Interior Dirichlet Problem and Existence of Results

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The undersigned hereby certify that they have read and recommend to the school of graduate studies for acceptance of a project entitled on **Interior Dirichlet Problem and Existence of Results** by Wodefit Shigute Mossa in partial fulfillment of the requirements for the degree of Master of Science in Mathematics.

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Abstract

In this project paper we investigated solution for interior Dirichlet problem as main parts. First, we showed uniqueness and continuity of Dirichlet problem in bounded region using maximum and minimum principle. Next, we transform the Laplace equation in two dimensional Cartesian coordinate to Laplace equation in two dimensional polar coordinate. Finally, the Dirichlet problem in bounded domain using separation of variable and Fourier series by appealing to annular region, rectangular region, heat equation and potential theory were discussed.

Contents

Notations

∇	Gradient
Ω	Omega
δ	Delta
$\mathbb{R}$	Real line
γ	Gamma
∂	Partial
θ	Theta
Δ	Laplace operator
$\mathbb{C}$	Complex number
α	Alpha
Φ	Phi
λ	Lambda
$\in$	Element of
ϵ	Epsilon
$\subset$	Subset
$\cap$	Intersection
$\cup$	Union
$\forall$	For all
$\notin$	Not an element of
∞	Infinity
$\sum$	Sum
π	Pi
$\neq$	Not equal to
$\Leftrightarrow$	Bi-implication
$\int$	Integration

Introduction

The Dirichlet problem is named after Peter Gustav Lejeune Dirichlet, who proposed a solution by a variation method which becomes known as Dirichlet principle. The existence of a unique solution is very plausible by the 'physical argument': any charge distribution on the boundary should, by the laws of electrostatics, determine an electrical potential as solution.

However, Karl Weierstass found a flaw in Dirichlet's argument, and a rigorous proof of existence was found only in 1900 by David Hilbert. It turns out that the existence of solution depends delicately on the smoothness of the boundary and the prescribed data.

The Dirichlet problem turned out to be fundamental in many areas of mathematics and physics, and the efforts to solve this problem led directly to many revolutionary ideas in mathematics. In mathematics, a Dirichlet problem is a problem of finding a function which solves a specified partial differential equation (PDE) in the interior of a given region that takes prescribed values on the boundary of the region.

The Dirichlet problem can be solved for many PDEs, although originally it was posed for Laplace's equation and the problem can be stated as follows:
Given a domain $\Omega \subset \mathbb{R}^n$ *and a function* $f : \partial\Omega \to \mathbb{R}$*, the Dirichlet problem is to find a function* u *satisfying*

$$\begin{cases} \Delta u = 0 & in\ \ \Omega \\ u = f & on\ \ \partial\Omega. \end{cases}$$

For a simple version of Dirichlet problem, we consider the following more general form of the problem. Let Ω be a region with boundary $\partial\Omega$, and let f be a real continuous function defined on $\partial\Omega$. The problem of Dirichlet consists in finding a function u satisfying following conditions:

1. u is continuous in $\Omega \cup \partial\Omega$.
2. u is harmonic in Ω.
3. $u = f$ on $\partial\Omega$

This requirement is called the ***Dirichlet boundary condition***. The main issue is to prove the existence of a solution; uniqueness can be proved using the maximum principle.

The purpose of this project paper is to show how to solve and solution to interior Dirichlet problem in bounded region for annular, rectangular, heat equation and potential theory.

The content of this project paper is organized as follows: In introduction, we visited about overview of the Dirichlet problem in bounded region. In chapter one, we confirmed the uniqueness and continuity, and the maximum and minimum principle. In chapter two, we discuss on the Fourier series and method of separation of variables. Finally, in chapter three, the Dirichlet problem in bounded region were discussed.

Chapter 1

Harmonic function and The Maximum Principle

1.1 Harmonic Function

Definition (Harmonic functions): Let $\Omega \subset \mathbb{C}$ be a domain. A function $u : \Omega \to \mathbb{R}$ is harmonic if it is twice continuously differentiable on Ω, and

$$u_{xx} + u_{yy} \equiv 0$$

Note:

- The first condition implies that the partial derivatives u_x, u_y, u_{xx}, u_{xy}, u_{yx}, u_{yy} exist and are continuous on Ω.
- The second-order partial differential equation defining the condition on u is called the Laplace condition. It can also be written

  $$\Delta u = 0$$

  where Δ stands for the set of partial derivatives above.

1.2 Boundary Value Problem

A boundary-value problem is finding a function which satisfies a given partial differential equation and particular boundary conditions. There are three basic types of boundary conditions that are usually associated with Laplaces equation. They are

Dirichlet Boundary Value Problem (BVP) : If the boundary condition (BC) are of Dirichlet type i.e., if the solution $u(x, y)$ to Laplace equation in a domain Ω is specified on the boundary $\partial\Omega$ i.e.,

$$u(x, y) = f(x, y) \; on \; \partial\Omega,$$

where $f(x, y)$ is a given function. The Laplace equation together with Dirichlet boundary condition (BC)are called **Dirichlet problem/Dirichlet boundary value problem (BVP).** The Dirichlet problem for the Laplace equation is of the form

$$\Delta u(x, y) = 0 \, in \, \Omega; \qquad u(x, y) = f(x, y) \, on \, \partial\Omega.$$

Neumann Boundary Value Problem (BVP) :The BC are of Neumann type if the directional derivative $\frac{\partial u}{\partial n}$ along the outward normal to the boundary is specified on $\partial\Omega$ i.e.,

$$\frac{\partial u}{\partial n}(x, y) = g(x, y) \;\; for \;\; (x, y) \in \partial\Omega.$$

In physical terms, the normal component of the solution gradient is known on the boundary. The Laplace equation together with Neumann BC are called the **Neumann BVP/ Neumann problem** which is written as

$$\Delta u = 0 \;\; in \;\; \Omega; \quad \frac{\partial u}{\partial n}(x, y) = g(x, y) \;\; for \;\; (x, y) \in \partial\Omega.$$

The Neumann problem will have no solution unless we assume that the average value of the function g on $\partial\Omega$ is zero. This assumption is known as the compatibility condition

$$\int_{\partial\Omega} \frac{\partial u}{\partial n} = \int_{\partial\Omega} g = 0.$$

Robin's Boundary Value Problem (BVP)*:* The boundary conditions are called Robins type or mixed type if Dirichlet BC are specified on part of the boundary $\partial\Omega$ and Neumann type BC are specified on the remaining part of the boundary $\partial\Omega$. For example,

$$\frac{\partial u}{\partial n} + c(u - g) = 0,$$

where c is a constant and g is a given function that can vary over the boundary. The Laplace equation together with the Rabins/Mixed BC known as **Rabins BVP / Mixed BVP**.

1.3 Maximum and Minimum Principles

Before we prove the uniqueness and continuity theorems for the interior Dirichlet problem for Laplace equation, we first prove the maximum and minimum principles.

Theorem 1.3.1. (The Maximum Principle) *Let* $\Omega \subseteq \mathbb{R}^2$ *be a smooth domain with a boundary* $\partial\Omega$. *Let* $\overline{\Omega} = \Omega \cup \partial\Omega$. *Let* u *satisfy*

$$u_{xx} + u_{yy} = 0 \qquad in\ \Omega \qquad\qquad (1.3.1)$$

and let u *be continuous on* $\overline{\Omega}$. *Then* u *attains its maximum on the boundary* $\partial\Omega$ *of* Ω.

Proof. Let

$$M = \max_{(x,y)\in\partial\Omega}\{u(x,y)\} = max\ \{u(x,y) : (x,y) \in \partial\Omega\}$$

Let us suppose that the maximum of u on $\overline{\Omega}$ is **not attained** on $\partial\Omega$. Let $P = (x_o, y_o) \in \Omega((x_o, y_o) \notin \partial\Omega)$ such that u attains its maximum at P. Let $M_0 = u(x_o, y_o)$. Then $M_0 > M$. Define

$$v(x,y) = u(x,y) + \frac{M_0 - M}{4R^2}\{(x - x_o)^2 + (y - y_o)^2\}, \qquad (x,y) \in \Omega$$

and R is the radius of a circle with center at (x_0, y_0) and containing Ω. Notice $v(x_0, y_0) = u(x_0, y_0) = M_0$. On $\partial\Omega$, we have

$$v(x,y) \leq M + \frac{M_0 - M}{4} < M_o$$

Now v attains its maximum in Ω (since $v(x_0, y_0) = M_0$ and $v(x,y) < M_0\ on\ \partial\Omega$), say at $Q(x_1, y_1)$. So

$$v_{xx} + v_{yy} \leq 0 \qquad at\ \ Q \qquad\qquad (1.3.2)$$

But on Ω

$$v_{xx} + v_{yy} = \frac{M_0 - M}{R^2} > 0 \qquad\qquad (1.3.3)$$

Now (1.3.2) and (1.3.3) leads to a contradiction (at Q). Thus, u attains its maximum on $\partial\Omega$.

Remarks: If u in Theorem 1.3.1 is not constant, we actually have proved that the maximum of u on $\overline{\Omega}$ is attained at point in $\partial\Omega$.

Theorem 1.3.2.(The Minimum Principle): *If $u(x,y)$ is harmonic in a bounded domain Ω and continuous in $\overline{\Omega} = \Omega \cup \partial\Omega$, then u attains its minimum on the boundary $\partial\Omega$ of Ω.*

Proof. The proof follows directly by applying the preceding theorem to the harmonic function $-u(x,y)$.

As a result of the above theorems, we see that $u =$ constant which is evidently harmonic attains the same value in the domain Ω as on the boundary $\partial\Omega$.

1.4 Uniqueness and Continuity Theorems

Theorem 1.4.1. (Uniqueness Theorem): *The solution u (if exists) of the Dirichlet problem,namely,*

$$\Delta u = 0 \; in \; \Omega \qquad and \qquad u = f \; on \; \partial\Omega$$

is unique.

Proof. Let $u_1(x,y)$ and $u_2(x,y)$ be two solutions of the Dirichlet problem. Then u_1 and u_2 satisfy

$$\Delta u_1 = 0, \quad \Delta u_2 = 0 \; in \; \Omega,$$
$$u_1 = f, \quad u_2 = f \; on \; \partial\Omega.$$

Since u_1 and u_2 are harmonic in Ω, $(u_1 - u_2)$ is also harmonic in Ω. But

$$u_1 - u_2 = 0 \;\; on \;\; \partial\Omega.$$

The maximum-minimum principle gives

$$u_1 - u_2 = 0$$

at all interior points of Ω. Thus, we have

$$u_1 = u_2$$

Therefore, the solution is unique.

Example 1.4.1: Let $D = (x, y) \,\epsilon\, \mathbb{R}^2 : x^2 + y^2 < 1 = (r, \theta) : 0 \leq \theta < 2\pi$. Consider the Dirichlet problem

$$\Delta u = 0 \; in \; \Omega$$

$$u(1, \theta) = \cos\theta - \sin\theta, \quad 0 \leq \theta < 2\pi$$

Consider $u(x, y) = x - y$. Obviously $\Delta u = 0$. On the boundary $x = \cos\theta$ and $y = \sin\theta$ and hence $u(x, y) = x - y = \cos\theta - \sin\theta$ on $\partial\Omega$. Thus, $u(x, y) = x - y$ is a solution of the Dirichlet problem. By uniqueness theorem, $u(x, y) = x - y$ is the unique solution of the given problem.

Theorem 1.4.2.(Continuity Theorem): *The solution of the Dirichlet problem depends continuously on the boundary data.*

Proof. Let u_1 and u_2 be the solutions of

$$\Delta u_1 = 0 \quad in \; \Omega,$$

$$u_1 = f_1 \quad on \; \partial\Omega,$$

and

$$\Delta u_2 = 0 \quad in \; \Omega,$$

$$u_2 = f_2 \quad on \; \partial\Omega,$$

If $v = u_1 - u_2$, then v satisfies

$$\Delta v = 0 \quad in \; \Omega,$$

$$v = f_1 - f_2 \quad on\ \partial\Omega.$$

By maximum and minimum principles, $f_1 - f_2$ attains the maximum and minimum of v on $\partial\Omega$. Thus, if $|f_1 - f_2| < \varepsilon$, then

$$-\varepsilon < v_{min} \leq v_{max} < \varepsilon \quad on\ \partial\Omega.$$

Thus, at any interior point in Ω, we have

$$-\varepsilon < v_{min} \leq v \leq v_{max} < \varepsilon.$$

Therefore, $|v| < \varepsilon$ in Ω. Hence,

$$|u_1 - u_2| < \varepsilon.$$

Theorem 1.4.3: *Let $\{u_n\}$ be a sequence of functions harmonic in Ω and continuous in Ω. Let f_i be the values of u_i on $\partial\Omega$. If a sequence u_n converges uniformly on $\partial\Omega$, then it converges uniformly in Ω.*

Proof: By hypothesis, $\{f_n\}$ converges uniformly on $\partial\Omega$. Thus, for $\varepsilon > 0$, there exists an integer N such that everywhere on $\partial\Omega$

$$|f_n - f_m| < \varepsilon, \ for\ n, \quad m > N.$$

It follows from the continuity theorem that for all $n,\ m > N$

$$|u_n - u_m| < \varepsilon$$

in Ω, and hence, the theorem is proved.

1.5 Laplace in Polar Coordinates

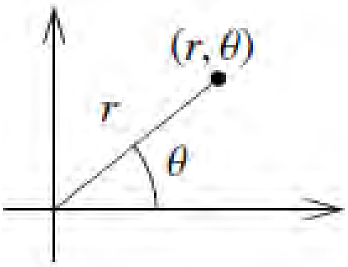
The Laplace's equation in two dimensional Cartesian coordinate is given by

$$u_{xx} + u_{yy} = 0$$

If the boundary of the region is a circle, then we must use polar coordinate (r, θ) . Let us find the equivalent Laplace's equation in polar coordinates.

$$u(x, y) \to u(r, \theta)$$

The first issue we face is that we do not know what the Laplacian is in polar coordinates.



Normally we would find u_{xx} and u_{yy} in terms of the derivatives in r and θ. We would need to solve for r and θ in terms of x and y. While this is certainly possible, it happens to be more convenient to work in reverse. Let us instead compute derivatives in r and θ in terms of derivatives in x and y and then solve. First

$$x_r = \cos\theta,\ x_\theta = -r\sin\theta,\ y_r = \sin\theta,\ y_\theta = r\cos\theta.$$

Next by chain rule we obtain

$$u_r = u_x u_r + u_y u_r = \cos(\theta)u_x + \sin(\theta)u_y,$$

$$u_{rr} = \cos(\theta)(u_{xx}x_r + u_{xy}y_r) + \sin(\theta)(u_{yx}x_r + u_{yy}y_r)$$

$$= \cos^2(\theta)u_{xx} + 2\cos(\theta)\sin(\theta)u_{xy} + \sin^2(\theta)u_{yy}.$$

Similarly for the θ derivative. Note that we have to use product rule for the second derivative.

$$u_\theta = u_x x_\theta + u_y y_\theta = -r\sin(\theta)u_x + r\cos(\theta)u_y,$$

$$\begin{aligned} u_{\theta\theta} &= -r\cos(\theta)(u_{xx}x_\theta + u_{xy}y_\theta) - r\sin(\theta)u_y + r\cos(\theta)(u_{yx}x_\theta + u_{yy}y_\theta) \\ &= -r\cos(\theta)u_x - r\sin(\theta)u_y + r^2\sin^2(\theta)u_{xx} - r^2 2\sin(\theta)u_{xy} + r^2\cos^2(\theta)u_{yy}. \end{aligned}$$

Let us now try to solve for $u_{xx} + u_{yy}$. We start with $\frac{1}{r^2}u_{\theta\theta}$ to get rid of those pesky r^2. If we add u_{rr} and use the fact that $\cos^2(\theta) + \sin^2(\theta) = 1$, we get

$$\frac{1}{r^2}u_{\theta\theta} + u_{rr} = u_{xx} + u_{yy} - \frac{1}{r}\cos(\theta)u_x - \frac{1}{r}\sin(\theta)u_y.$$

But all we are lacking is $\frac{1}{r}u_r$. Adding it we obtain the *Laplacian in polar coordinates:*

$$\frac{1}{r^2}u_{\theta\theta} + \frac{1}{r}u_r + u_{rr} = u_{xx} + u_{yy} = \Delta u. \tag{1.5.2}$$

Therefore,

$$\begin{array}{ccc} u_{xx} + u_{yy} = 0 & \iff & u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \\ (In\ \ Cartesian\ \ Coordinate) & & (In\ \ Polar\ \ Coordinate) \end{array}$$

Chapter 2

Fourier Series and Method of Separation of variables

2.1 Fourier Series

Fourier series is a very powerful tool in connection with various problems involving partial differential equations.

The functions

$$1, cosx, sinx, ..., cos2x, sin2x, ...$$

are mutually orthogonal to each other in the interval $[-\pi, \pi]$ and are linearly independent. Thus, we formally associate a trigonometric series with any piece wise continuous periodic function $f(x)$ of period 2π and write

$$\begin{aligned} f(x) &= a_0 + \sum_{n=1}^{\infty}(a_n cosnx + b_n sinnx) \\ &= a_0 + a_1 cosx + a_2 cos2x + a_3 cos3x + ... + b_1 sinx + b_2 cos2x + b_3 cos3xb + ... \end{aligned}$$

Definition 2.1.1: Function f is a periodic function if there is an $a > 0$ such that

$$f(x + a) = f(x), \quad \forall x \in \mathbb{R}. \qquad (2.1.1)$$

If this is the case a is called a period for f . Note that the period is not unique, but if there is a smallest such a, it is called the prime period of f .

Let f be a function of period 2π. We would like to get an expansion for f of the form

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos(nx) + b_n \sin(nx)\right], \qquad (2.1.2)$$

where the a_n and b_n are constants. Remember that $\sin(nx)$ and $\cos(nx)$ are periodic with period 2π.

If we integrate (2.1.2) term-by-term, we get

$$\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2}a_0 \int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty}\left[a_n \int_{-\pi}^{\pi} \cos(nx)dx + b_n \int_{-\pi}^{\pi} \sin(nx)dx\right]. \tag{2.1.3}$$

Since

$$\int_{-\pi}^{\pi} dx = 2\pi, \qquad \int_{-\pi}^{\pi} \cos(nx)dx = 0, \qquad \int_{-\pi}^{\pi} \sin(nx)dx = 0, \tag{2.1.4}$$

solving for a_0 gives

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx. \tag{2.1.5}$$

Lemma 2.1.1 Let $n, m \in \mathbb{N}$. We then have the following equalities:

$$\int_{-\pi}^{\pi} \sin(mx)\cos(nx)dx = 0, \tag{2.1.6}$$

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)dx = \pi\delta_{nm}, \tag{2.1.7}$$

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)dx = \pi\delta_{nm}, \tag{2.1.8}$$

where δ_{nm} is the Kronecker delta defined by

$$\delta_{nm} = \begin{cases} 0 & n \neq m, \\ 1 & n = m. \end{cases}$$

Proof. Equation (2.1.6) is trivial as $\sin(mx)\cos(nx)$ is odd. For equation (2.1.7) we compute, for $n \neq m$,

$$\begin{aligned} \int_{-\pi}^{\pi} \sin(mx)\sin(nx)dx &= \frac{1}{2}\int_{-\pi}^{\pi} \left[-\cos\{(m+n)x\} + \cos\{(m-n)x\}\right] dx \\ &= \frac{1}{2}\left[\frac{-\sin\{(m+n)x\}}{m+n} + \frac{\sin\{(m-n)x\}}{m-n}\right]_{-\pi}^{\pi} \\ &= 0. \end{aligned} \tag{2.1.9}$$

If $n = m$ we have

$$\begin{aligned}\int_{-\pi}^{\pi} \sin(nx)\sin(nx)dx &= \int_{-\pi}^{\pi} \left[\frac{1-\cos(2nx)}{2}\right] dx \\ &= \frac{1}{2}\left[x - \frac{\sin(2nx)}{2n}\right]_{-\pi}^{\pi} \\ &= \pi. \end{aligned} \tag{2.1.10}$$

Similar computations yield equation (2.1.8).

Thus, to find a_n and b_n, multiplying both sides of equation (2.1.2) by $\cos(mx)$ and integrating term-wise gives

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\cos(mx)dx &= \frac{1}{2}a_0 \int_{-\pi}^{\pi} \cos(mx)dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos(nx)\cos(mx)dx \\ &+ \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin(nx)\cos(mx)dx. \end{aligned} \tag{2.1.11}$$

The first term on the right-hand side is trivially zero for $m \neq 0$. Using Lemma 2.1.1 for the remaining terms give

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\cos(mx)dx &= \sum_{n=1}^{\infty} a_n \pi \delta_{nm} \\ &= \pi a_m \end{aligned} \tag{2.1.12}$$

and hence

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)dx \tag{2.1.13}$$

Note that this also holds for $m = 0$ (which is the reason for the factor of 1/2).

Multiplying equation (2.1.2) by $\sin(mx)$ and integrating term-wise, we similarly obtain

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)dx \tag{2.1.14}$$

Definition 2.1.2: Suppose f is such that

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx \tag{2.1.15}$$

exists. Then we shall write

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx)\right], \tag{2.1.16}$$

and call the series on the right-hand side the *Fourier series* for f , whether or not it converges to f . The constants a_n and b_n are called the Fourier coefficients of f.

2.2 Superposition Principle

A linear PDE can be written in the form

$$L[u] = f, \tag{2.2.1}$$

where $L[u]$ denotes a linear combination of u and some of its partial derivatives, with coefficients which are given functions of the independent variables.

Definition: (*Superposition principle*): Let u_1 be a solution of the linear PDE

$$L[u] = f_1$$

and let u_2 be a solution of the linear PDE

$$L[u] = f_2.$$

Then, for any constants c_1 and c_2, $c_1u_1 + c_2u_2$ is a solution of

$$L[u] = c_1f_1 + c_2f_2.$$

That is,

$$L[c_1u_1 + c_2u_2] = c_1f_1 + c_2f_2. \tag{2.2.2}$$

In particular, when $f_1 = 0$ and $f_2 = 0$, implies that if $u_1$and u_2 are solutions of the homogeneous linear PDE $L[u] = 0$, then $c_1u_1 + c_2u_2$ will also be a solution of $L[u] = 0$.

Remark: Note that the principle of superposition is not valid for nonlinear partial differential equations.

2.3 Method of Separation of variables

In this section we discuss the application of the method of separation of variables in the solution of PDEs. In developing a solution to a partial differential equation by separation of variables, one assumes that it is possible to separate the contributions of the independent variables into separate functions that each involve only one independent variable. Thus, the method consists of the following steps

1. Factorize the (unknown) dependent variable of the PDE into a product of functions, each of the factors being a function of one independent variable. That is, $u(x, y) = X(x)Y(y)$.
2. Substitute into the PDE, and divide the resulting equation by $X(x)Y(y)$.
3. Then the problem turns into a set of separated ODEs one for $X(x)$ and one for $Y(y)$.
4. The general solution of the ODEs is found, and boundary initial conditions are imposed.
5. $u(x, y)$ is formed by multiplying together $X(x)$ and $Y(y)$

Chapter 3

The Dirichlet Problem in Bounded Region

3.1 Dirichlet Problem and Principle

The Dirichlet problem for Laplaces equation on $\Omega \subset \mathbb{R}^n$ consists of finding a function u which satisfies the following conditions

$$\begin{cases} \Delta u(x) = 0, & x \in \Omega; \\ u(x) = g(x), & x \in \partial\Omega \end{cases} \qquad (3.1.1).$$

The inhomogeneous Laplace equation is known as Poissons equation. The Dirichlet problem for Poissons equation is to find a function $u \in C^2(\Omega)$, such that for a given function, $f \in C^0(\Omega)$, the following conditions hold

$$\begin{cases} \Delta u(x) = f(x), & x \in \Omega \\ u(x) = 0 & x \in \partial\Omega \end{cases} \qquad (3.1.2).$$

Note that by adding a solution of the Dirichlet problem for the Laplace equation on the same set one gets a solution for an inhomogeneous continuous boundary condition.

The problem (3.1.1) can be considered as a minimization problem according to the following principle.

The Dirichlet principle: *Let*

$$S = \{w \in C^2(\Omega),\ w = g \ \ on \ \ \partial\Omega\}$$

and let

$$I(w) = \frac{1}{2}\int_\Omega |\nabla w|^2 dx.$$

A function $u \in S$ is a solution to the Dirichlet problem for Laplaces equation on Ω, (3.1.1), if and only if

$$I(u) = \min_{w \in S} I(w).$$

Proof: Suppose u is a solution of (3.1.1). Then for any $w \in S$,

$$\begin{aligned} 0 &= \int_\Omega \Delta u(u-w)dx \\ &= -\int_\Omega |\nabla u|^2 dx + \int_\Omega \nabla u \nabla w dx \\ &\leq -\int_\Omega |\nabla u|^2 dx + \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + |\nabla w|^2\right) dx \\ &= -\frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{2}\int_\Omega |\nabla w|^2 dx. \end{aligned}$$

Therefore

$$\int_\Omega |\nabla u|^2 dx \leq \int_\Omega |\nabla w|^2 dx,$$

and since $w \in S$ was arbitrary

$$I(u) = \min_{w \in S} I(w).$$

Next suppose that $w \in S$ minimizes I. Let $v \in C^2$ be such that $v \equiv 0$ for $x \in \partial\Omega$. Then $u + \epsilon v \in S$ for all ϵ. Define

$$J(\epsilon) = I(u + \epsilon v).$$

Since u minimizes I, J must have minimum for $\epsilon = 0$. Now

$$J(\epsilon) = \int_\Omega \left(|\nabla u|^2 + 2\epsilon \nabla u \nabla v + \epsilon^2 |\nabla v|^2\right) dx,$$

implies

$$J'(\epsilon) = \int_\Omega \nabla u \nabla v + 2\epsilon |\nabla v|^2 dx,$$

so that

$$J'(0) = -\int_\Omega (\Delta u) v dx + \int_{\partial\Omega} \frac{\partial u}{\partial v} v d\sigma(x) = -\int_\Omega (\Delta u) dx$$

Now $J'(0) = 0$ implies

$$0 = \int_{\Omega} (\Delta u) v dx,$$

for all $v \in C^2(\Omega)$ such that $v = 0$ for $x \in \partial\Omega$, thus $\Delta u = 0$.

3.2 The Dirichlet Problem for the Annuli

We formulate the Dirichlet problem for the annulus in polar coordinates as follows:

$$PDE: \; u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \; r_1 < r < r_2, \tag{3.2.1}$$

$$BC: \; u(r_2, \theta) = f(\theta), \; f(\theta + 2\pi) = f(\theta),$$

$$u(r_1, \theta) = g(\theta), \; g(\theta + 2\pi) = g(\theta),$$

$$PC: \; u(r, \theta + 2\pi) = u(r, \theta), \; r_1 < r < r_2,$$

where $-\infty < \theta < \infty$ and PC stands for "periodically condition". Here, f and g are continuous periodic function with period 2π .

Using separation of variables, we seek solutions of the form

$$u(r, \theta) = R(r)T(\theta).$$

Substituting this into the PDE in (3.2.1) and separating variables, we obtain

$$R^{''}(r)T(\theta) + r^{-1}R^{'}T(\theta) + r^{-2}R(r)T^{''}(\theta) = 0$$

$$\Longrightarrow \frac{r^2R^{''}(r) + rR^{'}(r)}{R(r)} = -\frac{T^{''}(\theta}{T(\theta)} = c = \pm b^2 (b > 0).$$

This leads to the two ODEs

$$T^{''}(\theta) + cT(\theta) = 0, \tag{3.2.2}$$

$$r^2R^{''}(r) + rR^{'}(r) - cR(r) = 0. \tag{3.2.3}$$

Note that we get periodic solutions of period 2π , when $b = n$ and $c = +b^2 = n^2$, for $n =$ 0,1,2,... In this case, solving (3.2.2) we obtain

$$T_n(\theta) = a_n \cos n\theta + b_n \sin n\theta, \quad n = 0, 1, 2, ..., \tag{3.2.4}$$

where a_n and b_n are arbitrary constants. With $c = n^2$, equation (3.2.3) for $R(r)$ is the Cauchy-Euler equation

$$r^2 R^{''} + rR^{'}(r) - n^2 R(r) = 0 \tag{3.2.5}$$

This equation can be solved by taking $R(r) = r^m$. Substituting this into the (3.2.5), we get

$$r^2 m(m-1)r^{m-2} + rmr^{m-1} - n^2 r^m = 0$$

or

$$(m^2 - n^2)r^m = 0$$

$$\Longrightarrow r^m \ \ is\ a\ solution\ if \ \ m = \pm n.$$

For $n \geq 1$, the general solution is

$$R_n(r) = \begin{cases} c_n r^n + d_n r^{-n}, & for\ n = 1, 2, 3, ..., \\ c_o + d_o log(r), & for\ n = 0. \end{cases}$$

Putting together the expressions for $T_n(\theta)$ and $u_n(r, \theta)$, we obtain

$$u_o(r, \theta) = a_o + \alpha_o log(r), \tag{3.2.6}$$

$$u_n(r, \theta) = (a_n r^n + \alpha_n r^{-n}) \cos(n\theta) + (b_n r^n + \beta_n r^{-n}) \sin(n\theta), \ n \geq 1. \tag{3.2.7}$$

By the superposition principle, we obtain a more general solution of (3.2.3) as

$$u(r, \theta) = u_o(r, \theta) + \sum_{n=1}^{\infty} u_n(r, \theta) \tag{3.2.8}$$

Suppose that $f(\theta)$ and $g(\theta)$ have Fourier series of the form

$$f(\theta) = \frac{A_o}{2} + \sum_{n=1}^{\infty} A_n \cos n\theta + B_n \sin n\theta, \tag{3.2.9}$$

$$g(\theta) = \frac{C_o}{2} + \sum_{n=1}^{\infty} C_n \cos n\theta + D_n \sin n\theta \tag{3.2.10}$$

Comparing Fourier coefficients in the equations $u(r_2, \theta) = f(\theta)$ and $u(r_1, \theta) = g(\theta)$, we obtain

$$a_o + \alpha_o \log(r_2) = \frac{A_o}{2}; \quad a_o + \alpha_o \log(r_1) = \frac{C_o}{2}, \tag{3.2.11}$$

$$a_n r_2^n + \alpha_n r_2^{-n} = A_n, \quad a_n r_1^n + \alpha_n r_1^{-n} = C_n, \tag{3.2.12}$$

$$b_n r_2^n + \beta_n r_2^{-n} = B_n, \quad b_n r_1^n + \beta_n r_1^{-n} = D_n, \quad n = 1, 2, 3, ... \tag{3.2.13}$$

Solving for a_o ,α_o from (3.2.11), a_n ,α_n from (3.2.12) and b_n ,β_n from (3.2.13), we obtain

$$a_o = \frac{\frac{1}{2}C_o \log r_2 - \frac{1}{2}A_o \log r_1}{\log Q}, \quad \alpha_o = \frac{\frac{1}{2}A_o - \frac{1}{2}C_o}{\log Q} \tag{3.2.14}$$

$$a_n = \frac{A_n r_1^{-n} - C_n r_2^{-n}}{Q^n - Q^{-n}}, \qquad \alpha_n = \frac{C_n r_2^n - A_n r_1^{-n}}{Q^n - Q^{-n}}, \tag{3.2.15}$$

$$b_n = \frac{B_n r_1^{-n} - D_n r_2^{-n}}{Q^n - Q^{-n}}, \qquad \beta_n = \frac{D_n r_2^n - B_n r_1^n}{Q^n - Q^{-n}} \tag{3.2.16}$$

where $Q = r_2/r_1$. This provides us with the constants a_n, b_n, c_n, d_n in terms of the given Fourier coefficients A_n, B_n, C_n, D_n of $f(\theta)$ and $g(\theta)$.

Thus, the solution of (3.2.1), where $f(\theta)$ and $g(\theta)$ are given by (3.2.9)-(3.2.10), is

$$u(r, \theta) = a_o + \alpha_o \log r + \sum_{n=1}^{\infty} \left\{ [a_n r^n + \alpha_n r^{-n}] \cos n\theta + [b_n r^n + \beta_n r^{-n}] \sin n\theta \right\}, \tag{3.2.17}$$

where $a_n, \alpha_n, b_n, \beta_n$ are defined by (3.2.15)-(3.2.16).

Example 3.2.1: Solve the following Dirichlet problem

$$u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2} = 0, \qquad 1 < r < 2$$

$$u(1,\theta) = 1 + 4\cos(2\theta),$$

$$u(2,\theta) = 2 + 5\sin(\theta),$$

$$u(r,\theta + 2\pi) = u(r,\theta), \qquad 1 < r < 2.$$

Solution: Using the formulas (3.2.17) with $A_o = 2$, $C_o = 4$, $A_2 = 4$, $C_1 = 5$ and all other A_n, B_n, C_n, and D_n equal to 0. Note that $Q = r_2/r_1 = 2$.

Equating the Fourier coefficients in the BC with those of $u(r,\theta)$ in (3.2.17), using $r_1 = 1$and $r_2 = 2$, we obtain

$$a_o + \alpha_o \log(1) = 1, \quad a_o + \alpha_o \log(2) = 2,$$

$$b_1 + \beta_1 = 0, \quad 2b_1 + \frac{1}{2}\beta_1 = 5,$$

$$a_2 + \alpha_2 = 4, \quad 2^2 a_2 + 2^{-2}\alpha_2 = 0.$$

Solving (3.2.14) for a_0 and α_0, (3.2.15) for b_1 and β_1 and (3.2.16) for a_2 and α_2, we obtain

$$a_o = 1, \quad \alpha_o = \frac{1}{\log(2)}, \quad b_1 = 10/3, \quad \beta_1 = -10/3, \quad a_2 = -4/15, \quad \alpha_2 = 64/15$$

All other systems in (3.2.14)-(3.2.16) have solutions zero. The solution of (3.2.17) is then

$$u(r,\theta) = 1 + \frac{\log(r)}{\log(2)} + \left(\frac{10r}{3} - \frac{10r^{-1}}{3}\right)\sin(\theta) + \left(\frac{-4r^2}{15} + \frac{64r^{-2}}{15}\right)\cos(2\theta).$$

3.3 The Dirichlet Problem for the Disk

The Dirichlet problem in a disk of radius r_o and center at $(0,0)$ can be expressed as

$$\begin{aligned} PDE: &\quad u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2} = 0, \quad 0 < r < r_o, \quad -\pi \le \theta \le \pi \qquad (3.3.1) \\ BC: &\qquad u(r_o, \theta) = f(\theta), \quad -\pi \le \theta \le \pi, \end{aligned}$$

where $f(\theta)$ is a given periodic, continuous function of period 2π $(f(\theta + 2\pi) = f(\theta))$. To solve the above problem, we use the method of separation of variables.

Step 1: (Writing the ODEs): Seek solutions of the form

$$u(r, \theta) = R(r)T(\theta),$$

where $0 < r < r_o$ and $-\pi \le r \le \pi$. Substituting (3.3.1) and separating variable yield

$$R''(r)T(\theta) + r^{-1}R'(r)T(\theta) + r^{-2}R(r)T''(\theta) = 0.$$

$$\Longrightarrow \frac{r^2R''(r) + rR'(r)}{R(r)} = -\frac{T''(\theta)}{T(\theta)} = k.$$

Which leads to the following two ODEs:

$$T''(\theta) + kT(\theta) = 0, \qquad (3.3.2)$$

$$r^2R''(r) + rR'(r) - kR(r) = 0. \qquad (3.3.3)$$

Step 2. (Solving the ODEs):

Case (a) : When $k < 0$, the general solution to (3.3.2) is the sum of two exponentials. Hence we have only trivial 2π-periodic solutions.

Case (b) : When $k = 0$, we find that $T(\theta) = A\theta + B$ is the solution to (3.3.2). This linear function is periodic only when $A = 0$, that is, $T_0(\theta) = B$ is the only 2π-periodic solution corresponding to $k = 0$.

Case (c): When $k > 0$, the general solution to (3.3.2) is

$$T(\theta) = A\cos\left(\sqrt{k}\theta\right) + B\sin(\sqrt{k}\theta).$$

In this case we get a nontrivial 2π-periodic solution only when $\sqrt{k} = n,\ n = 1, 2, 3, ...$ .Hence, we obtain the nontrivial 2π-periodic solutions

$$T_n(\theta) = A_n\cos(n\theta) + B_n\sin(n\theta) \tag{3.3.4}$$

corresponding to $\sqrt{k} = n, \quad n = 1, 2, 3, ...$

Now for $k = n^2,\ n = 0, 1, 2, ...,$ equation (3.3.3) is the Cauchy-Euler equation

$$r^2R^{''}(r) + rR^{'}(r) - n^2R(r) = 0. \tag{3.3.5}$$

When $n = 0$, the general solution is

$$R_o(r) = C + D\ln r.$$

Since $\ln r \to \infty$ as $r \to 0^+$ this solution is unbounded near $r = 0$ when $D \neq 0$. Therefore, we must choose $D = 0$ if $u(r, \theta)$ is to be continuous at $r = 0$. We now have $R_o(r) = C$ and so $u_o(r, \theta) = R_o(r)T_o(\theta) = CB$. For convenience, we write $u_o(r, \theta)$ in the form

$$u_o(r, \theta) = \frac{A_o}{2}, \tag{3.3.6}$$

where A_o is an arbitrary constant.

When $k = n^2,\ n = 1, 2, ...,$the general solution of (3.3.3) is given by

$$R_n(r) = C_nr^n + D_nr^{-n}.$$

Since $r^{-n} \to \infty$ as $r \to 0^+$, we must set $D_n = 0$ in order to be bounded at $r = 0$. Thus

$$R_n(r) = C_nr^n$$

Now for each $n = 1, 2, \ldots$, we have the solutions

$$u(r, \theta) = R_n(r)T_n(\theta) = C_n r^n \left[A_n \cos(n\theta) + B_n \sin(n\theta)\right].$$

By superposition principle, we write

$$u(r, \theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} C_n r^n \left[A_n cos(n\theta) + B_n sin(n\theta)\right].$$

This series may be written in the equivalent form

$$u(r, \theta) = \frac{A_o}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{r_o}\right)^n \left[A_n \cos(n\theta) + B_n \sin(n\theta)\right], \tag{3.3.7}$$

where the A_n's and b_n's are constants. These constants can be determined from the boundary condition. With $r = r_o$ in (3.3.7), we have

$$f(\theta) = \frac{A_o}{2} + \sum_{n=1}^{\infty} \left[A_n \cos(n\theta) + B_n \sin(n\theta)\right].$$

Since $f(\theta)$ is 2π-periodic, we recognize that A_n, B_n are Fourier coefficients. Thus

$$A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos(n\theta) d\theta, \quad n = 0, 1, \ldots, \tag{3.3.8}$$

$$B_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin(n\theta) d\theta, \quad n = 1, 2, \ldots, \tag{3.3.9}$$

We now summarize the Dirichlet problem for a disk as follows.

In the Dirichlet problem (3.3.1), if

$$f(\theta) = \frac{A_o}{2} + \sum_{n=1}^{\infty} \left[A_n \cos(n\theta) + B_n \sin(n\theta)\right].$$

then the solution is given by

$$u(r, \theta) = \frac{A_o}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{r_o}\right)^n \left[A_n \cos(n\theta) + B_n \sin(n\theta)\right],$$

where A_n and B_n are given by (3.3.8) and (3.3.9), respectively.

Example 3.3.1: *Solve the following BVP*

$$PDE: \quad u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2} = 0, \quad 0 \le r < 1,$$

$$BC: \qquad u(1,\theta) = f(\theta),$$

where $f(\theta) = 1 + r\sin\theta + \frac{r^3}{2}\sin(3\theta) + r^4\cos(4\theta)$.

Solution: Here $r_o = 1$. Note that $f(\theta)$ is already in the form of Fourier series, with

$$A_n = \begin{cases} 2 \quad for \;\; n = 0 \quad and \quad 1 \quad for \;\; n = 4 \\ 0 \quad for \;\; other \;\; n \end{cases} \qquad B_n = \begin{cases} 1 \quad n = 1 \\ \frac{1}{2} \quad n = 3 \\ 0 \quad for \;\; other \;\; n \end{cases}$$

The solution of the BVP is

$$\begin{aligned} u(r,\theta) &= \frac{A_o}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{r_o}\right)^n [A_n\cos(n\theta) + B_n\sin(n\theta)] \\ &= 1 + r\sin\theta + \frac{r^3}{2}\sin(3\theta) + r^4\cos(4\theta). \end{aligned}$$

3.4 The Dirichlet Problem for a Rectangle

Consider the following Dirichlet problem in a rectangle:

$$PDE: u_{xx} + u_{yy} = 0, \quad 0 < x < a, \quad 0 < y < b, \tag{3.4.1}$$

$$BC: u(x,0) = f_1(x), \quad u(x,b) = f_2(x), \quad 0 \le x \le a, \tag{3.4.2}$$

$$u(0,y) = g_1(y), \quad u(a,y) = g_2 \quad 0 \le y \le b.$$

Let us show how the method of separation of variables is still applicable for the BVP. Since the BC are nonhomogeneous, we are required to do some preliminary work.

By the principle of superposition, we seek the solution of the above BVP (3.4.1)-(3.4.2) as

$$u(x, y) = u_1(x, y) + u_2(x, y) + u_3(x, y) + u_4(x, y),$$

where each of u_1, u_2, u_3 and u_4 satisfies the PDE with one of the original nonhomogeneous BC, and the homogeneous versions of the remaining three BC. These problems are then solved by the method of separation of variables.

Let us consider solving the following problem, as an example:

Example 3.4.1 *Solve the Dirichlet BVP:*

$$PDE : u_{xx} + u_{yy} = 0, \quad 0 < x < a, \quad 0 < y < b, \tag{3.4.3}$$

$$BC : u(x, 0) = f(x), \quad u(x, b) = 0, \quad 0 \le x \le a, \tag{3.4.4}$$

$$u(0, y) = 0, \quad u(a, y) = 0, \quad 0 \le y \le b.$$

Applying the method of separation of variables to solve this problem.

Step 1: (Reducing ODEs)

Separating variables, we seek for a solution of the form

$$u(x, y) = X(x)Y(y).$$

Substituting this into (3.4.3), we obtain

$$X^{''}(x)Y(y) + X(x)Y^{''}(y) = 0$$

and hence,

$$\frac{X^{''}(x)}{X(x)} = -\frac{Y^{''}(y)}{Y(y)} = k,$$

for some constant k, which is called the separation constant. This leads two ODEs

$$X^{''}(x) - kX(x) = 0, \tag{3.4.5}$$

$$Y^{''}(y) + kY(y) = 0. \tag{3.4.6}$$

Step 2:(Solving the resulting ODEs)

***Case I*:** When $k > 0$, set $k = \lambda^2$, where $\lambda \neq 0$. In this case, the solutions of ODEs are

$$\begin{aligned} X(x) &= [Ae^{\lambda x} + Be^{-\lambda x}], \\ Y(y) &= [C\cos(\lambda y) + D\sin(\lambda y)]. \end{aligned}$$

Therefore, the solutions of PDE $u(x, y)$ are given by

$$u(x, y) = [Ae^{\lambda x} + Be^{-\lambda x}][C\cos(\lambda y) + D\sin(\lambda y)].$$

***Case II*:** When $k = 0$, the solution of ODEs are linear are given by

$$X(x) = (A + Bx), \quad Y(y) = (C + Dy).$$

Therefore,

$$u(x, y) = (A + Bx)(C + Dy).$$

***Case III*:** Suppose $k < 0$, set $k = -\lambda^2$, where $\lambda > 0$. The solutions of ODEs are given by

$$\begin{aligned} X(x) &= [A\cos(\lambda x) + B\sin(\lambda x)] \\ Y(y) &= [Ce^{\lambda y} + De^{-\lambda y}] \end{aligned}$$

Thus , the solution of PDE is

$$u(x, y) = [A\cos(\lambda x) + B\sin(\lambda x)]\left[Ce^{\lambda y} + De^{-\lambda y}\right].$$

Step 3: (Applying the BC)

Using the boundary conditions $u(0, y) = 0$ and $u(a, y) = 0$ for the product solution obtained for the *case I* $(k > 0)$ leads to the equations

$$A + B = 0, \quad Ae^{\lambda a} + Be^{-\lambda a} = 0,$$

which has a trivial solution $A = 0$ and $B = 0$. Thus, only the trivial solution $u(x, y) = 0$ is possible. Similarly, use of boundary conditions $u(0, y) = 0$ and $u(a, y) = 0$ also leads to a trivial solution $u(x, y) = 0$ for the *case II* ($k = 0$). Let us examine the product solution obtained in *case III* (for $k < 0$) i.e.,

$$u(x, y) = [A\cos(\lambda x) + B\sin(\lambda x)][Ce^{\lambda y} + De^{-\lambda y}]$$

Using the boundary condition $u(0, y) = 0$ yields $A = 0$. The condition $u(x, y) = 0$ gives

$$[B\sin(\lambda a)][Ce^{\lambda y} + De^{-\lambda y}] = 0.$$

For a non-trivial solution,

$$\begin{aligned} B \neq 0 \quad &\Longrightarrow \quad \sin\lambda a = 0 \\ &\Longrightarrow \quad \lambda a = n\pi \;\; or \;\; \lambda = \frac{n\pi}{a}, \quad n = 1, 2, 3, ... \end{aligned}$$

Therefore, the sequence of non-trivial is given by

$$u_n(x, y) = \sin\left(\frac{n\pi x}{a}\right)\left[C_n e^{\frac{n\pi y}{a}} + D_n e^{-\frac{n\pi y}{a}}\right]$$

Applying the BC $u(x, b) = 0$, we obtain

$$\begin{aligned} \sin\left(\frac{n\pi x}{a}\right)\left[C_n e^{\frac{n\pi b}{a}} + D_n e^{-\frac{n\pi b}{a}}\right] &= 0 \\ &\Longrightarrow \; C_n e^{\frac{n\pi b}{a}} + D_n e^{-\frac{n\pi b}{a}} = 0 \\ &\Longrightarrow \; D_n = -C_n \frac{e^{\frac{n\pi b}{a}}}{e^{-\frac{n\pi b}{a}}}, \quad n = 1, 2, 3, ... \end{aligned}$$

Therefore, the solution now takes the form

$$\begin{aligned} u_n(x, y) &= \sin\left(\frac{n\pi x}{a}\right)\frac{2C_n}{e^{-\frac{n\pi b}{a}}}\left\{e^{\frac{n\pi(y-b)}{a}} - e^{-\frac{n\pi(y-b)}{a}}\right\}/2 \\ &= \frac{2C_n}{e^{-\frac{n\pi b}{a}}}\sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi(y-b)}{a}\right). \end{aligned}$$

Setting $c_n = \frac{2C_n}{e^{-\frac{n\pi b}{a}}}$ and using superposition principle, we obtain

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi(y-b)}{a}\right).$$

To satisfy the remaining nonhomogeneous BC, we must have

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{-n\pi b}{a}\right),$$

which is a half-range Fourier series. Therefore,

$$c_n \sinh\left(\frac{-n\pi b}{a}\right) = \frac{2}{a}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)dx,$$

and this implies

$$c_n = \frac{2}{a\sinh\left(\frac{-n\pi b}{a}\right)}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)dx. \tag{3.4.7}$$

Therefore, the required solution to the problem (3.4.3)-(3.4.4) is

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi(y-b)}{a}\right)$$

with the coefficients c_n computed from (3.4.7).

3.5 The Mixed Boundary Value Problem for a Rectangle

Let us consider the following Neumann problem for a rectangle:

$$PDE : u_{xx} + u_{yy} = 0, \quad 0 < x < a, \quad 0 < y < b, \tag{3.5.1}$$

$$\begin{aligned} BC : u_y(x,0) = f(x), \quad u_y(x,b) = g(x), 0 \le x \le a, \qquad (3.5.2)\\ u_x(0,y) = h(y), \quad u_x(a,y) = k(y) \quad 0 \le y \le b. \end{aligned}$$

This problem has no solution, unless the following compatibility condition holds:

$$\int_0^a g(x)dx - \int_0^a f(x)dx + \int_0^b k(y)dy - \int_0^b h(y)dy = 0$$

Solution: If $u(x, y)$ is a solution of (3.5.1), then

$$\begin{aligned}
0 &= \int_0^b \int_0^a (u_{xx} + u_{yy})\, dxdy \\
&= \int_0^b \int_0^a u_{xx} dxdy + \int_0^a \int_0^b u_{yy} dydx \\
&= \int_0^b [u_x(a, y) - u_x(0, y)]\, dy + \int_0^a [u_y(x, b) - u_y(x, 0)]\, dx \\
&= \int_0^b k(y)dy - \int_0^b h(y)dy + \int_0^a g(x)dx - \int_0^a f(x)dx,
\end{aligned}$$

Remark: *The compatibility condition is an immediate consequence of the following special case of Greens theorem*

$$\int_C \nabla u.\boldsymbol{n} ds = \int_C u_x dy - u_y dx = \int \int_R (u_{xx} + u_{yy})\, dxdy,$$

i.e., the flux of the gradient of u through the boundary is the integral of Δu *in the interior. Note that we only require that* u_x *and* u_y *be continuous on the closed rectangle.*

3.6 Heat Equation

Consider a thin bar placed along the x-axis of homogeneous material. Let $u(x, t)$ represent the temperature (of cross section) of the bar at position x and at time t.

We study the temperature of the bar where heat flow is occurring. Using the separation of variables method combined with Fourier series, to solve the heat equation

$$\begin{aligned}
PDE: &\quad 4u_{xx} = u_t, \quad 0 < x < \pi, \quad t > 0; && (3.6.1) \\
BC: &\quad u(0, t) = 0, \quad t > 0; && (3.6.2) \\
&\quad u(\pi, t) = 0, \quad t > 0; && (3.6.3) \\
IC: &\quad u(x, 0) = f(x), \quad 0 \le x \le \pi, && (3.6.4)
\end{aligned}$$

Thus we have, respectively; a PDE; two boundary conditions; and an initial temperature condition.

The idea is to assume the PDE (3.6.1) has a special form, namely

$$u(x,t) = X(x)T(t) \tag{3.6.5}$$

and then to suitably adjust the form by incorporating the conditions(3.6.2), (3.6.3) and then, in the final step, we employ (3.6.4).

From (3.6.5) we have: $u_x = X'(x)T(t)$; $u_{xx} = X''(x)T(t)$ and $u_t = X(x)T'(t)$, , so our PDE (3.6.1) becomes

$$4u_{xx} = u_t \Longrightarrow 4X"(x)T(t) = X(x)T'(t).$$

Separate the variables to form

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{4T(t)} \tag{3.6.6}$$

Now since left hand side of (3.6.6) only depend on x and since right hand side depend on t, we must have a constant γ such that

$$\frac{X''}{X} = \frac{T'}{4T} = \gamma. \tag{3.6.7}$$

we form two ODEs from (3.6.7)

$$T' = 4\gamma T \to T(t) = Ce^{4\gamma t}, \quad C - constant$$

$$X'' = \gamma X \to \quad Charcterstic\ equation \ \ \lambda^2 = \gamma$$

Thus

$$X(x) = \begin{cases} Ax + B, & \gamma = 0 \\ Ae^{\sqrt{\gamma}x} + Be^{-\sqrt{\gamma}x,} & \gamma > 0 \\ A\cos\sqrt{-\gamma}x + B\sin\sqrt{-\gamma}x, & \gamma < 0 \end{cases}$$

We now introduce (3.6.1) and (3.6.2) to determine which values of γ leads to nontrivial solutions. The BCs (3.6.1) and (3.6.2) yield

$$\begin{aligned} u(0,t) &= X(0)T(t) = 0 \to X(0) = 0 \\ u(\pi,t) &= X(\pi)T(t) = 0 \to X(\pi) = 0 \end{aligned} \tag{3.6.8}$$

Case I, $\gamma = 0$**:** $X(x) = Ax + B$ and (3.6.8) yield

$$\begin{aligned} &X(0) = A.0 + B = 0 \quad so \quad B = 0 \quad and \\ &X(\pi) = A\pi + 0 = 0 \quad so \quad A = 0. \end{aligned}$$

But this yields the trivial solution.

Case II, $\gamma > 0$**:** $X(x) = Ae^{\sqrt{\gamma}x} + Be^{-\sqrt{\gamma}x}$ and (3.6.8) yield

$$\begin{aligned} &X(0) = A + B = 0 \quad so \quad A = -B \\ &X(\pi) = Ae^{\sqrt{\gamma}\pi} + Be^{-\sqrt{\gamma}\pi} = 0 \quad we \;\; form \\ &A\left(e^{\sqrt{\gamma}\pi} - e^{-\sqrt{\gamma}\pi}\right) = 0, \quad so \;\; A = 0 \end{aligned}$$

But this yields the trivial solution.

Case III, $\gamma < 0$**:** $X(x) = A\cos\sqrt{-\gamma}x + B\sin\sqrt{-\gamma}x$ and (3.6.8) yield

$$X(0) = A = 0 \quad so \quad A = 0.$$

Ignore case $B = 0$ (trivial solution) and consider $\sin\sqrt{-\gamma}\pi = 0$ which holds when $\sqrt{-\gamma}\pi = n\pi, \;\; n = 1, 2, 3, ...$ so we have $\sqrt{-\gamma} = n, \;\; n = 1, 2, 3, ...$

Thus we have formed a sequence of functions

$$X_n(x) = B_n \sin nx, \quad n = 1, 2, 3, .. \qquad B_n = constants$$

From $T(t) = ce^{4\gamma t}$ to form

$$T_n(t) = C_n e^{-4n^2 t}, \quad n = 1, 2, 3, ... \qquad C_n = constants$$

From our assumed form $u(x,t) = X(x)T(t)$, we can form a sequence

$$\begin{aligned} u_n(x,t) &= X_n(x)T_n(t) \quad n = 1,2,3,... \\ &= (B_n \sin nx)\left(C_n e^{-4n^2 t}\right) \\ &= b_n e^{-4n^2 t} \sin nx \qquad\qquad (b_n \approx B_n C_n) \end{aligned}$$

Since (3.6.1), (3.6.2) and (3.6.3) are linear and homogeneous, every sum of solution is a solution. Hence we can form

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} b_n e^{-4n^2 t} \sin nx$$

To determine b_n we use (3.6.4)

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

From the above, we see that if we can write our initial temperature $u(x,0) = f(x)$ as an infinite sum involving then we may calculate the b_n and hence determine the exact solution to the problem.

If $u(x,0) = f(x) = \sin x - 2\sin 3x$ then we may write it as

$$\begin{aligned} u(x,0) &= f(x) = \sin x - 2\sin 3x = \sum_{n=1}^{\infty} b_n \sin nx \\ &= b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + b_4 \sin 4x + ... \end{aligned}$$

and equating the coefficients we see that:$b_1 = 1; b_3 = -2$ and all the other b_i must be zero. Thus, our solution

$$\sum_{n=1}^{\infty} b_n e^{-4n^2 t} \sin nx$$

simplifies to

$$u(x,t) = e^{-4t} \sin x - 2e^{-36t} \sin 3x.$$

3.7 Potential Theory

Let $\Phi(x)$ denote the fundamental solution of Laplaces equation. That is, let

$$\Phi(x) = \begin{cases} -\frac{1}{2}\ln|x| & n = 2 \\ \frac{1}{n(n-2)\alpha(n)} \cdot \frac{1}{|x|^{n-2}} & n \geq 3. \end{cases}$$

Let h be a continuous function on $\partial\Omega$. The **single layer potential** with moment h is defined as

$$\bar{u}(x) = -\int_{\partial\Omega} h(y)\Phi(x-y)dS(y). \tag{3.7.1}$$

The **double layer potential** with moment h is defined as

$$\bar{\bar{u}}(x) = -\int_{\partial\Omega} h(y)\frac{\partial\Phi}{\partial v_y}(x-y)dS(y). \tag{3.7.2}$$

we will prove that for a continuous function h, (3.7.1) and (3.7.2) are harmonic functions for all $x \notin \partial\Omega$.

Theorem 3.7.1: *For h a continuous function on $\partial\Omega$,*

1. *$\bar{u}$ and $\bar{\bar{u}}$ are defined for all $x \in \mathbb{R}^n$.*
2. *$\Delta\bar{u}(x) = \Delta\bar{\bar{u}}(x) = 0$ for all $x \notin \partial\Omega$.*

Proof:

1. We prove that $\bar{\bar{u}}$ is defined for all $x \in \mathbb{R}^n$. A similar proof works for $\bar{u}$. First, suppose $x \notin \partial\Omega$. Therefore, $\frac{\partial\Phi}{\partial v_y}(x-y)$ is difined for all $y \in \partial\Omega$. Consequently, for all $x \notin \partial\Omega$, we have

$$|\bar{\bar{u}}(x)| \leq |h(y)|_{L\infty(\partial\Omega)} \int_{\partial\Omega} \left|\frac{\partial\Phi}{\partial v_y}(x-y)\right| dS(y) \leq C.$$

Next, consider the case when x is in $\partial\Omega$. In this case, the term $\frac{\partial\Phi}{\partial v_y}(x-y)$ in the integrand is undefined at $x = y$. We prove $\bar{\bar{u}}$ is defined at this point x by showing that the integral in (3.7.2) still converges. We need to look for a bound on

$$-\int_{\partial\Omega} h(y)\frac{\partial\Phi}{\partial v_y}(x-y)dS(y).$$

Recall

$$\Phi(x-y) = \begin{cases} -\frac{1}{2}\ln|x-y| & n=2 \\ \frac{1}{n(n-2)\alpha(n)}\cdot\frac{1}{|x-y|^{n-2}} & n \geq 3. \end{cases}$$

Therefore,

$$\Phi_{y_i}(x-y) = \frac{x_i - y_i}{n\alpha(n)\left|y-x\right|^n},$$

and,

$$\begin{aligned} \frac{\partial \Phi}{\partial v_y}(x-y) &= \nabla_y \Phi(x-y).v(y) \\ &= \frac{(x-y).v_y}{n\alpha(n)\left|y-x\right|^n}, \end{aligned}$$

where $v(y)$ is the unit normal to $\partial\Omega$ at y.

***Claim*:** Fix $x \in \partial\Omega$. For all $y \in \partial\Omega$ there exists a constant $C > 0$ such that

$$|(x-y).v(y)| \leq C\left|x-y\right|^2.$$

Proof of Claim: By assumption, $\partial\Omega$ is C^2. This means at each point $x \in \partial\Omega$, there exists an $r > 0$ and a C^2 function $f : \mathbb{R}^{n-1} \to \mathbb{R}$ such that - upon relabeling and reorienting if necessary - we have

$$\Omega \cap B(x,r) = \{z \in B(x,r) | z_n > f(z_1, \ldots, z_{n-1})\}.$$

Without loss of generality (by reorienting if necessary), we may assume $x = 0$ and $v(x) = (0, \ldots, 0, 1)$. Using the fact that our boundary is C^2, we know there exists an $r > 0$ and a C^2 function $f : B(0,r) \subset \mathbb{R}^{n-1} \to \mathbb{R}$.

First, consider $y \in \partial\Omega$ such that $|x-y| \geq r$. In this case,

$$|(x-y).v(y)| \leq |x-y| \leq \frac{1}{r}\left|x-y\right|^2 = C(r)\left|x-y\right|^2.$$

Second, consider $y \in \partial\Omega$ such that $|x-y| \leq r$. In this case, we use the fact that

$$|(x-y).v(y)| \quad = \quad |(x-y).(v(x)+v(y)-v(x))|$$

$$\begin{aligned} &\leq |(x-y).v(x)| + |(x-y).(v(y)-v(x))| \\ &= |y_n| + |(x-y).(v(y)-v(x))|. \end{aligned}$$

Now,

$$y_n = f(y_1, \ldots, y_{n-1})$$

where $f \in c^2$, $f(0) = 0$ and $\nabla f(0) = 0$. Therefore, by Taylor's Theorem, we have

$$\begin{aligned} |y_n| &= |f(y_1, \ldots, y_{n-1})| \\ &\leq C\,|(y_1, \ldots, y_{n-1})|^2 \\ &\leq C\,|y|^2 \\ &= C\,|x-y|^2, \end{aligned}$$

where the constant C depends only on the bound on the second partial derivatives of $f(y_1, \ldots, y_{n-1})$ for $|(y_1, \ldots, y_{n-1})| \leq r$, but this is bounded because by assumption $f \in C^2(\overline{B(0,r)})$. Next, we look at $|(x-y).(v(y)-v(x))|$. By assumption, $\partial\Omega$ is C^2 and consequently, v is a C^1 function and therefore, there exists a constant $C > 0$ such that

$$|v(y) - v(x)| \leq C\,|y-x|.$$

Therefore,

$$|(x-y).(v(y)-v(x))| \leq C\,|y-x|^2.$$

Consequently, our claim is proven. We remark that the constant C will depend on r, but once x is chosen r is fixed.

Therefore, we conclude that for $x \in \partial\Omega$, all $y \in \partial\Omega$,

$$\begin{aligned} \left|\frac{\partial \Phi}{\partial v_y}(x-y)\right| &= \left|\frac{(x-y).v(y)}{n\alpha(n)\,|y-x|^n}\right| \\ &\leq C\frac{|x-y|^2}{|x-y|^n} \\ &= \frac{C}{|x-y|^{n-2}}. \end{aligned}$$

Therefore,

$$\begin{aligned}\left|-\int_{\partial\Omega} h(y)\frac{\partial\Phi}{\partial v_y}(x-y)dS(y)\right| &\leq |h(y)|_{L\infty(\partial\Omega)}\int -\partial\Omega\left|\frac{\partial\Phi}{\partial v_y}(x-y)\right|dS(y)\\ &\leq C\int_{\partial\Omega}\frac{1}{|x-y|^{n-2}}dS(y)\leq C\end{aligned}$$

using the fact that $\partial\Omega$ is of dimension $n-1$. Therefore, we conclude that $\bar{\bar{u}}$ is defined for all $x \in \partial\Omega$ and consequently for all $x \in \mathbb{R}^n$ as claimed.

2. Next, we will prove that $\Delta\bar{\bar{u}}(x) = 0$ for all $x \in \Omega$. A similar proof works to prove that $\Delta\bar{u}(x) = 0$. Fix $x \in \Omega$. We note that for all $y \in \partial\Omega$, $\frac{\partial\Phi}{\partial v_y}(x-y)$ is a smooth function. Further, using the fact that $\Phi(x-y)$ is harmonic for all $x \neq y$, we conclude that $\Delta_x\frac{\partial\Phi}{\partial v_y}(x-y) = 0$ for all $y \in \partial\Omega$. Therefore, using the fact that our integral is finite and $\frac{\partial\Phi}{\partial v_y}(x-y)$ is smooth, we conclude that

$$\begin{aligned}\Delta_x\bar{\bar{u}}(x) &= -\Delta_x\int_{\partial\Omega} h(y)\frac{\partial\Phi}{\partial v_y}(x-y)dS(y)\\ &= -\int_{\partial\Omega} h(y)\Delta_x\frac{\partial\Phi}{\partial v_y}(x-y)dS(y)\\ &= 0.\end{aligned}$$

In the above theorem, we showed that as long as h is a continuous function on $\partial\Omega$, then $\bar{u}$ and $\bar{\bar{u}}$, defined in (3.7.1) and (3.7.2), respectively, are harmonic functions on Ω. Consequently, if we can choose h appropriately so that our initial condition will be satisfied, then we can find a solution of our particular problem. For a moment, consider the interior Dirichlet problem

$$\begin{cases}\Delta u = 0, & x \in \Omega\\ u = g, & x \in \partial\Omega\end{cases}$$

As proven above, for h a continuous function on $\partial\Omega$, $\bar{\bar{u}}$ defined in (3.7.2) is harmonic. Now, if we can choose h appropriately, such that for all $x \in \partial\Omega$,

$$\lim_{x\in\Omega\to x_0}\bar{\bar{u}}(x) = g(x_0)$$

then we will have found a solution of the interior Dirichlet problem.

Theorem 3.7.2: *Let h be a continuous function on $\partial\Omega$. Define the double layer potential*

$$\bar{\bar{u}}(x) = -\int_{\partial\Omega} h(y)\frac{\partial\Phi}{\partial v_y}(x-y)dS(y).$$

Let $x_o \in \partial\Omega$. Then

$$\lim_{x\in\Omega\to x_o} \bar{\bar{u}}(x) = \frac{1}{2}h(x_o) + \bar{\bar{u}}(x_o)$$

Proof: Let $x \in \Omega$, $x_0 \in \partial\Omega$. we have

$$\begin{aligned}
\bar{\bar{u}}(x) &= -\int_{\partial\Omega} h(y)\frac{\partial\Phi}{\partial v_y}(x-y)dS(y) \\
&= -\int_{\partial\Omega} h(y)\frac{\partial\Phi}{\partial v_y}(x-y)dS(y) + h(x_0)\int_{\partial\Omega}\frac{\partial\Phi}{\partial v_y}(x-y)dS(y) - h(x_0)\int_{\partial\Omega}\frac{\partial\Phi}{\partial v_y}(x-y)dS(y) \\
&= -\int_{\partial\Omega} [h(y)-h(x_0)]\frac{\partial\Phi}{\partial v_y}(x-y)dS(y) + h(x_0) \\
&\equiv I(x) + h(x_0),
\end{aligned}$$

using the fact that

$$-\int_{\partial\Omega}\frac{\partial\Phi}{\partial v_y}(x-y)dS(y) = 1 \quad for\ x \in \Omega,$$

and also Gaus's Lemma. Similarly,

$$\begin{aligned}
\bar{\bar{u}}(x_0) &= -\int_{\partial\Omega} h(y)\frac{\partial\Phi}{\partial v_y}(x_0-y)dS(y) \\
&= -\int_{\partial\Omega} [h(y)-h(x_0)]\frac{\partial\Phi}{\partial v_y}(x_0-y)dS(y) - h(x_0)\int_{\partial\Omega}\frac{\partial\Phi}{\partial v_y}(x_0-y)dS(y) \\
&= -\int_{\partial\Omega} [h(y)-h(x_0)]\frac{\partial\Phi}{\partial v_y}(x_0-y)dS(y) + \frac{1}{2}h(x_0) \\
&\equiv I(x_0) + \frac{1}{2}h(x_0),
\end{aligned}$$

again using Gaus's Lemma. Therefore,

$$\bar{\bar{u}}(x) - \bar{\bar{u}}(x_0) = I(x) + h(x_0) - I(x_0) - \frac{1}{2}h(x_0),$$

which implies

$$\bar{\bar{u}}(x) = I(x) - I(x_0) + \frac{1}{2}h(x_0) + \bar{\bar{u}}(x_0).$$

Therefore, to prove our theorem, we need only show that

$$\lim_{x\in\Omega\to x_0}[I(x)-I(x_0)]=0,$$

where

$$I(x)\equiv-\int_{\partial\Omega}[h(y)-h(x_0)]\frac{\partial\Phi}{\partial v_y}(x-y)dS(y).$$

Now,

$$I(x)-I(x_0)=-\int_{\partial\Omega}[h(y)-h(x_0)]\left[\frac{\partial\Phi}{\partial v_y}(x-y)-\frac{\partial\Phi}{\partial v_y}(x_0-y)\right]dS(y).$$

We need to show that for all $\epsilon>0$ there exists a $\delta>0$ such that $|I(x)-I(x_0)|<\epsilon$ for $|x-x_0|<\delta$. By assumption, h is continuous, and as we know $\Phi(x-y)$ is smooth for $y\neq x$. Therefore, to get a bound on $|I(x)-I(x_0)|$, we divide $\partial\Omega$ into two pieces:

1. $B(x_0,\gamma)\cap\partial\Omega$
2. $\partial\Omega-\{B(x_0,\gamma)\cap\partial\Omega\}$.

We look at these two pieces below. First for (1),

$$\begin{aligned}&\left|-\int_{B(x_0,\gamma)\cap\partial\Omega}[h(y)-h(x_0)]\left[\frac{\partial\Phi}{\partial v_y}(x-y)-\frac{\partial\Phi}{\partial v_y}(x_0-y)\right]dS(y)\right|\\&\qquad\leq|h(y)-h(x_0)|_{L^\infty(B(x_0,\gamma)\cap\partial\Omega)}\int_{B(x_0,\gamma)\cap\partial\Omega)}\left|\frac{\partial\Phi}{\partial v_y}(x-y)-\frac{\partial\Phi}{\partial v_y}(x_0-y)\right|dS(y).\end{aligned}$$

By assumption, h is continuous. Therefore, for all $\tilde{\epsilon}>0$ there exists a γ such that $|h(y)-h(x_0)|<\tilde{\epsilon}$ if $|y-x_0|<\gamma$. In addition,

$$\int_{B(x_0,\gamma)\cap\partial\Omega)}\left|\frac{\partial\Phi}{\partial v_y}(x-y)-\frac{\partial\Phi}{\partial v_y}(x_0-y)\right|dS(y)\leq C$$

using the fact that $\bar{\bar{u}}$ is defined for all $x\in\mathbb{R}$. Therefore, we conclude that for any $\tilde{\epsilon}>0$,

$$|(1)|\leq C_1\tilde{\epsilon}$$

for γ chosen appropriately small.

Next, for (2), we use the fact that $\frac{\partial\Phi}{\partial v_y}(x-y)$ is continuous in x for x away from y. Consequently, we have

$$\left|-\int_{\partial\Omega-B(x_0,\gamma)\cap\partial\Omega}[h(y)-h(x_0)]\left[\frac{\partial\Phi}{\partial v_y}(x-y)-\frac{\partial\Phi}{\partial v_y}(x_0-y)\right]dS(y)\right|$$
$$\leq |h(y)-h(x_0)|_{L^\infty}\left|\frac{\partial\Phi}{\partial v_y}(x-y)-\frac{\partial\Phi}{\partial v_y}(x_0-y)\right|_{L^\infty(\partial\Omega-\{B(x_0,\gamma)\cap\partial\Omega\}}\left|\int dS(y)\right|.$$

Now, first h is bounded on $\partial\Omega$. Therefore, $|h(y)-h(x_0)|\leq C$. Next, $\left|\int dS(y)\right|\leq C$ Lastly, using the fact that $\frac{\partial\Phi}{\partial v_y}(x-y)$ is continuous in x uniformly for y, we conclude that there exists a $\delta>0$ such that

$$\left|\frac{\partial\Phi}{\partial v_y}(x-y)-\frac{\partial\Phi}{\partial v_y}(x_0-y)\right|_{L^\infty(\partial\Omega-\{B(x_0,\gamma)\cap\partial\Omega\}}\leq\tilde{\epsilon},$$

for $|x-x_0|<\delta$. Therefore,

$$|(2)|\leq C_2\tilde{\epsilon}$$

if $|x-x_0|<\delta$ where δ is chosen appropriately small.

Consequently, for $\epsilon>0$ choose $\tilde{\epsilon}>0$ such that

$$C_1\tilde{\epsilon}+C_2\tilde{\epsilon}<\epsilon.$$

Then choosing $\gamma>0$ sufficiently small such that

$$|(1)|\leq C_2\tilde{\epsilon}$$

and $\delta>0$ sufficiently small such that

$$|(2)|\leq C_2\tilde{\epsilon}$$

when $|x-x_0|<\delta$, we conclude that

$$|I(x)-I(x_0)|\leq C_1\tilde{\epsilon}+C_2\tilde{\epsilon}\leq\epsilon,$$

for $|x-x_0|<\delta$, as claimed.

Therefore, we have shown that

$$\lim_{x \to x_0} [I(x) - I(x_0)] = 0.$$

Consequently,

$$\begin{aligned} \lim_{x \in \Omega \to x_0} \bar{\bar{u}}(x) &= \lim_{x \in \Omega \to x_0} [I(x) - I(x_0)] + \frac{1}{2}h(x_0) + \bar{\bar{u}}(x) \\ &= \frac{1}{2}h(x_0) + \bar{\bar{u}}(x) \end{aligned}$$

as claimed.

Now, let us use this theorem to construct solutions of the interior Dirichlet problems as well as the Neumann problems.

We begin by considering the *Interior Dirichlet Problem*,

$$\begin{cases} \Delta u = 0, & x \in \Omega \\ u = g, & x \in \partial\Omega \end{cases}$$

For a given function h, define the double-layer potential $\bar{\bar{u}}(x)$ associated with h as

$$\bar{\bar{u}}(x) = -\int h(y) \frac{\partial \Phi}{\partial v_y}(x - y) dS(y).$$

We already proved that $\bar{\bar{u}}$ is a harmonic function in Ω. In addition, we proved that for $x_o \in \partial\Omega$,

$$\lim_{x \in \Omega \to x_o} \bar{\bar{u}}(x) = \frac{1}{2}h(x_o) + \bar{\bar{u}}(x_o).$$

Therefore, if we can find a continuous function h such that for all $x_o \in \partial\Omega$

$$g(x_o) = \frac{1}{2}h(x_o) - \int_{\partial\Omega} h(y) \frac{\partial \Phi}{\partial v_y}(x - y) dS(y)$$

and we define

$$\bar{\bar{u}}(x) = -\int_{\partial\Omega} h(y) \frac{\partial \Phi}{\partial v_y}(x - y) dS(y),$$

for that choice of h, then $\bar{\bar{u}}$ will give us a solution of our interior Dirichlet problem.

Now, we consider the Neumann problems. We will find solutions below as single-layer potentials. Consider the *Interior Neumann Problem* ,

$$\begin{cases} \Delta u = 0, & x \in \Omega \\ \frac{\partial u}{\partial v} = g, & x \in \partial\Omega. \end{cases}$$

First, we note a compatibility condition on the boundary data in order for a solution to exist. By the Divergence Theorem, we know

$$\int_{\partial\Omega} \Delta u = \int_{\partial\Omega} \frac{\partial u}{\partial v} dS(y).$$

Therefore, in order for a solution to exist, we need

$$\int_{\partial\Omega} g(y) dS(y) = 0.$$

For a continuous function h, define the single-layer potential

$$\bar{u}(x) = -\int_{\partial\Omega} h(y)\Phi(y-x)dy.$$

From **Theorem 3.7.1**, we know that $\bar{u}$ is harmonic in Ω. In order to choose h appropriately so that our boundary condition will be satisfied, we extend the notion of *normal derivative* to points not in $\partial\Omega$ as follows. Let $x_o \in \partial\Omega$. Let $v(x_o)$ be the outer unit normal to Ω at x_o. For $t < 0$, such that $x_o + tv(x_o)$ is in Ω, we define

$$i^{x_o}(t) = \nabla\overline{u}(x_o + tv(x_o)).v(x_o).$$

In a manner similar to the proof of **Theorem 3.7.2**, we can show that

$$\begin{aligned} \lim_{t\to 0^-} i^{x_o}(t) &= -\frac{1}{2}h(x_o) + \frac{\partial \bar{u}}{\partial v}(x_o) \\ &= -\frac{1}{2}h(x_o) - \int_{\partial\Omega} h(y)\frac{\partial\Phi}{\partial v_x}(x_o - y)dS(y). \end{aligned}$$

Therefore, if we can find a continuous function h such that for all $x_o \in \partial\Omega$,

$$g(x_0) = -\frac{1}{2}h(x_0) - \int_{\partial\Omega} h(y)\frac{\partial\Phi}{\partial v_x}(x_0 - y)dS(y),$$

then by defining the single-layer potential

$$\bar{u}(x) = -\int_{\partial\Omega} h(y)\Phi(x-y)dS(y),$$

for that choice of h, $\bar{u}$ will give us a solution of our interior Neumann problem.

Summary

Given a function f that has values everywhere on the boundary of a region in $\mathbb{R}^n$, there is a unique continuous function u twice continuously differentiable in the interior and continuous on the boundary, such that u is harmonic in the interior and $u = f$ on the boundary.

We established the existence of the solution of Dirichlet problem in different bounded regions like rectangular , annular , heat equation and potential theory. For instance, for the interior of annular region

$$PDE: \quad u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad r_1 < r < r_2$$

$$BC: \qquad u(r_2, \theta) = f(\theta), \quad f(\theta + 2\pi) = f(\theta),$$

$$u(r_1, \theta) = g(\theta), \quad g(\theta + 2\pi) = g(\theta)$$

$$PC: \quad u(r, \theta + 2\pi) = u(r, \theta), \quad r_1 < r < r_2$$

by separation of variables. The solution to the interior Dirichlet problem is

$$u(r, \theta) = \sum_{n=0}^{\infty} r^n \left[a_n \cos(n\theta) + b_n \sin(n\theta)\right].$$

For a given function f, which is periodic and continuous on given boundary condition, we conclude that the Dirichlet problem for bounded region have a solution.

Bibliography

1. T. Myint-U and L. Debnath, Linear Partial Differential Equations for Scientists and Engineers, Fourth Edition (2007).
2. N.H. Asmar, Partial Differential Equations with Fourier Series and Boundary Value Problems, Second Edition (2005).
3. J. Lebl, Differential Equations for Engineers (2013).
4. D. Caratelli, J. Gielis, I. Tavkhelidze and P. E Ricci, The Dirichlet problem for the Laplace equation in super shaped annuli; a Springer open journal Boundary Value Problems, 2013:113.
5. B. Goodwine, Engineering Differential Equations: Theory and Applications (2005).
6. H. M. Lieberstein, Theory of Partial Differential Equations, Volme 93 (1972).
7. Ü. Aksoy and A.O. elebi, Dirichlet problem for a generalized inhomogeneous polyharmonic equation in an annular domain; Complex Variables and Elliptic Equations 2012: 57.
8. J. D. Moore, Introduction to Partial Differential Equations (2003).
9. V.P. Pikulin, S.I. Pohozaev and A. Iacob, Equations in Mathematical Physics: A practical course (2001)
10. D. Bleecker and G. Csordas, Basic Partial Differential Equations, Second Edition (1995).
11. S. Meda, The Dirichlet Problem for the Laplace Operator (2013).