DEPARTMENT OF MATHEMATICS

primary ideals in non commutative semirings

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The undersigned here by certify that they have read and recommend to the school of graduate studies for acceptance of a project entitled **Primary Ideals in Noncommutative Semirings** by Heanok Tsegaberhane in partial fulfillment of the requirements for the degree of master of Science.

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Abstract

From an algebraic point of view, semirings provide the most natural gener- alizations of the theory of rings. In this paper, the ring theoretic results of concerning the primary ideals and their radicals to noncommutative semir- ings are generalized. The derived results are further carried over to a Gel'fand semiring.

Chapter 1

Introduction and Preliminaries

In [8], Sharma et.al. generalized the primary ideals from commutative rings to noncommutative rings by replacing the role of elements by ideals. This definition of primary ideals coincides with the definition of primary ideals given in [11] through their associated primes under the assumption that the ring is Noetherian. The former approach enabled them to generalize some basic results for primary ideals and prime radicals in commutative rings [13] to a noncommutative setting without resorting to the Noetherian restriction.For basic results of noncommutative rings one can refer to T.Y. Lam [6] and some results on radicals can be found in [12]. Following [8], we define a primary ideal of a semiring in terms of its ideals and prove some basic results concerning the primary ideals and their radicals for noncommutative semirings. We also define a subtractive primary ideal of a Gelfand semiring and prove certain results concerning the subtractive primary ideals and G-subtractive primary ideals.

In the absence of additive inverses, we have to impose the restriction of subtractiveness to prove many basic results concerning the operations on ideals in semirings (c.f. Lemma 3.1((ii)(b), (vii)), Lemma 3.2 ((iv),(v),(viii), (x)(a)(b)(c)), Theorem 3.4((ii), (iv)) and Theorem 3.21). For more details about semirings, their ideals and homomorphisms etc. one can refer to [4].

1.1 Groups with examples

In this section we just recapitulate the definition of semiring,monoides and some of its properties as the concept of semiring is made use of in studying several properties of primery ideals in noncommutativ semirings.

Definition 1.1.1. *A nonempty set S with an associative binary operation $\triangle$ is said to be a semiring if it satisfies the following conditions*
i,$(S, \triangle)$ is an algeberaic structure.
ii,$\triangle$ is associative with respect to elements of S.

Definition 1.1.2. *A semigroup with identity is called a monoid.*

Example 1.1.1. $(R, +)$, $(R, \times)$ *are monoides,where R is the usual real number and* $+$,$\times$ *are the usual addition and multiplication.*

1.2 Semigroups

Definition 1.2.1. *A nonempty set R together with two binary operations called addition and multiplication is a semiring if*
(i) (R, +) is a commutative monoid with identity element 0*;*
(ii) (R, .) is a monoid with identity element 1*;*
(iii) Multiplication distributes over addition from either side;
(iv) $0r = 0 = 0r$*, for all* $r \in R$*;*
(v) $1 \neq 0$*.*

Example 1.2.1. *Let* $R_1 = \{0\}$*,then* $(R_1, +, \times)$ *is a ring but it is not a semiring with respect to the usual adition (+) and multiplication*$(\times)$*.*

Example 1.2.2. $R_2 = (R, +, \times)$ *is a semiring.*

Definition 1.2.2. *For a semiring R, let*$G(R) = \{r \in R : 1+r \in U(R)\}$*, where U(R) is the set of all units of R. Then R is said to be a Gel'fand semiring if and only if R = G(R).Or if every prime ideal of a semi ring R is contained in a unique maximal ideal then R is called a Geĺfand semiring.*

Definition 1.2.3. *Let R be a semiring. Then R is yoked if for a, b* $\in$ *R, there exists an element r of R such that a + r =b or b + r = a.*

Definition 1.2.4. *A left ideal I of a semiring R is a nonempty subset of R satisfying the following conditions:*
(i). If a, b$\in$ *I; then a + b*$\in$ *I;*
(ii). If a $\in$ *I and r*$\in$ *R, then ra*$\in$ *I.*
(iii). I$\neq$ *R.*

A right ideal of R is defined in the analogous manner and an ideal of R is a subset which is both a left ideal and a right ideal.

Definition 1.2.5. *A nonempty subset A of a semiring R is subtractive if a* $\in$ *A and a + b*$\in$ *A,implies that b*$\in$ *A.*

Definition 1.2.6. *A nonempty ideal (subset) A of a semiring R is strong if and only if a + b*$\in$ *A implies that a*$\in$ *A and b*$\in$ *A: Clearly every strong ideal (subset) A of a semiring R is subtractive.*

Definition 1.2.7. *An ideal P of a semiring R is prime if and only if whenever A.B* $\subseteq$ *P, for ideals A and B of R, we must have either A*$\subseteq$ *P or B*$\subseteq$ *P.*

Definition 1.2.8. *A nonempty subset S of a semiring R is an m-system if and only if a, b*$\in$ *S implies that there exists an element r* $\in$ *R such that a.r.b*$\in$ *S.*

Definition 1.2.9. *An ideal I of a semiring R is said to be semiprime if and only if for any ideal A of R, we have $A^2 \in I$ implies $A \in I$. Clearly, prime ideals are surely semiprime.*

Definition 1.2.10. *An ideal M of a semi ring R is said to be a maximal ideal, for any ideal P of R if $M \subseteq P \subseteq R$ then $M = P$ or $P = R$.*

Chapter 2

Primary Ideals Of Semirings

Lemma 2.1. *1. Let R be a semiring and A,B,C are right or left ideals of R.Then*
(i)$A\bigcap B$ *is an ideal of R.*
(ii)A.B is an ideal of R and $A.B\subseteq A\bigcap B$.
(iii). If R is yoked and A and B is strong then $A\bigcup B$ *is an ideal of R.*
(iv). If R is yoked and A and B is strong then $A + B$ *is an ideal of R.*

Proof. we need to show
(i). A $\bigcap$ B is an ideal of R .
Let A and B are right ideals of R .
Then both A and B they consist an element 0.
A $\bigcap$ B$\neq \varnothing$ .Since 0 $\in$ A $\bigcap$B.
1 is not member of A and B. since both A and B are ideals of R
$\Rightarrow$ 1 is not member of A$\bigcap$B.
$\Rightarrow$ A $\bigcap$ B $\subset$ R.
Let x and y *in* A$\bigcap$B implies that x and y $\in$ A,B.
$\Rightarrow$ x + y $\in$ A and B.
$\Rightarrow$ x + y $\in$ A $\bigcap$ B.
Let x $\in$ A$\bigcap$B and r $\in$ R. Then x $\in$ A and B. $\Rightarrow$ x.r $\in$ A and B.
$\Rightarrow$ x.r $\in$ A$\bigcap$ B. Therefor A $\bigcap$ B is an ideal R.

,(ii). A.B is an ideal of R and A.B $\subseteq A\bigcap B$.
Proof. We need to show A $\times$ B is an ideal of R.
since A and B are ideals of R ,then 0$\in$ A and B.
$\Rightarrow$ 0=0 $\times$ 0 $\in$ A $\times$ B.
$\Rightarrow$ 0 $\in$ A $\times$ B and A $\times$ B $\neq \varnothing$.

Let x $\in$ A $\times$ B,implies that x=$\sum_{i=1}^{n} a_i b_i$,$a_i$$\in$A and b_i $\in$B.
$a_i.b_i$$\in$ A and B.Since A and B are ideals of R.
$\Rightarrow x \in A\bigcap B$
$\Rightarrow A \times B \subseteq A\bigcap B$.
Since $A\bigcap B \subset$ R, implies that A $\times$ B $\subset$ R.

Let x and y $\in$ A $\times$ B.Then x=$\sum_{i=1}^{n} a_i b_i$. Then x=$\sum_{i=1}^{n} a_i b_i$ and y=$\sum_{j=1}^{n} c_j d_j$ where a_i,$c_j$$\in$ A and b_i,$d_j$$\in$ B.

x + y=$\sum a_ib_i$ + $\sum c_jd_j$.
$\Rightarrow\sum a_ib_i + c_jd_j$ $\in$A$\times$B.
$\Rightarrow$ x + y $\in$ A$\times$B.
Let x $\in$A.B and r$\in$R.Then x.($\sum_{i=1}^{n} a_ib_i$) and x.r =($\sum a_ib_i$).r
$\Rightarrow$ xr $\in$ A.B.Since a_i $\in$ A and $(b_ir)$$\in$ B.
$\therefore$ A.B is an ideal of R. □

Proof. (iii) If R is yoked and A and B is strong then A $\bigcup$ B is an ideal of R.
We need to show A $\bigcup$ B is an ideal of R if R is yoked and A and B are strong ideals of R. clearly A $\bigcup$ B $\neq\phi$
Since A and B are ideals of R , then 1 is not member of A and B.
$\Rightarrow$ 1 is not member of A $\bigcup$ B, implies that A $\bigcup$ B$\subset$ R.
a)If x,y $\in$ A$\bigcup$B, the x $\in$ A or y$\in$ B or both x, y$\in$ A,B.
.If x,y $\in$ A or B then it is clear that x + y$\in$ A $\bigcup$B.
b) Assume that x $\in$ A and y$\in$ B ,then there is r $\in$ R such that x=y + r or y = x + r.
Let consider x=y+r then y+r $\in$ A Since x $\in$ A and x=y + r.
$\Rightarrow$ y,r $\in$ A since A is a strong ideal of R.
$\Rightarrow$ x+y $\in$ A and x+y $\in$ A $\subseteq$ A$\bigcup$B.
$\Rightarrow$ x+y $\in$ A $\bigcup$ B.
Let x $\in$ A$\bigcup$B, and r$\in$ R.Then x$\in$ A or x$\in$ B.
$\Rightarrow$ x.r$\in$ A or x.r$\in$ B, x.r$\in$ A$\bigcup$B.
$\therefore$ A$\bigcup$B is an ideal of R.

□

Proof. (iv). If R is yoked and A and B is strong then A + B is an ideal of R. Proof. We need to show A + B is an ideal of R .If R is yoked and A,B are strong ideals of R. A$\subseteq$ A+B and B$\subseteq$A+B . Implies that A$\bigcup$B $\subseteq$ A+B.
Let x$\in$ A + B ,then there exist a$\in$ A and b$\in$ B such that x=a + b. Since a, b$\in$ A,B respectively then a, b$\in$ A $\bigcup$ B and a + b$\in$ A$\bigcup$B.Since A,B are ideals of R.
$\Rightarrow$ a + b=x $\in$ A $\bigcup$ B.$\Rightarrow$A + B $\subseteq$ A$\bigcup$B.
$\therefore$ since A$\bigcup$B is an ideal of a semi ring R , then A + B is an ideal of R. □

Lemma 2.2. *Let R be a semiring and A,B,C,U,U$'$,A_i and B_i are right ideals of R. Then (i) Let A : B =*$\{x \in R : B.x \subseteq A\}$.
(a) A : B is an ideal of R, if B $\nsubseteq$ A. Further, if A is an ideal of R, then A :B$\supseteq$ A.
(b) If A is subtractive, then A : B is subtractive.
(ii) If A$\subseteq$ B, then U : A$\supseteq$ U : B .
(iii) If U$\subseteq$ U' then U : A $\subseteq$ U' : A.
(iv) $(\bigcap_{i=1}^{n} A_i)$;B=$\bigcap_{i=1}^{n}(A_{i=1} : B)$.
(v) A :$\sum_{i=1}^{n} B_i$=$\bigcap_{i=1}^{n}(A : B_i)$. *(vi) If A $\supseteq$ B, then A* $\bigcap(B+C)\supseteq B+(A\bigcap C)$. *Equality holds if A is subtractive.*

Proof. i). Let A : B =$\{x \in R : B.x \subseteq A\}$.
(a)A : B is an ideal of R, if B$\nsubseteq$ A. Further, if A is an ideal of R, then A : B $\supseteq$ A.
(b) If A is subtractive, then A : B is subtractive.
Proof. We need to show (a) A : B is an ideal of R and A$\subseteq$ A : B, if A and B are right ideals of R. A : B =$\{r \in R$: $Br \subseteq A$.
$\Rightarrow$ 0$\in$ A : B, since 0$\in$ A and B.0$\subseteq$ A.

$\Rightarrow$ A : B$\neq\emptyset$.
Suppose that A : B = R, implies that $\forall$r $\in$ R,B.r$\subseteq$ A.
$\Rightarrow$ B$\subseteq$ A. This is a contradiction with B$\nsubseteq$ A given in the above.
There for A : B$\neq$ R and A : B$\subseteq$ R.
Let x, y$\in$ A : B $\Rightarrow$ B.x$\subseteq$ A and B.y$\subseteq$ A.
b.x$\in$ A,$\forall$b$\in$ B and b.y$\in$ A ,$\forall$b$\in$ B.
$\Rightarrow$ b.x + b.y $\in$ A,$\forall$b$\in$ B, since A is an ideal.
$\Rightarrow$ b(x + y)$\in$ A,$\forall$b $\in$ B.
$\Rightarrow$ B(x + y)$\in$ A. Implies that x + y $\in$ A : B.
Let x $\in$ A : B and r$\in$ R, then B.x$\subseteq$ A,$\Rightarrow$ b.x$\in$ A;$\forall$b $\in$ B.
Implies that b.x.r $\in$ A,$\forall$b $\in$ B.
$\Rightarrow$ B.x.r$\subseteq$ A.
$\Rightarrow$ x.r $\in$ A : B.
(b) If A is subtractive, then A : B is subtractive. Let x and x + y $\in$ A : B;we went to show y$\in$ A : B. $\Rightarrow$ B.x $\subseteq$ A and B(x + y)$\subseteq$ A,
$\Rightarrow$ b.x $\in$ A and b(x + y) $\in$ A;$\forall$b $\in$ B.
$\Rightarrow$ b.x, b.x + b.y $\in$ A,$\forall$b $\forall$ B.
$\Rightarrow$ b.y $\in$ A,$\forall$b $\in$ B, since A is subtractive.
$\Rightarrow$ B.y $\subseteq$ A; implies that y $\in$ A : B.
$\therefore$ A : B is subtractive. □

Proof. (ii) If A$\subseteq$ B, then We went to shew U : A$\supseteq$ U : B .
Let x$\in$ U : B, implies that B.x $\subseteq$ U.
b.x $\in$ $\forall$b$\in$ B
$\Rightarrow$ a.x$\in$ U,$\forall$a$\in$ A; since A $\subseteq$ B.
$\Rightarrow$ A.x$\subseteq$ U.
$\Rightarrow$ x$\in$ U : A.
$\Rightarrow$ U : B $\subseteq$ U:A.
□

Proof. (iii) If U $\subseteq$ $U^{,}$ then we need to show U : A$\subseteq$ $U^{,}$: A.
Let x $\in$ U : A $\Rightarrow$ A.x$\subseteq$ U, $\Rightarrow$ A.x $\subseteq$ $U^{,}$, x$\in$ $U^{,}$: A.
$\Rightarrow$ U : A $\subseteq$ $U^{,}$: A.
□

Proof. (iv) $(\bigcap_{i=1}^{n} Ai)$: B = $\bigcap_{i=1}^{n}$(Ai : B).
Let x $\in$ $\bigcap_{i=1}^{n}$ $(A_i$: B).$\Rightarrow$x $\in$ $(A_1 \bigcap A_2 \bigcap \ldots\ldots \bigcap A_n)$:B.// B.x$\subseteq$ $(A_1 \bigcap A_2 \bigcap \ldots\ldots \bigcap A_n)$.
B.x $\subseteq$ A_1,B.x $\subseteq$ A_2,......B.x $\subseteq$ A_n:.
$\Rightarrow$ x$\in A_1$: B, x$\in$ A_2: B,.........,x$\in$ A_n:B.
$\Rightarrow$ x $\in$ A_1:B $\bigcap$ A_2 : B $\bigcap$........$\bigcap A_n$:B.
$\Rightarrow$ x $\in$ $\bigcap_{i=1}^{n}(A_i : B)$.
$\Rightarrow$ $(\bigcap_{i=1}^{n} A_i)$:B $\subseteq$ $\bigcap_{i=1}^{n} A_i : B$................♦
Let x $\in$ $\bigcap_{i=1}{}^{n}$ $(A_i : B)$.
, then we need to show x $\in$ $(\bigcap_{i=1}^{n} Ai)$: B,
$\Rightarrow$ x $\in$ $(A_1$: B)$\bigcap$ $(A_2 : B)\bigcap$........$\bigcap(A_n : B)$.
$\Rightarrow$ B.x $\subseteq$ A_1,B.x $\subseteq$ A_2........,B.x $\subseteq$ A_n.
$\Rightarrow$ B.x $\subseteq$ A_1, A_2, A_3........A_n.
$\Rightarrow$ B.x $\subseteq$ $A_1 \bigcap A_2 \bigcap \ldots\ldots \bigcap A_n$ $\Rightarrow$ B.x $\subseteq$ $(\bigcap_{i=1}^{n} Ai$.

x $\in$ ($\bigcap A_i$) : B.
$\Rightarrow \bigcap_{i=1}^{n}(A_i : B)\subseteq$ x$\in$ ($\bigcap_{i=1}^{n} Ai$) : B,...........♦♦
$\therefore$ from ♦ and ♦♦ ($\bigcap_{i=1}^{n} A_i$):B =$\bigcap_{i=1}^{n}(A_i : B)$. □

Proof. (v). We need to Show A :$\sum_{i=1}^{n} B_i \subseteq \bigcap_{i=1}^{n}(A : B_{i=1})$. And $\bigcap_{i=1}^{n}(A : B_i) \subseteq$ A :$\sum_{i=1}^{n} B_i$.
Let x$\in$ A : $\sum_{i}^{n} B_i$, then ($\sum B_{i=1}).x\subseteq$ A.
$\Rightarrow(B_1 + B_2 +$ .... $+ B_n)$.x $\subseteq$ A.
$\Rightarrow B_1.x$ + $B_2.x$+........+ $A_n.x\subseteq$ A
$\Rightarrow B_1.x\subseteq$ A,$B_2.x\subseteq$ A,..........,$B_n.x \subseteq$ A, since A is a right ideal of R.
$\Rightarrow$ x$\in$ A : B_1, x$\in$ $A : B_2$,..........,x$\in$ A:B_n).
$\Rightarrow$ x $\in (A : B_1)\bigcap(A : B_2\bigcap$ .... $\bigcap(A : B_n)$
$\Rightarrow$ x $\in \bigcap_{i=1}^{n}(A : B_i)$.
$\Rightarrow$ A:$\sum_{i=1}^{n} B_i \subseteq \bigcap_{i=1}^{n}(A : B_i)$.
Let x $\in \bigcap_{i=1}^{n}$(A:B_i), then x$\in$ A:B_1,A : B_2,.....A : B_n.
$\Rightarrow B_1.x \subseteq$ A,$B_2.x \subseteq$ A,........,$B_n.x \subseteq$ A.
$\Rightarrow b_1.x \in$ A,$\forall b_1 \in B_1$, $b_2.x \in A,\forall b_2 \in B_2$,.........,$b_n.x \in$ A,$\forall b_n \in B_n$
. $(b_1 + b_2 + b_3+$ ::::::::::: $+b_n).x \in$ A; $\forall b_i$,i = 1, 2, 3,.......,n$\in$ B.
$\Rightarrow$ (B1 + B2 + B3 + ... + Bn).x $\subseteq$ A.
$\Rightarrow$ ($\sum_{i=1}^{n} B_i$).x $\subseteq$ A.Implies that x$\in$ $(A : \sum_{i=1}^{n} B_i)$.
$\therefore \bigcap_{i=1}^{n} A : B_i) \subseteq$ A:$\sum_{i=1}^{n} B_i)$
□

Proof. If A $\supseteq$ B, then $A\bigcap(B + C) \supseteq$ B + $(A\bigcap C)$. Equality holds if A is subtractive. Obviously,$A\bigcap(B + C) \supseteq$ B + $(A\bigcap C)$. Conversely, let x $\in A\bigcap(B + C)$. Then x $\in$ A and x = b + c with b $\in$ B,c $\in$ C.
As B $\subseteq$ A, we get b $\in$ A. Since x = b+c $\in$ A and A is subtractive, therefore b $\in$ A implies that c $\in$ A.
Hence A $\bigcap(B + C) \subseteq$ B + $(A\bigcap C)$.
□

Definition 2.0.11. *a)Let R ,R' be semirings*
A function T : R → R' is said to be a homomorphism function, if it satisfied the following conditions;
(i). T(x + y) = T(x) + T(y), ∀ x,y ∈ R.
(ii). T(x.y) = T(x).T(y), ∀ x, y ∈ R.
(iii). T(0) = 0 for 0 ∈ R.
(iv) T(1) = 1 for 1 ∈ R.
b Let R and R' be two semi rings and T : R → R' an onto homomorphism.
Let K_T = x ∈ R: ∃ a, b ∈ R : x = a + b and T(a) = T(b).

Lemma 2.3. *Let T : R → R' be an onto homomorphism. If A and B are right ideals of R and A',B' right ideals of R' then*
(i) (a) A ⊆ B implies that T(A) ⊆ T(B).
(b)T(A) is a right ideal of R'.
(ii) T(A + B) = T(A) + T(B).
(iii) T(A.B) = T(A)T(B).
(iv) T(A⋂B) ⊆ T(A)⋂T(B). Equality holds if A is subtractive containing K_T or B

is subtractive containing K_T *.*
(v) T(A : B) $\subseteq$ *T(A) : T(B). Equality holds if A is subtractive containing* K_T *.*
(vi) $A^{,} \subseteq B^{,}$ *implies that* $T^{-1}(A^{,}) \subseteq T^{-1}(B^{,})$*.*
(vii) $T(T^{-1}(A)) = A$
(viii) $A \subseteq T^{-1}(T(A))$*, with equality if A is subtractive containing* K_T*.*
(ix) (a) $T^{-1}(A^{,} + B^{,}) \supseteq T^{-1}(A^{,}) + T^{-1}(B^{,})$*.*
(b) $T^{-1}(A^{,}.B^{,}) \supseteq T^{-1}(A^{,})T^{-1}(B^{,})$*.*
(x) (a) If $A^{,}$ *is subeteractiv; then* $T^{-1}(A^{,})$ *is subtractive.*
(b) $T^{-1}(A^{,}B^{,})$ $T^{-1}(A^{,} \bigcap T^{-1}(B^{,})$*; with quality if* $A^{,}$ *and* $B^{,}$ *are subtractive such that* $T^{-1}(A^{,}) \supseteq K_T$ *and* $T^{-1}(B^{,} \supseteq K_T$*.*
(c) $T^{-1}(A^{,} : B^{,}) \supseteq T^{-1}(A^{,})$ *:* $T^{-1}(B^{,}$*, with equality if* $A^{,}$ *is subtractive ideal of* $R^{,}$ *such that* $T^{-1}(A^{,}) \subseteq K_T$ *.*

Proof. (i). (a) A $\subseteq$ B implies that T(A) $\subseteq$ T(B).
(a) Let A $\subseteq$ B, WE need to show T(A) $\subseteq$ T(B).
if x $\in$ A, then T(x) $\in$ T(A), and x $\in$ B, since A $\subseteq$ B.
$\Rightarrow$ T(x) $\in$ T(B), since x $\in$ B. This implies T(A) $\subseteq$ T(B).
(b) Since 0 $\in$ A, then T(0) $\in$ T(A). Implies that T(A) $\neq \emptyset$.
Since A is an ideal of R', then 1 $\notin$ A and T(1)=1 is not member of T(A).
$\therefore$ T(A) $\neq \emptyset$ and T(A) $\subsetneq R'$
Let x , y $\in$ T(A). We need to show x + y $\in$ T(A).
Then $\exists$ a , b $\in$ A : a = T^{-1}(x) and b = T^{-1}(y) $\in$ A. Since T : R $\rightarrow$ R' is an on to homomorphism function.
$\Rightarrow$ a + b $\in$ A , since A is a an ideal of R. Then T(a + b) $\in$ T(A),implies that T(a) + T(b) $\in$ T(A)
$\Rightarrow$ x + y $\in$ T(A).
Let x $\in$ T(A)and $r' \in R'$, $\exists$ a $\in$ A and r $\in$ R : T(a)= x and T(r) = r'.
$\Rightarrow$ ar $\in$ A, since A is an ideal of R.
$\Rightarrow$ T(ar) $\in$ T(A).
$\Rightarrow$ T(a).T (r) $\in$ T(A).
$\Rightarrow$ x.r $\in$ T(A).
$\therefore$ T(A) is an ideal of R.
(ii). T(A + B) = T(A) + T(B)
We went to show T(A + B) $\subseteq$ T(A) + T(B) and T(A) + T(B) $\subseteq$ T(A + B).
Let x $\in$ T(A + B); $\exists$ a $\in$ A and b $\in$ B: a+b $\in$ A + B and T(a + b) $\in$ T(A + B).
$\Rightarrow$ T(a + b) = T(a) + T(b) = x $\in$ T(A) + T(B) and implies T(A + B) $\subseteq$ T(A) + T(B).
Let y $\in$ T(A) + T(B) $\exists$ a $\in$ A; and b $\in$ B : T(a + b) = y.
Since T(a + b) $\in$ T(A + B) and y = T(a + b) then y $\in$ T(A + B).
$\therefore$ T(A) + T(B) $\subseteq$ T(A + B), and implies T(A + B) = T(A) + T(B).
(iii). T(AB) = T(A)T(B). We need to show T(A.B) $\subseteq$ T(A)T(B).
and T(A)T(B) $\subseteq$ T(A.B)
Let x $\in$ T(A.B),then there exist a $\in$ A and b $\in$ B : T(a.b) = x. Implies that T(a)T(b) = x; since T is an on to homomorphism function.
Since a $\in$ A; then T(a) $\in$ T(A); and b $\in$ B; then T(b) $\in$ T(B)and T(a):T (b) $\in$ T(A)T(B).
$\Rightarrow$ T(a)T(b) = x $\in$ T(A)T(B).

$\Rightarrow$ T(A.B) $\subseteq$ T(A).T(B)..................★ Let y $\in$ T(A)T(B), then there exist a $\in$ A and b $\in$ T(B) : y = T(a)T(b), implies that y = T(a.b) $\in$ T(A.B), since T is an on to homomorphism function. $\Rightarrow$ T(A)(B) $\subseteq$ T(AB).............★★. T(A.B) = T(A)T(B) from ★ and ★★

(iv). T(A $\bigcap$ B) $\subseteq$ T(A)$\bigcap$T(B). Equality holds if A is subtractive containing K_T or B is subtractive containing K_T .

Obviously T(A $\bigcap$ B) $\subseteq$ T(A)$\bigcap$ T(B). On the other hand, let x $\in$ T(A)$\bigcap$T(B).
. Then there exist $y_1 \in$ A and $y_2 \in$ B such that T(y_1) = T(y_2) = x so that $y_1 + y_2 \in K_T$.
But $K_T \subseteq$ A or $K_T \subseteq$ B, consider that $K_T \subseteq$ A. Thus we have $y_1 \in$ A and $y_2 \in$ A.
Since A is subtractive, we get $y_2 \in$ A.
Therefore,$y_2 \in$ A $\bigcap$ B and so T(y_2) = x implies that x $\in T(A \bigcap B)$.
so T(y_2) =x implies that x $\in$ T(A B).
(v) The inclusion T(A : B) $\subseteq$ T(A) : T(B) is obvious. Suppose z $\in$ T(A) : T(B).
Then T(B).z $\subseteq$ T(A). Therefore, T(y).z $\subseteq$ T(A), for all y $\in$ B.
This implies that T(y).z = T(a) for some a $\in$ A.
Thus there exists x $\in$ R, (T(x) = z) such that T(yx) = T(a).
So yx + a $\in$ K $\subseteq$ A. Since, A is subtractive, we get yx $\in$ A, for all y $\in$ B.
That is, Bx $\subseteq$ A, this implies that z = T(x) $\in$ T(A : B).
yx $\in$ A, for all y $\in$ B. That is, B.x $\in$ A, this implies that z = T(x) $\in$ T(A : B).
(Vi) $A^{'} \subseteq B^{'}$, we need to show $T^{-1}(A^{'}) \subseteq$ T$^{-1}(B^{'})$
Let a $\in T^{-1}(A^{'})$, then $\exists$ x $\in A^{'}$ such that T(a) = x
Since x in $A^{'}$ and T(a) = x $\in A^{'} \subseteq B^{'}$,implies that T(a) $\in B^{'}$.
$\Rightarrow$ a $\in B^{'}$.
$\Rightarrow T^{-1}(A^{'}) \subseteq T^{-1}(B^{'})$.

(vii) proof We went to show Suppose that x $\in$ T($T^{-1}(A^{'})$) = $A^{'}$, implies that $a \in$ T($T^{-1}(A)$) such that T(a) = x and also there exist $a\prime \in A\prime$: $T^{-1}(a^{'})$ = a,
$\Rightarrow$ T(a) = $a^{'}$ and T(a) = x.
$\Rightarrow$ x = $a^{'} \in A^{'}$
$\Rightarrow T(T^{-1}(A)) \subseteq A^{'}$ ....★
Suppose s $\in A^{'}$, $T^{-1}(s) \in T^{-1}(A^{'})$
$\Rightarrow$ T($T^{-1}(A)$) $\subseteq A^{'}$ ....★
Suppose s $\in A^{'} \Rightarrow T^{-1}(s) \in T^{-1}(A^{'})$
$\Rightarrow T(T^{-1}(s^{'}) \in T(^{-1}(A^{'}))$
$\Rightarrow$ s $\in T(T^{-1}A^{'})$
$A^{'} \subseteq T(T^{-1}(A^{'}))$..............★★
From ★ and ★★ $A^{'} = T(T^{-1}(A^{'}))$

(viii) The inclusion A $\subseteq T^{-1}(T(A))$ is obvious. Now suppose that A is subtractive and A $\supseteq K_T$.
If x $\in T^{-1}(T(A))$, there exists y $\in$ A such that T(x) = T(y).
Hence x + y $\in K_T \subseteq$ A and therefore x $\in$ A, proving $T^{-1}(T(A)) \subseteq$ A.

(ix)(a) Let $T^{-1}(A^{'})$ = A and $T^{-1}(B)$ = B.
By (viii) and (ii), we have A + B $\subseteq T^{-1}(T(A + B)) = T^{-1}(T(A) + T(B))$. That is, $T^{-1}(A) + T^{-1}(B) \subseteq T^{-1}(A + B)$.

, using (vii) Similarly using (viii), (iii) and (vii) we get (b).

(x)(a) This follows easily as T(x), T(x) + T(y) $\in A^{'}$ (subtractive) implies T(y) $\in A^{'}$.
(b) The inclusion is obvious. Suppose $A^{'}$ and $B^{'}$ are subtractive such that $T^{-1}(A^{'}) \subseteq K_T$ and $T^{-1}(B^{'}) \subseteq K_T$.

It is easy to see that $A^{'} \bigcap B^{'}$ is also subtractive, so by (x)(a), $T^{-1}(A^{'}) \bigcap T^{-1}(B)$ is subtractive and contains K_T . Now equality follows using (viii) and (iv).

(c) The inclusion follows using (viii) and (v). Suppose $A^{'}$ is subtractive ideal of $R^{'}$ and $T^{-1}(A^{'}) \supseteq K_T$.
Then by (x)(a) and Lemma 3.1 ((ii)(a) and (b)), $K_T \subseteq T^{-1}(A^{-1}) \subseteq T^{-1}(A) : T^{-1}(B^{'})$ and both $T^{-1}(A^{'})$ and $T^{-1}(A^{'}) : T^{-1}(B^{'})$ are subtractive. Now the result follows from (viii) and (v).

□

Theorem 2.0.1. *Let $T : R \rightarrow R^{'}$ be an onto homomorphism. Let A be an ideal of R and $A^{'}$ an ideal of $R^{'}$.*
(i) $A^{'}$ is prime if and only if $T^{-1}(A^{'})$ is prime.
(ii) A is subtractive and $A \supseteq K_T$, then A is prime if and only if T(A) is prime.
(iii) If $T^{-1}(A^{'})$ is maximal, then A is maximal. (iv) Let A be subtractive containing K_T . If A is maximal, then T(A) is maximal.

Proof. . i) Since T is onto, we have T(1) = 1. So $A^{'} \subsetneq R^{'}$ implies that $T^{-1}(A^{'}) \subsetneq$ R. Now the proof follows using Lemma 3.2 ((i),(iii),(vi),(vii)and (viii)).
(ii) Let T(A) = $A^{'}$. Since A is subtractive containing K_T , we have $T^{-1}(A^{'})$ =A by Lemma 3.2 (viii).Thus, by (i) A is prime if and only if T(A) is prime.
(iii) This follows using (vi) and (vii) of Lemma 3.2. (iv) This follows using. (iii) as (ii) follows using (i).

□

Proof. . (i) Since T is onto, we have T(1) = 1. So $A^{'} \subsetneq R^{'}$ implies that $T^{-1}(A^{'}) \subsetneq$ R. Now the proof follows using Lemma 3.2 ((i),(iii),(vi),(vii) and (viii)).

(ii) Let T(A) = A0 . Since A is subtractive containing KT , we have T1(A0) =A by Lemma 3.2 (viii).Thus, by (i) A is prime if and only if T(A) is prime.
(iii) This follows using (vi) and (vii) of Lemma 3.2. (iv) This follows using (iii) as (ii) follows using (i).

□

Theorem 2.0.2. *For an ideal P of R, the following statements are equivalent:*
(i) P is prime.
(ii) For a, $b \in R$, $(a)(b) \subseteq P$ implies that $a \subseteq P$ or $b \in P$.
(iii) For a, $b \in R$, $aRb \subseteq P$ implies that $a \in P$ or $b \in P$.
(iv) For any right (left) ideals A,B of R, $AB \subseteq P$ implies that $A \subseteq P$ or $B \subseteq P$.

Proof. (i) ⇒ (ii) From (i) P- is prime from definition of a prime ideal , for any two ideals A and B of R, if A.B ⊆ P ⇒ A ⊆ P or B ⊆ P.
.From (ii) (a).(b) ⊆ P, for a, b ∈ R , We need to show a ∈ P or b ∈ P Since (a) and (b) are a principal ideals which is generated by a single element a and b of R respectively. Then (a) ⊆ P or (b) ⊆ P from (i)) a ∈ (a) ⊆ P or b ∈ (b) ⊆ P.
since 1 ∈ R, 1.a = a.1 = a ∈ (a) ⊆ P,or b.1 = 1.b = b ∈ (b) ⊆ P.
⇒ a ∈ P or b ∈ P.
ii ⇒ iii For a, b ∈ R, aRb ⊆ P, we need to show that a ∈ P or b ∈ P since aRb ⊆ P
⇒ (aRb)R ⊆ ⊆ , since P is an ideal of a semiring R.
⇒ (R.(aRb).R) ⊆ P.
⇒ (RaRbR) ⊆ P,since R-is an associative.
⇒ (RaR).(RbR) ⊆ P.
⇒ RaR ⊆ P or RbR ⊆ P, since RaR and RbR are ideals and P is a prime.
⇒ (a).(b) ⊆ P, since 1 ∈ R and RaR = (a) and RbR = (b) .
⇒ a ∈ P or b ∈ P from ii.

iii ⇒ iv ∀ right or left ideals A,BofR. if AB ⊆ P.// We need to show that A ⊆ P or B ⊆ P .Suppose that AB ⊆ P and A ⊊ .Where A,B are ideals R.Fix an element a ∈ A¶.For any b∈ B.We have aRb=(aR)b ⊆ AB P,so by (iii) b ∈ P.

□

Corollary 2.1. *1 An ideal P of R is prime if and only if R¶is an m-system.*

Proof. Assume P - is a prime ideal.
We need to show that R¶is an m- system.
Let a,b ∈ R¶.⇒ a,b ∈ R but a,b is not member P.
If arb ∈ P,∀ r ∈ R,then aRb ⊆ P.
⇒ a ∈ P or b ∈ P.This is contradiction from a,b not elements of P.
∃ r ∈ R:arb ∈ R but arb is not an element of *P*.
⇒ arb ∈ R¶.
⇒ R¶is an m-system.

conversely suppose not.
⇒ a,b ∈ R: aRb ⊆ P but a is not an element P or b ∈ is not an element P.
⇒ a,b ∈ aRb ⊆ P but a ∈ M and b ∈ M.
a,b ∈ M, arb is not an element of M ,∀ r ∈ R.
⇒ Which is a contradiction with the fact that M is an m- system.
There for P is Prime.

□

Proposition 2.1. *1. For any ideal I of R,the following statements are equivalent:*
(i) I is semiprime.
(ii) For a ∈ R, $(a)^2$ ⊆ I implies that a ∈ I.
(iii) For a ∈ R, aRa ⊆ I implies that a ∈ I.
(iv) For any right (left) ideals A of R, A ^{2} ⊆ I implies that A ⊆ I.

Proof. Proof i ⇒ ii. I- is semi prime from i, from definition for any ideal A of R, if A^2 ⊆ I ⇒ A ⊆ I.
For a ∈ R, if $(a)^2$ ⊆ I.
We need to show that a ⊆ I.

Since (a) is a principal ideal generated by single element a. Then $(a)^2 \subseteq$ I $\Rightarrow$ (a).(a) $\subseteq$ I.
$\Rightarrow$ (a) $\subseteq$ I, since I is a semi prime ideal.
$\Rightarrow$ a $\in$ (a) $\subseteq$ I , since 1 $\in$ R and 1.a = a.1 = a $\in$ (a) $\Rightarrow$ a $\in$ I.

ii $\Rightarrow$ iii For a $\in$ R, aRa $\subseteq$ I.
We need to show that a $\in$ I,Since aRa $\subseteq$ I $\Rightarrow$ R(aRa) $\subseteq$ I.
$\Rightarrow$ (RaRa)R $\subseteq$ I.
$\Rightarrow$ RaRaR $\subseteq$ I.
$\Rightarrow$ (RaR).(RaR) $\subseteq$ I.
$\Rightarrow$ (a).(a) $\subseteq$ I.
$\Rightarrow$ a $\in$ I, from ii.

iii $\Rightarrow$ iv For any left (right) ideal A of R, $A^2 \subseteq$ I implies that A $\subseteq$ I. We need to show A $\subseteq$ I.
Proof Assume that A is a right ideal and A $\subsetneq$ I.
Let fix an element a $\in$ A$\bigcap$I. Then (aR)a $\subseteq$ A.A $\subseteq$ I.
$\Rightarrow$ aRa $\subseteq A^2 \subseteq$ I.
$\Rightarrow$ aRa $\subseteq$ I, from iii, $\Rightarrow$ a $\in$ I. It is contradiction. Therefore A $\subseteq$ I.
iV $\Rightarrow$ i, For any right or left ideal A of R, if $A^2 \subseteq$ I.
Implies that A $\subseteq$ I. This implies that I- is semi prime from definition.

□

In [2], the authors considered a semiring R with 1 and for any ideal I of R defined $\sqrt{I}$ = a $\in$ R : $a^n \in$ I , for n $\geq$ 1 and called it radical of I. The radical so defined as a nice property that if I is subtractive then $\sqrt{I}$ is subtractive (c.f. [2, Proposition 2.19]). But in view of non commutativity, we follow J.S Golan [4] and have the following definition.

Proof. Assume that I is subtractive , I need to show $\sqrt{I}$ is subtractive.
let x and x + y $\in \sqrt{I}$,then x', $(x+y)' \in$ I.
Implies that y $\in$ I,since I is subtractive and $(x+y) \in$ I.

□

Definition 2.0.12. *Let R be a semiring. For an ideal A of R, the radical of A is defined as follows:*
$\sqrt{A}$ *= s $\in$ R : every m - system M containing s meets A $\subseteq \sqrt{A}$ where $\sqrt{A}$ is defined in [2].*
In the special case when R is a commutative semiring, the inclusion $\subseteq$ above is actually an equality and both the radicals are same. The following ring theoretic result holds in the case of a semiring. For the proof of (i), (ii), and (iii), one can see [4,proposition 7.27].

Proposition 2.2. *Let A and B be two ideals of a semiring R. Then*
(i) $A \subseteq B$, implies that $\sqrt{A} \subseteq \sqrt{B}$.
(ii)$\sqrt{\sqrt{A}} = \sqrt{A}$.
(iii)$\sqrt{A+B} = \sqrt{\sqrt{A}+\sqrt{B}}$

(iv)$\sqrt{AB} = \sqrt{A \bigcap B} = \sqrt{A} \bigcap \sqrt{B}$

Proof. (i) given A $\subseteq$ B.
Let x $\in \sqrt{A} \Rightarrow$ every m-system M- containing x-meets A.
$\Rightarrow$ M $\bigcap$ A $\neq \varnothing$.
$\Rightarrow$ M $\bigcap$ B $\neq \Rightarrow$,since A $\subseteq$ B.
$\Rightarrow$ every m-system M containing x-meets B. since A $\subseteq$ B.
$\Rightarrow$ x $\in \sqrt{B}$
There for $\sqrt{A} \subseteq \sqrt{B}$.

(ii) We need to show (a) $\sqrt{A} \subseteq \sqrt{\sqrt{A}}$ and (b) $\sqrt{\sqrt{A}} \subseteq \sqrt{A}$.
For (a) Since A $\subseteq \sqrt{A}$, from(i) $\sqrt{A} \subseteq \sqrt{\sqrt{A}}$
For (b).let x $\in \sqrt{\sqrt{A}} \Rightarrow$ every m - system M containing x meets $\sqrt{A}$.
$\Rightarrow$ M $\bigcap \sqrt{A} \neq \emptyset$.
$\Rightarrow$ let d $\in$ M $\bigcap$ A.
$\Rightarrow$ d $\in$ M and d $\in \sqrt{A}$. Since d $\in \sqrt{A}$, every m-system containing-d-meets A.
$\Rightarrow$ M $\bigcap$ A $\neq \emptyset$.
$\Rightarrow$ x $\in \sqrt{A}$, since x $\in$ M and M is an m- system.
$\Rightarrow \sqrt{\sqrt{A}} \subseteq \sqrt{A}$: From a and b $\sqrt{\sqrt{A}} \subseteq \sqrt{A}$:

(iii)We went to show $\sqrt{A+B} = \sqrt{\sqrt{A}+\sqrt{B}}$.
a) $\sqrt{A+B} \subseteq \sqrt{\sqrt{A}+\sqrt{B}}$.
(b) $\sqrt{\sqrt{A}+\sqrt{B}} \subseteq \sqrt{A+B}$.
(a). Since A $\subseteq \sqrt{A}$ and B $\subseteq \sqrt{B}$, then A + B $\subseteq \sqrt{A}$ + $\sqrt{B}$.
$\Rightarrow \sqrt{A+B} \subseteq \sqrt{\sqrt{A}+\sqrt{B}}$.
(b) Let x $\in \sqrt{\sqrt{A}+\sqrt{B}}$.Then every m - system M containing x meets $\sqrt{A}+\sqrt{B}$.
$\Rightarrow$ M $\bigcap(\sqrt{A}+\sqrt{B}) \neq \emptyset$.
Let d $\in$ M $\bigcap$ $(\sqrt{A}+\sqrt{B}) \Rightarrow$ d $\in$ M and d $\in (\sqrt{A}+\sqrt{B})$.
Then there exist a $\in \sqrt{A}$ and b $\sqrt{B}$: d = a + b.Assume that S is an m system containing a and T be an m system containing b, then S $\bigcap$ A $\neq \emptyset$ and T $\bigcap$ B $\neq \emptyset$.
Since A $\subseteq \sqrt{A}$ $\sqrt{A}+\sqrt{B}$ and A $\bigcap$ S $\neq \emptyset$,implies that S $\bigcap$ $(\sqrt{A}+\sqrt{B}) \neq \emptyset$.
In similar way B$\bigcap(\sqrt{A}+\sqrt{B}) \neq \emptyset$,
$\Rightarrow$ S and T meets $(\sqrt{A}+\sqrt{B})$, d $\in$ S and T.Then M $\bigcap$ S $\bigcap$ T $\neq \emptyset$.
$\Rightarrow$ M meets A, B and (A + B), since A,B $\subseteq$ A + B.
$\Rightarrow$ x $\in \sqrt{A+B}$. From a and b $\sqrt{(A+B)} = \sqrt{\sqrt{A}+\sqrt{B}}$.

(iv) First we went to show $\sqrt{A \bigcap B} = (\sqrt{A}+\sqrt{B})$.
(a) $\Rightarrow \sqrt{A \bigcap B} \subseteq \sqrt{A} \bigcap \sqrt{B}$, and $\sqrt{A} \bigcap \sqrt{B} \bigcap \sqrt{A \bigcap B}$.
A $\subseteq \sqrt{A}$ and B $\subseteq \sqrt{B}$.
Let x $\in \sqrt{A \bigcap B}$ from definition every m-system M containing x meets $A \bigcap B$. $\Rightarrow$ M $\bigcap$ $(A \bigcap B) \neq \emptyset$.
$\Rightarrow (M \bigcap A)$ $(M \bigcap B) \neq \emptyset$
Let s $\in (M \bigcap A) \bigcap \bigcap (M \bigcap B)$,implies that s $\in M \bigcap A$ and s $\in M \bigcap B$.
$\Rightarrow$ s $\in$ M and s $\in$ A and also s $\in$ M and s $\in$ B.
$\Rightarrow$ M meets A and M meets B $\Rightarrow$ x $\in \sqrt{A}$ and x $\in \sqrt{B}$.

$\Rightarrow$ x $\in \sqrt{A} \bigcap \sqrt{B} \Rightarrow \sqrt{A \bigcap B} \subseteq \sqrt{A} \bigcap \sqrt{B}$.
.............($\bigstar$)
(b) Let x $\in (\sqrt{A} \bigcap \sqrt{B}$, then x $\in \sqrt{A}$ and x $\in \sqrt{B}$.
$\Rightarrow$ M $\bigcap$ $(A \bigcap B) \neq \emptyset$.
$\Rightarrow$ t $\in \sqrt{A \bigcap B}$...................$\bigstar\bigstar$
$\Rightarrow \sqrt{A} \bigcap \sqrt{B} \subseteq \sqrt{A \bigcap B}$.
$\Rightarrow \sqrt{A \bigcap B}$= $\sqrt{\sqrt{A} \bigcap \sqrt{B}}$ from ($\bigstar$) and ($\bigstar\bigstar$).

Next we need to show that $\sqrt{A.B} = \sqrt{A \bigcap B}$.
Since $A.B \subseteq A \bigcap B$, impelies that $\sqrt{A.B} \subseteq \sqrt{A \bigcap B}$ from (i) ......$\bigstar$
let x $\in \sqrt{A \bigcap B}$, then every m- system M containing x- meets $A \bigcap B$
$\Rightarrow$ M meets A and B.There for M $\bigcap$ $(A \bigcap B) \neq \phi$.
$\Rightarrow$ Let r $\in$ R: r $\in$ M, r $\in$ A and r $\in$ B,then since $A.B$ is an ideal and M-is an M system,r.r$\in A.B$ and $\exists 1 \in$ R:r.1.r $\in$ M.
$\Rightarrow r^2 \in$ AB and $r^2 \in$ M.
$\Rightarrow$ AB $\bigcap$ M $\neq \emptyset$.
$\Rightarrow$ M meets A.B.
$\Rightarrow x^2$ $\sqrt{A.B}$.
Therefore $\sqrt{A \bigcap B} \subseteq \sqrt{A.B}$....................$\bigstar\bigstar$
From ($\bigstar$) and ($\bigstar\bigstar$) $\sqrt{A.B}$ =$\sqrt{A \bigcap B}$ =$\sqrt{A}$ $\bigcap$ $\sqrt{B}$:

□

(Zorn‘s Lemma) If A- is a nonempty partially ordered set. Every chain in A has an upper bound in A. Then A is contain a maximal element.Or let M denotes a nonempty collection of subsets of some fixed set R. A subsect C of M is said to be a chain in M, if for A,B $\subseteq$ C, either A $\subseteq$ B or B $\subseteq$ A. By union of the chain C- we mean the union of the subsects of R-which are elements of C.
Zorn‘s lemma —= Let M- be a nonempty collection of subsets of a fixed set R, if the union of each chain in M is an element of M then M- contains one or more maximal elements.

Remark 1. If R- is a ring and S- is the set of all ideals I of R Such that IR then S is partially ordered by set theoretic inclusion. M-is maximal ideal if and only if M- is maximal element in partially ordered set S.

Theorem 2.0.3. *If R is a semiring with unity 1 and N is an ideal in R. Then there exists in R a maximal ideal M such that $N \subseteq M$. Proof. First let us recall that a maximal ideal in R, means an ideal which is maximal in the set of all ideals in R other than R itself.*
Let $M = x : x$ is an ideal in R which containing N Since $N \supseteq N \Rightarrow N \subseteq M$
$\Rightarrow M \neq \emptyset$
We now assert that the union S of an arbitrary chain C in M is an element of M.
For a, $b \in \bigcup$, $\exists$ A,B $\in C : a^2$ A and $b \in B$. But since C is a chain we have $A \subseteq B$ or $B \subseteq A$.
suppose $A \subseteq B$. Implied that a, $b \in B$ and since B is an ideal in R, we have $a + b \in B$, a.r or r.a $\in B$;$\forall$ $r \in R$. Thus a + b, a.r or r.b $\in \bigcup$.

$\Rightarrow \bigcup$ is an ideal in R.
$\Rightarrow$ by the above definition of M,N $\subseteq \bigcup$. More over since no element of M (and therefore Of C) contains 1, we have $1 \in \bigcup$. This shows that $\bigcup \in$ M and zorn's lemma states that M is a maximal element P since $1 \in M$.
$\Rightarrow M \neq R$. Now let Q be any ideal in R such that $M \subseteq Q$. since P is maximal in M, Q is not in M. But $N \subseteq M \subseteq Q$. so if 1 is not in Q, then Q is not in M , a contradiction hence 1 is not in Q and (1) = Q = R thus M is a maximal ideal in R.

Theorem 2.0.4. *If R is a commutative ring (semi ring) such that $R^2 = R$ (in particular if R has an identity) then every maximal ideal M in R-is prime.*
Suppose a,b $\in$ M but a is not in M and b not in M, then each of the ideals M + (a) and M + (b) properly contains M. By maximality M + (a) = R =M + (b).
Since R-is commutative and a,b $\in$ M. $(a)(b) \subseteq (ab) \subseteq M$. This is a contradict with the fact that $M \neq R$ (since M- is maximal) there for $a \in M$ or $b \in M$.Hence M- is prime.

Example 2.0.3. . *I=(0) is a prime ideal of the set of integers Z but it is not a maximal ideal in the set of Z (integers,) because (2) is a principal ideal of Z and $(0) \subseteq (2) \subset \subset Z$.*

Note that A primary ideal in case of commutative rings and semi rings is defined interreges of elementes.But here we define a primary ideals of a semi ring (not necessarily commutative) interims of its elements.
Lemma . Let R be a semi ring and A any ideal of R. Then $\sqrt{A}$ equals the intersection of all the prime ideals of R containing A. In particular, $\sqrt{A}$ is an ideal of R.

Proof. . Let s $\in \sqrt{A}$ and P be any prime ideal of R containing A. Consider the m system R¶. This m system cannot contain s, for otherwise it meets A and hence also P. Therefore we have, s $\in$ P.

Conversely, assume that s is not an element of $\sqrt{A}$. Then by Definition 3.8, there exists an m system S containing s which is disjoint from A. By Zorns Lemma, there exists an ideal P $\supseteq$ A, which is maximal among all those ideals of R disjoint from S. Hence P is a prime ideal (c.f. [4,Proposi-tion 7.12]) and we have s is not a member of P.

□

Definition 2.0.13. *An ideal P of R is said to be primary if for any two ideals A and B of R, $A.B \subseteq P$ implies that $A \subseteq P$ or $B \subseteq \sqrt{P}$.*
Clearly, every prime ideal is primary.

Proposition 2.3. *For any ideal P of R , the following statements are equivalent:*
(i) P is primary.
(ii) For a, b $\in$ R, $(a)(b) \subseteq P$ implies that $a \in P$ or $b \in \sqrt{P}$, where (a) = RaR, (b) = RbR.
(iii) For a, b $\in$ R, $a.R.b \subseteq P$ implies that $a \in P$ or $b \in \sqrt{P}$.

Proof. of this proposition is similar with proof of theorem (2) page (12) of this paper. □

Lemma 2.4. . *For any ideal A of R,$\sqrt{A}$ is the smallest semi prime ideal of R which contains A.*

Proof. Suppose that T=P : P is prime ideals of R containing A
$\sqrt{A} = \bigcap P$, ∀ P ∈ T, from the above lemma(4).
First we need to show that $\sqrt{A}$ is a semi prim ideal of R and second $\sqrt{A} \subseteq$ Q ,for any semiprim ideal Q of R.
Since $\sqrt{A}$ is an ideal of R ,then $\sqrt{A} \subset$ R and $\bigcap P$ ∀P ∈ T is also an ideal from lemma (1).
Let I be any ideal of R,if $I^2 \subseteq \sqrt{A}$ then $I^2 \subseteq$ P; ∀ P ∈ T.
Implies that I ⊆ P, ∀ P, since P is a prime ideal of R.
Implies that I ⊆ ⋂ P; ∀ P ∈ T.
Implies that I ⊆ $\sqrt{A}$.There for $\sqrt{A}$ is a semi prime ideal of R.
Suppose that Q be any semi prime ideal of a semi ring R containing A.If x ∈ $\sqrt{A}$, then from definition ∃ an m system M ⊆ R such that M containing x meets A.
⇒ M ⋂ Q ≠ Ø.
⇒ x ∈ $\sqrt{Q}$ = Q ⇒ $\sqrt{A} \subseteq$ Q.
⇒ $\sqrt{A}$ is the smallest semi prime ideal of R containing A.

□

Proposition 2.4. *for any ideal A of R , the following statements are equivalent.*
(i) A is semiprime.
(ii) A is an intersection of prime ideals containing A.
(iii) $\sqrt{A}$.

Proof. . i. from given A is primary for any two ideals I and J of R_1IJ ⊆ A.
⇒ I ⊆ A or J ⊆ $\sqrt{A}$. And B is radical of A.
⇒ B = $\sqrt{A}$. since A ⊆ $\sqrt{A}$, then A ⊆ B——-(*)
let I be any ideal of R, if $I_2 \subseteq$ B,
⇒ $I_2 \subseteq \sqrt{A}$.
⇒ I ⊆ $\sqrt{A}$ since $\sqrt{A}$ is the smallest semi prime ideal containing A.
⇒ I ⊆ $\sqrt{a}$ =by substitution . ⇒ I ⊆ B I ⊆ B is a semi prime ideal ————(**)
therefore, from * and ** we get A ⊆ B and B is a semi prime ideal.
i. assume that b ∈ B then we want to show that M ∩ A ≠ Ø and M is an M-system containing A. i.e b ∈ B ⇒ b ∈ $\sqrt{A}$. which implies m-system M containing b meets A from definition b ∈ M.
iii. for any a,b ∈ R, aRb ⊆ A. then we need to show that ain or b ∈ B since A is primary and aRb ⊆ A.
→ a ∈ A or b ∈ B $\sqrt{A}$
→ a ∈ A or B ∈ B
as we see from the above i, ii, and iii,are satisfied. after showing these next we want to show that.
1. p-is primary and B is radical.
then let A ⊆ B and from (i) we have $\sqrt{A} \subseteq \sqrt{B}$ which implies $\sqrt{A} \subseteq$, since B is a semi

prime———(*) and if b $\in$ B $\to$ b $\in$ $\sqrt{A}$
hence B $\in$ $\sqrt{A}$————————————————————(**) therefore, from * and ** we get $\sqrt{A}$ =B.
2. for all b $\in$ R, aRb $\subseteq$ A $\Rightarrow$ a $\in$ A or bin$\sqrt{B}$ now we need to show A is primary which implies aRb $\subseteq$ A $\Rightarrow$ a $\in$ A or b $\in$ B $\Rightarrow$ a $\subseteq$ A or b $\in$ $\sqrt{A}$, since $\sqrt{A}$ = B for a,b $\in$ R, aRb $\subseteq$ A $\Rightarrow$ a $\in$ A or b $\in$ $\sqrt{A}$. hence A is a primary ideal.

□

Proposition 2.5. . *let A,B be any two ideals of R,such that.*
i. $A \in B$.
ii. if $b \in B$, then every m-system containing b meets A.
iii. B is maximal.
then A primary and B is its radical.

Proof. . i. A $\subseteq$ B $\Rightarrow$ if b $\in$ $\sqrt{A}$ $\Rightarrow$ B $\subseteq$ $\sqrt{A}$. since B and $\sqrt{A}$ are two ideals of R and B is a maximal ideal then,
$\Rightarrow$ B $\subseteq$ $\sqrt{A}$ $\subsetneq$ R.
$\Rightarrow$ B = $\sqrt{A}$.
thus B is equal to radical of A.
ii. for a,b $\in$ R, assume that aRb $\subseteq$ A, we want to show that a $\in$ A or b $\in$ $\sqrt{A}$ since aRb $\subseteq$ A $\subseteq$ A $\subseteq$ B $\to$ aRb $\in$ B or b $\in$ $\sqrt{A}$ B since B is a maximal ideal and it is a prime ideal?? $\Rightarrow$ a $\in$ $\sqrt{A}$ or b $\in$ $\sqrt{A}$, since B = $\sqrt{A}$.

□

Theorem 2.0.5. . *(i) let P_1, P_2,..., P_n be primary ideals of R such that $\sqrt{pi}$ = B ($\imath$ = 1,2,...,n). then*
(a) $P = \bigcap_n^{\imath}$ =1 $P_\imath$ is primary and $\sqrt{P}$ =B.
(b) let P =P_1,P_2 ....P_n. if B is ,aximal ,then $\sqrt{P}$ = B and P is primary.
(c)let P =$\sum_{\imath}^{n}$ =1 $P_\imath$. if B is maximal,
then $\sqrt{P}$ = B and P is primary.
(ii) if A is primary, B =$\sqrt{A}$ and U is an ideal such that $U \nsubseteq A$,
then A : U is primary and $\sqrt{A : U}$ = B, where A : U = $r \in R : U_r \subseteq A$.

Proof. . (ai) we need to show condition i, ii, iii of proposition (3.17) those are P $\subseteq$ B and B is a semi prime ideal.
b $\in$ B, then every M system containing b meets P
for a, b $\subseteq$ R, if aRb P $\Rightarrow$ a $\in$ P or b $\in$
let x $\in$ P $\Rightarrow$ x $\in$ $P_1 \bigcap P_2 \bigcap$ - - - - - - - - - $\bigcap P_n$, since P = $\bigcap_{\imath=1^n}$ P
$\Rightarrow$ x $\in$ P_1, x $\in$ P_2, x $\in$ p_3 —————,x $\in$ P_n
$\Rightarrow$ x $\in$ $P_1 \subseteq \sqrt{P_1}$,x $\in$ $P_2 \subseteq \sqrt{P_2}$,.........x $\in$ $P_n \subseteq \sqrt{P_n}$
$\Rightarrow$ x $\in$ $\sqrt{P_1}$ = B $\to$ x $\in$ B P $\subseteq$ B ———-(*)
we need to show that B is a semi prime ideal.
for any ideal I of R assume that $I^2 \subseteq$ B and B = $\sqrt{P_1}$
since $I^2 \subseteq$ B $\to$ $I^2 \subseteq \sqrt{P_1}$
$\Rightarrow$ $\bigcap Q_\imath$, where $Q_\imath$ are primary ideals of a semiring R containing P.
$\Rightarrow$ I $\subseteq$ $\bigcap Q_\imath$.
$\Rightarrow$ I $\subseteq$ $\sqrt{P_1}$

$\Rightarrow$ I $\subseteq$ B
B- is a semi prime ideal——————(**)
from (*) and (**) P $\subseteq$ B and B- is a semi prime ideal.
to prove (a ii), let b $\in$ B and S be and an M-system containing b.
since $\sqrt{P_\imath}$ =B, for all i =1, 2, .., n, S intersects each $P_\imath$.
let$d_1 \in$ S $\bigcap P_1$, $d_2 \in$ S $\bigcap P_2$,... $d_n \in$ S $\bigcap P_n$.
for d_1, d_2,...,$d_2 \in$ S, there exist r_1, r_2,..., $r_{n-1} \in$ R such that d = d_1 r_1 d_2...d_{n-1} r_{n-1} $d_n \in$ S. clearly, d $\in$ P and hence S intersects P,
now to prove (iii), let a $\in$ R such that a not P and b is not a member of B.
this implies that a is not member of an element of $P_\imath$ for some i. thus, aRb $\not\subseteq P_\imath$, = B and hence aRb $\not\subseteq$ P.
(bi) we need to show i, P $\subseteq$ B.
ii, if b $\in$ B, then every m-system containing b-meets P.
let x $\in$ P x$\in$ P_1, P_2, P_3,- $P_n \subseteq P_1 \bigcap P_2 \bigcap$ - - - - - - - - - - - - - - $\bigcap P_n$, since A.B $\subseteq$ A $\bigcap$ B for any two A and B. $\Rightarrow$ x $\in P_1 \bigcap P_2 \bigcap P_3 \bigcap$ - - - - - - - - - - - $\bigcap P_n$.
$\Rightarrow$ x $\in P_1 \subseteq \sqrt{P_1} \Rightarrow$ x $\in \sqrt{P_1}$ = B$\Rightarrow$ x $\in$ B
$\Rightarrow \in P_2 \subseteq \sqrt{P_2} \Rightarrow$ x $\in \sqrt{P_2}$ = B $\Rightarrow$ x $\in P_n \subseteq \sqrt{P_{n)}}$
$\Rightarrow$ x $\in \sqrt{P_n}$ = B $\Rightarrow$ x $\in$ B.
$\Rightarrow$ x $\in$ B
P $\subseteq$ B————————(* (b ii), let b $\in$ B S - be any m-system containing P.
we need to show that S $\bigcap$ P $\neq \neq \emptyset$ or S- meets P.
since B = $\sqrt{P_1}$, B = $\sqrt{P_2}$,———,B = $\sqrt{P_n}$ and b $\in$ B.then S intersects P_1, P_2 —— ——-P_n.
$\rightarrow$ s $\bigcap P_1 \neq \emptyset$. S $\bigcap P_2 \neq \emptyset$, ———-, S $\bigcap P_n \neq \emptyset$.
let $d_1 \in$ S and $d_1 \in P_1$.
$d_2 \in$ S $\bigcap P_2$.
, $d_3 \in$ S $\bigcap P_3 \Rightarrow d_3 \in$ S and $d_3 \in P_3$.

$d_3 \in$ S $\bigcap P_n \Rightarrow d_n \in$ S and $d_n \in P_n$.
there exist $_1$, $_2$, $_3$, - - - - - - - - - - - - -, $_n$, $\in P_n$,
$\Rightarrow$ d = $(d_1 r_1).(d_2 r_2)$ - - - - - - - - - - - - -.$(d_{n-1}.r_{n-1} d_n) \in$ S and p_1, p_2, - - - - - - - , p_n.
$\Rightarrow$ d $\in$ S $\Rightarrow d_i \in P_i \Rightarrow d_i.r_i \in P_i$,$\forall$ $r_i \in$ R and i=1, 2, 3, ..., n since p1, P_2, ..., P_n are ideals of R, then d = $(d_1.r_1).(d_2.r_2)$.....$(d_n,r_n) \in P_1$, P_2, P_3,..........P_n.
$\Rightarrow$ d $\in$ P, since each d_i,$r_i \in P_i$. $\Rightarrow$ d $\in$ P and S.implies that S intersects P.
$\Rightarrow$ $\forall$b $\in$ B, every m system containing b meets P
(iii) B maximal ideal is given.
$\therefore$ P - is primary and $\sqrt{P}$ = B by the above proposition.
(c i) we need to show i, P $\subseteq$ B.
ii. if b $\in$ B then every m-system containing b-meets P.
let m$\in$ p $\Rightarrow$ m $\in P_1 + P_2$ P_3 + - - - - - - - - - - - + P_n, for all $a_i \in P_i$, for i = 1,2——n
$\Rightarrow$ m = $a_1 + a_2 + a_3$ + - - - - - - - - - - + a_n
since $a_1 \in P_i \Rightarrow a_i \in p_i \subseteq \sqrt{P_i} \Rightarrow a_i \in \sqrt{P_i}$ for i=1,2,3,——,n.
since $p_1 \subseteq \sqrt{P_1}$ = B, $\Rightarrow a_i \subseteq$ B, and $a_i \in P_1 \in P_1$ then $a_1 \in$ B. $P_2 \subseteq \sqrt{P_2}$ = B, $\Rightarrow P_2 \subseteq$ B, if $a_2 \in P_2$ then $a_2 \in$ B.
$P_3 \subseteq \sqrt{P_3}$ = B, $\Rightarrow P_3 \subseteq$ B then if $a_3 \in P_3$ then $a_3 \in$ B.
———— $P_n \subseteq \sqrt{P_n}$ = B, $\Rightarrow P_n \subseteq$ B and if $a_n \in \P_n$ then $a_n \in$ B.

$\Rightarrow$ m = a_1 + a_1 + a_2 + a_3 - - - - - - - - - + a_n $\in$ B $\forall$ $a_i \in P_i$ and since B - is an ideal of a semi ring R containing each a_i .
$\Rightarrow$ m $\in$ B. and hence P $\subseteq$ B ————-(*)

WNTS(ii) , if b$\in$ B, then every m system containing b
that b $\in$ B. and S be any m-system containin b
,then since b $\in$ B = $\sqrt{P_i}$, for i=1, 2, 3,————, n, and b $\in$ S m-system
then S meets P_1, P_2 —–,P_n,
$\Rightarrow$ S $\bigcap$ $P_1 \neq \emptyset$, S $\bigcap$ $P_2 \neq \emptyset$, S $\bigcap$ $P_3 \doteq \emptyset$,, ———, S $\bigcap$ $P_n \neq \emptyset$,
$\Rightarrow$ let $d_i \in$ S $\bigcap$ $P_i \Rightarrow d_i \in$ S and $d_i \in P_i$,
for i=1, 2, 3,........., n, then there exist r_1,r_2, r_3,——————–, r_{n-1}, $\in$ R :
d = $(d_1 r_1)$ - - - - - - - - - - - - $(d_{n-1} r_{n-1}).(d_n)$ $\in$ S, P_1, P_2, ———— P_n
, since S is an m-system and P_1, P - 2, ——————P_n are ideals of a semiring R
d = $(d_1 r_1 d_2 r_2$ - - - - - - - $d_{n-1} r_{n-1} d_n)$ + 0 + 0 + 0 .... + 0 $\in P_1 + P_2$ + - - - - - - - - - + P_n,
since d, 0 $\in P_i$ $\forall$ i= 1, 2, 3, n,.
since d=$(d_1.r_1.d_2.r_2.$- - - - - - - - - -$.d_{n-1}.r_{n-1}.d_n)$ $\in P_1$ and 0 $\in P_2 + P_3$ + - - - - - - - - - - - - + P_n $\Rightarrow$ d $\in$ S And d $\in$ $(P_1 + P_2$ + - - - - - $P_n)$ = P,
$\Rightarrow$ d $\in$ S and d $\in$ P $\Rightarrow$ S $\bigcap$ P $\neq \emptyset$
$\Rightarrow$ S meets P. there for m system containing b meets P. since B- is maximal P- is primary and $\sqrt{P}$
= B (ii) it again sufficts to verify the three condition of the above proposition of A : U and B.
to show that A : U $\subseteq$ B, we observe that U.(a : U)
subseteq A and U $\subsetneq$ A. therefor A : U $\sqrt{A}$ = B, because A is primary.
clearly B being radical of A is prime.
to prove (ii), let b $\in$ B and S be any m system containing b. then S intersects A as A is primary and B = $\sqrt{A}$ (c.f proposition 3.17),
that is, there exist d $\in$ S $\bigcap$ A. since, d $\in$ A, U.d $\subseteq$ A and so d $\subseteq$ A : U.
this shows that S intersects A : U.
finaly, to prove (iii), let a is not an element of A : U and b is not an element of B. the former implies that there exist c $\in$ U such that c.a is not an element of A. therefore using the above proposition 4(iii) for A and B there exist r $\in$ R such that c.a.r.b $\in$ A and hence a.r.b is not an element A : B, that is , $_aR_b$ $\subsetneq$ A : U . □

Theorem 2.0.6. *let T :R $\rightarrow$ R' be an onto homomorphism. suppose that A is an ideal of R such that both A and $\sqrt{A}$ are subtractive and K_T = $\{x \in R : x = a + b$, and T(a) = T(b)$\}$ $\subseteq$ A.*
then A is primary if and only if T (A) is primary in R, when this is so $\sqrt{T(A)}$ = T ($\sqrt{A}$).

Proof. since A $\subseteq$ $\sqrt{A}$ $\subseteq$,clearly $T(A)$ $\subseteq$ T($\sqrt{A}$) (from lemma 4(i)). To show that T ($\sqrt{A}$) is semi prime ,
let a $\in$ R, such that a $\in$ T($\sqrt{A}$) . Then no pre image of a is in $\sqrt{A}$.
since $\sqrt{A}$ is semi prime, there exist r $\in$ R such that c.r.c is not in $\sqrt{A}$. If x is any pre image of T(a,r,c) = T(c).T(r).T(c) = a .T(r).a, then T(c.r.c) = T(x) implies that x + c.r.c $\in K_T \subseteq \sqrt{A}$. Therefor x is not in $\sqrt{A}$, for otherwise subtractive character of $\sqrt{A}$

gives the contradiction c.r.c $\in \sqrt{A}$.

consequently, a.T(r).a is not in T($\sqrt{A}$. Thus, T($\sqrt{A}$) is semiprime and we are through with the verification of condition (i) of proposition 3.

To verify the condition (ii) of preposition 3.
, let b $\in$ T($\sqrt{a}$, and $S^{'}$, be an m -system in $R^{'}$ containing b. And
Let S = $T^{-1}(S^{'}) \subseteq$ R then S is an m -system in R. Since b $\in$ T($\sqrt{A}$) , there exists a $\in \sqrt{A}$ such that T(a) = b. Moreover, b $\in S^{'}$ implies that c $\in$ S such that T(c) = b. Therefore we have T(a) = T(c) implying a + c $\in K_T \subseteq$ A $\subseteq \sqrt{A}$. Since a + c $\in \sqrt{A}$ and $\sqrt{A}$ is subtractive, we get c $\in \sqrt{A}$. Thus every m - system containing c, in particular S, intersects A nontrivially. That is, there exists d $\in$ S $\bigcap$ A and hence T(d) $\in S^{'} \bigcap$ T(A) and we have verified condition (ii) of Proposition 2.

Finally, for condition (iii) of Proposition 3, let a $\in$ T(A) and b not in $T(\sqrt{A})$
Then obviously no preimage l of a is in A and no preimage m of b is in $\sqrt{A}$. Therefore by primary character of A, there exists r $\in$ R such that l.r.m not in A. If y is any preimage of T(l.r.m) =T(l)T(r)T(m) = a.T(r).b, then T(l.r.m)= T(y) implies that y + l.r.m $in\ K_T \subseteq$ A.Therefore y $\notin$ A , for otherwise subtractive character of A gives the contradiction l.r.m A.
Consequently, a.T(r).b $\notin$ T(A). Thus T(A) is primary.

For the converse, suppose that T(A) is primary and $\sqrt{(T(A)} = T(\sqrt{A})$.
To show A is primary, let a, b $\in$ R such that a $\notin$ A and $\sqrt{A}$. Then as above using subtractive character of A together with $K_T \subseteq$ A, we get T(a)$\notin T(\sqrt{A})$.By a similar argument T(b) $\notin T(\sqrt{A})$ as $\sqrt{A}$ is also subtractive.Since T(A) is primary and $\sqrt{T(A)}$ = T($\sqrt{A}$), we have $r^{'} \in R^{'}$; such that T(a).$r^{'}$.T(b) $\notin$ T(A). As T is onto, there exists r $\in$ R such that T(r) = $r^{'}$.
Therefore, T(a.r.b) = T(a).T(r).T(b) = T(a)$r^{'}$.T(b) $\notin$ T(A).
Thus, we have shown that a.r.b $\notin$ A, that is, a.R.b $\nsubseteq$ A, proving that A is primary.The condition $K_T \subseteq$ A in the above theorem is sufficient but not necessary.

□

Chapter 3

SUBTRACTIVE PRIMARY IDEALS OF SEMIRINGS

Definition 3.0.14. *Let A be an ideal of a semiring R. We define the subtractive radical of A, denoted as* $\sqrt[s]{A}$*,by* $\sqrt[s]{A} = \bigcap \{P_i \in \wp_s(A)\}$*,where* $\wp_s(A) = \{P \subset R$ *:P is prime and subtractive ideal of R containing* $\{A\}$*.*
clearly $\sqrt{A} \subseteq \sqrt[s]{A}$*.If R is a Geĺfand semi ring.*

Proof. Suppose x $\in \sqrt{A}$, C=$\{P : PisprimeidealofR, containingA\}$ and D = $\{Q : QisprimeandsubtractiveidealofRcontainingA\}$.
$\Rightarrow \sqrt{A} = \bigcap$P, where P $\in$ C.
$\Rightarrow$ x $\in \bigcap$P and also $\bigcap$P $\subseteq \bigcap$Q, since P $\in$ C and Q $\in$ D.
$\Rightarrow$ x $\in \bigcap$Q, $\forall$ Q $\in$ D.
$\Rightarrow$ x $\in \sqrt[s]{A}$ and $\sqrt{A} \subseteq \sqrt[s]{A}$.

□

Lemma 3.1. *If A is an ideal of a semi ring R,then* $\sqrt[s]{A}$ *is a subtractive ideal of R containing A.*

Proof. since $\sqrt[s]{A}$ is nonempty as every maximal ideal M of a Gelfand semiring is strong and so subtractive (c.f. [4, Propositions 6.59, 6.62 and 7.13 By lemma 4 of this project paper $\sqrt[s]{A}$ is an ideal, since $\sqrt[s]{A} = \bigcap$P, where P $\in$ C = $\{P : PisprimeandsubtractiveidealofRcontainin$
Next suppose that a and a+b $\in \sqrt[s]{A}$ then a and a+b $\in \bigcap$P.Implies that a and a+b $\in$ $\forall$P $\in$ C.
Since P is subtractive b $\in$ C, $\forall$P $\in$ C ,implies that b $\in \bigcap$P, implies that b $\in \sqrt[s]{A}$.
$\Rightarrow \sqrt[s]{A}$ is subtractive ideal of R containing A.

□

Definition 3.0.15. *A is an ideal of a semiring R, then* $\sqrt[s]{A}$ *is a subtractive ideal of R containing A.As* $\sqrt{P} \subseteq \sqrt[s]{P}$*, every primary ideal of a semiring R is subtractive primary.*

Lemma 3.2. *Let A and B be two ideals of a semiring R. Then*
(i) $A \subseteq B$ *implies* $\sqrt[s]{A} \subseteq \sqrt[s]{B}$*.*
(ii) $\sqrt[s]{A} = \sqrt[s]{\sqrt[s]{A}}$*.*

(iii) $\sqrt[s]{A+B} = \sqrt[s]{\sqrt[s]{A}} + \sqrt[s]{\sqrt[s]{B}}$.
(iv) $\sqrt[s]{A \bigcap B} = \sqrt[s]{A} \bigcap \sqrt[s]{B} = \sqrt[s]{A.B}$.

Proof. (i) A $\subseteq$ B ,We went to show that $\sqrt[s]{A} \subseteq \sqrt[s]{B}$
Suppose that x $\in \sqrt[s]{A}$ and, C = $\{P : PisprimeandsubtractiveidealofRcontainingA\}$.
D = $\{Q : QisprimeandsubtractiveidealofRcontainingB\}$.
Then x $\in \bigcap$ P, where P $\in$ C.
Since A $\subseteq$ B, x $\in \bigcap$Q, Q $\in$ D.
Implies that x $\in \sqrt[s]{B}$ and $\sqrt[s]{A} \subseteq \sqrt[s]{B}$
(ii) we went to show $\sqrt[s]{A} \subseteq \sqrt[s]{\sqrt[s]{A}}$ and $\sqrt[s]{\sqrt[s]{A}} \subseteq \sqrt[s]{A}$.
Since A $\subseteq \sqrt{A}$ and $\sqrt{A} \subseteq \sqrt[s]{A}$ then A $\subseteq \sqrt[s]{A}$.
$\Rightarrow \sqrt[s]{A} \subseteq \sqrt[s]{\sqrt[s]{A}}$ ....★

Suppose that x $\in \sqrt[s]{\sqrt[s]{A}}$ and, C = $\{P : PisprimeandsubtractiveidealofRcontaining\sqrt[s]{A}\}$.
then x $\bigcap$ P, where P $\in$ C, since A $\subseteq \sqrt[s]{A}$ and $\sqrt[s]{A} \subseteq \bigcap$ P.
Implies that A $\subseteq \bigcap$ P and x $\in \sqrt[s]{A}$,since A $\subseteq \bigcap$ P. Implies that $\sqrt[s]{\sqrt[s]{A}} \subseteq \sqrt[s]{A}$...............★★
$\Rightarrow \sqrt[s]{A} = \sqrt[s]{\sqrt[s]{A}}$ from ★ and ★★.
$\therefore$ x $\in \sqrt[s]{A}$ and $\sqrt[s]{A} = \sqrt[s]{\sqrt[s]{A}}$

(iii) we went to Show that $\sqrt[s]{A+B} \subseteq \sqrt[s]{\sqrt[s]{A}} + \sqrt[s]{\sqrt[s]{B}}$ and $\sqrt[s]{\sqrt[s]{A}} + \sqrt[s]{\sqrt[s]{B}} \subseteq \sqrt[s]{A+B}$
Since A, B, $\sqrt[s]{A}$ and $\sqrt[s]{B}$ are ideals of R, then A + B $\subseteq \sqrt[s]{A} + \sqrt[s]{A}$ and $\sqrt[s]{A+B} \subseteq \sqrt[s]{\sqrt[s]{A} + \sqrt[s]{A}}$

uppose that x $\in \sqrt[s]{\sqrt[s]{A} + \sqrt[s]{A}}$, then x $\in \bigcap$ P,P is prime and subtractive ideal of R containing $\sqrt[s]{A} + \sqrt[s]{A}$, implies that p consists A + B $\subseteq \sqrt[s]{A} + \sqrt[s]{A}$. Implies that x $\in \sqrt[s]{A+B}$
Implies that x $\in \sqrt[s]{A+B}$
There for $\sqrt[s]{A+B} = \sqrt[s]{\sqrt[s]{A}} + \sqrt[s]{\sqrt[s]{B}}$.

□

Theorem 3.0.7. *Let* $\{A_i\}$, $i \in I$ *be a family of strong ideals of a semiring R.Then*
(i) $\bigcap_{i\in I} A_i$ *is a strong ideal of R.*
(ii) $\bigcup_{i\in I}$ *is a strong ideal of R, if R is yoked.*
(iii) $\bigcup_{i\in I} A_i = \sum i \in IA_i$, *if R is yoked.*

Proof. (i) Let $a+b \in \bigcap_{i\in I} A_i$. *Then* $a+b \in A_i$ *for all i. Since each* A_i *is strong, a* $\in A_i$ *and* $b \in A_i$ *for all i proving that* $\bigcap_{i\in I} A_i$ *is strong.*

(ii) First we prove that $\bigcup_{i\in I}$ *is an ideal of R. Let a, b* $\in \bigcup_{i\in I}$. *Then* $a \in A_i$, $b \in A_j$ *for some i, j* $\in$ *I. Since R is yoked,there exists an element* $r \in R$ *such that either* $a+r = b$ *or* $b+r = a$.
Suppose $a+r = b$, *then* $a+r \in A_j$ *and the strong character of* A_j *will yield* $a \in A_j$ *and* $r \in A_j$.
Hence $a+b \in A_j \subseteq \bigcup_{i\in I}$ *.If* $b + r = a$, *then we get* $a + b \in A_i \subseteq \bigcup_{i\in I}$.
Also for $r \in R$, $a \in \bigcup_{i\in I}$, *we get* $r.a \in A_i$ *for some* $i \in I$ *and so belongs to* $\bigcup_{i\in I} A_i$.

Moreover since A_i are ideals of R, $1 \notin A_i$ for any i. Hence $1 \notin \bigcup_{i\in I} A_i$ and so $\bigcup_{i\in I} A_i$ is an ideal of R.Now let $a+b \in \bigcup_{i\in I} A_i$. This implies $a+b \in A_i$ for some i. Since A_i is strong, $a \in A_i$ and b in A_i. So $a \in \bigcup_{i\in I} A_i$ and $b \in \bigcup_{i\in I} A_i$ proving that $\bigcup_{i\in I} A_i$ is strong.

(iii) Let $a \in \bigcup_{i\in I} A_i$. Then $a = \sum x_i$, where $x_i \in A_i$. This implies $x_i \bigcup A_i$ for all i and since $\bigcup_{i\in I} A_i$ is an ideal of R, $a = \sum x_i \in \bigcup \sum_{i\in I} A_i$. Hence $i \in I$ $A_i \subseteq \bigcup_{i\in I} A_i$. Converse follows from the fact that $A_i \subseteq \sum_{i\in I} A_i$ for all i and therefore $\bigcup_{i\in I} A_i \subseteq \sum_{i\in I} A_i$.

Theorem 3.0.8. *Let P_1, P_2, ..., P_n be subtractive primary ideals of R such that $\sqrt[s]{P_i}$= B, i = 1, 2, ..., n.Then*
(i) $P = \bigcap P_i$ is subtractive primary and $\sqrt[s]{P} = B$.
(ii) If $P = p_1.p_1.p_2............p_n$ then P is subtractive primary and $\sqrt[s]{P} = B$.
(iii) If P_1, P_2, ..., P_n and B are strong ideals of R and $\sqrt[s]{P_i} = B$, then
(a)$\sum_{i=1}^{n} \sqrt[s]{p_i} = \sqrt[s]{\sum_{i=1}^{n} P_i}$.
(b)$\sum_{i=1}^{n}$ is subtractive primary and $\sqrt[s]{\sum_{i=1}^{n} P_i}$=B.
(iv) If A is subtractive primary, then A : B is subtractive primary and $\sqrt[s]{A} \subseteq \sqrt[n]{A : B}$.

Proof. (i) Let C, D be two ideals of R such that C.D ⊆ P $=\bigcap_{i=1}^{n} P_i$. This implies C.D $\subseteq P - i$ for all i. Since P_i is subtractive primary, either C $\subseteq P - i$ or D $\subseteq \sqrt[s]{P - i}$.If C $\subseteq P - i$ for all i, then we get C ⊆ P. Suppose C $\nsubseteq P - j$ for some j. Then we have D $\subseteq \sqrt[s]{P - j}$= B for some j. Since $\sqrt[s]{P - i}$ = B
for all i=1, 2, 3, ...,n by lemma 4.2(iii) we get D ⊆
$\bigcap_{i=1}^{n} \sqrt[s]{P_{i=1}} = \sqrt[s]{\bigcap_{i=1}^{n} P_{i=1}} = \sqrt[s]{P}$.
This proves that $\bigcap_{i=1}^{n} P - i$ is subtractive primary.

(ii) Using the fact that $\bigcap_{i=1}^{n} \sqrt[s]{P - i} = \sqrt[s]{\bigcap_{i=1}^{n} P - i = 1} = \sqrt[s]{P - 1}.P - 2...........P - n$(c.f. Lemma 4.2)

(iii) the proof follows exactly as in (i). (iii)(a) Since $\sqrt[s]{P - i}$ are strong,by using Lemma 4.5 (iii) we get $\sqrt[s]{\sum_{i=1}^{n} P - i} = \sqrt[s]{\bigcup -i = 1^n \sqrt{P - i}} = \sqrt[s]{\sqrt[s]{P - i}} = \sqrt[s]{P - i} = \bigcup -i = 1^n \sqrt{P - i} = \sum -i = 1^n \sqrt[s]{P_i}$. As each $\sqrt[s]{P_i}$ = B.

(b) Assume that P_1, P_2................P_n and B are strong ideals of R. Let C and D be two ideals of R such that C.D $\subseteq \sum_{i=1}^{n} P_i = \bigcup_{i=1}^{n} P_i$ (c.f. Lemma 4.5 (iii)).
This implies C.D $\subseteq P_i$ for some i. The subtractive primary character of P_i will then yield that, either C $\subseteq P_i$ or D $\subseteq \sqrt[s]{P_i}$. If C $\subseteq P_i$, then C $\subseteq \bigcup_i nP_i = \sum_{i=1}^{n} P_i$.
Suppose C $\nsubseteq P_i$ for any i. Then we have D $\subseteq \sqrt[s]{P_i}$, then the result follows from (a)

(iv) Assume that A is a subtractive primary ideal of R.Let C and D be two ideals of R such that C.D ⊆ A : B. Then B.(C.D) = (B.C).D ⊆ A. Since A is subtractive primary, either B.C ⊆ A or D $\subseteq \sqrt[s]{A}$. If B.C ⊆ A, then C ⊆ A : B by the definition of A : B. If D $\subseteq \sqrt[s]{A}$, then D $\subseteq \sqrt[s]{A : B}$ as A ⊆ A : B (c.f. Lemma 4.2 (i)). Hence A : B is subtractive primary.

□

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