Interaction of Subharmonic Light Modes with Three-Level Atom

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DECLARATION

I hereby declare that this PhD dissertation is my original work and has not been presented for a degree in any other university, and that all sources of material used for the dissertation have been duly acknowledged.

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by

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Approved by the Examination Committee

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Abstract

In this dissertation we have studied the photon statistical and the quadrature squeezing properties of the two-mode cavity light available following the interaction of subharmonic light modes, emerging from a nonlinear crystal pumped by coherent light, with a single three-level atom in a closed cavity. The cavity is coupled to a vacuum reservoir via a single port mirror. We have considered the case in which the top and bottom levels of the three-level atom are not coupled by the coherent light emerging from the nonlinear crystal. Employing the pertinent Hamiltonian and master equations, we have obtained the equations of evolution of the expectation values for the cavity mode and atomic operators.

Applying the steady-state solutions of the equations of evolution of the expectation values for the cavity mode and atomic operators, we have calculated the global mean photon number for the two-mode cavity light. The global mean photon number of the two-mode cavity light is found to be the sum of the mean photon number due to the subharmonic generation and the mean photon number emitted and absorbed by the atom. We have observed that the effect of the interaction of the subharmonic light modes with the three-level atom is to decrease the global mean photon number of the two-mode cavity light. This is due to the fact that the mean number of photons absorbed is greater than the mean number of photons emitted.

Moreover, applying the time dependent solutions of the equations of evolution of the expectation values for the cavity mode and atomic operators, we have also obtained the local mean photon number for the two-mode cavity light. Our anal-

ysis shows that the local mean photon number of the two-mode cavity light increases with frequency and eventually approaches to the global mean photon number within a relatively small frequency interval.

In addition, we have calculated the global photon-number variance for the two-mode cavity light. We have noticed that the photon statistics of the two-mode cavity light is supper poissonian. Our analysis indicates that the global photon-number variance in the presence of the interaction is less than that in the absence of the interaction. This implies that the effect of the interaction is to decrease the global photon-number variance.

Furthermore, we have determined the global quadrature squeezing for the two-mode cavity light with arbitrary ordering of the two-mode vacuum reservoir noise operators. We have established that the two-mode cavity light is in squeezed state and the squeezing occurs in the plus quadrature. We have also found that the global quadrature squeezing in the presence of the interaction (43.5%) is less than the global quadrature squeezing (50%) in the absence of the interaction. In addition, We have seen that the maximum local quadrature squeezing for the two-mode cavity light is 60.8% at $\lambda = 0.17$ and eventually approaches to the global quadrature squeezing as the frequency increases.

Finally, applying the steady-state solutions of the equations of evolution of the expectation values for the cavity mode operators, we have determined the global quadrature squeezing for the two-mode cavity light when the two-mode vacuum reservoir noise operators are in normal order. We have found that the global quadrature squeezing for the two-mode cavity light is 100% for $\gamma_c = \frac{16}{15} \approx 1.067$ and is 88.3% for $\gamma_c = 1.25$ with the vacuum noise operators in normal order.

Contents

List of Figures

1

Introduction

Quadrature squeezing is one of the nonclassical features of light that has attracted a great deal of interest. Several authors have carried out the analysis of the quantum properties of the squeezed light generated by various quantum optical systems [1-15]. In squeezed light the noise in one quadrature is below the vacuum-state level at the expense of enhanced fluctuations in the conjugate quadrature, with the product of the uncertainties in the two quadratures satisfying the uncertainty relation [16-19]. Squeezed light has potential applications in low-noise optical communications, precision measurements, and weak signal detections [2-5].

It has been predicted that three-level atoms in a closed cavity can produce under certain conditions squeezed light [16-18]. Fesseha [16, 18] has studied the squeezing and statistical properties of the light produced by the three-level atoms available in a closed cavity and pumped by electron bombardment. He has found that the light emitted by the atoms is in a squeezed state, with the maximum quadrature squeezing being 50% below the vacuum-state level. In addition, he has considered three-level atoms in a closed cavity pumped by coherent light [18]. For this case he has found that the maximum quadrature squeezing is 43.4% below the vacuum-state level. The analysis of the aforementioned cases have been carried out by normally ordering the reservoir noise operators.

On the other hand, it has been shown theoretically [18-31] and subsequently confirmed experimentally [32-34] that a subharmonic generator, a nonlinear crystal pumped by coherent light, produces squeezed light. It is found that the maximum quadrature squeezing of the superposed subharmonic light modes at threshold and at steady-state is 50% below the vacuum-state level [18, 19].

Furthermore, some authors have studied the statistical and squeezing properties of the light produced by the interaction of subharmonic light modes with three-level atoms, using the usual commutation relation [35-40]. However, it appears to be difficult to believe the results obtained in this manner to be correct in light of the discussion given in Ref. [41].

In this PhD dissertation we seek to investigate the interaction of subharmonic light modes (emerging from a nonlinear crystal pumped by coherent light) with a three-level atom. We consider the case in which the nonlinear crystal and the three-level atom are in a closed cavity coupled to a vacuum reservoir via a single port mirror. Our interest is to analyze the squeezing and statistical properties of the two-mode cavity light available following this interaction. We carry out our analysis applying the master equation for the cavity modes and atomic operators. We consider the case in which the top and bottom levels of the three-level atom are not coupled by the coherent light emerging from the nonlinear crystal. The large-time approximation scheme is used to decouple the equations of evolution for the cavity mode operators.

Employing the steady-state solutions of the equations of evolution for the expectation values of the cavity modes and atomic operators, we calculate the global mean photon number, the global photon number variance, the global quadrature variance, and the global quadrature squeezing. Furthermore, applying the time dependent solutions of the equations of evolution for the expectation values of the cavity modes and atomic operators, we determine the local mean photon number and the local quadrature squeezing.

In addition, we wish to determine the quadrature squeezing of the two-mode cavity light available following the interaction of the subharmonic light modes with the three-level atom by putting the two-mode vacuum reservoir noise operators in normal order.

2

Operator Dynamics

We consider here the case in which a three-level atom, in a cascade configuration, and a nonlinear linear crystal driven by coherent light are available in a closed cavity coupled to a vacuum reservoir via a single port mirror. Moreover, we consider the case in which the top and bottom levels of the three-level atom are not coupled by the coherent light emerging from the nonlinear crystal. This is physically realized by covering the right-side of the nonlinear crystal by a screen which can absorb the coherent light [36]. We denote the top, intermediate, and bottom levels of the atom by $|a\rangle$, $|b\rangle$, and $|c\rangle$, respectively. We assume the transitions between levels $|a\rangle$ and $|b\rangle$ and between levels $|b\rangle$ and $|c\rangle$ to be dipole allowed, with direct transition between levels $|a\rangle$ and $|c\rangle$ to be dipole forbidden [17, 18].

When a three-level atom undergoes a transition from the top energy level $|a\rangle$ to the intermediate energy level $|b\rangle$, it emits a photon of frequency ω_{ab}. Furthermore, the atom undergoes a transition from the intermediate energy level $|b\rangle$ to the bottom energy level $|c\rangle$ by emitting a photon of frequency ω_{bc}. We prefer to represent the light emitted by the three-level atom from the top energy level by mode a and the light emitted from the intermediate energy level by mode b. These light modes are at resonance with the two transitions $|a\rangle$ to $|b\rangle$ and $|b\rangle$ to $|c\rangle$.

The process of subharmonic generation taking place inside the nonlinear crystal

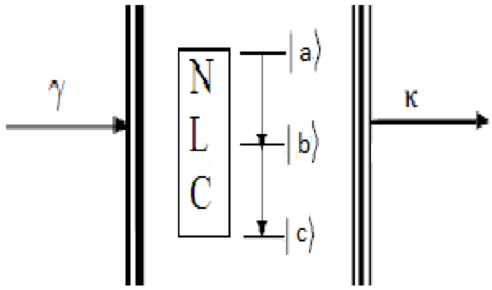

Figure 2.1: A three-level atom with a nonlinear crystal (NLC).

can be described by the Hamiltonian

$$\hat{H}' = i\lambda(\hat{c}^\dagger\hat{a}\hat{b} - \hat{c}\hat{a}^\dagger\hat{b}^\dagger), \tag{2.1}$$

where the operators $\hat{a}$ and $\hat{b}$ represent the subharmonic light modes, λ is the coupling constant between the coherent light and light mode a or b and $\hat{c}$ is the annihilation operator for the coherent light. In order to have a manageable mathematical analysis, we replace the operator $\hat{c}$ by γ which is taken to be real, positive, and constant. we can then write the Hamiltonian as

$$\hat{H}' = i\varepsilon(\hat{a}\hat{b} - \hat{a}^\dagger\hat{b}^\dagger), \tag{2.2}$$

where $\varepsilon = \lambda\gamma$. In addition, the interaction of the subharmonic light modes with the three-level atom at resonance can be described by the Hamiltonian

$$\hat{H}'' = ig(\hat{\sigma}_a^\dagger\hat{a} - \hat{a}^\dagger\hat{\sigma}_a + \hat{\sigma}_b^\dagger\hat{b} - \hat{b}^\dagger\hat{\sigma}_b), \tag{2.3}$$

where g is the coupling constant between the atom and light mode a or b, $\hat{\sigma}_a$ and $\hat{\sigma}_b$ are lowering atomic operators defined by

$$\hat{\sigma}_a = |b\rangle\langle a|, \tag{2.4}$$

$$\hat{\sigma}_b = |c\rangle\langle b|. \tag{2.5}$$

Thus the interactions involving the cavity light modes are described by

$$\hat{H}_c = i\varepsilon(\hat{a}\hat{b} - \hat{a}^\dagger\hat{b}^\dagger) + ig(\hat{\sigma}_a^\dagger\hat{a} - \hat{a}^\dagger\hat{\sigma}_a + \hat{\sigma}_b^\dagger\hat{b} - \hat{b}^\dagger\hat{\sigma}_b) \tag{2.6}$$

and the interaction involving the three-atom is given by

$$\hat{H}_a = ig(\hat{\sigma}_a^\dagger\hat{a} - \hat{a}^\dagger\hat{\sigma}_a + \hat{\sigma}_b^\dagger\hat{b} - \hat{b}^\dagger\hat{\sigma}_b). \tag{2.7}$$

In accordance with Ref. [41], we fix the Hamiltonian given by Eqs. (2.6) and (2.7) at the initial time wether we are working in the Schrodinger or Heisenberg picture. We assume that the two-mode cavity light is coupled to a two-mode vacuum reservoir via a single-port mirror. Moreover, we carry out our analysis by putting the noise operators associated with the vacuum reservoir in arbitrary order.

2.1 Equations of evolution of cavity mode operators

Now we seek to obtain the equations of evolution of the expectation values for the cavity mode operators employing the master equation [18]

$$\begin{aligned}\frac{d}{dt}\hat{\rho}(t) &= -i\big[\hat{H}_c(0), \hat{\rho}(t)\big] + \frac{\kappa}{2}\bigg(2\hat{a}(0)\hat{\rho}(t)\hat{a}^\dagger(0) - \hat{a}^\dagger(0)\hat{a}(0)\hat{\rho}(t) - \hat{\rho}(t)\hat{a}^\dagger(0)\hat{a}(0)\bigg)\\ &\quad+\frac{\kappa}{2}\bigg(2\hat{b}(0)\hat{\rho}(t)\hat{b}^\dagger(0) - \hat{b}^\dagger(0)\hat{b}(0)\hat{\rho}(t) - \hat{\rho}(t)\hat{b}^\dagger(0)\hat{b}(0)\bigg).\end{aligned} \tag{2.8}$$

Then on account of Eq. (2.6), we have

$$\begin{aligned}\frac{d}{dt}\hat{\rho}(t) &= -i\big[i\varepsilon(\hat{a}(0)\hat{b}(0) - \hat{a}^\dagger(0)\hat{b}^\dagger(0)) + ig(\hat{\sigma}_a^\dagger(0)\hat{a}(0) - \hat{a}^\dagger(0)\hat{\sigma}_a(0) + \hat{\sigma}_b^\dagger(0)\hat{b}(0)\\ &\quad-\hat{b}^\dagger(0)\hat{\sigma}_b)(0), \hat{\rho}(t)\big] + \frac{\kappa}{2}\bigg(2\hat{a}(0)\hat{\rho}(t)\hat{a}^\dagger(0) - \hat{a}^\dagger(0)\hat{a}\hat{\rho}(t) - \hat{\rho}(t)\hat{a}^\dagger(0)\hat{a}(0)\bigg)\\ &\quad+\frac{\kappa}{2}\bigg(2\hat{b}(0)\hat{\rho}(t)\hat{b}^\dagger(0) - \hat{b}^\dagger(0)\hat{b}(0)\hat{\rho}(t) - \hat{\rho}(t)\hat{b}^\dagger(0)\hat{b}(0)\bigg).\end{aligned} \tag{2.9}$$

We recall that the equation of evolution for an operator $\hat{A}$ is expressible as

$$\frac{d}{dt}\langle\hat{A}\rangle = Tr\bigg(\frac{d}{dt}\hat{\rho}(t)\hat{A}\bigg). \tag{2.10}$$

Hence employing this relation along with Eq. (2.8), one can write

$$\frac{d}{dt}\langle\hat{a}(t)\rangle = Tr\bigg(-i\big[\hat{H}_c(0),\hat{\rho}(t)\big]\hat{a}(0)+\frac{\kappa}{2}\bigg(2\hat{a}(0)\hat{\rho}(t)\hat{a}^\dagger(0)-\hat{a}^\dagger(0)\hat{a}(0)\hat{\rho}(t) -\hat{\rho}(t)\hat{a}^\dagger(0)\hat{a}(0)\bigg)\hat{a}(0). \quad (2.11)$$

We note that

$$Tr\bigg(-i\big[\hat{H}_c(0),\hat{\rho}(t)\big]\hat{a}(0)\bigg) = Tr\bigg(-i\big(\hat{H}_c(0)\hat{\rho}(t)\hat{a}(0)-\hat{\rho}(t)\hat{H}_c(0)\hat{a}(0)\big)\bigg). \quad (2.12)$$

Applying the cyclic property of the trace operation, we find

$$Tr\bigg(-i\big[\hat{H}_c(0),\hat{\rho}(t)\big]\hat{a}(0)\bigg) = Tr\bigg(-i\hat{\rho}(t)\big[\hat{a}(0),\hat{H}_c(0)\big]\bigg). \quad (2.13)$$

In view of (2.6), the above equation turns out to be

$$Tr\bigg(-i\hat{\rho}(t)\big[\hat{a}(0),\hat{H}_c(0)\big]\bigg) = Tr\bigg(\varepsilon\hat{\rho}(t)\bigg(\big[\hat{a}(0),\hat{a}(0)\hat{b}(0)\big]-\big[\hat{a}(0),\hat{a}^\dagger(0)\hat{b}^\dagger(0)\big]\bigg) +g\hat{\rho}(t)\bigg(\big[\hat{a}(0),\hat{\sigma}_a^\dagger(0)\hat{a}(0)\big]-\big[\hat{a}(0),\hat{a}^\dagger(0)\hat{\sigma}_a(0)\big] +\big[\hat{a}(0),\hat{\sigma}_b^\dagger(0)\hat{b}(0)\big]-\big[\hat{a}(0),\hat{b}^\dagger(0)\hat{\sigma}_b(0)\big]\bigg)\bigg). \quad (2.14)$$

Assuming that $[\hat{a}(0),\hat{a}^\dagger(0)]=1$ [41] and taking into consideration the fact that $\hat{a}$ and $\hat{b}$ commute, we get

$$Tr\bigg(-i\hat{\rho}(t)\big[\hat{a}(0),\hat{H}_c(0)\big]\bigg) = -\varepsilon\langle\hat{b}^\dagger(t)\rangle - g\langle\hat{\sigma}_a(t)\rangle. \quad (2.15)$$

In view of this result and the fact that

$$\frac{\kappa}{2}\bigg(2\hat{a}(0)\hat{\rho}(t)\hat{a}^\dagger(0)-\hat{a}^\dagger(0)\hat{a}(0)\hat{\rho}(t)-\rho(t)\hat{a}^\dagger(0)\hat{a}(0)\bigg)\hat{a}(0) = -\frac{\kappa}{2}\langle\hat{a}(t)\rangle, \quad (2.16)$$

(2.11) takes the form

$$\frac{d}{dt}\langle\hat{a}(t)\rangle = -\frac{\kappa}{2}\langle\hat{a}(t)\rangle - \varepsilon\langle\hat{b}^\dagger(t)\rangle - g\langle\hat{\sigma}_a(t)\rangle. \quad (2.17)$$

Following the similar procedure, one can also readily obtain

$$\frac{d}{dt}\langle\hat{b}(t)\rangle = -\frac{\kappa}{2}\langle\hat{b}(t)\rangle - \varepsilon\langle\hat{a}^\dagger(t)\rangle - g\langle\hat{\sigma}_b(t)\rangle, \quad (2.18)$$

$$\frac{d}{dt}\langle\hat{a}^{\dagger}(t)\hat{a}(t)\rangle = -\kappa\langle\hat{a}^{\dagger}(t)\hat{a}(t)\rangle - \varepsilon\big\langle\hat{b}(t)\hat{a}(t) + \hat{a}^{\dagger}(t)\hat{b}^{\dagger}(t)\big\rangle - g\big\langle\hat{\sigma}_a^{\dagger}(t)\hat{a}(t) + \hat{a}^{\dagger}(t)\hat{\sigma}_a(t)\big\rangle, \tag{2.19}$$

$$\frac{d}{dt}\langle\hat{a}(t)\hat{a}^{\dagger}(t)\rangle = -\kappa\langle\hat{a}(t)\hat{a}^{\dagger}(t)\rangle - \varepsilon\big\langle\hat{a}(t)\hat{b}(t) + \hat{b}^{\dagger}(t)\hat{a}^{\dagger}(t)\big\rangle - g\big\langle\hat{a}(t)\hat{\sigma}_a^{\dagger}(t) + \hat{\sigma}_a(t)\hat{a}^{\dagger}(t)\big\rangle + \kappa, \tag{2.20}$$

$$\frac{d}{dt}\langle\hat{b}^{\dagger}(t)\hat{b}(t)\rangle = -\kappa\langle\hat{b}^{\dagger}(t)\hat{b}(t)\rangle - \varepsilon\big\langle\hat{a}(t)\hat{b}(t) + \hat{b}^{\dagger}(t)\hat{a}^{\dagger}(t)\big\rangle - g\big\langle\hat{\sigma}_b^{\dagger}(t)\hat{b}(t) + \hat{b}^{\dagger}(t)\hat{\sigma}_b(t)\big\rangle, \tag{2.21}$$

$$\frac{d}{dt}\langle\hat{b}(t)\hat{b}^{\dagger}(t)\rangle = -\kappa\langle\hat{b}(t)\hat{b}^{\dagger}(t)\rangle - \varepsilon\big\langle\hat{b}(t)\hat{a}(t) + \hat{a}^{\dagger}(t)\hat{b}^{\dagger}(t)\big\rangle - g\big\langle\hat{b}(t)\hat{\sigma}_b^{\dagger}(t) + \hat{\sigma}_b(t)\hat{b}^{\dagger}(t)\big\rangle + \kappa, \tag{2.22}$$

$$\frac{d}{dt}\langle\hat{a}(t)\hat{b}(t)\rangle = -\kappa\langle\hat{a}(t)\hat{b}(t)\rangle - \varepsilon\big\langle\hat{a}^{\dagger}(t)\hat{a}(t) + \hat{b}^{\dagger}(t)\hat{b}(t)\big\rangle - g\big\langle\hat{\sigma}_a(t)\hat{b}(t) + \hat{a}(t)\hat{\sigma}_b(t)\big\rangle - \varepsilon, \tag{2.23}$$

$$\frac{d}{dt}\langle\hat{b}(t)\hat{a}(t)\rangle = -\kappa\langle\hat{b}(t)\hat{a}(t)\rangle - \varepsilon\big\langle\hat{a}^{\dagger}(t)\hat{a}(t) + \hat{b}^{\dagger}(t)\hat{b}(t)\big\rangle - g\big\langle\hat{b}(t)\hat{\sigma}_a(t) + \hat{\sigma}_b(t)\hat{a}(t)\big\rangle - \varepsilon, \tag{2.24}$$

$$\frac{d}{dt}\langle\hat{a}^{2}(t)\rangle = -\kappa\langle\hat{a}^{2}(t)\rangle - \varepsilon\big\langle\hat{a}(t)\hat{b}^{\dagger}(t) + \hat{b}^{\dagger}(t)\hat{a}(t)\big\rangle - g\big\langle\hat{a}(t)\hat{\sigma}_a(t) + \hat{\sigma}_a(t)\hat{a}(t)\big\rangle, \tag{2.25}$$

$$\frac{d}{dt}\langle\hat{b}^{2}(t)\rangle = -\kappa\langle\hat{b}^{2}(t)\rangle - \varepsilon\big\langle\hat{b}(t)\hat{a}^{\dagger}(t) + \hat{a}^{\dagger}(t)\hat{b}(t)\big\rangle - g\big\langle\hat{b}(t)\hat{\sigma}_b(t) + \hat{\sigma}_b(t)\hat{b}(t)\big\rangle, \tag{2.26}$$

$$\frac{d}{dt}\langle\hat{a}^{\dagger}(t)\hat{b}(t)\rangle = -\kappa\langle\hat{a}^{\dagger}(t)\hat{b}(t)\rangle - \varepsilon\big\langle\hat{a}^{\dagger 2}(t) + \hat{b}^{2}(t)\big\rangle - g\big\langle\hat{\sigma}_a^{\dagger}(t)\hat{b}(t) + \hat{a}^{\dagger}(t)\hat{\sigma}_b(t)\big\rangle, \tag{2.27}$$

and

$$\frac{d}{dt}\langle\hat{b}(t)\hat{a}^\dagger(t)\rangle = -\kappa\langle\hat{b}(t)\hat{a}^\dagger(t)\rangle - \varepsilon\big\langle\hat{a}^{\dagger 2}(t) + \hat{b}^2(t)\big\rangle$$
$$-g\big\langle\hat{b}(t)\hat{\sigma}_a^\dagger(t) + \hat{\sigma}_b(t)\hat{a}^\dagger(t)\big\rangle. \tag{2.28}$$

We see that Eqs. (2.19)-(2.28) are nonlinear differential equations and hence it is not possible to find the exact time-dependent solutions of these equations. To overcome this problem, we apply the large-time approximation scheme [18]. To this end, we rewrite Eqs. (2.17) and (2.18) as

$$\frac{d}{dt}\hat{a}(t) = -\frac{\kappa}{2}\hat{a}(t) - \varepsilon\hat{b}^\dagger(t) - g\hat{\sigma}_a(t) + \hat{F}_a(t), \tag{2.29}$$

$$\frac{d}{dt}\hat{b}(t) = -\frac{\kappa}{2}\hat{b}(t) - \varepsilon\hat{a}^\dagger(t) - g\hat{\sigma}_b(t) + \hat{F}_b(t), \tag{2.30}$$

where $\hat{F}_a(t)$ and $\hat{F}_b(t)$ are noise operators with vanishing mean and associated with the cavity mode operators $\hat{a}(t)$ and $\hat{b}(t)$, respectively. Now applying the large-time approximation scheme to Eqs. (2.29) and (2.30), we get the following approximately valid relations

$$\hat{a}(t) = -\frac{2\varepsilon}{\kappa}\hat{b}^\dagger(t) - \frac{2g}{\kappa}\hat{\sigma}_a(t) + \frac{2}{\kappa}\hat{F}_a(t) \tag{2.31}$$

and

$$\hat{b}(t) = -\frac{2\varepsilon}{\kappa}\hat{a}^\dagger(t) - \frac{2g}{\kappa}\hat{\sigma}_b(t) + \frac{2}{\kappa}\hat{F}_b(t). \tag{2.32}$$

It then follows that

$$\hat{a}(t) = \frac{4\varepsilon\kappa g}{\kappa^2 - 4\varepsilon^2}\left(\frac{\hat{\sigma}_b^\dagger(t)}{\kappa} - \frac{\hat{\sigma}_a(t)}{2\varepsilon} + \frac{\hat{F}_a(t)}{2\varepsilon g} - \frac{\hat{F}_b^\dagger(t)}{\kappa g}\right) \tag{2.33}$$

and

$$\hat{b}(t) = \frac{4\varepsilon\kappa g}{\kappa^2 - 4\varepsilon^2}\left(\frac{\hat{\sigma}_a^\dagger(t)}{\kappa} - \frac{\hat{\sigma}_b(t)}{2\varepsilon} + \frac{\hat{F}_b(t)}{2\varepsilon g} - \frac{\hat{F}_a^\dagger(t)}{\kappa g}\right). \tag{2.34}$$

Upon substituting Eqs. (2.33) and (2.34) together with their adjoint into Eqs. (2.19)-(2.28), we readily obtain

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}^{\dagger}(t)\hat{a}(t)\rangle &= -\kappa\langle\hat{a}^{\dagger}(t)\hat{a}(t)\rangle - \varepsilon\big\langle\hat{b}(t)\hat{a}(t) + \hat{a}^{\dagger}(t)\hat{b}^{\dagger}(t)\big\rangle \\ &\quad -\frac{4\varepsilon\kappa g^2}{\kappa^2-4\varepsilon^2}\Bigg[\frac{\langle\hat{\sigma}_c(t)+\hat{\sigma}_c^{\dagger}(t)\rangle}{\kappa} - \frac{\langle\hat{\eta}_a(t)\rangle}{\varepsilon} \\ &\quad +\Bigg\langle\hat{\sigma}_a^{\dagger}(t)\bigg(\frac{\hat{F}_a(t)}{2\varepsilon g} - \frac{\hat{F}_b^{\dagger}(t)}{\kappa g}\bigg) + \bigg(\frac{\hat{F}_a^{\dagger}(t)}{2\varepsilon g} \\ &\quad -\frac{\hat{F}_b(t)}{\kappa g}\bigg)\hat{\sigma}_a(t)\Bigg\rangle\Bigg], \qquad (2.35)\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}(t)\hat{a}^{\dagger}(t)\rangle &= -\kappa\langle\hat{a}(t)\hat{a}^{\dagger}(t)\rangle - \varepsilon\big\langle\hat{a}(t)\hat{b}(t) + \hat{b}^{\dagger}(t)\hat{a}^{\dagger}(t)\big\rangle \\ &\quad -\frac{4\varepsilon\kappa g^2}{\kappa^2-4\varepsilon^2}\Bigg[-\frac{\langle\hat{\eta}_b(t)\rangle}{\varepsilon} + \Bigg\langle\bigg(\frac{\hat{F}_a(t)}{2\varepsilon g} - \frac{\hat{F}_b^{\dagger}(t)}{\kappa g}\bigg)\hat{\sigma}_a^{\dagger}(t) \\ &\quad +\hat{\sigma}_a(t)\bigg(\frac{\hat{F}_a^{\dagger}(t)}{2\varepsilon g} - \frac{\hat{F}_b(t)}{\kappa g}\bigg)\Bigg\rangle\Bigg] + \kappa, \qquad (2.36)\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}\langle\hat{b}^{\dagger}(t)\hat{b}(t)\rangle &= -\kappa\langle\hat{b}^{\dagger}(t)\hat{b}(t)\rangle - \varepsilon\big\langle\hat{a}(t)\hat{b}(t) + \hat{b}^{\dagger}(t)\hat{a}^{\dagger}(t)\big\rangle \\ &\quad -\frac{4\varepsilon\kappa g^2}{\kappa^2-4\varepsilon^2}\Bigg[-\frac{\langle\hat{\eta}_b(t)\rangle}{\varepsilon} + \Bigg\langle\hat{\sigma}_b^{\dagger}(t)\bigg(\frac{\hat{F}_b(t)}{2\varepsilon g} \\ &\quad -\frac{\hat{F}_a^{\dagger}(t)}{\kappa g}\bigg) + \bigg(\frac{\hat{F}_b^{\dagger}(t)}{2\varepsilon g} - \frac{\hat{F}_a(t)}{\kappa g}\bigg)\hat{\sigma}_b(t)\Bigg\rangle\Bigg], \qquad (2.37)\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}\langle\hat{b}(t)\hat{b}^{\dagger}(t)\rangle &= -\kappa\langle\hat{b}(t)\hat{b}^{\dagger}(t)\rangle - \varepsilon\big\langle\hat{b}(t)\hat{a}(t) + \hat{a}^{\dagger}(t)\hat{b}^{\dagger}(t)\big\rangle \\ &\quad -\frac{4\varepsilon\kappa g^2}{\kappa^2-4\varepsilon^2}\Bigg[\frac{\langle\hat{\sigma}_c(t)+\hat{\sigma}_c^{\dagger}(t)\rangle}{\kappa} - \frac{\langle\hat{\eta}_c(t)\rangle}{\varepsilon} \\ &\quad +\Bigg\langle\bigg(\frac{\hat{F}_b(t)}{2\varepsilon g} - \frac{\hat{F}_a^{\dagger}(t)}{\kappa g}\bigg)\hat{\sigma}_b^{\dagger}(t) + \hat{\sigma}_b(t)\bigg(\frac{\hat{F}_b^{\dagger}(t)}{2\varepsilon g} \\ &\quad -\frac{\hat{F}_a(t)}{\kappa g}\bigg)\Bigg\rangle\Bigg] + \kappa, \qquad (2.38)\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}(t)\hat{b}(t)\rangle &= -\kappa\langle\hat{a}(t)\hat{b}(t)\rangle - \varepsilon\big\langle\hat{a}^{\dagger}(t)\hat{a}(t) + \hat{b}^{\dagger}(t)\hat{b}(t)\big\rangle \\ &\quad -\frac{4\varepsilon\kappa g^2}{\kappa^2-4\varepsilon^2}\Bigg[\frac{2\langle\hat{\eta}_b(t)\rangle}{\kappa} + \Bigg\langle\hat{\sigma}_a(t)\bigg(\frac{\hat{F}_b(t)}{2\varepsilon g} \\ &\quad -\frac{\hat{F}_a^{\dagger}(t)}{\kappa g}\bigg) + \bigg(\frac{\hat{F}_a(t)}{2\varepsilon g} - \frac{\hat{F}_b^{\dagger}(t)}{\kappa g}\bigg)\hat{\sigma}_b(t)\Bigg\rangle\Bigg] - \varepsilon, \qquad (2.39)\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}\langle\hat{b}(t)\hat{a}(t)\rangle &= -\kappa\langle\hat{b}(t)\hat{a}(t)\rangle - \varepsilon\big\langle\hat{a}^\dagger(t)\hat{a}(t) + \hat{b}^\dagger(t)\hat{b}(t)\big\rangle \\ &-\frac{4\varepsilon\kappa g^2}{\kappa^2-4\varepsilon^2}\Bigg[\frac{\langle\hat{\eta}_a(t)+\hat{\eta}_c(t)\rangle}{\kappa} - \frac{\langle\hat{\sigma}_c(t)\rangle}{\varepsilon} \\ &+\Bigg\langle\bigg(\frac{\hat{F}_b(t)}{2\varepsilon g} - \frac{\hat{F}_a^\dagger(t)}{\kappa g}\bigg)\hat{\sigma}_a(t) + \hat{\sigma}_b(t)\bigg(\frac{\hat{F}_a(t)}{2\varepsilon g} \\ &-\frac{\hat{F}_b^\dagger(t)}{\kappa g}\bigg)\Bigg\rangle\Bigg] - \varepsilon,\end{aligned} \tag{2.40}$$

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}^2(t)\rangle &= -\kappa\langle\hat{a}^2(t)\rangle - \varepsilon\big\langle\hat{a}(t)\hat{b}^\dagger(t) + \hat{b}^\dagger(t)\hat{a}(t)\big\rangle \\ &-\frac{4\varepsilon\kappa g^2}{\kappa^2-4\varepsilon^2}\Bigg[\Bigg\langle\bigg(\frac{\hat{F}_a(t)}{2\varepsilon g} - \frac{\hat{F}_b^\dagger(t)}{\kappa g}\bigg)\hat{\sigma}_a(t) \\ &+\hat{\sigma}_a(t)\bigg(\frac{\hat{F}_a(t)}{2\varepsilon g} - \frac{\hat{F}_b^\dagger(t)}{\kappa g}\bigg)\Bigg\rangle\Bigg],\end{aligned} \tag{2.41}$$

$$\begin{aligned}\frac{d}{dt}\langle\hat{b}^2(t)\rangle &= -\kappa\langle\hat{b}^2(t)\rangle - \varepsilon\big\langle\hat{b}(t)\hat{a}^\dagger(t) + \hat{a}^\dagger(t)\hat{b}(t)\big\rangle \\ &-\frac{4\varepsilon\kappa g^2}{\kappa^2-4\varepsilon^2}\Bigg[\Bigg\langle\bigg(\frac{\hat{F}_b(t)}{2\varepsilon g} - \frac{\hat{F}_a^\dagger(t)}{\kappa g}\bigg)\hat{\sigma}_b(t) \\ &+\hat{\sigma}_b(t)\bigg(\frac{\hat{F}_b(t)}{2\varepsilon g} - \frac{\hat{F}_a^\dagger(t)}{\kappa g}\bigg)\Bigg\rangle\Bigg],\end{aligned} \tag{2.42}$$

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}^\dagger(t)\hat{b}(t)\rangle &= -\kappa\langle\hat{a}^\dagger(t)\hat{b}(t)\rangle - \varepsilon\big\langle\hat{a}^{\dagger 2}(t) + \hat{b}^2(t)\big\rangle \\ &-\frac{4\varepsilon\kappa g^2}{\kappa^2-4\varepsilon^2}\Bigg[\Bigg\langle\hat{\sigma}_a^\dagger(t)\bigg(\frac{\hat{F}_b(t)}{2\varepsilon g} - \frac{\hat{F}_a^\dagger(t)}{\kappa g}\bigg) \\ &+\bigg(\frac{\hat{F}_a^\dagger(t)}{2\varepsilon g} - \frac{\hat{F}_b(t)}{\kappa g}\bigg)\hat{\sigma}_b(t)\Bigg\rangle\Bigg],\end{aligned} \tag{2.43}$$

$$\begin{aligned}\frac{d}{dt}\langle\hat{b}(t)\hat{a}^\dagger(t)\rangle &= -\kappa\langle\hat{b}(t)\hat{a}^\dagger(t)\rangle - \varepsilon\big\langle\hat{a}^{\dagger 2}(t) + \hat{b}^2(t)\big\rangle \\ &-\frac{4\varepsilon\kappa g^2}{\kappa^2-4\varepsilon^2}\Bigg[\Bigg\langle\bigg(\frac{\hat{F}_b(t)}{2\varepsilon g} - \frac{\hat{F}_a^\dagger(t)}{\kappa g}\bigg)\hat{\sigma}_a^\dagger(t) \\ &+\hat{\sigma}_b(t)\bigg(\frac{\hat{F}_a^\dagger(t)}{2\varepsilon g} - \frac{\hat{F}_b(t)}{\kappa g}\bigg)\Bigg\rangle\Bigg],\end{aligned} \tag{2.44}$$

where

$$\hat{\sigma}_c = |c\rangle\langle a|, \tag{2.45}$$

$$\hat{\eta}_a = |a\rangle\langle a|, \tag{2.46}$$

$$\hat{\eta}_b = |b\rangle\langle b|, \tag{2.47}$$

and

$$\hat{\eta}_c = |c\rangle\langle c|. \tag{2.48}$$

Since the atomic and the noise operators are not correlated, one can write [17]

$$\langle \hat{F}(t)\hat{\sigma}(t)\rangle = \langle \hat{F}(t)\rangle\langle\hat{\sigma}(t)\rangle = 0. \tag{2.49}$$

Hence Eqs. (2.35)-(2.44) take the form

$$\begin{aligned}\frac{d}{dt}\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle &= -\kappa\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle - \varepsilon\big\langle \hat{b}(t)\hat{a}(t) + \hat{a}^\dagger(t)\hat{b}^\dagger(t)\big\rangle \\ &\quad -\frac{4\varepsilon\kappa g^2}{\kappa^2-4\varepsilon^2}\left[\frac{\langle\hat{\sigma}_c(t)+\hat{\sigma}_c^\dagger(t)\rangle}{\kappa} - \frac{\langle\hat{\eta}_a(t)\rangle}{\varepsilon}\right],\end{aligned} \tag{2.50}$$

$$\begin{aligned}\frac{d}{dt}\langle \hat{a}(t)\hat{a}^\dagger(t)\rangle &= -\kappa\langle \hat{a}(t)\hat{a}^\dagger(t)\rangle - \varepsilon\big\langle \hat{a}(t)\hat{b}(t) + \hat{b}^\dagger(t)\hat{a}^\dagger(t)\big\rangle \\ &\quad +\frac{4\varepsilon\kappa g^2}{\kappa^2-4\varepsilon^2}\frac{\langle\hat{\eta}_b(t)\rangle}{\varepsilon} + \kappa,\end{aligned} \tag{2.51}$$

$$\begin{aligned}\frac{d}{dt}\langle \hat{b}^\dagger(t)\hat{b}(t)\rangle &= -\kappa\langle \hat{b}^\dagger(t)\hat{b}(t)\rangle - \varepsilon\big\langle \hat{a}(t)\hat{b}(t) + \hat{b}^\dagger(t)\hat{a}^\dagger(t)\big\rangle \\ &\quad +\frac{4\varepsilon\kappa g^2}{\kappa^2-4\varepsilon^2}\frac{\langle\hat{\eta}_b(t)\rangle}{\varepsilon},\end{aligned} \tag{2.52}$$

$$\begin{aligned}\frac{d}{dt}\langle \hat{b}(t)\hat{b}^\dagger(t)\rangle &= -\kappa\langle \hat{b}(t)\hat{b}^\dagger(t)\rangle - \varepsilon\big\langle \hat{b}(t)\hat{a}(t) + \hat{a}^\dagger(t)\hat{b}^\dagger(t)\big\rangle \\ &\quad -\frac{4\varepsilon\kappa g^2}{\kappa^2-4\varepsilon^2}\left[\frac{\langle\hat{\sigma}_c(t)+\hat{\sigma}_c^\dagger(t)\rangle}{\kappa} - \frac{\langle\hat{\eta}_c(t)\rangle}{\varepsilon}\right] + \kappa,\end{aligned} \tag{2.53}$$

$$\begin{aligned}\frac{d}{dt}\langle \hat{a}(t)\hat{b}(t)\rangle &= -\kappa\langle \hat{a}(t)\hat{b}(t)\rangle - \varepsilon\big\langle \hat{a}^\dagger(t)\hat{a}(t) + \hat{b}^\dagger(t)\hat{b}(t)\big\rangle \\ &\quad -\frac{4\varepsilon\kappa g^2}{\kappa^2-4\varepsilon^2}\frac{2\langle\hat{\eta}_b(t)\rangle}{\kappa} - \varepsilon,\end{aligned} \tag{2.54}$$

$$\begin{aligned}\frac{d}{dt}\langle \hat{b}(t)\hat{a}(t)\rangle &= -\kappa\langle \hat{b}(t)\hat{a}(t)\rangle - \varepsilon\big\langle \hat{a}^\dagger(t)\hat{a}(t) + \hat{b}^\dagger(t)\hat{b}(t)\big\rangle \\ &\quad -\frac{4\varepsilon\kappa g^2}{\kappa^2-4\varepsilon^2}\left[\frac{\langle\hat{\eta}_a(t)+\hat{\eta}_c(t)\rangle}{\kappa} - \frac{\langle\hat{\sigma}_c(t)\rangle}{\varepsilon}\right] - \varepsilon,\end{aligned} \tag{2.55}$$

$$\frac{d}{dt}\langle \hat{a}^2(t)\rangle = -\kappa\langle \hat{a}^2(t)\rangle - \varepsilon\big\langle \hat{a}(t)\hat{b}^\dagger(t) + \hat{b}^\dagger(t)\hat{a}(t)\big\rangle, \tag{2.56}$$

$$\frac{d}{dt}\langle \hat{b}^2(t)\rangle = -\kappa\langle \hat{b}^2(t)\rangle - \varepsilon\big\langle \hat{b}(t)\hat{a}^\dagger(t) + \hat{a}^\dagger(t)\hat{b}(t)\big\rangle, \tag{2.57}$$

$$\frac{d}{dt}\langle \hat{a}^\dagger(t)\hat{b}(t)\rangle = -\kappa\langle \hat{a}^\dagger(t)\hat{b}(t)\rangle - \varepsilon\big\langle \hat{a}^{\dagger 2}(t) + \hat{b}^2(t)\big\rangle, \tag{2.58}$$

and

$$\frac{d}{dt}\langle \hat{b}(t)\hat{a}^\dagger(t)\rangle = -\kappa\langle \hat{b}(t)\hat{a}^\dagger(t)\rangle - \varepsilon\big\langle \hat{a}^{\dagger 2}(t) + \hat{b}^2(t)\big\rangle. \tag{2.59}$$

The steady-state solutions of Eqs. (2.50)-(2.59) are found to be

$$\langle \hat{a}^\dagger\hat{a}\rangle = -\frac{\varepsilon}{\kappa}\big\langle \hat{b}\hat{a} + \hat{a}^\dagger\hat{b}^\dagger\big\rangle - \frac{4g^2}{\kappa^2 - 4\varepsilon^2}\left[\frac{\varepsilon\langle\hat{\sigma}_c + \hat{\sigma}_c^\dagger\rangle}{\kappa} - \langle\hat{\eta}_a\rangle\right], \tag{2.60}$$

$$\langle \hat{a}\hat{a}^\dagger\rangle = -\frac{\varepsilon}{\kappa}\big\langle \hat{a}\hat{b} + \hat{b}^\dagger\hat{a}^\dagger\big\rangle + \frac{4g^2}{\kappa^2 - 4\varepsilon^2}\langle\hat{\eta}_b\rangle + 1, \tag{2.61}$$

$$\langle \hat{b}^\dagger\hat{b}\rangle = -\frac{\varepsilon}{\kappa}\big\langle \hat{a}\hat{b} + \hat{b}^\dagger\hat{a}^\dagger\big\rangle + \frac{4g^2}{\kappa^2 - 4\varepsilon^2}\langle\hat{\eta}_b\rangle, \tag{2.62}$$

$$\langle \hat{b}\hat{b}^\dagger\rangle = -\frac{\varepsilon}{\kappa}\big\langle \hat{b}\hat{a} + \hat{a}^\dagger\hat{b}^\dagger\big\rangle - \frac{4g^2}{\kappa^2 - 4\varepsilon^2}\left[\frac{\varepsilon\langle\hat{\sigma}_c + \hat{\sigma}_c^\dagger\rangle}{\kappa} - \langle\hat{\eta}_c\rangle\right] + 1, \tag{2.63}$$

$$\langle \hat{a}\hat{b}\rangle = -\frac{\varepsilon}{\kappa}\big\langle \hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b}\big\rangle - \frac{4g^2}{\kappa^2 - 4\varepsilon^2}\frac{2\varepsilon\langle\hat{\eta}_b\rangle}{\kappa} - \frac{\varepsilon}{\kappa}, \tag{2.64}$$

$$\langle \hat{b}\hat{a}\rangle = -\frac{\varepsilon}{\kappa}\big\langle \hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b}\big\rangle - \frac{4g^2}{\kappa^2 - 4\varepsilon^2}\left[\frac{\varepsilon\langle\hat{\eta}_a + \hat{\eta}_c\rangle}{\kappa} - \langle\hat{\sigma}_c\rangle\right] - \frac{\varepsilon}{\kappa}, \tag{2.65}$$

$$\langle \hat{a}^2\rangle = -\frac{\varepsilon}{\kappa}\left\langle \hat{a}\hat{b}^\dagger + \hat{b}^\dagger\hat{a}\right\rangle, \tag{2.66}$$

$$\langle \hat{b}^2\rangle = -\frac{\varepsilon}{\kappa}\left\langle \hat{b}\hat{a}^\dagger + \hat{a}^\dagger\hat{b}\right\rangle, \tag{2.67}$$

$$\langle \hat{a}^\dagger\hat{b}\rangle = -\frac{\varepsilon}{\kappa}\left\langle \hat{a}^{\dagger 2} + \hat{b}^2\right\rangle, \tag{2.68}$$

and

$$\langle \hat{b}\hat{a}^\dagger\rangle = -\frac{\varepsilon}{\kappa}\left\langle \hat{a}^{\dagger 2} + \hat{b}^2\right\rangle. \tag{2.69}$$

Now with the aid of Eq. (2.67) and the complex conjugate of Eq. (2.66), Eqs. (2.68) and (2.69) can be written as

$$\langle\hat{a}^\dagger\hat{b}\rangle\left(\frac{\kappa^2-2\varepsilon^2}{2\varepsilon^2}\right)=\langle\hat{b}\hat{a}^\dagger\rangle \tag{2.70}$$

and

$$\langle\hat{b}\hat{a}^\dagger\rangle\left(\frac{\kappa^2-2\varepsilon^2}{2\varepsilon^2}\right)=\langle\hat{a}^\dagger\hat{b}\rangle. \tag{2.71}$$

Then from Eqs. (2.70) and (2.71), we obtain

$$\langle\hat{a}^\dagger\hat{b}\rangle=\left(\frac{\kappa^2-2\varepsilon^2}{2\varepsilon^2}\right)^2\langle\hat{a}^\dagger\hat{b}\rangle. \tag{2.72}$$

This shows that

$$\langle\hat{a}^\dagger\hat{b}\rangle=0. \tag{2.73}$$

Moreover, on the basis of Eqs. (2.70) and (2.71), one can also easily establish that

$$\langle\hat{b}\hat{a}^\dagger\rangle=\left(\frac{\kappa^2-2\varepsilon^2}{2\varepsilon^2}\right)^2\langle\hat{b}\hat{a}^\dagger\rangle, \tag{2.74}$$

so that

$$\langle\hat{b}\hat{a}^\dagger\rangle=0. \tag{2.75}$$

Therefore, in view of Eqs. (2.73) and (2.75) along with their complex conjugates, Eqs. (2.66) and (2.67), become

$$\langle\hat{a}^2\rangle=\langle\hat{b}^2\rangle=0. \tag{2.76}$$

Furthermore, using Eqs.(2.64) and (2.65) along with their complex conjugates in Eqs. (2.60) and (2.62), we easily find

$$\begin{aligned}\langle\hat{a}^\dagger\hat{a}\rangle &= \frac{2\varepsilon^2}{\kappa^2-2\varepsilon^2}+\frac{2\varepsilon^2}{\kappa^2-2\varepsilon^2}\langle\hat{b}^\dagger\hat{b}\rangle+\frac{4g^2}{(\kappa^2-2\varepsilon^2)(\kappa^2-4\varepsilon^2)}\Big(2\varepsilon^2\langle\hat{\eta}_c\rangle\\ &\quad+\langle\hat{\eta}_a\rangle(2\varepsilon^2+\kappa^2)-2\varepsilon\kappa\langle\hat{\sigma}_c+\hat{\sigma}_c^\dagger\rangle\Big)\end{aligned} \tag{2.77}$$

and

$$\langle \hat{b}^\dagger \hat{b}\rangle = \frac{2\varepsilon^2}{\kappa^2-2\varepsilon^2}+\frac{2\varepsilon^2}{\kappa^2-2\varepsilon^2}\langle \hat{a}^\dagger\hat{a}\rangle+\frac{4g^2}{(\kappa^2-2\varepsilon^2)(\kappa^2-4\varepsilon^2)}\Big(4\varepsilon^2 +\kappa^2\Big)\langle\hat{\eta}_b\rangle. \tag{2.78}$$

On account of Eqs. (2.77) and (2.78), we readily obtain

$$\langle \hat{a}^\dagger \hat{a}\rangle = \frac{2\varepsilon^2}{\kappa^2-4\varepsilon^2}+\frac{4g^2}{\kappa^2(\kappa^2-4\varepsilon^2)^2}\Big((\kappa^4-4\varepsilon^4)\langle\hat{\eta}_a\rangle +2\varepsilon^2(4\varepsilon^2+\kappa^2)\langle\hat{\eta}_b\rangle+2\varepsilon^2(\kappa^2-2\varepsilon^2)\langle\hat{\eta}_c\rangle -2\varepsilon\kappa(\kappa^2-2\varepsilon^2)\big\langle\hat{\sigma}_c+\hat{\sigma}_c^\dagger\big\rangle\Big) \tag{2.79}$$

and

$$\langle \hat{b}^\dagger \hat{b}\rangle = \frac{2\varepsilon^2}{\kappa^2-4\varepsilon^2}+\frac{4g^2}{\kappa^2(\kappa^2-4\varepsilon^2)^2}\Big(2\varepsilon^2(\kappa^2+2\varepsilon^2)\langle\hat{\eta}_a\rangle +(\kappa^2+4\varepsilon^2)(\kappa^2-2\varepsilon^2)\langle\hat{\eta}_b\rangle+4\varepsilon^4\langle\hat{\eta}_c\rangle -4\varepsilon^3\kappa\big\langle\hat{\sigma}_c+\hat{\sigma}_c^\dagger\big\rangle\Big). \tag{2.80}$$

In addition, employing Eqs. (2.64) and (2.65) along with their complex conjugates, Eqs. (2.61) and (2.63) can be put in the form

$$\langle \hat{a}\hat{a}^\dagger\rangle = 1+\frac{2\varepsilon^2}{\kappa^2-4\varepsilon^2}+\frac{4g^2}{\kappa^2(\kappa^2-4\varepsilon^2)^2}\Big(2\varepsilon^2(\kappa^2+2\varepsilon^2)\langle\hat{\eta}_a\rangle +(\kappa^2+4\varepsilon^2)(\kappa^2-2\varepsilon^2)\langle\hat{\eta}_b\rangle+4\varepsilon^4\langle\hat{\eta}_c\rangle -4\varepsilon^3\kappa\big\langle\hat{\sigma}_c+\hat{\sigma}_c^\dagger\big\rangle\Big) \tag{2.81}$$

and

$$\langle \hat{b}\hat{b}^\dagger\rangle = 1+\frac{2\varepsilon^2}{\kappa^2-4\varepsilon^2}+\frac{4g^2}{\kappa^2(\kappa^2-4\varepsilon^2)^2}\Big(4\varepsilon^2(\kappa^2-\varepsilon^2)\langle\hat{\eta}_a\rangle +2\varepsilon^2(4\varepsilon^2+\kappa^2)\langle\hat{\eta}_b\rangle+(\kappa^4-2\varepsilon^2(\kappa^2+2\varepsilon^2))\langle\hat{\eta}_c\rangle -2\varepsilon\kappa(\kappa^2-2\varepsilon^2)\big\langle\hat{\sigma}_c+\hat{\sigma}_c^\dagger\big\rangle\Big). \tag{2.82}$$

2.2 Equations of evolution of atomic operators

Next we seek to obtain the equations of evolution of the expectation values for the atomic operators. Since we consider the case in which the three-level atom doesn't interact with the vacuum reservoir, the master equation for the atom can be written as [42]

$$\frac{d}{dt}\hat{\rho}(t) = -i\big[\hat{H}_a(0), \hat{\rho}(t)\big]. \tag{2.83}$$

The equation of evolution for the expectation value of an atomic operator $\hat{\sigma}(t)$ is expressible as

$$\frac{d}{dt}\langle\hat{\sigma}(t)\rangle = Tr\bigg(\frac{d}{dt}\hat{\rho}(t)\hat{\sigma}(0)\bigg), \tag{2.84}$$

so that in view of Eq. (2.83), there follows

$$\frac{d}{dt}\langle\hat{\sigma}(t)\rangle = -iTr\bigg(\big[\hat{H}_a(0), \hat{\rho}(t)\big]\hat{\sigma}(0)\bigg). \tag{2.85}$$

Now, using the cyclic property of the trace operation, we can write this equation as

$$\frac{d}{dt}\langle\hat{\sigma}(t)\rangle = -iTr\bigg(\hat{\rho}(t)\big[\hat{\sigma}(0), \hat{H}_a(0)\big]\bigg). \tag{2.86}$$

Employing this relation along with Eq. (2.7), we readily get

$$\frac{d}{dt}\langle\hat{\sigma}_a(t)\rangle = g\bigg\langle\big(\hat{\eta}_b(t) - \hat{\eta}_a(t)\big)\hat{a}(t) + \hat{b}^\dagger(t)\hat{\sigma}_c(t)\bigg\rangle, \tag{2.87}$$

$$\frac{d}{dt}\langle\hat{\sigma}_b(t)\rangle = g\bigg\langle -\hat{a}^\dagger(t)\hat{\sigma}_c(t) + \big(\hat{\eta}_c(t) - \hat{\eta}_b(t)\big)\hat{b}(t)\bigg\rangle, \tag{2.88}$$

$$\frac{d}{dt}\langle\hat{\sigma}_c(t)\rangle = g\bigg\langle\hat{\sigma}_b(t)\hat{a}(t) - \hat{\sigma}_a(t)\hat{b}(t)\bigg\rangle, \tag{2.89}$$

$$\frac{d}{dt}\langle\hat{\eta}_a(t)\rangle = g\bigg\langle\hat{\sigma}_a^\dagger(t)\hat{a}(t) + \hat{a}^\dagger(t)\hat{\sigma}_a(t)\bigg\rangle, \tag{2.90}$$

$$\frac{d}{dt}\langle\hat{\eta}_b(t)\rangle = g\bigg\langle\hat{\sigma}_b^\dagger(t)\hat{b}(t) + \hat{b}^\dagger(t)\hat{\sigma}_b(t) - \big(\hat{\sigma}_a^\dagger(t)\hat{a}(t) + \hat{a}^\dagger(t)\hat{\sigma}_a(t)\big)\bigg\rangle, \tag{2.91}$$

$$\frac{d}{dt}\langle\hat{\eta}_c(t)\rangle = -g\bigg\langle\hat{\sigma}_b^\dagger(t)\hat{b}(t) + \hat{b}^\dagger(t)\hat{\sigma}_b(t)\bigg\rangle. \tag{2.92}$$

In view of Eqs. (2.33) and (2.34) together with their adjoints, Eqs. (2.87)-(2.92) take the form

$$\begin{aligned}\frac{d}{dt}\langle\hat{\sigma}_a(t)\rangle &= \frac{\kappa^2\varepsilon\gamma_c}{\kappa^2-4\varepsilon^2}\bigg(\bigg\langle-\frac{\hat{\sigma}_a(t)}{2\varepsilon}+\frac{\hat{\sigma}_b^\dagger(t)}{\kappa}\bigg\rangle+\bigg\langle\big(\hat{\eta}_b(t)-\hat{\eta}_a(t)\big)\\&\times\bigg(\frac{\hat{F}_a(t)}{2g\varepsilon}-\frac{\hat{F}_b^\dagger(t)}{g\kappa}\bigg)+\bigg(\frac{\hat{F}_b^\dagger(t)}{2g\varepsilon}-\frac{\hat{F}_a(t)}{g\kappa}\bigg)\hat{\sigma}_c(t)\bigg\rangle\bigg),\end{aligned} \tag{2.93}$$

$$\begin{aligned}\frac{d}{dt}\langle\hat{\sigma}_b(t)\rangle &= \frac{\kappa^2\gamma_c\varepsilon}{\kappa^2-4\varepsilon^2}\bigg(\frac{-1}{2\varepsilon}\langle\hat{\sigma}_b(t)\rangle+\bigg\langle-\bigg(\frac{\hat{F}_a^\dagger(t)}{2g\varepsilon}-\frac{\hat{F}_b(t)}{g\kappa}\bigg)\hat{\sigma}_c(t)\\&+\big(\hat{\eta}_c(t)-\hat{\eta}_b(t)\big)\bigg(\frac{\hat{F}_b(t)}{2g\varepsilon}-\frac{\hat{F}_a^\dagger(t)}{g\kappa}\bigg)\bigg\rangle\bigg),\end{aligned} \tag{2.94}$$

$$\begin{aligned}\frac{d}{dt}\langle\hat{\sigma}_c(t)\rangle &= \frac{\kappa^2\varepsilon\gamma_c}{4\varepsilon^2-\kappa^2}\bigg(\bigg\langle\frac{\hat{\sigma}_c(t)}{2\varepsilon}+\frac{1}{\kappa}\big(\hat{\eta}_b(t)-\hat{\eta}_c(t)\big)\bigg\rangle\\&+\bigg\langle\hat{\sigma}_b(t)\bigg(\frac{\hat{F}_a(t)}{2g\varepsilon}-\frac{\hat{F}_b^\dagger(t)}{g\kappa}\bigg)\\&-\hat{\sigma}_a(t)\bigg(\frac{\hat{F}_b(t)}{2g\varepsilon}-\frac{\hat{F}_a^\dagger(t)}{g\kappa}\bigg)\bigg\rangle\bigg),\end{aligned} \tag{2.95}$$

$$\begin{aligned}\frac{d}{dt}\langle\hat{\eta}_a(t)\rangle &= \frac{\kappa^2\varepsilon\gamma_c}{4\varepsilon^2-\kappa^2}\bigg(\bigg\langle-\frac{1}{\kappa}\big(\hat{\sigma}_c^\dagger(t)+\hat{\sigma}_c(t)\big)+\frac{\hat{\eta}_a}{\varepsilon}\bigg\rangle\\&+\bigg\langle\hat{\sigma}_a^\dagger(t)\bigg(\frac{\hat{F}_a(t)}{2g\varepsilon}-\frac{\hat{F}_b^\dagger(t)}{g\kappa}\bigg)\\&+\bigg(\frac{\hat{F}_a^\dagger(t)}{2g\varepsilon}-\frac{\hat{F}_b(t)}{g\kappa}\bigg)\hat{\sigma}_a(t)\bigg\rangle\bigg),\end{aligned} \tag{2.96}$$

$$\begin{aligned}\frac{d}{dt}\langle\hat{\eta}_b(t)\rangle &= \frac{\kappa^2\varepsilon\gamma_c}{4\varepsilon^2-\kappa^2}\bigg(\bigg\langle\frac{1}{\kappa}\big(\hat{\sigma}_c^\dagger(t)+\hat{\sigma}_c(t)\big)+\frac{1}{\varepsilon}\big(\hat{\eta}_b(t)-\hat{\eta}_a(t)\big)\bigg\rangle\\&+\bigg\langle\hat{\sigma}_b^\dagger(t)\bigg(\frac{\hat{F}_b(t)}{2g\varepsilon}-\frac{\hat{F}_a^\dagger(t)}{g\kappa}\bigg)+\bigg(\frac{\hat{F}_b^\dagger(t)}{2g\varepsilon}-\frac{\hat{F}_a(t)}{g\kappa}\bigg)\hat{\sigma}_b(t)\\&-\big(\hat{\sigma}_a^\dagger(t)\bigg(\frac{\hat{F}_a(t)}{2g\varepsilon}-\frac{\hat{F}_b^\dagger(t)}{g\kappa}\bigg)\\&+\bigg(\frac{\hat{F}_a^\dagger(t)}{2g\varepsilon}-\frac{\hat{F}_b(t)}{g\kappa}\bigg)\hat{\sigma}_a(t)\big)\bigg\rangle\bigg),\end{aligned} \tag{2.97}$$

$$\begin{aligned}\frac{d}{dt}\langle\hat{\eta}_c(t)\rangle &= -\frac{\kappa^2\varepsilon\gamma_c}{4\varepsilon^2-\kappa^2}\bigg(\bigg\langle\frac{\hat{\eta}_b(t)}{\epsilon}\bigg\rangle+\bigg\langle\hat{\sigma}_b^\dagger(t)\bigg(\frac{\hat{F}_b(t)}{2g\varepsilon}-\frac{\hat{F}_a^\dagger(t)}{g\kappa}\bigg)\\&+\bigg(\frac{\hat{F}_b^\dagger(t)}{2g\varepsilon}-\frac{\hat{F}_a(t)}{g\kappa}\bigg)\hat{\sigma}_b(t)\bigg\rangle\bigg),\end{aligned} \tag{2.98}$$

where $\gamma_c = \frac{4g^2}{\kappa}$ is the stimulated emission decay constant [18]. On account of Eq. (2.49), the above equations reduce to

$$\frac{d}{dt}\langle\hat{\sigma}_a(t)\rangle = -\frac{\kappa^2\varepsilon\gamma_c}{\kappa^2-4\varepsilon^2}\bigg\langle\frac{\hat{\sigma}_a(t)}{2\varepsilon} - \frac{\hat{\sigma}_b^\dagger(t)}{\kappa}\bigg\rangle, \tag{2.99}$$

$$\frac{d}{dt}\langle\hat{\sigma}_b(t)\rangle = -\frac{1}{2}\frac{\kappa^2\gamma_c}{\kappa^2-4\varepsilon^2}\langle\hat{\sigma}_b(t)\rangle, \tag{2.100}$$

$$\frac{d}{dt}\langle\hat{\sigma}_c(t)\rangle = \frac{\kappa^2\varepsilon\gamma_c}{4\varepsilon^2-\kappa^2}\bigg\langle\frac{\hat{\sigma}_c(t)}{2\varepsilon} + \frac{1}{\kappa}\big(\hat{\eta}_b(t)-\hat{\eta}_c(t)\big)\bigg\rangle, \tag{2.101}$$

$$\frac{d}{dt}\langle\hat{\eta}_a(t)\rangle = \frac{\kappa^2\varepsilon\gamma_c}{4\varepsilon^2-\kappa^2}\bigg\langle -\frac{1}{\kappa}\big(\hat{\sigma}_c^\dagger(t)+\hat{\sigma}_c(t)\big) + \frac{\hat{\eta}_a}{\varepsilon}\bigg\rangle, \tag{2.102}$$

$$\frac{d}{dt}\langle\hat{\eta}_b(t)\rangle = \frac{\kappa^2\varepsilon\gamma_c}{4\varepsilon^2-\kappa^2}\bigg\langle\frac{1}{\kappa}\big(\hat{\sigma}_c^\dagger(t)+\hat{\sigma}_c(t)\big) + \frac{1}{\varepsilon}\big(\hat{\eta}_b(t)-\hat{\eta}_a(t)\big)\bigg\rangle, \tag{2.103}$$

$$\frac{d}{dt}\langle\hat{\eta}_c(t)\rangle = -\frac{\kappa^2\varepsilon\gamma_c}{4\varepsilon^2-\kappa^2}\bigg\langle\frac{\hat{\eta}_b(t)}{\epsilon}\bigg\rangle. \tag{2.104}$$

The steady-state solutions of Eqs. (2.101)-(2.103) are found to be of the form

$$\langle\hat{\sigma}_c\rangle = \frac{2\varepsilon}{\kappa}\bigg(\langle\hat{\eta}_c\rangle - \langle\hat{\eta}_b\rangle\bigg), \tag{2.105}$$

$$\langle\hat{\sigma}_c^\dagger\rangle + \langle\hat{\sigma}_c\rangle = \frac{\kappa}{\varepsilon}\langle\hat{\eta}_a\rangle, \tag{2.106}$$

$$\langle\hat{\eta}_a\rangle - \langle\hat{\eta}_b\rangle = \frac{\varepsilon}{\kappa}\bigg(\langle\hat{\sigma}_c^\dagger\rangle + \langle\hat{\sigma}_c\rangle\bigg). \tag{2.107}$$

Now from Eqs. (2.106) and (2.107), one can easily get

$$\langle\hat{\eta}_b\rangle = 0. \tag{2.108}$$

Upon substituting Eq. (2.105) along with its complex conjugate into Eq. (2.106) and taking into account Eq.(2.108), we readily obtain

$$\langle\hat{\eta}_c\rangle = \frac{\kappa^2}{4\varepsilon^2}\langle\hat{\eta}_a\rangle. \tag{2.109}$$

The completeness relation for the three-level atom is given by

$$\langle\hat{\eta}_a\rangle + \langle\hat{\eta}_b\rangle + \langle\hat{\eta}_c\rangle = 1, \tag{2.110}$$

where $\langle\hat{\eta}_a\rangle$, $\langle\hat{\eta}_b\rangle$, and $\langle\hat{\eta}_c\rangle$ are the probabilities to find the atom in the top, intermediate, and bottom levels, respectively. Therefore, applying Eqs. (2.108) and (2.109), we arrive at

$$\langle\hat{\eta}_a\rangle = \frac{4\varepsilon^2}{\kappa^2 + 4\varepsilon^2} \tag{2.111}$$

and

$$\langle\hat{\eta}_c\rangle = \frac{\kappa^2}{\kappa^2 + 4\varepsilon^2}. \tag{2.112}$$

Finally, combination of Eqs. (2.105), (2.108), and (2.112) leads to

$$\langle\hat{\sigma}_c\rangle = \frac{2\varepsilon\kappa}{\kappa^2 + 4\varepsilon^2}. \tag{2.113}$$

3

Photon Statistics

Applying the steady-state solutions of the equations of evolution of the expectation values for the cavity mode and atomic operators, we seek to obtain the global mean photon number and photon number variance for the two-mode cavity light. In addition, using the time dependent solutions of the equations of evolution of the expectation values, we obtain the local mean photon number.

3.1 Global mean photon number

We seek to calculate the global mean photon number for the two-mode cavity light, which is the superposition of subharmonic light modes a and b. The annihilation operator $\hat{c}$ for the two-mode cavity light can be defined by [18]

$$\hat{c} = \hat{a} + \hat{b}. \tag{3.1}$$

The mean photon number for the two-mode cavity light is given by

$$\bar{n} = \langle \hat{c}^{\dagger} \hat{c} \rangle. \tag{3.2}$$

On the basis of Eq. (3.1) and its adjoint, we arrive at

$$\bar{n} = \langle \hat{a}^{\dagger} \hat{a} \rangle + \langle \hat{b}^{\dagger} \hat{b} \rangle + \langle \hat{a}^{\dagger} \hat{b} \rangle + \langle \hat{b}^{\dagger} \hat{a} \rangle. \tag{3.3}$$

Then on account of Eq. (2.73) together with its complex conjugate, we note that

$$\bar{n} = \langle \hat{a}^{\dagger} \hat{a} \rangle + \langle \hat{b}^{\dagger} \hat{b} \rangle. \tag{3.4}$$

We see that the mean photon number of the two-mode cavity light is the sum of the mean photon numbers of light modes a and b. Thus applying Eqs. (2.79) and (2.80), we get

$$\bar{n} = \frac{4\varepsilon^2}{\kappa^2 - 4\varepsilon^2} + \frac{\kappa\gamma_c}{(\kappa^2 - 4\varepsilon^2)^2}\bigg((2\varepsilon^2 + \kappa^2)\langle\hat{\eta}_a\rangle + (4\varepsilon^2 + \kappa^2)\langle\hat{\eta}_b\rangle + 2\varepsilon^2\langle\hat{\eta}_c\rangle - 2\varepsilon\kappa\Big\langle\hat{\sigma}_c + \hat{\sigma}_c^\dagger\Big\rangle\bigg), \tag{3.5}$$

so that employing Eqs. (2.106) and (2.108), Eq. (3.5) can be put in the form

$$\bar{n} = \frac{4\varepsilon^2}{\kappa^2 - 4\varepsilon^2} - \frac{\kappa\gamma_c}{(\kappa^2 - 4\varepsilon^2)^2}\bigg((\kappa^2 - 2\varepsilon^2)\langle\hat{\eta}_a\rangle - 2\varepsilon^2\langle\hat{\eta}_c\rangle\bigg). \tag{3.6}$$

Now on account of Eqs. (2.108) and (2.110), Eq. (3.6) takes the form

$$\bar{n} = \frac{4\varepsilon^2}{\kappa^2 - 4\varepsilon^2} + \frac{\kappa\gamma_c}{(\kappa^2 - 4\varepsilon^2)^2}\bigg(2\varepsilon^2 - \kappa^2\langle\hat{\eta}_a\rangle\bigg). \tag{3.7}$$

With the aid of Eqs. (2.111) and (2.112), Eq. (3.7) can be put in the form

$$\bar{n} = \frac{4\varepsilon^2}{\kappa^2 - 4\varepsilon^2} + \frac{\kappa\gamma_c}{(\kappa^2 - 4\varepsilon^2)^2}\bigg(\frac{\kappa^2 + 4\varepsilon^2}{2}\langle\hat{\eta}_a\rangle - 4\varepsilon^2\langle\hat{\eta}_c\rangle\bigg). \tag{3.8}$$

It proves to be more convenient to rewrite Eq. (3.8) as

$$\bar{n} = \frac{4\varepsilon^2}{\kappa^2 - 4\varepsilon^2} + f_e(\varepsilon) - f_a(\varepsilon), \tag{3.9}$$

in which

$$f_e(\varepsilon) = \frac{\kappa\gamma_c}{(\kappa^2 - 4\varepsilon^2)^2}\bigg(\frac{\kappa^2 + 4\varepsilon^2}{2}\bigg)\langle\hat{\eta}_a\rangle \tag{3.10}$$

and

$$f_a(\varepsilon) = \frac{4\varepsilon^2\kappa\gamma_c}{(\kappa^2 - 4\varepsilon^2)^2}\langle\hat{\eta}_c\rangle. \tag{3.11}$$

We realize that $f_e(\varepsilon)$ and $f_a(\varepsilon)$ are the mean number of photons emitted and the mean number of photons absorbed, respectively. From the plots in Fig.3.1, we observe that the effect of the interaction of the subharmonic light modes with the three-level atom is to decrease the mean photon number of the two-mode cavity

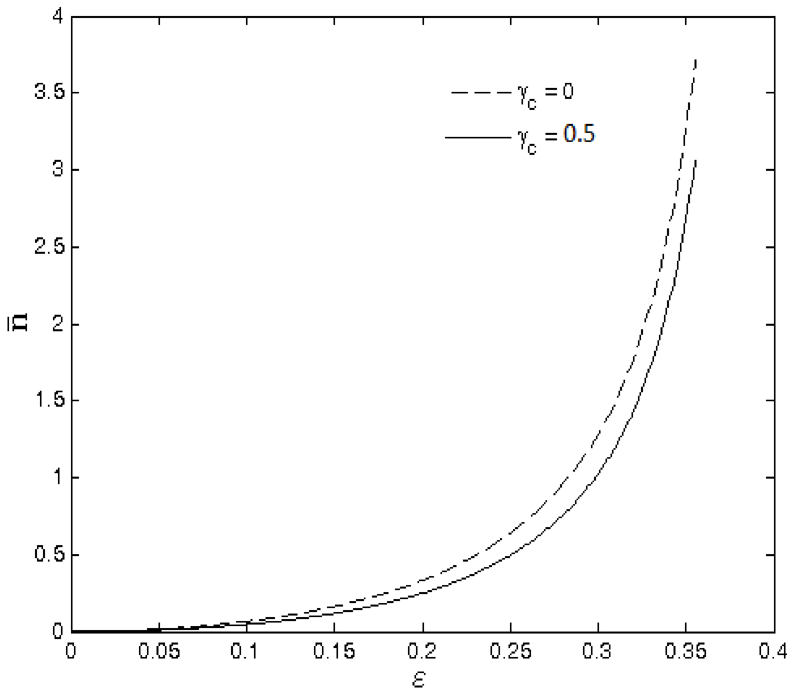

Figure 3.1: Plots of the global mean photon number [Eq. (3.8)] versus ε for $\kappa = 0.8$.

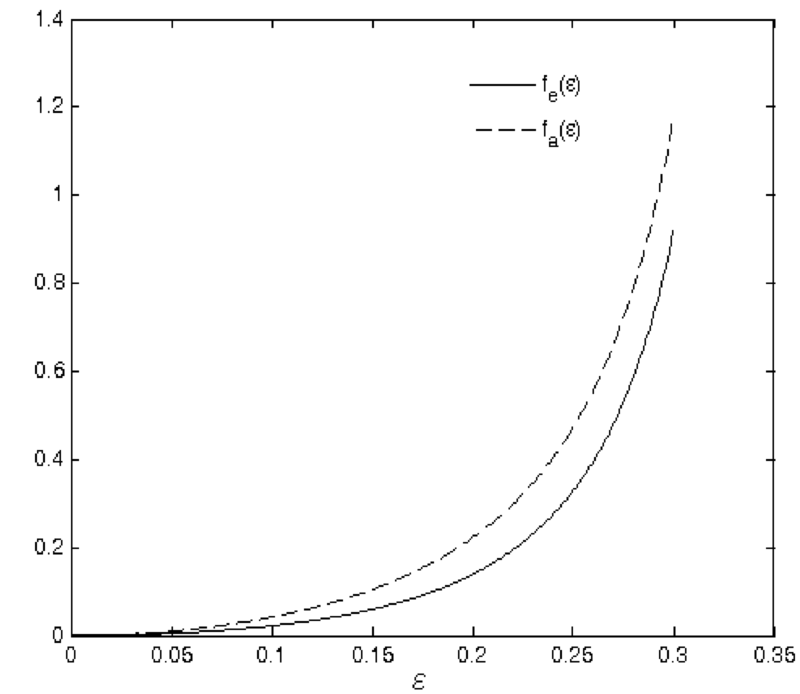

Figure 3.2: Plots of the mean photon number emitted and absorbed [Eqs. (3.10) and (3.11)] versus ε for $\kappa = 0.8$ and $\gamma_c = 0.5$.

light. As seen from the plots in Fig.3.2, the decrease in the mean photon number is due to the fact that the mean number of photons emitted is less than the mean number of photons absorbed.

We now consider the special case in which the subharmonic light modes don't interact with the three-level atom. Hence upon setting $\gamma_c = 0$, Eq. (3.8) reduces to [19]

$$\bar{n} = \frac{4\varepsilon^2}{\kappa^2 - 4\varepsilon^2}. \quad (3.12)$$

This represents the mean photon number of the two-mode subharmonic light in the absence of interaction with the three-level atom.

3.2 Local mean photon number

To determine the local mean photon number in a given frequency interval, we first obtain the power spectrum for the two-mode cavity light when both light modes a and b have the same central frequency ω_0. The power spectrum for the two-mode cavity light is expressible as

$$P(\omega) = \frac{1}{\pi} Re \int_0^\infty d\tau exp\Big(i(\omega - \omega_0)\tau\Big)\langle \hat{c}^\dagger(t)\hat{c}(t+\tau)\rangle_{ss}. \quad (3.13)$$

Upon integrating both sides of Eq. (3.13) over ω, we readily get

$$\int_{-\infty}^{\infty} P(\omega) d\omega = \bar{n}, \quad (3.14)$$

in which $\bar{n}$ is the steady-state mean photon number of the two-mode cavity light. From this result, we observe that $P(\omega)d\omega$ is the stead-state mean photon number of the two-mode cavity light in the frequency interval between ω and $\omega + d\omega$ [17-18].

Now we proceed to determine the two-time correlation function that appears in Eq. (3.13) for the two-mode cavity light. To this end, applying the large-time approximation scheme to Eq. (2.100), we get

$$\langle\hat{\sigma}_b(t)\rangle = 0. \tag{3.15}$$

Then employing the complex conjugate of this result, one can put Eq. (2.99) in the form

$$\frac{d}{dt}\langle\hat{\sigma}_a\rangle = -\frac{1}{2}\frac{\kappa^2\gamma_c}{\kappa^2 - 4\varepsilon^2}\langle\hat{\sigma}_a(t)\rangle. \tag{3.16}$$

We notice that the solution of this equation of evolution is expressible as

$$\langle\hat{\sigma}_a(t)\rangle = \langle\hat{\sigma}_a(0)\rangle exp\left(\frac{-\mu}{2}t\right), \tag{3.17}$$

where

$$\mu = \frac{\kappa^2\gamma_c}{\kappa^2 - 4\varepsilon^2}. \tag{3.18}$$

With the atom considered to be initially in the bottom level, Eq. (3.17) turns out to be

$$\langle\hat{\sigma}_a(t)\rangle = 0. \tag{3.19}$$

Moreover, the solution of Eq. (2.100) is found to be

$$\langle\hat{\sigma}_b(t)\rangle = \langle\hat{\sigma}_b(0)\rangle exp\left(\frac{-\mu}{2}t\right) \tag{3.20}$$

and with the atom considered to be initially in the same level, we have

$$\langle\hat{\sigma}_b(t)\rangle = 0. \tag{3.21}$$

Now substituting the complex conjugate of the expectation value of Eq. (2.32) into Eq. (2.17), we get

$$\frac{d}{dt}\langle\hat{a}(t)\rangle = -\left(\frac{\kappa^2 - 4\varepsilon^2}{2\kappa}\right)\langle\hat{a}(t)\rangle + g\left(\frac{2\varepsilon}{\kappa}\langle\hat{\sigma}_b^\dagger(t)\rangle - \langle\hat{\sigma}_a(t)\rangle\right) \tag{3.22}$$

and on taking into account Eq. (3.15), we have

$$\frac{d}{dt}\langle\hat{a}(t)\rangle = -\left(\frac{\kappa^2 - 4\varepsilon^2}{2\kappa}\right)\langle\hat{a}(t)\rangle - g\langle\hat{\sigma}_a(t)\rangle. \tag{3.23}$$

The solution of this equation is a well-behaved function provided that

$$\frac{\kappa^2 - 4\varepsilon^2}{2\kappa} > 0. \tag{3.24}$$

It then follows that

$$\varepsilon < \frac{\kappa}{2}. \tag{3.25}$$

We identify this relation as the threshold condition. We realize that the the solution of Eq. (3.23) is expressible for $\varepsilon < \frac{\kappa}{2}$ as

$$\begin{aligned} \langle \hat{a}(t) \rangle &= \langle \hat{a}(0) \rangle exp\bigg(-\frac{\beta}{2} t \bigg) \\ &\quad -g \int_0^t exp\bigg(-\frac{\beta}{2}(t-t') \bigg) \langle \hat{\sigma}_a(t') \rangle dt', \end{aligned} \tag{3.26}$$

in which

$$\beta = \frac{\kappa^2 - 4\varepsilon^2}{\kappa}, \tag{3.27}$$

Hence with the assumption that the cavity light is initially in a vacuum state and in view of Eq. (3.19), Eq. (3.26) turns out to be

$$\langle \hat{a}(t) \rangle = 0. \tag{3.28}$$

Applying the large-time approximation scheme to Eq. (2.99), we obtain

$$\langle \hat{\sigma}_a(t) \rangle = \frac{2\varepsilon}{\kappa} \langle \hat{\sigma}_b^\dagger(t) \rangle. \tag{3.29}$$

Using the complex conjugate of the expectation value of Eq. (2.31) into Eq.(2.18), we obtain

$$\frac{d}{dt}\langle \hat{b}(t) \rangle = -\bigg(\frac{\kappa^2 - 4\varepsilon^2}{2\kappa} \bigg) \langle \hat{b}(t) \rangle + g\bigg(\frac{2\varepsilon}{\kappa} \langle \hat{\sigma}_a^\dagger(t) \rangle - \langle \hat{\sigma}_b(t) \rangle \bigg) \tag{3.30}$$

and employing the complex conjugate of Eq. (3.29), we arrive at

$$\frac{d}{dt}\langle \hat{b}(t) \rangle = -\bigg(\frac{\kappa^2 - 4\varepsilon^2}{2\kappa} \bigg) \langle \hat{b}(t) \rangle - \frac{g}{\kappa^2} \bigg(\kappa^2 - 4\varepsilon^2 \bigg) \langle \hat{\sigma}_b(t) \rangle. \tag{3.31}$$

Moreover, we realize that the solution of the above equation of evolution is expressible as

$$\begin{aligned}\langle \hat{b}(t)\rangle &= \langle \hat{b}(0)\rangle exp\Big(-\frac{\beta}{2}t\Big)\\ &-\frac{g\gamma_c}{\mu}\int_0^t exp\Big(-\frac{\beta}{2}(t-t')\Big)\langle\hat{\sigma}_b(t')\rangle dt',\end{aligned} \tag{3.32}$$

and with the aid of Eq. (3.21) and with the assumption that the cavity light is initially in a vacuum state, Eq. (3.32) is found to be

$$\langle \hat{b}(t)\rangle = 0. \tag{3.33}$$

Therefore, on account of Eqs. (3.28) and (3.33), Eq. (3.1) takes the form

$$\langle \hat{c}(t)\rangle = 0. \tag{3.34}$$

In addition, on adding Eqs. (3.23) and (3.31), we find

$$\frac{d}{dt}\langle \hat{c}(t)\rangle = -\Big(\frac{\kappa^2-4\varepsilon^2}{2\kappa}\Big)\langle \hat{c}(t)\rangle - g\langle\hat{\sigma}_0(t)\rangle, \tag{3.35}$$

where

$$\hat{\sigma}_0(t) = \hat{\sigma}_a(t) + \frac{\gamma_c}{\mu}\hat{\sigma}_b(t). \tag{3.36}$$

In view of Eqs. (3.34) and (3.35), the annihilation operator of the two-mode cavity light $\hat{c}$ is a Gaussian variable with zero mean. Now one can rewrite Eqs. (2.100), (3.16) and (3.35) as

$$\frac{d}{dt}\hat{\sigma}_b(t) = -\frac{\mu}{2}\hat{\sigma}_b(t) + \hat{f}_b(t), \tag{3.37}$$

$$\frac{d}{dt}\hat{\sigma}_a(t) = -\frac{\mu}{2}\hat{\sigma}_a(t) + \hat{f}_a(t) \tag{3.38}$$

and

$$\frac{d}{dt}\hat{c}(t) = -\frac{\beta}{2}\hat{c}(t) - g\hat{\sigma}_0 + \hat{F}_c(t), \tag{3.39}$$

where $\hat{f}_b$ and $\hat{f}_a$ are the noise operators associated with the atomic operators $\hat{\sigma}_b$ and $\hat{\sigma}_a$, respectively. And $\hat{F}_c$ is the noise operator associated with the cavity mode operator $\hat{c}$. Then we realize that the solutions of Eqs. (3.37), (3.38), and (3.39) are given by

$$\begin{aligned}\hat{\sigma}_b(t+\tau) &= \hat{\sigma}_b(t)exp\Big(-\frac{1}{2}\mu\tau\Big)+\int_0^\tau d\tau' exp\Big(-\frac{1}{2}\mu(\tau-\tau')\Big)\\ &\quad\times\hat{f}_b(t+\tau'),\end{aligned} \tag{3.40}$$

$$\begin{aligned}\hat{\sigma}_a(t+\tau) &= \hat{\sigma}_a(t)exp\Big(-\frac{1}{2}\mu\tau\Big)+\int_0^\tau d\tau' exp\Big(-\frac{1}{2}\mu(\tau-\tau')\Big)\\ &\quad\times\hat{f}_a(t+\tau')\end{aligned} \tag{3.41}$$

and

$$\begin{aligned}\hat{c}(t+\tau) &= \hat{c}(t)exp\Big(-\frac{1}{2}\beta\tau\Big)-g\int_0^\tau d\tau' exp\Big(-\frac{1}{2}\beta(\tau-\tau')\Big)\\ &\quad\times\Big[\hat{\sigma}_0(t+\tau')-\frac{1}{g}\hat{F}_c(t+\tau')\Big].\end{aligned} \tag{3.42}$$

Moreover, in view of Eq. (3.36), Eq. (3.42) takes the form

$$\begin{aligned}\hat{c}(t+\tau) &= \hat{c}(t)exp\Big(-\frac{1}{2}\beta\tau\Big)-g\int_0^\tau d\tau' exp\Big(-\frac{1}{2}\beta(\tau-\tau')\Big)\\ &\quad\times\Big[\hat{\sigma}_a(t+\tau')+\frac{1}{\mu}\hat{\sigma}_b(t+\tau')-\frac{1}{g}\hat{F}_c(t+\tau')\Big],\end{aligned} \tag{3.43}$$

so that employing Eqs. (3.40) and (3.41), we get

$$\begin{aligned}\hat{c}(t+\tau) &= \hat{c}(t)exp\Big(-\frac{1}{2}\beta\tau\Big)-g\int_0^\tau d\tau' exp\Big(-\frac{1}{2}\beta(\tau-\tau')\Big)\\ &\quad\times\Big[exp\Big(-\frac{1}{2}\mu\gamma_c\tau'\Big)\Big(\hat{\sigma}_a(t)+\frac{1}{\mu}\hat{\sigma}_b(t)\Big)\\ &\quad+\int_0^{\tau'} d\tau'' exp\Big(\frac{-1}{2}\gamma_c\mu(\tau'-\tau'')\Big)\Big(\hat{f}_a(t+\tau'')\\ &\quad+\frac{1}{\mu}\hat{f}_b(t+\tau'')\Big)-\frac{1}{g}\hat{F}_c(t+\tau')\Big].\end{aligned} \tag{3.44}$$

Now multiplying the above equation on the left by $\hat{c}^\dagger(t)$ and taking the expectation value of the resulting equation, we have

$$\begin{aligned}\langle\hat{c}^\dagger(t)\hat{c}(t+\tau)\rangle &= \langle\hat{c}^\dagger(t)\hat{c}(t)\rangle exp\biggl(-\frac{1}{2}\beta\tau\biggr) - g\int_0^\tau d\tau' exp\biggl(-\frac{1}{2}\beta(\tau-\tau')\biggr)\\ &\times\biggl[exp\biggl(-\frac{1}{2}\mu\gamma_c\tau'\biggr)\biggl\langle\hat{c}^\dagger(t)\biggl[\hat{\sigma}_a(t)+\frac{1}{\mu}\hat{\sigma}_b(t)\biggr]\biggr\rangle\\ &+\int_0^{\tau'} d\tau'' exp\biggl(-\frac{1}{2}\gamma_c\mu(\tau'-\tau'')\biggr)\biggl\langle\hat{c}^\dagger(t)\biggl(\hat{f}_a(t+\tau'')\\ &+\frac{1}{\mu}\hat{f}_b(t+\tau'')\biggr)\biggr\rangle - \frac{1}{g}\langle\hat{c}^\dagger(t)\hat{F}_c(t+\tau')\rangle\biggr] \qquad (3.45)\end{aligned}$$

In view of the fact that the noise operators $\hat{F}_c(t+\tau)$ at a certain time does not affect the cavity mode operator at earlier time, we can write

$$\langle\hat{c}^\dagger(t)\hat{F}_c(t+\tau')\rangle = \langle\hat{c}(t)\rangle\langle\hat{F}_c^\dagger(t+\tau')\rangle = 0. \qquad (3.46)$$

On the other hand since the atomic noise operators and the cavity mode operators are not correlated, one can write

$$\langle\hat{c}^\dagger(t)\hat{f}_a(t+\tau')\rangle = \langle\hat{c}^\dagger(t)\rangle\langle\hat{f}_a(t+\tau')\rangle = 0 \qquad (3.47)$$

and

$$\langle\hat{c}^\dagger(t)\hat{f}_b(t+\tau')\rangle = \langle\hat{c}^\dagger(t)\rangle\langle\hat{f}_b(t+\tau')\rangle = 0. \qquad (3.48)$$

Then Eq. (3.45) reduces to

$$\begin{aligned}\langle\hat{c}^\dagger(t)\hat{c}(t+\tau)\rangle &= \langle\hat{c}^\dagger(t)\hat{c}(t)\rangle exp\biggl(-\frac{1}{2}\beta\tau\biggr) - g\int_0^\tau d\tau' exp\biggl(-\frac{1}{2}\beta(\tau-\tau')\biggr)\\ &\times exp\biggl(-\frac{1}{2}\mu\gamma_c\tau'\biggr)\biggl\langle\hat{c}^\dagger(t)\biggl(\hat{\sigma}_a(t)+\frac{1}{\mu}\hat{\sigma}_b(t)\biggr)\biggr\rangle. \qquad (3.49)\end{aligned}$$

Upon applying the large-time approximation to Eq. (3.39), we get the approximately valid relation

$$\frac{\beta}{2}\hat{c}(t) - \hat{F}_c(t) = -g\biggl(\hat{\sigma}_a(t)+\frac{1}{\mu}\hat{\sigma}_b(t)\biggr) = -g\hat{\sigma}_0(t), \qquad (3.50)$$

so that on account of this result, Eq. (3.49) takes the form

$$\begin{aligned} \langle \hat{c}^{\dagger}(t)\hat{c}(t+\tau)\rangle &= \langle \hat{c}^{\dagger}(t)\hat{c}(t)\rangle exp\Big(-\frac{1}{2}\beta\tau\Big) + \int_0^{\tau} d\tau' exp\Big(-\frac{1}{2}\beta(\tau-\tau')\Big) \\ &\times exp\Big(-\frac{1}{2}\mu\tau'\Big)\Big\langle \hat{c}^{\dagger}(t)\Big(\frac{\beta}{2}\hat{c}(t) - \hat{F}_c(t)\Big)\Big\rangle. \end{aligned} \tag{3.51}$$

Furthermore, we notice that the solution of Eq. (3.39) to gether with Eq. (3.36) is given by

$$\begin{aligned} \hat{c}(t) &= \hat{c}(0) exp\Big(-\frac{1}{2}\beta t\Big) - g\int_0^{t} dt' exp\Big(-\frac{1}{2}\beta(t-t')\Big) \\ &\times \Big[\hat{\sigma}_0(t') - \frac{1}{g}\hat{F}_c(t')\Big] \end{aligned} \tag{3.52}$$

and taking the adjoint of this equation, we have

$$\begin{aligned} \hat{c}^{\dagger}(t) &= \hat{c}^{\dagger}(0) exp\Big(-\frac{1}{2}\beta t\Big) - g\int_0^{t} dt' exp\Big(-\frac{1}{2}\beta(t-t')\Big) \\ &\times \Big[\hat{\sigma}_0^{\dagger}(t') - \frac{1}{g}\hat{F}_c^{\dagger}(t')\Big]. \end{aligned} \tag{3.53}$$

Now multiplying this resulting equation by $\hat{F}_c(t)$ from the right side, we arrive at

$$\begin{aligned} \hat{c}^{\dagger}(t)\hat{F}_c(t) &= \hat{c}^{\dagger}(0)\hat{F}_c(t) exp\Big(-\frac{1}{2}\beta t\Big) - g\int_0^{t} dt' exp\Big(-\frac{1}{2}\beta(t-t')\Big) \\ &\times \Big[\hat{\sigma}_0^{\dagger}(t') - \frac{1}{g}\hat{F}_c^{\dagger}(t')\Big]\hat{F}_c(t) \end{aligned} \tag{3.54}$$

Upon taking the expectation value of the above equation, we see that

$$\begin{aligned} \langle \hat{c}^{\dagger}(t)\hat{F}_c(t)\rangle &= \langle \hat{c}^{\dagger}(0)\hat{F}_c(t)\rangle exp\Big(-\frac{1}{2}\beta t\Big) - g\int_0^{t} dt' exp\Big(-\frac{1}{2}\beta(t-t')\Big) \\ &\times \Big\langle \hat{\sigma}_0^{\dagger}(t')\hat{F}_c(t) - \frac{1}{g}\hat{F}_c^{\dagger}(t')\hat{F}_c(t)\Big\rangle \end{aligned} \tag{3.55}$$

Since the system under consideration is coupled to a vacuum reservoir, then

$$\langle \hat{F}_c^{\dagger}(t')\hat{F}_c(t)\rangle = 0 \tag{3.56}$$

and applying Eqs. (2.49) and (3.46), the expectation value of the Eq. (3.55) reduces to

$$\langle \hat{c}^{\dagger}(t)\hat{F}_c(t)\rangle = 0 \tag{3.57}$$

Therefore, taking into account Eq. (3.57), Eq. (3.51) can be put in the form

$$\langle \hat{c}^{\dagger}(t)\hat{c}(t+\tau)\rangle = \langle \hat{c}^{\dagger}(t)\hat{c}(t)\rangle exp\Big(-\frac{1}{2}\beta\tau\Big)\Big[1+\frac{\beta}{2}\int_{0}^{\tau} d\tau' exp\Big(\frac{1}{2}(\beta -\mu)\tau'\Big)\Big], \tag{3.58}$$

so that on carrying out the integration, one readily gets

$$\langle \hat{c}^{\dagger}(t)\hat{c}(t+\tau)\rangle_{ss} = \bar{n}\Big[\frac{\beta}{\beta-\mu}exp\Big(-\frac{1}{2}\mu\tau\Big) -\frac{\mu}{\beta-\mu}exp\Big(-\frac{1}{2}\beta\tau\Big)\Big]. \tag{3.59}$$

On introducing Eq. (3.59) into Eq. (3.13), the power spectrum of the two-mode cavity light takes the form

$$P(\omega) = \frac{\bar{n}}{\pi}Re\Big[\frac{\beta}{\beta-\mu}\int_{0}^{\infty} d\tau exp\Big(-(\frac{1}{2}\mu - i(\omega-\omega_0))\tau\Big) -\frac{\mu}{\beta-\mu}\int_{0}^{\infty} d\tau exp\Big(-(\frac{1}{2}\beta - i(\omega-\omega_0))\tau\Big)\Big], \tag{3.60}$$

so that on carrying out the integration, there follows

$$P(\omega) = \frac{\beta\bar{n}}{\beta-\mu}\Big(\frac{\frac{\mu}{2\pi}}{(\frac{\mu}{2})^2+(\omega-\omega_0)^2}\Big) - \frac{\mu\bar{n}}{\beta-\mu}\Big(\frac{\frac{\beta}{2\pi}}{(\frac{\beta}{2})^2+(\omega-\omega_0)^2}\Big). \tag{3.61}$$

We recall that the mean photon number in the interval between $\omega' = -\lambda$ and $\omega' = \lambda$ is expressible as [18]

$$\bar{n}_{\pm\lambda} = \int_{-\lambda}^{\lambda} d\omega' P(\omega'), \tag{3.62}$$

in which $\omega' = \omega - \omega_0$. Therefore, upon substituting Eq. (3.61) into Eq. (3.62) and carrying out the integration, we arrive at [17]

$$\bar{n}_{\pm\lambda} = \bar{n}\Big[\frac{\frac{2\beta}{\pi}}{\beta-\mu}tan^{-1}\Big(\frac{2\lambda}{\mu}\Big) - \frac{\frac{2\mu}{\pi}}{\beta-\mu}tan^{-1}\Big(\frac{2\lambda}{\beta}\Big)\Big]. \tag{3.63}$$

We note that the mean photon number of the two-mode cavity light in the frequency interval between $\omega' = -\lambda$ and $\omega' = \lambda$ is expressible as

$$\bar{n}_{\pm\lambda} = \bar{n}z(\lambda), \tag{3.64}$$

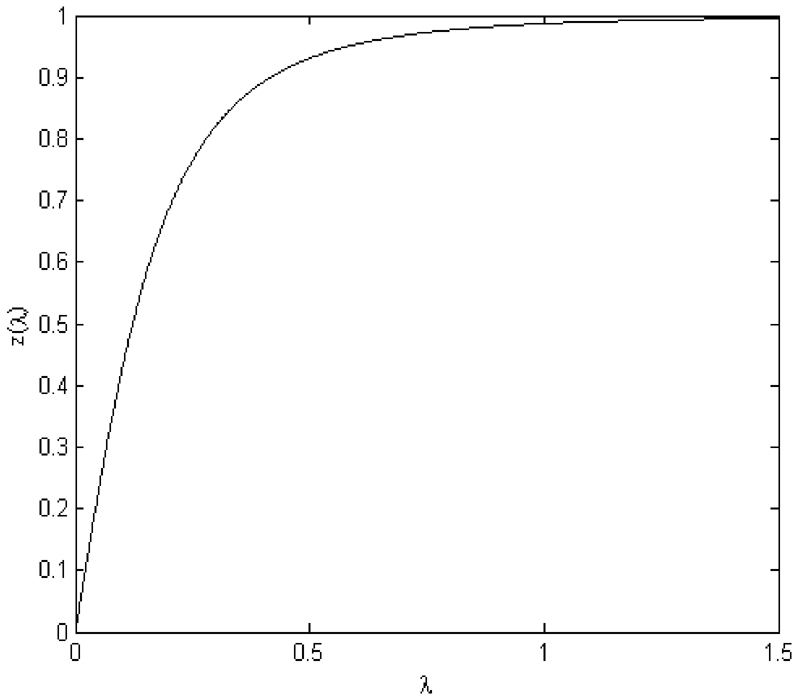

Figure 3.3: Plot of the local mean photon number [Eq. (3.65)] versus λ for $\varepsilon = 0.3$, $\kappa = 0.8$, and $\gamma_c = 0.5$.

where the function $z(\lambda)$ is given by

$$z(\lambda) = \frac{\frac{2\beta}{\pi}}{\beta - \mu} tan^{-1}\left(\frac{2\lambda}{\mu}\right) - \frac{\frac{2\mu}{\pi}}{\beta - \mu} tan^{-1}\left(\frac{2\lambda}{\beta}\right). \tag{3.65}$$

We observe from the plot in Fig.3.3, that the local mean photon number increases with λ. In addition, form the plot along with Eq. (3.64), we see that the local mean photon number approaches to the global mean photon number. On the other hand, we notice that a large part of the total mean photon number is confined within a relatively small frequency interval.

3.3 Global photon number variance

The global photon number variance of the two-mode cavity light at steady-state is defined by

$$(\Delta n)^2 = \langle (\hat{c}^\dagger \hat{c})^2 \rangle - \langle \hat{c}^\dagger \hat{c} \rangle^2. \tag{3.66}$$

Since $\hat{c}$ is a Gaussian variable with zero mean, the photon number variance of the two-mode cavity light can be put in the form [18]

$$(\Delta n)^2 = \langle \hat{c}^\dagger \hat{c} \rangle \langle \hat{c} \hat{c}^\dagger \rangle + \langle \hat{c}^{\dagger 2} \rangle \langle \hat{c}^2 \rangle. \tag{3.67}$$

Now we proceed to obtain the explicit expressions for $\langle \hat{c}\hat{c}^\dagger \rangle$ and $\langle \hat{c}^2 \rangle$. To this end, applying Eq. (3.1), we easily obtain

$$\langle \hat{c}\hat{c}^\dagger \rangle = \langle \hat{a}\hat{a}^\dagger + \hat{b}\hat{b}^\dagger + \hat{a}\hat{b}^\dagger + \hat{b}\hat{a}^\dagger \rangle, \tag{3.68}$$

so that on account of Eq. (2.75) and its complex conjugate there follows

$$\langle \hat{c}\hat{c}^\dagger \rangle = \langle \hat{a}\hat{a}^\dagger + \hat{b}\hat{b}^\dagger \rangle. \tag{3.69}$$

Then upon substituting Eqs. (2.81) and (2.82) into this equation, we readily get

$$\begin{aligned} \langle \hat{c}\hat{c}^\dagger \rangle &= 2 + \frac{4\varepsilon^2}{\kappa^2 - 4\varepsilon^2} + \frac{\kappa\gamma_c}{(\kappa^2 - 4\varepsilon^2)^2}\Big(6\varepsilon^2 \langle \hat{\eta}_a \rangle + (4\varepsilon^2 + \kappa^2)\langle \hat{\eta}_b \rangle \\ &\quad + (\kappa^2 - 2\varepsilon^2)\langle \hat{\eta}_c \rangle - 2\varepsilon\kappa \big\langle \hat{\sigma}_c + \hat{\sigma}_c^\dagger \big\rangle\Big). \end{aligned} \tag{3.70}$$

With the aid of Eqs. (2.106) and (2.108), Eq. (3.70) can be written as

$$\begin{aligned} \langle \hat{c}\hat{c}^\dagger \rangle &= 2 + \frac{4\varepsilon^2}{\kappa^2 - 4\varepsilon^2} + \frac{\kappa\gamma_c}{(\kappa^2 - 4\varepsilon^2)^2}\Big((6\varepsilon^2 - 2\kappa^2)\langle \hat{\eta}_a \rangle \\ &\quad + (\kappa^2 - 2\varepsilon^2)\langle \hat{\eta}_c \rangle\Big). \end{aligned} \tag{3.71}$$

In addition, applying Eq. (3.1), we easily get

$$\langle \hat{c}^2 \rangle = \langle \hat{a}\hat{b} + \hat{b}\hat{a} + \hat{a}^2 + \hat{b}^2 \rangle \tag{3.72}$$

and with the help of Eq. (2.76), we find

$$\langle \hat{c}^2 \rangle = \langle \hat{a}\hat{b} + \hat{b}\hat{a} \rangle. \tag{3.73}$$

Hence using Eqs. (2.64) and (2.65) in Eq. (3.73), we obtain

$$\begin{aligned} \langle \hat{c}^2 \rangle &= -\frac{2\varepsilon}{\kappa}\big\langle \hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b} \big\rangle - \frac{4g^2}{\kappa^2 - 4\varepsilon^2}\left[\frac{2\varepsilon\langle \hat{\eta}_b \rangle}{\kappa} + \frac{\varepsilon\langle \hat{\eta}_a + \hat{\eta}_c \rangle}{\kappa} - \langle \hat{\sigma}_c \rangle\right] \\ &\quad -\frac{2\varepsilon}{\kappa}. \end{aligned} \tag{3.74}$$

Moreover, on account of Eqs. (2.79) and (2.80), there follows

$$\begin{aligned}\langle\hat{c}^2\rangle &= -\frac{2\varepsilon}{\kappa}-\frac{2\varepsilon}{\kappa}\left(\frac{4\varepsilon^2}{\kappa^2-4\varepsilon^2}\right)-\frac{\kappa\gamma_c}{(\kappa^2-4\varepsilon^2)^2}\Big(3\varepsilon\kappa\langle\hat{\eta}_a\rangle+4\varepsilon\kappa\langle\hat{\eta}_b\rangle\\ &\quad+\varepsilon\kappa\langle\hat{\eta}_c\rangle-[4\varepsilon^2\langle\hat{\sigma}_c+\hat{\sigma}_c^\dagger\rangle+(\kappa^2-4\varepsilon^2)\langle\hat{\sigma}_c\rangle]\Big) \qquad (3.75)\end{aligned}$$

and in view of Eqs. (2.105), (2.106), and (2.108), we arrive at

$$\begin{aligned}\langle\hat{c}^2\rangle &= -\frac{2\varepsilon}{\kappa}-\frac{2\varepsilon}{\kappa}\left(\frac{4\varepsilon^2}{\kappa^2-4\varepsilon^2}\right)-\frac{\kappa\gamma_c}{(\kappa^2-4\varepsilon^2)^2}\Big(-\varepsilon\kappa\langle\hat{\eta}_a\rangle\\ &\quad+(\varepsilon\kappa-\frac{2\varepsilon}{\kappa}(\kappa^2-4\varepsilon^2))\langle\hat{\eta}_c\rangle\Big) \qquad (3.76)\end{aligned}$$

Furthermore, we note that

$$\langle\hat{c}^{\dagger 2}\rangle=\langle\hat{a}^\dagger\hat{b}^\dagger+\hat{b}^\dagger\hat{a}^\dagger+\hat{a}^{\dagger 2}+\hat{b}^{\dagger 2}\rangle \qquad (3.77)$$

and with the aid of the complex conjugate of Eq. (2.76), we find

$$\langle\hat{c}^{\dagger 2}\rangle=\langle\hat{a}^\dagger\hat{b}^\dagger+\hat{b}^\dagger\hat{a}^\dagger\rangle. \qquad (3.78)$$

Then on applying the complex conjugates of Eqs. (2.64) and (2.65), this equation turns out to be

$$\begin{aligned}\langle\hat{c}^{\dagger 2}\rangle &= -\frac{2\varepsilon}{\kappa}\langle\hat{a}^\dagger\hat{a}+\hat{b}^\dagger\hat{b}\rangle-\frac{4g^2}{\kappa^2-4\varepsilon^2}\left[\frac{2\varepsilon\langle\hat{\eta}_b\rangle}{\kappa}+\frac{\varepsilon\langle\hat{\eta}_a+\hat{\eta}_c\rangle}{\kappa}-\langle\hat{\sigma}_c^\dagger\rangle\right]\\ &\quad-\frac{2\varepsilon}{\kappa} \qquad (3.79)\end{aligned}$$

and on account of Eqs. (2.79) and (2.80), there follows

$$\begin{aligned}\langle\hat{c}^{\dagger 2}\rangle &= -\frac{2\varepsilon}{\kappa}-\frac{2\varepsilon}{\kappa}\left(\frac{4\varepsilon^2}{\kappa^2-4\varepsilon^2}\right)-\frac{\kappa\gamma_c}{(\kappa^2-4\varepsilon^2)^2}\Big(3\varepsilon\kappa\langle\hat{\eta}_a\rangle+4\varepsilon\kappa\langle\hat{\eta}_b\rangle\\ &\quad+\varepsilon\kappa\langle\hat{\eta}_c\rangle-[4\varepsilon^2\langle\hat{\sigma}_c+\hat{\sigma}_c^\dagger\rangle+(\kappa^2-4\varepsilon^2)\langle\hat{\sigma}_c^\dagger\rangle]\Big) \qquad (3.80)\end{aligned}$$

Employing Eqs. (2.106), (2.108), and the complex conjugate of (2.105), we readily obtain

$$\begin{aligned}\langle\hat{c}^{\dagger 2}\rangle &= -\frac{2\varepsilon}{\kappa}-\frac{2\varepsilon}{\kappa}\left(\frac{4\varepsilon^2}{\kappa^2-4\varepsilon^2}\right)-\frac{\kappa\gamma_c}{(\kappa^2-4\varepsilon^2)^2}\Big(-\varepsilon\kappa\langle\hat{\eta}_a\rangle\\ &\quad+(\varepsilon\kappa-\frac{2\varepsilon}{\kappa}(\kappa^2-4\varepsilon^2))\langle\hat{\eta}_c\rangle\Big). \qquad (3.81)\end{aligned}$$

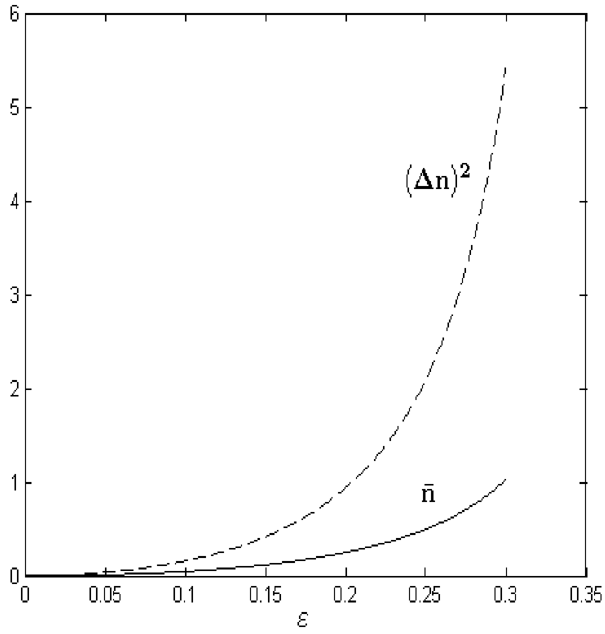

Figure 3.4: Plots of the global mean photon number[Eq. (3.8)] and photon number variance [Eq. (3.82)] versus ε for $\kappa = 0.8$ and $\gamma_c = 0.5$.

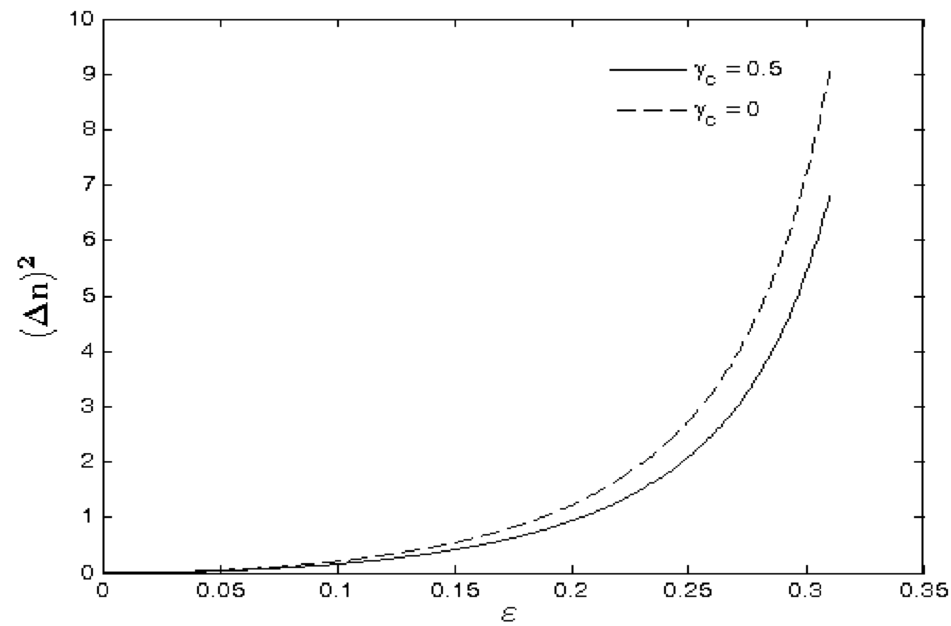

Figure 3.5: Plots of the global photon number variance [Eq. (3.82)] versus ε for $\kappa = 0.8$, $\gamma_c = 0.5$, and $\gamma_c = 0$.

Now substituting Eqs. (3.8), (3.71), (3.76), and (3.81) into Eq. (3.67), we readily find

$$\begin{aligned}(\Delta n)^2 \;=\; & \left(\frac{4\varepsilon^2}{\kappa^2-4\varepsilon^2}+\frac{\kappa\gamma_c}{(\kappa^2-4\varepsilon^2)^2}\left(\frac{\kappa^2+4\varepsilon^2}{2}\langle\hat{\eta}_a\rangle-4\varepsilon^2\langle\hat{\eta}_c\rangle\right)\right)\left(2+\frac{4\varepsilon^2}{\kappa^2-4\varepsilon^2}\right.\\ & +\left.\frac{\kappa\gamma_c}{(\kappa^2-4\varepsilon^2)^2}\left((6\varepsilon^2-2\kappa^2)\langle\hat{\eta}_a\rangle+(\kappa^2-2\varepsilon^2)\langle\hat{\eta}_c\rangle\right)\right)\\ & +\left(-\frac{2\varepsilon}{\kappa}-\frac{2\varepsilon}{\kappa}\left(\frac{4\varepsilon^2}{\kappa^2-4\varepsilon^2}\right)-\frac{\kappa\gamma_c}{(\kappa^2-4\varepsilon^2)^2}\left(-\varepsilon\kappa\langle\hat{\eta}_a\rangle\right.\right.\\ & \left.\left.+(\varepsilon\kappa-\frac{2\varepsilon}{\kappa}(\kappa^2-4\varepsilon^2))\langle\hat{\eta}_c\rangle\right)\right)^2, \end{aligned} \tag{3.82}$$

where $\langle\hat{\eta}_a\rangle$ and $\langle\hat{\eta}_c\rangle$ are given by Eqs. (2.111) and (2.112), respectively. From the plots in fig.3.4, we notice that the photon statistics of the two-mode cavity light is supper poissonian. Moreover, from the plots in Fig3.5, we observe that the effect of the interaction of the subharmonic light modes with the three-level atom is to decreases the photon number variance.

4

Quadrature Squeezing

In this chapter we seek to study the quadrature squeezing for the two-mode cavity light. Applying the steady-state solutions of the equations of evolution of the expectation values for the cavity mode and atomic operators, we first obtain the global quadrature variance. We then determine the global quadrature squeezing. Moreover, employing the time dependent solutions of the equations of evolution of the expectation values for the cavity mode and atomic operators, we calculate the local quadrature squeezing.

4.1 Global quadrature squeezing

The squeezing properties of the two-mode cavity light are described by two quadrature operators defined by [18]

$$\hat{c}_+ = \hat{c}^\dagger + \hat{c} \tag{4.1}$$

and

$$\hat{c}_- = i(\hat{c}^\dagger - \hat{c}), \tag{4.2}$$

where $\hat{c}_+$ and $\hat{c}_-$ are Hermitian operators representing observables called the plus and minus quadratures, respectively [17]. With the help of Eqs. (4.1) and (4.2), one can establish that the two quadrature operators satisfy the commutation relation

$$\big[\hat{c}_+, \hat{c}_-\big] = 2i\big[\hat{c}, \hat{c}^\dagger\big]. \tag{4.3}$$

Then the uncertainty relation for these physical observables is given by [42]

$$\Delta c_- \Delta c_+ \geq \left|\left\langle \left[\hat{c}, \hat{c}^\dagger\right] \right\rangle\right|. \tag{4.4}$$

Now we proceed to obtain the explicit expression for the commutator

$$\left[\hat{c}, \hat{c}^\dagger\right] = \hat{c}\hat{c}^\dagger - \hat{c}^\dagger \hat{c}. \tag{4.5}$$

Then applying Eqs. (3.71) and (3.8), we readily find

$$\left[\hat{c}, \hat{c}^\dagger\right] = 2 + \frac{\kappa\gamma_c}{\kappa^2 - 4\varepsilon^2}\left(\hat{\eta}_c - \hat{\eta}_a\right). \tag{4.6}$$

We identify the first and second terms in this equation to be due to the ordering of the vacuum noise and atomic operators, respectively. Then on substituting of Eq. (4.6) into Eq. (4.4), we get

$$\Delta c_- \Delta c_+ \geq \left|2 + \frac{\kappa\gamma_c}{\kappa^2 - 4\varepsilon^2}\left(\langle\hat{\eta}_c\rangle - \langle\hat{\eta}_a\rangle\right)\right|. \tag{4.7}$$

Thus on account of Eqs. (2.111) and (2.112), the uncertainty relation for the plus and minus quadratures takes the form

$$\Delta c_- \Delta c_+ \geq 2 + \frac{\kappa\gamma_c}{\kappa^2 + 4\varepsilon^2}. \tag{4.8}$$

Furthermore, upon setting $\varepsilon = 0$, the above equation reduces to

$$\Delta c_- \Delta c_+ \geq 2 + \frac{\gamma_c}{\kappa}. \tag{4.9}$$

This is the uncertainty relation for the quadrature operators when a three-level atom is interacting with a vacuum cavity mode.

Next we proceed to calculate the global quadrature variance of the two-mode cavity light. The quadrature variance of the two-mode cavity light is defined by [17,18]

$$(\Delta c \pm)^2 = \langle\hat{c}_\pm^2\rangle - \langle\hat{c}_\pm\rangle^2, \tag{4.10}$$

so that in view of Eqs. (4.1) and (4.2), we have

$$(\Delta c\pm)^2 = \pm\langle(\hat{c}^\dagger \pm \hat{c})^2\rangle \mp (\langle\hat{c}^\dagger\rangle \pm \langle\hat{c}\rangle)^2. \tag{4.11}$$

Moreover, on account of Eq. (3.34), the quadrature variance of the two-mode cavity light takes the form

$$(\Delta c\pm)^2 = \langle\hat{c}^\dagger\hat{c}\rangle + \langle\hat{c}\hat{c}^\dagger\rangle \pm \big(\langle\hat{c}^{\dagger 2}\rangle + \langle\hat{c}^2\rangle\big). \tag{4.12}$$

Now upon substituting Eqs. (3.8), (3.71), (3.76), and (3.81) into Eq. (4.12), one obtains

$$\begin{aligned}(\Delta c_+)^2 &= 2 - \frac{4\varepsilon}{\kappa+2\varepsilon} + \frac{\kappa\gamma_c}{(\kappa^2-4\varepsilon^2)^2}\Bigg(\langle\hat{\eta}_a\rangle(8\varepsilon^2+\kappa^2-6\varepsilon\kappa) + 2(4\varepsilon^2+\kappa^2-4\varepsilon\kappa)\langle\hat{\eta}_b\rangle \\ &\quad +(\kappa-2\varepsilon)\kappa\langle\hat{\eta}_c\rangle + (\kappa(\kappa-4\varepsilon)+4\varepsilon^2)\langle\hat{\sigma}_c+\hat{\sigma}_c^\dagger\rangle\Bigg)\end{aligned} \tag{4.13}$$

and

$$\begin{aligned}(\Delta c_-)^2 &= 2 + \frac{4\varepsilon}{\kappa-2\varepsilon} + \frac{\kappa\gamma_c}{(\kappa^2-4\varepsilon^2)^2}\Bigg(\langle\hat{\eta}_a\rangle(8\varepsilon^2+\kappa^2+6\varepsilon\kappa) + 2(4\varepsilon^2+\kappa^2+4\varepsilon\kappa)\langle\hat{\eta}_b\rangle \\ &\quad +(\kappa+2\varepsilon)\kappa\langle\hat{\eta}_c\rangle - (\kappa(\kappa+4\varepsilon)+4\varepsilon^2)\langle\hat{\sigma}_c+\hat{\sigma}_c^\dagger\rangle\Bigg).\end{aligned} \tag{4.14}$$

Furthermore, on introducing Eqs. (2.108), (2.111), (2.112), and (2.113) into Eqs. (4.13) and (4.14), the global plus and minus quadrature variances of the two-mode cavity light are found to be

$$\begin{aligned}(\Delta c_+)^2 &= 2 - \frac{4\varepsilon}{\kappa+2\varepsilon} + \frac{\kappa\gamma_c}{(\kappa^2+4\varepsilon^2)(\kappa^2-4\varepsilon^2)^2}\Bigg[4\varepsilon^2\Bigg(8\varepsilon^2+\kappa^2-6\varepsilon\kappa\Bigg) + \kappa^3(\kappa-2\varepsilon) \\ &\quad +4\varepsilon\kappa\Bigg(\kappa(\kappa-4\varepsilon)+4\varepsilon^2\Bigg)\Bigg]\end{aligned} \tag{4.15}$$

and

$$\begin{aligned}(\Delta c_-)^2 &= 2 + \frac{4\varepsilon}{\kappa-2\varepsilon} + \frac{\kappa\gamma_c}{(\kappa^2+4\varepsilon^2)(\kappa^2-4\varepsilon^2)^2}\Bigg[4\varepsilon^2\Bigg(8\varepsilon^2+\kappa^2+6\varepsilon\kappa\Bigg) + \kappa^3(\kappa+2\varepsilon) \\ &\quad -4\varepsilon\kappa\Bigg(\kappa(\kappa+4\varepsilon)+4\varepsilon^2\Bigg)\Bigg],\end{aligned} \tag{4.16}$$

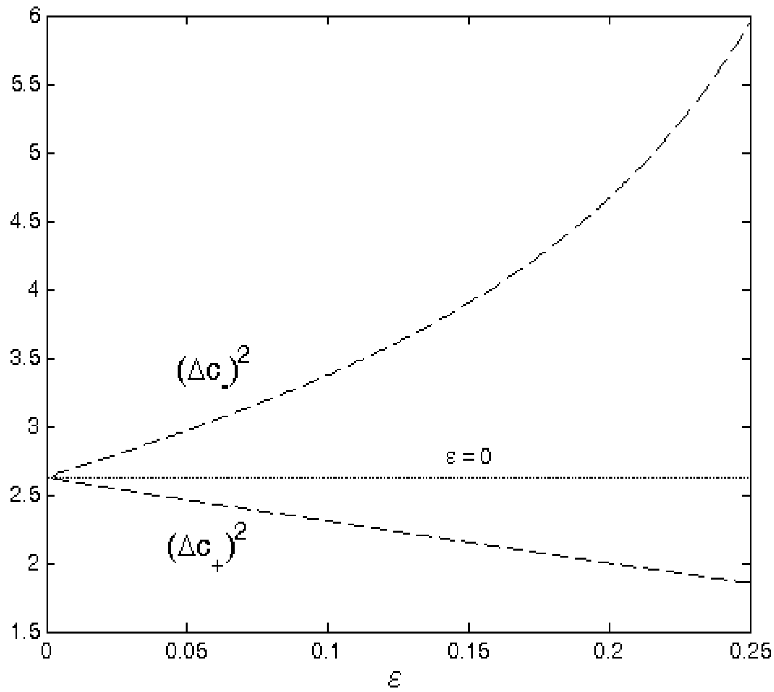

Figure 4.1: Plots of the global plus and minus quadrature variances [Eqs. (4.15), (4.16), and (4.17)] versus ε for $\kappa = 0.8$ and $\gamma_c = 0.5$.

where we have taken into account the fact that $\langle\hat{\sigma}_c^\dagger\rangle = \langle\hat{\sigma}_c\rangle$. In addition, we note that for $\varepsilon = 0$, Eqs. (4.15) and (4.16) reduce to

$$(\Delta c_\pm)^2_{vac} = 2 + \frac{\gamma_c}{\kappa}. \tag{4.17}$$

This indeed represents the quadrature variance for a two-mode cavity vacuum state in the presence of the interaction of subharmonic light modes with a three-level atom, with the effect of the noise operators included. From Eqs. (4.9) and (4.17), we also notice that for $\varepsilon = 0$ the uncertainty in the plus and minus quadratures are equal and satisfy the minimum uncertainty relation. From the plots in Fig.4.1, we observe that the two-mode cavity light is in a squeezed state and the squeezing occurs in the plus quadrature.

Now we calculate the global quadrature squeezing for the two-mode cavity light relative to the global quadrature variance of the two-mode cavity vacuum state. We

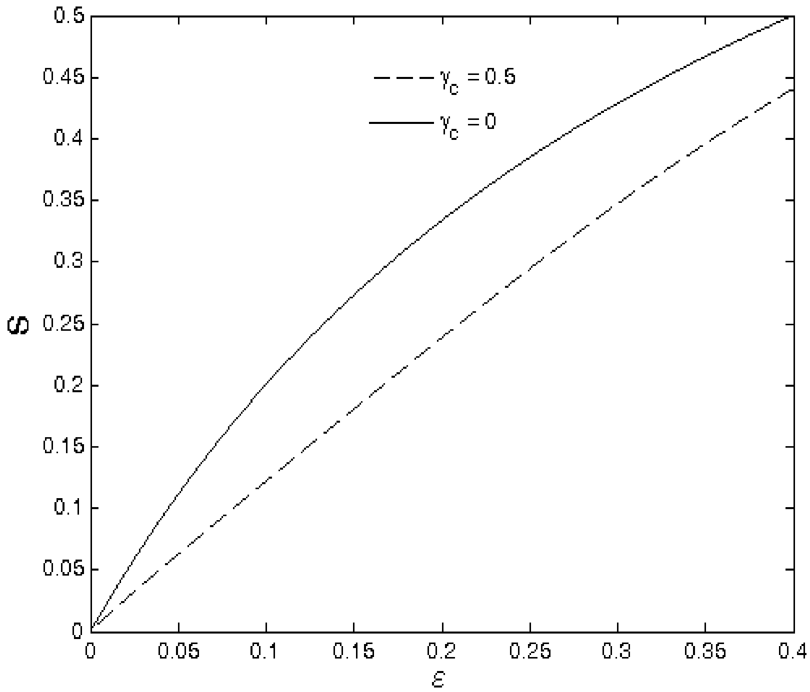

Figure 4.2: Plots of the global quadrature squeezing [Eqs. (4.20) and (4.21)] versus ε for $\kappa = 0.8$.

then define the quadrature squeezing for the two-mode cavity light by [18]

$$S = \frac{(\Delta c_+)^2_{vac} - (\Delta c_+)^2}{(\Delta c_+)^2_{vac}}. \tag{4.18}$$

It then follows that

$$S = 1 - \frac{(\Delta c_+)^2}{(\Delta c_+)^2_{vac}}. \tag{4.19}$$

On using Eqs. (4.15) and (4.17) in Eq. (4.19), the steady-state global quadrature squeezing is found to be

$$\begin{aligned} S &= 1 - \frac{\kappa}{\gamma_c + 2\kappa}\bigg[2 - \frac{4\varepsilon}{\kappa + 2\varepsilon} + \frac{\kappa\gamma_c}{(\kappa^2 + 4\varepsilon^2)(\kappa^2 - 4\varepsilon^2)^2} \\ &\quad \times \bigg[4\varepsilon^2\bigg(8\varepsilon^2 + \kappa^2 - 6\varepsilon\kappa\bigg) + \kappa^3(\kappa - 2\varepsilon) \\ &\quad + 4\varepsilon\kappa\bigg(\kappa(\kappa - 4\varepsilon) + 4\varepsilon^2\bigg)\bigg]\bigg]. \end{aligned} \tag{4.20}$$

This represents the global quadrature squeezing in the presence of the interaction with the three-level atom. We then notice that the global quadrature squeezing in the absence of interaction ($\gamma_c = 0$) Eq. (4.20) reduces to [19]

$$S = \frac{2\varepsilon}{\kappa + 2\varepsilon}. \tag{4.21}$$

This is just the global quadrature squeezing for the superposed subharmonic light modes. From the plots in Fig.4.2, we observe that the effect of the interaction is to decrease the global quadrature squeezing.

4.2 Local quadrature squeezing

We finally seek to obtain the local quadrature squeezing for the two-mode cavity light. To this end, we first determine the spectrum of quadrature fluctuations of the two-mode cavity light. We define this spectrum of quadrature fluctuations for a two-mode cavity light by

$$S_{\pm}(\omega) = \frac{1}{\pi} Re \int_0^{\infty} d\tau \langle \hat{c}_{\pm}(t), \hat{c}_{\pm}(t+\tau)\rangle_{ss} exp\bigg(i(\omega - \omega_0)\tau\bigg), \tag{4.22}$$

where ω_0 is the central frequency of either light mode a or b. Upon integrating both sides of the above equation, we get

$$\int_{-\infty}^{\infty} d\omega S_{\pm}(\omega) = (\Delta c_{\pm})^2, \tag{4.23}$$

in which

$$(\Delta c_{\pm})^2 = \langle \hat{c}_{\pm}(t), \hat{c}_{\pm}(t)\rangle_{ss} \tag{4.24}$$

is the quadrature variance of the two-mode cavity light at steady-state. On the basis of the result given by Eq. (4.23), we assert that $S_{\pm}(\omega)d\omega$ is the quadrature variance of the two-mode cavity light at steady-state in the frequency interval between ω and $\omega + d\omega$ [18].

Now we proceed to determine the two-time correlation function that appears in Eq. (4.22). To this end, with the aid of Eqs. (4.1) and (4.2) along with Eq. (4.12), we find

$$\langle \hat{c}_{\pm}(t), \hat{c}_{\pm}(t+\tau)\rangle = \Big\langle \hat{c}^{\dagger}(t)\hat{c}(t+\tau) + \hat{c}(t)\hat{c}^{\dagger}(t+\tau) \pm \hat{c}(t)\hat{c}(t+\tau) \pm \hat{c}^{\dagger}(t)\hat{c}^{\dagger}(t+\tau)\Big\rangle. \tag{4.25}$$

Next we seek to find the explicit expressions for each term that appears in Eq. (4.25). Therefore, taking the adjoint of Eq. (3.44), we have

$$\begin{aligned}\hat{c}^{\dagger}(t+\tau) &= \hat{c}^{\dagger}(t)exp\Big(-\frac{1}{2}\beta\tau\Big) - g\int_0^{\tau} d\tau' exp\Big(-\frac{1}{2}\beta(\tau-\tau')\Big) \\ &\times\Big[exp\Big(-\frac{1}{2}\mu\gamma_c\tau'\Big)\Big(\hat{\sigma}_a^{\dagger}(t) + \frac{1}{\mu}\hat{\sigma}_b^{\dagger}(t)\Big) \\ &+\int_0^{\tau'} d\tau'' exp\Big(\frac{-1}{2}\gamma_c\mu(\tau'-\tau'')\Big)\Big(\hat{f}_a^{\dagger}(t+\tau'') \\ &+\frac{1}{\mu}\hat{f}_b^{\dagger}(t+\tau'')\Big) - \frac{1}{g}\hat{F}_c^{\dagger}(t+\tau')\Big].\end{aligned} \tag{4.26}$$

Now multiplying the above equation on the left by $\hat{c}(t)$ and taking the expectation value of the resulting equation, we have

$$\begin{aligned}\langle\hat{c}(t)\hat{c}^{\dagger}(t+\tau)\rangle &= \langle\hat{c}(t)\hat{c}^{\dagger}(t)\rangle exp\Big(-\frac{1}{2}\beta\tau\Big) - g\int_0^{\tau} d\tau' exp\Big(-\frac{1}{2}\beta(\tau-\tau')\Big) \\ &\times\Big[exp\Big(-\frac{1}{2}\mu\gamma_c\tau'\Big)\Big\langle\hat{c}(t)\Big[\hat{\sigma}_a^{\dagger}(t) + \frac{1}{\mu}\hat{\sigma}_b^{\dagger}(t)\Big]\Big\rangle \\ &+\int_0^{\tau'} d\tau'' exp\Big(-\frac{1}{2}\gamma_c\mu(\tau'-\tau'')\Big)\Big\langle\hat{c}(t)\Big(\hat{f}_a^{\dagger}(t+\tau'') \\ &+\frac{1}{\mu}\hat{f}_b^{\dagger}(t+\tau'')\Big)\Big\rangle - \frac{1}{g}\langle\hat{c}(t)\hat{F}_c^{\dagger}(t+\tau')\rangle\Big].\end{aligned} \tag{4.27}$$

In view of Eqs. (3.46)-(3.48), Eq. (4.27) reduces to

$$\begin{aligned}\langle\hat{c}(t)\hat{c}^{\dagger}(t+\tau)\rangle &= \langle\hat{c}(t)\hat{c}^{\dagger}(t)\rangle exp\Big(-\frac{1}{2}\beta\tau\Big) - g\int_0^{\tau} d\tau' exp\Big(-\frac{1}{2}\beta(\tau-\tau')\Big) \\ &\times exp\Big(-\frac{1}{2}\mu\gamma_c\tau'\Big)\Big\langle\hat{c}(t)\Big(\hat{\sigma}_a^{\dagger}(t) + \frac{1}{\mu}\hat{\sigma}_b^{\dagger}(t)\Big)\Big\rangle.\end{aligned} \tag{4.28}$$

We note that the adjoint of Eq. (3.50) is given by

$$\frac{\beta}{2}\hat{c}^{\dagger}(t) - \hat{F}_c^{\dagger}(t) = -g(\hat{\sigma}_a^{\dagger}(t) + \frac{1}{\mu}\hat{\sigma}_b^{\dagger}(t)), \tag{4.29}$$

so that using this equation, Eq. (4.28) can be written as

$$\langle\hat{c}(t)\hat{c}^{\dagger}(t+\tau)\rangle = \langle\hat{c}(t)\hat{c}^{\dagger}(t)\rangle exp\Big(-\frac{1}{2}\beta\tau\Big) + \int_0^{\tau} d\tau' exp\Big(-\frac{1}{2}\beta(\tau-\tau')\Big) \times exp\Big(-\frac{1}{2}\mu\tau'\Big)\Big\langle\hat{c}(t)\Big(\frac{\beta}{2}\hat{c}^{\dagger}(t) - \hat{F}_c^{\dagger}(t)\Big)\Big\rangle. \tag{4.30}$$

Furthermore, we notice that the solution of Eq. (3.39) can be put in the form

$$\hat{c}(t) = \hat{c}(0)exp\Big(-\frac{1}{2}\beta t\Big) - g\int_0^t dt' exp\Big(-\frac{1}{2}\beta(t-t')\Big) \times\Big[\hat{\sigma}_0(t') - \frac{1}{g}\hat{F}_c(t')\Big]. \tag{4.31}$$

On multiplying this equation by $\hat{F}_c^{\dagger}(t)$ from the right side, we get

$$\hat{c}(t)\hat{F}_c^{\dagger}(t) = \hat{c}(0)\hat{F}_c^{\dagger}(t)exp\Big(-\frac{1}{2}\beta t\Big) - g\int_0^t dt' exp\Big(-\frac{1}{2}\beta(t-t')\Big) \times\Big[\hat{\sigma}_0(t') - \frac{1}{g}\hat{F}_c(t')\Big]\hat{F}_c^{\dagger}(t) \tag{4.32}$$

and taking the expectation value, we arrive at

$$\langle\hat{c}(t)\hat{F}_c^{\dagger}(t)\rangle = \langle\hat{c}(0)\hat{F}_c^{\dagger}(t)\rangle exp\Big(-\frac{1}{2}\beta t\Big) - g\int_0^t dt' exp\Big(-\frac{1}{2}\beta(t-t')\Big) \times\Big\langle\hat{\sigma}_0(t')\hat{F}_c^{\dagger}(t) - \frac{1}{g}\hat{F}_c(t')\hat{F}_c^{\dagger}(t)\Big\rangle. \tag{4.33}$$

Since the two-mode cavity light is coupled to a two-mode vacuum reservoir, we see that

$$\langle\hat{F}_c(t')\hat{F}_c^{\dagger}(t)\rangle = 2\kappa\delta(t-t'). \tag{4.34}$$

On applying Eqs. (2.49), (3.46), and (4.34), Eq. (4.33) can be put in the form

$$\langle\hat{c}(t)\hat{F}_c^{\dagger}(t)\rangle = 2\kappa\int_0^t dt' exp\Big(-\frac{1}{2}\beta(t-t')\Big)\delta(t-t'). \tag{4.35}$$

Now making use of the fact that

$$\int_0^t dt' exp\Big(-\frac{1}{2}\beta(t-t')\Big)\delta(t-t') = \frac{1}{2}, \tag{4.36}$$

so that Eq. (4.35) takes the form

$$\langle\hat{c}(t)\hat{F}_c^{\dagger}(t)\rangle = \kappa. \tag{4.37}$$

Hence with the aid of this result, Eq. (4.30) can be written as

$$\begin{aligned}\langle\hat{c}(t)\hat{c}^{\dagger}(t+\tau)\rangle &= \langle\hat{c}(t)\hat{c}^{\dagger}(t)\rangle exp\Big(-\frac{1}{2}\beta\tau\Big)\Big[1+\frac{\beta}{2}\int_0^{\tau}d\tau' exp\Big(\frac{1}{2}(\beta-\mu)\tau'\Big)\Big]\\ &\quad -(\kappa)exp\Big(-\frac{1}{2}\beta\tau\Big)\int_0^{\tau}d\tau' exp\Big(\frac{1}{2}(\beta-\mu)\tau'\Big) \qquad (4.38)\end{aligned}$$

and on evaluating the integral, one readily gets

$$\begin{aligned}\langle\hat{c}(t)\hat{c}^{\dagger}(t+\tau)\rangle &= \langle\hat{c}(t)\hat{c}^{\dagger}(t)\rangle\Big[\frac{\beta}{\beta-\mu}exp\Big(-\frac{1}{2}\mu\tau\Big)-\frac{\mu}{\beta-\mu}exp\Big(-\frac{1}{2}\beta\tau\Big)\Big]\\ &\quad -\frac{2\kappa}{\beta-\mu}\Big[exp\Big(-\frac{1}{2}\mu\tau\Big)-exp\Big(-\frac{1}{2}\beta\tau\Big)\Big]. \qquad (4.39)\end{aligned}$$

In addition, on multiplying Eq. (3.44) on the left by $\hat{c}(t)$ and taking the expectation value of the resulting equation, we have

$$\begin{aligned}\langle\hat{c}(t)\hat{c}(t+\tau)\rangle &= \langle\hat{c}(t)\hat{c}(t)\rangle exp\Big(-\frac{1}{2}\beta\tau\Big)-g\int_0^{\tau}d\tau' exp\Big(-\frac{1}{2}\beta(\tau-\tau')\Big)\\ &\quad\times\Big[exp\Big(-\frac{1}{2}\mu\gamma_c\tau'\Big)\Big\langle\hat{c}(t)\Big[\hat{\sigma}_a(t)+\frac{1}{\mu}\hat{\sigma}_b(t)\Big]\Big\rangle\\ &\quad+\int_0^{\tau'}d\tau'' exp\Big(-\frac{1}{2}\gamma_c\mu(\tau'-\tau'')\Big)\Big\langle\hat{c}(t)\Big(\hat{f}_a(t+\tau'')\\ &\quad+\frac{1}{\mu}\hat{f}_b(t+\tau'')\Big)\Big\rangle-\frac{1}{g}\langle\hat{c}(t)\hat{F}_c(t+\tau')\rangle\Big]. \qquad (4.40)\end{aligned}$$

On account of Eqs. (3.46)-(3.48), Eq. (4.40) takes the form

$$\begin{aligned}\langle\hat{c}(t)\hat{c}(t+\tau)\rangle &= \langle\hat{c}(t)\hat{c}(t)\rangle exp\Big(-\frac{1}{2}\beta\tau\Big)-g\int_0^{\tau}d\tau' exp\Big(-\frac{1}{2}\beta(\tau-\tau')\Big)\\ &\quad\times exp\Big(-\frac{1}{2}\mu\gamma_c\tau'\Big)\Big\langle\hat{c}(t)\Big(\hat{\sigma}_a(t)+\frac{1}{\mu}\hat{\sigma}_b(t)\Big)\Big\rangle. \qquad (4.41)\end{aligned}$$

With the aid of Eq. (3.50), Eq. (4.41) can be rewritten as

$$\begin{aligned}\langle\hat{c}(t)\hat{c}(t+\tau)\rangle &= \langle\hat{c}(t)\hat{c}(t)\rangle exp\Big(-\frac{1}{2}\beta t\Big)+\int_0^{\tau}d\tau' exp\Big(-\frac{1}{2}\beta(\tau-\tau')\Big)\\ &\quad\times exp\Big(-\frac{1}{2}\mu\tau'\Big)\Big\langle\hat{c}(t)\Big(\frac{\beta}{2}\hat{c}(t)-\hat{F}_c(t)\Big)\Big\rangle. \qquad (4.42)\end{aligned}$$

Moreover, upon multiplying Eq. (4.31) by $\hat{F}_c(t)$ from the right side and taking the expectation value of the resulting equation, we have

$$\begin{aligned}\langle\hat{c}(t)\hat{F}_c(t)\rangle &= \langle\hat{c}(0)\hat{F}_c(t)\rangle exp\Big(-\frac{1}{2}\beta t\Big)-g\int_0^{\tau}d\tau' exp\Big(-\frac{1}{2}\beta(t-t')\Big)\\ &\quad\times\Big\langle\hat{\sigma}_0(t')\hat{F}_c(t)-\frac{1}{g}\hat{F}_c(t')\hat{F}_c(t)\Big\rangle. \qquad (4.43)\end{aligned}$$

With the help of the fact that for vacuum reservoir

$$\langle \hat{F}_c(t')\hat{F}_c(t)\rangle = 0. \tag{4.44}$$

And on applying Eqs. (2.49), (3.46), and (4.44), Eq. (4.43) becomes

$$\langle \hat{c}(t)\hat{F}_c(t)\rangle = 0. \tag{4.45}$$

Then upon taking into account Eq. (4.45), Eq. (4.41) can be put in the form

$$\begin{aligned}\langle \hat{c}(t)\hat{c}(t+\tau)\rangle &= \langle \hat{c}^2(t)\rangle exp\bigg(-\frac{1}{2}\beta\tau\bigg)\bigg[1+\frac{\beta}{2}\int_0^\tau d\tau' exp\bigg(\frac{1}{2}(\beta \\ &\quad -\mu)\tau'\bigg)\bigg].\end{aligned} \tag{4.46}$$

Thus on carrying out the integration, one readily finds

$$\begin{aligned}\langle \hat{c}(t)\hat{c}(t+\tau)\rangle &= \langle \hat{c}^2(t)\rangle\bigg[\frac{\beta}{\beta-\mu}exp\bigg(-\frac{1}{2}\mu\tau\bigg) \\ &\quad -\frac{\mu}{\beta-\mu}exp\bigg(-\frac{1}{2}\beta\tau\bigg)\bigg].\end{aligned} \tag{4.47}$$

On taking the complex conjugate of (4.47), we have

$$\begin{aligned}\langle \hat{c}^\dagger(t)\hat{c}^\dagger(t+\tau)\rangle &= \langle \hat{c}^{\dagger 2}(t)\rangle\bigg[\frac{\beta}{\beta-\mu}exp\bigg(-\frac{1}{2}\mu\tau\bigg) \\ &\quad -\frac{\mu}{\beta-\mu}exp\bigg(-\frac{1}{2}\beta\tau\bigg)\bigg].\end{aligned} \tag{4.48}$$

On substituting Eqs. (3.59), (4.39), (4.47), and (4.48) into Eq. (4.25), we obtain

$$\begin{aligned}\langle \hat{c}_\pm(t),\hat{c}_\pm(t+\tau)\rangle &= \bigg(\Big\langle \hat{c}^\dagger(t)\hat{c}(t)+\hat{c}(t)\hat{c}^\dagger(t)\pm\hat{c}(t)\hat{c}(t)\pm\hat{c}^\dagger(t)\hat{c}^\dagger(t)\Big\rangle\bigg) \\ &\quad \times\bigg[\frac{\beta}{\beta-\mu}exp\bigg(-\frac{1}{2}\mu\tau\bigg)-\frac{\mu}{\beta-\mu}exp\bigg(-\frac{1}{2}\beta\tau\bigg)\bigg] \\ &\quad -\frac{2\kappa}{\beta-\mu}\bigg(exp\bigg(-\frac{1}{2}\mu\tau\bigg)-exp\bigg(-\frac{1}{2}\beta\tau\bigg)\bigg).\end{aligned} \tag{4.49}$$

Now using Eq. (4.12), this equation can be written, at steady-state, as

$$\begin{aligned}\langle \hat{c}_\pm(t),\hat{c}_\pm(t+\tau)\rangle_{ss} &= (\Delta c_\pm)^2\bigg[\frac{\beta}{\beta-\mu}exp\bigg(-\frac{1}{2}\mu\tau\bigg)-\frac{\mu}{\beta-\mu}exp\bigg(-\frac{1}{2}\beta\tau\bigg)\bigg] \\ &\quad -\frac{2\kappa}{\beta-\mu}\bigg(exp\bigg(-\frac{1}{2}\mu\tau\bigg)-exp\bigg(-\frac{1}{2}\beta\tau\bigg)\bigg).\end{aligned} \tag{4.50}$$

Now on introducing this equation into Eq. (4.22), the spectrum of the plus quadrature fluctuation of the two-mode cavity light takes the form

$$\begin{aligned} S_+(\omega) &= \frac{1}{\pi}Re\Bigg[(\Delta c_+)^2\Bigg[\frac{\beta}{\beta-\mu}\int_0^\infty d\tau exp\Bigg(-(\frac{1}{2}\mu - i(\omega-\omega_0))\tau\Bigg) \\ &-\frac{\mu}{\beta-\mu}\int_0^\infty d\tau exp\Bigg(-(\frac{1}{2}\beta - i(\omega-\omega_0))\tau\Bigg)\Bigg] \\ &-\frac{2\kappa}{\beta-\mu}\Bigg(\int_0^\infty d\tau exp\Bigg(-(\frac{1}{2}\mu - i(\omega-\omega_0))\tau\Bigg) \\ &-\int_0^\infty d\tau exp\Bigg(-(\frac{1}{2}\beta - i(\omega-\omega_0))\tau\Bigg)\Bigg)\Bigg], \end{aligned} \quad (4.51)$$

so that on carrying out the integration, there follows

$$\begin{aligned} S_+(\omega) &= (\Delta c_+)^2\Bigg[\frac{\beta}{\beta-\mu}\Bigg(\frac{\frac{\mu}{2\pi}}{(\frac{\mu}{2})^2+(\omega-\omega_0)^2}\Bigg) \\ &-\frac{\mu}{\beta-\mu}\Bigg(\frac{\frac{\beta}{2\pi}}{(\frac{\beta}{2})^2+(\omega-\omega_0)^2}\Bigg)\Bigg] \\ &-.\frac{2\kappa}{\beta-\mu}\Bigg(\frac{\frac{\mu}{2\pi}}{(\frac{\mu}{2})^2+(\omega-\omega_0)^2} \\ &-\frac{\frac{\beta}{2\pi}}{(\frac{\beta}{2})^2+(\omega-\omega_0)^2}\Bigg). \end{aligned} \quad (4.52)$$

We recall that the local quadrature variance in the interval between $\omega' = -\lambda$ and $\omega' = \lambda$ is expressible as [18]

$$(\Delta c_+)^2_{\pm\lambda} = \int_{-\lambda}^{\lambda} d\omega' S_+(\omega'), \quad (4.53)$$

in which $\omega' = \omega - \omega_0$. Therefore, upon substituting Eq. (4.52) into Eq. (4.53), we have

$$\begin{aligned} (\Delta c_+)^2_{\pm\lambda} &= (\Delta c_+)^2\int_{-\lambda}^{\lambda} d\omega'\Bigg[\frac{\beta}{\beta-\mu}\Bigg(\frac{\frac{\mu}{2\pi}}{(\frac{\mu}{2})^2+(\omega-\omega_0)^2}\Bigg) \\ &-\frac{\mu}{\beta-\mu}\Bigg(\frac{\frac{\beta}{2\pi}}{(\frac{\beta}{2})^2+(\omega-\omega_0)^2}\Bigg)\Bigg] \\ &-.\frac{2\kappa}{\beta-\mu}\int_{-\lambda}^{\lambda} d\omega'\Bigg(\frac{\frac{\mu}{2\pi}}{(\frac{\mu}{2})^2+(\omega-\omega_0)^2} \\ &-\frac{\frac{\beta}{2\pi}}{(\frac{\beta}{2})^2+(\omega-\omega_0)^2}\Bigg) \end{aligned} \quad (4.54)$$

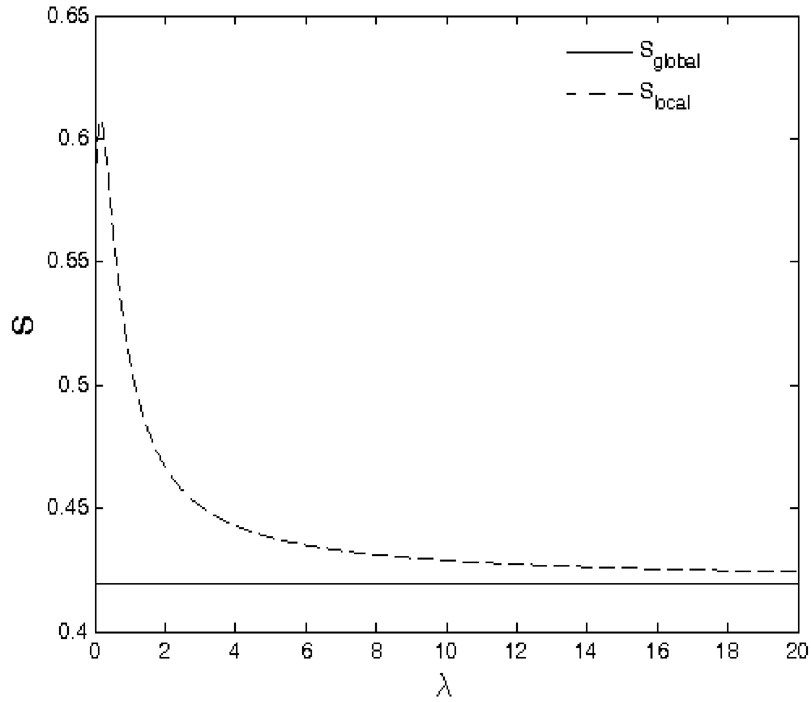

Figure 4.3: Plots of local quadrature squeezing (dashed line) and global quadrature squeezing (solid line) [Eqs. (4.59) and (4.20)] versus λ for $\varepsilon = 0.35$, $\kappa = 0.8$, and $\gamma_c = 0.3$.

and on evaluating the integral, the local quadrature variance is found to be

$$\begin{aligned}(\Delta c_+)^2_{\pm\lambda} &= (\Delta c_+)^2\left[\frac{\frac{2\beta}{\pi}}{\beta-\mu}tan^{-1}\left(\frac{2\lambda}{\mu}\right) - \frac{\frac{2\mu}{\pi}}{\beta-\mu}tan^{-1}\left(\frac{2\lambda}{\beta}\right)\right]\\ &-\frac{4\kappa}{\pi(\beta-\mu)}\left[tan^{-1}\left(\frac{2\lambda}{\mu}\right) - tan^{-1}\left(\frac{2\lambda}{\beta}\right)\right]. \end{aligned} \tag{4.55}$$

On setting $\varepsilon = 0$ in Eq. (4.55), we have

$$\begin{aligned}\bigg((\Delta c_+)^2_{\pm\lambda}\bigg)_{vac} &= \frac{\gamma_c+2\kappa}{\kappa}\left[\frac{\frac{2\kappa}{\pi}}{\kappa-\gamma_c}tan^{-1}\left(\frac{2\lambda}{\gamma_c}\right) - \frac{\frac{2\gamma_c}{\pi}}{\kappa-\gamma_c}tan^{-1}\left(\frac{2\lambda}{\kappa}\right)\right]\\ &-\frac{4\kappa}{\pi(\kappa-\gamma_c)}\left[tan^{-1}\left(\frac{2\lambda}{\gamma_c}\right) - tan^{-1}\left(\frac{2\lambda}{\kappa}\right)\right]. \end{aligned} \tag{4.56}$$

This represents the local quadrature variance for a two-mode vacuum state.

Now we define the local quadrature squeezing for the two-mode cavity light in the $\pm\lambda$ frequency interval by

$$S_{\pm\lambda} = \frac{\bigg((\Delta c_+)^2_{\pm\lambda}\bigg)_{vac} - (\Delta c_+)^2_{\pm\lambda}}{\bigg((\Delta c_+)^2_{\pm\lambda}\bigg)_{vac}}, \tag{4.57}$$

from which follows

$$S_{\pm\lambda} = 1 - \frac{(\Delta c_+)^2_{\pm\lambda}}{\left((\Delta c_+)^2_{\pm\lambda}\right)_{vac}}. \tag{4.58}$$

Hence on account of Eqs. (4.55) and (4.56), there emerges

$$S_{\pm\lambda} = 1 - \frac{(\Delta c_+)^2 A - \frac{4\kappa}{\pi(\beta-\mu)}\left[tan^{-1}\left(\frac{2\lambda}{\mu}\right) - tan^{-1}\left(\frac{2\lambda}{\beta}\right)\right]}{B - \frac{4\kappa}{\pi(\kappa-\gamma_c)}\left[tan^{-1}\left(\frac{2\lambda}{\gamma_c}\right) - tan^{-1}\left(\frac{2\lambda}{\kappa}\right)\right]}, \tag{4.59}$$

where

$$A = \frac{\frac{2\beta}{\pi}}{\beta-\mu}tan^{-1}\left(\frac{2\lambda}{\mu}\right) - \frac{\frac{2\mu}{\pi}}{\beta-\mu}tan^{-1}\left(\frac{2\lambda}{\beta}\right), \tag{4.60}$$

$$B = \frac{\gamma_c + 2\kappa}{\kappa}\left[\frac{\frac{2\kappa}{\pi}}{\kappa-\gamma_c}tan^{-1}\left(\frac{2\lambda}{\gamma_c}\right) - \frac{\frac{2\gamma_c}{\pi}}{\kappa-\gamma_c}tan^{-1}\left(\frac{2\lambda}{\kappa}\right)\right]. \tag{4.61}$$

From the plots in Fig.4.3, we see that the maximum local quadrature squeezing of the two-mode cavity light is 60.8% at $\lambda = 0.17$. In addition, we note that the local quadrature squeezing eventually approaches to the global quadrature squeezing as λ increases.

5

Quadrature Squeezing with Normally Ordered Noise Operators

In this chapter we seek to determine the quadrature squeezing for the two-mode cavity light by putting the reservoir noise operators in normal order [16, 18]. Applying the steady-state solutions of the equations of evolution of the expectation values for the cavity mode operators, we obtain the quadrature variance by normally ordering the vacuum reservoir noise operators. We then calculate the normally ordered quadrature squeezing relative to the normally ordered quadrature variance for the two-mode vacuum state.

Now we proceed to calculate the quadrature variance by normally ordering the reservoir noise operators. Thus the normally ordered quadrature variance at steady-state can be expressed as

$$\bigg(:(\Delta c_{\pm})^2:\bigg)_F = \langle : \hat{c}^{\dagger}\hat{c} : \rangle_F + \langle : \hat{c}\hat{c}^{\dagger} : \rangle_F \pm \langle : \hat{c}^2 : + : \hat{c}^{\dagger 2} : \rangle_F, \tag{5.1}$$

in which we have taken into consideration Eq. (3.34) and where $(::)_F$ stands for normal ordering of the reservoir noise operators. Upon using Eq. (3.1) and its adjoint in Eq. (5.1), we obtain

$$\Big(:(\Delta c_{\pm})^2: \Big)_F = \langle : \hat{a}^{\dagger}\hat{a} : \rangle_F + \langle : \hat{b}^{\dagger}\hat{b} : \rangle_F + \langle : \hat{a}^{\dagger}\hat{b} : \rangle_F + \langle : \hat{b}^{\dagger}\hat{a} : \rangle_F + \langle : \hat{a}\hat{a}^{\dagger} : \rangle_F + \langle : \hat{b}\hat{b}^{\dagger} : \rangle_F$$
$$+ \langle : \hat{b}\hat{a}^{\dagger} : \rangle_F + \langle : \hat{a}\hat{b}^{\dagger} : \rangle_F \pm \Big(\langle : \hat{a}^2 : + : \hat{a}^{\dagger 2} : + : \hat{b}^2 : + : \hat{b}^{\dagger 2} : \rangle_F$$
$$+ \langle : \hat{a}\hat{b} : \rangle_F + \langle : \hat{b}\hat{a} : \rangle_F + \langle : \hat{a}^{\dagger}\hat{b}^{\dagger} : \rangle + \langle : \hat{b}^{\dagger}\hat{a}^{\dagger} : \rangle_F \Big). \tag{5.2}$$

In view of Eqs. (2.73), (2.75) and, (2.76) along with their complex conjugate, Eq. (5.2) turns out to be

$$\Big(:(\Delta c_{\pm})^2: \Big)_F = \langle : \hat{a}^{\dagger}\hat{a} : \rangle_F + \langle : \hat{b}^{\dagger}\hat{b} : \rangle_F + \langle : \hat{a}\hat{a}^{\dagger} : \rangle_F + \langle : \hat{b}\hat{b}^{\dagger} : \rangle_F$$
$$\pm \Big(\langle : \hat{a}\hat{b} : \rangle_F + \langle : \hat{b}\hat{a} : \rangle_F + \langle : \hat{a}^{\dagger}\hat{b}^{\dagger} : \rangle$$
$$+ \langle : \hat{b}^{\dagger}\hat{a}^{\dagger} : \rangle_F \Big). \tag{5.3}$$

We now proceed to determine the various expectation values that appear in this equation. To this end, employing the relation

$$\frac{d}{dt}\big\langle \hat{a}^{\dagger}(t)\hat{a}(t) \big\rangle = \Big\langle \frac{d\hat{a}^{\dagger}(t)}{dt}\hat{a}(t) \Big\rangle + \Big\langle \hat{a}^{\dagger}(t)\frac{d\hat{a}(t)}{dt} \Big\rangle \tag{5.4}$$

together with Eq. (2.29) and its adjoint, we arrive at

$$\frac{d}{dt}\big\langle \hat{a}^{\dagger}(t)\hat{a}(t) \big\rangle = -\kappa \langle \hat{a}^{\dagger}(t)\hat{a}(t) \rangle - \varepsilon \Big(\langle \hat{b}\hat{a} \rangle + \langle \hat{a}^{\dagger}\hat{b}^{\dagger} \rangle \Big) - g \Big(\big\langle \hat{a}^{\dagger}(t)\hat{\sigma}_a(t) \big\rangle$$
$$+ \big\langle \hat{\sigma}_a^{\dagger}(t)\hat{a}(t) \big\rangle \Big) + \big\langle \hat{a}^{\dagger}(t)\hat{F}_a(t) \big\rangle + \big\langle \hat{F}_a^{\dagger}(t)\hat{a}(t) \big\rangle. \tag{5.5}$$

This equation can be written as

$$\frac{d}{dt}\big\langle : \hat{a}^{\dagger}(t)\hat{a}(t) : \big\rangle_F = -\kappa \langle : \hat{a}^{\dagger}(t)\hat{a}(t) : \rangle_F - \varepsilon \Big(\langle : \hat{b}\hat{a} : \rangle_F + \langle : \hat{a}^{\dagger}\hat{b}^{\dagger} : \rangle_F \Big)$$
$$-g \Big(\big\langle : \hat{a}^{\dagger}(t)\hat{\sigma}_a(t) : \big\rangle_F + \big\langle : \hat{\sigma}_a^{\dagger}(t)\hat{a}(t) : \big\rangle_F \Big)$$
$$+ \big\langle : \hat{a}^{\dagger}(t)\hat{F}_a(t) : \big\rangle_F + \big\langle : \hat{F}_a^{\dagger}(t)\hat{a}(t) : \big\rangle_F. \tag{5.6}$$

On substituting Eq. (2.33) and its adjoint into Eq. (5.6), we have

$$\begin{aligned}\Big\langle :\hat{a}^{\dagger}(t)\hat{\sigma}_a(t):\Big\rangle_F + \Big\langle :\hat{\sigma}_a^{\dagger}(t)\hat{a}(t):\Big\rangle_F &= \frac{4\kappa g}{\kappa^2-4\varepsilon^2}\bigg[\frac{\varepsilon\langle\hat{\sigma}_b(t)\hat{\sigma}_a(t)\rangle}{\kappa} - \frac{\langle\hat{\sigma}_a^{\dagger}(t)\hat{\sigma}_a(t)\rangle}{2}\\ &+\frac{\langle\hat{F}_a^{\dagger}(t)\hat{\sigma}_a(t)\rangle}{2g} - \frac{\varepsilon\langle\hat{F}_b(t)\hat{\sigma}_a(t)\rangle}{\kappa g}\\ &+\frac{\varepsilon\langle\hat{\sigma}_a^{\dagger}(t)\hat{\sigma}_b^{\dagger}(t)\rangle}{\kappa} - \frac{\langle\hat{\sigma}_a^{\dagger}(t)\hat{\sigma}_a(t)\rangle}{2}\\ &+\frac{\langle\hat{\sigma}_a^{\dagger}(t)\hat{F}_a(t)\rangle}{2g} - \frac{\varepsilon\langle\hat{\sigma}_a^{\dagger}(t)\hat{F}_b^{\dagger}(t)\rangle}{\kappa g}\bigg]. \qquad (5.7)\end{aligned}$$

Hence on account of Eq. (2.49), Eq. (5.7) takes the form

$$\begin{aligned}\Big\langle :\hat{a}^{\dagger}(t)\hat{\sigma}_a(t):\Big\rangle_F + \Big\langle :\hat{\sigma}_a^{\dagger}(t)\hat{a}(t):\Big\rangle_F &= \frac{4\kappa g}{\kappa^2-4\varepsilon^2}\bigg[\frac{\varepsilon\langle\hat{\sigma}_b(t)\hat{\sigma}_a(t)\rangle}{\kappa} - \frac{\langle\hat{\sigma}_a^{\dagger}(t)\hat{\sigma}_a(t)\rangle}{2}\\ &+\frac{\varepsilon\langle\hat{\sigma}_a^{\dagger}(t)\hat{\sigma}_b^{\dagger}(t)\rangle}{\kappa} - \frac{\langle\hat{\sigma}_a^{\dagger}(t)\hat{\sigma}_a(t)\rangle}{2}\bigg]. \qquad (5.8)\end{aligned}$$

Making use of Eqs. (2.4) and (2.5), one can readily establish that

$$\frac{\varepsilon\langle\hat{\sigma}_b(t)\hat{\sigma}_a(t)\rangle}{\kappa} - \frac{\langle\hat{\sigma}_a^{\dagger}(t)\hat{\sigma}_a(t)\rangle}{2} = \frac{\varepsilon}{\kappa}\Big\langle\hat{\sigma}_c(t)\Big\rangle - \frac{1}{2}\Big\langle\hat{\eta}_a(t)\Big\rangle \qquad (5.9)$$

and

$$\frac{\varepsilon\langle\hat{\sigma}_a^{\dagger}(t)\hat{\sigma}_b^{\dagger}(t)\rangle}{\kappa} - \frac{\langle\hat{\sigma}_a^{\dagger}(t)\hat{\sigma}_a(t)\rangle}{2} = \frac{\varepsilon}{\kappa}\Big\langle\hat{\sigma}_c^{\dagger}(t)\Big\rangle - \frac{1}{2}\Big\langle\hat{\eta}_a(t)\Big\rangle. \qquad (5.10)$$

Now in view of Eqs. (2.111) and (2.113), Eqs. (5.9) and (5.10) turn out to be

$$\frac{\varepsilon\langle\hat{\sigma}_b(t)\hat{\sigma}_a(t)\rangle}{\kappa} - \frac{\langle\hat{\sigma}_a^{\dagger}(t)\hat{\sigma}_a(t)\rangle}{2} = 0 \qquad (5.11)$$

and

$$\frac{\varepsilon\langle\hat{\sigma}_a^{\dagger}(t)\hat{\sigma}_b^{\dagger}(t)\rangle}{\kappa} - \frac{\langle\hat{\sigma}_a^{\dagger}(t)\hat{\sigma}_a(t)\rangle}{2} = 0. \qquad (5.12)$$

Now combination of Eqs. (5.11), (5.12), and (5.8) leads to

$$\Big\langle :\hat{a}^{\dagger}(t)\hat{\sigma}_a(t):\Big\rangle_F + \Big\langle :\hat{\sigma}_a^{\dagger}(t)\hat{a}(t):\Big\rangle_F = 0. \qquad (5.13)$$

Moreover, one can rewrite Eqs. (3.23) and (3.31) as

$$\frac{d}{dt}\hat{a}(t) = -\bigg(\frac{\kappa^2-4\varepsilon^2}{2\kappa}\bigg)\hat{a}(t) - g\hat{\sigma}_a(t) + \hat{F}_a(t) \qquad (5.14)$$

and

$$\frac{d}{dt}\hat{b}(t) = -\bigg(\frac{\kappa^2 - 4\varepsilon^2}{2\kappa}\bigg)\hat{b}(t) - \frac{g}{\kappa^2}\bigg(\kappa^2 - 4\varepsilon^2\bigg)\hat{\sigma}_b(t) + \hat{F}_b(t). \quad (5.15)$$

We notice that the the solutions of Eqs. (5.14) and (5.15) are expressible as

$$\begin{aligned}\hat{a}(t) &= \hat{a}(0)exp\bigg(-\frac{1}{2}\beta t\bigg) + \int_0^t dt' exp\bigg(-\frac{1}{2}\beta(t-t')\bigg)\\ &\quad\times\bigg[-g\hat{\sigma}_a(t') + \hat{F}_a(t')\bigg] \end{aligned} \quad (5.16)$$

and

$$\begin{aligned}\hat{b}(t) &= \hat{b}(0)exp\bigg(-\frac{1}{2}\beta t\bigg) + \int_0^t dt' exp\bigg(-\frac{1}{2}\beta(t-t')\bigg)\\ &\quad\times\bigg[-\frac{g}{\kappa^2}\bigg(\kappa^2 - 4\varepsilon^2\bigg)\hat{\sigma}_b(t') + \hat{F}_b(t')\bigg], \end{aligned} \quad (5.17)$$

where β is defined by Eq. (3.27). Now multiplying Eq.(5.16) by $\hat{F}_a^\dagger(t)$ from the left and taking the expectation value of the resulting equation, we have

$$\begin{aligned}\langle\hat{F}_a^\dagger(t)\hat{a}(t)\rangle &= \langle\hat{F}_a^\dagger(t)\hat{a}(0)\rangle exp\bigg(-\frac{1}{2}\beta t\bigg) + \int_0^t dt' exp\bigg(-\frac{1}{2}\beta(t-t')\bigg)\\ &\quad\times\bigg[-g\langle\hat{F}_a^\dagger(t)\hat{\sigma}_a(t')\rangle + \langle\hat{F}_a^\dagger(t)\hat{F}_a(t')\rangle\bigg]. \end{aligned} \quad (5.18)$$

Therefor, with the help of Eqs. (2.49) and (3.46), Eq. (5.18) can be put in the form

$$\langle\hat{F}_a^\dagger(t)\hat{a}(t)\rangle = \int_0^t dt' exp\bigg(-\frac{1}{2}\beta(t-t')\bigg)\big\langle\hat{F}_a^\dagger(t')\hat{F}_a(t)\big\rangle. \quad (5.19)$$

We can now rewrite

$$\langle: \hat{F}_a^\dagger(t)\hat{a}(t) :\rangle_F = \int_0^t dt' exp\bigg(-\frac{1}{2}\beta(t-t')\bigg)\big\langle : \hat{F}_a^\dagger(t')\hat{F}_a(t) : \big\rangle_F. \quad (5.20)$$

We note that

$$\langle: \hat{F}_a^\dagger(t)\hat{a}(t) :\rangle_F = \int_0^t dt' exp\bigg(-\frac{1}{2}\beta(t-t')\bigg)\big\langle\hat{F}_a^\dagger(t)\hat{F}_a(t')\big\rangle. \quad (5.21)$$

We recall that for a cavity mode coupled to a vacuum reservoir

$$\big\langle\hat{F}_a^\dagger(t)\hat{F}_a(t')\big\rangle_F = 0. \quad (5.22)$$

Thus Eq. (5.21) turns out to be

$$\langle : \hat{F}_a^\dagger(t)\hat{a}(t) :\rangle_F = 0. \tag{5.23}$$

Hence on account of Eqs. (5.13) and (5.23) along with its complex conjugate, Eq. (5.6) takes the form

$$\frac{d}{dt}\Big\langle : \hat{a}^\dagger(t)\hat{a}(t) : \Big\rangle_F \;=\; -\kappa\langle : \hat{a}^\dagger(t)\hat{a}(t) :\rangle_F - \varepsilon\big(\langle : \hat{b}\hat{a} :\rangle_F + \langle : \hat{a}^\dagger\hat{b}^\dagger :\rangle_F\big). \tag{5.24}$$

The steady-state solution of the above equation is found to be

$$\langle : \hat{a}^\dagger\hat{a} :\rangle_F \;=\; -\frac{\varepsilon}{\kappa}\bigg(\langle : \hat{b}\hat{a} :\rangle_F + \langle : \hat{a}^\dagger\hat{b}^\dagger :\rangle_F\bigg). \tag{5.25}$$

Furthermore, on applying the relation

$$\frac{d}{dt}\Big\langle\hat{a}(t)\hat{a}^\dagger(t)\Big\rangle = \left\langle\hat{a}(t)\frac{d\hat{a}^\dagger(t)}{dt}\right\rangle + \left\langle\frac{d\hat{a}(t)}{dt}\hat{a}^\dagger(t)\right\rangle \tag{5.26}$$

along with Eq. (2.29) and its adjoint, we easily obtain

$$\begin{aligned}\frac{d}{dt}\Big\langle\hat{a}(t)\hat{a}^\dagger(t)\Big\rangle \;&=\; -\kappa\langle\hat{a}(t)\hat{a}^\dagger(t)\rangle - \varepsilon\bigg(\langle\hat{a}\hat{b}\rangle + \langle\hat{b}^\dagger\hat{a}^\dagger\rangle\bigg)\\ &\quad -g\bigg(\Big\langle\hat{\sigma}_a(t)\hat{a}^\dagger(t)\Big\rangle + \Big\langle\hat{a}(t)\hat{\sigma}_a^\dagger(t)\Big\rangle\bigg)\\ &\quad +\Big\langle\hat{F}_a(t)\hat{a}^\dagger(t)\Big\rangle + \Big\langle\hat{a}(t)\hat{F}_a^\dagger(t)\Big\rangle.\end{aligned} \tag{5.27}$$

We can now write

$$\begin{aligned}\frac{d}{dt}\Big\langle : \hat{a}(t)\hat{a}^\dagger(t) : \Big\rangle_F \;&=\; -\kappa\langle : \hat{a}(t)\hat{a}^\dagger(t) :\rangle_F - \varepsilon\bigg(\langle : \hat{a}\hat{b} :\rangle_F + \langle : \hat{b}^\dagger\hat{a}^\dagger :\rangle_F\bigg)\\ &\quad -g\bigg(\Big\langle : \hat{\sigma}_a(t)\hat{a}^\dagger(t) : \Big\rangle_F + \Big\langle : \hat{a}(t)\hat{\sigma}_a^\dagger(t) : \Big\rangle_F\bigg)\\ &\quad +\Big\langle : \hat{F}_a(t)\hat{a}^\dagger(t) : \Big\rangle_F + \Big\langle : \hat{a}(t)\hat{F}_a^\dagger(t) : \Big\rangle_F.\end{aligned} \tag{5.28}$$

Next employing Eq. (2.33) and its adjoint, we have

$$\Big\langle : \hat{\sigma}_a(t)\hat{a}^\dagger(t) : \Big\rangle_F + \Big\langle : \hat{a}(t)\hat{\sigma}_a^\dagger(t) : \Big\rangle_F = \frac{4\kappa g}{\kappa^2 - 4\varepsilon^2}\Bigg[\frac{\varepsilon\langle\hat{\sigma}_a(t)\hat{\sigma}_b(t)\rangle}{\kappa} - \frac{\langle\hat{\sigma}_a(t)\hat{\sigma}_a^\dagger(t)\rangle}{2} + \frac{\langle\hat{\sigma}_a(t)\hat{F}_a^\dagger(t)\rangle}{2g} - \frac{\varepsilon\langle\hat{\sigma}_a(t)\hat{F}_b(t)\rangle}{\kappa g} + \frac{\varepsilon\langle\hat{\sigma}_b^\dagger(t)\hat{\sigma}_a^\dagger(t)\rangle}{\kappa} - \frac{\langle\hat{\sigma}_a(t)\hat{\sigma}_a^\dagger(t)\rangle}{2} + \frac{\langle\hat{F}_a(t)\hat{\sigma}_a^\dagger(t)\rangle}{2g} - \frac{\varepsilon\langle\hat{F}_b^\dagger(t)\hat{\sigma}_a^\dagger(t)\rangle}{\kappa g}\Bigg]. \tag{5.29}$$

In addition, in view of Eq. (2.49), Eq. (5.29) takes the form

$$\Big\langle : \hat{\sigma}_a(t)\hat{a}^\dagger(t) : \Big\rangle_F + \Big\langle : \hat{a}(t)\hat{\sigma}_a^\dagger(t) : \Big\rangle_F = \frac{4\kappa g}{\kappa^2 - 4\varepsilon^2}\Bigg[\frac{\varepsilon\langle\hat{\sigma}_a(t)\hat{\sigma}_b(t)\rangle}{\kappa} - \frac{\langle\hat{\sigma}_a(t)\hat{\sigma}_a^\dagger(t)\rangle}{2} + \frac{\varepsilon\langle\hat{\sigma}_b^\dagger(t)\hat{\sigma}_a^\dagger(t)\rangle}{\kappa} - \frac{\langle\hat{\sigma}_a(t)\hat{\sigma}_a^\dagger(t)\rangle}{2}\Bigg]. \tag{5.30}$$

Moreover, applying Eqs. (2.4) and (2.5), one can readily establish that

$$\frac{\varepsilon\langle\hat{\sigma}_a(t)\hat{\sigma}_b(t)\rangle}{\kappa} - \frac{\langle\hat{\sigma}_a(t)\hat{\sigma}_a^\dagger(t)\rangle}{2} = -\frac{1}{2}\Big\langle\hat{\eta}_b(t)\Big\rangle \tag{5.31}$$

and

$$\frac{\varepsilon\langle\hat{\sigma}_b^\dagger(t)\hat{\sigma}_a^\dagger(t)\rangle}{\kappa} - \frac{\langle\hat{\sigma}_a(t)\hat{\sigma}_a^\dagger(t)\rangle}{2} = -\frac{1}{2}\Big\langle\hat{\eta}_b(t)\Big\rangle. \tag{5.32}$$

On introducing Eq. (2.108) into Eqs. (5.31) and (5.32), there follows

$$\frac{\varepsilon\langle\hat{\sigma}_a(t)\hat{\sigma}_b(t)\rangle}{\kappa} - \frac{\langle\hat{\sigma}_a(t)\hat{\sigma}_a^\dagger(t)\rangle}{2} = 0 \tag{5.33}$$

and

$$\frac{\varepsilon\langle\hat{\sigma}_b^\dagger(t)\hat{\sigma}_a^\dagger(t)\rangle}{\kappa} - \frac{\langle\hat{\sigma}_a(t)\hat{\sigma}_a^\dagger(t)\rangle}{2} = 0. \tag{5.34}$$

Hence on substituting Eqs. (5.33) and (5.34) into Eq. (5.30), we get

$$\Big\langle : \hat{\sigma}_a(t)\hat{a}^\dagger(t) : \Big\rangle_F + \Big\langle : \hat{a}(t)\hat{\sigma}_a^\dagger(t) : \Big\rangle_F = 0. \tag{5.35}$$

Furthermore, multiplying Eq. (5.16) by $\hat{F}_a^\dagger(t)$ from the right and taking the expectation value of the resulting equation, we have

$$\langle\hat{a}(t)\hat{F}_a^\dagger(t)\rangle = \langle\hat{a}(0)\hat{F}_a^\dagger(t)\rangle exp\Big(-\frac{1}{2}\beta t\Big) + \int_0^t dt' exp\Big(-\frac{1}{2}\beta(t-t')\Big) \times\Big[-g\langle\hat{\sigma}_a(t')\hat{F}_a^\dagger(t)\rangle + \langle\hat{F}_a(t')\hat{F}_a^\dagger(t)\rangle\Big]. \quad (5.36)$$

With the aid of Eqs. (2.49) and (3.46), Eq. (5.36) can be put in the from

$$\langle\hat{a}(t)\hat{F}_a^\dagger(t)\rangle = \int_0^t dt' exp\Big(-\frac{1}{2}\beta(t-t')\Big)\big\langle\hat{F}_a(t')\hat{F}_a^\dagger(t)\big\rangle. \quad (5.37)$$

And this equation can be rewritten as

$$\big\langle :\hat{a}(t)\hat{F}_a^\dagger(t):\big\rangle_F = \int_0^t dt' exp\Big(-\frac{1}{2}\beta(t-t')\Big)\big\langle :\hat{F}_a(t')\hat{F}_a^\dagger(t):\big\rangle_F. \quad (5.38)$$

Thus we notice that

$$\big\langle :\hat{a}(t)\hat{F}_a^\dagger(t):\big\rangle_F = \int_0^t dt' exp\Big(-\frac{1}{2}\beta(t-t')\Big)\big\langle\hat{F}_a^\dagger(t)\hat{F}_a(t')\big\rangle. \quad (5.39)$$

In view of Eq. (5.22), Eq. (5.39) goes over into

$$\big\langle :\hat{a}(t)\hat{F}_a^\dagger(t):\big\rangle_F = 0. \quad (5.40)$$

Then employing Eqs. (5.35) and (5.40) along with its complex conjugate, Eq. (5.28) can be written as

$$\frac{d}{dt}\big\langle :\hat{a}(t)\hat{a}^\dagger(t):\big\rangle_F = -\kappa\langle :\hat{a}(t)\hat{a}^\dagger(t):\rangle_F - \varepsilon\Big(\langle :\hat{a}\hat{b}:\rangle_F + \langle :\hat{b}^\dagger\hat{a}^\dagger:\rangle_F\Big). \quad (5.41)$$

The steady-state solution of the above equation is found to be

$$\langle :\hat{a}\hat{a}^\dagger:\rangle_F = -\frac{\varepsilon}{\kappa}\Big(\langle :\hat{a}\hat{b}:\rangle_F + \langle :\hat{b}^\dagger\hat{a}^\dagger:\rangle_F\Big). \quad (5.42)$$

Following a similar procedure, one can readily establish that

$$\langle :\hat{b}^\dagger\hat{b}:\rangle_F = -\frac{\varepsilon}{\kappa}\Big(\langle :\hat{a}\hat{b}:\rangle_F + \langle :\hat{b}^\dagger\hat{a}^\dagger:\rangle_F\Big). \quad (5.43)$$

and

$$\langle :\hat{b}\hat{b}^\dagger:\rangle_F = -\frac{\varepsilon}{\kappa}\bigg(\langle :\hat{b}\hat{a}:\rangle_F + \langle :\hat{a}^\dagger\hat{b}^\dagger:\rangle_F\bigg) + \frac{\kappa\gamma_c}{4\varepsilon^2+\kappa^2}. \tag{5.44}$$

We observe that the effect of the noise operators doesn't appear in equations (5.25), (5.42), (5.43) and, (5.44). This is because we have calculated the result given by these equations by normally ordering the vacuum reservoir noise operators.

Now using Eqs. (2.108) and (2.111)-(2.113) in Eqs. (2.79) and (2.80), we readily obtain

$$\langle :\hat{a}^\dagger\hat{a}:\rangle_F = \frac{2\varepsilon^2}{\kappa^2-4\varepsilon^2} - \left(\frac{\gamma_c}{\kappa}\right)\frac{2\varepsilon^2(\kappa^2-2\varepsilon^2)}{(\kappa^2-4\varepsilon^2)(\kappa^2+4\varepsilon^2)} \tag{5.45}$$

and

$$\langle :\hat{b}^\dagger\hat{b}:\rangle_F = \frac{2\varepsilon^2}{\kappa^2-4\varepsilon^2} - \left(\frac{\gamma_c}{\kappa}\right)\frac{4\varepsilon^4}{(\kappa^2-4\varepsilon^2)(\kappa^2+4\varepsilon^2)}. \tag{5.46}$$

The second term in Eq. (5.45) or (5.46) represents the effect of the interaction of the subharmonic light modes with the three-level atom. From Eqs. (5.25) and (5.42)-(5.44), we note that

$$\langle :\hat{a}\hat{a}^\dagger:\rangle_F = \langle :\hat{b}^\dagger\hat{b}:\rangle_F \tag{5.47}$$

and

$$\langle :\hat{b}\hat{b}^\dagger:\rangle_F = \langle :\hat{a}^\dagger\hat{a}:\rangle_F + \frac{\kappa\gamma_c}{\kappa^2+4\varepsilon^2}. \tag{5.48}$$

Upon adding Eqs. (5.25) and (5.43), one easily finds

$$-\frac{\kappa}{\varepsilon}\bigg(\langle :\hat{a}^\dagger\hat{a}:\rangle_F + \langle :\hat{b}^\dagger\hat{b}:\rangle_F\bigg) = \langle :\hat{b}\hat{a}:\rangle_F + \langle :\hat{a}\hat{b}:\rangle_F + \langle :\hat{a}^\dagger\hat{b}^\dagger:\rangle_F + \hat{b}^\dagger\hat{a}^\dagger:\rangle_F. \tag{5.49}$$

On account of Eqs. (5.47)-(5.49), Eq. (5.3) can be put in the form

$$\bigg(:(\Delta c_+)^2:\bigg)_F = \frac{\kappa\gamma_c}{\kappa^2+4\varepsilon^2} + 2\big(\langle :\hat{a}^\dagger\hat{a}:\rangle_F + \langle :\hat{b}^\dagger\hat{b}:\rangle_F\big) - \frac{\kappa}{\varepsilon}\bigg(\langle :\hat{a}^\dagger\hat{a}:\rangle_F + \langle :\hat{b}^\dagger\hat{b}:\rangle_F\bigg) \tag{5.50}$$

and

$$\Big(: (\Delta c_-)^2 : \Big)_F = \frac{\kappa\gamma_c}{\kappa^2+4\varepsilon^2} + 2\big(\langle: \hat{a}^\dagger\hat{a} :\rangle_F + \langle: \hat{b}^\dagger\hat{b} :\rangle_F\big) + \frac{\kappa}{\varepsilon}\Big(\langle: \hat{a}^\dagger\hat{a} :\rangle_F + \langle: \hat{b}^\dagger\hat{b} :\rangle_F\Big). \tag{5.51}$$

Then upon substituting Eqs. (5.45) and (5.46) into Eqs. (5.50) and (5.51), we readily arrive at

$$\Big(: (\Delta c_+)^2 : \Big)_F = \frac{\kappa\gamma_c}{\kappa^2+4\varepsilon^2}\Big(\frac{\kappa+4\varepsilon}{\kappa+2\varepsilon}\Big) - \frac{4\varepsilon}{\kappa+2\varepsilon} \tag{5.52}$$

and

$$\Big(: (\Delta c_-)^2 : \Big)_F = \frac{\kappa\gamma_c}{\kappa^2+4\varepsilon^2}\Big(\frac{\kappa-4\varepsilon}{\kappa-2\varepsilon}\Big) + \frac{4\varepsilon}{\kappa-2\varepsilon}. \tag{5.53}$$

These results represent the plus and minus quadrature variance for the two-mode cavity light obtained by normally ordering the noise operators. The first term in Eq. (5.52) or (5.53) represents the effect of the interaction of the subharmonic light modes with the three-level atom. Whereas the second term is due to the subharmonic generation. Moreover, we note that for $\varepsilon = 0$, Eqs. (5.52) and (5.53) reduce to

$$\Big(: (\Delta c_+)^2_{vac} : \Big)_F = \Big(: (\Delta c_-)^2_{vac} : \Big)_F = \frac{\gamma_c}{\kappa}. \tag{5.54}$$

This indeed represents the quadrature variance for a two-mode cavity vacuum state with the noise operators in normal order. From the plots in Fig. 5.1 we see that the quadrature variance is negative for $\gamma_c < \frac{16}{15}$, is zero for $\gamma_c = \frac{16}{15}$, and is positive for $\gamma_c > \frac{16}{15}$ for values of ε close to 0.4. Moreover, from the plots in Fig. 5.1, we observe that the two-mode cavity light is in a squeezed state for any value of $\gamma_c > 0$.

Now we seek to determine the uncertainty relation for the plus and minus quadratures with normal ordering of the vacuum reservoir noise operators. Applying Eq. (3.1) along with Eqs. (2.73) and (2.75), Eq. (4.4) can be written as

$$\Big(: \Delta c_+ \Delta c_- : \Big)_F \geq \Big| \big\langle : \hat{a}\hat{a}^\dagger :\rangle_F + \big\langle : \hat{b}\hat{b}^\dagger : \big\rangle_F - \big(\big\langle : \hat{a}^\dagger\hat{a} : \big\rangle_F + \big\langle : \hat{b}^\dagger\hat{b} : \big\rangle_F\big)\Big|, \tag{5.55}$$

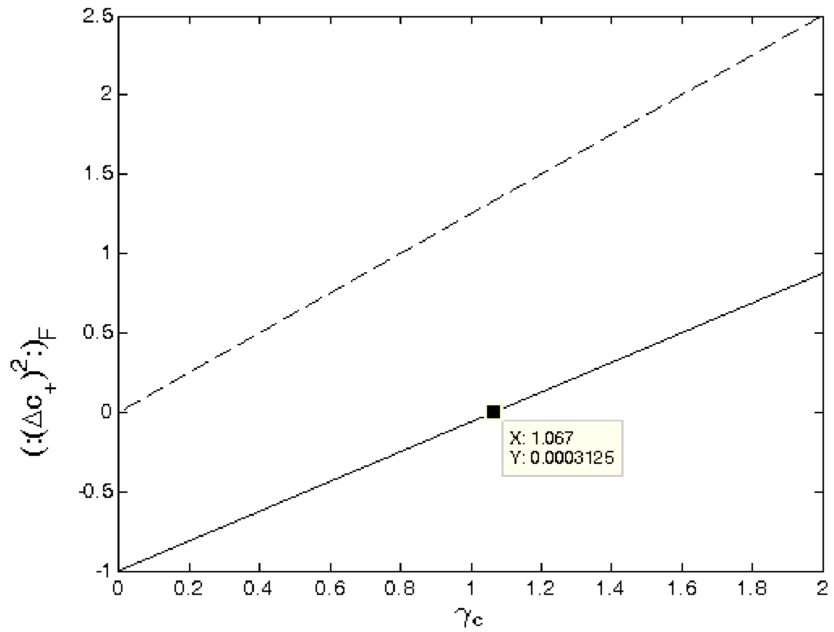

Figure 5.1: Plots of the plus quadrature variance [Eqs. (5.52) and (5.54)] versus γ_c for $\kappa = 0.8$, $\varepsilon \approx 0.4$ (solid line), and $\varepsilon = 0$ (dashed line).

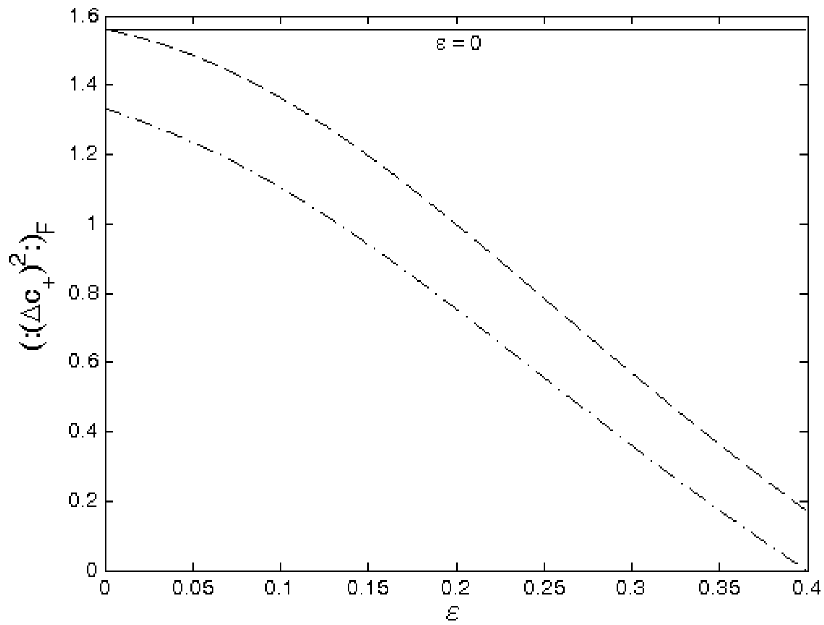

Figure 5.2: Plots of the plus quadrature variance [Eqs. (5.52) and (5.54)] versus ε for $\kappa = 0.8$, $\gamma_c = \frac{16}{15}$ (dotted line), and $\gamma_c = 1.25$ (dashed line).

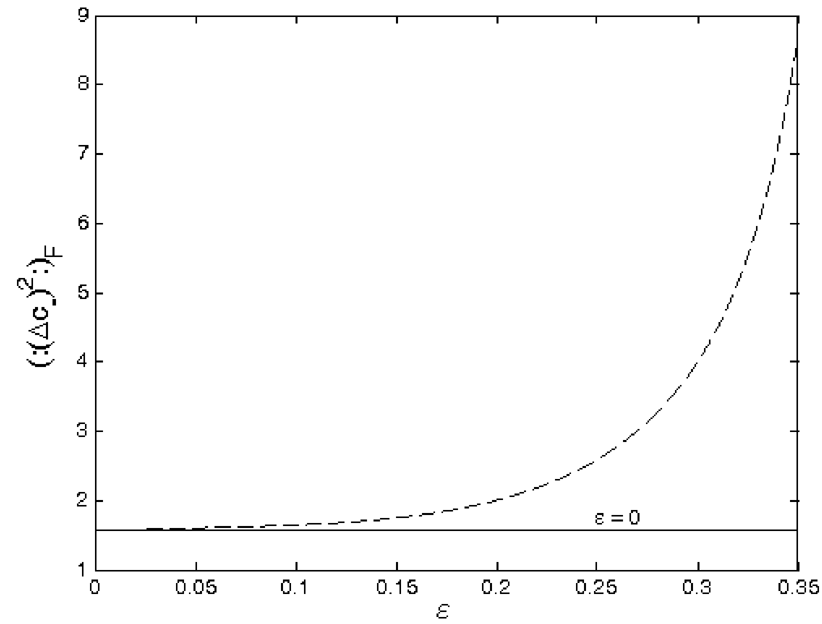

Figure 5.3: Plots of the minus quadrature variance [Eqs. (5.52) and (5.54)] versus ε for $\kappa = 0.8$ and $\gamma_c = \frac{16}{15}$ (dashed line).

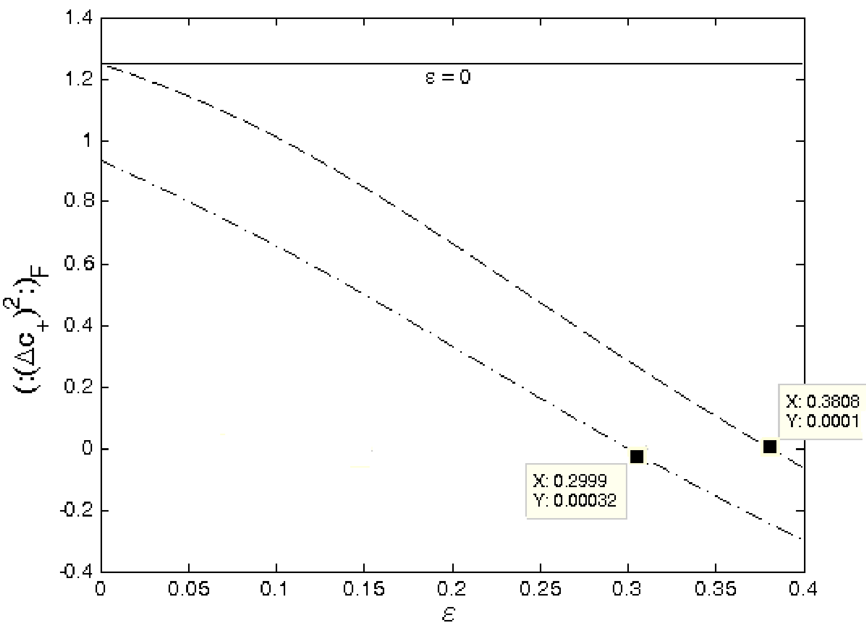

Figure 5.4: Plots of the plus quadrature variance [Eqs. (5.52) and (5.54)] versus ε for $\kappa = 0.8$, $\gamma_c = 0.75$ (doted line), and $\gamma_c = 1$ (dashed line).

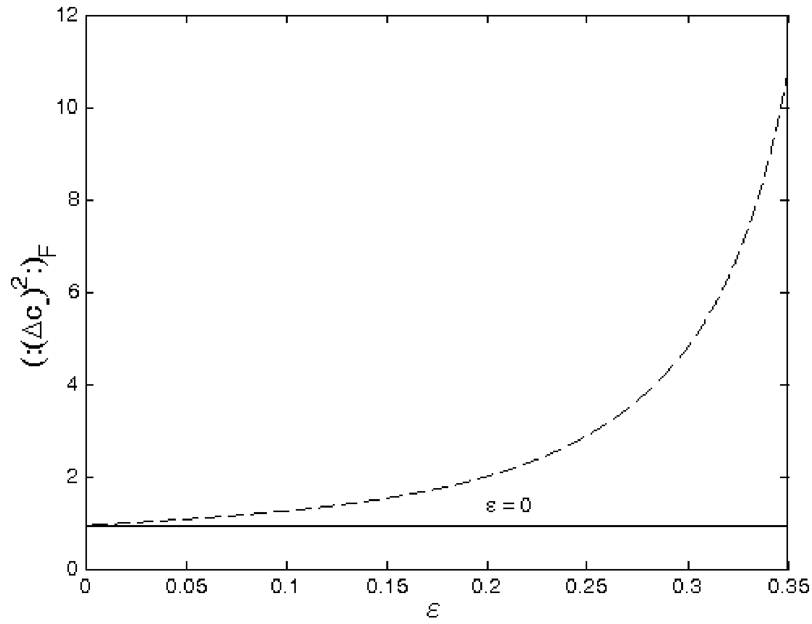

Figure 5.5: Plots of the minus quadrature variance [Eqs. (5.53) and (5.54)] versus ε for $\kappa = 0.8$ and $\gamma_c = 0.75$.

so that making use of Eqs. (5.45) - (5.48), we readily find

$$\bigg(: \Delta c_- \Delta c_+ : \bigg)_F \geq \frac{\kappa\gamma_c}{\kappa^2 + 4\varepsilon^2}. \tag{5.56}$$

Now upon setting $\varepsilon = 0$, the above equation reduces to

$$\bigg(: \Delta c_- \Delta c_+ : \bigg)_F \geq \frac{\gamma_c}{\kappa}. \tag{5.57}$$

This represents the uncertainty relation for the vacuum state with the noise operators in normal order. It proves to be more convenient to set

$$f_b(\varepsilon) = \bigg(: \Delta c_- \Delta c_+ : \bigg)_F, \tag{5.58}$$

so that employing Eqs. (5.52) and (5.53), we have

$$\begin{aligned} f_b(\varepsilon) \;=\; & \Bigg[\bigg(\frac{\kappa\gamma_c}{\kappa^2 + 4\varepsilon^2}\bigg(\frac{\kappa + 4\varepsilon}{\kappa + 2\varepsilon}\bigg) - \frac{4\varepsilon}{\kappa + 2\varepsilon}\bigg) \times \\ & \bigg(\frac{\kappa\gamma_c}{\kappa^2 + 4\varepsilon^2}\bigg(\frac{\kappa - 4\varepsilon}{\kappa - 2\varepsilon}\bigg) + \frac{4\varepsilon}{\kappa - 2\varepsilon}\bigg)\Bigg]^{\frac{1}{2}}. \end{aligned} \tag{5.59}$$

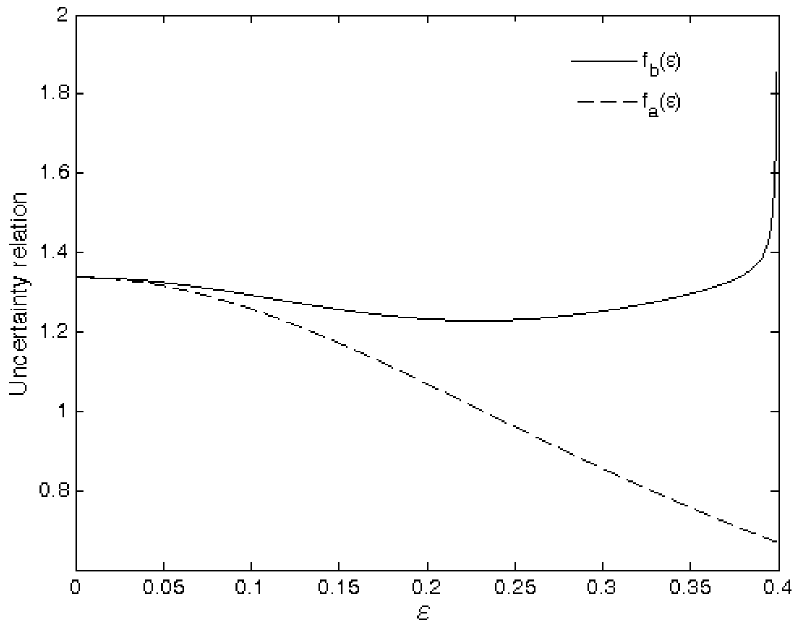

Figure 5.6: Plots of the uncertainty relation [Eqs. (5.59) and (5.60)] versus ε for $\kappa = 0.8$ and $\gamma_c = \frac{16}{15}$.

In addition we set

$$f_a(\varepsilon) = \frac{\kappa\gamma_c}{\kappa^2 + 4\varepsilon^2}. \tag{5.60}$$

From the plots in Fig.5.6, we see that the uncertainty relation for the two-mode cavity light with the normally ordered noise operators holds perfectly.

Finally we calculate the quadrature squeezing for the two-mode cavity light with the noise operators in normal order. We define the quadrature squeezing for the two-mode cavity light by

$$\Big(: S : \Big)_F = \frac{\Big(: (\Delta c_+)^2_{vac} : \Big)_F - \Big(: (\Delta c_+)^2 : \Big)_F}{\Big(: (\Delta c_+)^2_{vac} : \Big)_F}. \tag{5.61}$$

This can be rewritten as

$$\Big(: S : \Big)_F = 1 - \frac{\Big(: (\Delta c_+)^2 : \Big)_F}{\Big(: (\Delta c_+)^2_{vac} : \Big)_F}. \tag{5.62}$$

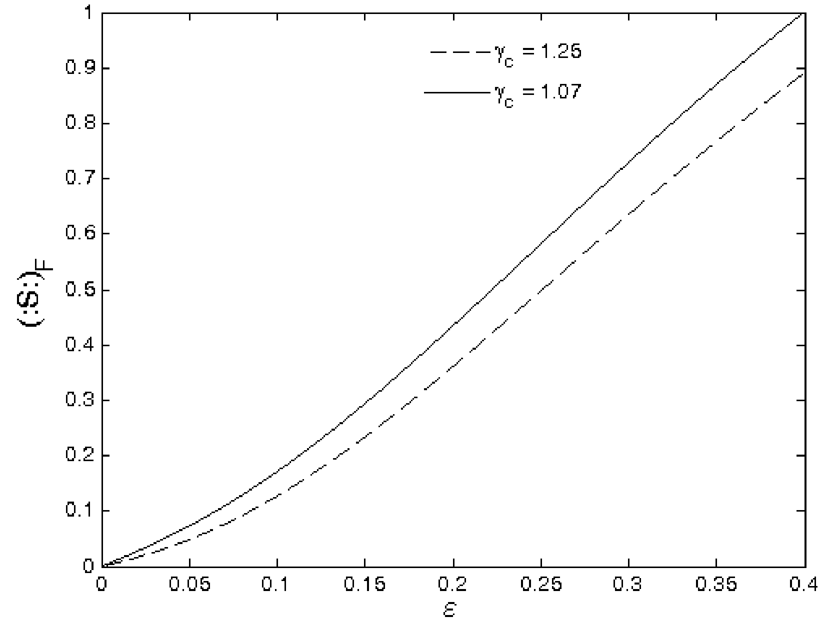

Figure 5.7: Plots of the quadrature squeezing [Eq. (5.63)] versus ε for $\kappa = 0.8$, $\gamma_c = \frac{16}{15}$, and $\gamma_c = 1.25$.

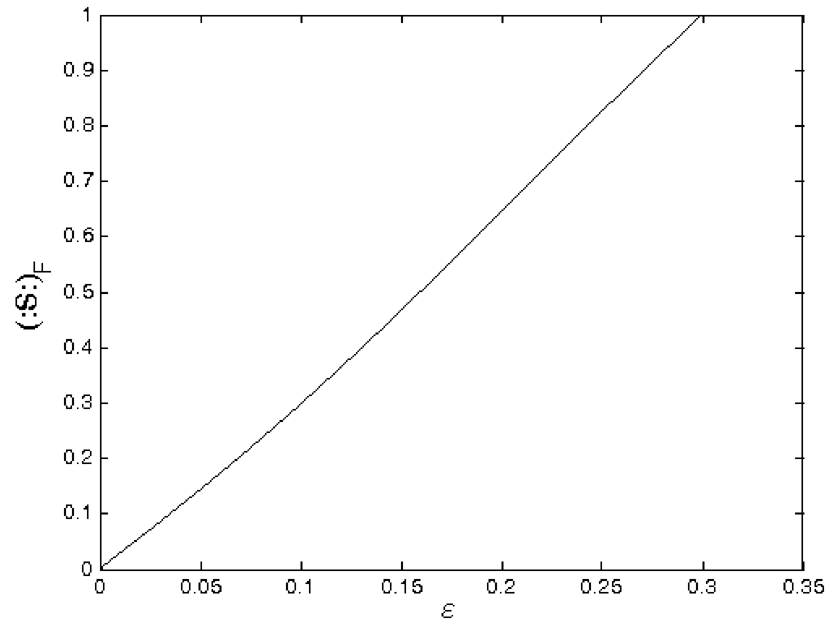

Figure 5.8: A plot of the quadrature squeezing [Eq. (5.63)] versus ε for $\kappa = 0.8$ and $\gamma_c = 0.75$.

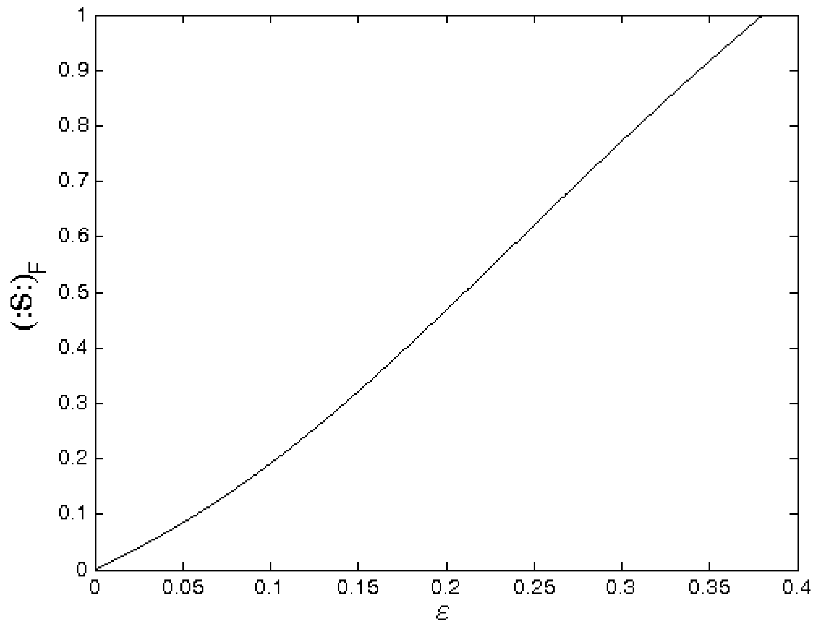

Figure 5.9: A plot of the quadrature squeezing [Eq. (5.63)] versus ε for $\kappa = 0.8$ and $\gamma_c = 1$.

Then on applying Eqs. (5.52) and (5.54) in Eq. (5.62), we readily get

$$\Big(:S:\Big)_F = 1 - \frac{\kappa}{\gamma_c}\left[\frac{\kappa\gamma_c}{\kappa^2+4\varepsilon^2}\left(\frac{\kappa+4\varepsilon}{\kappa+2\varepsilon}\right) - \frac{4\varepsilon}{\kappa+2\varepsilon}\right]. \tag{5.63}$$

This represents the quadrature squeezing for the two-mode cavity light when the noise operators are in normal order. From the plots in Fig.5.7, we notice that the maximum quadrature squeezing is 100% for $\gamma_c = \frac{16}{15}$ and is 88.3% for $\gamma_c = 1.25$ at values of ε close to 0.4. Moreover, from the plots in Fig.5.8 and Fig.5.9, we also observe that the maximum quadrature squeezing is 100% for $\gamma_c = 0.75$ and for $\gamma_c = 1$ at values of ε close to 0.3 and 0.38, respectively.

6

Conclusion

In this dissertation we have studied the photon statistical and quadrature squeezing properties of the two-mode light available in the cavity following the interaction of the subharmonic light modes with the three-level atom. We have considered the case in which the vacuum reservoir noise operators are in arbitrary order as well as in normal order. Employing the pertinent Hamiltonian and master equations, we have obtained the equations of evolution of the expectation values for the cavity mode and atomic operators.

Making use of the steady-state solutions of the equations of evolution of the expectation values for the cavity mode and atomic operators, we have calculated the global mean photon number for the two-mode cavity light. The global mean photon number of the two-mode cavity light consists of the mean photon number due to the subharmonic generation, the mean photon number emitted and absorbed by the atom. We have observed that the effect of the interaction of the subharmonic light modes with the three-level atom is to decrease the global mean photon number of the two-mode cavity light. This is due to the fact that the mean number of photons absorbed is greater than the mean number of photons emitted.

Furthermore, applying the time dependent solutions of the equations of evolution of the expectation values for the cavity mode and atomic operators, we have also obtained the local mean photon number for the two-mode cavity light. Our analysis shows that the local mean photon number of the two-mode cavity light increases with frequency and eventually approaches to the global mean photon number within a relatively small frequency interval.

In addition, we have determined the global photon-number variance for the two-mode cavity light. We have noticed that the photon statistics of the two-mode cavity light is supper poissonian. Moreover, our analysis indicates that the global photon-number variance in the presence of the interaction is less than that in the absence of the interaction. This implies that the effect of the interaction is to decrease the global photon-number variance.

On the other hand, we have determined the global and local quadrature variance for the two-mode cavity light with arbitrary ordering of the noise operators associated with the two-mode vacuum reservoir. We have noticed that the two-mode cavity light is in squeezed state and the squeezing occurs in the plus quadrature. We have found that the maximum global quadrature squeezing in the presence of the interaction is 43.35% and 50% in the absence of the interaction. From this result, we see that the presence of the interaction is to decrease the global quadrature squeezing. In addition, we have seen that the maximum local quadrature squeezing for the two-mode cavity light is 60.8% and occurs at $\lambda = 0.17$. We have also noticed that the local quadrature squeezing decreases with the frequency and eventually approaches to the global quadrature squeezing.

Finally we have considered the global quadrature squeezing for the two-mode cavity light with the two-mode vacuum reservoir noise operators in normal order. It so happens that the quadrature variance is negative for $\gamma_c < \frac{16}{15} \approx 1.07$, is zero for $\gamma_c = \approx 1.07$, and is positive for $\gamma_c > \frac{16}{15} \approx 1.07$. In addition, we have noted that the two-mode cavity light is in squeezed state for arbitrary values of γ_c. It is not hard to realize that the emergence of negative quadrature variance is due to normal ordering of the noise operators. We have found that the maximum global quadrature squeezing for $\gamma_c = \frac{16}{15} \approx 1.07$ is 100% and for $\gamma_c = 1.25$ is 88.3% at values of ε close to 0.4. Besides, it is found that the maximum global quadrature squeezing for $\gamma_c = 0.75$ and for $\gamma_c = 1$ is 100% at values of ε close to 0.3 and 0.38, respectively. Unfortunately, we have not been able to introduce an appropriate procedure with the aid of which the amount of squeezing available can be detrmined when the quadrature variance is below the vacuum level and at the same time negative.

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