ADDIS ABABA UNIVERSITY
COLLAGE OF NATURAL SCIENCES
DEPARTMENT OF MATHEMATICS

PROJECT ON

CENTER MANIFOLD ANALYSIS FOR STATIC BIFURCATION AND NORMAL FORMS

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Submitted in partial fulfillment of the requirements for the degree of Master of Science in Mathematics

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Addis Ababa, Ethiopia
February, 2012

PERMISSION

This is to certify that this project is compiled by Muluken Admasu in the Department of Mathematics, Addis Ababa University, under my supervision. I hereby also confirm that the project can be submitted for evaluation by examiners and eventual defense.

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ABSTRAT

This paper addresses the dependency of solution of differential equation on initial data and control parameter. To reduce the dimension of the system, we use center manifold reduction method. Mainly, the paper focus on the cases of static bifurcations occur and non bifurcation. We transformed the fixed points and control parameter to zero. This greatly helps us to use lower order approximation of Taylor series. Finally, by using center manifold reduction and Taylor series expansion we construct the normal form of such types of bifurcations.

ACKNOWLEDGEENT

First and at most I want to praise my GOD, the Almighty GOD, who has passed me many Situations throughout my life from the beginning until this time and about my future.

I would like to say thank my advisor Dr.Tadesse for his help during the study of this project. My sincere thanks also forwarded to my brother, Luleseged Admasu for his help in many ways and encouragement.

Special thank should be given to my Love, Abayneshe Fissha for her encouragement and assistance.

Finally, I would like to thank Melaknesh Assefa she lend her personal desktop computer for writing the paper and my heartfelt gratitude to my family.

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INTRODUTION

Nowadays, the word “bifurcation” has come to mean not only the splitting of rest points (regardless of the number of rest points involved), but any change in the qualitative behavior of a system of differential equations as a parameter is varied. Many important problems in the operation of physical systems can be interpreted as static bifurcations, i.e., bifurcations associated with a change in the equilibrium point structure of the underlying equations. This change occurs when a control parameter is varied in state space.

When one thinks of simplifying dynamical systems, two approaches come to mind. Namely, reduction of the dimensionality of the system and elimination of the nonlinearity. Two rigorous mathematical techniques that allow substantial progress along both lines of approach are center manifold theory and the method of normal forms. These techniques are the most important, generally applicable methods available in the local theory of dynamical systems. The idea of normal forms in the study of bifurcations is similar to the idea of normal forms for matrices (e.g. the Jordan Normal Form) to obtain a simple representation of the dynamical system. i.e., make bifurcations easy to represent and provide a useful framework for theory building.

CHAPTER 1

OVERVEIW OF BASIC CONCEPTS

In this chapter we introduce some basic facts and notations that are essential for the subsequent work.

1.1 CONTINUOUS DEPENDENCY OF SOLUTIONS

Consider the system of differential equations

$$\dot{x} = F(x,\mu) \tag{1.1}$$

Where $F\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and μ is parameter. A solution of this system is a function

$x\colon I \longrightarrow \mathbb{R}^n$ defined on some open interval $I \subseteq \mathbb{R}$ such that, for all $t \in I$,

$$\dot{x}(t) = F(x(t),\mu)$$

Geometrically, $x(t)$ is a curve in $\mathbb{R}^n$ whose tangent vector $\dot{x}(t)$ exists for all $t \in I$ and equals $F(x(t))$.

Definition Let $F\colon I \times U \longrightarrow \mathbb{R}^n$, where $U \subseteq \mathbb{R}^n$ be open set and I be open interval, then F is said to be

$i)$ $Uniformly\ Liptschitz$ continuous w.r.t x iff $\exists\ L \in \mathbb{R}$ such that
$\|F(t,x) - F(t,y)\| \leq L\|x - y\|$, $\forall x,y \in U$ and $\forall t \in I$

$ii)$ $Lipschitz\ continuous\ \ w.r.t\ x$ iff $\forall (t,x) \in I \times U$ $\exists$ a neighborhood $J \times V$ such that $F/_{(I\times V)\cap(J\times V)}$ is uniformly Lipschitz continuous.

Recall, Picard-LindelÖf theorem, if F is $Lipschitz\ continuous\ \ w.r.t\ x$ then the initial value problem

$$\begin{cases} \dot{x} = F(x,\mu) \\ x(t_0) = x_0 \end{cases}$$

has unique solution.

Theorem Every continuously differentiable function in $\mathbb{R}^n$ is Lipschitz continuous.

Proof Let $F: \mathbb{R}^n \to \mathbb{R}^n$ be C^1

Fix any two points $x\,, y \in \mathbb{R}^n$ and define the function $f: [0,1] \to \mathbb{R}^n$ as

$$f(\theta) = F(x + \theta(y - x))$$

then, $f(0) = (x)$ and $f(1) = F(y)$

Furthermore, by chain rule f is differentiable and

$$\frac{d}{d\theta} f(\theta) = \nabla F(x + \theta(y - x))(y - x)$$

Where, $\nabla F(w)$ is the Jacobean matrix of f at w.

$$\nabla F(w) = \begin{bmatrix} \frac{\partial f_1(w)}{\partial w1} & \cdots & \frac{\partial f_1(w)}{\partial wn} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n(w)}{\partial w1} & \cdots & \frac{\partial f_n(w)}{\partial wn} \end{bmatrix}$$

Where, $\frac{\partial f_i(w)}{\partial wj}$ is the partial derivative of the $i-th$ component of f with respect to the $j-th$ coordinate.

By definition of f and then by fundamental theorem of calculus, we have

$$F(y) - F(x) = f(1) - f(0) = \int_0^1 f'(\theta) d\theta$$

$$= \left(\int_0^1 \nabla F(x + \theta(y - x) d\theta \right)(y - x)$$

$$= \left(\int_0^1 \frac{\partial f_i}{\partial wj}(x + \theta(y - x) d\theta \right)(y - x)$$

Where, $i, j = 1, 2, \ldots \ldots, n$

$$\text{Now} \quad \|F(y) - F(x)\| = \left\|\left(\int_0^1 \nabla F(x + \theta(y - x)d\theta\right)(y - x)\right\|$$

$$\leq \left\|\int_0^1 \nabla F(x + \theta(y - x)d\theta\right\| \|y - x\|$$

We use the notation $\|.\|$ for both the vector norm and its associated matrix norm.Notice that, by triangular inequality for the integrals,

We have,

$$\left\|\int_0^1 \nabla F\big(x + \theta(u - x)\big)d\theta\right\| \leq \int_0^1 \|\nabla F(x + \theta(u - x)\| \, d\theta$$

$$\leq \sup_{\theta \epsilon [0.1]} \|\nabla F(x + \theta(v - x)\| \int_0^1 d\theta$$

$$\leq \sup_{\theta \epsilon [0.1]} \|\nabla F(x + \theta(v - x)\|$$

Furthermore, by equivalence of matrix norm, we have

$$\exists \, c_0 > 0 : \|A\| \leq c_0 \|A\|_\infty \, , \ \ \forall \, A \ \in \ \mathbb{R}^{n \times n}$$

Thus, to establish Lipchitz continuity, we fix an arbitrary point $z \in \mathbb{R}^n$ and estabilish a bounded for $\int_0^1 \nabla F(x + \theta(y - x)d\theta$ in an appropriate neighborhood B of z.

Let $B = \mathrm{B}L\,(z)$, with L arbitrary.

Since, f is continuously differentiable on $\bar{B}$, which is closed and bounded, then, there exists λ_0 such that

$$\sup_{w \epsilon B} \left\|\frac{\partial f_i}{\partial wj}(w)\right\| \leq \lambda_0 \, , \qquad \forall i, j \, \epsilon \, [1, n]$$

Since, each continuous function (∂wif_i) is bounded on a closed and bounded set B (Weierstrass theorem). Now, given $x, y \epsilon B$, it follows that $x + \theta(y - x) \in B$ for all $\theta \epsilon [0, 1]$ because B is a ball,

$$\text{So, } \sup_{\theta\in[0,1]} \|\nabla F(x + \theta(u - x))\| \leq c_0 n \lambda_0 = \lambda$$

Thus, $\|(y) - F(x)\| \leq \lambda\|y - x\|, \forall y, x \in B$

Where, λ is a Lipschitz constant, then F is Lipchitz continuous on $B_L(z)$

Since, z is arbitrary from $\mathbb{R}^n$

Therefore, F is Lipschitz continuous.

Theorem Consider the first order system

$$\dot{x} = F(x)$$

Where, $F: \mathbb{R}^n \to \mathbb{R}^n$ is C^1.

Suppose $x(t)$ is a solutions of this system on the closed interval $[t_0, t_1]$ with initial condition $x(t_0) = x_0$, then there is a neighborhood $U \subset R^n$ of x_0 and constant k such that, if $y_0 \in U$, then there is unique solution also defined on $[t_0, t_1]$ with $y(0) = y_0$.

Moreover $y(t)$ satisfies,

$$|y(t) - x(t)| \leq |y_0 - x_0| e^{k(t-t_0)}, \forall\, t \epsilon [t_0, t_1]$$

i.e the solution depends continuously on the initial data.

Proof Since, F is C^1 function by uniqueness and existence theorem, there exists a unique solution on $[t_0, t_1]$. It also belongs to the maximal interval of existence. To prove continuous dependency, we need the following auxiliary lemma.

Lemma (Gronwall inequality): Let $g(t)$ be continuous and non negative real valued function on the interval $[a, b]$, Now suppose that there exist positive constant k_1 and k_2 such that $\forall t \epsilon [a, b]$ the following holds $g(t) \leq k_1 + k_2 \int_a^t g(s) ds$ then, for all $t \epsilon [a, b]$,

$g(t) \leq k_1 \mathrm{e}^{(k_2(t-a))}$.

Proof Let $U(t)$ be a function which satisfies $\mathrm{U(t)} = k_1 + k_2 \int_{\mathrm{a}}^{\mathrm{t}} \mathrm{g(s)ds}$. But then $U(t)$ is a strictly positive differentiable function on [a, b] and

$$U'(t) = k_2\, g(t) \leq k_2\, U(t)$$

$$\Rightarrow \quad \frac{U'(t)}{U(t)} \leq k_2$$

$$\Rightarrow \log U(t) - \log U(a) \leq k_2(t-a)$$

$$\Rightarrow \log U(t) \leq k_2(t-a) + \log U(a)$$

$$\Rightarrow g(t) \leq U(t) \leq U(a)\, \mathrm{e}^{k_2(t-a)}$$

$$\therefore \quad g(t) \leq k_1\, \mathrm{e}^{k_2(t-a)}$$

Proof Let $x(\mathrm{t})$ and $y(t)$ be solutions for the system over the interval $I = [t_0, t_1]$ with initial condition $x(t_0) = x_0$ and $y(t_0) = y_0$, respectively.

Let $Q = \max_{t \in I} |x(t) - y(t)|$

Assume the maximum is attained at some point $a \in I$ and

$$Q = |x(a) - y(a)| \leq \left| \int_{t_0}^{a} (x'(s) - y'(s)) ds \right|$$

$$\leq \int_{t_0}^{a} \left| x'(s) - y'(s) \right| ds$$

$$\leq \int_{t_0}^{a} \left| F\big(x(s)\big) - F(x(s)) \right| ds$$

$$\leq \int_{t_0}^{a} k |x(s) - y(s)|\, ds$$

Where, k is Lipschitz constant.

Now

$$y(t) - x(t) = y(t_0) - x(t_0) + \int_{t_0}^{a} [F(y(s)) - F(x(s))] ds$$

If we let $v(t) = |y(t) - x(t)|$

$$\text{Then, } v(t) \leq v(t_0) + \int_{t_0}^{t} kv(s) ds$$

By the Gronwall's inequality leads to

$$v(t) \leq v(t_0) e^{k(t-t_0)}$$

$$\text{Hence} \quad |y(t) - x(t)| \leq |y(t_0) - x(t_0)| e^{k(t-t_0)}$$

$$\leq |y_0 - x_0| e^{k(t-t_0)}$$

This completes proof of the theorem.

Theorem Let $\dot{x} = F(t, x, \mu)$ be a system of first order differential equation for which F is C^1 in both x and μ. Then the flow of this system depends continuously on the parameter μ as well.

Proof Consider the system

$$\begin{aligned} \dot{x}_\iota &= F(t, x_i, \mu_i) \\ x_i(t_0) &= a \end{aligned}$$

Since F is C^1 function then, F is Lipschiz continuous. There exists a unique solution $x(t)$ with initial condition $x(t_0) = a$

Let $x_1(t)$ and $x_2(t)$ be solutions of the system

W.L.O.G, assume $t \geq t_0$

$$|x_1(t) - x_2(t)| = \left|\int_{t_0}^{t}[F(s, x_1(s), \mu_1) - F(s, x_2(s), \mu_2)]ds\right|$$

$$\leq \int_{t_0}^{t}|F(s, x_1(s), \mu_1) - F(s, x_2(s), \mu_2)|\, ds$$

$$\leq \int_{t_0}^{t}[|F(s, x_1(s), \mu_1) - F(s, x_1(s), \mu_2)| + |F(s, x_1(s), \mu_2) - F(s, x_2(s), \mu_2)|]ds$$

$$\leq \int_{t_0}^{t}[L_2|\mu_1 - \mu_2| + L_1|x_1(s) - x_2(s)]|ds$$

Let $X(t) = L_2|\mu_1 - \mu_2| + L_1|x_1(t) - x_2(t)|$

$$\Longrightarrow X(t) \leq L_2|\mu_1 - \mu_2| + L_1\int_{t_0}^{t} X(s)\, ds$$

By Gronwall's inequality, we have

$$X(t) \leq L_2|\mu_1 - \mu_2|\mathrm{e}^{L_1\,(t-t_0)}$$

Consequently ,

$$X(t) = L_2|\mu_1 - \mu_2| + L_1|x_1(t) - x_2(t)| \leq L_2|\mu_1 - \mu_2|e^{L_1(t-t_0)}$$

$$\Rightarrow \qquad L_1|x_1(t) - x_2(t)| \leq L_2|\mu_1 - \mu_2|e^{L_1(t-t_0)} - L_2|\mu_1 - \mu_2|$$

$$\leq L_2|\mu_1 - \mu_2|\left(e^{L_1(t-t_0)} - 1\right)$$

$$\therefore \quad |x_1(t) - x_2(t)| \leq \frac{L_2}{L_1}|\mu_1 - \mu_2|\left(e^{L_1(t-t_0)} - 1\right)$$

This completes the proof of the theorem.

i.e it shows that the solution depends continuously on the parameter.

1.2 FIXED POINTS AND LINEARIZATION

The fixed points of the autonomous system

$$\dot{x} = F(x, \mu)$$

Is a point $x^* \in \mathbb{R}^n$; such that $F(x^*, \mu) = \mathbf{0}$.

A location in the state space where this condition is satisfied is called a *singular point*. At such a point, the integral curve of the vector field F corresponds to the point itself. Also, an orbit of a fixed point is the fixed point itself.

Suppose that we have a non linear autonomous system of ODE's written in vector form

$$\dot{x} = F(x, \mu) \tag{1.2}$$

Where, $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is C^1 and $\mu \in \mathbb{R}$. Suppose that x^* is a fixed point.

i.e $F(x^*, \mu) = \mathbf{0}$

Now take $\frac{\partial F}{\partial x} = \mathrm{D}F$

The right hand side of (1.2)

$$\dot{x} = F(x^*, \mu) + \left.\frac{\partial F}{\partial x}\right|_{x^*} (x - x^*) + \cdots$$

$$= \left.\frac{\partial F}{\partial x}\right|_{x^*} (x - x^*) + \cdots \tag{1.3}$$

The partial derivative in the above equation is to be interpreted as the Jacobean matrix. If the components of the state vector x are $(x_1, x_2, \dots, x_n)^T$ and components of a rate vector F are $(F_1, F_2, \dots, F_n)^T$, then the Jacobean is

$$D = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_n} \end{bmatrix}$$

Now define $\delta x = x - x^*$

Taking derivatives implies $\dot{\delta} x = \dot{x}$

If δx is small, then only the first term in equation (3.2) is significant. Since the higher order term involves powers of our small displacement from the fixed point, and then we have,

$$\dot{\delta} x = D^* \delta x$$

Where, D^* is the Jacobean evaluated at the fixed point. The matrix D^* is constant. So this is just a linear system of differential equation. According to the theory of linear differential equations, the solution can be written as a superposition of the terms of the form $e^{\lambda_j t}$.

Where $\{\lambda_j\}$ is the set of eigenvalue of the Jacobean.

The eigenvalues of the Jacobean are in general complex numbers.

$$\text{Let } \lambda_j = \mu_j + i v_j$$

Where, μ_j *and* v_j are respectively the real and imaginary parts of the eigenvalues, then

$$e^{\lambda_j t} = e^{\mu_j t} . e^{i v_j t}$$

The complex exponential in turn can be written

$$e^{i v_j t} = \cos(v_j t) + i sin(v_j t)$$

The complex part of eigenvalue, therefore only contributes on oscillatory components to the solution.

It is the real part that matters. If any one of the $\mu_j > 0$, then $e^{\mu_j t}$ grows with time and hence the solution of the system will tend to move away from the fixed point.

Definition A fixed point x^* is called hyperbolic or non degenerate when the Jacobean D^* has no eigenvalue with zero real part, otherwise non hyperbolic.

Definition A fixed point x^* of the differential equation (1.2) is stable if all the eigenvalues of D^* (the Jacobean evaluated at x^*) have negative real part.

Definition A fixed point x^* of the differential equation (1.2) is said to be unstable, if at least one of the eigenvalues of D^* have positive real part.

Example Given the ODE

$$\dot{x} = y$$

$$\dot{y} = x - x^3 - y$$

The fixed points are (0,0),(1,0) and (-1,0).

The Jacobean matrix is

$$D = \begin{bmatrix} \frac{d\dot{x}}{dx} & \frac{d\dot{x}}{dy} \\ \frac{d\dot{y}}{dx} & \frac{d\dot{y}}{dy} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 - 3x^2 & -1 \end{bmatrix}$$

Evaluating the Jacobean at the fixed point (0,0),we get

$$D^* = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

The eigenvalues are

$$\lambda_1 = \frac{-1+\sqrt{5}}{2} ,$$

$$\lambda_2 = \frac{-1-\sqrt{5}}{2}$$

Then the fixed point (0, 0) is unstable.

Evaluating the Jacobean at the fixed points (1,0) and (-1,0), we get the eigenvalues,

$$\lambda_1 = \frac{-1-\sqrt{7}\,i}{2} ,$$

$$\lambda_2 = \frac{-1+\sqrt{7}\,i}{2}$$

Therefore, the fixed points (1,0) and (-1,0) are stable fixed points.

Note If some of the eigenvalues have zero real part while the others are all negative, this can’t be decided on linear stability analysis.

Example Consider the following example

$$\begin{cases} \dot{x} = x^2 \\ \dot{y} = -y \end{cases}$$

The only steady-state solution for this system is the origin. All other solutions (except those on the *y*-axis) move to the right and toward the *x*-axis. On the *y*-axis, solutions tend along this straight line to the origin. Hence the phase portrait is as shown in below.

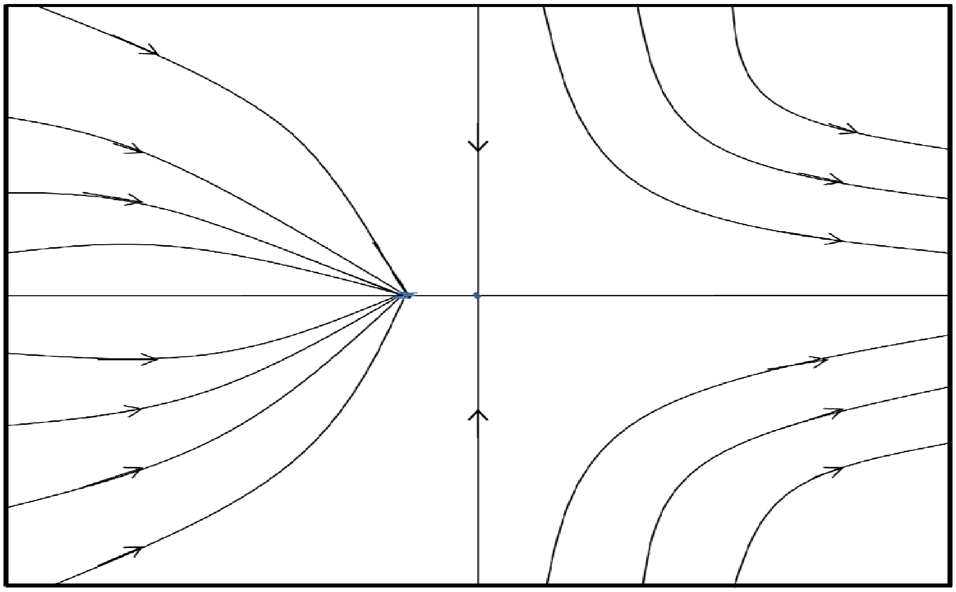

Figure-1 $x'=x^2$
$y'=-y$

The linearization of the system at the origin is,

$$\begin{cases} \dot{x} = 0 \\ \dot{y} = -1 \end{cases}$$

For which all points on the x-axis are fixed points, and all other solutions lie on vertical lines $x = c,\ c \in \mathbb{R}$. The problem here is that the fixed point for the linearization system at the origin is not hyperbolic. When a linear planar system has a zero eigenvalue or a center, the addition of nonlinear terms often completely changes the phase portrait.

1.3 CLASSIFICATION OF FIXED POINTS

When all of the eigenvalues of D have nonzero real parts, the corresponding fixed point is called a hyperbolic fixed point, irrespective of the values of the imaginary parts; otherwise, it is called a non hyperbolic fixed point.

There are three types of hyperbolic fixed points: *sinks*, *sources*, and *saddles*. If all of the eigenvalues of D have negative real parts then, x approaches to the fixed point x^* of (1.2) as $t \to \infty$. Therefore, the fixed point x^* of (1.2) is attractive.

An attractive fixed point is called a *sink*. If the matrix D associated with a sink has complex eigenvalues, the sink is also called a *stable focus*. On the other hand, if all of the eigenvalues of the matrix D associated with a sink are real, the sink is also called a *stable node*. A sink is stable in forward time (i.e. $t \to \infty$.,) but unstable in reverse time (i.e. $t \to -\infty$).

If one or more of the eigenvalues of D have positive real parts, then x moves away from the fixed point x^* of (1.2) as t increases. In this case, the fixed point x^* is said to be unstable.

When all of the eigenvalues of D have positive real parts, x^* is said to be a *source*. If the matrix D associated with a source has complex eigenvalues, such a source is called an *unstable focus*.

On the other hand, if all of the eigenvalues of the matrix D associated with a source are real, the source is also called an *unstable node*. A source is unstable in forward time but stable in reverse time. Because trajectories move away from a source in forward time, the source is an example of a repellor.

When some, but not all, of the eigenvalues have positive real parts while the rest of the eigenvalues have negative real parts, the associated fixed point is called a *saddle point*. Because a saddle point is unstable in both forward and reverse times, it is called a *non stable fixed point*.

A non hyperbolic fixed point is unstable if one or more of the eigenvalues of D have positive real parts. If some of the eigenvalues of D have negative real parts while the rest of the eigenvalues are distinct and have zero real parts, the fixed point $x = x^*$ of (1.2) is said to be *neutrally* or *marginally stable*.

If all of the eigenvalues of D are distinct, nonzero, and purely imaginary, the corresponding fixed point is called a *center*.

CHAPTER 2

NORMAL FORMS

We will construct normal forms of smooth continuous systems depending on a scalar control parameter μ near their fixed point. Previously we have seen linearization is not a guarantee to decide the stability of non hyperbolic fixed point (if at least one of the eigenvalue real part is zero and the other is negative). Now we transform the fixed point to the origin and use higher order Taylor series expansion and center of manifold. Before non linear system let us see first linear system

2.1 LINEAR SYSREM

We consider the linear system

$$\dot{x} = Ax \tag{2.1}$$

Where, $A \in \mathbb{R}^{n\times n}$. In this case, the trivial solution $x^* = 0$ is a fixed point of this linear system.

We denote the eigenvalues of A by $\lambda_i, i = 1, 2, \ldots, n$, and the corresponding eigenvectors (generalized eigenvectors) by $p_i, i = 1, 2, \ldots, n$. The eigenvalues are the roots of the characteristic equation.

$$det(A - \lambda I) = 0 \tag{2.2}$$

The eigenvector p_i corresponding to a distinct eigenvalue λ_i is given by

$$Ap_i = \lambda_i p_i \tag{2.3}$$

The generalized eigenvectors corresponding to an eigenvalue λ_m with multiplicity n_m are the nontrivial solutions of

$$(A - \lambda_m I)p = 0, \ (A - \lambda_m I)^2 p = 0, \ \ldots. \ , \ (A - \lambda_m I)^{n_m} p = 0 \qquad (2.4)$$

If an eigenvalue is complex-valued, then its corresponding eigenvector and generalized eigenvectors are also complex-valued.

Introduce the transformation

$$x = PY \qquad (2.5)$$

Where, $P \in \mathbb{R}^{n \times n}$, $P = [p_1 \, p_2, \ldots, p_n]$ and p_i is an eigenvector

$$P\dot{Y} = APY \qquad (2.6)$$

Multiplying (2.6) from the left with the inverse P^{-1} of P, we obtain the normal of (2.1) as

$$\dot{Y} = JY \qquad (2.7)$$

Where, $J = P^{-1}AP$ is called the Jordan canonical form of A.

2.1.1 System with Distinct Eigenvalues

If the eigenvalues of A are distinct, then J is a diagonal matrix with entries $\lambda_i, i = 1,2, \ldots, n$; that is

$$J = \begin{bmatrix} \lambda_1 & 0 & . & . & . & 0 \\ 0 & \lambda_2 & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & \lambda_n \end{bmatrix} \qquad (2.8)$$

Then, (2.7) can be rewritten as

$$\dot{Y}^{(m)} = \lambda_m Y^{(m)}, \quad m = 1,2 \dots, n \tag{2.9}$$

Where, $Y^{(m)}$ is the m-th component of Y. Therefore,

$$Y^{(m)} = c_m e^{\lambda_m t}, \; m = 1,2, \dots, n \tag{2.10}$$

Where, c_m is a constant.

It follows from (2.10) that $Y^{(m)} \longrightarrow 0$ as $t \longrightarrow \infty$, when λ_m lies in the left-half of the complex plane, $Y^{(m)} \longrightarrow \infty$ as $t \longrightarrow \infty$, when λ_m lies in the Right-half of the complex plane, and $Y^{(m)} = c_m$ for all time when λ_m lies on the imaginary axis.

Therefore, the origin of (2.1) is attractive if all of the eigenvalues λ_m of the matrix A lie in the left-half of the complex plane, unstable if one or more λ_m lie in the right-half of the complex plane, and neutrally or marginally stable if one or more λ_m lie on the imaginary axis with the rest of the eigenvalues being in the left-half of the complex plane.

2.1.2 System with Repeated Eigenvalues

If the number of distinct eigenvalues of A is $k < n$, then J is a block if all of the p_i are eigenvectors; otherwise, it has the form

$$J = \begin{bmatrix} J_1 & \emptyset & . & . & . & \emptyset \\ \emptyset & J_2 & . & . & . & \emptyset \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ \emptyset & \emptyset & . & . & . & J_k \end{bmatrix} \tag{2.11}$$

Where, $\emptyset$ represents a matrix with zero entries and

$$J_m = \begin{bmatrix} \lambda_m & 1 & 0 & . & . & . & 0 \\ 0 & \lambda_m & 1 & . & . & . & 0 \\ 0 & 0 & \lambda_m & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & \lambda_m \end{bmatrix} \tag{2.12}$$

We consider a system with repeated roots and a non diagonal J.

Example $$A = \begin{bmatrix} a & \frac{1}{2}(a-b) \\ -\frac{1}{2}(a-b) & b \end{bmatrix}$$

Where, $b \neq a$ the eigenvalues of this matrix are $\lambda = \frac{1}{2}(a+b)$ with algebraic multiplicity two.

The corresponding eigenvector and generalized eigenvector are

$$p_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \text{and} \quad p_2 = \begin{bmatrix} -1 \\ 1 + \frac{2}{b-a} \end{bmatrix}$$

Then, $$P^{-1}AP = J = \begin{bmatrix} \frac{1}{2}(a+b) & 1 \\ 0 & \frac{1}{2}(a+b) \end{bmatrix}$$

And (2.7) yields

$$\begin{cases} \dot{Y}^{(1)} = \lambda Y^{(1)} + Y^{(2)} \\ \dot{Y}^{(2)} = \lambda Y^{(2)} \end{cases} \tag{2.13}$$

The general solutions of (2.13) can be expressed as

$$Y^{(1)} = c_1 e^{\lambda t} + t c_2 e^{\lambda t} \text{ and}$$

$$Y^{(2)} = c_2 e^{\lambda t}$$

2.2 NONLINEAR SYSTEM

Consider the nonlinear autonomous system of $ODE'S$

$$\dot{x} = F(x, \mu) \tag{2.14}$$

To investigate the behavior of a fixed point (x^*, μ^*), where $x^* \in \mathbb{R}^n$and, $\mu^* \in \mathbb{R}$

We apply a small perturbation Y and obtain,

$$x = x^* + Y \tag{2.15}$$

Substituting (2.15) in to (2.14) yields

$$\dot{Y} = F(x^* + Y, \mu) \tag{2.16}$$

We note that the fixed point $x = x^*$ of (2.14) has been transformed into the fixed point $Y = 0$ of (2.16).

Assuming that F is at least twice continuously differentiable (i.e.,C^2), expanding (2.16) in a Taylor series about x^*, and retaining only linear terms in the perturbation leads to

$$\dot{Y} = F(x^*, \mu^*) + D_x(x^*, \mu^*)Y + O(\|Y\|^2)$$

Or

$$\dot{Y} \approx D_x(x^*, \mu^*)Y \equiv AY \tag{2.17}$$

Where, A is the matrix of first partial derivatives evaluated at the fixed point (x^*, μ^*), is called the *Jacobean matrix.*

If the components of F are

$$F_1(x_1, x_2, \ldots, x_n), F_2(x_1, x_2, \ldots, x_n), \ldots, F_n(x_1, x_2, \ldots, x_n)$$

then,

$$A = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \cdot & \cdot & \cdot & \frac{\partial F_1}{\partial x_n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{\partial F_n}{\partial x_1} & \frac{\partial F_n}{\partial x_2} & \cdot & \cdot & \cdot & \frac{\partial F_n}{\partial x_n} \end{bmatrix}$$

We have transformed the problem of determining the local behavior of the fixed point $x^*$of (3.17) in to that of determining the behavior of the trivial solution of the linear system (2.17). We say $local\ behavior$ because we have considered a small perturbation and linearization of the vector field.

The Hartman–Grobman theorem states that the trajectories in the vicinity of a hyperbolic fixed point $x = x^*$ of (2.14) are qualitatively similar to those in the vicinity of the hyperbolic fixed point $Y = 0$ of (2.17).

The Shoshitaishvili theorem is applicable to non hyperbolic fixed points. From these theorems, it follows that,

(a) The fixed point $x = x^*$of the nonlinear system (2.14) is stable when the fixed point $Y = 0$ of the linear system (2.17) is asymptotically stable.

(b) The fixed point $x = x^*$ of the nonlinear system (2.14) is unstable when the fixe point $Y = 0$ of the linear system (2.17) is unstable, and

(c) Linearization cannot determine the stability of neutrally stable fixed points (including centers) of (2.14).

In case of neutrally stable fixed points, a nonlinear analysis is necessary to determine the stability of x^* It will be necessary to retain quadratic and, sometimes, higher-order terms in the disturbance Y in the Taylor-series expansion of (2.17).

According to the Hartman–Grobman theorem, there exists a continuous coordinate transformation (i.e., a homeomorphism) that transforms the nonlinear flow into the linear flow in the vicinity of a hyperbolic fixed point.

Further, such a coordinate transformation would be a differentiable one because the method of normal forms yields transformations in the form of power-series expansions.

CHAPTER 3

CENTER MANIFOLD REDUCTION

One of the main methods of simplifying dynamical system is to reduce the dimension of the system. Center manifold Theory is a rigorous mathematical technique that makes this reduction possible, at least near fixed points.

3.1 CENTER MANIFOLD

Definition A bijective map $f\colon M \longrightarrow N$ is called diffeomorphism, if both f and f^{-1} are differentiable(if $f, f^{-1} \in C^r, r \geq 1$, f is called C^r- diffeomorphism).

Definition A manifold is a subset of $\mathbb{R}^n$ such that, for some fixed integer $k \geq 0$, each point in the subset has a neighborhood that is essentially the same as the Euclidean space $\mathbb{R}^k$.

To make this definition precise we will have to define what is meant by a neighborhood in the subset, and we will also have to understand the meaning of the phrase “essentially the same as $\mathbb{R}^k$.” However, these notions should be intuitively clear In effect, a neighborhood in the manifold is an open subset that is diffeomorphic to $\mathbb{R}^k$.

Definition Two manifold M and N are diffeomorphic, if there is a smooth bijective map f from M to N with a smooth inverse. They are C^r- diffeomorphic, if there is C^r map between them whose inverse is also C^r.

Points, lines, planes, arcs, spheres, and tori are examples of manifolds. Let us recall that a curve is a smooth function from an open interval of real numbers into $\mathbb{R}^n$.An arc is the image of a curve. Every solution of a differential equation is a curve; the corresponding orbit is an arc. Thus, every orbit of a differential equation is a manifold.

Consider the differential equation

$$\dot{x} = f(x)\,,\ x \in \mathbb{R}^n \tag{*}$$

with flow ϕ_t, and let S be a subset of $\mathbb{R}^n$ that is a union of orbits of this flow. If a solution has its initial condition in S, then the corresponding orbit stays in S for all time, past and future.

Definition A set $S \subseteq \mathbb{R}^n$ is called an *invariant set* for the differential equation (*) if, for each $x \in S$, the solution $t \longmapsto \phi_t(x)$, defined on its maximal interval of existence, has its image in S. Alternatively, the orbit passing through each $x \in S$ lies in S. If, in addition, S is a manifold, then S is called an *invariant manifold.*

A curve $y = h(x)$ defined for $|x|$ small ($|x| < \delta$), is said to an invariant manifold for the system of differential equations

$$\begin{aligned} \dot{x} &= f(x, y) \\ \dot{y} &= g(x, y) \end{aligned} \tag{3.1}$$

If the solution (x(t),y(t)) of (3.1) through $(x_0, h(x_0))$ lies on the curve $y = h(x)$ as long as $x(t)$ remains small.

Consider the system

$$\begin{cases} \dot{x} = Ax + f(x, y) \\ \dot{y} = By + g(x, y) \end{cases} \tag{3.2}$$

Where, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times m}$ such that $Re(\lambda) = 0\ \forall(\lambda) \in \sigma(A)$ and

$Re(\lambda) < 0, \forall(\lambda) \in \sigma(B)$. The function f and g are C^2 functions with $f(0,0) = \mathbf{0}$, $\nabla f(0,0) = \mathbf{0}$, $g(0,0) = \mathbf{0}, \nabla g(0,0) = \mathbf{0}$.

If f and g are identically zero then, (3.2) has the two obvious invariant manifolds $x = 0$ and $y = 0$. The invariant manifold $x = 0$ is called the stable manifold and on the stable manifold all solution decay to zero exponentially fast.

The invariant manifold $y = 0$ is called the center manifold. In general, an invariant manifold $y = h(x)$ for defined for small $|x|$ with $h(0) = 0, h^{'}(0) = 0$ is called a center manifold.

Definition The stable manifold of a fixed point x_0 for an autonomous differential equation with (local defined) flow ϕ_t is the set of all points x in the domain of definition of ϕ_t such that $\lim_{t\to\infty} \phi_t(x) = x_0$. The unstable manifold of a fixed point x_0 is the set of all points x in the domain of definition of ϕ_t such that $\lim_{t\to-\infty} \phi_t(x) = x_0$.

We consider the local dynamics near a non hyperbolic fixed point $x^*$of the nonlinear system (2.14), where F is an analytic vector function of x. According to the center-manifold theorem, there exists a $C^r, r \geq 2$ local center manifold for the nonlinear system (2.14) near x^*.Furthermore, if none of the eigenvalues of this fixed point lies in the right-half of the complex plane, the long-time dynamics of (2.14) can be reduced to determining the dynamics on the center manifold.

Theorem consider the first order differential equation (3.2) then, we have a local center manifold $y = h(x), |x| < \delta$ where, h is C^2

Proof Let $\psi: \mathbb{R}^n \longrightarrow [0,1]$ be a C^∞ function with $\psi(x) = 1$ when $|x| \leq 1$ and $\psi(x) = 0$ when$|x| \geq 2$.

For $\varepsilon > 0$ define F and G by

$$F(x,y) = f\left(x\psi\left(\frac{x}{\varepsilon}\right), y\right), \quad G(x,y) = g(x\psi\left(\frac{x}{\varepsilon}\right), y)$$

The reason that the cut-off function ψ is only a function of x is that the proof of the existence of a centre manifold generalizes in an obvious way to infinite dimensional problems.

We prove that the system

$$\begin{cases} \dot{x} = Ax + F(x,y) \\ \dot{y} = By + G(x,y) \end{cases} \tag{3.3}$$

has a centre manifold $y = h(x)$, $x \in \mathbb{R}^n$ for small enough ε. Since F and G agree with f and g in a neighborhood of the origin, this proves the existence of local centre manifold for (3.2). Moreover, we can show that the existence of lipschitz center manifold for (3.3) and h is C^2 function.

N.B In general equation (3.2)

i) Does not have a unique center manifold.

ii) If f and g are C^k, (k≥ 2), then h is C^k. But, if f and g are analytic, then does not have an analytic center manifold.

For each eigenvalue λ of D there is an associated subspace of $\mathbb{R}^n$-the eigenspace E_λ. For simplicity we assume D is diagonalizable then our definition of E_λ depends only on whether λ is real or complex.

When λ is real, E_λ is simply the subspace spanned by the eigenvectors,$\lambda \in \mathbb{R}$

$$E_\lambda = \{v \in \mathbb{R}^n | (D - \lambda I) \cdot v = \mathbf{0}\}$$

When λ is complex, then the eigenvectors are also complex; furthermore, since D is assumed to be a real matrix, if $v_1 + iv_2$ is the eigenvector for λ, the complex-conjugated vector $v_1 - iv_2$ is an eigenvector for $\bar{\lambda}$.

The eigenspace E_λ in this case is spanned by the real and imaginary parts of the eigenvectors for λ .

$$E_\lambda = \left\{v \in \mathbb{R}^n \middle| (D - \lambda I)\left(D - \bar{\lambda} I\right) \cdot v = \mathbf{0}\right\}$$

An eigenvalue λ corresponds to a "mode" of the system that is stable, unstable, or neutral, depending on whether $Re\lambda < 0$, $Re\lambda > 0$ or $Re\lambda = 0$, respectively.

We divide the eigenvectors (and generalized eigenvectors) of D into three sets according to these possibilities and form the stable subspace E^s, unstable subspace E^u, and center subspace E^c.

$$E^s = span\,\{v|v \in E_\lambda \; and \; Re\lambda < 0\,\}$$

$$E^u = span\,\{v|v \in E_\lambda \; and \; Re\lambda > 0\,\}$$

$$E^c = span\,\{v|v \in E_\lambda \; and \; Re\lambda = 0\,\}$$

These subspaces span the phase space, $\mathbb{R}^n = E^s \oplus E^u \oplus E^c$ and they are invariant.

If the solution $x(t) \in E^s$, then as $t \longrightarrow +\infty$ the trajectories converges to fixed point. If $x(t) \in E^u$, then the trajectories converges to fixed points as $t \longrightarrow -\infty$.

3.2 APPROXIMATION OF CENTER MANIFOLD

Next, we describe how to construct the center manifold in the neighborhood of a non hyperbolic fixed point with some of the real part of the eigenvalues being zero and the real parts of all of the other eigenvalues being negative.

We assume that the fixed point has been shifted to the origin and that the linear part has been transformed into a Jordan canonical form; that is, we consider the system,

$$\dot{x}_k = Jx_k + F(x)$$

Where, $J \in \mathbb{R}^{n\times n}$

$$\Rightarrow \quad \begin{cases} \dot{x} = Ax + f(x, Y) & (3.4) \\ \dot{Y} = BY + G(x, Y) & (3.5) \end{cases}$$

Where, $A \in \mathbb{R}^{m\times m}$, $B \in \mathbb{R}^{(n-m)\times(n-m)}$ such that $Re(\lambda) = 0\ \forall(\lambda) \in \sigma(A)$ and $Re(\lambda) < 0$ $\forall(\lambda) \in \sigma(B)$ and f and G are vector-valued nonlinear functions of $x\ and\ Y$.

According to the center-manifold theorem, there exists a center manifold

$$Y = h(x) \tag{3.6}$$

Moreover, the dynamics of the system (3.4) and (3.5) is qualitatively similar to the dynamics on this manifold; that is,

$$\dot{x} = A(x) + f(x, h(x))$$

Substituting (3.6) into (3.5)

$$\dot{h}(x) = A(x) + Bh(x) + G(x, h(x))$$

$$\Rightarrow \quad h'(x)f[x, h(x)] = A(x) + Bh(x) + G(x, h(x)) \tag{3.7}$$

Then after construct approximate solutions of (3.7).

Example Consider the system

$$\dot{x} = a_1xy$$

$$\dot{y} = -y + a_2x^2$$

Where, a_1 and a_2 are constants. Clearly, the origin is a fixed point, and its associated eigenvalues are $\lambda = 0, and\ \lambda = -1$, the origin is non hyperbolic and the center and stable subspaces are one-dimensional. Consequently, the center manifold is one-dimensional.

To calculate the center manifold and the dynamics on this manifold, we note that

$$A = 0, B = -1, f(x,y) = a_1 xy, and\ G(x,y) = a_2 x^2$$

Hence, (4.5) becomes

$$h'(x) a_1 x h(x) = -h(x) + a_2 x^2 \qquad (3.8)$$

We seek an approximate solution of (3.8)

$$h(x) = ax^2 + \cdots \qquad (3.9)$$

$$\Longrightarrow \quad 2a^2 a_1 x^4 + ax^2 - a_2 x^2 + \cdots = 0$$

Equating to zero the coefficient of x^2, we obtain $a = a_2$.Hence, the center manifold is given by

$$\dot{x} = a_1 a_2 x^3 + \cdots \qquad (3.10)$$

We note that linearization is not sufficient for determining the stability of the origin because its associated eigenvalues are $\lambda = 0\ and\ \lambda = -1$.

However, including the nonlinear terms, we find from (3.10) that the origin is unstable when $a_1 a_2 > 0$ and stable when $a_1 a_2 < 0$.

CHAPTER 4

STATIC BIFURCATIONS

We know that the matrix A in (2.17) and the associated eigenvalues are functions of the control parameter μ. Let us suppose that, as μ is slowly varied, a fixed point x^* becomes non hyperbolic at a certain location in the state-control space. Then, if the phase portraits before and after this location are qualitatively different, this location is called a *bifurcation point*, and the accompanying qualitative change is called a *bifurcation*.

Consider the first order system

$$\dot{x} = f(x, \mu) \tag{4.1}$$

Where, $\mu \in \mathbb{R}$ and $x \in \mathbb{R}^n$.

A bifurcation point of (4.1) is a value of μ at which the solutions of (4.1) change their behavior. In other word, $\mu = \mu^*$ is a bifurcation point of (4.1) if the phase portrait of (4.1) for $\mu < \mu^*$ and $\mu > \mu^*$ are different.

Definition A bifurcation is said to be local bifurcation if the bifurcation occur in the vicinity of fixed point. In other words, we have considered a small disturbance Y on the fixed point and linearized the vector field.

Definition A static bifurcation is local bifurcations of fixed points occurs at a certain value of μ_0 say $= \mu_0$, if the following conditions are satisfied.

a) $F(x_0, \mu_0) = 0$

b) The Jacobian matrix ∇F at (x_0, μ_0) has a zero eigenvalue while all of its other eigenvalues have negative real parts.

According to the center-manifold theorem, analysis of the the dynamics of an n-dimensional continuous system near one of its fixed points can be reduced to the analysis of the dynamics on its center manifold.

In our case (static bifurcation), the system is reduced to one-dimensional center manifold.

Now, we consider bifurcations of the one-dimensional dynamical system.

$$\dot{x} = f(x, \mu) \tag{4.2}$$

Where, x and f are scalar and $f(x, \mu)$ is a smooth function of x and μ.

We assume that $(x = 0, \mu = 0)$ is a non hyperbolic fixed point of (4.2); that is, $f(0,0) = 0$ and $f_x(0,0) = 0$. Expanding $f(x, \mu)$ in a Taylor series for small x and μ.

We obtain,

$$\dot{x} = f_\mu \mu + \frac{1}{2}\left(f_{\mu\mu}\mu^2 + 2f_{x\mu}x\mu + f_{xx}x^2\right) \tag{4.3}$$

$$+ \frac{1}{6}(f_{\mu\mu\mu}\mu^3 + 3f_{x\mu\mu}x\mu^2 + 3f_{xx\mu}x^2\mu + f_{xxx}x^3) + \cdots$$

Next, we consider the following four cases:

$$(a)\ f_\mu \neq 0\ and\ f_{xx} \neq 0\ ; \qquad (b)\ f_\mu \neq 0\ ,\ f_{xx} = 0\ and\ f_{xxx} \neq 0$$

$$(c)\ f_\mu = 0\ and\ f_{xx} \neq 0\ \ and \quad (d)\ f_\mu = 0\ and\ f_{xx} = 0$$

We want to show that the first case corresponds to saddle – node bifurcation, the second case corresponds to a non bifurcation, the third case correspond to transcritical bifurcation, and the fourth case corresponds to pitchfork bifurcation.

4.1 SADDLE-NODE BIFURCATION

We consider the case $f_\mu \neq 0 \; and \; f_{xx} \neq 0$. As $x \longrightarrow 0 \; and \; \mu \longrightarrow 0$ (4.3) tends to

$$\dot{x} = f_\mu \mu + \frac{1}{2} f_{xx} x^2 + \cdots \tag{4.4}$$

When $f_\mu \mu \neq 0$, the fixed points of (4.4) are

$$x^* = \pm\sqrt{\frac{-2f_\mu \mu}{f_{xx}}} \tag{4.5}$$

Which exist only when $f_\mu f_{xx} \mu < 0$. The eigenvalue associated with this two fixed points are

$$\lambda = \pm f_{xx}\sqrt{\frac{-2f_\mu \mu}{f_{xx}}} \tag{4.6}$$

It follows from (4.5) that there are two branches of fixed points in the neighborhood of $(x, \mu) = (0,0) \; for \; f_\mu \mu < 0 \; if \; f_{xx} > 0 \;\; and \; f_\mu \mu > 0 \; if \; f_{xx} < 0$ then, it follows from (4.6) the upper branch is stable and the lower branch is unstable if $f_{xx} < 0$ and that the upper branch is unstable and the lower branch is stable if $f_{xx} > 0$.

This bifurcation of the non hyperbolic fixed point at the origin as μ passes through zero is called $fold$ or $tangent$ or $saddle - node \; bifurcation$.

Example Consider the system

$$\dot{x} = f(x, \mu) = \mu + ax^2 \tag{4.7}$$

Clearly, $f_\mu = 1 \; and \; f_{xx} = a = \pm 1$

When $a = -1$, the system doesn't have any fixed points for $\mu < 0$ but has the two non trivial fixed points

$$x = \sqrt{\mu} \;\; and \;\; x = -\sqrt{\mu}$$

For $\mu > 0$ The Jacobean matrix has a single eigenvalue given by

$$\lambda = -2x$$

Thus the fixed point $x = \sqrt{\mu}$ is $stable\ node$ because $\lambda < 0$, and the fixed point $x = -\sqrt{\mu}$ is an $unstable\ node$ because $\lambda > 0$

On the other hand, when $a = 1$, (4.7) doesn't have any fixed point for $\mu > 0$ but has the two nontrivial fixed points.

$$x = \sqrt{-\mu} \;\; and \;\; x = -\sqrt{-\mu}$$

for $\mu < 0$, the Jacobean matrix has a single eigenvalue given by

$$\lambda = 2x$$

Thus, the fixed point $x = \sqrt{-\mu}$ is $unstable\ node$ because $\lambda > 0$, and the fixed point $x = -\sqrt{-\mu}$ is a $stable\ node$ because $\lambda < 0$.

In Figure –2 (a) and (b), display the different fixed-point solutions of (4.7) and their stability in the $x - \mu$ space. We use broken and solid lines to depict branches of unstable and stable fixed points, respectively.

We note that $f(x, \mu) = 0$ and $f_x(x, \mu) = 0\ at\ (0,0)$, and hence there is a non hyperbolic fixed point at $\mu = 0$.

Moreover, we note that there is a change in the number of fixed points from zero to two as μ passes through zero. Thus, the origin of the $x-\mu$ space is a static bifurcation. It is clear from Figure -2 that the stable and unstable branches meet at the bifurcation point and have the same tangent.

Therefore, this bifurcation is called a tangent bifurcation. Although branches of stable and unstable nodes meet at this bifurcation point, the tangent bifurcation is also called saddle-node bifurcation because, in higher-dimensional systems, branches of saddle points and stable nodes meet at such static bifurcation points.

Equation (4.7) is the *normal form for a generic saddle-node bifurcation of a fixed point of a continuous system*

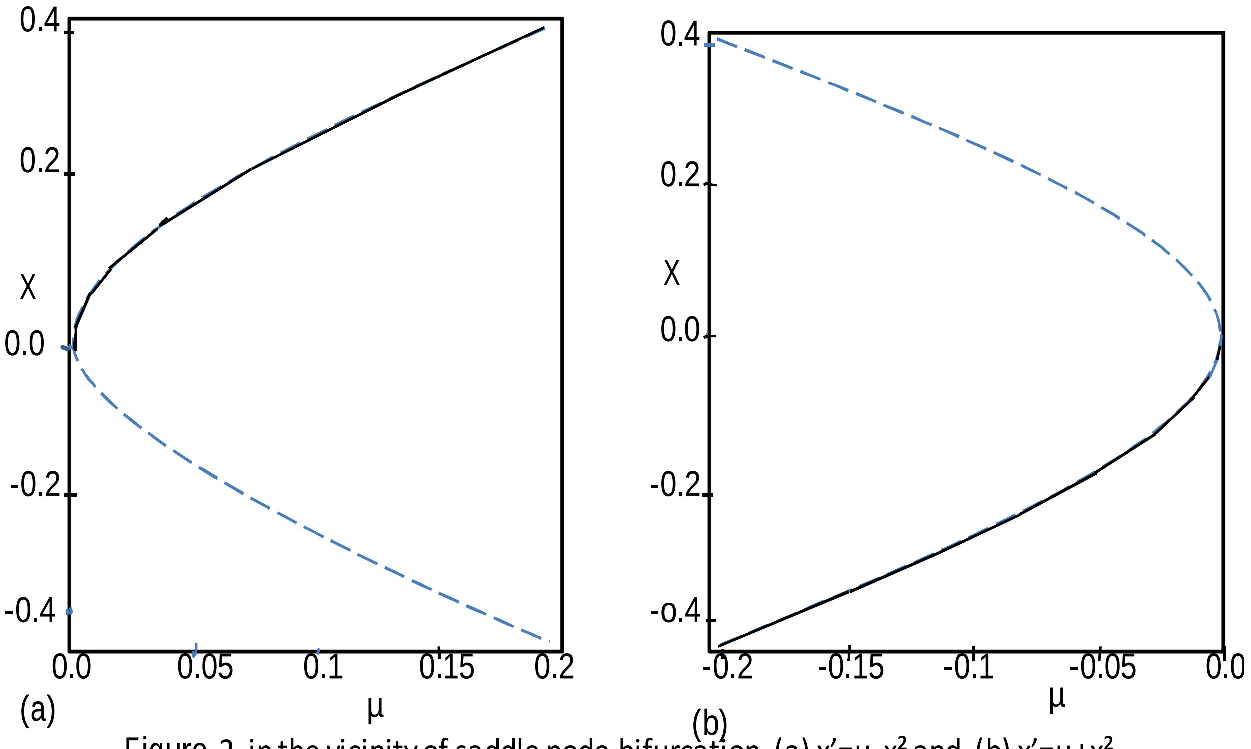

Figure-2 ,in the vicinity of saddle node bifurcation (a) $x'=\mu-x^2$ and (b) $x'=\mu+x^2$

4.2 NON BIFURICATION

We consider the case $f_\mu \neq 0, f_{xx} = 0$ and $f_{xxx} \neq 0$. As $x \longrightarrow 0$ and $\mu \longrightarrow 0$ (4.3) tends to

$$\dot{x} = f_\mu \mu + \frac{1}{6} f_{xxx} x^3 + \cdots \tag{4.8}$$

It has only one nontrivial fixed point when $f_\mu \mu \neq 0$ given by

$$x^* = \left(\frac{-6 f_\mu \mu}{f_{xxx}}\right)^{\frac{1}{3}} \tag{4.9}$$

The eigenvalue associated with this fixed point is

$$\lambda = \frac{1}{2} f_{xxx} \left(\frac{-6 f_\mu \mu}{f_{xxx}}\right)^{\frac{2}{3}} \tag{4.10}$$

It follows from (4.9) that this fixed point exists on both sides of $f_\mu \mu$ and it follows from (4.10) that it is stable when $f_{xxx} < 0$ and unstable when $f_{xxx} > 0$.

Although, the fixed point is non hyperbolic because $f(x, \mu) = 0$ and $\lambda = 0$ at $\mu = 0$, this fixed point is not a bifurcation point because there is no qualitative change either in the number of fixed-point solutions or in the stability of the fixed-point solutions as μ passes through zero in the state-control space.

Example Consider the system

$$\dot{x} = f(x, \mu) = \mu - x^3$$

Clearly, $f_\mu = 1$ and $f_{xxx} = -6$

We have only one fixed point namely

$$x^* = \mu^{\frac{1}{3}}$$

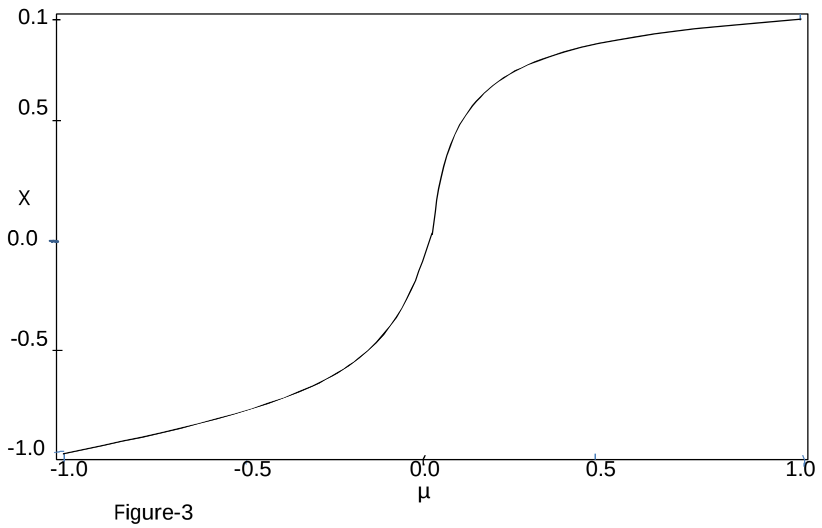

Figure-3

This solution is depicted in Figure 3.At the origin of the $x - \mu$ space, $f(x, \mu) = 0$ and f_x has a zero eigenvalue, implying that $x = 0$ is a non hyperbolic fixed point at $\mu = 0$.

However, $(0, 0)$ is not a bifurcation point because there is no qualitative change either in the number of fixed-point solutions or in the stability of the fixed point solutions as μ passes through zero in the state-control space.

4.3 TRANSCRITICAL BIFURCATION

We consider the case $f_\mu = 0, f_{xx} \neq 0$. As $x \longrightarrow 0 \ and \ \mu \longrightarrow 0$ (4.3) tends to

$$\dot{x} = \frac{1}{2}(f_{\mu\mu}\mu^2 + 2f_{x\mu}x\mu + f_{xx}x^2) + \cdots \tag{4.11}$$

Whose fixed points are,

$$x_1^* = \frac{-f_{x\mu} + \sqrt{f_{x\mu}^2 - f_{\mu\mu}f_{xx}}}{f_{xx}}\mu \ and$$

$$x_2^* = \frac{-f_{x\mu} - \sqrt{f_{x\mu}^2 - f_{\mu\mu}f_{xx}}}{f_{xx}}\mu \tag{4.12}$$

They exist when $f_{x\mu}^2 - f_{\mu\mu} f_{xx} > 0$. Their associated eigenvalues are

$$\boldsymbol{\lambda}(x_1^*) = \sqrt{f_{x\mu}^2 - f_{\mu\mu} f_{xx}}\,\mu \quad and$$

$$\boldsymbol{\lambda}(x_2^*) = -\sqrt{f_{x\mu}^2 - f_{\mu\mu} f_{xx}}\,\mu \qquad (4.13)$$

Therefore, x_1^* is stable when $\mu < 0$ and unstable when $\mu > 0$. On the other hand, x_2^* is unstable when $\mu < 0$ and stable when $\mu > 0$. In this case, we have two branches of fixed points that interchange stability at $(0, 0)$. Therefore, the bifurcation of the non-hyperbolic fixed point at the origin as μ passes through zero is transcritical bifurcation.

Example Consider the system

$$\dot{x} = f(x, \mu) = \mu x - x^2 \qquad (4.14)$$

$$\Longrightarrow \quad f_{\mu\mu} = 0, f_{\mu x} = 1 \; and \; f_{xx} = -2$$

There are two fixed points

$$x = 0 \qquad trivial\; fixed\; point$$

$$x = \mu \qquad non\; trivial\; fixed\; point$$

The Jacobian matrix $f_x = \mu - 2x$ has a single eigenvalue

$$\lambda = \mu \;\; at\; x = 0$$

$$\lambda = -\mu \;\; at\; x = \mu$$

The fixed point $x = 0$ is a non hyperbolic fixed point at $\mu = 0$. At this point, a static bifurcation occurs because there is an exchange of stability between the trivial and nontrivial branches.

The bifurcation point in Figure-5 is a *transcritical bifurcation point*. We point out that all of the branches that meet at this bifurcation point do not have the same tangent.

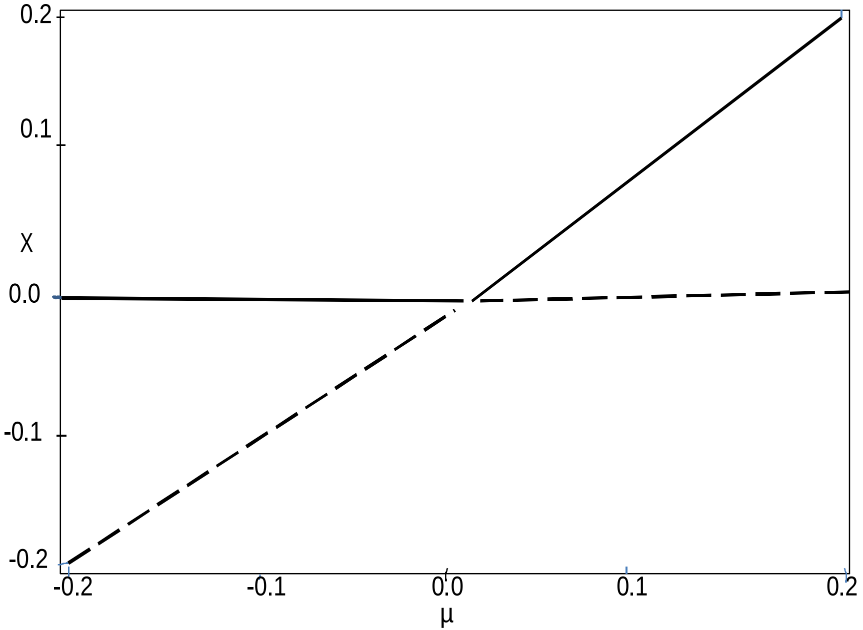

Figure -5 In the vicinity of a transcritical bifurcation.

Equation (4.14) is the *normal form for a generic transcritical bifurcation of a fixed point of a continuous system*

4.4 PITCHFORK BIFURCATION

We consider the case $f_\mu = 0, f_{xx} = 0$. As $x \longrightarrow 0$ and $\mu \longrightarrow 0$ (5.2) tends to

$$\dot{x} = f_{x\mu} x\mu + \frac{1}{6} f_{xxx} x^3 + \cdots \tag{4.15}$$

Whose fixed points are

$$x_1 = 0,\ x_2 = \sqrt{\frac{-6f_{x\mu}\mu}{f_{xxx}}} \quad and \quad x_3 = -\sqrt{\frac{-6f_{x\mu}\mu}{f_{xxx}}} \tag{4.16}$$

The second and the third fixed points exist only when $f_{x\mu}f_{xxx} < \mu$.

The Jacobian matrix in this case $f_x = f_{\mu x}\mu + \frac{1}{2}f_{xxx}x^2$

Has the single eigenvalue

$$\lambda(x_1) = f_{\mu x}\mu,\ \lambda(x_2) = -2f_{\mu x}\mu \ \ and\ \ \lambda(x_3) = -2f_{\mu x}\mu \tag{4.17}$$

Consequently, the trivial fixed point is stable when $f_{\mu x}\mu < 0$ and unstable when $f_{\mu x}\mu > 0$. On the other hand, the nontrivial fixed points exist and are stable when $f_{xxx} < 0$ and $f_{\mu x}\mu > 0$and exist and are unstable when $f_{xxx} > 0\ and\ f_{\mu x}\mu < 0$.

Example Consider the system

$$\dot{x} = \mu x + ax^3 \tag{4.18}$$

$$\Longrightarrow \quad f_{\mu\mu} = 0, f_{\mu x} = 1\ and\ \frac{1}{6}f_{xxx} = a = \pm 1$$

Where, μ is again the scalar control parameter.

There are three fixed points,

$$x = 0\,; \quad trivial\ fixed\ point$$

$$x = \pm\sqrt{\frac{-\mu}{a}}\ ;\ nontrivial\ fixed\ point$$

In this case, the Jacobian matrix

$$f_x = \mu + 3ax^2$$

We have single eigenvalue

$$\lambda = 0 \quad at\, x = 0$$

$$\lambda = -2\mu \;\; at \;\; x = \sqrt{\frac{-\mu}{a}}$$

Consequently, the trivial fixed point is stable when $\mu < 0$ and unstable when $\mu > 0$.

On the other hand, when $\alpha < 0$, nontrivial fixed points exist only when $\mu > 0$ and they are stable. However, when $\alpha > 0$, nontrivial fixed points exist only when $\mu < 0$ and they are unstable.

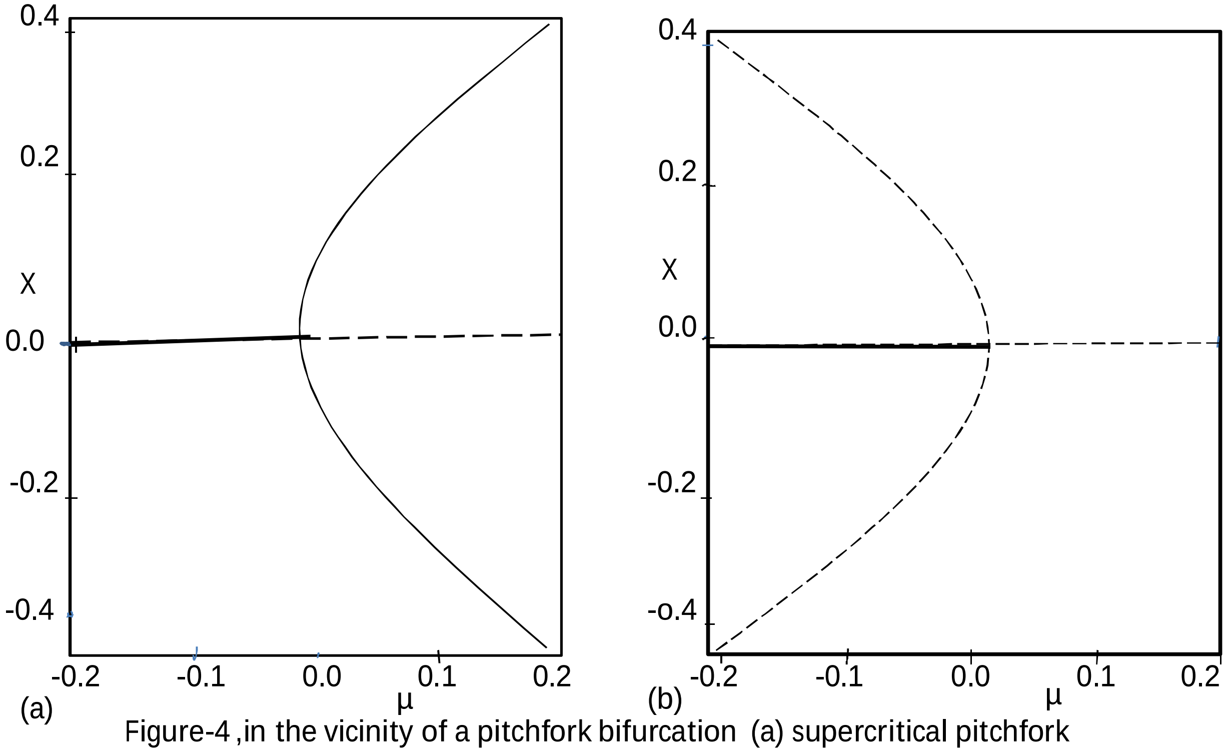

Figure-4 ,in the vicinity of a pitchfork bifurcation (a) supercritical pitchfork bifurcation (a=-1) and (b) subcritical pitchfork bifurcation (a=1)

The bifurcation diagrams correspond to $a = -1$ *and* $a = 1$. In both cases, we note the following at $(0,0)$.

$(a)\ f(x,\mu) = 0$

(b) f_x has a zero eigenvalue,

(c) The number of fixed-point solutions for $\mu < 0$ is different from that for $\mu > 0$, and

(d) There is a change in the stability of the trivial fixed point as we pass through $\mu = 0$.

Hence, the origin of the state control space is a bifurcation point.

When $a = -1$, two stable branches of fixed points $x = \sqrt{\mu}$ and $x = -\sqrt{\mu}$ bifurcate from the bifurcation point, as shown in Figure -4 (a). When $a = 1$, two unstable branches of fixed points $x = \sqrt{\mu}$ and $x = -\sqrt{\mu}$ bifurcate from the bifurcation point, as shown in Figure –4(b).

The bifurcations observed in Figure -3 (a),(b) are called *pitchfork bifurcations* because the bifurcating nontrivial branches have the geometry of a pitchfork at (0, 0).

Specifically, the bifurcation in Figure-3 (a) is called a *supercritical pitchfork bifurcation*, and the bifurcation in Figure -3(b) is called a *subcritical* or *reverse pitchfork bifurcation*.

In the case of a supercritical pitchfork bifurcation, locally we have a branch of stable fixed points on one side of the bifurcation point and two branches of stable fixed points and a branch of unstable fixed points on the other side of the bifurcation point.

In the case of a subcritical pitchfork bifurcation, locally we have two branches of unstable fixed points and a branch of stable fixed points on one side of the bifurcation point and a branch of unstable fixed points on the other side of the bifurcation point.

Equation (4.18) is the *normal form for a generic pitchfork bifurcation of a fixed point of a continuous system*

CHAPTER 5

NORMAL FORM OF STATIC BIFURCATION

Consider reduction of the nonlinear continuous system (2.14) near a static bifurcation fixed point to its normal form. We assume that the fixed point has been shifted to $x = 0$ and its corresponding control parameter has been shifted to $\mu = 0$.

Moreover, we expand (2.14) in a Taylor series for small $x \; and \; \mu$ obtain

$$\dot{x} = Ax + b\mu + B\mu x + Q(x, x) + C(x, x, x) + \cdots \tag{5.1}$$

Where $A = D_x F, B = D_{\mu x} F, and \; b = D_\mu F \; at \; (x, \mu) = (0, \mu) \; and \; Q(x, x) \; and \; C(x, x, x)$ are bilinear and trilinear column vectors involving quadratic and cubic terms, respectively.

Because the fixed point $(x, \mu) = (0,0)$ is a static bifurcation point, one of the eigenvalues of the matrix A is zero and all of its other eigenvalues are in the left-half of the complex plane.

Next, we will see how can use the method of center-manifold reduction to compute the normal forms of saddle-node, transcritical, and pitchfork bifurcations.

To compute the normal form of the static bifurcation of (5.1) at the origin

First, calculate the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and eigenvectors (generalized eigenvectors) $p_1, p_2, \ldots, p_n$ of matrix A.

Second, we arrange the eigenvalue of A so that $\lambda_1 = 0$ and p_1 be the corresponding eigenvector.

Third, introduce the transformation $x = PY$ and obtain

$$\dot{Y} = JY + P^{-1}b\mu + P^{-1}BPY\mu + P^{-1}Q(PY, PY) + P^{-1}C(PY, PY, PY) \tag{5.2}$$

Where, $P^{-1}AP = J \in \mathbb{R}^{n \times n}$ and it can be rewritten as,

$$J = \begin{bmatrix} 0 & 0 \\ 0 & J_s \end{bmatrix} \tag{5.3}$$

Where, $J_s \in \mathbb{R}^{(n-1)\times(n-1)}$ whose eigenvalues are $\lambda_2, \dots, \lambda_n$

Fourth, let $Y_s \in \mathbb{R}^{(n-1)}$ and $Y_s = (Y_2, \dots, Y_n)$.

We can write (5.2) as

$$\dot{Y}_1 = \hat{b}_1\mu + \sum_{i=1}^{n} \xi_i\, Y_i\mu + f_2(Y_1, Y_s) + f_3(Y_1, Y_s) \tag{5.4}$$

And

$$\dot{Y}_s = J_sY_s + \hat{b}_s\mu + \zeta Y_s\mu + \hat{B}Y\mu + F_2(Y_1, Y_s) + F_3(Y_1, Y_s) \tag{5.5}$$

Where, $\hat{b} = P^{-1}b$, f_2 and F_2 are the first and last $(n-1)$ components of $P^{-1}Q(PY, PY)$ and f_3 and F_3 the first and last $(n-1)$ components of $P^{-1}C(PY, PY, PY)$.

To determine the dependence of the center manifold on μ, we augment (5.4) and (5.5) with the additional equation

$$\dot{\mu} = 0 \tag{6.6}$$

Because the f_i and F_i are polynomials and hence infinitely differentiable, there exists a *local center manifold* of the form

$$Y_s = h(Y_1, \mu) \tag{5.7}$$

Where h is a polynomial function of Y_1 and μ such that

$$h(0,0) = 0,\ D_{Y_1}h_i(0,0) = 0,\ D_\mu h_i(0,0) = 0 \tag{5.8}$$

Where, h_i are the scalar components of h

Substituting (5.7) into (5.5) yields,

$$h^{'}(Y_1)\left[\hat{b}_1\mu + \xi_1 Y_1\mu + \textstyle\sum_{i=2}^{n} \xi_i\, h_i(Y_1)\mu + f_2(Y_1, h(Y_1))\right]$$

$$= J_s h(Y_1) + \hat{b}_s\mu + \zeta Y_1\mu + \hat{B}h(Y_1)\mu + F_2[Y_1, h(Y_1)] + \cdots \tag{5.9}$$

To solve (5.9), one approximates the components of $h(Y_1, \mu)$ with polynomials.The polynomial approximations are usually taken to be quadratic to the first approximation and do not contain constant and linear terms so that the conditions (5.8) are satisfied. Substituting the assumed quadratic polynomial approximations into (5.9) and equating the coefficients of the different terms in the polynomials on both sides, one obtains a system of algebraic equations for the coefficients of the polynomials. Solving these equations, we obtain a first approximation to the center manifold $Y_s = h(Y_1, \mu)$.

Finally, substituting this approximation into (5.4), we obtain the one-dimensional dynamical system.

$$\dot{Y}_1 = \hat{b}_1\mu + \xi_1 Y_1\mu + \textstyle\sum_{i=2}^{n} \xi_i\, Y_i\mu + f_2[Y_1, h(Y_1)] + f_3[Y_1, h(Y_1)] \tag{5.10}$$

Describing the dynamics of the system (5.1) on the center manifold.

Example Construct a normal form of a system

$$\begin{cases} \dot{x}_1 = b_1\mu + 2x_1 - 2x_2 + \mu x_1 + a_1 {x_1}^2 & (5.11) \\ \dot{x}_2 = b_2\mu + 4x_1 - 4x_2 - 4\mu x_2 + a_2 x_1 x_2 & (5.12) \end{cases}$$

The coefficient matrix of this system at (0, 0, 0) is given by

$$A = \begin{bmatrix} 2 & -2 \\ 4 & -4 \end{bmatrix}$$

Its eigenvalues are $\lambda_1 = 0 \; and \; \lambda_2 = -2$ and their corresponding eigenvectors are

$$p_1 = \begin{bmatrix}1\\1\end{bmatrix},\ p_2 = \begin{bmatrix}1\\2\end{bmatrix}$$

Hence

$$p = \begin{bmatrix}1 & 1\\1 & 2\end{bmatrix} \quad \text{and} \quad p^{-1} = \begin{bmatrix}2 & -1\\-1 & 1\end{bmatrix}$$

Next, we introduce the transformation

$$\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}Y_1\\Y_2\end{bmatrix}$$

Substitute in to (5.11) and (5.12) and obtain

$$\dot{Y}_1 = (2b_1 - b_2)\mu + 6Y_1\mu + 10Y_2\mu + (2a_1 - a_2)Y_1^2 \tag{5.13}$$

$$+(4a_1 - 3a_2)Y_1Y_2 + 2(a_1 - a_2)Y_2^2$$

$$\dot{Y}_2 = -2Y_2 + (b_2 - b_1)\mu - 5Y_1\mu - 9Y_2\mu \tag{5.14}$$

$$-(a_1 - a_2)Y_1^2 + (2a_1 - 3a_2)Y_1Y_2 - (a_1 - 2a_2)Y_2^2$$

We seek the center manifold of (5.13) and (5.14) in the form $Y_2(t,\mu) = h(Y_1,\mu)$ and rewrite (5.13) as

$$\dot{Y}_1 = (2b_1 - b_2)\mu + 6Y_1\mu + 10h\mu + (2a_1 - a_2)Y_1^2 \tag{5.15}$$

$$+(4a_1 - 3a_2)Y_1h + 2(a_1 - a_2)h^2$$

Substituting (5.15) and the expression for the center manifold into (6.14) yields

$$h'[(2b_1 - b_2)\mu + 6Y_1\mu + 10h\mu + (2a_1 - a_2)Y_1^2] = -2h + (b_2 - b_1)\mu$$

$$-5Y_1\mu - 9h\mu - (a_1 - a_2)Y_1^2 + (2a_1 - 3a_2)Y_1h - (a_1 - 2a_2)h^2 + \cdots \tag{5.16}$$

Whose approximate solution can be expressed as

$$h = \tfrac{1}{2}(b_2 - b_1)\mu - \tfrac{1}{2}(a_1 - a_2)Y_1^2 + \cdots \tag{5.17}$$

Substituting (5.17) into (5.15) yields the following one-dimensional equation describing the dynamics on the center manifold.

$$\dot{Y}_1 = (2b_1 - b_2)\mu + \left[5(b_2 - b_1) + \frac{1}{4}(a_1 - a_2)(b_2 - b_1)^2\right]\mu^2 + (2a_1 - a_2)Y_1^2$$

$$+[6+\tfrac{1}{2}(b_2 - b_1)(4a_1 - 3a_2)]Y_1\mu - \tfrac{1}{2}(a_1 - a_2)(4a_1 - 3a_2)Y_1^3 \qquad (5.18)$$

There are two possibilities: (a) $b_2 \neq 2b_1$ and (b) $b_2 = 2b_1$. In the first case as:

$Y_1 \longrightarrow 0$ and $\mu \longrightarrow 0$ (5.18) tends to

$$\dot{Y}_1 = (2b_1 - b_2)\mu + (2a_1 - a_2)Y_1^2 \qquad (5.19)$$

Therefore, the origin of (5.11) and (5.12) undergoes a *saddle node bifurication* as μ *passes through zero* when $a_2 \neq 2a_1$.When $a_2 = 2a_1$, the origin of (5.11) and (5.12) is *not a bifurcation point*.

When $b_2 = 2b_1$ *as* $Y_1 \longrightarrow 0$ *and* $\mu \longrightarrow 0$ (5.18) tends to

$$\dot{Y}_1 = \left[5b_1 + \tfrac{1}{4}(a_1 - a_2)b_1^2\right]\mu^2 + [6+\tfrac{1}{2}b_1(4a_1 - 3a_2)]Y_1\mu + (2a_1 \qquad (5.20)$$

And therefore, the origin of (5.11) and (5.12) under goes a *transcritical bifurication* as μ *passes through zero* when $a_2 \neq 2a_1$

When $a_2 = 2a_1$ *as* $Y_1 \longrightarrow 0$ *and* $\mu \longrightarrow 0$ (5.18) tends to

$$\dot{Y}_1 = (6 - a_1b_1)Y_1\mu - a_1^2Y_1^2 \qquad (5.21)$$

And therefore, the origin of (5.11) and (5.12) under goes *supercritical pitchfork bifurication* μ *passes through zero*.

CONLUSTION

In this paper we have seen the cases bifurcations occur when one of the eigenvalue being zero. The dimension of the system is reduced to one dimensional center manifold. It depends on the coefficient of Fourier series expansion. Saddle node bifurcation occur when $f_\mu \neq 0 \; and \; f_{xx} \neq 0$, transcritical bifurcation occur when $f_\mu = 0 \; and \; f_{xx} \neq 0$ and pitchfork bifurcation occur when $f_\mu = 0 \; and \; f_{xx} = 0$.

Finally we can construct normal forms for each type of bifurcations. Thus, $\dot{x} = \mu + ax^2$ $\dot{x} = \mu x - x^2$ and $\dot{x} = \mu x + ax^3$ are the normal form saddle-node, transcritical and pitchfork bifurcations, respectively.

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