in } \Omega$$

and

$$\frac{\partial v}{\partial n} + \alpha v = \frac{\alpha}{\alpha_0}\Phi + \frac{F}{2n}\left(-2xn + \alpha\left(\frac{1+R^2}{\alpha_0} + R^2 - |x|^2\right)\right)$$

$$\ge \Phi + \frac{F}{2n}(-2xn + 1 + R^2) \ge \Phi \ \text{ on } \partial\Omega$$

Where n is the unit exterior normal vector of $\partial\Omega$ with $w = v \pm u$

We have

$$\Delta w + cw \le 0 \text{ in } \Omega$$

$$\frac{\partial y}{\partial x} + \alpha w \ge 0 \quad \text{on } \partial\Omega$$

It is easy to prove that $w \ge 0$ in Ω or $|u| \le v$ in Ω.

This yields the desired results.

Summary

Overall, it is important to understand the power of various form of maximum principle in the study of PDEs. Using relatively simple and only few technical tools, one can achieve a wide variety of results concerning the behavior of solutions and properties of elliptic differential operators. Many of the proof use similar approaches and the results obtained extend well to more general cases for the most parts.

In this paper, we shall use this idea in the study of second order elliptic equation. The maximum principle for second order elliptic equations is well known. Here we shall show that properties of weak and strong maximum principle in linear operator. When we see weak maximum principle can be, prove for inequalities of the form $\sum_{i,j=1}^{n} a_{ij}u_{x_i x_j} \geq 0$. On the other hand, strong maximum principle can be proving by using interior sphere condition.
In general, we see the uniqueness and existence of solution of some boundary problem especially in the Dirichlet problem in a bounded domain, which is a consequence of comparison principle. Furthermore, possible application of maximum principle is to obtain a prior estimate for the gradients of the solution $u \; and \; \Delta u$.

Bibliography

[1] D.Gilberg et al.: *Elliptic partial differential equation of second order*; Springer (1977)

[2] P.R.GARBEDIAN: *Partial differential equations*; WILLEY (1964)

[3] Paul W.Berg: *Elementary partial differential equations;* Holden-Day (1966)

[4] Richard Bellman: *Partial differential equation;* D.Reide (1980)

[5] R.P. Agarwal*: Boundary value problem for higher order differential equations*; World scientific (1986)