APPLICATION OF VARIATIONAL METHODS ON THE THEORY OF SUPERCONDUCTIVITY

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By

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Table of Contents

List of Figures

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Abstract

There are two main theories in superconductivity: i) Microscopic or (BCS) theory which describes why materials are superconducting, ii) The phenomenological theory which describes the properties of superconductors in magnetic fields; Unlike the microscopic (BCS) theory the phenomenological theory is based on a so-called phenomenological order parameter.

Most physical problems in physics can not be solved exactly, and hence they need to be dealt with approximately. The principles of variations as developed for providing an elegant description of a wide variety of physical phenomena; Here we are using this method to solve problems related to the phenomenological theory of superconductivity, which includes London's equation, first and second Ginzburg-Landau equation and in the case of weak superconductivity, the penetration of the magnetic field on Josephson's contact between two superconductors, when there is an insulator between them, i.e Ferrell-Prange equation and the physical phenomena on the junction.

Introduction

The calculus of variation involves problems in which the quantity to be minimized (or maximized). The function y needs to be determined from a class described by an infinitesimal parameter "δy". As the simplest case , let

$$J = \int f(y, y_x, x) dx. \tag{0.0.1}$$

Here J is the quantity that depends on the analytic form of f, which is a known function of variables $y(x), y_x(x) = \partial y(x)/\partial x$ and x.

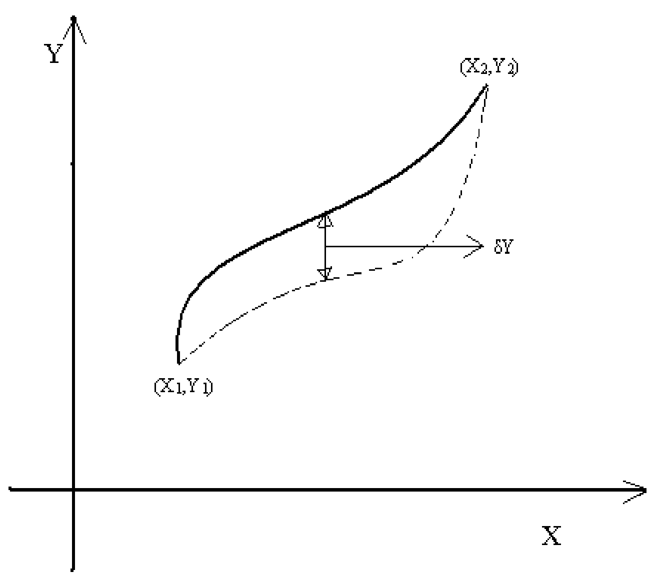


Figure 1: Concept of variation on the action minimization.

This means that although the integral is from x_1 to x_2 , the exact path of integration is not known. We are to choose the path of integration through points (x_1, y_1) to (x_2, y_2) to minimize J. Strictly speaking, we determine stationary values of J-minima, maxima or saddle points. In most cases of physical interest the stationary value will be a minimum. The difference between these two paths for a given x is called the variation of y "δy".

The principles of variation is applicable for a wide variety of physical phenomena. Such as : in classical mechanics, for the derivation of Lagrangian equation from Hamilton's principles, i.e if the integrand f is taken to be a Lagrangian L which is defined as the difference of kinetic energy T and potential energy V of the system, then by using time as an independent variable.

The equation $\delta J = 0$ is then a mathematical statement of Hamilton's principles of classical mechanics[1].

$$\delta \int_{t_1}^{t_2} L(x_1, x_2, ...x_n, \dot{x_1}, \dot{x_2}, ...\dot{x_n}, t)dt = 0. \tag{0.0.2}$$

In words, Hamilton's principles asserts that the motion of the particles from t_1 to t_2 is such that the time integral of the lagrangian, or action, has a stationary value. The resulting equation will be,

$$\frac{d}{dx}\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = 0. \tag{0.0.3}$$

In relativistic mechanics, that is, to show the Lagrangian

$$L = m_0 c^2 (1 - \sqrt{1 - \frac{v^2}{c^2}}) - V(r). \tag{0.0.4}$$

and this leads to a relativistic form of Newton's second law of motion,

$$\frac{d}{dt}(\frac{m_0 v_i}{\sqrt{1 - \frac{v^2}{c^2}}}) = F_i. \tag{0.0.5}$$

in which the force components are $F_i = -\frac{\partial V}{\partial x_i}$ and electromagnetic theory for instance, if the Lagrangian (per unit volume) of an electromagnetic field with a charge density ρ is given by[1]

$$\pounds = \frac{1}{2}(\epsilon_0 \vec{E}^2 - \frac{1}{\mu_0}\vec{B}^2) - \rho\phi + \rho\vec{V} \cdot \vec{A}. \tag{0.0.6}$$

This leads to two Maxwell's equation, the remaining two are a consequence of the definition of $\vec{E}$ and $\vec{B}$ in terms of $\vec{A}$ and ϕ. The convenience should not be minimized, but at the same time we should be aware that these cases the calculus of variations has only provided an alternate description of what was already known. However, the situation does change with incomplete theories. If the basic physics is not yet known a postulated variational principle can be a useful starting point[1,2].

Here we are using this method in the theory of superconductivity by describing the proper functionals of each theories. For the derivation of Londons' equation and Ginzbuge-Landau equation our functional is the free energy of the superconductors but with different approaching and by taking variation with respect to magnetic field for the case of Londons' equation and where as for the case of Ginzbuge-Landau equation taking variation with respect to the order parameter and magnetic field and in the Ferrell-Prange equation when we take our functional is the energy stored in the Josephson's junction and we have to take variation with respect to the wave phase difference of the superconducting electrons across the junction.

The aim of the thesis is self consistently using variational method to drive the Ferrell-Prange equation and determining the first magnetic critical field of the josephsons junction where the first vortex appearing in the junction and to show the penetration of the magnetic field on Josephson's contact between two superconductors, when there is an insulator between them.

Chapter 1

BASIC EXPERIMENTAL SURVEY OF SUPERCONDUCTIVITY

1.1 The discovery of superconductivity

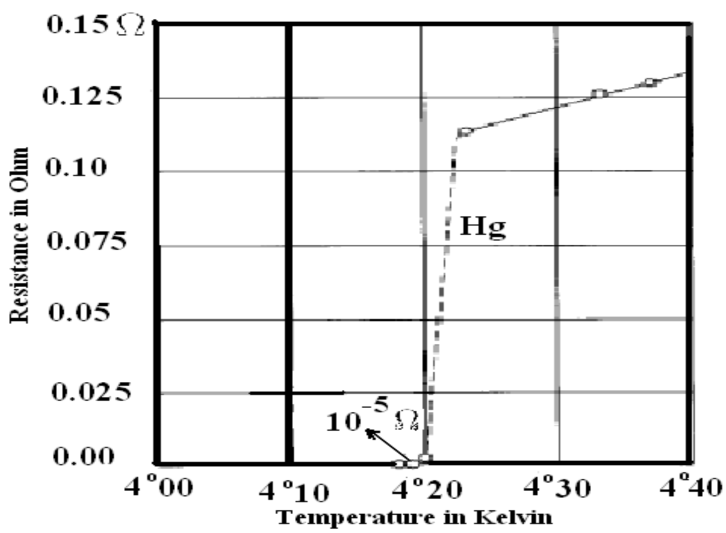


Figure 1.1: Resistance in ohms of a specimen of mercury versus absolute temperature. This plot by Kamerlingh Onnes marked the discovery of superconductivity.

The electrical resistivity of many metals and alloys suddenly drops to zero when the specimen is cooled to a sufficiently low temperature, often a temperature in the liquid helium

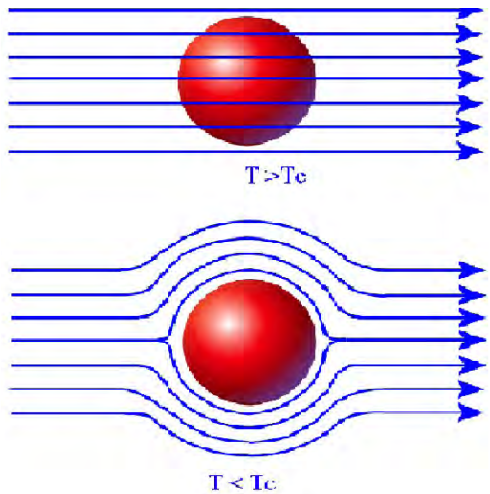


Figure 1.2: Meissner effect

range this phenomenon, called low temperature superconductivity, was first observed by Kamerlign Onnes in Leideen in 1911, three years after he first liquified helium.

In general the complete disappearance of all resistance to the flow of a direct current means that a current induced in a superconducting ring will flow indefinitely. Experimentally the decay time is certainly greater than 10^5 years. A number of other properties are also changed drastically at and below T_c, where T_c is the critical temperature which is characteristic of the metal. The magnetic properties exhibited by superconductors are as dramatic as their electrical properties. The magnetic properties cannot be accounted for by the assumption that a superconductor is a normal conductor with zero electrical resistivity.

It is an experimental fact that a bulk of superconductor in a weak magnetic field will act as a perfect diamagnet, with zero magnetic induction in the interior. When a specimen is placed in a magnetic field and is then cooled through the transition temperature for superconductivity, the magnetic flux originally present is ejected from the specimen. The unique magnetic properties of superconductors are central to the characterization of the superconducting state[3].

1.2 Occurrence of superconductivity

Superconductivity occurs in many metallic elements of the periodic system and also in alloys, intermetallic compounds and doped semiconductors. So far more than twenty-five elements and many thousands of alloys and compounds have been shown to be superconducting. The alkalis, magnetically ordered metals such as Iron Nickel and Cobalt and the noble metals (Copper, Silver, Gold) do not become superconducting. Elements representing most types of crystal structure can be superconductors.

Compound	T_c(k)	H_c(G)	ξ(Å)	λ_L(Å)
Al	7.18	105	1300-1600	160-500
$As(P = 14Gpa)$	0.5			
$Ba(P = 20Gpa)$	53			
Be	0.02			
$Bi(P = 8Gpa)$	8.5			
Cd	0.52	30	7600	1100
$Ce(P = 5Gpa)$	1.7			
$Cs(P = 13Gpa)$	1.6			
Ga	1.09	58.9		
Hf	0.02			
Hg	3.95			
In	3.41	289	2400-3500	390-640
Ir	0.10	20.1		
La	6.0	1096		
Lu	0.1			
Mg	0.0005			
Mo	0.92	98		
Nb	9.3	1980	380	390
$P(P = 17Gpa)$	5.8			

Pb	7.20	803	510-960	390-630
Ru	0.49	47		
$Se(P = 13Gpa)$	6.9			
$Si(P = 12Gpa)$	7.1			
Sn	3.7	308	1000-3000	340-750
Ta	4.46	831		
Tc	7.8	1410		
$Te(P = 8Gpa)$	4.3			
Th	1.37	162		
Ti	0.42	56		
Tl	2.4	180	4200	
U	1.8			
W	0.02	1.07		
$Y(P = 17Gpa)$	2.7			
Zn	0.85	52		
Zr	0.53	47		
Nb_3Sn	18.5	28	34	1600
$YBa_2Cu_30_{7-x}$	92	500	4-8	900-8000
$HgBa_2Ca_2Cu_3O_y$	135			

Table1.1: Properties of superconductors. (*critical temperature T_c, lower magnetic field H_c where flux first penetrates sample at zero temperature, coherence length ξ and London penetration depth λ_L for selected superconductors. Measurements are at atmospheric pressure unless otherwise indicated[4]).*

The range of transition temperatures best confirmed as of 2006 extends from the highest-temperature superconductor (at ambient pressure) is mercury thallium barium calcium copper oxide ($Hg_{12}T_{l3}Ba_{30}Ca_{30}Cu_{45}O_{125}$), at $138K$ and is held by a cuprate-perovskite material, possibly 164 K under high pressure[5], to below $0.001K$ for the compound Rh[3], although there are values of T_c, which is about 128K. This is for the complex compound $Tl_2Ba_2Ca_2Cu_3O_{10-\delta}$. The actual value of T_c depends very little on the particular sample

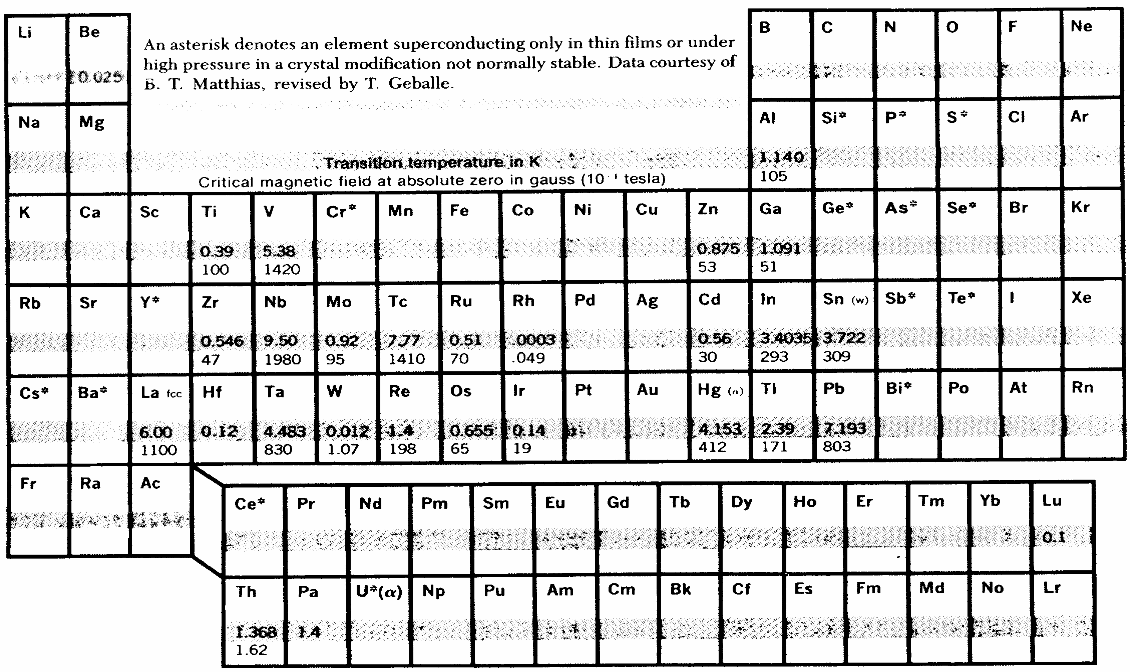


Figure 1.3: The periodic table showing the superconducting elements. The transition temperature and critical magnetic field for each superconductor is shown in the middle and lower parts of each box respectively [3].

of the metal which is measured, unless it contains magnetic impurities, when even small traces can depress T_c very considerably. The concept of high temperature superconductivity was first introduced in 1986 by Karl Muller and Johannes Bednorz. Although the width of the temperature range over which the resistance drops to zero is much narrower for pure, single-crystal specimens[6,7,8].

1.3 What causes superconductivity?

The physical mechanism of superconductivity became clear only 46 years after the phenomenon had been discovered, when Bardeen, Cooper and Schrieffcr published their theory (the BCS theory). The resistance of metals at low temperature is caused by the

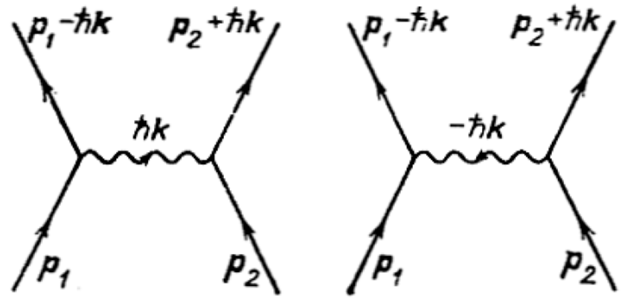


Figure 1.4: Electron - electron interaction mediated by the emission and absorption of a virtual phonon. Solid lines for electrons, waved lines for phonons[9]

scattering of the conduction electrons by impurity atoms. In no way can the quasi-free electron model be modified so that the interaction with impurities can be reduced to zero. These impurities are always present and they will always scatter the electrons. A completely fresh approach to the problem is necessary in order to explain the disappearance of the resistivity.

The fundamental idea underlying the modern theory of superconductivity is that the electrons pair up with one another due to a special type of attractive interaction. The zero resistivity pair up can only be scattered if the energy involved is sufficient to break up into two single electrons. In general this energy will not be available and so the electron pair passed on, undeviated by impurities.

Of course the ordinary coulomb interaction between electrons produce a repulsion, and the attractive pair interaction is much more subtle. It is an indirect interaction and it is caused by the way a positive ion in the crystal responds to the passage of electrons in its vicinity.

Let us consider an electron passing close to an ion Fig.1.4. There will be a momentary attraction between them which might slightly modify the vibrations of the ion. This in turn could interact with a second electron nearby which will be attracted to the ion but the net effect of these two interaction is that there is an apparent attractive force between the two electrons and this would not have arisen if the ion and not been present. In the language of field theory the interaction is said to be due to the exchange of a virtual phonon between the two electrons[9,10].

1.4 Destruction of superconductivity

A sufficiently strong magnetic field will destroy superconductivity. The threshold or critical value of the applied magnetic field for the destruction of superconductivity is denoted by $H_c(T)$ and is a function of the temperature. At the critical temperature the critical field is zero ; $H_c(T_c) = 0$.

$$H_c(T) = H_c(0)(1 - (T/T_c)^2). \tag{1.4.1}$$

The threshold curve separates the superconducting state in the lower left of the Fig.1.5

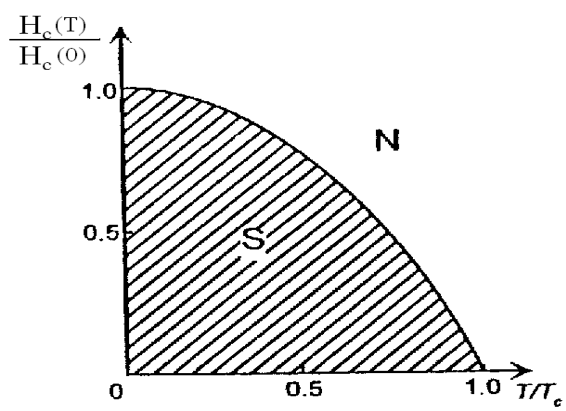


Figure 1.5: Temperature dependence of the critical field $\vec{H}_c$.

from the normal state in the upper right.

The magnetization curve expected for a superconductor under the conditions of the Meissner-Ochsenfeld experiment is sketched in Fig.1.7a . This applies quantitatively to a specimen in the form of along solid cylinder placed in a longitudinal magnetic field. Pure specimens of many material except Niobium (Nb) exhibit a magnetization curve behaving like this are called type-I superconductor, formerly, soft conductor. The value of $\vec{H}_c$ are always too low for type-I superconductor to have any useful in technical application in coils for superconducting magnets. Other materials exhibit a magnetization curve of Fig.1.7b

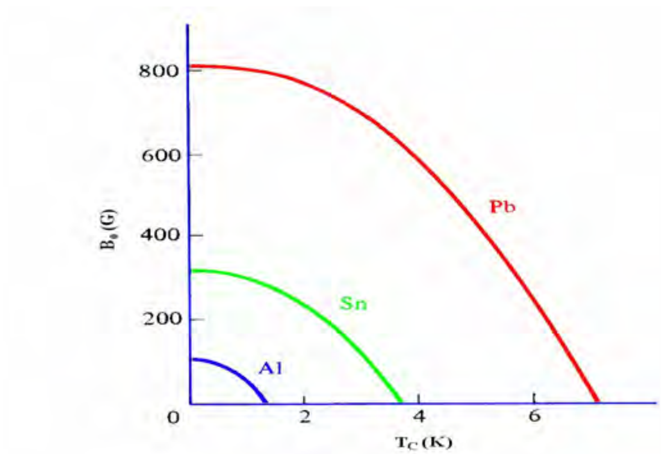


Figure 1.6: Experimental threshold curves of the critical field $H_c(T)$ versus temperature for some elements[3].

and are known as type-II superconductors. They tend to be alloys or transition metals with high values of the electrical resistivity in the normal state; that is, the electronic mean free path in the normal state is short. We shall see later why the mean free path is involved in the " magnetization" of superconductors. Type-II superconductors have superconducting electrical properties up to a field denoted by $\vec{H}_{c2}$. Between the lower critical field $\vec{H}_{c1}$ and the upper critical field $\vec{H}_{c2}$ the flux density $\vec{H} \neq 0$ and the Meissner effect is said to be incomplete. The value of $\vec{H}_{c2}$ may be 100 times or more higher than the value the critical field $\vec{H}_c$ calculated from the thermodynamics of the transition. In the

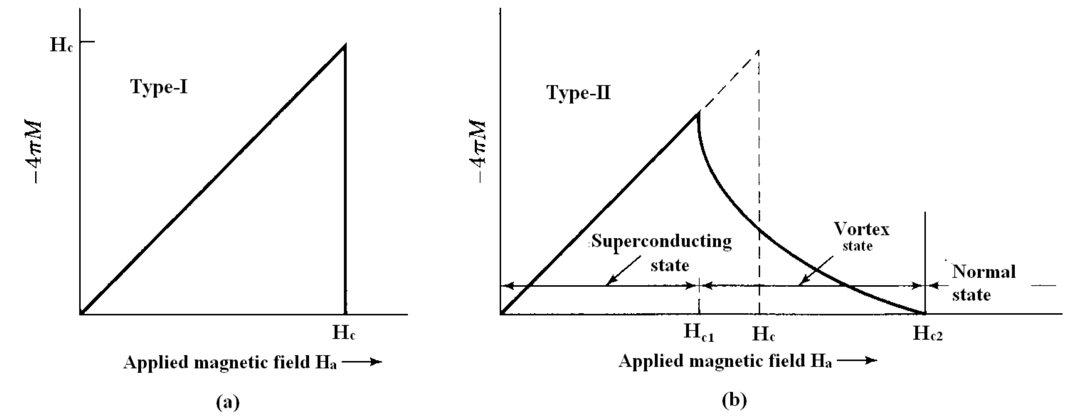


Figure 1.7: Magnetization versus applied magnetic field for a bulk superconductor. (a) corresponds to type-I and (b) corresponds to type-II superconductors[3].

region between $\vec{H}_{c1}$ and $\vec{H}_{c2}$ the superconductor is threaded by flux lines and said to be in the Vortex State. A field $\vec{H}_{c2}$ of 410KG(41 Tesla) has been attained in an alloy of Nb, Al, and Ge at the boiling point of Helium, and 540KG has been reported for $PbMo_6S_8$[6]. Basically there is no difference in the mechanism of superconductivity in type-I and type-II superconductors. Both types have similar thermal properties at the superconductor-normal transition in zero magnetic field. But the Meissner effect is entirely different.

A good type-I superconductor excludes a magnetic field until superconductivity is destroyed suddenly, and then the field penetrates completely. A good type-II superconductor excludes the field completely up to a field $\vec{H}_{c1}$. Above $\vec{H}_{c1}$ the field is partially excluded, but the specimen remains electrically superconducting. At a much higher field, $\vec{H}_{c2}$, the flux penetrates completely and superconductivity vanishes. An outer layer of the specimen may remain superconducing up to a still higher field $\vec{H}_{c3}$[11,12].

Chapter 2

LONDON'S EQUATION

2.1 Introduction

We have said that the zero value of $\vec{H}_{int}$ is a result of induced surface currents which themselves produce a field to oppose $\vec{H}_{ext}$. These currents cannot flow in an infinitely thin surface layer because otherwise the critical current density would be exceeded, and so they must extend into the material slightly. Only completely under this layer will the full influence of the surface currents be felt so that $\vec{H}_{int} = 0$. In the layer itself the field will not be zero and hence an external field is not completely canceled.

The Meissner effect and the field penetration cannot be deduced from Maxwell's electrodynamics equation alone.

Fritz London and Heinz London first examined in quantitative way that the fundamental fact that a metal in the superconducting state permits no magnetic field in its interior. Their analysis starts with the two fluid model of Gorter and Casimir: in order to understand the behavior of a superconductor in an external electromagnetic field, we use this two-fluid model. We assume that all free electrons of the superconductor are divided into two groups: superconducting electrons of density n_s and normal electrons of density n_n. The total density of free electrons is $n = n_s + n_n$. As the temperature increases from 0 to T_c, the density n_s decreases from n to 0[13,14].

The only crucial assumption of this model that we shall use is that in a superconductor at temperature $T < T_c$, only a fraction $\frac{n_s(T)}{n_n}$ of the total number of conduction electrons are participating in a supercurrent. The supercurrent flows with no resistance. Whatever it will carry the entire current induced by any small transitory electric field, and the normal electrons will remain quit inert.

2.2 Thermodynamics of superconductors

Let us consider in more detail about the basic properties of those thermodynamic potentials that we shall often deal with when studying superconductivity. These potentials are equally suitable for analysis of any sort of matter, superconductors in particular.

Suppose one takes a long thin cylinder of suerconducting material and places it in an external magnetic field of strength H that points along the the cylinder axis. The long cylinder shape is desirable because its demagnetizing factor is zero, and the normal component of the magnetic field automatically vanishes along most of its surface. Experimentally, Meissner and Ochsenfeld (1933) found that at critical applied field H_c the superconductivity is destroyed and magnetic flux penetrates the sample. The transition is completely reversible, so right at this critical field the superconducting state and normal state of the material must be in equilibrium. The free energy per unit volume of the metal in the normal state must be just:

$$F = F_{normal} + \frac{H_c^2}{8\pi}.$$

Because the superconducting state expels the magnetic induction, its free energy is just $F_{superconducting}$.

There now appears to be a problem. The free energy of the normal state is greater than that of the superconducting one, and the magnetic field energy is also positive. Yet when the critical field H_c is applied the above equation is supposed to equal $F_{superconducting}$.

What has gone wrong? The answer is that thermodynamics is being used improperly. The reversible experiments are carried out by slowly varying external currents, not by controlling $\vec{H}$ directly. In the instant that the normal metal expels $\vec{H}$ and becomes superconducting, a host of magnetic flux lines blasts off to infinity. The work done during this instant must be accounted for; equivalently, one must choose to work with thermodynamic variables that really do vary smoothly during the change from normal metal to superconductor. The current thermodynamic potential is then,

$$G = F - \frac{H}{4\pi} \cdot \frac{\delta F}{\delta H} = F_{normal} - \frac{H_c^2}{8\pi}.$$

Therefore the difference in free energy per volume at any given temperature between superconducting and normal metal must be exactly,

$$\triangle F \equiv F_{normal} - F_{superconducting} = \frac{H_c^2}{8\pi}.$$

One should remember that the magnetic field $\vec{H}$ inside a long thin cylinder with axis along the field is always spatially uniform and equal to the externally applied field. It is magnetic induction $\vec{H}$ that is expelled from superconducting samples, not the magnetic field $\vec{H}$.

By differentiating the above equation with respect to temperature, one finds immediately that the excess entropy of normal metal to the the superconductor must be,

$$\triangle S = \frac{\partial}{\partial T}\triangle F = \frac{H_c}{4\pi}\frac{\partial H_c}{\partial T}.$$

Meissner and Ochsenfeld determined that as temperature rises towards the critical temperature T_c where superconductivity vanishes in zero magnetic field, the critical field H_c also goes to zero. It follows from above equation that that the latent heat of transformation is zero, and therefore the transition is second order[4].

2.3 London's Equation

The kinetic energy of superconducting electrons per unit volume is:

$$K = \frac{1}{2} m n_s v_s^2. \tag{2.3.1}$$

Where n_s is the number of superconducting electrons per unit volume and it is a parameter of the theory .

v_s is the drift velocity of superconducting electrons,

m is the mass of electron,

The current density $\vec{j}$ is :

$$\vec{j}_s = e n_s \vec{v}_s. \tag{2.3.2}$$

$$\vec{v}_s = \frac{\vec{j}_s}{n_s e}. \tag{2.3.3}$$

where e is charge of electron.

From equation (2.2.1) and (2.2. 3),

$$K = \frac{1}{2} m n_s (\frac{j_s}{n_s e})^2. \tag{2.3.4}$$

From Maxwell's equation,

$$\nabla \times \vec{H} = \frac{4\pi}{c} \vec{j}_s. \tag{2.3.5}$$

It follows that,

$$\vec{j}_s = \frac{c}{4\pi} (\nabla \times \vec{H}). \tag{2.3.6}$$

Here equations (2.2.4) and (2.2.6) give us;

$$K = \frac{1}{32} \frac{mc^2}{n_s \pi^2 e^2} (\nabla \times \vec{H})^2. \tag{2.3.7}$$

The energy density per unit volume due to the magnetic field is: $\frac{\vec{H}^2}{8\pi}$. Therefore the free energy of the superconductor will be,

$$F_s = F_0 + \int (K + \frac{\vec{H}^2}{8\pi}) dv. \tag{2.3.8}$$

but $K=\frac{1}{32}\frac{mc^2}{n_s\pi^2e^2}(\nabla\times\vec{H})^2$ and let's λ be,

$$\lambda^2=\frac{mc^2}{4\pi e^2n_s}. \tag{2.3.9}$$

$$F_s=F_0+\int[\frac{\lambda^2}{8\pi}(\nabla\times\vec{H})^2+\frac{\vec{H}^2}{8\pi}]dv. \tag{2.3.10}$$

The expression (2.2.10) can be considered as a functional and F_s depend on a particular form of $\vec{H}(r)$. It is known that the free energy has a minimum at the heat equilibrium state. We will find a minimum of the functional equation (2.2.10) by applying the variation of the magnetic field $\vec{H}\rightarrow\vec{H}+\delta\vec{H}$. The superconducting state is a new phase thermodynamically of a normal metal as it is well know the free energy is minimized,

$$\delta F_s=\frac{1}{4\pi}\int[\lambda^2(\nabla\times\vec{H})(\nabla\times\delta\vec{H})+\vec{H}\cdot\delta\vec{H}]dv. \tag{2.3.11}$$

We demand δF_s= 0 and use the vector identity;

$$\nabla\cdot(\vec{a}\times\vec{b})=\vec{b}\cdot(\nabla\times\vec{a})-\vec{a}\cdot(\nabla\times\vec{b}). \tag{2.3.12}$$

Let $\vec{a}=\nabla\times\vec{H}$, $\vec{b}=\delta\vec{H}$, then equation (2.3.11) becomes;

$$\frac{1}{4\pi}\int dv[\lambda^2\nabla\times\nabla\times\vec{H}+\vec{H}]\delta\vec{H}-\frac{1}{4\pi}\int\nabla\cdot(\nabla\times\vec{H}\times\delta\vec{H})dv=0. \tag{2.3.13}$$

From the divergence theorem,

$$\frac{1}{4\pi}\int\nabla\cdot(\nabla\times\vec{H}\times\delta\vec{H})dv=\frac{1}{4\pi}\oint(\nabla\times\vec{H}\times\delta\vec{H})\cdot d\vec{s}. \tag{2.3.14}$$

Here we use $d\vec{s}=\hat{n}ds$ where $\hat{n}$ is the unit vector normal to the surface of the superconductor and at the surface of our superconductor the magnetic field is equal to the external magnetic field, this means that $\delta\vec{H}|_s$= 0,then equation (2.2.13) takes the form:

$$\frac{1}{4\pi}\int dv[\lambda^2\nabla\times\nabla\times\vec{H}+\vec{H}]\delta\vec{H}=0. \tag{2.3.15}$$

Our volume is arbitrary, therefore the integrand of (2.2.15) must be zero. We present it in the form;

$$\lambda^2[\nabla(\nabla\cdot\vec{H})-(\nabla\cdot\nabla)\vec{H}]+\vec{H}=0. \tag{2.3.16}$$

Noting that,

$$\nabla \cdot \vec{H} = 0.$$

Therefore the distribution of the magnetic field $\vec{H}(x, y, z)$ in the superconductor satisfies the equation,

$$\vec{H} - \lambda^2 \nabla^2 \vec{H} = 0. \tag{2.3.17}$$

This is the London's Equation. Sometimes it can be written as;

$$\nabla \times \vec{j_s} = -\frac{n_s e^2}{mc} \vec{H},$$

$$\vec{j_s} = -\frac{n_s e^2}{mc} \vec{A}. \tag{2.3.18}$$

Equation(2.3.18) can be regarded as a replacement for Ohm's law, $\vec{j} = \sigma \vec{E}$, as a description of the behavior of the superconducting electrons[15].

2.4 London penetration depth and Meissner effect

From variational methods we have derived the London's equation $\vec{H} - \lambda^2 \nabla^2 \vec{H}$=0. This equation is seen to account for the Meissner effect, because it doesn't allow a solution uniform in space, so that a uniform magnetic field cannot exist in superconductor i.e. $\vec{H}(r)$=$\vec{H}_0$, which is not a solution of this equation unless the constant $\vec{H}_0$=0.

In the pure superconducting state the only field allowed is exponentially damped as we go in from an external surface. Let a semi-infinite superconductor occupy the space on the positive side of the X-axis. Therefore the London's equation in one dimension will be;

$$H - \lambda^2 \frac{d^2}{dx^2} H = 0, \qquad \lambda^2 = \frac{mc^2}{4\pi e^2 n_s}. \tag{2.4.1}$$

The solution of the London's equation in one dimension using boundary conditions, let it be at $x = 0$, H=$H(0) = H_0$,

$$H(x) = A\ e^{-x/\lambda} + B\ e^{x/\lambda}. \tag{2.4.2}$$

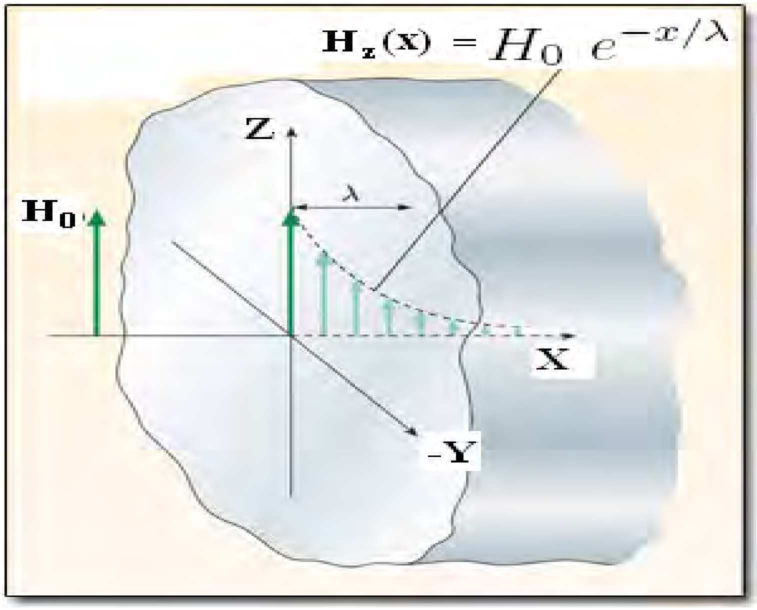


Figure 2.1: The Meissner effect and Penetration depth

$B = 0$, we ignore increasing field

$H(x)$=$A\ e^{-x/\lambda}$.

$H(0) = A = H_0$.

$$H(x) = H_0\ e^{-x/\lambda}. \tag{2.4.3}$$

Now to find the current in the superfluid condition, let's assume the H field is parallel to z-axis.

From Maxwell's equation

$$\frac{4\pi}{c}\vec{j_s} = -\frac{d}{dx}H(x)\ \ \hat{y}.$$

$$\vec{j_s} = -\frac{c}{4\pi}\frac{d}{dx}(H_0\ e^{-x/\lambda})\ \ \hat{y}.$$

$$\vec{j_s} = \frac{c}{4\pi\lambda}(H_0\ e^{-x/\lambda})\ \ \hat{y}. \tag{2.4.4}$$

The current density is in the y-direction, but an applied magnetic field H will penetrate a thin film fairly uniformly if the thickness is much less than λ. Thus in the Meissner effect is not complete for thin superconducting film[16]. If a superconducting film or filament which is thinner than 10nm its properties are significantly different from those of the bulk materials. In particular, the value of the critical magnetic increases as the thickness decrease, and the special property of type-II superconductors arises from this[17].

Since the perfect diamagnetism of a superconductor is dependant on $\vec{H}$ being zero throughout the material, the surface penetration of the field will reduce the diamagnetic effect. Thus measurements of the susceptibility can be used to investigate the behavior of λ. A straight forward technique is to measure the value of a mutual inductance in which the primary and secondary wings are wound around the specimen is prepared in the form of very small particles or fine wires or films, in which one or more dimensions are of the same order or smaller than λ. The diamagnetic effect will then be very much reduced.

Here $\lambda^2 = \frac{mc^2}{4\pi e^2 n_s}$ and we will clarify its physical meaning latter, measures the depth of penetration of the magnetic field ; it is known as the London penetration depth. Actual penetration depths are not described precisely by λ alone, for the London equation is now known to be some what over simplified and λ is in the range between $10^{-8}m$ to $10^{-7}m$ or 100Å to 1000Å. As the transition temperature is approached the number of paired electron decreases and so λ will increase.

Lets take values and check it with the experimental value.

$m = 10^{-27}g$, $c = 3 \times 10^{10}\frac{cm}{sec}$, $e = 4.8 \times 10^{-10}cgs$, $n_s = 10^{22}cm^{-3}$, then $\lambda = 5.58 \times 10^{-6}cm$ $\approx$ 600Å, which is consistent with the experimental value.

λ is the distance for $\vec{H}$ to fall from $\vec{H}_0$ at the surface to $\vec{H}_0/e$. The penetration depth does not have a fixed value but varies with temperature, at low temperature it is nearly independent of temperature, however, the penetration depth increase rapidly and approaches infinity as the temperature approaches the transition temperature. The variation penetration depth with temperature is found to fit very closely the relation[20]

$$\lambda = \frac{\lambda_0}{[1-(T/T_c)^4]^{1/2}} \tag{2.4.5}$$

Chapter 3

LANDAU-GINZBURG EQUATION

3.1 Ginzburg-Landau Theory

Calculation of surface energy (σ) between normal metal and superconductor interface with the help of London's equation gives that σ is negative, this means that it is energetically profitable to increase area of interface between normal state and superconducting phases. This is true for type-II superconductor and completely does not correspond to Meissner effect like for the type-I superconducting state, because in the type-I superconductor σ is positive thus the Ginzburg-Landau equation removes this problem. The

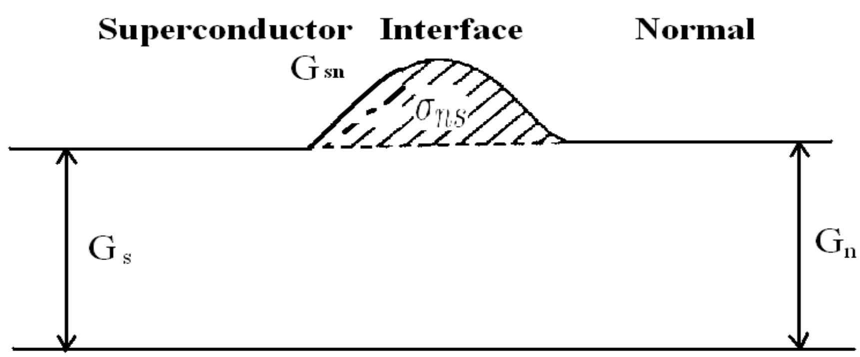


Figure 3.1: Density of the Gibbs free energy G_{sn} in the vicinity of a normal metal-superconductor interface.

London theory (Chap.2) did not take into account quantum effects. The first quantum (phenomenological) theory of superconductivity was the Ginzburg-Landau (GL) theory. In physics Ginzburg-Landau theory is a mathematical theory to model superconductivity. It does not purport to explain the microscopic mechanisms giving rise to superconductivity. This theory sometimes called phenomenological as it describes some of the phenomena of superconductivity with out explaining the underling microscopic mechanism.
Ginzburg and Landau asserted that the superconducting state could be characterized by a complex " order parameter " Ψ which vanishes above T_c, where T_c is the critical temperature of the superconductor.
The order parameter is expected to have the following properties

1) $\Psi(r)$ is zero above T_c.
2) $\Psi(r)$ is not zero below T_c.
3) $\Psi(r)$ can approach zero continuously as T approaches T_c from below (in the case of a transition which is not first order).
4) $\Psi(r)$ is not completely determined in the absence of external fields if T is below T_c.

We have to construct the functional which depends on $\Psi(r)$ and $\vec{H}$ or the vector potential $\vec{A}$ ($\vec{H} = \bigtriangledown \times \vec{A}$). Now by using variational methods we can drive the Ginzburg-Landau equations and the boundary conditions to these equations at the surface of superconductor, because the most probable value of $\Psi(r)$ is to be determined by minimization of the free energy density.
Consider the superconducting state as a condensate state of electron pairs (Cooper pairs). These electrons have the same wave function.

$$\Psi(r) = (\frac{n_s}{2})^{1/2} e^{i\theta(r)}. \tag{3.1.1}$$

Here the order parameter $\Psi(r)$ with the property $\Psi(r)^*\Psi(r)=\frac{n_s(r)}{2}$ and $\theta(r)$ is the phase of the collective superconducting wave function. This is the local concentration of a superconducting electrons.

We first set up a form for the Gibbs free energy F_s in a superconductor with no magnetic field near T_c can be expanded in powers of the order parameter $|\Psi|^2$.

$$F_s = F_n + \alpha|\Psi|^2 + \frac{\beta}{2}|\Psi|^4 + ... \tag{3.1.2}$$

Here F_s is the Gibbs free energy density of the superconductor in the absence of magnetic field, F_n, is its free energy density in the normal state α and β are some phenomenological expansion coefficients which are characteristics of the material.

The expression (3.1.2) valid near the critical temperatures T_c , $\frac{T_c - T}{T_c} << 1$, where the order parameter Ψ is small.

In the thermal equilibrium,

$$\frac{\partial F_s}{\partial|\Psi|^2} = \alpha + \beta|\Psi|^2 = 0. \tag{3.1.3}$$

$$|\Psi_0|^2 = -\frac{\alpha}{\beta} > 0. \tag{3.1.4}$$

Ginzburg and Landau proposed that

1) *Parameter* $\alpha \propto (T - T_c)$,
2) *Parameter* $\alpha = 0$ if $T > T_c$,
3) *Parameter* $\alpha < 0$ at $T < T_c$,
4) *Parameter* β does not depend on T,

Indeed, as follows from equation(3.1.4), at $T < T_c$ and $\alpha < 0$, $|\Psi_0|^2$ can be positive only if $\beta > 0$. On the other hand, if $T > T_c$ (and consequently, from $\alpha \propto (T - T_c)$, let's assume $\alpha > 0$) and $\beta > 0$, the energy F_s reaches its minimum at $|\Psi_0|^2 = 0$. This means that there is no superconducting state at $T > T_c$ as indeed ought to be the case. Thus we have $\beta > 0$ both at $T < T_c$ and at $T > T_c$. Therefore, to a first approximation in $(T_c - T)$, we may assume β =const. Then equation(3.1.2) becomes

$$F_s = F_n + \alpha(\frac{-\alpha}{\beta}) + \frac{\alpha^2}{2\beta}. \tag{3.1.5}$$

Where F_n is the thermodynamic potential of the normal state with out magnetic field.

$$F_s = F_n - \frac{\alpha^2}{2\beta}. \tag{3.1.6}$$

Here $\frac{\alpha^2}{2\beta} = \frac{\vec{H}^2_{cm}}{8\pi}$.

$$\vec{H}^2_{cm} = \frac{4\pi\alpha^2}{\beta}. \tag{3.1.7}$$

Now we introduce the magnetic field. In this case Ψ is coordinate dependant. Thus we should add both the energy of the magnetic field $\frac{\vec{H}^2}{8\pi}$ and the energy connected with the inhomogeneity. Near the critical point it is enough to add $|\Psi|^2$. We know that Cooper pairs are charged particles, consequently because of gauge invariance, only combination $(-i\hbar\nabla + \frac{2e}{c}\vec{A})$ is possible to make a proper dimension we write the corresponding term as $\frac{1}{4m}|(-i\hbar\nabla + \frac{2e}{c}\vec{A})\Psi|^2$. Finally we now switch on the magnetic field in the Gibb's free energy density in the quantum mechanical state can be expanded in powers of Ψ near T_c

$$\mathtt{G}_s = \mathtt{G}_n + \alpha|\Psi|^2 + \frac{\beta}{2}|\Psi|^4 + \frac{1}{4m}|(-i\hbar\nabla + \frac{2e}{c}\vec{A})\Psi|^2 + \frac{\vec{H}^2}{8\pi} - \frac{\vec{H}\cdot\vec{H}_0}{4\pi}.$$

Here $\alpha|\Psi|^2 + \frac{\beta}{2}|\Psi|^4$ is the typical Landau form for the expansion of the free energy in terms of an order parameter that vanishes at a second order phase transition, the term $\frac{1}{4m}|(-i\hbar\nabla + \frac{2e}{c}\vec{A})\Psi|^2$ represents an increase in energy by a spatial variation of the order parameter. It has the form of the kinetic energy in quantum mechanics, the last two terms $\frac{\vec{H}^2}{8\pi}$ and $\frac{\vec{H}\cdot\vec{H}_0}{4\pi}$ represents the external magnetic energy density and the increase in the superconducting free energy caused by the expulsion of magnetic flux from the superconductor respectively. The Gibbs free energy of a superconductor as a whole is

$$G_s = G_n + \int dv[\alpha|\Psi|^2 + \frac{\beta}{2}|\Psi|^4 + \frac{1}{4m}|(-i\hbar\nabla + \frac{2e}{c}\vec{A})\Psi|^2 + \frac{\vec{H}^2}{8\pi} - \frac{\vec{H}\cdot\vec{H}_0}{4\pi}]. \tag{3.1.8}$$

where the integration is carried out over the entire volume of the superconductor. This is the functional of Ginzburg-Landau theory. The G-L theory consists of the order parameter Ψ and the vector potential $\vec{A}$ $(i.e\vec{H} = \nabla\times\vec{A})$. We will seek a minimum of G-L functional under variation of Ψ and $\vec{A}$ i.e $\Psi \to \Psi + \delta\Psi$ and $\vec{A} \to \vec{A} + \delta\vec{A}$.

3.2 First Ginzburg-Landau Equation

Our task now is to find equations for the functions $\Psi(r)$ and $\vec{A}(r)$ such that their solutions, when substituted in (3.1.8), give the minimum value of G_s. First let's take the variation of the order parameter Ψ^* on the free energy to obtained the first G-L equation, by keeping $\vec{H}$ to be invariant.

$$\delta G_s = \int dv[\alpha\Psi\delta\Psi^* + \beta|\Psi|^2\Psi\delta\Psi^* \\ +\frac{1}{4m}(-i\hbar\nabla\Psi + \frac{2e}{c}\vec{A}\Psi)\cdot(i\hbar\nabla\delta\Psi^* + \frac{2e}{c}\vec{A}\delta\Psi^*)]. \tag{3.2.1}$$

Now consider $\delta\Psi^\star$ under ∇ sign and let's focused on the third term of the right hand side of equation(3.2.1)

$$\frac{1}{4m}(-i\hbar\nabla\Psi + \frac{2e}{c}\vec{A}\Psi)\cdot(i\hbar\nabla\delta\Psi^* + \frac{2e}{c}\vec{A}\delta\Psi^*) = \\ \frac{1}{4m}(-i\hbar\nabla\Psi + \frac{2e}{c}\vec{A}\Psi)i\hbar\nabla\delta\Psi^* \\ +\frac{1}{4m}(-i\hbar\nabla\Psi + \frac{2e}{c}\vec{A}\Psi)\frac{2e}{c}\vec{A}\delta\Psi^*. \tag{3.2.2}$$

We have the vector identity

$$\nabla\cdot(\vec{V}\phi) = \phi\nabla\cdot\vec{V} + \vec{V}\nabla\phi.$$

Here our $\vec{V}=\frac{1}{4m}(-i\hbar\nabla\Psi + \frac{2e}{c}\vec{A}\Psi)$ and $\phi=i\hbar\delta\Psi^*$.

Therefore the right hand side of equation (3.2.2) will be :

$$\frac{1}{4m}(-i\hbar\nabla\Psi + \frac{2e}{c}\vec{A}\Psi)i\hbar\nabla\delta\Psi^* + \frac{1}{4m}(-i\hbar\nabla\Psi + \frac{2e}{c}\vec{A}\Psi)\frac{2e}{c}\vec{A}\delta\Psi^* = \\ -i\hbar\delta\Psi^*\nabla\cdot[\frac{1}{4m}(-i\hbar\nabla\Psi + \frac{2e}{c}\vec{A}\Psi)] \\ +\nabla\cdot[\frac{1}{4m}(-i\hbar\nabla\Psi + \frac{2e}{c}\vec{A}\Psi)\cdot -i\hbar\delta\Psi^*] \\ +\frac{1}{4m}(-i\hbar\nabla\Psi + \frac{2e}{c}\vec{A}\Psi)\frac{2e}{c}\vec{A}\delta\Psi^*. \tag{3.2.3}$$

Now let's back to equation(3.2.1)

$$\delta G_s = \int dv[\alpha\Psi\delta\Psi^* + \beta|\Psi|^2\Psi\delta\Psi^* + \nabla\cdot[\frac{1}{4m}(-i\hbar\nabla\Psi + \frac{2e}{c}\vec{A}\Psi)i\hbar\delta\Psi^*]$$
$$-i\hbar\delta\Psi^*\nabla\cdot\frac{1}{4m}(-i\hbar\nabla\Psi + \frac{2e}{c}\vec{A}\Psi) + \frac{1}{4m}(i\hbar\nabla\Psi + \frac{2e}{c}\vec{A}\Psi)\frac{2e}{c}\vec{A}\delta\Psi^*]. \qquad (3.2.4)$$

At the thermal equilibrium the Gibbs potential is a minimum, we demand that;

$$\delta G_s = 0. \qquad (3.2.5)$$

$$0 = \int dv[\alpha\Psi + \beta|\Psi|^2\Psi + \frac{1}{4m}(-i\hbar\nabla\Psi + \frac{2e}{c}\vec{A}\Psi)\frac{2e}{c}\vec{A}$$
$$-\nabla\cdot\frac{i\hbar}{4m}(-i\hbar\nabla\Psi + \frac{2e}{c}\vec{A}\Psi)]\delta\Psi^*$$
$$+\int dv\nabla\cdot[\frac{1}{4m}(-i\hbar\nabla\Psi + \frac{2e}{c}\vec{A}\Psi)i\hbar\delta\Psi^*]. \qquad (3.2.6)$$

From divergence theorem

$$\int \nabla\cdot\vec{A}dv = \oint_s \vec{A}\cdot d\vec{s}.$$

Therefore

$$\int dv\nabla\cdot[\frac{1}{4m}(-i\hbar\nabla\Psi + \frac{2e}{c}\vec{A}\Psi)i\hbar\delta\Psi^*] =$$
$$\oint_s \frac{1}{4m}(-i\hbar\nabla\Psi + \frac{2e}{c}\vec{A}\Psi)i\hbar\delta\Psi^*\cdot d\vec{s}. \qquad (3.2.7)$$

And our boundary condition on the superconductor surface will be

$$[-i\hbar\nabla\Psi + \frac{2e}{c}\vec{A}\Psi]\cdot\hat{n} = 0. \qquad (3.2.8)$$

where $\hat{n}$ is the unit vector normal to the surface of the superconductor

Therefore equation (3.2.6) will be written as :

$$\int dv[\alpha\Psi + \beta|\Psi|^2\Psi + \frac{1}{4m}(-i\hbar\nabla\Psi + \frac{2e}{c}\vec{A}\Psi)\frac{2e}{c}\vec{A}$$
$$-\nabla\cdot\frac{i\hbar}{4m}(-i\hbar\nabla\Psi + \frac{2e}{c}\vec{A}\Psi)]\delta\Psi^* = 0. \qquad (3.2.9)$$
$$\alpha\Psi + \beta|\Psi|^2\Psi + \frac{1}{4m}(-i\hbar\nabla + \frac{2e}{c}\vec{A})^2\Psi = 0. \qquad (3.2.10)$$

This is the first G-L equation with the boundary condition (3.2.8) on the superconductor surface. If we take variation w.r.t Ψ we will get the complex conjugate of equation (3.2.10)

3.3 Second Ginzburg-Landau Equation

We have obtained the equation for the order parameter Ψ. Now we consider the variation of G-L functional with respect to $\vec{A}$ to obtained the equation for the quantum mechanical current density or $\vec{A}$ and recall equation(3.1.8) and take variation of $\vec{A}$, i.e $\vec{A} \rightarrow \vec{A} + \delta\vec{A}$ with the usage of $\nabla \times \nabla \times \vec{A}$=$\frac{4\pi}{c}\vec{j}_s$

The variation of $\vec{H}^2$ leads to

$$\delta(\nabla \times \vec{A})^2 = 2(\nabla \times \vec{A})(\nabla \times \delta\vec{A}). \tag{3.3.1}$$

Then we can use the relation $div(\vec{A} \times \vec{B})$=$\vec{B}(\nabla \times \vec{A}) - \vec{A}(\nabla \times \vec{B})$. The G-L functional i.e equation(3.1.8) can be written as follows

$$G_s = G_n + \int dv[\alpha|\Psi|^2 + \frac{\beta}{2}|\Psi|^4 + \frac{1}{4m}|(-i\hbar\nabla\Psi + \frac{2e}{c}\vec{A}\Psi)|^2 + \frac{(\nabla \times \vec{A})^2}{8\pi} - \frac{(\nabla \times \vec{A}) \cdot \vec{H}_0}{4\pi}].$$

The free energy can be calculated in a straight forward way, and equating the total variation to zero we obtain,

$$\begin{aligned}\delta G_s = \int dv[\frac{1}{4m}(i\hbar\nabla\Psi^* + \frac{2e}{c}\vec{A}\Psi^*)\frac{2e}{c}\Psi\delta\vec{A} + \\ \frac{1}{4m}(-i\hbar\nabla\Psi + \frac{2e}{c}\vec{A}\Psi)\frac{2e}{c}\Psi^*\delta\vec{A} + \\ \frac{1}{4m}(\nabla \times \vec{A})(\nabla \times \delta\vec{A}) - \frac{H_0}{4\pi} \cdot (\nabla \times \delta\vec{A})].\end{aligned} \tag{3.3.2}$$

$$\begin{aligned}\int dv[\frac{ie\hbar}{2mc}\Psi\nabla\Psi^* + \frac{e^2}{mc^2}\vec{A}|\Psi|^2 - \frac{ie\hbar}{2mc}\Psi^*\nabla\Psi + \frac{e^2}{mc^2}\vec{A}|\Psi|^2 + \frac{1}{4\pi}(\nabla \times \nabla \times \vec{A})]\delta\vec{A} + \\ \int \frac{1}{4\pi}\nabla \cdot (\vec{H}_o \times \delta\vec{A} - \nabla \times \vec{A} \times \delta\vec{A})dv = 0.\end{aligned} \tag{3.3.3}$$

From divergence theorem

$$\frac{1}{4\pi}\int \nabla \cdot (\vec{H}_o \times \delta\vec{A} - \nabla \times \vec{A} \times \delta\vec{A})dv = \oint_s [(\vec{H}_o \times \delta\vec{A} - \nabla \times \vec{A} \times \delta\vec{A})] \cdot d\vec{s}.$$

Our boundary condition will be,

$$[\vec{H}_o \times \delta\vec{A} - \nabla \times \vec{A} \times \delta\vec{A}] \cdot \hat{n} = 0. \tag{3.3.4}$$

$$\int dv[\frac{ie\hbar}{2mc}(\Psi\nabla\Psi^* - \Psi^*\nabla\Psi) + \frac{2e^2}{mc^2}\vec{A}|\Psi|^2 + \frac{1}{4\pi}(\nabla \times \vec{H})]\delta\vec{A} = 0. \tag{3.3.5}$$

$$\frac{ie\hbar}{2mc}(\Psi\nabla\Psi^* - \Psi^*\nabla\Psi) + \frac{2e^2}{mc^2}\vec{A}|\Psi|^2 + \frac{1}{4\pi}(\frac{4\pi}{c}\vec{j}_s) = 0. \tag{3.3.6}$$

where $\vec{j}_s$=$\frac{c}{4\pi}(\nabla \times \vec{H})$

$$\vec{j}_s = \frac{ie\hbar}{2m}(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*) - \frac{2e^2}{mc}\vec{A}|\Psi|^2. \tag{3.3.7}$$

This is just the quantum mechanical expression for the current for a particle with charge $(-2e)$ and the mass $2m$.

If we introduce dimensionless wavefunction $\psi(r)$ by setting $\psi(r) = \frac{\Psi}{\Psi_0} = \frac{\Psi}{\sqrt{n_s/2}}$, where $\Psi_0 = \sqrt{n_s/2} = \sqrt{\frac{|\alpha|}{\beta}}$ then system of G-L equations can be presented in the form by using the two fundamental lengths, that are the London's penetration length λ (3.4.1) $\lambda^2 = \frac{mc^2}{4\pi n_s e^2} = \frac{mc^2\beta}{8\pi e^2|\alpha|}$ and the coherence length ξ (3.4.2) $\xi^2 = \frac{\hbar^2}{4m|\alpha|}$, and $\Phi_0 = \frac{\pi hc}{e} = \frac{hc}{2e}$ which is the quantum of magnetic flux, like this;

$$\xi^2(-i\nabla + \frac{2\pi}{\Phi_0}\vec{A})^2\psi - \psi + \psi|\psi|^2 = 0. \tag{3.3.8}$$

$$\nabla \times \nabla \times \vec{A} = i\frac{\Phi_0}{4\pi\lambda^2}(\psi^*\nabla\psi - \psi\nabla\psi^*) - \frac{|\psi|^2}{\lambda^2}\vec{A}. \tag{3.3.9}$$

With the boundary condition for ψ

$$[-i\nabla\psi + \frac{2\pi}{\Phi_0}\vec{A}\psi] \cdot \hat{n} = 0. \tag{3.3.10}$$

This is a possible choice, which assures that no current passes through the surface used by GL, and it is appropriate at an insulating surface. Using the microscopic theory, de Gennes has shown that for a metal- superconductor interface with no current, equation(3.3.10) must be generalized to

$$[-i\nabla\psi + \frac{2\pi}{\Phi_0}\vec{A}\psi] \cdot \hat{n} = \frac{i\hbar}{b}\psi. \tag{3.3.11}$$

where b is a real constant. As shown in Fig. 3.2, if $\vec{A}_n = 0$, b is the extrapolation length to the point outside the boundary at which ψ would go to zero if it maintained the slope it had at the surface. The value of b will depend on the nature of the material to which contact is made, approaching zero for a magnetic material and infinity for an insulator, with normal metals lying in between[17].

Furthermore, if ψ is written as $\psi = |\psi|\ e^{i\theta}$, the second G-L equation becomes

$$\nabla \times \nabla \times \vec{A} = \frac{1}{\lambda^2}(\frac{\Phi_0}{2\pi}\nabla\theta - \vec{A}) \tag{3.3.12}$$

Here α, β, λ, ξ and n_s are the phenomenological parameters in a simply connected superconducting surfaces, and now we clarify the physical meaning of the parameter ξ. Consider Fig.3.2 below, the order parameter penetrates the normal region to the depth $b = \xi_n$ while decaying exponentially over this distance.

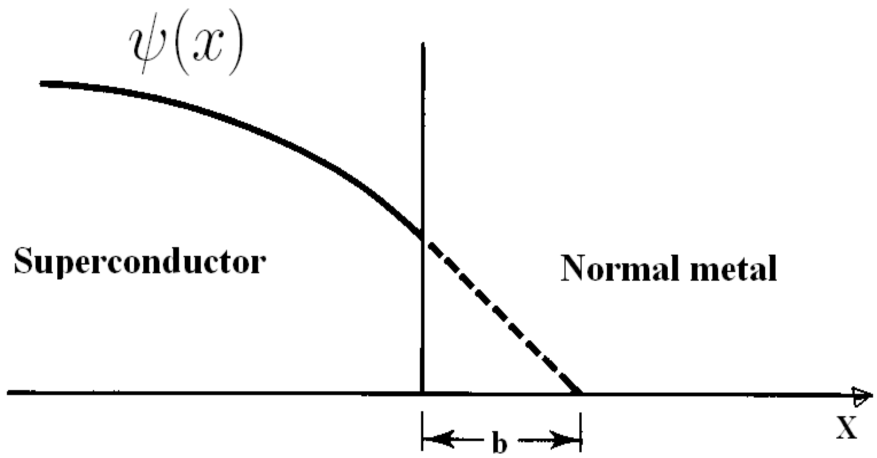


Figure 3.2: Schematic diagram illustrating the boundary condition (3.3.10) at an interface characterized by an extrapolation length b.

For this geometry assume $[\epsilon(x) << 1]$ and $\psi(x) = 1 - \epsilon(x)$ if there is no magnetic field $\vec{A} = 0$, we note that ψ can be chosen real because we are dealing with a simply connected superconductor, and the first G-L equation in one dimension takes the form;

$$-\xi^2 \frac{d^2(1-\epsilon)}{dx^2} - (1-\epsilon) + (1-\epsilon)^3 = 0. \tag{3.3.13}$$

We have to neglect the higher order terms $3\epsilon^2 - \epsilon^3$

$$\xi^2 \frac{d^2(\epsilon)}{dx^2} - 2\epsilon = 0. \tag{3.3.14}$$

The solution of this differential equation will be

$$\epsilon = A\ e^{-\frac{\sqrt{2}}{\xi}x} + B\ e^{\frac{\sqrt{2}}{\xi}x}. \tag{3.3.15}$$

But the term $B\ e^{\frac{\sqrt{2}}{\xi}x}$ will be neglected because we want only the decaying exponential. Allowing for the fact that as $x \to \infty$ the order parameter $\psi \to 1$, we have $\epsilon(\infty) = 0$. Then the obvious solution of equation(3.3.12) when $A = \epsilon(0)$ is

$$\epsilon(x) = \epsilon(0)\ e^{-\frac{x}{\xi/\sqrt{2}}}. \tag{3.3.16}$$

$(\xi/\sqrt{2})$ is the typical scale of variation of the order parameter and ξ is called the coherence length. At the same time it gives a typical size of the cooper pair.

3.4 Application of G-L theory to simple geometries

The theory has been developed in order to study the properties of surface layers and the boundary between normal and superconductivity regions and this is just where one would expect an order parameter to vary. Now in any physical problem, if parameters change rapidly extra energy is usually involved. The minimum free energy " G " therefore contains a length, the coherence length over which Ψ only varies slowly. A full development of the theory enables us to calculate the effect of a field on the penetration depth λ, and the value of λ for very small samples. Its most remarkable achievement, however, was to predict the behavior of type-II superconductors with their two critical fields, $\vec{H}_{c1}$ and $\vec{H}_{c2}$, and the criterion which determines whether a material is type-I or type-II[6].

3.4.1 Coherence length

The London penetration depth λ is fundamental length that characterizes a superconductor. An independent length is the Coherence length ξ. The coherence length is a measure of the distance within which the superconducting electron concentration cannot change drastically in a spatially-varying magnetic field.

The London equation is a local equation: it relates the current density at a point r to the vector potential at the same point. So long as $\vec{j}(r)$ is given as a constant times $\vec{A}(r)$, the current is required to follow exactly any variation in the vector potential. But the coherence length ξ is a measure of the range over which we should average $\vec{A}$ to obtain $\vec{j}$. It is also a measure of the minimum spatial extent of a transition layer between normal and superconductor[3].

Ginzburg-Landau equations produce many interesting and valid results. Perhaps the most important of these is its prediction of the existence of this two characteristic lengths in a superconductor. The first is the penetration depth which we already know from the London's equation λ, given by

$$\lambda = \sqrt{\frac{mc^2}{8\pi e^2 \Psi_0^2}} \tag{3.4.1}$$

where $\Psi_0 = n_s/2$ is the equilibrium value of the order parameter in the absence of an electromagnetic field. The penetration depth describes the depth to which an external magnetic field can penetrate the superconductor. The second is a coherence length ξ, given by

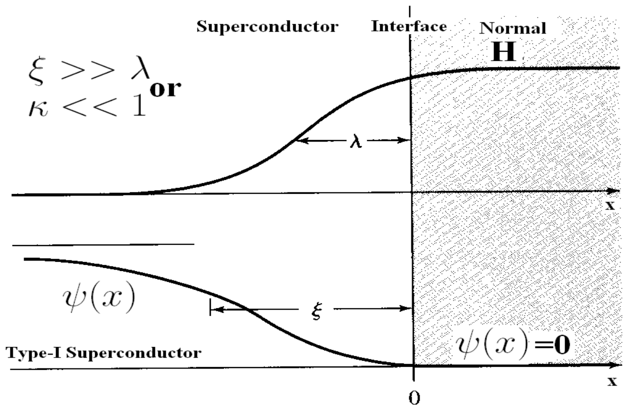
$$\xi = \sqrt{\frac{\hbar^2}{4m|\alpha|}} \tag{3.4.2}$$

which describes the size of thermodynamic fluctuations in the superconducting phase.

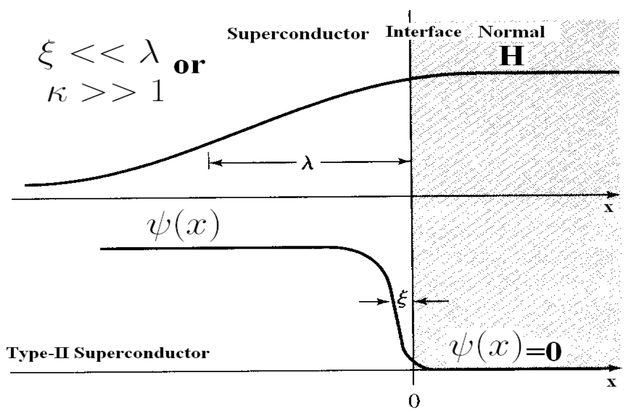
The ratio $\kappa = \frac{\lambda}{\xi}$ is known as the Ginzburg-Landau parameter. It has been shown that type-I superconductors are those with $0 < \kappa < 1/\sqrt{2}$, and type-II superconductors those with $\kappa > 1/\sqrt{2}$. For type-II superconductors, the phase transition from the normal state is of second order, for type-I superconductors it is of first order. This is proved by deriving a dual Ginzburg-Landau theory for the superconductor. The most important finding from Ginzburg-Landau theory was made by Alexei Abrikosov in 1957. In a type-II superconductor in a high magnetic field - the field penetrates in quantized vortexes of the magnetic flux , which are most commonly arranged in a hexagonal arrangement[19,20].

3.4.2 The criterion for type-I and type-II superconductivity

We have already described that observations of the intermediate state in type-I superconductors (which is resistive) show that a reality small number of normal and superconducting regions are formed and this would suggest that the surface energy at the boundary between the two phases is positive. In type-II materials the penetration of a magnetic field is not accompanied by only resistance and the specimen breaks up into a fine filamentary structure of normal regions. It is believed that each of these filaments is at the center of one flux quantum and this induces a persistent current which circulates around the filament. This very fine filamentary structure would suggest that in type-II materials the boundary energy would be negative since the surface area appears to be maximized. We now wish to show why the sign of the surface energy is controlled by the relative values of two dimensions which we have already seen in some detail: the penetration depth λ and the coherence length ξ. If $\frac{\xi}{\lambda}$ is greater than unity or $\sqrt{2}$ the surface energy will be positive and if it is less than unity or $\sqrt{2}$ it will be negative. To see how this arises, consider the boundary between a normal and superconducting region in a field $\vec{H}_{ext}$. The two phases are in thermodynamic equilibrium and so in the bulk their free energies must be equal. In the normal region the field is $\vec{H}_c$, with in the boundaries of the superconducting phase there are circulating currents which oppose $\vec{H}_{ext}$. These will increase the free energy, but this increase will be exactly compensated by the reduction in energy due to the electron pairing. In the boundary region, however, this compensation is not exact. The increase in energy due to the circulating current will occur over a distance of the order of their penetration depth λ, where as the decrease due to electron pairing will extend over the coherence length ξ Fig.3.3(a) shows the situation if $\xi > \lambda$. There is an energy maximum within the interphase region. The case $\xi < \lambda$ is shown in the Fig.3.3(b) and now there is



(a)



(b)

Figure 3.3: Spatial variations of the order parameter ψ and the magnetic field H in the vicinity of the Normal-superconductor interface[19].

an energy minimum, which is negative with respect to the bulk energy boundary layer. This relative magnitude of λ and ξ determines the sign of the surface energy and hence whether a material is type-I or type-II.

High temperature superconductors are extreme type-II materials with a ratio of penetration depth to coherence length of $\sim 10^4$ i.e $\frac{\lambda}{\xi} \sim 10^4$ assume $\lambda = 1000$Å, then $\xi = 0.1$Å[6].

Chapter 4

Josephson's effect

4.1 Josephson superconducting tunneling

Consider two metals separated by an insulator, the insulator normally acts as a barrier to the flow of conduction electrons from one metal to the other. If the barrier is sufficiently thin (less than 10 or 20Å) there is a significant probability that an electron which impinges on the barrier will pass from one metal to the other.

Under suitable conditions we observe a remarkable effects associated with the tunneling of superconducting electron pairs from a superconductor through a layer of an insulator in to another superconductor. Such a junction is called a weak link. The effects of pair tunneling includes; A dc current flows across the junction in the absence of any electric or magnetic field, and a dc voltage applied across the junction causes rf current oscillating across the junction[15].

In 1962 Brian Josephson, then a 22-year old graduate student, made a remarkable prediction that two superconductors separated by a thin insulating barrier should give rise to a spontaneous (zero voltage) DC current, $I_s = I_c \sin\varphi$ where φ is the difference in phase across the junction. And that if a finite (DC) voltage were applied, an AC current with frequency $\omega = \frac{2eV}{\hbar}$ would flow. I_c is called the Josephson critical current. There is a myth that Brian Josephson did his calculation (1962) and won the Nobel prize (1973) as part of the solution to a homework problem of Phil Anderson's. The truth is that

Anderson was a lecturer on sabbatical at Cambridge in 1961-62, and he gave a series of lectures in which he mentioned the problem of tunneling between two superconductors, which Josephson then promptly solved. The idea was opposed at first by John Bardeen, who felt that pairing could not exist in the barrier region. Thus much of the early debate centered on the nature of the tunneling process, whereas in fact today we know that the Josephson effect occurs in a variety of situations whenever two superconductors are separated by a "weak link", which can be an insulating region, normal metal, or short, narrow constriction[21,22].

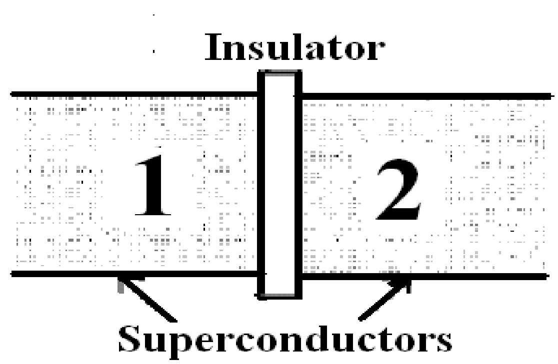


Figure 4.1: Insulator sandwiched by two superconductors[10].

4.2 Stationary (dc) Josephson's effect

Let us start with the first Josephson effect which is adc effect. The physics behind it is essentially as follows. A sufficiently small current can pass through a weak link (Josephson junction) without dissipation (dissipativeless current). In other words, when such a current passes through the weak link, no voltage is generated across the junction. Since the current is always small, the magnetic field generated by it can be neglected and, following the Ginzburg-Landau theory, we can argue that the density of the current is determined by the phase gradient $\nabla\theta$ of the superconducting electron wavefunction.

An important characteristic of the weak link is that the phase gradient $\nabla\theta$ is very large compared to the phase gradients inside the bulk superconductors (in the following we shall refer to them as 'electrodes'). For a tunnel junction, strictly speaking, the expression 'phase gradient' is inappropriate and one should rather speak of a discontinuous phase jump across the junction. Therefore, from now on we shall always analyze the phase difference across the weak link:

$\varphi = \theta_2 - \theta_1$, where θ_2 and θ_1 are the phases of the superconducting electron wave functions in the first and second superconducting electrodes respectively. Here our task is to show the relation between the current through the weak link, I_s, and the phase difference φ.

Now let us propose that

$$I_s(\varphi) = I_c \sin\varphi, \tag{4.2.1}$$

where I_c is the maximum dissipation-free current through the junction often referred to as the critical current.

Let us prove that our proposal is indeed appropriate. But first a short foray into quantum mechanics. The evolution of a quantum-mechanical system with time is described by a wavefunction $\psi(t)$ which is the solution of the Schrodinger equation[15]:

$$i\hbar\frac{\partial\Psi}{\partial t} = H\Psi. \tag{4.2.2}$$

where $\Psi = \sum_\alpha C_\alpha\psi_\alpha$ and this implies that

$$i\hbar\sum_\alpha \frac{dC_\alpha}{dt}\psi_\alpha = \sum_\alpha C_\alpha\hat{H}\psi_\alpha. \tag{4.2.3}$$

Now let's multiply both sides of equation(4.2.3) by $\int \psi_\beta^* d\tau$

$$i\hbar\sum_\alpha \frac{dC_\alpha}{dt}\int \psi_\beta^*\psi_\alpha d\tau = \sum_\alpha C_\alpha\int \psi_\beta^*\hat{H}\psi_\alpha d\tau. \tag{4.2.4}$$

The term $\int \psi_\beta^* \psi_\alpha d\tau = \delta_{\alpha\beta}$ and $\int \psi_\beta^* \hat{H} \psi_\alpha d\tau = H_{\beta\alpha}$, then

$$i\hbar \frac{dc_\beta}{dt} = \sum_\alpha C_\alpha H_{\beta\alpha}. \tag{4.2.5}$$

Expressing the amplitudes C_1 and C_2 as $C_1 \sim \sqrt{n_s}\ e^{i\theta_1}$ and $C_2 \sim \sqrt{n_s}\ e^{i\theta_2}$, therefore

$$i\hbar \frac{dc_1}{dt} = H_{11}C_1(t) + H_{12}C_2(t),$$

$$i\hbar \frac{dc_2}{dt} = H_{21}C_1(t) + H_{22}C_2(t). \tag{4.2.6}$$

If there is a potential difference V across the junction, when a pair tunnel through the junction, there is an energy change of $2eV$. We can phenomenologically write H_{11} as eV and H_{22} as $-eV$ at the junction: and $H_{12} = H_{21} = K$, which is the weak link coupling K:

$$i\hbar \frac{\partial C_1}{\partial t} = eVC_1(t) + KC_2(t), \tag{4.2.7}$$

and

$$i\hbar \frac{\partial C_2}{\partial t} = -eVC_2(t) + KC_1(t). \tag{4.2.8}$$

Since $|C(t)|^2 = n$, we can write $C_1 = \sqrt{n_1}\ e^{i\theta_1}$ and $C_2 = \sqrt{n_2}\ e^{i\theta_2}$, therefore equation (4.2.7) and (4.2.8) can be written as:

$$i\hbar[\frac{1}{2\sqrt{n_1}}\ e^{i\theta_1}\frac{\partial}{\partial t}n_1 + i\sqrt{n_1}\ e^{i\theta_1}\frac{\partial}{\partial t}\theta_1]$$

$$= eV\sqrt{n_1}\ e^{i\theta_1} + K\sqrt{n_2}\ e^{i\theta_2}. \tag{4.2.9}$$

$$\Rightarrow i\hbar[\frac{1}{2}\frac{\partial}{\partial t}n_1 + in_1\frac{\partial}{\partial t}\theta_1] = eVn_1 + K\sqrt{n_1 n_2}\ e^{i(\theta_2-\theta_1)}. \tag{4.2.10}$$

and

$$i\hbar[\frac{1}{2\sqrt{n_2}}\ e^{i\theta_2}\frac{\partial}{\partial t}n_2 + i\sqrt{n_2}\ e^{i\theta_2}\frac{\partial}{\partial t}\theta_2]$$

$$= -eV\sqrt{n_2}\ e^{i\theta_2} + K\sqrt{n_1}\ e^{i\theta_1}. \tag{4.2.11}$$

$$\Rightarrow i\hbar[\frac{1}{2}\frac{\partial}{\partial t}n_2 + in_2\frac{\partial}{\partial t}\theta_2] = -eVn_2 + K\sqrt{n_1 n_2}\ e^{i(\theta_1-\theta_2)}. \tag{4.2.12}$$

Equating real and imaginary parts:

$$\frac{\hbar}{2}\frac{\partial}{\partial t}n_1 = K\sqrt{n_1 n_2}\sin(\theta_2 - \theta_1). \tag{4.2.13}$$

$$-\hbar n_1\frac{\partial}{\partial t}\theta_1 = eVn_1 + K\sqrt{n_1 n_2}\cos(\theta_2 - \theta_1). \tag{4.2.14}$$

and

$$\frac{\hbar}{2}\frac{\partial}{\partial t}n_2 = K\sqrt{n_1 n_2}\sin(\theta_1 - \theta_2). \tag{4.2.15}$$

$$-\hbar n_2\frac{\partial}{\partial t}\theta_2 = -eVn_1 + K\sqrt{n_1 n_2}\cos(\theta_1 - \theta_2). \tag{4.2.16}$$

If $\varphi = \theta_2 - \theta_1$

$$\frac{\hbar}{2}\frac{\partial}{\partial t}n_1 = K\sqrt{n_1 n_2}\sin\varphi. \tag{4.2.17}$$

and

$$\frac{\hbar}{2}\frac{\partial}{\partial t}n_2 = -K\sqrt{n_1 n_2}\sin\varphi. \tag{4.2.18}$$

The current through the tunnel junction is proportional to dn_s/dt. Indeed, when the current is switched on, the superconducting electron density starts to vary at the rate dn_s/dt thereby giving rise to a current $I_s = dn_s/dt$. This means that electrons start to leave the superconducting electrode. But this process is immediately compensated by the arrival of new electrons from an external current source, because the junction is a part of a closed electric circuit. Therefore, the density n_s remains constant due to electroneutrality of the system as a whole. However, for defining the supercurrent, it is sufficient to assume $I_s \sim dn_s/dt$. Then from equation (4.2.17) and (4.2.18) we immediately obtain the equation for the dc Josephson effect or we can write equation (4.2.17) and (4.2.18) by subtracting one from the other.

$$J = e\frac{d}{dt}(n_1 - n_2) = \frac{2e}{\hbar}[K\sqrt{n_1 n_2}\sin\varphi - (-K\sqrt{n_1 n_2}\sin\varphi)].$$

$$J = \frac{4Ke\sqrt{n_1 n_2}}{\hbar}\sin\varphi.$$

$$J = J_c \sin\varphi. \tag{4.2.19}$$

where $J_c = \frac{4Ke\sqrt{n_1 n_2}}{\hbar}$.

In other words,

Josephson super current ⇔ phase difference across the junction

dc-Josephson super current ⇔ constant phase difference across the junction

It is more common to use Josephson current than current density. $I = JA$, and we can write similarly[23,24,25],

$$I = I_c \sin\varphi. \tag{4.2.20}$$

4.3 Non-Stationary (ac) Josephson's effect

Let a dc voltage be applied across the junction. We can do this because the junction is an insulator. An electron pair experiences a potential energy difference qV on passing across the junction, where $q = -2e$. We can say that a pair on one side is at potential energy $-eV$ and a pair on the other side is eV.

The phase difference across the two sides of Josephson junction will oscillate and this gives rise to an oscillating supercurrent. We can write equation (4.2.14) and (4.2.16) by subtracting one from the other like this;

$$\frac{\partial}{\partial t}\varphi = [\frac{eV}{\hbar} + \frac{K}{\hbar}\sqrt{\frac{n_2}{n_1}}\cos\varphi] - [-\frac{eV}{\hbar} + \frac{K}{\hbar}\sqrt{\frac{n_1}{n_2}}\cos\varphi]. \tag{4.3.1}$$

where $\varphi = \theta_2 - \theta_1$, if $n_1 \sim n_2$, then

$$\frac{\partial}{\partial t}\varphi = \frac{2eV}{\hbar}. \tag{4.3.2}$$

This is equation for the non stationary Josephson's effect. We see by integrating (4.3.2) with a dc voltage across the junction the relative phase of the probability varies as

$$\varphi(t) = \varphi(0) + \frac{2eVt}{\hbar}. \tag{4.3.3}$$

The superconducting current is given by

$$J = J_c \sin[\varphi(0) + \frac{2eVt}{\hbar}]. \tag{4.3.4}$$

Now, what happens to a Josephson junction if we apply a constant current $I > I_c$ to it? Since the supercurrent cannot exceed I_c, it is obvious that a current of normal electrons I_n must start flowing through the junction, in addition to the supercurrent. This conclusion leads us directly to the so- called resistively shunted model of the Josephson junction (RSJ) in which the latter is considered as a circuit made up of the Josephson junction itself and a normal resistance connected in parallel (Fig. 4.3). From Ohm's law $I = \frac{V}{R}$

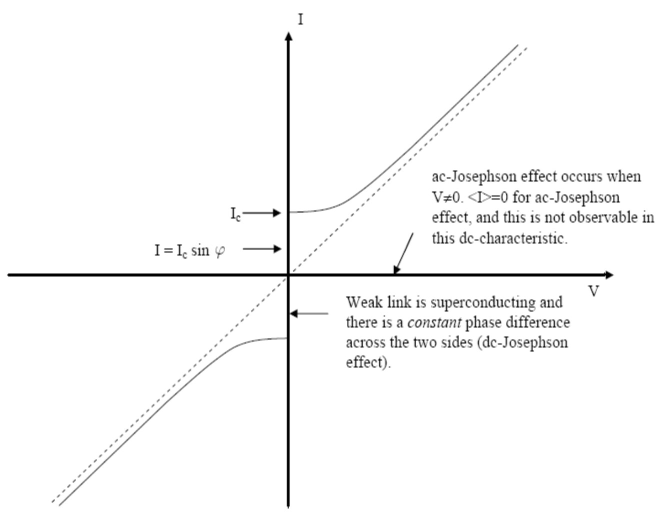


Figure 4.2: Current-voltage characteristics of a Jossephson junction[3].

and $V = \frac{\hbar}{2e}\frac{d\varphi}{dt}$, therefor the total current I is then a sum of the normal current V/R and the supercurrent $I_s = I_c \sin\varphi$:

$$I = I_c \sin\varphi + \frac{\hbar}{2eR}\frac{d\varphi}{dt}. \tag{4.3.5}$$

where R is the normal-state resistance of the junction. This differential equation can be easily integrated. Substituting the solution into equation (4.3.2), we obtain the voltage across the junction as

$$V(t) = R\frac{I^2 - I_c^2}{I + I_c \cos\omega t}. \tag{4.3.6}$$

Oscillating frequency $\omega = \frac{2eV}{\hbar}$. This says that a photon of energy $\hbar\omega = 2eV$ isa emitted

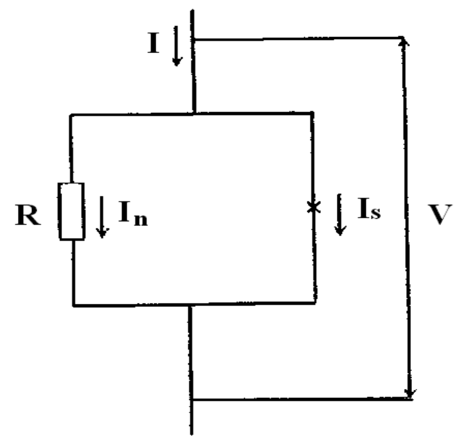


Figure 4.3: Resistively shunted model of a Josephson junction. The supercurrent through the junction is $I_s = I_c \sin\varphi$.

or absorbed when an electron pair crosses the barrier.

If $V = 1\mu V$, then $\omega = \frac{2eV}{\hbar} = \frac{2\times1\times10^{-6}\times1.0602\times10^{-12}}{1.055\times10^{-34}} = 3.037 \times 10^9 GHz$

4.4 Penetration of magnetic field in the Josephson's contact (Ferrell-Prange equation)

In this section we shall restrict our attention to Josephson junctions without concentration of current, that is, to sandwiches and tunnel junctions. Suppose, for example, that such a junction consists of two bulk superconducting slabs separated by a thin insulating layer, as in Fig. 4.4. If this system is placed in an external magnetic field parallel to the plane of the junction, a screening supercurrent will be generated at the outer surfaces of the slabs. This current circulates within a surface layer of thickness λ and, in doing so, has to cross the weak link, where the critical current density is very small. Therefore, in

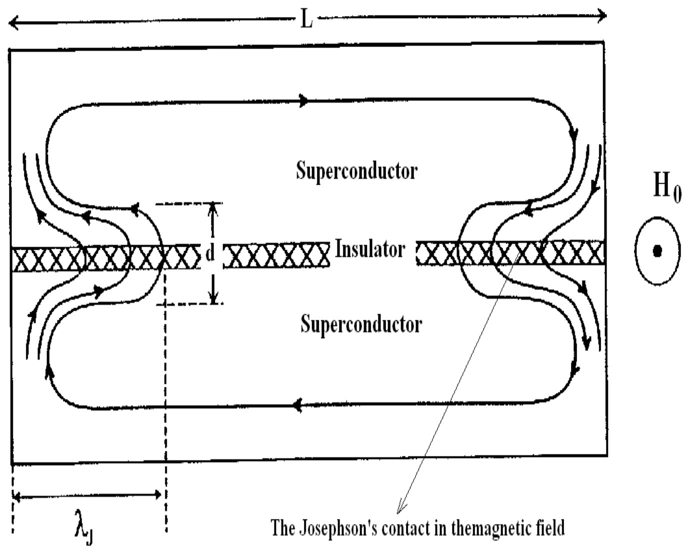


Figure 4.4: Tunnel Josephson junction in a magnetic field H_0. The distribution of the screening (Meissner) supercurrent is shown by lines with arrows

order to maintain its dissipation-free flow, the current must spread over a rather wide area protruding into the junction. This situation is illustrated in Fig. 4.4. Let us try to describe it in mathematical terms.

Consider a Josephson junction in an external magnetic field. The plane of the junction coincides with the X-axis and the magnetic field is directed along the Z-axis. The length along the Y-axis of the region penetrated by the current and the magnetic field is $d = 2\lambda + t$ (see Fig. 4.4). Here t is the thickness of the insulating layer. Consider two closely spaced pairs of points in the vicinity of the junction: 1, 2 and 3, 4, as in Fig. 4.5.

If the superconducting current increase in the Josephson's contact from 0 to j_s during time t there are some energy will be stored in the contact per unit area of the contact, this energy is

$$W_J = \int_0^t j_s V dt. \tag{4.4.1}$$

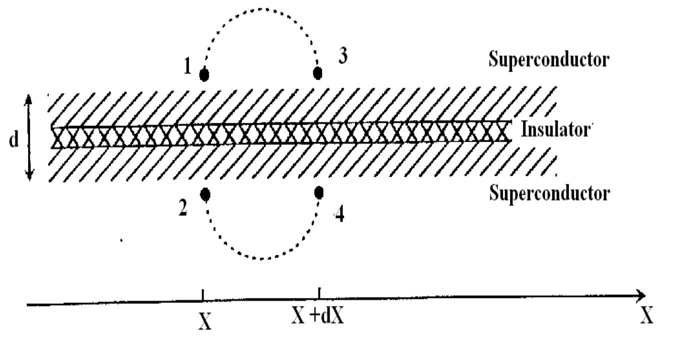


Figure 4.5: Section near the edge of the Josephson junction. The dashed area is the region penetrated by the magnetic field[8]

where V is the voltage drop. Substituting $j_s = j_c \sin\varphi$ and $V = \frac{\hbar}{2e}\frac{d\varphi}{dt}$ we get

$$W_J = \frac{\hbar}{2e} j_c (1 - \cos\varphi). \tag{4.4.2}$$

The magnetic field penetrates in to the contact with magnetic energy per unit length is;

$$W_H = \frac{H^2}{8\pi} d. \tag{4.4.3}$$

Consider the G-L equation

$$\nabla \times \nabla \times \vec{A} = \frac{|\psi|^2}{\lambda^2}\left(\frac{\Phi_0}{2\pi}\nabla\theta - \vec{A}\right). \tag{4.4.4}$$

This equation in the bulk of the superconductor where $\vec{A}$ is constant and $|\psi|^2 = 1$ which gives

$$\frac{\Phi_0}{2\pi}\nabla\theta = \vec{A}. \tag{4.4.5}$$

Now we integrate this equation over the contour 1 3 4 2 Fig. 4.5

$$\frac{\hbar c}{2e}\left[\int_1^3 \nabla\theta dl + \int_4^2 \nabla\theta dl\right] = \oint \vec{A} \cdot d\vec{l}. \tag{4.4.6}$$

The distance d is assumed to be negligibly small. Then the right-hand side of (4.4.6) can be approximated by

$$\oint \vec{A} \cdot d\vec{l} = d\Phi. \tag{4.4.7}$$

where $d\Phi$ is the magnetic flux enclosed by the contour comprising the dashed curves 1-3 and 4-2 which are supplemented with missing sections 3-4 and 2-1. Carrying out the integration on the left-hand side we get

$$\hbar(\theta_3 - \theta_1 + \theta_2 - \theta_4) = \frac{2e}{c} d\Phi. \tag{4.4.8}$$

Taking into account that $\theta_3 - \theta_4 = \varphi(x+dx)$ and $\theta_1 - \theta_2 = \varphi(x)$, we have $\varphi(x+dx) - \varphi(x) = \frac{2e}{c} d\Phi$ or, recalling that $\Phi_0 = \pi\hbar c/e$,

$$\frac{d\varphi}{dx} = \frac{2\pi}{\Phi_0}\frac{d\Phi}{dx}. \tag{4.4.9}$$

where $\Phi_0 = \frac{\hbar\pi c}{e}$ is the quanta of magnetic flux and noting that $H = \frac{1}{d}\frac{d\Phi}{dx}$ is the magnetic field H at the point x of the junction, we can write

$$H = \frac{\Phi_0}{2\pi d}\frac{d\varphi}{dx}. \tag{4.4.10}$$

Our functional is the energy stored in the contact;

$$\begin{aligned} W(\varphi) &= \int_0^L dx[\frac{H^2}{8\pi}d + \frac{\hbar}{2e}j_c(1-\cos\varphi)] \\ &= \int_0^L dx[\frac{\Phi_0^2}{32\pi^3 d}(\frac{d\varphi}{dx})^2 + \frac{\hbar}{2e}j_c(1-\cos\varphi)]. \end{aligned} \tag{4.4.11}$$

Lets take variation with respect to φ, where $\varphi(x)$ give the phase distribution along the contact.

$$\delta W = \int_0^L dx[\frac{\Phi_0^2}{32\pi^3 d}2\frac{d\varphi}{dx}\frac{d}{dx}\delta\varphi + \frac{\hbar}{2e}j_c \sin\varphi\delta\varphi]. \tag{4.4.12}$$

$$\Rightarrow \int_0^L \frac{d\varphi}{dx}\frac{d}{dx}\delta\varphi = \int_0^L \frac{d}{dx}(\frac{d\varphi}{dx}\delta\varphi) - \int_0^L \frac{d^2\varphi}{dx^2}\delta\varphi. \tag{4.4.13}$$

But $\int_0^L \frac{d}{dx}(\frac{d\varphi}{dx}\delta\varphi) = 0$

$$\delta W = \int_0^L [\frac{-\Phi_0^2}{16\pi^3 d}\frac{d^2\varphi}{dx^2} + \frac{\hbar}{2e}j_c\sin\varphi]dx\delta\varphi. \tag{4.4.14}$$

We demand $\delta W = 0$ and using $\Phi_0 = \frac{\pi\hbar c}{e}$

$$\frac{d^2\varphi}{dx^2} = \frac{1}{\lambda_J^2}\sin\varphi. \tag{4.4.15}$$

where

$$\lambda_J = (\frac{c\Phi_0}{8\pi^2 j_c d})^{1/2}. \qquad (4.4.16)$$

One can see that the quantity λ_J, which has the dimensions of length, represents the typical length of magnetic field in to the contact, i.e the depth of magnetic field penetration into the Josephson junction. λ_J is usually referred to as the Josephson penetration depth. Now equation (4.4.15) will take the form if $\varphi << 1$,

$$\frac{d^2\varphi}{dx^2} = \frac{1}{\lambda_J^2}\varphi. \qquad (4.4.17)$$

This is Ferrell-Prange equation. The solutions of the Ferrell-Prange equation can be easily solved

$$\varphi = \varphi_0 \quad e^{-\frac{x}{\lambda_J}} \qquad (4.4.18)$$

Substituting this solution into equation(4.4.10), we find the magnetic field in the junction:

$$\vec{H}(x) = \vec{H}_0(x) \quad e^{-\frac{x}{\lambda_J}} \qquad (4.4.19)$$

Now, what happens if the external field increases? It turned out that the behavior of a Josephson junction in an external magnetic field is reminiscent of the behavior of a type-II superconductor. Just as with type-II superconductors, when the external field exceeds a certain critical value H_{c1}, which is a characteristic of the junction, the field starts to penetrate into the junction in the form of superconducting vortices, each carrying one magnetic flux quantum Φ_0. In the case of a Josephson junction, they are called Josephson vortices.

Indeed, one of the solutions of the Ferrell-Prange equation (4.4.18) has the form by not small φ

$$\varphi(x) = 4\arctan[exp(x/\lambda_J)] \qquad (4.4.20)$$

One can easily verify that this solution satisfies equation (4.4.18). The functions $\varphi_0(x)$, $d\varphi_0/dx \sim \vec{H}$, and $d\varphi_0^2/dx^2 \sim \vec{j}_s$ are shown in Fig. 4.6.

Thus, beginning at the field H_{c1}, penetration of Josephson vortices into the junction becomes energetically favorable. Having penetrated into the junction, the vortices form a linear chain and the junction goes into the mixed state. So far we have a complete analogy with a type-II superconductor. But this analogy is not really complete. Because unlike an Abrikosov vortex (see Chap. 1 and chap. 3), a Josephson vortex does not have a normal core. The existence of the normal core in type- II superconductors relates to the second critical field H_{c2}: Superconductivity disappears when an external magnetic field presses the vortices so close together that their normal cores come into contact. The absence of a normal core for Josephson vortices implies that there is no upper critical field for a Josephson junction. Nevertheless, as we shall soon find out, the dependence of the maximum current through the junction on magnetic field can be rather curious[16,27].

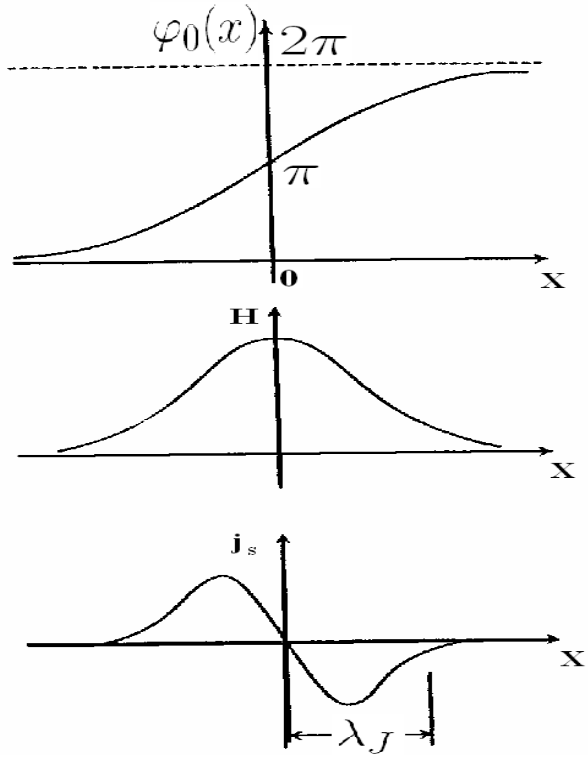


Figure 4.6: Phase difference $\varphi_0(x)$, magnetic field $H(x)$, and supercurrent $j_s(x)$ for a Josephson vortex[4]

Now the energy of a single vortex appearing in the contact can be calculated by using equation (4.4.18) and equation (4.4.11);

$$W = \frac{2\hbar j_c \lambda_L}{e} \tag{4.4.21}$$

But this is half of the energy of the vortex

$$E_{vortex} = \frac{4\hbar j_c \lambda_L}{e} = \frac{4\Phi_0 j_c \lambda_L}{\pi c}$$

If we apply the external field H_0 to the contact the Gibb's free energy of one single Vortex will be

$$G_o = E_{vortex} - \Phi_0 H_0 / 4\pi \tag{4.4.22}$$

At week field $G_o > 0$ and penetration of the vortex in the contact is not energy profitable but at $G_o > 0$ the vortex appearing, therefore the critical field where the vortex appearing will be

$$H_c = \frac{4\pi E_{vortex}}{\Phi_0} = \frac{2\Phi_0}{\pi^2 \lambda_J d} \tag{4.4.23}$$

Chapter 5

CONCLUSION

In this work we discussed the application of the variational method to the theory of superconductivity. By constructing the proper functionals, we obtained the phenomenological equations,

i) London's equation

$$\vec{H} - \lambda^2 \nabla^2 \vec{H} = 0.$$

ii) Ginzburg-Landau equations

$$\alpha\Psi + \beta|\Psi|^2\Psi + \frac{1}{4m}(-i\hbar\nabla + \frac{2e}{c}\vec{A})^2\Psi = 0.$$

$$\vec{j}_s = \frac{ie\hbar}{2m}(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*) - \frac{2e^2}{mc}\vec{A}|\Psi|^2.$$

with boundary condition

$$[-i\hbar\nabla\Psi + \frac{2e}{c}\vec{A}\Psi] \cdot \hat{n} = 0.$$

iii) Ferrell-Prange equation

$$\frac{d^2\varphi}{dx^2} = \frac{1}{\lambda_J^2}\varphi.$$

This equations contain the phenomenological parameters

1) London penetration depth $\lambda^2 = \frac{mc^2}{4\pi e^2 n_s}$.

2) Penetration depth and coherence length $\lambda = \sqrt{\frac{mc^2\beta}{8\pi e^2|\alpha|}}$ and $\xi = \sqrt{\frac{\hbar^2}{4m|\alpha|}}$.

3) Penetration depth of the magnetic field in to the Josephson contact (Josephson's penetration depth) $\lambda_J = (\frac{c\Phi_0}{8\pi^2 j_s d})^{1/2}$.

The energy stored and the critical magnetic field for the first vortex appearing in the Josephson junction is $W = \frac{2\hbar j_c \lambda_L}{e}$ and $H_c = \frac{4\pi E_{vortex}}{\Phi_0} = \frac{2\Phi_0}{\pi^2 \lambda_J d}$ respectively.

In 1957 Bardeen, Cooper, Schrieffer laid the basis for quantum theory of superconductivity and give an answer for the question why materials are superconductor ?

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DECLARATION

I here under signed declare that the thesis is my original work, has not been presented for a degree in any other university and that all sources of material used for the thesis have been dully acknowledged.

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This thesis has been submitted for examination with my approve as a university advisor.

Name:- Professor V.N. Mal'nev

Signature:- ——————