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On Strongly Regular near rings

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The undersigned here by certify that they have read and recommend to the school of graduate studies for acceptance of a project entitled **on strongly regular near rings** by Dereje Teshome in partial fulfillment of the requirements for the degree of master of Science.

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Abstract

As the name suggests, a near-ring is a generalized ring more precisely the commutativity of addition is not required and just one of the distributive laws is postulated. Many parts of the well established theory of rings were transferred to near-rings and now near-ring specific features were discovered, building up a theory of near-rings step by step. Clearly, every ring is a near-ring. But we can give examples of near-rings which are not rings. The most common example is the set of all mappings of a group (not necessarily abelian) into itself, under point wise addition and composition of maps. The present study focusses mainly on strongly regular near-rings. An attempt is made in this paper to concentrate on characterizations and generalizations of strongly regular near-rings.

Notations

N Near rings.

\subseteq Inclusion.

\subset strict inclusion.

\in An element of N .

$C(N) = \{ n \in N : nx = xn \text{ for all } x, y \in N \}$

Introduction

The paper consists of three chapters. chapter 1 presents some basic definitions and result near rings. which are used in the subsequent chapters. In chapter 2, a relation between strong regularity and strong reducedness is established. We show that a near-ring is left strongly regular if and only if it is strongly reduced and regular. In the beginning part of this chapter, we establish the equivalence of left regularity, left strong regularity and right regularity and the significance of this equivalence is the role of strong reducedness for proving it. We prove that for certain special near-rings, these three concepts are equivalent to right strong regularity. In chapter 3, we find some properties and characterizations of strongly regular near-rings. We show that a near-ring N is strongly regular if and only if N is right semicentral and regular if and only if N is strongly reduced and regular if and only if $A = \sqrt{A}$, for every N -subgroup A of N . Also, we show that a zero-symmetric near-ring N is strongly regular if and only if N is a regular IFP near-ring.

Chapter 1

PRELIMINARY DEFINITIONS AND RESULTS

In the first section we just recall the definition of near rings and give some examples. Section two is devoted to the introduction of N-groups, homomorphism and ideal like substructures.

1.1 Definitions

Definition 1.1.1. *A near-ring is a set N together with two binary operations addition $(+)$ and multiplication (\cdot) such that*

- i) $(N, +)$ is a group, (not necessarily abelian),*
- ii) (N, \cdot) is a semigroup,*
- iii) $(x + y) \cdot z = x \cdot z + y \cdot z$, for every $x, y, z \in N$ (right distributive law).*

Then $(N, +, \cdot)$ is said to be right near rings.

Instead of (iii) If N satisfies

- iv) $x \cdot (y + z) = x \cdot y + x \cdot z$, for every $x, y, z \in N$, then $(N, +, \cdot)$ is said to be a left near-ring.*
- we will write xy for $x \cdot y$ just for simplicity of notation.*

Here after a near-ring means a right near-ring only.

Example 1.1.1. *Let \mathbb{Z} be the set of positive and negative integers with 0. $(\mathbb{Z}, +)$ is a group.*

Define " \cdot " on \mathbb{Z} by $a \cdot b = a$ for all a, b in \mathbb{Z} .

Then $(\mathbb{Z}, +, \cdot)$ is a near-ring.

To show this

i) $(\mathbb{Z}, +)$ is a group.

ii) let a, b, c in \mathbb{Z} then $a(bc) = a = ab = (ab)c$.

Therefore $a(bc) = (ab)c$.

Hence (\mathbb{Z}, \cdot) is semigroup.

iii) $(a + b)c = a + b = ac + bc$ for every a, b, c in \mathbb{Z} .

Therefore right distributive law holds.

From i,ii and iii $(\mathbb{Z}, +, \cdot)$ is near rings.

Example 1.1.2. Let $\mathbb{Z}_{12} = \{ 0, 1, 2, \dots, 11 \}$. $(\mathbb{Z}_{12}, +)$ is a group under $(+)$ modulo 12.

Define (\cdot) on \mathbb{Z}_{12} by $a \cdot b = a$ for all $a \in \mathbb{Z}_{12}$.

Clearly $(\mathbb{Z}_{12}, +, \cdot)$ is a near-ring.

Example 1.1.3. Let G be an additively written (but not necessarily abelian) group with zero 0 . Then the following sets of mappings from G into G are near ring. Addition is defined point wise and Multiplication is composition of maps.

$$M(G) = \{ f : G \rightarrow G \}.$$

$$M_0(G) = \{ f : G \rightarrow G : f(0) = 0 \}.$$

$$M_c(G) = \{ f : G \rightarrow G : f \text{ is constant} \}$$

The additive identity of $M(G)$ that is the zero mapping is denoted by 0 .

Consider in example 1.1.3 In this near ring the left distribution law fails to hold.

Define $(+)$ and (\circ) on $M(G)$ as:

$$(f + g)(x) = f(x) + g(x) \text{ and } (f \circ g)(x) = f(g(x)) \text{ for all } x \in G.$$

,show that $(M(G), +, \circ)$ is a near ring but not a ring.

To verify this fact, take a, b, c , in G and $a \neq 0$ Define:

$$f_a : G \rightarrow G \text{ by } f_a(g) = a \text{ for all } g \text{ in } G,$$

$$f_b : G \rightarrow G \text{ by } f_b(g) = b \text{ for all } g \text{ in } G,$$

$$f_c : G \rightarrow G \text{ by } f_c(g) = c \text{ for all } g \text{ in } G .$$

(i) Let $g \in G$ then

$$\begin{aligned} [(f_a + f_b) + f_c](g) &= (f_a + f_b)(g) + f_c(g) \\ &= (f_a(g) + f_b(g)) + f_c(g) \\ &= (a + b) + c \\ &= a + (b + c) \\ &= f_a(g) + (f_b(g) + f_c(g)) \\ &= f_a(g) + (f_b + f_c)(g) \\ &= (f_a + (f_b + f_c))(g) \end{aligned}$$

Therefore $(f_a + f_b) + f_c = f_a + (f_b + f_c)$

(ii) Existence of identity .

The zero mapping is the identity element that is 0.

(iii) Existence of inverse.

For all g in G there exist $f_x(g) \in M(G)$ such that $f_a(g) + f_x(g) = 0 = f_x(g) + f_a(g)$.

$$\Rightarrow a + x = 0$$

$$\Rightarrow x = -a = -f_a(g)$$

Therefore the inverse of $f_a(g)$ is $-f_a(g)$.

Thus from (i, ii, and iii) $(M(G), +)$ is a group.

(iv) Associativity of $(M(G), \circ)$.

Let $g \in G$ then

$$\begin{aligned} [f_a \circ (f_b \circ f_c)](g) &= f_a(f_b \circ f_c(g)) \\ &= f_a(f_b(f_c(g))) \\ &= f_a(f_b(c)) \\ &= f_a(b) \\ &= a \end{aligned}$$

$$\begin{aligned}
[(f_a \circ f_b) \circ f_c](g) &= f_a \circ f_b(f_c(g)) \\
&= f_a(f_b(f_c(g))) \\
&= f_a(f_b(c)) \\
&= f_a(b) \\
&= a
\end{aligned}$$

Therefore $f_a \circ (f_b \circ f_c) = (f_a \circ f_b) \circ f_c$

v) *Right distributive law.*

Let $g \in G$ then

$$\begin{aligned}
[(f_a + f_b) \circ f_c](g) &= (f_a + f_b)(f_c(g)) \\
&= f_a(f_c(g)) + f_b(f_c(g)) \\
&= f_a(c) + f_b(c) \\
&= a + b
\end{aligned}$$

Also,

$$\begin{aligned}
[(f_a \circ f_c) + (f_b \circ f_c)](g) &= (f_a \circ f_c)(g) + (f_b \circ f_c)(g) \\
&= f_a(f_c(g)) + f_b(f_c(g)) \\
&= f_a(c) + f_b(c) \\
&= a + b
\end{aligned}$$

Therefore $f_a \circ (f_b + f_c) = (f_a \circ f_c) + (f_b \circ f_c)$

Therefore right distributive holds.

Thus it is right near ring.

vi) *Left distributive law*

Let $g \in G$

Now,

$$\begin{aligned}
[f_a \circ (f_b + f_c)](g) &= f_a(f_b + f_c)(g) \\
&= f_a(f_b(g) + f_c(g)) \\
&= f_a(a + b) \\
&= a
\end{aligned}$$

Also,

$$\begin{aligned}
 [(f_a \circ f_b) + (f_a \circ f_c)](g) &= (f_a \circ f_b)(g) + (f_a \circ f_c)(g) \\
 &= f_a(f_b(g)) + f_a(f_c(g)) \\
 &= f_a(b) + f_a(c) \\
 &= a + a \neq a
 \end{aligned}$$

since $a \neq 0$.

Therefore $f_a \circ (f_b + f_c) \neq (f_a \circ f_b) + (f_a \circ f_c)$.

This show that N fails to satisfy the left distributive law.

This provides an example of a right near ring that is not a left near ring.

Also this provides an example of near ring that is not a ring.

Definition 1.1.2. A subgroup M of a near-ring N with $M.M \subseteq M$ is called a subnear-ring of N . It is denoted by $M \leq N$.

Definition 1.1.3. Let N be a near-ring.

If $(N, +)$ is abelian, we call N an abelian near-ring.

If (N, \cdot) is commutative we call N itself a commutative near-ring.

If all non-zero elements of N are left (right) cancelable, we say that N fulfills the left (right) cancellation law.

Proposition 1.1.1. Let N be a near ring.

For all $n, m \in N$,

i. $0n = 0$ and $(-n)m = -nm$

ii. $(a + b) = -b - a$ for $a, b \in N$.

Proof. i) Let n in N , Now $0n = (0 + 0)n = 0n + 0n$. (by right distributive)

This implies $0 + 0n = 0n + 0n$

$\Rightarrow 0 = 0n$ for all $n \in N$ (by right cancelation law in $(N, +)$)

Let $n, m \in N$, Now $(-n)m + nm = (-n + n)m = 0m = 0$. (by right distributive)

Which implies $(-n)m + nm + -nm = 0 + -nm$.

This means $(-n)m = -nm$.

ii) Take $a, b \in N$, Now $(a + b) + -b) + -a) = a + (b - b) - a = (a + -a) = 0$. by (i)
 $-(a+b) = -b-a$.

Therefore $-(a + b) = -b - a$. □

Definition 1.1.4. Given a near-ring N ,

- i. $N_0 = \{n \in N : n0 = 0\}$, which is called the zero-symmetric part of N ,
- ii. $N_c = \{n \in N : n0 = n\} = \{n \in N : nm = n \text{ for every } m \text{ in } N\}$, which is called the constant part of N .

Note that $\{n \in N : n0 = n\} = \{n \in N : \text{for all } m \in N, nm = n\}$.

Proof. let $n \in \{n \in N : n0 = n\}$

we have to show that $n \in \{n \in N : \text{for all } m \in N, nm = n\}$.

let $m \in N$, Now $nm = (n0)m = n(0m) = n0 = n$. (by definition of constant part of N).

Therefore, $\{n \in N : n0 = n\} \subseteq \{n \in N : \text{for all } m \text{ in } N, nm = n\}$.

let $n \in \{n \in N : \text{for all } m \text{ in } N, nm = n\}$.

since $0 \in N$, We have $n \in \{n \in N : n0 = n\}$. (by definition of constant part of N)

Therefore $\{n \in N : \text{for all } m \text{ in } N, nm = n\} \subseteq \{n \in N : n0 = n\}$.

Hence $\{n \in N : n0 = n\} = \{n \in N : \text{for all } m \text{ in } N, nm = n\}$. □

Remark 1.1. $n0$ need not be equal to 0

Proof. For this consider $M_c(G)$, where G is a non zero additive group.

let $a \neq 0$,

Define a mapping $f_a : G \rightarrow G$ by $f_a(x) = a$ for all x in G .

Now f_a in $M(G)$ and $(f_a 0)(x) = f_a(0x) = f_a(0) = a \neq 0 = 0x$.

This shows that $f_a 0 \neq 0$.

So if we write $n = f_a$ then $n0 \neq 0$. □

Remark 1.2. N_0 and N_c are subnear-rings of N .

Proof. First we show that N_0 is a subgroup of N .

Let $x, y \in N_0$, Then $x0 = 0$ and $y0 = 0$.

Now, $(x - y)0 = x0 - y0 = 0$.

Therefore $x - y \in N_0$.

Hence $(N_0, +)$ is subgroup of $(N, +)$.

Take $n, m \in N_0$.

Now, $(nm)0 = n(m0) = n0 = 0$.

Therefore $n, m \in N_0$ and so $N_0 N_0 \subseteq N_0$.

Hence, N_0 is a subnear rings of N .

Next we show that N_c is a sub near ring of N .

Let $x, y \in N_c$, this implies that $(x - y)0 = x0 - y0 = x - y$.

This means $x - y \in N_c$.

So $(N_c, +)$ is subgroup of N .

Let $n, m \in N_c$,

This implies $(nm)0 = n(m0) = nm$ and so $nm \in N_c$.

Hence $N_c N_c \subseteq N_c$.

Therefore N_c is a subnear ring of N .

Thus N_0 and N_c are subnear ring of N . □

Definition 1.1.5. A near-ring N is called

i) a zero-symmetric, if $N = N_0$ and

ii) a constant near-ring, if $N = N_c$.

Definition 1.1.6. An element n in N is said to be an idempotent if $n^2 = n$.

An element n in N is called nilpotent if there exist a least positive integer k such that $n^k = 0$.

Example 1.1.4. i) In near ring $(\mathbb{Z}_6, +, \cdot)$, the element 0, 1, 3 and 4 are idempotent (because $0^2 = 0, 1^2 = 1, 3^2 = 3, 4^2 = 4$).

ii) In near ring $(\mathbb{Z}_8, +, \cdot)$, the element 0, 2 and 4 are nilpotent (because $0^1 = 2^3 = 4^2 = 0$).

1.2 N-groups, Homomorphism and ideal-like subsets

In this section we recall the definition of N-groups, N-homomorphism and ideals in a near-ring and illustrate with examples.

Definition 1.2.1. Let $(G, +)$ be a group with 0 and let N be a near-ring.

G is said to be an N -group if there exists a mapping $N \times G \rightarrow G$ (the image of (n, g) in $N \times G$ is denoted by ng), satisfying the following conditions

- i) $(n+m)g = ng + mg$ and
- ii) $(nm)g = n(mg)$ for all g in G and m, n in N

we denote this N -group by N^G .

Example 1.2.1. Let N be a near-ring. Then $\mu : N \times N \rightarrow N$ with $\mu(n, n) = nn$ makes $(N, +)$ into an N -group, denoted by N^N

Definition 1.2.2. Let G be an N -group.

A subgroup $(H, +)$ of $(G, +)$ is said to be an N -subgroup of G if $NH \subseteq H$ (this is denoted by $H \leq N^G$)

Example 1.2.2. Let $(\mathbb{R}, +, \cdot)$ be a near-ring. Where \mathbb{R} denote the set of real numbers.

Then \mathbb{R} is a N -group.

\mathbb{Q} the set of rational is a N -subgroup of \mathbb{R} .

Definition 1.2.3. Let N and N_1 be two near-rings. P and P_1 be two N -groups.

A mapping,

- i) $h : N \rightarrow N_1$ is called a near-ring homomorphism if for all $m, n \in N$, $h(m + n) = h(m) + h(n)$ and $h(mn) = h(m)h(n)$.
- ii) $h : P \rightarrow P_1$ is called N -homomorphism if for all $p, q \in P$ and for all $n \in N$, $h(p + q) = h(p) + h(q)$ and $h(np) = nh(p)$.

Definition 1.2.4. A non empty subset I of N is called

- i) a left ideal of N , if I is a normal subgroup of $(N, +)$ and $n(m + i) - nm \in I$, for every $n, m \in N$ and $i \in I$.
- ii) a right ideal of N , if I is a normal subgroup of $(N, +)$ and $IN \subseteq I$.
- iii) an ideal of N , if I is both left and right ideal of N .

Definition 1.2.5. Let A, B be subsets of N . Define $(A:B) = \{ n \in N : nB \subseteq A \}$.

Theorem 1.2.1. *If A is a left ideal of N and B is a subset of N , then $(A:B)$ is always a left ideal of N .*

Proof. (i) Let $x, y \in (A:B)$. So, $xB \subseteq A$ and $yB \subseteq A$.

Now, $(x - y)b = xb - yb \in A$, for every $b \in B$.

Thus, $x - y \in (A:B)$.

Hence, $(A:B)$ is an additive subgroup of N .

ii) Let $n \in N$ and $x \in (A:B)$. We have, $xB \subseteq A$.

Now, $(n + x - n)b = nb + xb - nb \in A$, since $xb \in A$ and A is a left ideal of N .

Thus, $x + n - x \in (A:B)$.

Hence, $(A:B)$ is a normal subgroup of N .

iii) Let $n, m \in N$ and $x \in (A:B)$. We have, $xB \subseteq A$.

Let $b \in B$. Now, $(n(m + x) - nm)b = n(mb + xb) - n(mb) \in A$,

since $xb \in A$ and A is a left ideal.

Thus, $n(m + x) - nm \in (A:B)$.

Hence, $(A:B)$ is a left ideal of N . □

Definition 1.2.6. *A near-ring N is said to be reduced, if it is without nonzero nilpotent elements.*

Chapter 2

STRONG REGULARITY AND STRONG REDUCEDNESS OF NEAR-RINGS

A regular rings play an important role in ring theory. They generalize some properties of near-fields to a much wider class of rings. This concept not only transfers to near-rings, it is also motivated by the fact,that some of the most important types of near-rings are regular but The purpose of this chapter is to find a relation between strong regularity and strong reducedness of near-rings. It is easy to see that every leftstrongly regular near-ring is strongly reduced, but the converse is nottrue in general. We show that a near-ring is left strongly regular if and only if it is strongly reduced and regular. This result can be viewed as an alternate definition for left strong regularity.

2.1 Regular Rings

Definition 2.1.1. *An element x of a ring R is called a regular element if there exists $y \in R$ such that $x = xyx$. Ring R is called a regular ring if every element of R is regular.*

Example 2.1.1. *In the ring \mathbb{Z} ,the only regular elements are $0,1$,and -1 .*

Thus, \mathbb{Z} is not a regular ring.

Example 2.1.2. *Let R be a division ring and $x \in R$.*

If $x = 0$,then $x = xxx$.

Suppose $x \neq 0$.Then $xx^{-1} = 1$, so $x = xx^{-1}x$.

Thus, R is a regular ring

Example 2.1.3. *Consider \mathbb{R} ,the field of real numbers and $\mathbb{R}^2 = \{(x,y):x,y \in \mathbb{R}\}$.*

Define $+$ and \cdot on \mathbb{R}^2 by

$$(x,y)+(z,w) = (x+z,y+w)$$

$$(x,y)\cdot(z,w) = (xz,yw)$$

for all $x,y,z,w \in \mathbb{R}$. Then \mathbb{R}^2 is a commutative ring with identity.

Now $(1,0),(0,1) \in \mathbb{R}^2$ and $(1,0)(0,1)=(0,0)$.

This shows that \mathbb{R}^2 contains zero divisors, so \mathbb{R}^2 is not a field.

We claim that \mathbb{R}^2 is regular.

Let $(x,y) \in \mathbb{R}^2$.

If $x = 0 = y$, then $(x,y)(x,y)(x,y) = (x,y)$.

If $x \neq 0$ and $y \neq 0$, then $(x,y)(x^{-1}, y^{-1})(x,y) = (x,y)$.

If $x = 0$, but $y \neq 0$, then $(x,y)(x, y^{-1})(x,y) = (x,y)$.

Similarly, if $x \neq 0$ and $y = 0$, then $(x,y)(x^{-1}, y)(x,y) = (x,y)$.

Thus, in any case, (x,y) is a regular element.

Hence, \mathbb{R}^2 is a regular ring.

Example 2.1.4. Let $M_2(R)$ be the set of all 2×2 matrices over R .

Now $M_2(R)$ is a noncommutative ring with 1,

where " $+$ " and " \cdot " are the usual matrix addition and multiplication, respectively.

We show that $M_2(R)$ is a regular ring.

Let $A = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ in $M_2(R)$

Case 1: $xw-zy \neq 0$. Then $B = \begin{pmatrix} \frac{w}{xw-zy} & \frac{-y}{xw-zy} \\ \frac{-z}{xw-zy} & \frac{x}{xw-zy} \end{pmatrix} \in M_2(R)$ and $A = ABA$.

Case 2: $xw-zy = 0$.

Subcase 2a: x, y, z, w are all zero.

In this case, $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, so for any $B \in M_2(R)$, $ABA = A$.

Subcase 2b: x,y,z,w are not all zero.

Suppose $x \neq 0$ and let $B = \begin{pmatrix} \frac{1}{x} & 0 \\ 0 & 0 \end{pmatrix}$ Then

$$\begin{aligned} ABA &= \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} \frac{1}{x} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ \frac{z}{x} & 0 \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} \\ &= \begin{pmatrix} x & y \\ z & \frac{zy}{x} \end{pmatrix} \\ &= \begin{pmatrix} x & y \\ z & w \end{pmatrix} \end{aligned}$$

because $xw - zy = 0$ and $x \neq 0$ implies $w = \frac{zy}{x}$.

If $y \neq 0$, then let $B = \begin{pmatrix} 0 & 0 \\ \frac{1}{y} & 0 \end{pmatrix}$ Then,

$$\begin{aligned} ABA &= \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \frac{1}{y} & 0 \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ \frac{w}{y} & 0 \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} \\ &= \begin{pmatrix} x & y \\ \frac{wx}{y} & w \end{pmatrix} \\ &= \begin{pmatrix} x & y \\ z & w \end{pmatrix} \end{aligned}$$

Similarly, if $z \neq 0$ or $w \neq 0$, then we can find B such that $ABA = A$.

Thus, $M_2(R)$ is a regular ring.

Because $M_2(R)$ is not a division ring, it follows that a regular ring need not be a division ring.

However, a division ring is a regular ring as shown in Example 2.1.1 In the next theorem, we show that a regular ring under a suitable condition becomes a division ring.

Theorem 2.1.1. Let R be a regular ring with more than one element. Suppose for all $x \in R$, there exists a unique $y \in R$ such that $x = xyx$. Then

- i) R has no zero divisors,
- ii) if $x \neq 0$ and $x = xyx$, then $y = yxy$ for all $x \in R$,
- iii) R has an identity,

iv) R is a division ring.

Proof. (i) Let x be a nonzero element of R and $xz = 0$ for some $z \in R$.

Now by the hypothesis, there exists a unique $y \in R$ such that $xyx = x$.

Thus, $x(y - z)x = xyx - xzx = xyx$.

Hence, by the uniqueness of y , $y - z = y$, so $z = 0$.

This proves that R has no zero divisors.

ii) Let $x \neq 0$ and $xyx = x$. Then $x(y - yxy) = xy - xyxy = xy - xy = 0$.

Because R has no zero divisors and $x \neq 0$, $y - yxy = 0$, so $yxy = y$.

iii) Let $0 \neq x \in R$. Then there exists a unique $y \in R$ such that $xyx = x$.

Let $e = yx$.

If $e = 0$, then $x = xyx = 0$, which is a contradiction. Therefore, $e \neq 0$.

Also, $e^2 = yxyx = y(xyx) = yx = e$.

Let $z \in R$. Then $(ze - z)e = ze^2 - ze = ze - ze = 0$.

Thus, by (i), either $ze - z = 0$ or $ze = z$. Similarly, $e(ez - z) = 0$ implies that $ez = z$.

Hence, e is the identity of R .

(iv) By (iii), R contains an identity element e .

To show R is a division ring, it remains to be shown that every nonzero element of R has an inverse in R .

Let x be a nonzero element in R . Then there exists a unique $y \in R$ such that $xyx = x$.

Thus, $xyx = xe$, that is, $x(yx - e) = 0$.

Because R has no zero divisors and $x \neq 0$, $yx - e = 0$,

so $yx = e$. Similarly, $xyx = ex$ implies $xy = e$.

Therefore, $xy = e = yx$.

Hence, R is a division ring. □

Definition 2.1.2. A near ring N is Regular Near ring if each element $a \in N$ then there exist an element x in N such that $a = axa$.

2.2 Definitions

Definition 2.2.1. *i) A near-ring N is called left strongly regular, if for every a in N , there exists x in N such that $a = xa^2$.*

ii) A near-ring N is called left regular, if for every a in N , there is an x in N such that $a = xa^2$ and $a = axa$.

iii) A near-ring N is called right strongly regular, if for every a in N , there exists x in N such that $a = a^2x$.

iv) A near-ring N is called right regular, if for every a in N , there is an x in N such that $a = a^2x$ and $a = axa$.

v) A near-ring N is said to be strongly regular, if it is both left and right strongly regular.

vi) A near-ring N is said to be strongly reduced, if for a in N , $a^2 \in N_c$ implies $a \in N_c$, or equivalently, for a in N and any positive integer n , a^n in N_c implies a in N_c , where N_c denotes the constant part of N .

vii) An idempotent e in N is right semi central, if $en = ene$, for each n in N .

viii) A near-ring in which every idempotent is right semicentral is called a right semi-central near-ring.

2.3 Equivalence of left regularity, left strong regularity and right regularity of near-rings

Lemma 2.3.1. *If N is a left strongly regular near-ring, then it has the following properties:*

i) for $a, b \in N$, $ab = 0$ implies $ba = b0$,

ii) for $a \in N$, $a^3 = a^2$ implies $a^2 = a$.

Proof. i) Since $ab = 0$, we have, $(ba)^2 = b0a = b0$.

So, it is enough to show that $ba = (ba)^2$.

Since N is left strongly regular, there exists x in N such that $ba = x(ba)^2$.

So, $ba = xb0$ and Hence

$$\begin{aligned} (ba)^2 &= (ba)(ba) \\ &= (xb0)(ba) \\ &= xb0 \\ &= ba \end{aligned}$$

(ii) Let $a \in N$ be such that $a^3 = a^2$.

Since N is left strongly regular, we can find an x in N such that $a = xa^2$.

Then,

$$\begin{aligned} a &= xa^2 \\ &= xaa \\ &= x(xa^2)a \\ &= x^2a^3 \\ &= x^2a^2 \end{aligned}$$

and so

$$a = x(xa^2) = xa.$$

$$\Rightarrow a = xa.$$

$$\Rightarrow aa = xaa.$$

$$\Rightarrow a^2 = xa^2.$$

$$\text{Hence } a^2 = xa^2 = a.$$

□

An interesting result is that strong reducedness is equivalent to the property (lemma 2.3.1) For proving it, we need some basic properties of strongly reduced near-rings.

Lemma 2.3.2. *If N is a strongly reduced near-ring, then it has the following properties:*

- i) If a, b in N with $ab \in N_c$, then $ba \in N_c$, and for every x in N , axb, bxa in N_c ,*
- ii) If a, b in N such that either ab^n in N_c or $a^n b$ in N_c , where n is a positive integer greater than 1, then ab in N_c .*

Proof. (i) Let $ab \in N_c$. So, $ab0 = ab$.

Now,

$$\begin{aligned} (ba)^2 &= (ba)(ba) \\ &= b(ab)a \\ &= b(ab0)a \\ &= bab0 \end{aligned}$$

But, $bab0 \in N_c$, since $(bab0)n = bab0$, for every n in N .

That is $(ba)^2$ in N_c .

Since N is strongly reduced, we get $ba \in N_c$.

Then we obtain xba in N_c for each x in N , since $NN_c \subseteq N_c$

Now,

$$\begin{aligned} (axb)^2 &= (axb)(axb) \\ &= a(xba)xb \end{aligned}$$

Since xba in N_c , $(xba)xb = xba$ and hence $(axb)^2 = a(xba)$.

Since $NN_c \subseteq N_c$, $a(xba)$ in N_c .

That is, $(axb)^2$ in N_c .

By the strongly reducibility of N , we obtain $axb \in N_c$, for each x in N .

Let $ba \in N_c$.

So $ba0 = ba$ since $ba \in N_c$

Now,

$$\begin{aligned} (ab)^2 &= (ab)(ab) \\ &= a(ba)b \\ &= a(ba0)b \\ &= aba0 \end{aligned}$$

But $aba0 \in N_c$, since $(aba0)n = aba0$ for every n in N . that is $(ab)^2 \in N_c$

since N is strongly reduced, we get $ab \in N_c$

Then we obtain $xab \in N_c$ for each $x \in N$, since $NN_c \subseteq N_c$.

Now,

$$\begin{aligned}(bxa)^2 &= (bxa)(bxa) \\ &= b(xab)xa\end{aligned}$$

Since xab in N_c , $(xab)xa = xab$ and

hence $(bxa)^2 = b(xab)$.

Since $NN_c \subseteq N_c$, $b(xab) \in N_c$

that is $(bxa)^2 \in N_c$.

By strongly reducibility of N , we obtain $axb \in N_c$ for each $x \in N$

ii) Let ab^n in N_c .

Then $(ab)^n \in N_c$, by (i).

Since N is strongly reduced, this implies $ab \in N_c$.

Similarly, we can prove that $a^n b$ in N_c implies ab in N_c . □

Theorem 2.3.1. *Let N be a near-ring. Then N is strongly reduced if and only if it has the property (lemma 2.3.1).*

Proof. Assume that N is strongly reduced.

Let $a^3 = a^2$.

We show that $a^2 = a$.

Now, $(a - a^2)a = 0 \in N_c$.

This implies $a(a - a^2)$, $a^2(a - a^2) \in N_c$, by Lemma 2.3.2(i).

Hence,

$$\begin{aligned}(a - a^2)^2 &= (a - a^2)(a - a^2) \\ &= a(a - a^2) - a^2(a - a^2) \in N_c\end{aligned}$$

Since N is strongly reduced, we obtain $a - a^2 \in N_c$. $a - a^2 = (a - a^2)a = 0$.

This shows that $a^2 = a$.

Thus, condition (ii) of property (lemma 2.3.1) holds.

Now, let $ab = 0$.

Then,

$$\begin{aligned}(ba)^2 &= (ba)(ba) \\ &= b(ab)a \\ &= b0a\end{aligned}$$

Since $ab = 0 \in N_c$, we obtain $ba, b0a \in N_c$, by Lemma 2.3.2(i).

Thus, $(ba)^2 - ba \in N_c$.

Then,

$$\begin{aligned}(ba)^2 - ba &= ((ba)^2 - ba)b \\ &= babab - bab \\ &= b0 - b0 \\ &= 0\end{aligned}$$

Hence,

$$\begin{aligned}ba &= (ba)^2 \\ &= b0\end{aligned}$$

This proves the first condition of property (lemma 2.3.1)

Conversely, assume that N has the property (lemma 2.3.1)

Let $a^2 \in N_c$.

Then,

$$\begin{aligned}(a)^3 &= (a)^2a \\ &= (a)^2\end{aligned}$$

By condition (ii) of property (lemma 2.3.1), this implies $a = a^2 \in N_c$.

Thus, N is strongly reduced. □

Lemma 2.3.3. *Let N be a left strongly regular near-ring. If $a = xa^2$, for some $a, x \in N$, then $a = axa$ and $ax = xa$.*

Proof. Since $a = xa^2$, we have $(a - axa)a = 0$ and hence by Lemma 2.3.1, $a(a - axa) = a0$ and $axa(a - axa) = axa0$.

Therefore,

$$\begin{aligned}(a - axa)^2 &= (a - axa)(a - axa) \\ &= a(a - axa) - axa(a - axa) \\ &= a0 - axa0 \\ &= (a0 - ax)a0\end{aligned}$$

Now,

$$\begin{aligned}(a - axa)^3 &= (a - axa)^2(a - axa) \\ &= (a0 - ax)a0(a - axa) \\ &= (a0 - ax)a0 \\ &= (a - axa)^2\end{aligned}$$

So, by Lemma 2.3.1, we have $(a - axa)^2 = a - axa$.

Consequently,

$$\begin{aligned}0 &= (a - axa)a \\ &= (a - axa)^2a \\ &= (a0 - ax)aa0a \\ &= (a0 - ax)a0 \\ &= (a - axa)^2 \\ &= a - axa\end{aligned}$$

Thus $a = axa$

Now,

$$\begin{aligned}(ax - xa)a &= axa - xa^2 \\ &= a - a \\ &= 0\end{aligned}$$

As in the first part, we can prove that

$$\begin{aligned}ax - xa &= (ax - xa)^2 \\ &= ax0 - xa0 \\ &= (ax - xa)0\end{aligned}$$

Therefore,

$$\begin{aligned}
0 &= (ax - xa)a \\
&= (ax - xa)0a \\
&= (ax - xa)0 \\
&= ax - xa
\end{aligned}$$

Thus $ax = xa$ □

Theorem 2.3.2. *If N is a left strongly regular near-ring, then it is regular, left regular and right strongly regular.*

Proof. Assume that N is left strongly regular.

Let $a \in N$.

Then there exists $x \in N$ such that $a = xa^2$.

Thus, we get $a = axa$ and $ax = xa$, by using Lemma 2.3.3.

This implies that $a^2x = axa = a$.

Thus, N is regular, left regular and right strongly regular. □

Theorem 2.3.3. *If N is a right strongly regular near-ring, then it is strongly reduced.*

Proof. Let $a \in N$ be such that $a^2 \in N_c$.

Since N is right strongly regular, there exists $x \in N$ such that $a = a^2x$.

Since $a^2 \in N_c$, $a^2x = a^2$.

Therefore, $a = a^2 \in N_c$.

Thus, N is strongly reduced. □

Corollary 2.3.1. *If N is a left strongly regular near ring, then it is strongly reduced.*

Proof. Assume N is left strongly regular near ring .

let $a \in N$ be such that $a^2 \in N_c$

since N is left strongly regular, it is right strongly regular by theorem 2.3.2.

Then for every $a \in N$ there is $x \in N$ such that $a = a^2x$.

since $a^2 \in N_c$, $a^2x = a^2$.

$\Rightarrow a^2x \in N_c$

$\Rightarrow a = a^2x$ so, $a^2 = a^2x = a$ and since $a^2 = a \in N$

Thus N is strongly reduced □

Theorem 2.3.4. *Let N be a near-ring.*

Then, N is right regular if and only if it is right semicentral and regular.

Proof. Assume that N is right regular.

Then, by definition, N is right strongly regular and regular.

Since N is right strongly regular, it is strongly reduced, by using Theorem 2.3.3.

Let $e^2 = e$.

Now, $(en - ene)e = 0 \in N_c$.

Therefore, $e(en - ene)$, $en(en - ene)$, $ene(en - ene) \in N_c$,

by Lemma 2.3.2(i).

Hence,

$$\begin{aligned}(en - ene)^2 &= (en - ene)(en - ene) \\ &= en(en - ene) - ene(en - ene) \in N_c\end{aligned}$$

Since N is strongly reduced, we obtain $en - ene \in N_c$.

Thus,

$$\begin{aligned}0 &= (en - ene)e \\ &= en - ene\end{aligned}$$

So, $en = ene$.

Thus, N is right semicentral.

Conversely, assume that N is right semicentral and regular.

Let $a \in N$.

Since N is regular, $a = axa$ for some $x \in N$.

Then, ax is idempotent, since

$$\begin{aligned}(ax)^2 &= (ax)(ax) \\ &= (axa)x \\ &= ax\end{aligned}$$

Now, $xa = x(axa) = x(ax)a$

Since ax is idempotent and N is right semicentral,

we have $(ax)a = (ax)a(ax)$ and hence $xa = x(ax)a = x(ax)a(ax)$.
Now,

$$\begin{aligned}
a &= axa \\
&= a(xa) \\
&= a(x(ax)a(ax)) \\
&= (axa)(xaax) \\
&= a(xaax) \\
&= (axa)ax \\
&= a^2x
\end{aligned}$$

Thus N is right regular □

Theorem 2.3.5. *Let N be a near-ring.*

Then, N is left regular if and only if it is right semicentral and regular.

Proof. Assume that N is left regular.

Then, by definition, N is left strongly regular and regular.

Since N is left strongly regular, it is strongly reduced, by Corollary 2.3.1.

As in the proof of Theorem 2.3.4, we obtain N is right semicentral and regular.

That is, let $e^2 = e$

Now, $(en - ene)e = 0 \in N_c$. Therefore, $e(en - ene), en(en - ene), ene(en - ene) \in N_c$, by Lemma 2.3.2(i).

Hence,

$$\begin{aligned}
(en - ene)^2 &= (en - ene)(en - ene) \\
&= en(en - ene) - ene(en - ene) \in N_c
\end{aligned}$$

Since N is strongly reduced, we obtain $en - ene \in N_c$.

Thus,

$$\begin{aligned}
0 &= (en - ene)e \\
&= en - ene
\end{aligned}$$

So, $en = ene$.

Thus, N is right semicentral.

Let $a \in N$.

Since N is left regular, it is regular by definition that is $a = axa$ for some $x \in N$.

Thus N is semicentral and regular.

Conversely, assume that N is right semicentral and regular.

Let $a \in N$.

Since N is regular, $a = axa$, for some $x \in N$.

Then, xa is idempotent, since

$$\begin{aligned} (xa)^2 &= (xa)(xa) \\ &= x(axa) \\ &= xa \end{aligned}$$

Now,

$$\begin{aligned} ax &= (axa)x \\ &= a(xa)x \end{aligned}$$

But, $(xa)x = (xa)x(xa)$, since xa is idempotent and N is right semicentral.

Hence $ax = a(xa)x = a(xa)x(xa)$. Now,

$$\begin{aligned} a &= axa \\ &= (ax)a \\ &= (a(xa)x(xa))a \\ &= (axa)x^2a^2 \\ &= ax^2a^2 \\ &= ya^2 \end{aligned}$$

where $y = ax^2 \in N$.

Thus, N is left strongly regular and consequently left regular □

Corollary 2.3.2. *A near-ring N is left regular if and only if it is right regular.*

Proof. Assume N is left regular.

We need to show that N is right regular.

Since N is left regular, it is right semi central and regular by theorem 2.3.5

Now let $a \in N$, since N is regular, $a = axa$ for some $x \in N$.

Then ax is idempotent ,since $(ax)^2 = (ax)(ax) = (axa)x = ax$
 Now,

$$\begin{aligned}
 a &= axa \\
 &= a(xa) \\
 &= a(x(ax)a(ax)) \\
 &= (axa)xaax \\
 &= a(xaax) \\
 &= (axa)ax \\
 &= a^2x
 \end{aligned}$$

Thus N is right regular .

conversely,Assume N is right regular .

We have to show that N is left regular .

since N is right regular it is right semi central and regular by theorem 2.3.4.

let $a \in N$.

since N is regular $a = axa$ for some $x \in N$.

Then $ax = (axa)x = a(xa)a$.

But $(xa)a = (xa)a(xa)$, since xa is idempotent and N is right semicentral.

Hence $ax = a(xa)x = a(xa)x(xa)$.

Now,

$$\begin{aligned}
 a &= axa \\
 &= (ax)a \\
 &= (a(xa)x(xa))a \\
 &= (axa)x^2a^2 \\
 &= ax^2a^2 \\
 &= ya^2
 \end{aligned}$$

where $y = ax^2 \in N$.

Thus, N is left strongly regular and consequently it is left regular. □

Theorem 2.3.6. *Let N be a near-ring.*

Then the following are equivalent.

- i) N is left strongly regular,*
- ii) N is left regular,*
- iii) N is right regular.*

Proof. (i) \Rightarrow (ii)

let N be left strongly regular

let $a \in N$ then exist x in N such that $a = xa^2$

thus we get $a = axa$ and $ax = xa$ by lemma 2.3.3

This implies that $a^2x = axa = a$

Thus N is regular and left regular .

(ii) \Rightarrow (i)

It is clear from the definition of left regularity.

The statements (ii)and(iii) are equivalent,using Corollary 2.3.2. □

2.4 Criterion for the equivalence of left strong regularity and right strong regularity of near-rings

In the first section,we can see that for an arbitrary near-ring, the concepts of left strong regularity,left regularity and right regularity are equivalent,and these imply the right strong regularity. The next theorem gives a criterion for the equivalence of these four concepts.

Definition 2.4.1. *The descending chain condition (dcc) on N -subgroup of N guaranties that the chain of N -subgroup $Na \supseteq Na^2 \supseteq Na^3 \supseteq \dots$ terminate after finite steps. so there is a natural number n such that $Na^n = Na^{n+1} = Na^{n+2} = \dots$*

Theorem 2.4.1. *Suppose that N satisfies dcc on N -subgroups of N .*

Then, N is right strongly regular if and only if it is left strongly regular.

Proof. Assume that N is right strongly regular.

Then, N is strongly reduced,by using Theorem 2.3.3.

Let $a \in N$ and consider the chain of N -subgroups $Na \supseteq Na^2 \supseteq \dots$

Since N has dcc on N -subgroups,there exists a positive integer $n > 1$ such that $Na^n = Na^{n+1} = Na^{n+2} = \dots$

Now, $a^{n+1} \in Na^n = Na^{n+2}$.

So, $a^{n+1} = xa^{n+2}$, for some $x \in N$.

Now,

$$\begin{aligned}
(a^n - xa^{n+1})a^n &= a^n a^n - xa^{n+1} a^n \\
&= a^{n+1} a^{n-1} - xa^{n+1} a^n \\
&= (xa^{n+2})a^{n-1} - xa^{n+1} a^n \\
&= 0 \in N_c
\end{aligned}$$

So, $a^n(a^n - xa^{n+1}), xa^{n+1}(a^n - xa^{n+1}) \in N_c$ by Lemma 2.3.2(i).

Hence,

$$\begin{aligned}
(a^n - xa^{n+1})^2 &= (a^n - xa^{n+1})(a^n - xa^{n+1}) \\
&= a^n(a^n - xa^{n+1}) - xa^{n+1}(a^n - xa^{n+1}) \in N_c
\end{aligned}$$

Since N is strongly reduced, we obtain $(a^n - xa^{n+1}) \in N_c$.

Hence,

$$\begin{aligned}
0 &= (a^n - xa^{n+1})a^n \\
&= (a^n - xa^{n+1})
\end{aligned}$$

This shows that $a^n = xa^{n+1}$.

Continuing in this way, we can show that $a = xa^2$.

Thus, N is left strongly regular.

The converse follows from Theorem 2.3.6 □

The following Corollary is a direct consequence of Theorems 2.3.6 and 2.4.1

Corollary 2.4.1. *If N satisfies dcc on N -subgroups of N , then the following are equivalent.*

- i) N is left strongly regular,
- ii) N is left regular,
- iii) N is right regular,
- iv) N is right strongly regular.

2.5 Relation between strong regularity and strong reducedness of near-rings

It can be seen from Corollary 2.3.1 that every left strongly regular near-ring is strongly reduced. But the converse need not be true in general, as the following example shows.

Example 2.5.1. Let $N = \mathbb{Z}$ the set of all integers with usual addition and multiplication given by $a.b = a|b|$, for $a, b \in N$.

Then, N is a strongly reduced near-ring, but it is not left strongly regular, since $a \in N$ with $a \neq 0, 1$ or -1 , there exists no $x \in N$ such that $a = xa^2$.

The following theorem shows that if N is strongly reduced and regular, then it becomes left strongly regular.

Theorem 2.5.1. Let N be a near-ring. Then, N is left strongly regular if and only if it is strongly reduced and regular.

Proof. Assume that N is left strongly regular.

Then, N is strongly reduced and regular, by using Theorem 2.3.2 and Corollary 2.3.1.

Conversely, assume that N is strongly reduced and regular.

Let $e^2 = e$.

As in the proof of Theorem 2.3.4, we can show that N is right semicentral.

Thus, N is right semicentral and regular.

This implies N is left strongly regular, by using Theorem 2.3.5. □

Example 2.5.2. Let $\mathbb{Z}_3 = \{ 0, 1, 2 \}$, with addition modulo 3 and define multiplication as follows.

.	0	1	2
0	0	0	0
1	0	1	1
2	0	2	2

Obviously, $(\mathbb{Z}_3, +, \cdot)$ is a strongly reduced near-ring, since the constant part of \mathbb{Z}_3 is 0. Also, it is regular.

Therefore, by using Theorem 2.5.1, this near-ring is left strongly regular.

Chapter 3

CHARACTERIZATIONS OF STRONGLY REGULAR NEAR-RINGS

3.1 Introduction

In this chapter, we find some properties and characterizations of strongly regular near-rings. We show that idempotents in a strongly regular zero-symmetric unital near-ring are central. Also, we show that a near-ring N is strongly regular if and only if N is right semicentral and regular if and only if N is strongly reduced and regular if and only if $A = \sqrt{A}$ for every N -subgroup A of N . Finally, we show that a zero-symmetric near-ring N is strongly regular if and only if it is a regular IFP near-ring.

3.2 Properties of strongly regular near-rings

Theorem 3.2.1. *If N is a strongly regular near-ring, then it is regular.*

Proof. Assume that N is left strongly regular

Since N is strongly regular, for every $a \in N$, then there is $x \in N$ such that $a = xa^2$

We have, $(a - axa)a = 0$

Since $a = xa^2$, we have $(a - axa)a = 0$ and hence by Lemma 2.3.1, $a(a - axa) = a0$ and $axa(a - axa) = axa0$.

Therefore,

$$\begin{aligned}
(a - axa)^2 &= (a - axa)(a - axa) \\
&= a(a - axa) - axa(a - axa) \\
&= a0 - axa0 \\
&= (a0 - ax)a0
\end{aligned}$$

Now,

$$\begin{aligned}
(a - axa)^3 &= (a - axa)^2(a - axa) \\
&= (a0 - ax)a0(a - axa) \\
&= (a0 - ax)a0 \\
&= (a - axa)^2
\end{aligned}$$

So,by Lemma 2.3.1, we have $(a - axa)^2 = a - xa$.

Consequently

$$\begin{aligned}
0 &= (a - axa)a \\
&= (a - axa)^2a \\
&= (a0 - ax)a0a \\
&= (a0 - ax)a0 \\
&= (a - axa)^2 \\
&= a - axa
\end{aligned}$$

Therefore $a = axa$

Thus N is regular. □

Definition 3.2.1. A near-ring N is said to be reduced,if it is without nonzero nilpotent elements.that is for each $a \in N$, $a^n = 0$ for some integer n implies $a = 0$

Theorem 3.2.2. If N is a strongly regular near-ring,then it is reduced.

Proof. Let $a \in N$ be such that $a^2 = 0$.

Since N is strongly regular,there exists $x \in N$ such that $a = xa^2$.

That is, $a = x0$.

Now,

$$\begin{aligned}0 &= a^2 \\ &= aa \\ &= x0a \\ &= x0 \\ &= a\end{aligned}$$

Thus, N is reduced. □

Definition 3.2.2. Let N be a near-ring. N is said to full the insertion of factors property (IFP) provided that for all $a, b, n \in N$ we have $ab = 0$ implies $anb = 0$.

Theorem 3.2.3. Let N be a zero-symmetric near-ring.

If N is strongly regular, then for $a, b \in N, ab = 0$ implies $ba = 0$.

Also, N has the IFP.

Proof. Since N is strongly regular, it is reduced by Theorem 3.2.2.

Let $a, b \in N$ with $ab = 0$.

Now,

$$\begin{aligned}(ba)^2 &= (ba)(ba) \\ &= b(ab)a \\ &= b0a \\ &= 0\end{aligned}$$

since N is reduced, $ba = 0$

Now for every $x \in N$ we have.

$$\begin{aligned}(axb)^2 &= (axb)(axb) \\ &= ax(ba)xb \\ &= ax0xb \\ &= 0\end{aligned}$$

Since N is reduced, we have $axb = 0$.

Thus, $ab = 0$ implies $axb = 0$, for every x in N .

Therefore, N has the IFP. □

Theorem 3.2.4. *Let N be a zero-symmetric near-ring.*

If N is strongly regular, then for every $a, b \in N$ and for every idempotent $e \in N$, we have $abe = aebe$.

Proof. Let $a, b \in N$ and $e^2 = e$.

Since $(a - ae)e = 0$, by IFP, we have $(a - ae)be = 0$.

implies that $abe - aebe = 0$

Thus, $abe = aebe$. □

Proposition 3.2.1. *A near-ring N has the strongly IFP if and only if $a, b \in N$ with $ab \in I$ implies $anb \in I$, for every $n \in N$, where I is an ideal of N .*

Proof. suppose N has strongly IFP and let I be an ideal of N .

let $\pi : N \rightarrow N/I$ be the canonical epimorphism

suppose that $a, b, n \in N$ and $ab \in I$.

To show that $anb \in I$, consider $\pi(a)\pi(b) = ab + I = 0 + I$.

Since N/I has IFP and π is onto, we have $\pi(a)(n)\pi(b) = 0$.

$\Rightarrow \pi(anb) = 0$

$\Rightarrow anb \in I$

Therefore $ab \in I$ implies $anb \in I$ for all $a, b, n, \in N$.

conversely

Suppose that for all ideals I of N , and for $a, b, n, \in N$

implies $anb \in I$.

we show that N has strongly IFP.

Let $f : N \rightarrow N'$ be a homomorphism.

Now, we show that $f(N)$ has IFP.

By first isomorphism theorem, N/I is isomorphic to $f(N)$, where $I = \ker f$

Therefore, we have to show that N/I has IFP.

Take $a + I, b + I, n + I \in N/I$ and

$(a + I)(b + I) = 0$.

This implies $ab + I = 0$.

$\Rightarrow ab \in I$.

$\Rightarrow anb \in I$ (by converse hypothesis).

$$\Rightarrow (a + I)(n + I)(b + I) = 0.$$

Hence, N has strongly IFP. □

Theorem 3.2.5. *Let N be a zero-symmetric near-ring.*

If N is strongly regular, then it has the strong IFP.

Proof. Let I be an ideal of N and let $a, b \in N$ with $ab \in I$.

Since N is strongly regular, it is regular by Theorem 3.2.1.

Then $b = byb$, for some $y \in N$. Now,

$$\begin{aligned} (by)^2 &= (by)(by) \\ &= (byb)y \\ &= by \end{aligned}$$

So by is idempotent.

Therefore, by Theorem 3.2.4, we have

$$\begin{aligned} anb &= an(byb) \\ &= an(by)b \\ &= a(by)n(by)b \\ &= abyn(byb) \\ &= abynb \in abN \subseteq IN \subseteq I \end{aligned}$$

for every n in N .

Hence, by Proposition 3.2.1, N has the strong IFP. □

Remark 3.1. *It is well known that in a reduced ring R , every idempotent element is central, that is, $e^2 = e$ implies $ex = xe$, for every x in R .*

In the case of near-rings, this assertion is not true.

For example, on the set $(Z_4, +)$ of residue classes modulo 4, define the multiplication by the following table.

.	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	1	2	3
3	0	1	2	3

$(Z_4, +, \cdot)$ is a reduced near-ring in which every element is idempotent, but 0 is the only central element.

Situation is essentially another, if the near-rings are zero-symmetric, unital and strongly regular.

Theorem 3.2.6. *Let N be a strongly regular zero-symmetric near-ring.*

Then the following are true.

- i) every distributive idempotent is central,*
- ii) if N is unital, all idempotents are central.*

Proof. Let $e^2 = e$ and x in N .

Now, $(ex - exe)e = 0$.

So, by Theorem 3.2.3, we have $e(ex - exe) = 0$ and $ex(ex - exe) = 0$.

Hence, $-exe(ex - exe) = (-ex)0 = 0$.

Therefore,

$$\begin{aligned} (ex - exe)^2 &= (ex - exe)(ex - exe) \\ &= ex(ex - exe) - exe(ex - exe) \\ &= 0 \end{aligned}$$

Hence, $ex - exe = 0$, since N is reduced, by Theorem 3.2.2.

This shows that $ex = exe$.

- i) Let e be a distributive element.

Then for every x in N , we have $e(xe - exe) = 0$.

Hence, by Theorem 3.2.3, we have $(xe - exe)e = 0$.

That is, $xe - exe = 0$, which implies that $xe = exe$.

From the first part of the proof, we have $ex = exe$.

Thus, $ex = exe = xe$, for every idempotent e .

Thus e is central.

- ii) Let $e^2 = e$.

If N has an identity 1, then $(1-e)e = 0$.

So, by IFP, we have $(1 - e)xe = 0$ for every x in N .

This implies that $xe = exe$.

By the first part of the proof, we have $ex = exe$.

Thus, $ex = exe = xe$, for every idempotent e .

Thus, e is central □

3.3 Characterizations of strongly regular near-rings

In this section, we prove some characterizations of strongly regular near-rings.

Theorem 3.3.1. *A near-ring N is strongly regular if and only if it is right semicentral and regular.*

Proof. Assume that N is strongly regular.

Then N is regular and strongly reduced, by Theorems 3.2.1 and 2.3.3.

Let $e^2 = e$.

Now, $(en - ene)e = 0 \in N_c$.

Hence, $en(en - ene), ene(en - ene) \in N_c$, by Lemma 2.3.2(i).

So,

$$\begin{aligned} (en - ene)^2 &= (en - ene)(en - ene) \\ &= en(en - ene) - ene(en - ene) \in N_c \end{aligned}$$

Since N is strongly reduced, we obtain $en - ene \in N_c$.

Hence $0 = (en - ene)e = en - ene$.

So, $en = ene$.

Thus, N is right semicentral.

Conversely, assume that N is right semicentral and regular.

Let $a \in N$.

Since N is regular, $a = axa$, for some $x \in N$.

Then xa is idempotent, since

$$\begin{aligned} (xa)^2 &= (xa)(xa) \\ &= x(axa) \\ &= xa \end{aligned}$$

Now, $ax = (axa)x = a(xa)x$

But, $(xa)x = (xa)x(xa)$, since xa is idempotent and N is right semicentral.

Hence, $ax = a(xa)x(xa)$.

Now,

$$\begin{aligned}
 a &= (ax)a \\
 &= (a(xa)x(xa))a \\
 &= (axa)x^2a^2 \\
 &= ax^2a^2 \\
 &= ya^2
 \end{aligned}$$

where $y = ax^2 \in N$

Thus, N is left strongly regular and consequently it is strongly regular. \square

Theorem 3.3.2. *A near-ring N is strongly regular if and only if it is strongly reduced and regular.*

Proof. Assume that N is left strongly regular.

Then, N is strongly reduced and regular, by using theorem 2.3.2 and corollary 2.3.1.

conversely assume that N is strongly reduced and regular .

Let $e^2 = e$.

As in the proof of Theorem 2.3.4, we can show that N is right semicentral .

Thus, N is semicentral and regular .

This implies N is left regular, by using theorem 2.3.5 \square

Definition 3.3.1. *For any subset A of a near-ring N , we define $\sqrt{A} = \{ x \in N : x^n \in A \text{ for some } n \}$*

Theorem 3.3.3. *A near-ring N is strongly regular if and only if $A = \sqrt{A}$ for every N -subgroup A of N .*

Proof. Assume that N is strongly regular.

Let A be an N -subgroup of N .

Clearly, $A \subseteq \sqrt{A}$.

Now, let $a \in \sqrt{A}$.

Then, $a^n \in A$, for some positive integer n .

Since N is strongly regular, there exists $x \in N$ such that

$$\begin{aligned}
 a &= xa^2 \\
 &= xaa \\
 &= x(xa^2)a \\
 &= x^2a^3 \\
 &\dots \\
 &\dots \\
 &= x^{n-1}a^n \in NA
 \end{aligned}$$

But, $NA \subseteq A$, since A is an N -subgroup of N . That is, $a \in A$.

Thus, $\sqrt{A} \subseteq A$

Hence, $A = \sqrt{A}$

Conversely, let $A = \sqrt{A}$, for every N -subgroup A of N .

Let $a \in N$.

Then Na^2 is an N -subgroup of N and by assumption $Na^2 = \sqrt{Na^2}$.

Now, $a^3 \in Na^2$ and therefore $a \in \sqrt{Na^2} = Na^2$.

That is, $a = xa^2$, for some $x \in N$.

Thus, N is strongly regular. □

Now, we have the main theorem of this chapter.

Theorem 3.3.4. *Let N be a near-ring. Then the following are equivalent.*

- i) N is strongly regular,
- ii) N is right semicentral and regular,
- iii) N is strongly reduced and regular,
- iv) $A = \sqrt{A}$ for every N -subgroup A of N .

Proof. It follows from Theorems 3.3.1, 3.3.2 and 3.3.3.

Finally, we have the following characterization of a strongly regular zero-symmetric near-ring. □

Theorem 3.3.5. *Let N be a zero-symmetric near-ring.*

Then, N is strongly regular if and only if it is a regular IFP near-ring.

Proof. Assume that N is strongly regular.

Then it is a regular IFP near-ring, by using Theorems 3.2.1 and 3.2.3.

Conversely, assume that N is a regular IFP near-ring.

Let $a \in N$.

Since N is regular, there exists $b \in N$ such that $a = aba$.

Then, ba is idempotent, since

$$\begin{aligned}(ba)^2 &= (ba)(ba) \\ &= b(aba) \\ &= ba\end{aligned}$$

Now, $ba = ba(ba) = b(ba)a(ba)$,

since ba is idempotent and by Theorem 3.2.4.

So,

$$\begin{aligned}ba &= (bba)(aba) \\ &= (bba)a \\ &= b^2a^2\end{aligned}$$

Therefore

$$\begin{aligned}a &= a(ba) \\ &= a(b^2a^2) \\ &= (ab^2)a^2 \\ &= xa^2\end{aligned}$$

where $x = ab^2 \in N$.

Thus, N is strongly regular. □

Summary

An attempt is made in this paper to concentrate on characterizations and generalizations of strongly regular near-rings. A relation between strong regularity and strong reducedness is established. We show that a near-ring is left strongly regular if and only if it is strongly reduced and regular. In the part of some chapter, we establish the equivalence of left regularity, left strong regularity and right regularity and the significance of this equivalence is the role of strong reducedness for proving it. We prove that for certain special near-rings, these three concepts are equivalent to right strong regularity

Bibliography

- [1] Y. V. REDDY and C. V. L. N. MURTY, *On strongly regular near-rings*, Proc. Edinburgh Math. Soc. 27 (1984), 61-64.
- [2] G. MASON, *Strongly regular near-rings*, Proc. Edinburgh Math. Soc. 23 (1980), 27-35.
- [3] G. PILZ, *Near-rings (North-Holland, Amsterdam, 1977)*.
- [4] FONG, Yuen, PILZ, Gunter, OSWALD, Alan and SMITH, Kirby C., *Near-ring Newsletter, National Cheng Kung Univ.*, 20 (1998).