ADDIS ABABA UNIVERSITY

COLLEGE OF NATURAL AND COMPUTATIONAL SCIENCES

DEPARTMENT OF MATHEMATICS

VARIATIONAL METHODS FOR SOLUTIONS OF LINEAR BOUNDARYVALUE PROBLEMS

BY: TIRHAS FISSEHA

ADVISOR: Dr. MENGISTU G

A thesis submitted to the college of Natural and Computational Science of the Addis Ababa University in partial fulfillment of the requirements for degree of Master of Science.

Approval

This thesis has been examined and approved as meeting the requirements for the partial fulfillment of Master of Science in Mathematics.

Examining board members

Name______________ Signature Date___

1. Dr. Mengistu Goa _______(Advisor)___________ 05/09/2018

2. Dr. Hunduma Legesse(Examiner)___________ 05/09/2018

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DECLARATION

I, Tirhas Fisseha Gebremariam, with student ID GSK/0526/04, hereby declarethat this thesis entitled "VARIATIONAL METHODS FOR SOLUTIONS OF LINEAR BOUNDARYVALUE PROBLEMS" has been compiled and organized by myself under the supervision of Dr. Mengistu Goa and that it has neverbeen submitted for completion of graduate qualification at any higherlearning institution. Any work done by others hasbeen acknowledged and referenced accordingly.

Tirhas Fisseha Gebremariam

Addis Ababa,Ethiopia

August 30, 2018

ACKNOWLEDGEMENT

First, I would like to express my deepest gratitude to God for giving me patience; I am also grateful to my advisor Dr.Mengistu Goa for his helpful discussion, comments and providing the necessary materials in the preparation of this project.

Secondly I would like to extend my thank to my families and all who encouraged me to complete my project and their heart full help while writing the paper. At the last, but not the least I would like thanks to Department of mathematics, Addis Ababa University for giving the necessary materials throughout the preparation of this paper

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Abstract

The variational formulation of boundary value problems is valuable in providing remarkably easy computational algorithms as well as an alternative framework with which to prove existence results. It is given that the conditions which guarantee that a given BVP can be reformulated in variational form. It is also shown that by applying the Lax-milgram theorem the existence and uniqueness of solution for the variational problem is proved.

INTRODUCTION

Variational methods provide a conceptual and computational framework for boundary value problems (BVPs). In the thesis we derive conditions which guarantee that a given BVP can be reformulated in variational form. This can be a way of proving existence for(BVPs) and it also provides a scheme for determination of a computational algorithm. In the case of nonlinear eigenvalue problems, these can be a way of producing outcomes like parameter path-following as has been produced by other means by computer packages.

The question as to which boundary value problems have a variational formulation in which all of the boundary conditions appear naturally is largely still an open one and is the subject of this paper. Boundary conditions appear naturally by the addition of suitable terms in the Hamiltonian (usually surface integrals on the boundary of the region of interest). It is usually difficult and sometimes impossible to determine these in a systematic way. Problems exist (like the two point second order boundary value problem in one dimension with Dirichlet boundary conditions mentioned below) for which this is impossible. In these cases the boundary conditions need to be imposed on the space of functions on which the stationary values of the Hamiltonian is found, and are called “essential boundary conditions". This can be of nuisance value in the determination of the subspaces of functions on which the optimization occurs. Thus it appears useful to be able to write the variational formulation with all of the boundary conditions appearing naturally.

This thesis has three parts. The first chapter introduces some basic concepts in functional analysis such as normed spaces particularly focusing on Hilbert spaces functions. The second part of this thesis introduces the variational formulation of linear boundary problems, the relevant functional spaces and the concept of weak solutions. Further, we will state the existence and uniqueness theorem in the framework of the Lax-Milgram theorem.

The third part of the thesis describes some applications ofvariationalmethodstoDirichletand Neumann boundary problems.

Unit 1

Functional Spaces and their Properties

In this chapter some essential tools are presented to facilitate understanding and manipulation of this particularly famous and efficient technique for the numerical analysis of equations having partial derivatives.

1.1 Functional spaces

The term domain and the symbol Ω shall be reserved for an open set in n-dimensional, real Euclidean space$\mathbb{R}^n$. We shall be concerned with differentiability and integrability of functions defined onΩ.

Definition1.1.1 By a functional on a vector space V we mean a scalar-valued function f defined on V. the functional f is linear provided

$$f(ax+by)=af(x)+bf(y), \qquad x,y\in V, \qquad a,b\in\mathbb{C}.$$

Definition 1.1.2 A norm on a vector space V is a real-valued functional f which satisfies

- **i.** $f(x)\geq 0\ forall x\in V$ with equality if and only if $x=0$,
- **ii.** $f(cx)=|c|f(x) forevery x\in V$ and $c\in\mathbb{C}$,
- **iii.** $f(x+y)\leq f(x)+f(y), forevery x,y\in V$

Definition 1.1.3 Let$V$be a normed space. A sequence $\{v_n\}\subset V$ is called a Cauchy sequence if for any $\epsilon>0$ there exists a number $N(\epsilon)$ such that

$$\|v_m-v_n\|<\epsilon\forall m,n>N(\epsilon)$$

Definition 1.1.4A normed space is said to be complete if every Cauchy sequence from the space converges to an element in the space. A complete normed space is called a Banach space.

Let A be a subset of a normed space V. Then,

i. The set A is said to be closed in V if and only if $v_n\in A and v_n\to v$imply that$v\in A$.

ii. The closure $\bar{A} of A$ is the smallest closed set in Vcontaining A.

iii. The set A is dense in V if for $v \in V$ there exists a sequence $v_n \in A$ such that$v_n \to v$.

iv. $A$is said to be bounded if for some constant $M, \|v\| \leq M foreveryv \in A$.

Definition 1.1.5 Let$V$be a vector space over the field K. A mapping $(.,.): V \times V \to K$ is said to be an inner product on V if and only if for all vectors $u, v\ and\ w$ and scalar α:

1. $(u + v, w) = (u, w) + (v, w)$
2. $(\alpha u, v) = \alpha(u, v)$
3. $(u, v) = \overline{(v, u)}$
4. $(u, u) \geq 0\ and\ (u, u) = 0 \Leftrightarrow u = 0$

The space V together with the inner product $(.,.)$ is called an inner product space. We simply say V is an inner product space. When$\mathbb{K} = \mathbb{R}$, V is called a real inner product space, whereas if $\mathbb{k} = \mathbb{C}$, V is called a complex inner product space. An inner product $(.,.)$ induces a norm through the formula

$$\|v\| = \sqrt{(v, v)} \qquad v \in V$$

Definition 1.1.6 A complete inner product space is called a$Hilbertspace$.

1.2 Adapted Functional Spaces and Their Properties:

This section recalls the definition of certain functional spaces that would be used to state certain fundamental results.

Definition1.2.1 LetΩ be an open subset of$\mathbb{R}^n$and $1 \leq p < \infty$. We define $L^p(\Omega)$ to be the space of all measurable functions $f: \Omega \to \mathbb{R}$ for which$\|f\|_{L^p(\Omega)} < \infty$, where

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f|^p dx\right)^{1/p} \text{(1.1)}$$

Definition1.2.2 Let Ω be an open domain of $\mathbb{R}^n$and the Sobolev space $H^1(\Omega)$ is defined as: $H^1(\Omega) = \left\{v: \Omega \to \mathbb{R}, v \in L^2(\Omega), \frac{\partial v}{\partial x_i} \in L^2(\Omega), (i = 1, n)\right\}$(1.2)

Definition (1.2.2): is then generalized by introducing the sobolev space $H^m(\Omega)$ as follows:

Definition 1.2.3$\forall$m $\in$N and the result is:

$$H^m(\Omega) = \left\{v: \Omega \subset \mathbb{R}^n \to \mathbb{R}, v \in L^2(\Omega), \frac{\partial^k v}{\partial x_{i_1} \dots \partial x_{i_k}} \in L^2(\Omega), (\forall k = 1, m)\right\} \text{(1.3)}$$

Theorem 1.2.1[4, 5]For any integer m, the Sobolev space $H^m(\Omega)$is a Hilbert space.

Space$\boldsymbol{\mathfrak{D}(\Omega)}$ for which the notion of support (noted $\boldsymbol{suppv}$) is introduced as the smallest closed subset containing all the points where a given function$\boldsymbol{v}$ is nonzero, is another functional space essential for the functional analysis of equations having partial derivatives:

$$suppv \equiv \overline{\{x \in \mathbb{R}^n / v(x) \neq 0\}} \quad (1.4)$$

To illustrate the notion of support, consider the example of a function of a real variable defined as:

$$H(x) = \begin{cases} 1\,, if\ 0 < x < 1 \\ 0\,, otherwise \end{cases} \quad (1.5)$$

In this case, the function H is non-zero at the open domain $(0,1)$ but having closed interval $[0,1]$ as support:

$$suppH \equiv \overline{\{x \in \mathbb{R} / H(x) \neq 0\}} = [0, 1]$$

Space$\mathfrak{D}(\Omega)$ is therefore defined as:

$$\mathfrak{D}(\Omega) = \{v: \Omega \subset \mathbb{R}^n \to \mathbb{R}, v \in C^\infty(\Omega), \mathrm{supp}v \subset \Omega\} \quad (1.6)$$

Terminology: The $\mathfrak{D}(\Omega)$ space is the space of functions C^∞over $\boldsymbol{\Omega}$ with a compact support strictly included in $\boldsymbol{\Omega}$.

The following fundamental density theorem is thus obtained:

Theorem1.2.2 Thespace $\mathfrak{D}(\Omega)$ is dense in $L^2(\Omega)$.

Finally, the closure of space $H^1(\Omega)$ in $\mathfrak{D}(\Omega)$is associated to the former and isdenoted as$H_0^1(\Omega)$. The following definition and property are thus obtained:

Definition 1.2.4

$$H_0^1(\Omega) = \overline{H^1(\Omega)} \quad (1.7)$$

Theorem 1.2.3

$$H_0^1(\Omega) = \{v: \Omega \subset \mathbb{R}^n \to \mathbb{R}, v \in H^1(\Omega), v = 0\ on \partial\Omega\} \quad (1.8)$$

The following theorem gives us about a variation of the Green formula.

Theorem 1.2.4 LetΩ be an open- bounded domain of $\mathbb{R}^{\boldsymbol{n}}$ with continuous boundary $\partial\Omega = \Gamma$ only admitting discontinuity of the first kind for the tangent vector (i.e. typical angular points). Given that$u$ *and* $v$are two functions of the defined variables $(x_1, x_2, \dots, x_n)$ on Ω having real values such that $u \in C^2(\Omega) \cap C^1(\overline{\Omega})andv \in C^1(\Omega) \cap C^0(\overline{\Omega})$, the result obtained is:

$$\int_\Omega \Delta u . v d\Omega = -\int_\Omega \nabla u . \nabla v d\Omega + \int_{\partial\Omega} \frac{\partial u}{\partial n} v d\Gamma \quad (1.9)$$

Where n denotes the normal vector external to open domain Ω and $\frac{\partial u}{\partial n}$ the projection of the gradient vector in the direction of normal$n$.

It is to be noted again that the use of formula (1.9) is valid for functions having a weaker regularity, namely for $u \in H^1(\Omega)$and$v \in H^1(\Omega)$.

1.3 A Set of Fundamental Inequalities:

We recall some fundamental inequalities that emerge from the analysis and that are used intensely within the framework of functional analysis of equations having partial derivatives.

Cauchy-Schwartz Inequality: let u *and* v be two functions be two functions belonging to $\mathcal{L}^2(\Omega)$. Then the result obtained is

$$\int_\Omega u . v d\Omega \leq \left[\int_\Omega u^2 d\Omega\right]^{\frac{1}{2}} . \left[\int_\Omega v^2 d\Omega\right]^{\frac{1}{2}} \quad (1.10)$$

Holder's Inequality:

Let p and q be twopositive real numbers that satisfy:$\frac{1}{p} + \frac{1}{q} = 1$. Let $u$be a function belonging to$L^P(\Omega)$ and v be a function belonging to$L^q(\Omega)$. The result

obtained is:

$$\int_\Omega u . v d\Omega \leq \left[\int_\Omega u^p d\Omega\right]^{\frac{1}{p}} . \left[\int_\Omega v^q d\Omega\right]^{\frac{1}{q}} \quad (1.11)$$

Poincare's Inequality:

Let Ω be an open-bounded domain of $\mathbb{R}^n$ and u a function belonging to Sobolev space$H_0^1(\Omega)$. The constant $C(\Omega)$ is such that:

$\int_\Omega |u|^2 d\Omega \leq C(\Omega) \int_\Omega |\nabla u|^2 d\Omega$(1.12)

Unit 2

Variational Formulations for Linear BoundaryValue Problems

2.1 Structure and Functional Framework of Equations Having Partial Derivatives

To give concrete expression to the demonstration, the two-dimensional problem of Laplace-Dirichlet is considered. More specifically, let Ω be a bounded open domain of $\mathbb{R}^2$and it is required to find function u defined from Ω to $\mathbb{R}$ and solution of:

$$(CP)\begin{cases}-\Delta u = f in\Omega, \\ u = 0 \quad on\partial\Omega\end{cases}(2.1)$$

Where$f$ is a given function. At this stage, it is important to note that such a formulation is incomplete because neither the nature of the regularity of boundary $\partial\Omega$ of the integration domain Ω nor that of the second member f is specified though the regularity of solution $u$of (CP) depends much upon it, as does the regularity of research perimeter V in which solution u can be considered.

In this way, for reasons that will be explained later, the integration domain Ω will be assumed to possess a boundary $\partial\Omega$ whose regularity is of the order of C^2.In other words, the curvature is a continuous function of the curvilinear abscissa that describes boundary$\partial\Omega$.

Moreover, assuming that the second member$f$ belongs to$C^0(\Omega)$, it is then legitimate to consider the search for solutions of (CP) as elements of $C^2(\Omega)$, thus ensuring that the Laplacian is itself continuous, (it then implies classical solutions).

In this case, the Poisson equation can be considered again, not in the form of a functional equation, but at each point MofΩ, in the form:

Find u belonging to $C^2(\Omega)$ which is a solution of

$$(CP)\begin{cases}-\Delta u(M) = f(M) & \forall M \in \Omega \\ u(M) = 0 & \forall M \in \partial\Omega\end{cases}(2.2)$$

It is obvious that the second member f does not always exhibit regularityC^0.For instance, consider the case where f belongs to$L^2(\Omega)$,in this case, the Laplacian of solution u (which is equal to) must also be an element of$L^2(\Omega)$.

This is why it is necessary to find solution u to the (CP) inSobolev space $H^2(\Omega)$because if this is the case, the Laplacian of u is indeed an element of$L^2(\Omega)$.

The Poisson equation can no more be considered, a priori, point-by-point as in the case of regularity C^0 for f but in the form of a functional equation. In the present case, the Poisson equation needs to be considered as equality in $L^2(\Omega)$that is, as a root mean square equality, or as an "energy" balance:

$$\left(\Delta u + f = 0 \; in L^2(\Omega)\right) \Leftrightarrow \left(\int_\Omega [\Delta u + f]^2 \, d\Omega = 0\right)(2.3)$$

The Poisson equation would nevertheless still be studied as a global equation (2.3) written in $L^2(\Omega)$ rather than as a local equation (2.2).

2.2Construction of a Variational Formulation:

The essential principles constituting the variational formulation will now be studied. The basic idea prevailing in this method is to consider the unknown u no more as a scalar field which, at each point$M$of Ω associates a real number $u(M)$ that needs to be determined, but as an element belonging to a space of functions$V$in which different research trajectories would be contemplated so as to lead to the identification of the solution.

Concerning the approximation, it is no more required to determine a numerical sequence$(\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_N)$ which provides an approximation of the finite differences type for values

$(u_1, u_2, \ldots, u_N)$ of solution to (CP) along points $M_j, (j = 1, N)$ that has been chosen on an that have been chosen on an adequate mesh and covering integration domain Ω.

However, it is more meaningful to elaborate a method that would lead to an approximation function $\tilde{u}$. It is obvious, in fine, that knowing solution,$u$or rather its approximation $\tilde{u}$, would facilitate the

evaluation of $\tilde{u}$ at any point M of domain Ω and this evaluation would not be limited to a set of points lying on an already defined mesh, as is the case for finite differences.

A second major characteristic of the finite elements method is the transformation of (CP) into an integral formulation known as variational (VP).

To proceed, a function v, called test function, defined from Ω to$\mathbb{R}$ and describing the functional space$V$ that will be elaborated later and that is not defined a priori.

Equation (2.1) is then multiplied by the test function v and the two members of the equation are integrated on Ω:

$$-\int_\Omega \Delta u.vd\Omega = \int_\Omega f.vd\Omega, \forall v \in V \quad (2.4)$$

Moreover, the transformation of the local writing of problem (CP) into a global or integral formulation (VP) is motivated by the need to reach a formalism that properly fits the concept of research trajectories in a functional space$V$.

This is precisely the case within an integral formulation, in so far as the functions do not directly reveal their numerical values at points M ofΩ and only the concept of the “average value” of the functions is apparent.

Applying Green formula(1.9), the enabling equation (2.4) to be written as:

$$\int_\Omega \nabla u.\nabla vd\Omega - \int_{\partial\Omega} \frac{\partial u}{\partial n} vd\Gamma = \int_\Omega f.vd\Omega, \forall v \in V \quad (2.5)$$

The present stage consists in the definition of the characteristics of space$V$.

A first point concerns the complete preservation of information between the writing of the formulation of the continuous problem (CP) and that of thevariational formulation (VP).

As such, it is observed that the Dirichlet condition $u = 0$ along boundary $\partial\Omega$ of Ω cannot be analyzed directly within the integral writing (2.5).

Considering that the future solution u of the variational problem (VP) must be

one of the functions v of$V$, it is compulsory that all functions v of$V$ satisfy the

Dirichlet condition:

$v = 0 \quad on\partial\Omega$(2.6)

This yields equation (2.5) written as

$$\int_\Omega \nabla u . \nabla v d\Omega = \int_\Omega f . v d\Omega , \forall v \in V \qquad (2.7)$$

The second point concerns the existence of the integrals of formulation (2.7).Is essential to impose sufficient conditions of convergence to the integrals of equation (2.7).

The convergence of the second member of equation (2.7) is easily obtained by verification, via the Cauchy-Schwartz inequality:

$$\left|\int_\Omega f . v d\Omega\right| \leq \int_\Omega |f . v| d\Omega \leq \left[\int_\Omega |f|^2 d\Omega\right]^{\frac{1}{2}} . \left[\int_\Omega |v|^2 d\Omega\right]^{\frac{1}{2}} (2.8)$$

Therefore, since f is a given function belonging to $L^2(\Omega)$, it is sufficient to consider that v is also an element of$L^2(\Omega)$, as to ensure the convergence of the second member of equation (2.7).

In the case of the convergence of the first integral of the left member of equation (2.7), the absolute convergence of the integral is always taken into consideration and the Cauchy-Schwartz inequality is once again used

$$\left|\int_{\Omega} \nabla u \cdot \nabla v d\Omega\right| \leq \int_{\Omega} |\nabla u \cdot \nabla v| d\Omega \leq \left[\int_{\Omega} |\nabla u|^2 d\Omega\right]^{\frac{1}{2}} \cdot \left[\int_{\Omega} |\nabla v|^2 d\Omega\right]^{\frac{1}{2}} \cdot (2.9)$$

Convergence of the first member of (2.7) is hence assured if the gradients of test function v

belonging to V are compulsorily elements that belong to $L^2(\Omega)$.

In conclusion, it has been proved that the sufficient conditions for the convergence of the integrals of equation (2.7) are:

$v \in L^2(\Omega)$and$\nabla v \in L^2(\Omega)$.

These reasons consequently explain the choice of the variationalspace as the Sobolev space $H^1(\Omega)$ and to which the homogenous Dirichlet condition (2.6) must necessarily be added.

In other words, the following is stated:

$$V \equiv H_0^1(\Omega) \equiv \{v: \Omega \longrightarrow \mathbb{R}, v \in L^2(\Omega), \nabla v \in L^2(\Omega), v = 0 \; on \partial\Omega\} (2.10)$$

All the results when grouped enable the expression of the variational formulation (VP) that will be considered in the sequel:

$$(\boldsymbol{VP}) \begin{cases} find u \in H_0^1(\Omega) \text{solution of} \\ \int_{\Omega} \nabla u . \nabla v d\Omega = \int_{\Omega} f . v d\Omega, \forall v \in H_0^1(\Omega) \end{cases} \qquad (2.11)$$

2.3 Equivalence Between Strong and Weak Formulations

An additional point concerning the equivalence between different formulations needs to be mentioned with in the whole transformation process that has been presented above.

More precisely, it is not obvious to declare that any solution to variational problem (VP) (2.11**)** is a solution to continuous problem (CP) (2.1)

The subtleties of the concept of equivalence between the two formulations may be tested by continuing to assume that the second member f is a function belonging to $L^2(\Omega)$ and it is then only necessary to believe that if the solution to the continuous problem is searched for in the Sobolev space $H^2(\Omega)$ then that of the variational problem (VP) is searched for in $H^1(\Omega)$ and

$H^2(\Omega) \subset H^1(\Omega)$.

In other words, any solution to the continuous problem may be a solution to a variational problem with regards to its regularity, whereas, a priori, there is no justification for a solution to a variational problem (VP) to be the solution to a continuous problem (CP).

In fact, the concept of equivalence between the two formulations is completely dependent on the functional frameworks governing the respective areas of research for solutions to a continuous problem (CP) on one hand and to a variational problem (VP) on the other hand.

2.4 Existence and Uniqueness of a Weak Solution

Concerning variational formulations, there is a sufficient general formalism for which, under some conditions, the existence and uniqueness of the solution may be guaranteed.

Definition 2.4.1A vector space V is said to be finite dimensional ifthere exists a finite maximal set of independent vectors $\{v_1, \ldots, v_n\}$; i.e. the set $\{v_1, \ldots, v_n\}$ is linearly independent , but

$\{v_1, \ldots, v_n, v_{n+1}\}$is linearly dependent for any $v_{n+1} \in V$. The set $\{v_1, \ldots, v_n\}$ is called a basis of the space. If such a finite basis does not exit, then V is said to be infinite dimensional.

Definition 2.4.2 Let$L$ be a function from one linear space V to another linear space W. We say L is a linear function if

(a). For all $u, v \in V, L(u + v) = L(u) + L(v)$;

(b). For all $v \in V and \alpha \in \mathbb{R} L(\alpha v) = \alpha L(v)$

Definition2.4.3 IfVand$W$are vector spaces, a bilinear form a:$V \times W \to \mathbb{R}$ is defined to be an operator with properties,

$$a(\alpha u + \beta w, v) = \alpha a(u, v) + \beta a(w, v) u, w \in V, v \in W$$

$$a(u, \alpha v + \beta w) = \alpha a(u, v) + \beta a(u, w) u \in V\ , v, w \in W$$

where$\alpha and \beta are real numbers$.

It is the objective of the Lax-Milgram theorem that is pointed out in the following form:

Theorem2.4.1(Lax – Milgram) Let$V$ be a Hilbert space in relation to a given norm $\|.\|, a(.,.)$ a bilinear form defined on $V \times V$ and L a linear form defined on V satisfying the following properties:

i. $a(.,.)$ is continuous: $\exists C_1 > 0$ such that:

$$|a(u, v)| \leq C_1 \|u\|. \|v\|, \forall (u, v) \in V \times V$$

ii. $a(.,.)$ is V-elliptical:$\exists C_2 > 0$ such that:$a(v, v) \geq C_2 \|v\|^2, \quad \forall v \in V$

iii. L is continuous: $\exists C_3 > 0$ such that $L(v) \leq C_3 \|v\| \forall v \in V$

Then, there is unique solution u belonging to V, solution to the variation problem

$$\begin{cases} find u \in V solution of : \\ a(u, v) = L(v), \quad \forall v \in V \end{cases} \qquad (2.12)$$

Proof:

1. For eachfixed element $u \in V$,the mapping $u \mapsto a(u, v)$ is a bounded linear functional on V;whence the Riesz representation theorem asserts the existence of a unique element $w \in V$ satisfying

$$a(u, v) = (w, \boldsymbol{v})(v \in V)(2.13)$$

Let us write $Au = w$ whenever (2.13) holds; so that

$a(u, v) = (Au, v) \qquad (u, v \in V)$.(2.14)

2. We first claim $A: V \rightarrow V$ is a bounded linear operator. Indeed if $\lambda_1, \lambda_2 \in \mathbb{R}$ and $u_1, u_2 \in V$, we see

$(A(\lambda_1 u_1 + \lambda_2 u_2), v) = a(\lambda_1 u_1 + \lambda_2 u_2, v)$by (2.14)

$$= \lambda_1 a(u_1, v) + \lambda_2 a(u_2, v)$$

$= \lambda_1 (Au_1, v) + \lambda_2 (Au_2, v)$again by (2.14)

$$= (\lambda_1 Au_1 + \lambda_2 Au_2, v)$$

This equality obtains for each $v \in V$ and so A is linear. Furthermore,

$$\|Au\|^2 = (Au, Au) = a(u, Au) \leq c_1 \|u\| \|Au\|$$

Consequently $\|Au\| \leq c_1 \|u\|$ for all$u \in V$, and so A is bounded.

3. Next we assert

$$\begin{cases} A is one to one, and \\ R(A), range of A, is closed in V \end{cases} \qquad (2.15)$$

To proof this, let us compare

$\|u\|^2 \leq a(u, u) = (Au, u) \leq \|Au\| \|u\|$.

Hence$c_2 \|u\| \leq \|Au\|$.

This inequality implies (2.15).

4. We demonstrate now

$R(A) = V$ (2.16)

For if not, then since $R(A)$ is closed, there would exist a non-zero element $w \in V$with

$w \in (R(A))^{\perp}$. But this fact in turn implies the contradiction

$$c_2\|w\|^2 \leq a(w,w) = (Aw,w) = 0$$

5. Next, we observe once more from the Riesz Representation theorem that

$$L(v) = (w,v) forallv \in V$$

For some element$w \in V$. We then utilize (2.15) and (2.16) to find $u \in V$satisfying$Au = w$. Then

6. Finally, we show there is at most one element $u \in V$ verifying (2.12) .

For if both $a(u,v) = L(v)$and$a(\tilde{u},v) = L(v)$, then$a(u-\tilde{u},v) = 0 (v \in V)$.

We set $u = u - \tilde{u}$ to find $c_2\|u-\tilde{u}\|^2 \leq a[u-\tilde{u}, u-\tilde{u}] = 0$. ■

Definition2.4.4 We say that $\partial\Omega$ is Lipschitz if there exists $r > 0$ with the property that for each point $x \in \partial\Omega$ there is Lipschitz functions$\gamma: \mathbb{R}^{d-1} \to \mathbb{R}$ such that, upon relabeling and reorienting the coordinate axis if necessary, we have

$$\Omega \cap B(x,r) = \{y \in B(x,r): y_d > \gamma(y_1, \dots, y_{d-1})\}$$

Theorem 2.4.2 (trace theorem) If $\partial\Omega$ is Lipschitz, then there exists bounded linear operator

$T: H^1(\Omega) \to L^2(\partial\Omega)$such that $Tu = u\Big|_{\partial\Omega}$ if $u \in C^1(\bar{\Omega})$

Proof:we establish the boundedness of T for the case where $\Omega \in \mathbb{R}^d$ is the half$\mathbb{R}^{d-1} \times [0,\infty)$.

Let T^0 be the linear map defined by

$T^0: u \in C_0^1(\mathbb{R}^{d-1} \times [0,\infty)) \to u\Big|_{\mathbb{R}^{d-1}}$,

With the convention that $C_0^1(\Omega)$ denotes the set of those functions with the property that the support of the trivially extended function $\bar{u}: \mathbb{R}^d \to \mathbb{R}$ is compact. The trivial extension $\bar{u}$

Assumes the value $o$outsideΩ.

Let us first show that

$\|T^0(u)\|_{L^2(\mathbb{R}^{d-1})} \leq \|u\|_{H^1\left(\mathbb{R}^{d-1}\times[0,\infty)\right)}$(2.17)

Since u has compact support, we have

$$\left|u(x',0)\right|^2 = -\int_0^\infty \frac{\partial}{\partial x_d}(u(x',x_d)^2)\,dx_d = -\int_0^\infty 2u(x',x_d)\frac{\partial}{\partial x_d}u(x',x_d)\,dx_d$$

Therefore, by Young's inequality

$\left|u(x',0)\right|^2 \leq \int_0^\infty (u(x',x_d)^2)\,dx + \int_0^\infty \left|\frac{\partial}{\partial x_d}u(x)\right|^2 dx_d$(2.18)

Integrating over $\mathbb{R}^{d-1}$ in x' and using Fubini's theorem one obtains

$$\int_{\mathbb{R}^{d-1}} \left|u(x',0)\right|^2 dx \leq \int_{\mathbb{R}^{d-1}\times[0,\infty)} u^2\,dx + \int_{\mathbb{R}^{d-1}\times[0,\infty)} \left|\frac{\partial}{\partial x_d}u(x)\right|^2 dx$$

Which gives (2.17)

Suppose now that $u \in H^1\left(\mathbb{R}^{d-1}\times[0,\infty)\right)$. By standard density results there exists a sequence $u_n \in H^1\left(\mathbb{R}^{d-1}\times[0,\infty)\right)$ converging to $u \in H^1\left(\mathbb{R}^{d-1}\times[0,\infty)\right)$. By construction $T(u) = T^0(u)$ if $u \in C_0^1\left(\mathbb{R}^{d-1}\times[0,\infty)\right)$ so that T is a linear extension of $T^0$to$H^1\left(\mathbb{R}^{d-1}\times[0,\infty)\right)$. By construction T is uniquely determined and linear and continuous from $H^1\left(\mathbb{R}^{d-1}\times[0,\infty)\right)toL^2(\mathbb{R}^{d-1})$. ■

UNIT 3

APPLICATIONS TO LINEAR BOUNDARY VALUE PROBLEMS

3.1Problems with the DirichletBoundary Conditions

A first application of the Lax-Milgram theorem is proposed in order to establish the existence and uniqueness of the solution to the variationalformulation (VP) defined by (2.11),

associated with the continuous problem (CP) defined by (2.1) when the given data f belongs to $L^2(\Omega)$.

Application of the Lax-Milgram theorem requires the identification of space V, the bilenear form $a(.,.)$and that of the linear form $L(.)$.

Variational formulation (VP) defined by (2.11) suggests the introduction of the following quantities:

Let V be the search space of solution u to the variational problem defined by:

$$V = H_0^1(\Omega)$$

Space $H_0^1(\Omega)$ is provided with the natural norm $\|.\|_{H_0^1(\Omega)}$ of functoons belonging to$H^1(\Omega)$.

Thus $\forall v \in H^1(\Omega)$, the following is written:

$$\|v\|^2_{H^1(\Omega)} = \int_\Omega v^2 d\Omega + \int_\Omega \left(\frac{\partial v}{\partial x}\right)^2 d\Omega + \int_\Omega \left(\frac{\partial v}{\partial y}\right)^2 d\Omega \quad (3.1)$$

This norm is Hilberian for space $H^1(\Omega)$, as well as for $H_0^1(\Omega)$ as a closed vectorial subspace $H^1(\Omega)$.

Let a be the bilinear form defined by:

$$a: V \times V \to \mathbb{R}$$

$$(u, v) \rightsquigarrow a(u, v) \equiv \int_\Omega \nabla u . \nabla v d\Omega (3.2)$$

Likewise, let L be the linear form defined by:

$$L: V \to \mathbb{R}$$

$$v \rightsquigarrow L(v) \equiv \int_\Omega \mathrm{f} v d\Omega (3.3)$$

Thus, variational formulation (VP) defined by (2.11) is written in the form:

$$\begin{cases} \text{Find } u \in V \text{a solution of:} \\ a(u, v) = L(v), \quad \forall v \in H_0^1(\Omega). \end{cases} (3.4)$$

Then, a verification of the clauses of the Lax-Milgram theorem is carried out.

1. $a(.,.)$ is a continuous bilinear form

As for its continuity, consider any two elements u and v belonging to $H_0^1(\Omega)$.

The following obtained:

$$|a(u, v)| \leq \int_\Omega |\nabla u . \nabla v| \leq \left(\int_\Omega |\nabla u|^2\right)^{\frac{1}{2}} . \left(\int_\Omega |\nabla v|^2\right)^{1/2}, \ (3.5)$$

Where the Cauchy-Schwartz inequality would have been used.

However,

$$\int_\Omega |\nabla u|^2 = \int_\Omega \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] = \left\|\frac{\partial u}{\partial x}\right\|^2_{L^2(\Omega)} + \left\|\frac{\partial u}{\partial y}\right\|^2_{L^2(\Omega)} (3.6)$$

Where $\|.\|_{L^2(\Omega)}$refers to the natural norm in$L^2(\Omega)$, namely:

$\forall u \in L^2(\Omega): \|u\|_{L^2(\Omega)} \equiv \left(\int_\Omega |\mathrm{u}|^2\right)^{1/2}$. (3.7)

The following is then inferred:

$\int_\Omega |\nabla u|^2 \leq \|u\|^2_{H^1(\Omega)}$.(3.8)

Inequality (3.4) then leads to:

$|a(u,v)| \leq \|u\|_{H^1(\Omega)}.\|v\|_{H^1(\Omega)}$(3.9)

And the continuity constant C_1 of Lax –milgram theorem is basically equal to one.

2. $a(.,.)$ is a V-elliptical form:

In order to establish the V-ellipticity of the bilinear $a(.,.)$ form, the quantity $a(v,v)$ needs to be minorated.

Also any function $H_0^1(\Omega)$ yields:

$$a(v,v) = \int_\Omega |\nabla v|^2 = \left\|\frac{\partial v}{\partial x}\right\|^2_{L^2(\Omega)} + \left\|\frac{\partial v}{\partial y}\right\|^2_{L^2(\Omega)} \quad (3.10)$$

In order to obtain a lower bound of $a(v,v)$ in relation to the $H^1(\Omega)$ norm, it is pointed out that for all functions v belonging to$H_0^1(\Omega)$, the Poincare inequality (1.12) is available.

In other words, a constant $C(\Omega) > 0$ exists such that:

$$\int_\Omega |v|^2 d\Omega \leq C(\Omega) \int_\Omega |\nabla v|^2 d\,\Omega \quad (3.11)$$

To each side of inequality (3.11), the square of norm $L^2(\Omega)$ of the module of ∇v is added so as to yield the square of norm $H^1(\Omega)$ of function v:

$\|v\|^2_{H^1(\Omega)} \equiv \|v\|^2_{L^2(\Omega)} + \left\|\frac{\partial v}{\partial x}\right\|^2_{L^2(\Omega)} + \left\|\frac{\partial v}{\partial y}\right\|^2_{L^2(\Omega)}$(3.12)

$\leq \left(1 + C(\Omega)\right)\left[\left\|\frac{\partial v}{\partial x}\right\|^2_{L^2(\Omega)} + \left\|\frac{\partial v}{\partial y}\right\|^2_{L^2(\Omega)}\right]$(3.13)

$\leq (1 + C(\Omega))a(u, v).$ (3.14)

It then becomes;

$a(v, v) \geq C_2\|v\|^2_{H^1(\Omega)}$, (3.15)

Where the V-ellipticity constant C_2 is defined by$C_2 = \frac{1}{1+C(\Omega)}$.

3. $L(.)$ is a continuous linear form:

The linearity of form L is obvious from its definition.

Control of linear form L is quite simple being given that f is a function belonging to$L^2(\Omega)$:

$|L(v)| \leq \int_\Omega |fv|d\Omega \leq \|f\|_{L^2(\Omega)}.\|v\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)}.\|v\|_{H^1(\Omega)}$ (3.16)

Continuity constant C_3 of linear form L is thus equal to$\|f\|_{L^2(\Omega)}$.

According to the Lax-Milgram theorem, only one function belongs to $H^1_0(\Omega)$ solution of the variational formulation (VP) defined by (3.3). In case the continuous problem (CP) defined by (2.1) is replaced by the Laplace-Neumann-Dirichlet problem then the boundaryΓ ofΩ is constituted of two complementary parts $\Gamma_1$andΓ_2, respectively dedicated to the definition of the Dirichlet and the Neumann conditions.

In such a case, the continuous problem (CP)takes the following form:

$$(CP)\begin{cases} -\Delta u = f \ \ in \ \ \Omega \\ u = 0 \quad on \ \ \Gamma_1 \\ \frac{\partial u}{\partial n} = g \quad on \ \Gamma_2 \end{cases} \quad (3.17)$$

Where it is assumed that f and g are two given functions respectively belonging to $L^2(\Omega)$and $L^2(\mathit{\Gamma}_2)$.

As a consequence, it is easily established that the new associated variational formulation is written as:

$$\begin{cases} find\ u \in H^1_{\Gamma_1}(\Omega)\ solution\ to \\ \int_\Omega \nabla u.\nabla v d\Omega = \int_\Omega f.v d\Omega + \int_{\Gamma_2} g.v d\Gamma\ ,\ \forall v \in H^1_{\Gamma_1}(\Omega) \end{cases} \qquad (3.18)$$

Where Sobolev$H^1_{\Gamma_1}(\Omega)$ space is defined by:

$$H^1_{\Gamma_1}(\Omega) = \{v: \Omega \to R, v \in L^2(\Omega), \nabla v \in [L^2(\Omega),]^2, v = 0\ on\ \Gamma_1\} \qquad (3.19)$$

In fact, in this case, the action of form L on any function v belonging to $H^1_{\Gamma_1}(\Omega)$ is expressed as:

$$L(v) \equiv \int_\Omega f.v\, d\Omega + \int_{\Gamma_2} g.v d\Gamma\ ,\ \forall v \in H^1_{\Gamma_1}(\Omega) \qquad (3.20)$$

The control of $L(v)$ is then carried out using:

$$L(v) \leq \int_\Omega |f.v|\, d\Omega + \int_{\Gamma_2} |g.v| d\Gamma \qquad (3.21)$$

$$\leq \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma_2)} \|v\|_{L^2(\Gamma_2)} (3.22)$$

Thus, a new difficulty results from the application of the g Neumann condition defined on the $\mathit{\Gamma}_2$boundary. Since, control of $L(v)$ should be performed only in relation to norm $H^1(\Omega)$ of function v. This is why the term resulting from the Neumann condition and providing a measure of v for norm $L^2(\mathit{\Gamma}_2)$ should consequently be modified.

The trace theoremmentioned above is the one that would enable a control over $L(v)$ in relation to the only measure of function$v$ for norm $H^1(\Omega)$. It is to be noted that $C_4$is the continuity constant of the trace application γ defined by(2.17).

Then, inequality (3.22) may be modified as follows:

$|L(v)| \leq \|f\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} + C_4\|g\|_{L^2(\Gamma_2)}\|v\|_{H^1(\Omega)}$(3.23)

$\leq C_5\|v\|_{H^1(\Omega)}$(3.24)

These are the essential points that needed to be specified for extending the Laplace-Dirichlet problem to that of Laplace-Neumann-Dirichlet.

Other minor modifications, that do not represent any major difficulties, concern the adaptation of the results while shifting the functional framework of $H_0^1(\Omega)$to $H_{\Gamma_1}^1(\Omega)$.

This is why, once the point about the control of linear form $L(.)$ defined by (3.20)is made, the application of Lax-Milgram theorem guarantees the existence and uniqueness of solution $u \in H_{\Gamma_1}^1(\Omega)$ to the variational problem **(**VP**)** defined by (3.18)

3.2 Problems with Neumann boundary conditions

Consider next the Neumann problem of determining u which satisfies

$$\begin{cases} -\Delta u + u = f \quad in\,\Omega \\ \dfrac{\partial u}{\partial v} = g \quad on\,\Gamma \end{cases} \qquad (3.25)$$

Hence $f \; and \; g$ are given functions in Ω and on Γ , respectively and $\partial/\partial v$ denotes the norma derivative on Γ. Again we first derive a weak formulation.

Assume $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is a classical solution of the problem (3.25). Multiplying (3.25) by an arbitrary test function v with certain smothness for the following calculations to make sence, integrating over Ω and performing an integration by parts, we obtain

$$\int_\Omega (\nabla u.\nabla v + uv)dx = \int_\Omega fvdx + \int_\Gamma \frac{\partial u}{\partial x} vds$$

Then, substitution of the Neumann boundary condition (3..25) in the boundary term leads to the relation

$$\int_{\Omega} (\nabla u.\nabla v + uv)dx = \int_{\Omega} fvdx + \int_{\Gamma} gvds$$

Assume $f \in L^2(\Omega)$, $g \in L^2(\Gamma)$. for each term in the above relation, choose the space $H^1(\Omega)$ for both the trial function u and the test function v. Thus, the weak formulation of the boundary value problem (3.25) is

$u \in H^1(\Omega) \int_{\Omega} (\nabla u.\nabla v + uv)dx = \int_{\Omega} fvdx + \int_{\Gamma} gvds \quad \forall v \in H^1(\Omega)$(3.26)

This problem has the form (2.12) , where $V = H^1(\Omega)$, $a(.,.)$ and $L(.)$ are defined by

$$a(u, v) = \int_{\Omega} (\nabla u.\nabla v + uv)dx$$

$$L(v) = \int_{\Omega} fvdx + \int_{\Gamma} gvds$$

respectively. Applying the Lax-Milgram lemma , we can show that the weak formulation (3.26) has a unique solution $u \in H^1(\Omega)$.

it is more delicate to study the pure Neumann problem for the poisson equation

$$\begin{cases} -\Delta u = f \quad in \;\; \Omega \\ \dfrac{\partial u}{\partial v} = g \; on \; \Gamma \end{cases} \quad (3.27)$$

Where $f \in L^2(\Omega)$, $g \in L^2(\Gamma)$ are given. In general, the problem (3.27) does not have, and when the problem has a solution u, any function of the form $u + c,\;\; c \in \mathbb{R}$ is a solution.This suggests that in formulating the weak version of this problem we should restrict ourselves to the subspace

$$V = \left\{ v \in H^1(\Omega) \colon \int_{\Omega} vdx = 0 \right\}$$

Formally, the corresponding weak equation is

$u \in H^1(\Omega) \int_{\Omega} (\nabla u.\nabla v)dx = \int_{\Omega} fvdx + \int_{\Gamma} gvds \quad \forall v \in H^1(\Omega)$(3.28)

An application of Equivalent Norm Theorem shows that over the space $V, |.|_1$ is a norm equivalent to the norm $\|.\|_1$. The bilinear form $(u, v) = \int_\Omega \nabla u.\nabla v dx$ is both continuous and V-elliptic . so there is a unique solution to the problem

$$u \in V \int_\Omega (\nabla u.\nabla v)dx = \int_\Omega fvdx + \int_\Gamma gvds \quad \forall\, v \in V$$

UNIT 4

CONCLUSION

To conclude variational formulation: (homogeneous) Dirichlet boundary conditions are enforced by the test-function space, which is included in H^1 for second order problems; we multiply the PDE by a test-function possibly assuming additional regularity on the solution and use integration by parts or Green's formula to obtain the variational problem. The bilinear form must be well-defined on the test-function space.

The second point is to check that the variational formulation actually gives rise to a solution of the boundary value problem. This point is usually itself in two steps: first obtain the PDE in the sense of distributions by using test-functions in $\mathfrak{D}$, second retrieve Neumann or Fourier boundary conditions by using the full test-function space. The first two points can appear somewhat formal because of the assumed regularity on the solution that is not always easily obtained in the end. The final third point is to try and apply the Lax-Milgram theorem, by making precise regularity and possibly sign assumptions on the various functions that act as data. Here we prove existence and uniqueness of the solution to the variational problem.

As a closing remark, this type of analysis of linear boundary problems using variational formulations is an accurate method of understanding the behaviors of solutions.

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