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Boundary-Domain Integral Equations for Variable-Coefficient Mixed BVP in $2-$Dimensional Unbounded Domain

A Thesis Submitted to the Department of Mathematics of Addis Ababa University
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We, the undersigned, hereby certify that we have read and examined this thesis on **Boundary-Domain Integral Equations for Variable-Coefficient Mixed BVP in** $2-$**Dimensional Unbounded Domain**, which is done by **Eshetu Seid** in partial fulfillment of the requirements for the degree of master of science and recommend to the school of graduate studies for acceptance of the thesis.

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Abstract

In this thesis, the direct segregated Boundary Domain Integral Equations (BDIEs) for the Mixed Boundary Value Problems (MBVPs) for a scalar second-order elliptic Partial Differential Equation (PDE) with variable coefficients in an unbounded (exterior) 2D domain is considered. In this thesis, we formulate the exterior 2D domain of the direct segregated systems of BDIEs for the MBVPs for a scalar second-order divergent elliptic PDE with variable coefficients. The aim of this work is to reduce the MBVPs to some direct segregated BDIEs with the use of an appropriate parametrix (Levi function). We examine the characteristics of corresponding parametrix-based integral volume and layer potentials in some weighted Sobolev spaces, as well as the unique solvability of BDIEs and their equivalence to the original MBVPs. This analysis is based on the corresponding properties of the MBVPs in weighted Sobolev spaces that are proved as well.

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Contents

Notations and Abbreviations

α	Multi-index in $\mathbb{N}_0^2, \alpha = (\alpha_1, \alpha_2)$.
$\lvert\alpha\rvert$	The length of the multi-index α, $\quad \lvert\alpha\rvert = \alpha_1 + \alpha_2$.
x	A point in the plane $\mathbb{R}^2$, $x = (x_1, x_2)$.
x^α	The multi-index notation for the power of x, $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2}$.
∂^α	The multi index notation for derivative operator, $\partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2}$.
Ω	An open set in $\mathbb{R}^2$.
$\partial\Omega$	Boundary of Ω.
$\mathbf{n}$	Outward normal to Ω.
∇	Gradient operator.
Δ	Laplace's operator.
$\mathcal{D}$	The set of test functions.
$\mathcal{D}^{'}$	The set of all continuous linear functionals over $\mathcal{D}$.
$\mathcal{S}$	The space of rapidly decreasing test functions.
$\mathcal{S}^{'}$	The space of linear continuous functionals on $\mathcal{S}$.
$\mathrm{supp}(u)$	The support of u.
δ	Dirac's delta function.
$\mathcal{F}$	Fourier transform operator.
r_{Γ_1}	Restriction operator on a set Γ_1.
$\mathbf{T}^{\pm}$	Co-normal derivative operator inside and outside of Ω.
$\gamma^{\pm}$	The trace operator inside and outside Ω .
$W^{k,p}(\Omega)$	Sobolev spaces of order k.

$H^s(\Omega)$	Bessel potential spaces.
$\mathcal{H}^s(\Omega)$	weighted Sobolev space .
PDE	Partial Differential Equation.
BVP	Boundary Value Problem.
BIE	Boundary Integral Equation.
BDIE	Boundary Domain Integral Equation.

Introduction

Mathematical modeling of inhomogeneous media (such as functionally graded materials or materials with damage-induced inhomogeneity) in solid mechanics, electromagnetics, thermoconductivity, fluid flows through porous media, and other branches of physics and engineering frequently involves partial differential equations (PDEs) with variable coefficients.

When the PDE coefficients are not constant, there are typically no explicit fundamental solutions available, which makes it impossible to solve Boundary Value Problems (BVPs) for such PDEs numerically. However, for an extensive range of variable-coefficient PDEs, an explicit parametrix (Levi function) linked to a fundamental solution of corresponding frozen coefficient PDEs can be utilized instead. This reduces BVPs for these PDEs in interior domains to BDIE systems for additional numerical solution of the latter. see e.g, [4,6,20,21,22,25].

The primary objective of this study is to demonstrate the reduction of mixed problems with variable coefficients in exterior domains to certain systems of BDIEs. Additionally, we aim to explore the distinct sovability of BDIEs and their equivalence to the original BVP in the weighted Sobolev spaces. To achieve this, we extend to exterior domains and weighted spaces, using the techniques developed in [25] for interior domains and standard Sobolev (Bessel potential) spaces.

The characteristics of the related boundary value problems are crucial to the BDIE analysis. Today, a lot of research has been done on variable-coefficient BVPs in bounded domains, see e.g. [10,13]. In particular, the analysis of segregated boundary-domain integral equations for variable-coefficient mixed boundary-value problems 3-dimensional unbounded domains can be found in [5]. Nonetheless, due to the logarithmic term in the parametrix of the related partial differential equation, the BDIEs in the two-dimensional case show unique characteristics when compared to the higher dimensions. As a result, in order to guarantee the invertibility of the layer potentials and, consequently, the BDIEs' unique solvability, we must impose requirements on the function spaces.

Chapter 1

Preliminaries

The fundamental definitions and characteristics of several topics that we utilize in the dissertation will be covered in the upcoming chapter. Since we deal in two dimensions, we concentrate on the theories in $\mathbb{R}^2$.Let $\partial_j = \partial_{x_j} := \frac{\partial}{\partial x_j}$ for $j = 1, 2$. A vector of the form $\alpha = (\alpha_1, \alpha_2)$, where each component α_i is a non-negative integer, is called a multi-index of order

$$|\alpha| = \alpha_1 + \alpha_2$$

Given a multi-index α, we define $\partial^\alpha u = D^\alpha u(x) = \frac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}} = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} u(x)$, $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2}$

1.1 Vector Space

Definition 1.1. *Let $\mathcal{F}$ be a field. A set V is called a vector space over $\mathcal{F}$ if there is an operation of addition*

$$(x, y) \mapsto x + y$$

on V , and a scalar multiplication function

$$(\lambda, x) \mapsto \lambda x$$

from $\mathcal{F} \times V$ to V , such that the following Axioms are satisfied:
Axioms for vector addition
A1. If u and v are in V, then u+v is in V .
A2. u+v = v+u for all u and v in V.
A3. u+ (v+w) = (u+v) +w for all u, v, and w in V .
A4. An element 0 in V exists such that v+0 = v = 0+v for every v in V.
A5. For each v in V, an element -v in V exists such that -v+v=0 and v+(-v)=0.
Axioms for scalar multiplication
S1. If v is in V, then av is in V for all a in $\mathcal{F}$.

S2. $a(v+w) = av+aw$ *for all* v *and* w *in* V *and all* a *in* $\mathcal{F}$.
S3. $(a+b)v = av+bv$ *for all* v *in* V *and all* a *and* b *in* $\mathcal{F}$.
S4. $a(bv) = (ab)v$ *for all* v *in* V *and all* a *and* b *in* $\mathcal{F}$.
S5. $1v = v$ *for all* v *in* V .

Sub Space

Definition 1.2. *Let V be a vector space, a nonempty subset U of V is called a subspace of V if U itself is a vector space under the operations of V.*

Theorem 1.1. *Let V be a vector space over a field* $\mathbb{R}$ *and U be a nonempty subset of V. then U is said to be a subspace of V if and only if for all* $u_1, u_2 \in U$ *and for all* $a, b \in \mathbb{R}, au_1 + bu_2 \in U$.

Proof. $\Rightarrow$ Since U is a subspace of V by assumption, it then implies U it self is a vector space under the operations in vector V. and hence , au_1 and $bu_2 \in U$ by S1 and $au_1 + bu_2 \in U$ by A1.

$\Leftarrow$ by assumption we have for all $u_1, u_2 \in U$ and for all $a, b \in \mathbb{R}, au_1 + bu_2 \in U$. then one can easily verify that the 10 axioms are satisfied; by taking suitable choose of a,b $\in \mathbb{N}$ □

1.2 Normed Vector Space

Definition 1.3. *Consider a vector space V. A norm is a real-valued functional* $||.||$ *on V that is given for each real number* α *and each* f *and* g *in V, so that,*

(The Triangle Inequality)

$$||f + g|| \leq ||f|| + ||g||$$

(Positive Homogeneity)

$$||\alpha f|| = |\alpha|||f||$$

(Nonnegativity)

$$||f|| \geq 0 \ and \ ||f|| = 0 \ if \ and \ only \ if \ f = 0.$$

By a normed vector space, we mean a vector space together with a norm, that is, $(V, ||.||)$.

Semi Normed Space

Definition 1.4. *Consider a vector space V. A semi norm is a real-valued functional* $|.|$ *on V that is given for each real number* α *and each f and g in V, so that*

(The Triangle Inequality)

$$|f+g| \leq |f| + |g|$$

(Positive Homogeneity)

$$|\alpha f| = |\alpha||f|$$

(Nonnegativity)

$$|f| \geq 0 \textit{ and } |f| = 0 \textit{ does not necessarily imply } f = 0.$$

By a semi normed space we mean a vector space together with a semi norm. that is $(V, |.|)$.

Example 1.1. *Define a function* $p{:}S \rightarrow [0,\infty)$ *by* $p(x) = |\lim_{n\to\infty} \psi_n|$;$x = \{\psi_n\}$ *for all convergent sequence* $\{\psi_n\}$. *One can easily verify that the function satisfies the three conditions. However, p is not a norm. This is because if we take the sequence* $x = \{\frac{1}{n}\}$ *then* $p(x) = |\lim_{n\to\infty} \frac{1}{n}| = 0$.

Banach Space

Definition 1.5 (Cauchy Sequence, Convergent Sequence)**.** *Assume that V is a Normed space and that* $\{S_n\}$ *is a sequence of points in V.*

I. A sequence $\{S_n\}$ *is considered to be Cauchy if, for each* $\epsilon > 0$, *there is a* $N \in \mathbb{N}$ *such that*

$$m, n \geq N \Rightarrow ||x_n - x_m|| < \epsilon$$

II. if

$$\lim_{n\to\infty} ||x_n - x|| = 0,$$

then $\{S_n\}$ *converges to a point* $x \in V$.

Definition 1.6. *If every Cauchy sequence in a normed space V converges, then V is referred to as a Banach space.*

1.2.1 L^p *Spaces*

Assume that the open set $\Omega \subset \mathbb{R}^2$ is non-empty. $L^p(\Omega)$ is the linear space of measurable functions $u : \Omega \to \mathbb{R}$ for $p \in [1, \infty)$ in such a way that

$$L^p(\Omega) = \left\{ u : \Omega \to \mathbb{R} : u \text{ is measurable and } \int_\Omega |u(\boldsymbol{x})|^p dx < \infty \text{ in } \Omega \right\}.$$

The norm in $L^p(\Omega)$ is defined by

$$\|u\|_{L^p(\Omega)} = \left[\int_\Omega |u(\boldsymbol{x})|^p dx \right]^{1/p} < \infty \tag{1.1}$$

The space $L^\infty(\Omega)$ consists of all essentially bounded measurable functions $u : \Omega \to \mathbb{N}$ i.e

$$L^\infty(\Omega) = \left\{ u : \Omega \to \mathbb{R} : u \text{ is measurable and } \|u\|_{L^\infty(\Omega)} < \infty \text{ in } \Omega \right\}.$$

The norm in $L^\infty(\Omega)$ is defined by

$$\|u\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{x \in \Omega} |u(x)| \tag{1.2}$$

Definition 1.7. *The $L^p_{loc}(\Omega)$ (p-locally integrable) consists of all measurable functions in Ω such that*

$$\int_{\Omega'} |u(x)|^p dx < \infty$$

$$L^p_{loc}(\Omega) = \{ u : u \in L^p(\Omega') \, \forall \Omega' \subset\subset \Omega, \Omega' \textit{ compact} \}$$

Definition 1.8. *Let α be a multi-index and $u \in L^1_{loc}(\Omega)$. If $w \in L^1_{loc}(\Omega)$ satisfies the following conditions, it is referred to as the α th -weak derivative of u.*

$$\int_\Omega u(x) D^\alpha \varphi(x) dx = (-1)^{|\alpha|} \int_\Omega w(x) \varphi(x) dx \quad \forall \varphi \in C_0^\infty(\Omega)$$

Remark 1.1. *We can use integration by part if $u(x)$ is sufficiently smooth to have a continuous derivative $D^\alpha u$.*

$$\int_\Omega u(x) D^\alpha \varphi(x) dx = (-1)^{|\alpha|} \int_\Omega D^\alpha u(x) \varphi(x) dx \quad \forall \varphi \in C_0^\infty(\Omega)$$

Hence, the classical derivatives is also a weak derivatives.

It is very easy to show that$\| \cdot \|_{L^p(\Omega)}$ is a norm for $p \in \{1, 2, \infty\}$. The primary challenge in showing that $\| \cdot \|_{L^p(\Omega)}$ is a norm for different values of p is proving the triangle inequality, also referred to as the Minkowski inequality (see Lemma 1.3 below). First, we use Young's inequality and then the Hölder inequality to

show the Minkowski inequality.
For $p \in [1, \infty]$, we define its conjugate q by the relation

$$\frac{1}{p} + \frac{1}{q} = 1 \tag{1.3}$$

Here we adopt the convention $1/\infty = 0$. It is easy to see that $1 \leq q \leq \infty$. Moreover, $1 < q < \infty$ if $1 < p < \infty, q = 1$ if $p = \infty$, and $q = \infty$ if $p = 1$. For $p \neq 1, \infty$, we obtain from the defining relation (1.3) that its conjugate is given by the formula

$$q = \frac{p}{p-1}$$

We first introduce Young's inequality.

Lemma 1.1 (Young's Inequality)**.** *Let $a, b \geq 0, 1 < p < \infty$, and q the conjugate of p. Then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Proof. Let us restate the above lemma in an equivalent form as follows:
Let $\alpha, \beta \in [0, \infty)$, and let $\lambda \in (0, 1)$. Then

$$\alpha^{\lambda} \beta^{1-\lambda} \leq \lambda\alpha + (1-\lambda)\beta.$$

is also Young's inequality. Define $f : [0, \infty) \to \mathbb{R}$ by

$$f(t) = (1-\lambda) + \lambda t - t^{\lambda}, \quad t \in [0, \infty).$$

Then for each $t \in (0, \infty)$

$$f'(t) = \lambda - \lambda t^{\lambda - 1} = \lambda \left(1 - t^{\lambda - 1}\right).$$

Since $\lambda - 1 < 0$, we have $f'(t) < 0$ for $t \in (0, 1)$ and $f'(t) > 0$ for $t \in (1, \infty)$. Hence

$$0 = f(1) \leq f(t) = 1 - \lambda + \lambda t - t^{\lambda} \quad \forall t \in [0, \infty). \tag{1.4}$$

If $\beta = 0$, then the Lemma holds trivially.
Assume $\beta \neq 0$. Then by substituting $t = \frac{\alpha}{\beta}$ in (1.4) we get

$$\left(\frac{\alpha}{\beta}\right)^{\lambda} \leq 1 - \lambda + \lambda \frac{\alpha}{\beta} \text{ or } \alpha^{\lambda} \beta^{-\lambda} \leq (1-\lambda) + \lambda \alpha \beta^{-1} \tag{1.5}$$

Multiplying both sides of (1.5) by $\beta > 0$, we have

$$\alpha^{\lambda} \beta^{1-\lambda} \leq \lambda\alpha + (1-\lambda)\beta.$$

This implies that,

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

□

Lemma 1.2 (Hölder's Inequality)**.** *Let $p \in [1, \infty]$ and q be its conjugate. Then for any $u \in L^p(\Omega)$ and any $v \in L^q(\Omega)$,*

$$\int_\Omega |u(\boldsymbol{x})v(\boldsymbol{x})|dx \leq \|u\|_p \|v\|_q$$

Proof. The equality follows trivially if either u=0 or v =0. Assume that $u \neq 0$ and $v \neq 0$, then we have the following:

$$\begin{aligned}
\frac{|u(\boldsymbol{x})v(\boldsymbol{x})|}{\|u\|_p\|v\|_q} = \left(\frac{|u(\boldsymbol{x})|^p}{\|u\|_p^p}\right)^{\frac{1}{p}} \left(\frac{|v(\boldsymbol{x})|^q}{\|v\|_q^q}\right)^{\frac{1}{q}} &\leq \frac{1}{p}\frac{|u(\boldsymbol{x})|^p}{\|u\|_p^p} + \frac{1}{q}\frac{|u(\boldsymbol{x})|^q}{\|u\|_q^q} \quad \textit{by Lemma } (1.1) \\
\frac{1}{\|u\|_p\|v\|_q}\int_\Omega |u(\boldsymbol{x})v(\boldsymbol{x})|dx &\leq \frac{1}{p\|u\|_p^p}\int_\Omega |u(\boldsymbol{x})|^p dx + \frac{1}{q\|v\|_q^q}\int_\Omega |v(\boldsymbol{x})|^q dx \\
&= \frac{1}{p\|u\|_p^p}\|u\|_p^p + \frac{1}{q\|v\|_q^q}\|v\|_q^q \\
&= \frac{1}{p} + \frac{1}{q} \\
&= 1.
\end{aligned}$$

□

Lemma 1.3 (Minkowski's Inequality)**.** *Let $1 \leq p \leq \infty$. If the functions u and v belongs to $L^p(\Omega)$, then,*

$$\|u+v\|_p \leq \|u\|_p + \|v\|_p \quad \forall u, v \in L^p(\Omega)$$

Proof.

$$\begin{aligned}
(\|u+v\|_p)^p = \int_\Omega |u(\boldsymbol{x})v(\boldsymbol{x})|^p dx &= \int_\Omega |u(\boldsymbol{x})v(\boldsymbol{x})||u(\boldsymbol{x})v(\boldsymbol{x})|^{p-1}dx \\
&\leq \int_\Omega |u(\boldsymbol{x})||u(\boldsymbol{x})v(\boldsymbol{x})|^{p-1}dx + \int_\Omega |v(\boldsymbol{x})||u(\boldsymbol{x})v(\boldsymbol{x})|^{p-1}dx \\
&\leq \|u\|_p\left(\int_\Omega |u(\boldsymbol{x})v(\boldsymbol{x})|^{(p-1)q}dx\right)^{\frac{1}{q}} + \|v\|_p\left(\int_\Omega |u(\boldsymbol{x})v(\boldsymbol{x})|^{(p-1)q}dx\right)^{\frac{1}{q}} \\
&= (\|u\|_p + \|v\|_p)\left(\int_\Omega |u(\boldsymbol{x})v(\boldsymbol{x})|^{(p-1)q}dx\right)^{\frac{1}{q}} \\
&= (\|u\|_p + \|v\|_p)\left(\int_\Omega |u(\boldsymbol{x})v(\boldsymbol{x})|^{p}dx\right)^{\frac{1}{q}} \\
&= (\|u\|_p + \|v\|_p)(\|u+v\|_p)^{\frac{p}{q}}
\end{aligned}$$

Hence $(\|u+v\|_p)^{p-\frac{p}{q}} \leq \|u\|_p + \|v\|_p$ but since $p - \frac{p}{q} = 1$ we get:
$\|u+v\|_p \leq \|u\|_p + \|v\|_p$ □

Theorem 1.2. *The Spaces L^p is a Banach space.*

Proof. Let $\{f_n\}_{n=1}^{\infty}$ be a Cauchy sequence in L^p, and consider a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ of $\{f_n\}$ with the following property $\left\|f_{n_{k+1}} - f_{n_k}\right\|_{L^p} \leq 2^{-k}$ for all $k \geq 1$. We now consider the series whose convergence will be seen below

$$f(x) = f_{n_1}(x) + \sum_{k=1}^{\infty} \left(f_{n_{k+1}}(x) - f_{n_k}(x)\right)$$

and

$$g(x) = |f_{n_1}(x)| + \sum_{k=1}^{\infty} \left|f_{n_{k+1}}(x) - f_{n_k}(x)\right|$$

and the corresponding partial sums

$$S_m(f)(x) = f_{n_1}(x) + \sum_{k=1}^{m} \left(f_{n_{k+1}}(x) - f_{n_k}(x)\right)$$

and

$$S_m(g)(x) = |f_{n_1}(x)| + \sum_{k=1}^{m} \left|f_{n_{k+1}}(x) - f_{n_k}(x)\right|$$

The triangle inequality for L^p implies

$$\begin{aligned}\|S_m(g)\|_{L^p} &\leq \|f_{n_1}\|_{L^p} + \sum_{k=1}^{m} \left\|f_{n_{k+1}} - f_{n_k}\right\|_{L^p} \\ &\leq \|f_{n_1}\|_{L^p} + \sum_{k=1}^{m} 2^{-k}.\end{aligned}$$

By allowing m to approach infinity and utilizing the monotone convergence theory *see* [9, *page* 83], it can be shown that $\int f^p < \infty$. Consequently, the series defining f and $f \in L^p$ converge almost everywhere. We now show that f is the sequence $\{f_n\}$'s intended limit. Given that the telescopic series construction yields a $(m-1)^{\text{th}}$ partial sum of this series that is exactly f_{n_m}, we may conclude that

$$f_{n_m}(x) \to f(x) \quad \text{a.e.}$$

To prove that $f_{n_m} \to f$ in L^p as well, we first observe that

$$\begin{aligned}|f(x) - S_m(f)(x)|^p &\leq [2\max(|f(x)|, |S_m(f)(x)|)]^p \\ &\leq 2^p|f(x)|^p + 2^p\,|S_m(f)(x)|^p \\ &\leq 2^{p+1}|g(x)|^p\end{aligned}$$

for all m. Then,by applying the dominated convergence theorem we obtain $\|f_{n_m} - f\|_{L^p} \to 0$ as m tends to infinity.

Finally, Since $\{f_n\}$ is Cauchy then for any $\epsilon > 0$, there exists N such that for all $n, m > N$ we have $\|f_n - f_m\|_{L^p} < \epsilon/2$. If n_m is chosen so that $n_m > N$, and $\|f_{n_m} - f\|_{L^p} < \epsilon/2$, then the triangle inequality implies

$$\|f_n - f\|_{L^p} \leq \|f_n - f_{n_m}\|_{L^p} + \|f_{n_m} - f\|_{L^p} < \epsilon$$

whenever $n > N$. This concludes the proof of the theorem.

□

1.3 Inner product Space

Definition 1.9. *Let V be a vector space over $\mathbb{C}$ (or over $\mathbb{R}$). Assume that the function*

$$\langle . \ , \ . \rangle : V \times V \mapsto \mathbb{C} \text{ or } \langle . \ , \ . \rangle : V \times V \mapsto \mathbb{R}$$

is defined for any $u, v \in V$ and satisfies the following axioms

1. $\langle u, v\rangle = \overline{\langle v, u\rangle}$ or, for the field $\mathbb{R}$, $\langle u, v\rangle = \langle v, u\rangle$;
2. $\langle \alpha u + \beta v, w\rangle = \alpha\langle u, w\rangle + \beta\langle v, w\rangle$;
3. $\langle u, u\rangle \geq 0$;
4. $\langle u, u\rangle = 0$ if and only if $u = 0$.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^2$, and let f,g and h be any element in $L^2(\Omega)$ then,*

$$\langle f \ , \ g\rangle := \int_\Omega f(\boldsymbol{x})g(\boldsymbol{x})dx \tag{1.6}$$

is an inner product on $L^2(\Omega)$

Proof.

$$\begin{aligned}\langle f \ , \ g\rangle := \int_\Omega f(\boldsymbol{x})g(\boldsymbol{x})dx &= \int_\Omega g(\boldsymbol{x})f(\boldsymbol{x})dx \\ &= \langle g \ , \ f\rangle\end{aligned}$$

$$\begin{aligned}\langle \alpha f + \beta g, h\rangle := \int_\Omega (\alpha f(\boldsymbol{x}) + \beta g(\boldsymbol{x}))h(\boldsymbol{x})dx &= \int_\Omega \alpha f(\boldsymbol{x})h(\boldsymbol{x}) + \beta g(\boldsymbol{x})h(\boldsymbol{x})dx \\ &= \alpha \int_\Omega f(\boldsymbol{x})h(\boldsymbol{x})dx + \beta \int_\Omega g(\boldsymbol{x})h(\boldsymbol{x}) \\ &= \alpha\langle f, h\rangle + \beta\langle g, h\rangle\end{aligned}$$

$$\begin{aligned}\langle f, f\rangle := \int_\Omega f(\boldsymbol{x})f(\boldsymbol{x})dx &= \int_\Omega f^2(\boldsymbol{x})dx \\ &\geq 0; \ \textit{since } f^2(x) \ \geq 0\end{aligned}$$

$$\begin{aligned}\langle f, f\rangle := \int_\Omega f(\boldsymbol{x})f(\boldsymbol{x})dx = \int_\Omega f^2(\boldsymbol{x})dx \\ = 0;\ if\ and\ only\ if\ f^2 = 0\ implies\ that\ f = 0.\end{aligned}$$

This concludes the proof of the theorem. □

Hilbert Space

Definition 1.10 (Cauchy Sequence, Convergent Sequence)**.** *Assume that V is an Inner Product space and that $\{S_n\}$ is a sequence of points in V.*

I. A sequence $\{S_n\}$ is considered to be Cauchy if, for each ϵ > 0, there is an N $\in \mathbb{N}$ such that

$$m, n \geq N \Rightarrow \langle y_n, y_m\rangle < \epsilon$$

II. if

$$\lim_{n\to\infty} \langle y_n, y\rangle = 0$$

then $\{S_n\}$ converges to a point $y \in V$.

Definition 1.11. *If every Cauchy sequence in an Inner Product space V converges, then V is referred to as a Hilbert space.*

Lemma 1.4. *The following equation is the L^2 norm induced by the inner product (1.6).*

$$\|f\| = \sqrt{\langle f, f\rangle} = \left[\int_\Omega |f(\boldsymbol{x})|^2 dx\right]^{1/2} = \|f\|_{L^2} \tag{1.7}$$

Proof. The case Positive Homogeneity and Nonnegativity is trivial. so let us check the triangle inequality case.
let f and g be in L^2 then,

$$\begin{aligned}\|f+g\| = \left(\int_\Omega |f(\boldsymbol{x}) + g(\boldsymbol{x})|^2 dx\right)^{1/2} &\leq \left(\int_\Omega |f(\boldsymbol{x})|^2 + 2|(fg)(\boldsymbol{x})| + |g(\boldsymbol{x})|^2 dx\right)^{1/2} \\ &= \left(\int_\Omega |f(\boldsymbol{x})|^2 dx + 2\int_\Omega |(fg)(\boldsymbol{x})| dx + \int_\Omega |g(\boldsymbol{x})|^2 dx\right)^{1/2} \\ &\leq \left(\|f\|^2 + 2\|f\|\|g\| + \|g\|^2\right)^{1/2} \quad by\ Holder's\ Inequality \\ &= \left(\|f+g\|^2\right)^{1/2} \\ &= \|f\| + \|g\|\end{aligned}$$

□

Corollary 1.1. *The Spaces L^2 is a Hilbert space.*

Proof. We have proven by $Theorem$ 1.2 that L^2 is a Banach space with respect to the norm (1.7) and hence L^2 is a Hilbert space. □

Chapter 2

Distributions

2.1 Test and Distributions

2.1.1 The space of test functions $\mathcal{D}(\Omega)$

We denote the space $C^k(\Omega)$, the space of k- times continuously differentiable functions equipped with the norm, for an open bounded domain $\Omega \subset \mathbb{R}^2$ and $k \in \mathbb{N}_0$.

$$\|f\|_{C^k(\Omega)} = \sum_{|\alpha| \leq k x \in \bar{\Omega}} \sup |\partial^\alpha f(x)|$$

The set of all functions that are infinitely differentiable in Ω is represented by $C^\infty(\Omega)$. That is: for any integer $k \geq 0$,

$$C^\infty(\Omega) = \big\{u : u \in C^k(\Omega) \quad \text{for all } k\big\}$$

The closure of the set where u does not vanish, $\overline{\{x \in \Omega : u(x) \neq 0\}}$, is the support of a continuous function u in Ω, and represented by supp u.The space of infinitely differentiable functions with compact support is denoted by $C_0^\infty(\Omega)$.

$$C_0^\infty(\Omega) = \{u \in C^\infty(\Omega) : \operatorname{supp} u \subset\subset \Omega\}$$

Definition 2.1. *The set of all infinitely differentiable functions in Ω with compact support is called the set of test functions $\mathcal{D}(\Omega)$. We also denoted the set $\mathcal{D}(\Omega)$ by $C_0^\infty(\Omega)$*

$$\mathcal{D}(\Omega) = \{\varphi \in C^\infty(\Omega) : \operatorname{supp}(\varphi) \ \textit{is compact in } \Omega\}.$$

Example 2.1. *For $x \in \mathbb{R}^2$, let*

$$\varphi(x) = \begin{cases} e^{\frac{-1}{1-|x|^2}} : & |x| \leq 1 \\ 0 : & |x| > 1 \end{cases}$$

We assert that, $\varphi^\alpha(x) = \frac{P_\alpha(x)\varphi(x)}{(1-|x|^2)^{|\alpha|}}$ *for any multi-indix* α*, for some polynomial* P_α*. For* $|x| < 1$*, We can distinguish and infer inductively in* α*, that* $\varphi^\alpha(x) = P_\alpha(x)e^{-t}t^{|\alpha|}$ *for some polynomial* P_α*, where* $t = \frac{1}{(1-|x|^2)^{|\alpha|}}$*. Moreover,* $\varphi^\alpha(x) = 0$ *for* $|x| > 1$*. Thus, the formula above for* φ^α *is verified in the case* $|x| \neq 1$*. Since the exponential increases faster than any finite power,* $\frac{P_\alpha(x)\varphi(x)}{(1-|x|^2)^k} = \frac{P_\alpha(x)t^{|\alpha|}|+k}{e^t} \to 0$ *as* $|x| \to 1$ *(i.e., as* $t \to \infty$ *). Applying these findings (inductively) with* $k = 0$ *shows that* φ^α *is continuous at* $|x| = 1$*, and using* $k = 1$ *shows that it is likewise differentiable and has a zero derivative there. Hence, the formula that we claimed holds for any* x*. In addition, we also observe from the fact that* φ^α *is continuous and bounded for any* α*. Hence,* $\varphi \in \mathcal{D}(\Omega)$ *for any open set* Ω *containing the closed unit disc. As a result* φ *is infinitely differentiable and we can show that its* n^{th} *derivative has the form* $\varphi^{(n)}(x) = p_n\left(x, \frac{1}{1-x^2}\right)\varphi(x)$ *for some polynomial* $p_n(s,t)$ *and Hence* $\varphi \in C_0^\infty\left(\mathbb{R}^2\right)$ *with compact support. supp* $\varphi = \left\{(x,y) : x^2 + y^2 \leq 1\right\}$ *and it is differentiable infinitely.*

Remark 2.1. *i. If* $\varphi_1(x)$ *and* $\varphi_2(x)$ *are test functions on* Ω*, so is* $C_1\varphi_1(x) + C_2\varphi_2(x)$ *for any* $C_1, C_2 \in \mathbb{R}$*. Hence, the space* $\mathcal{D}(\Omega)$ *is a real linear space.*
ii. If $a \in C^\infty(\Omega)$ *and* $\varphi \in \mathcal{D}(\Omega)$*, then* $a\varphi \in \mathcal{D}(\Omega)$

Convergence in $\mathcal{D}(\Omega)$: if there is a compact subset E of Ω with supp $(\varphi_n) \subset E$, $n \in \mathbb{N}$ and $\partial^\alpha\varphi_n \to \partial^\alpha\varphi$ uniformly on $E, \forall\alpha \in \mathbb{N}_0^2$, then we call a sequence $\{\varphi_n\}_{n=1}^\infty$ in $\mathcal{D}(\Omega)$ converges in $\mathcal{D}(\Omega)$ to φ in $\mathcal{D}(\Omega)$.

2.1.2 The space of distributions $\mathcal{D}'(\Omega)$

Definition 2.2. *Let* Ω *be a domain in* $\mathbb{R}^2$ *and* $\mathcal{D}(\Omega)$ *be the space of test functions. The set of distributions(or generalized functions) is a collection of all complex valued linear continuous functionals* u *over* $\mathcal{D}(\Omega)$*.*

The above definition can be interpreted as follows:

1. A distribution f is a functional over $\mathcal{D}(\Omega)$, meaning that there is a (complex-valued) number associated with each $\varphi \in \mathcal{D}(\Omega)$.
$$\langle f, \varphi\rangle = f(\varphi)$$
2. A distribution f is said to be linear functional over $\mathcal{D}(\Omega)$, if $\psi, \varphi \in \mathcal{D}(\Omega)$ and λ, μ are complex numbers such that
$$\langle f, \lambda\varphi + \mu\psi\rangle = \lambda\langle f, \varphi\rangle + \mu\langle f, \psi\rangle$$
3. A distribution f is said to be a continuous functional over $\mathcal{D}(\Omega)$, if $\varphi_k \to \varphi$ as $k \to \infty$ in $\mathcal{D}(\Omega)$ such that
$$\langle f, \varphi_k\rangle \to \langle f, \varphi\rangle, \quad k \to \infty$$

The set of all distribution specified in Ω are denoted by $\mathcal{D}'(\Omega)$
Convergence in $\mathcal{D}'(\Omega)$: if $\langle u_n, \varphi\rangle \to \langle u, \varphi\rangle$ in $\mathbb{C}$ as $n \to \infty$, $\forall\varphi \in \mathcal{D}(\Omega)$

,then a sequence of distributions $\{u_n\}$ said to be convergent to u in $\mathcal{D}'(\Omega)$. And it is expressed as $u_n \to u$ in $\mathcal{D}'(\Omega)$.
Distributions, which are definable in terms of locally integrable functions are said to be **regular** distributions. The remaining generalized functions are called **singular** distributions.

Example 2.2. *The Dirac delta distribution* δ, *which is defined by* $\langle \delta, \varphi \rangle = \varphi(0), \quad \forall \varphi \in \mathcal{D}(\Omega)$ *is singular distribution. The Dirac delta distribution cannot be evaluated at points, it makes sense to say that it vanishes except at origin. Thus,* $\operatorname{supp}(\boldsymbol{\delta}) = \{0\}$.

Remark 2.2. *Since every test functions are locally integrable,* $L^1_{loc}(\Omega) \subset \mathcal{D}'(\Omega)$ *and hence* $\mathcal{D}(\Omega) \subset \mathcal{D}'(\Omega)$

- *We define multiplication of a distribution f, by* $a \in C^\infty(\Omega)$ *as* $\langle af, \varphi \rangle = \langle f, a\varphi \rangle$ *for all* $\varphi \in \mathcal{D}(\Omega)$.
- *However multiplication of two arbitrary distributions is not possible in general. For instance,*

$$0 = 0 \cdot \mathcal{P}\frac{1}{x} = (x\delta(x))\mathcal{P}\frac{1}{x} = (\delta(x)x)\mathcal{P}\frac{1}{x} = \delta(x)\left(x\mathcal{P}\frac{1}{x}\right) = \delta(x)$$

Definition 2.3. *If* $\langle u, \varphi \rangle = 0$ *for all* $\varphi \in \mathcal{D}(E)$, *then we say that a distribution* $u \in \mathcal{D}'(\Omega)$ *vanishes on an open set* $E \subset \Omega$. *If their difference vanishes on* E, *then the distributions are said to be equal on E. The support of u is the complement of the greatest open set on which u vanishes in* Ω.

Definition 2.4. *Let* $u \in C^k(\Omega)$, *where k is a positive integer. Then for multi-index* $\alpha, |\alpha| \leq k$ *and* $\varphi \in \mathcal{D}(\Omega)$, *we have the following integration by parts formula:*

$$\langle D^\alpha u, \varphi \rangle = \int D^\alpha u(x)\varphi(x)dx = (-1)^{|\alpha|} \int u(x) D^\alpha \varphi(x) dx = (-1)^{|\alpha|} \langle u, D^\alpha \varphi \rangle$$

Example 2.3. *Let* $f \in \mathcal{D}'(\Omega)$. *Then the derivative of f with respect to* $x_j, j = 1, 2$ *is defined as:*

$$\left\langle \frac{\partial f}{\partial x_j}, \varphi \right\rangle = -\left\langle f, \frac{\partial \varphi}{\partial x_j} \right\rangle \qquad \text{for all } \varphi \in \mathcal{D}(\Omega)$$

This motivates the definition of the derivative $D^\alpha u$ of the distribution $u \in \mathcal{D}'(\Omega)$:

$$\langle D^\alpha u, \varphi \rangle = (-1)^{|\alpha|} \langle u, D^\alpha \varphi \rangle \tag{2.1}$$

Since $u \in \mathcal{D}'(\Omega)$ the functional $D^\alpha u$, defined on the right hand side of equation (2.1), is linear and continuous :

$$\langle D^\alpha u, \varphi_k\rangle = (-1)^{|\alpha|}\langle u, D^\alpha\varphi_k\rangle \to (-1)^{|\alpha|}\langle u, D^\alpha\varphi\rangle = \langle D^\alpha u, \varphi\rangle$$

for, if $\varphi_k \to \varphi$ as $k \to \infty$ in $\mathcal{D}$, then also $\partial^\alpha\varphi_k \to \partial^\alpha\varphi$ as $k \to \infty$ in $\mathcal{D}$. Therefore, $D^\alpha u \in \mathcal{D}'$.

Example 2.4. *If $0 \in \Omega$ and $\delta \in \mathcal{D}'(\Omega)$ is the Dirac distribution, then $D^\alpha\delta$ given by*

$$\langle D^\alpha\delta, \varphi\rangle = (-1)^{|\alpha|}\langle \delta, D^\alpha\varphi\rangle = D^\alpha\varphi(0)$$

2.1.3 The space $\mathcal{S}\left(\mathbb{R}^2\right)$ and $\mathcal{S}'\left(\mathbb{R}^2\right)$

Definition 2.5. *The space of basic functions $\mathcal{S}\left(\mathbb{R}^2\right)$ is the collection of all functions infinitely differentiable in $\mathbb{R}^2$ that decrease together with all their derivatives, as $|x| \to \infty$, faster than any power of $|x|^{-1}$ i.e.*

$$\mathcal{S}\left(\mathbb{R}^2\right) = \left\{\varphi \in C^\infty\left(\mathbb{R}^2\right) : \sup_{x\in\mathbb{R}^2}\left|x^\alpha\partial^\beta\varphi(x)\right| < \infty \textit{ for all multi-indices } \quad \alpha\&\beta.\right\}$$

We define the norm in $\mathcal{S}\left(\mathbb{R}^2\right)$ as:

$$\|\varphi\|_p = \sup_{|\alpha|\le p}\left(1+|x|^2\right)^{\frac{p}{2}}|D^\alpha\varphi(x)|, \varphi \in \mathcal{S}\left(\mathbb{R}^2\right), \quad p \in \mathbb{W}$$

Clearly, $\|\varphi\|_0 \le \|\varphi\|_1 \le \|\varphi\|_2 \le \cdots$

The space $\mathcal{S}\left(\mathbb{R}^2\right)$ is usually called the Schwartz space. The space $\mathcal{S}\left(\mathbb{R}^2\right)$ is a larger class of functions than the space $\mathcal{D}\left(\mathbb{R}^2\right)$, i.e., $\mathcal{D}\left(\mathbb{R}^2\right) \subset \mathcal{S}\left(\mathbb{R}^2\right)$ $\left(\mathcal{S}\left(\mathbb{R}^2\right)\right.$ does not coincides with $\left.\mathcal{D}\left(\mathbb{R}^2\right)\right)$. An example of this would be the function, $\phi(x) = e^{-|x|^2}$ which doesn't contain a compact support so it does not belong to $\mathcal{D}\left(\mathbb{R}^2\right)$ but belongs $\mathcal{S}\left(\mathbb{R}^2\right)$.

Convergence in $\mathcal{S}\left(\mathbb{R}^2\right)$: if for all multi-indices α, β

$$x^\alpha\partial^\beta(\phi_n(x) - \phi(x)) \to 0 \text{ uniformly for } \quad x \in \mathbb{R}^2$$

then a sequence of basic functions $\{\phi_n\}$ in $\mathcal{S}\left(\mathbb{R}^2\right)$ said to be convergent to ϕ in $\mathcal{S}\left(\mathbb{R}^2\right)$.

Since the basic functions from $\mathcal{S}$ are locally integrable in $\mathbb{R}^2$, then the operation of Fourier Transform $\mathcal{F}(\varphi)$ and $\mathcal{F}^{-1}(\varphi)$ are defined in $\mathcal{S}\left(\mathbb{R}^2\right)$.

Definition 2.6. *Let $\varphi(\xi) \in \mathcal{S}\left(\mathbb{R}^2\right)$. Then the integral transformation*

$$\hat{\varphi} = \mathcal{F}[\varphi](x) = \mathcal{F}_{\xi\to x}[\varphi(\xi)] := \int_{\mathbb{R}^2}\varphi(\xi)e^{-i2\pi x\cdot\xi}d\xi, \quad x \in \mathbb{R}^2$$

is called the Fourier transform of φ and

$$\mathcal{F}^{-1}[\varphi](x) = \mathcal{F}^{-1}_{\xi\to x}[\varphi(\xi)] := \int_{\mathbb{R}^2}\varphi(\xi)e^{i2\pi x\cdot\xi}d\xi, \quad x \in \mathbb{R}^2$$

is the inverse Fourier transform of φ

The following proof illustrates that Fourier transform can be differentiated under the integral sign any number of times which implies that they belongs to the class of C^{∞}

$$\partial^{\alpha}\mathcal{F}[\varphi](x)=\int_{\mathbb{R}^2}(-i2\pi\xi)^{\alpha}\varphi(\xi)e^{-i2\pi\xi\cdot x}d\xi=\mathcal{F}\left[(-i2\pi\xi)^{\alpha}\varphi\right](x) \tag{2.2}$$

Hence it follows that $\mathcal{F}[\varphi]\in C^{\infty}\left(\mathbb{R}^2\right)$.
Furthermore,

$$\mathcal{F}\left[\partial^{\alpha}\varphi\right](x)=\int_{\mathbb{R}^2}\partial^{\alpha}\varphi(\xi)e^{-i2\pi\xi\cdot x}d\xi=(i2\pi x)^{\alpha}\mathcal{F}[\varphi](x) \tag{2.3}$$

From equations (2.2) and (2.3), for $\beta\leq\alpha$ we obtain

$$x^{\beta}D^{\alpha}\mathcal{F}[\varphi](x)=x^{\beta}\mathcal{F}\left[(-i2\pi\xi)^{\alpha}\varphi\right](x)=\mathcal{F}\left[D^{\beta}\left((-2i\pi)^{\alpha-\beta}\xi^{\alpha}\varphi\right)\right](x) \tag{2.4}$$

It follows from equation (2.4) that α and β the magnitudes $x^{\beta}D^{\alpha}\mathcal{F}[\varphi](x)$ are uniformly bounded with respect to $x\in\mathbb{R}^2$

$$\left|x^{\beta}D^{\alpha}\mathcal{F}[\varphi](x)\right|\leq(2\pi)^{\alpha-\beta}\int\left|D^{\beta}\left(\xi^{\alpha}\varphi\right)\right|d\xi$$

This means that $\mathcal{F}[\varphi]\in\mathcal{S}$. So the Fourier transform maps the space $\mathcal{S}$ into itself.

Note That 2.1. *if $\varphi\in\mathcal{S}\left(\mathbb{R}^2\right)$, then $\mathcal{F}\varphi\in\mathcal{S}\left(\mathbb{R}^2\right),\mathcal{F}^{-1}\mathcal{F}\varphi=\varphi$ and $\mathcal{F}\mathcal{F}^{-1}\varphi=\varphi$. where*

$$\begin{aligned}\mathcal{F}^{-1}[\varphi](x)&=\int_{\mathbb{R}^2}\varphi(\xi)e^{i2\pi x\cdot\xi}d\xi\\&=\int_{\mathbb{R}^2}\varphi(\xi)e^{-i2\pi x\cdot-(\xi)}d\xi\\&=\mathcal{F}[\varphi](-x).\end{aligned}$$

It is known that $\hat{u}\in\mathcal{S}\left(\mathbb{R}^2\right)$ and if $u\in\mathcal{S}\left(\mathbb{R}^2\right)$ then, $\mathcal{F}:\mathcal{S}\left(\mathbb{R}^2\right)\to\mathcal{S}\left(\mathbb{R}^2\right)$ is a linear operator.

Definition 2.7. *A distribution of slow growth, or tempered distribution, is any continuous linear functional on the space $\mathcal{S}\left(\mathbb{R}^2\right)$ of basic functions; and is denoted by $\mathcal{S}'\left(\mathbb{R}^2\right)$.*

Note That 2.2. *if $f\in\mathcal{S}'\left(\mathbb{R}^2\right)$, then $\langle f,\varphi\rangle\in\mathbb{R}$ or $\mathbb{C}$ for all $\varphi\in\mathcal{S}\left(\mathbb{R}^2\right)$. Since $\mathcal{D}\left(\mathbb{R}^2\right)\subset\mathcal{S}\left(\mathbb{R}^2\right)$, we can observe that $\mathcal{S}'\left(\mathbb{R}^2\right)\subset\mathcal{D}'\left(\mathbb{R}^2\right)$. The Fourier transformation is extended to the class $\mathcal{S}'\left(\mathbb{R}^2\right)$. Let $u\in\mathcal{S}'\left(\mathbb{R}^2\right)$, a functional $\mathcal{F}[u]\in\mathcal{S}'\left(\mathbb{R}^2\right)$ is called the Fourier image of u, if*

$$\langle\mathcal{F}[u],\varphi\rangle=\langle u,\mathcal{F}[\varphi]\rangle,\forall\varphi\in\mathcal{S}\left(\mathbb{R}^2\right)$$

Definition 2.8. *Let $u \in \mathcal{S}'\left(\mathbb{R}^2\right)$. Then the Fourier transform $\mathcal{F}[u]$ and the inverse Fourier transform $\mathcal{F}^{-1}[u]$ are given by*

$$\langle \mathcal{F}[u], \varphi \rangle = \langle u, \mathcal{F}[\varphi] \rangle \quad \text{and} \quad \left\langle \mathcal{F}^{-1}[u], \varphi \right\rangle = \left\langle u, \mathcal{F}^{-1}[\varphi] \right\rangle \quad \forall \varphi \in \mathcal{S}\left(\mathbb{R}^2\right)$$

Example 2.5. *Let $\varphi \in \mathcal{S}\left(\mathbb{R}^2\right)$. Then*

$$\langle \mathcal{F}[\delta], \varphi \rangle = \langle \delta, \mathcal{F}[\varphi] \rangle = \mathcal{F}[\varphi](0)$$

But by the definition of the Fourier transform

$$\mathcal{F}[\varphi](0) = \int_{\mathbb{R}^2} \varphi(x) dx = \langle 1, \varphi \rangle$$

which implies $\mathcal{F}\delta = 1$

Note That 2.3. *if $f \in \mathcal{S}'\left(\mathbb{R}^2\right)$, then $\mathcal{F}f \in \mathcal{S}'\left(\mathbb{R}^2\right), \mathcal{F}^{-1}\mathcal{F}f = f$ and $\mathcal{F}\mathcal{F}^{-1}f = f$.*

Theorem 2.1. *$\mathcal{F}$ and $\mathcal{F}^{-1}$ are continuous on $\mathcal{S}'\left(\mathbb{R}^2\right)$.*

Proof. Suppose that $f_n \to f$ in $\mathcal{S}'\left(\mathbb{R}^2\right)$. Then $\langle f_n, \varphi \rangle \to \langle f, \varphi \rangle$ for every $\varphi \in \mathcal{S}\left(\mathbb{R}^2\right)$. Hence

$$\left\langle \hat{f}_n, \varphi \right\rangle = \langle f_n, \hat{\varphi} \rangle \to \langle f, \hat{\varphi} \rangle = \langle \hat{f}, \varphi \rangle$$

so, $\hat{f}_n \to \hat{f}$ in $\mathcal{S}'\left(\mathbb{R}^2\right)$.

A similar argument holds for the inverse Fourier transform. □

Definition 2.9. *Let φ and ψ be functions in $\mathcal{S}\left(\mathbb{R}^2\right)$, the convolution $\varphi * \psi$ is defined by*

$$(\varphi * \psi)(x) = \int_{\mathbb{R}^2} \varphi(x-y)\psi(y) dy$$

Remark 2.3. *For φ and ψ be functions in $\mathcal{S}\left(\mathbb{R}^2\right)$, then $\mathcal{F}(\varphi * \psi) = \mathcal{F}\varphi\mathcal{F}\psi$.*

Proof. Now from definition of Fourier transform,we have:

$$\begin{aligned}
\mathcal{F}(\varphi * \psi) &= \int_{\mathbb{R}^2} (\varphi * \psi) e^{-i2\pi x\cdot\xi} d\xi \\
&= \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} \varphi(x-y)\psi(y)dy \right) e^{-i2\pi x\cdot\xi} d\xi \\
&= \int_{\mathbb{R}^2} \psi(y) \left(\int_{\mathbb{R}^2} \varphi(x-y) e^{-i2\pi x\cdot\xi} d\xi \right) dy \qquad \textit{(by interchanging the order)} \\
&= \int_{\mathbb{R}^2} \psi(y) \left(\int_{\mathbb{R}^2} \varphi(u) e^{-i2\pi (u+y)\cdot\xi} du \right) dy \qquad \textit{(by replacing } u = \xi - y \textit{ and } d\xi = du\textit{)} \\
&= \int_{\mathbb{R}^2} \psi(y) \left(\int_{\mathbb{R}^2} \varphi(u) e^{-i2\pi u\cdot\xi} du \right) e^{-i2\pi y\cdot\xi} dy \\
&= \left(\int_{\mathbb{R}^2} \varphi(u) e^{-i2\pi u\cdot\xi} du \right) . \left(\int_{\mathbb{R}^2} \psi(y) e^{-i2\pi y\cdot\xi} dy \right) \\
&= \mathcal{F}(\varphi * \psi) = \mathcal{F}\varphi\mathcal{F}\psi
\end{aligned}$$

□

As the product of two distributions are not generally defined, the convolution of tempered distributions is also not generally defined. However if one factor is in $\mathcal{S}\left(\mathbb{R}^2\right)$*, these problem will be solved. that is if* $f \in \mathcal{S}'\left(\mathbb{R}^2\right)$ *and* $\psi \in \mathcal{S}\left(\mathbb{R}^2\right)$*, then* $f * \psi$ *is defined by*

$$\langle f * \psi, \varphi \rangle = \langle f, \hat{\psi} * \varphi \rangle$$

Definition 2.10. *Let* $f, g \in \mathcal{D}'\left(\mathbb{R}^2\right)$*. Then the convolution of* $f(x)$ *and* $g(x)$ *is defined by:*

$$\langle (f*g), \varphi \rangle = \langle f(x)g(y), \varphi(x+y) \rangle = \langle f(x), \langle g(y), \varphi(x+y) \rangle \rangle = \langle g(y), \langle f(x), \varphi(x+y) \rangle \rangle$$

Note That 2.4. *This definition is meaningful under the conditions: either* f *or* g *has compact support or in one dimension, the support of* f *and* g *are bounded from the same side.*

Remark 2.4. *The operator convolution is commutative i.e* $f * g = g * f$*. The convolution of any distribution* f *with the* δ *function is always equal to* f

$$f * \delta = \delta * f = f$$

Indeed, for all $\varphi \in \mathcal{D}\left(\mathbb{R}^2\right)$*,*

$$\begin{aligned}
\langle \boldsymbol{\delta} * f, \boldsymbol{\varphi} \rangle &= \langle \boldsymbol{\delta}(x) f(y), \boldsymbol{\varphi}(x+y) \rangle = \langle f(y), \langle \boldsymbol{\delta}(x), \varphi(x+y) \rangle \rangle \\
&= \langle f(y), \varphi(y) \rangle = \langle f, \boldsymbol{\varphi} \rangle
\end{aligned}$$

Hence, $\boldsymbol{\delta} * f = f$*.*

If the convolution $f * g$ exist, then the convolution $D^\alpha f * g$ and $f * D^\alpha g$ exists and

$$D^\alpha f * g = D^\alpha (f * g) = f * D^\alpha g.$$

By the definition of distributional derivative

$$\begin{aligned}\langle D^\alpha (f * g), \varphi \rangle &= (-1)^{|\alpha|} \langle f * g, D^\alpha \varphi \rangle \\ &= (-1)^{|\alpha|} \langle g(y) f(x), D^\alpha \varphi(x+y) \rangle \\ &= (-1)^{|\alpha|} \langle g(y), \langle f(x), D^\alpha \varphi(x+y) \rangle \rangle \\ &= \langle g(y), \langle D^\alpha f(x), \varphi(x+y) \rangle \rangle = \langle D^\alpha f * g, \varphi \rangle\end{aligned}$$

Thus, $D^\alpha f * g = D^\alpha (f * g)$ and using commutativity, we also find $f * D^\alpha g = D^\alpha (f * g)$.
For more detailed information on this topic, readers are encouraged to refer to [28].

Chapter 3

Sobolev space

We will use distributions to define an important class of function spaces for PDEs, the Sobolev space. As we have seen, every function in $L^p(\Omega)$ is actually a distribution, therefore it has a distributional derivative. Sobolev spaces are useful tools in the analysis of boundary value problem.

3.1 Integer and Fractional order Sobolev spaces

3.1.1 Integer order Sobolev spaces

We always interprete $f \in L^p$ with $1 \leq p \leq \infty$ as tempered distribution. In particular, $D^\alpha f \in \mathcal{S}'$ make sense for all $\alpha \in \mathbb{N}_0^2$.

Definition 3.1. *Let k be a non negative integer and $1 \leq p < \infty$ and let Ω be nonempty open subset of $\mathbb{R}^2$. Then we define Sobolev space $W_p^k(\Omega)$ of order k to be the set of all distribution $u \in L^p(\Omega)$ such that $D^\alpha u \in L^p(\Omega)$ for $|\alpha| \leq k$. That is*

$$W_p^k(\Omega) = \left\{ u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega), \forall \alpha \in \mathbb{N}_0^2, |\alpha| \leq k \right\}$$

Remark 3.1. *Here, of course, $D^\alpha u$ is viewed as a distributional derivative of u on Ω.*

In $W_p^k(\Omega)$, we define a norm by

$$\|u\|_{W_p^k(\Omega)} = \begin{cases} \left(\int_\Omega \sum_{|\alpha| \leq k} |D^\alpha u|^p \, dx \right)^{\frac{1}{p}} & 1 \leq p < \infty \\ \sum_{|\alpha| \leq k} \operatorname{esssup}_\Omega |D^\alpha u| & p = \infty \end{cases}$$

Example 3.1. *Let* $1 \leq p < \infty$*, we have*

$$\|u\|^p_{W^1_p(\Omega)} = \int_\Omega \left(|u|^p + \sum_{i=1}^{2} \left| \frac{\partial u}{\partial x_i} \right|^p \right) dx$$

$$\|u\|^p_{W^2_p(\Omega)} = \int_\Omega \left(|u|^p + \sum_{i=1}^{2} \left| \frac{\partial u}{\partial x_i} \right|^p + \sum_{i,j=1}^{2} \left| \frac{\partial^2 u}{\partial x_i \partial x_j} \right|^p \right) dx$$

$$\|u\|^p_{W^3_p(\Omega)} = \int_\Omega \left(|u|^p + \sum_{i=1}^{2} \left| \frac{\partial u}{\partial x_i} \right|^p + \sum_{i,j=1}^{2} \left| \frac{\partial^2 u}{\partial x_i \partial x_j} \right|^p + \sum_{i,j=1}^{2} \left| \frac{\partial^3 u}{\partial x_i^2 \partial x_j} \right|^p \right) dx$$

3.1.2 Fractional order Sobolev spaces

To define the Sobolev space of fractional order, we denote the Slobodeckii seminorm by

$$|u|^p_{\lambda,p,\Omega} = \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{2+p\lambda}} dxdy \text{ for } 0 < \lambda < 1$$

Definition 3.2. *For* $s = \lambda + r$ *with a real number* $\lambda \in (0,1)$ *and an integer* $r \geq 0$*, we define the Fractional order Sobolev spaces as:*

$$W^s_p(\Omega) = \left\{ u \in W^r_p(\Omega) : |\partial^\alpha u|_{\lambda,p,\Omega} < \infty \textit{ for } |\alpha| = r \right\}$$

In $W^s_p(\Omega)$ *we define the norm*

$$\|u\|_{W^s_p(\Omega)} = \left(\|u\|_{W^r_p(\Omega)} + \sum_{|\alpha|=r} |\partial^\alpha u|^p_{\lambda,p,\Omega} \right)^{\frac{1}{p}}$$

Sobolev space-second definition

Definition 3.3. *For* $s \in \mathbb{R}$*, we define a continuous linear operator* $\mathcal{J}^s : \mathcal{S}\left(\mathbb{R}^2\right) \to \mathcal{S}\left(\mathbb{R}^2\right)$ *called the Bessel potential of order s, by*

$$\mathcal{J}^s u(x) = \int_{\mathbb{R}^2} \left(1 + |\xi|^2\right)^{\frac{s}{2}} \hat{u}(\xi) e^{i2\pi\xi\cdot x} d\xi \quad x \in \mathbb{R}^2 \tag{3.1}$$

In this way, we have

$$\mathcal{F}_{x\to\xi}\left\{\mathcal{J}^s u(x)\right\} = \left(1 + |\xi|^2\right)^{\frac{s}{2}} \hat{u}(\xi)$$

Indeed

$$\begin{aligned}\langle \mathcal{F}_{x\to\xi}\left(\mathcal{J}^s u(x)\right), \varphi\rangle = \langle \mathcal{J}^s u(x), \mathcal{F}_{\xi\to x}(\varphi(\xi))\rangle &= \int_{\mathbb{R}^2} \mathcal{J}^s u(x)\mathcal{F}_{\xi\to x}[\varphi(\xi)]dx \\ &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left(1+|\xi|^2\right)^{\frac{s}{2}} \hat{u}(\xi)e^{i2\pi\xi\cdot x}d\xi\hat{\varphi}(x)dx \\ &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left(1+|\xi|^2\right)^{\frac{s}{2}} \hat{u}(\xi)\hat{\varphi}(x)e^{i2\pi\xi\cdot x}dxd\xi \\ &= \int_{\mathbb{R}^2} \left(1+|\xi|^2\right)^{\frac{s}{2}} \hat{u}(\xi)\varphi(\xi)d\xi = \left\langle \left(1+|\xi|^2\right)^{\frac{s}{2}} \hat{u}(\xi), \varphi(\xi)\right\rangle\end{aligned}$$

Therefore,

$$\mathcal{F}_{x\to\xi}\left\{\mathcal{J}^s u(x)\right\} = \left(1+|\xi|^2\right)^{\frac{s}{2}} \hat{u}(\xi)$$

It follow from (3.1) that $\langle \mathcal{J}^s u, v\rangle = \langle u, \mathcal{J}^s v\rangle$ for all $u, v \in \mathcal{S}\left(\mathbb{R}^2\right)$. Since,

$$\begin{aligned}\langle \mathcal{J}^s u, v\rangle &= \int_{\mathbb{R}^2} \mathcal{J}^s u(x)v(x)dx = \int_{\mathbb{R}^2}\left[\int_{\mathbb{R}^2}\left(1+|\xi|^2\right)^{\frac{s}{2}} \hat{u}(\xi)e^{i2\pi\xi\cdot x}d\xi\right]v(x)dx \\ &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left(1+|\xi|^2\right)^{\frac{s}{2}} \hat{u}(\xi)e^{i2\pi\xi\cdot x}v(x)dxd\xi \\ &= \int_{\mathbb{R}^2}\left(1+|\xi|^2\right)^{\frac{s}{2}} \hat{u}(\xi)e^{i4\pi\xi\cdot x}\hat{v}(\xi)d\xi \\ &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left(1+|\xi|^2\right)^{\frac{s}{2}} u(x)e^{i2\pi\xi\cdot x}\hat{v}(\xi)d\xi dx \\ &= \int_{\mathbb{R}^2} u(x)\mathcal{J}^s v(x)dx = \langle u, \mathcal{J}^s v\rangle\end{aligned}$$

Therefore, $\langle \mathcal{J}^s u, v\rangle = \langle u, \mathcal{J}^s v\rangle$.

Note That 3.1. *for all $s, t \in \mathbb{R}$, we have the following:*

$$\mathcal{J}^{s+t} = \mathcal{J}^s\mathcal{J}^t, \quad \left(\mathcal{J}^s\right)^{-1} = \mathcal{J}^{-s}, \quad \mathcal{J}^0 = \textit{ identity operator}$$

Using the definition of Bessel potential we have

$$\begin{aligned}\left\langle \mathcal{J}^s\mathcal{J}^t u, v\right\rangle &= \left\langle \mathcal{J}^t u, \mathcal{J}^s v\right\rangle \\ &= \int_{\mathbb{R}^2} \mathcal{J}^t u(x)\mathcal{J}^s v(x)dx \\ &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left(1+|\xi|^2\right)^{\frac{s}{2}} \hat{u}(\xi)e^{i2\pi\xi\cdot x}d\xi\int_{\mathbb{R}^2}\left(1+|\xi|^2\right)^{\frac{t}{2}} \hat{v}(\xi)e^{i2\pi\xi\cdot x}d\xi dx \\ &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left(1+|\xi|^2\right)^{\frac{s}{2}} \hat{u}(\xi)e^{i2\pi\xi\cdot x}d\xi\left(1+|\xi|^2\right)^{\frac{t}{2}}\left(\int_{\mathbb{R}^2}\hat{v}(\xi)e^{i2\pi\xi\cdot x}d\xi\right)dx \\ &= \int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\left(1+|\xi|^2\right)^{\frac{s+t}{2}} \hat{u}(\xi)e^{i2\pi\xi\cdot x}d\xi\right)v(x)dx = \int_{\mathbb{R}^2}\mathcal{J}^{s+t}u(x)v(x)dx \\ &= \left\langle \mathcal{J}^{s+t}u, v\right\rangle\end{aligned}$$

which implies that $\mathcal{J}^{s+t} = \mathcal{J}^s \mathcal{J}^t$.
And also,

$$\mathcal{J}^0 u(x) = \int_{\mathbb{R}^2} \left(1+|\xi|^2\right)^{\frac{0}{2}} \hat{u}(\xi) e^{i2\pi\xi\cdot x} d\xi = \int_{\mathbb{R}^2} \hat{u}(\xi) e^{i2\pi\xi\cdot x} d\xi = u(x)$$

which follows $\mathcal{J}^0 =$ identity operator.
Finally, let $(\mathcal{J}^s)^{-1}$ be the inverse of operator $\mathcal{J}^s$, then

$$\langle u, v\rangle = \left\langle \mathcal{J}^s \left(\mathcal{J}^s\right)^{-1} u, v \right\rangle = \left\langle \left(\mathcal{J}^s\right)^{-1} u, \mathcal{J}^s v \right\rangle \tag{3.2}$$

Using the identity operator $\mathcal{J}^0$

$$\langle u, v\rangle = \left\langle \mathcal{J}^0 u, v\right\rangle = \left\langle \mathcal{J}^{s+(-s)} u, v \right\rangle = \left\langle \mathcal{J}^s \mathcal{J}^{-s} u, v\right\rangle = \left\langle \mathcal{J}^{-s} u, \mathcal{J}^s v\right\rangle \tag{3.3}$$

From (3.2) and (3.3) follows that

$$\left\langle \left(\mathcal{J}^s\right)^{-1} u, \mathcal{J}^s v \right\rangle = \left\langle \mathcal{J}^{-s} u, \mathcal{J}^s v\right\rangle$$

which implies $(\mathcal{J}^s)^{-1} = \mathcal{J}^{-s}$.

The Space $H^s\left(\mathbb{R}^2\right)$

For any $s \in \mathbb{R}$, we define the Sobolev space $W_2^s(\mathbb{R}^2) = H^s\left(\mathbb{R}^2\right)$ of order s as follows:

Definition 3.4. *For $s \in \mathbb{R}$, we denote by $H^s\left(\mathbb{R}^2\right)$ the space of distributions $u \in \mathcal{S}'\left(\mathbb{R}^2\right)$ such that $\left(1+|\xi|^2\right)^{\frac{s}{2}} \hat{u}(\xi) \in L_2\left(\mathbb{R}^2\right)$.*

$$\begin{aligned} H^s\left(\mathbb{R}^2\right) &= \left\{u \in \mathcal{S}'\left(\mathbb{R}^2\right) : \mathcal{F}_{x\to\xi}\left\{\mathcal{J}^s u(x)\right\} \in L_2\left(\mathbb{R}^2\right)\right\} \\ &= \left\{u \in \mathcal{S}'\left(\mathbb{R}^2\right) : \left(1+|\xi|^2\right)^{\frac{s}{2}} \hat{u}(\xi) \in L_2\left(\mathbb{R}^2\right)\right\} \end{aligned}$$

We equip this space with the inner product

$$\langle u, v\rangle_{H^s(\mathbb{R}^2)} = \langle \mathcal{J}^s u, \mathcal{J}^s v\rangle = \int_{\mathbb{R}^2} \left(1+|\xi|^2\right)^s \hat{u}(\xi)\overline{\hat{v}}(\xi) d\xi$$

and with the associated norm $\|\cdot\|_{H^s(\mathbb{R}^2)}$ defined by

$$\|u\|_{H^s(\mathbb{R}^2)} = \sqrt{\langle u, u\rangle_{H^s(\mathbb{R}^2)}} = \|\mathcal{J}^s u\|_{L_2(\mathbb{R}^2)}$$

Note That 3.2. *The Bessel potential $\mathcal{J}^s : H^s\left(\mathbb{R}^2\right) \to L_2\left(\mathbb{R}^2\right)$ is a unitary isomorphic. In particular, since $\mathcal{J}^0 =$ identity operator so that $\mathcal{J}^0 u = u$, and*

$$H^0\left(\mathbb{R}^2\right) = L_2\left(\mathbb{R}^2\right)$$

Remark 3.2. *1.* $W_p^0(\Omega) = L^p(\Omega)$.
2. In particular p=2, we have $W_2^k(\Omega) = W^k(\Omega)$.

The case $p = 2$, which is denoted by $W_2^k(\Omega) = W^k(\Omega) = H^k(\Omega)$ is of special importance because these are even Hilbert spaces with the inner products given by

$$\langle u, v\rangle_{H^k(\Omega)} = \sum_{|\alpha|\leq k} \int_\Omega D^\alpha u(x)\overline{D^\alpha v(x)}dx$$

In particular, for $k = 1$, the space $H^1(\Omega)$, expressed as follows,

$$H^1(\Omega) = \left\{u : u \in L^2(\Omega) \quad \text{and} \quad \frac{\partial u_i}{\partial x_i} \in L^2(\Omega), i = 1, 2\right\}$$

And the inner product in $H^1(\Omega)$ is given by

$$\langle u, v\rangle_{H^k(\Omega)} = \sum_{|\alpha|\leq 1} \int_\Omega \overline{D^\alpha u(x)} D^\alpha v(x)dx = \int_\Omega \overline{u(x)}v(x)dx + \sum_{i=1}^{2} \int_\Omega \overline{\frac{\partial u}{\partial x_i}} \frac{\partial v}{\partial x_i}dx$$
$$= \int_\Omega \overline{u(x)}v(x)dx + \int_\Omega \overline{\nabla u} \cdot \nabla v dx$$

while the norm is given by

$$\|u\|^2_{H^1(\Omega)} = \|u\|^2_{L_2(\Omega)} + \|\nabla u\|^2_{L_2(\Omega)}$$

If $t \geq s > 0$, then we can see that, $u \in H^t(\Omega)$ implies that $u \in H^s(\Omega)$. We have the inclusions $u \in H^t(\Omega) \subset H^s(\Omega) \subset H^0(\Omega) = L_2(\Omega)$.

Theorem 3.1. *[18,page 98]* $H^{-s}\left(\mathbb{R}^2\right)$ *is the dual space of* $H^s\left(\mathbb{R}^2\right)$*, i.e.,* $H^{-s}\left(\mathbb{R}^2\right)$ *is the space of all linear functionals on* $H^s\left(\mathbb{R}^2\right)$*.*

Since $\mathcal{D}\left(\mathbb{R}^2\right)$ is dense subset of $H^s\left(\mathbb{R}^2\right)$, the dual space of $H^s\left(\mathbb{R}^2\right)$ is space of distributions, i.e, $H^{-s}\left(\mathbb{R}^2\right) \subset \mathcal{D}'\left(\mathbb{R}^2\right)$. For any non-empty open set, $\Omega \subset \mathbb{R}^2$, the space $H^s(\Omega)$ consists of restriction on Ω of distributions from $H^s\left(\mathbb{R}^2\right)$, i.e.,

$$H^s(\Omega) = \left\{u : u = v|_\Omega \text{ for some} \quad v \in H^s\left(\mathbb{R}^2\right)\right\}$$

equipped with norm

$$\|u\|_{H^s(\Omega)} := \inf_{v\in H^s(\mathbb{R}^2), u=v|_\Omega} \|u\|_{H^s(\mathbb{R}^2)}$$

We denote by $\tilde{H}^s(\Omega)$ the closure of $\mathcal{D}(\Omega)$ in $H^s\left(\mathbb{R}^2\right)$, which can be characterized as

$$\tilde{H}^s(\Omega) = \left\{g : g \in H^s\left(\mathbb{R}^2\right), \operatorname{supp}(g) \subset \bar{\Omega}\right\}$$

(see e.g., [13,Theorem 3.29]. The space $H^s(\Omega)$ consists of restrictions on Ω of distributions from $H^s\left(\mathbb{R}^2\right)$,

$$H^s(\Omega)=\left\{\left.g\right|_\Omega : g \in H^s\left(\mathbb{R}^2\right)\right\}$$

and $H_0^s(\Omega)$ the closure of $\mathcal{D}(\Omega)$ in $H^s(\Omega)$.

Note That 3.3. *For* $s \geq 0$, *we can identify* $\tilde{H}^s(\Omega)$ *with the subset of functions from* $H^s(\Omega)$, *whose extensions by zero outside* Ω *belong to* $\tilde{H}^s\left(\mathbb{R}^2\right)$, *i.e., identify functions* $u \in \tilde{H}^s(\Omega)$ *with their restrictions,* $\left.u\right|_\Omega \in H^s(\Omega)$ *(see, [8], [23], [17]).*

3.2 Weighted Sobolev Spaces

Definition 3.5. *Let* Ω *be an open set in* $\mathbb{R}^2$. *We denote by* $\mathcal{W}(\Omega)$ *the set of all measurable function a.e. in* Ω *positive, finite and locally integrable functions.*

Elements of $\mathcal{W}(\Omega)$ will be called weight functions. Every weight gives rise to a measure on the measurable subsets E of Ω through integration. This measure will also be denoted by ω. Thus, $\omega(E)=\int_E \omega dx$ for measurable set $E \subset \Omega$.

Definition 3.6. *Let* $\Omega \subset \mathbb{R}^2$ *an open set and* $\omega \in \mathcal{W}(\Omega)$. *For* $1 \leq p<\infty$, *we define* $L^p(\Omega, \omega)$ *as the set of measurable functions* f *on* Ω *such that*

$$\|f\|_{L^p(\Omega, \omega)}=\left(\int_{\Omega}|f|^p \omega dx\right)^{1 / p}<\infty$$

For $\omega(x) \equiv 1$ *we obtain the usual Lebesgue space* $L^p(\Omega)$.

Definition 3.7. *Let* $1 \leq p<\infty$ *and let* ω *be a weight.We say that* $\omega \in \mathcal{B}_p(\Omega)$*if*
(a) $\omega^{-1/(p-1)}$ *is locally integrable, when* $p>1$;
(b) ess $\sup_{x \in B} \frac{1}{\omega(x)}<\infty$ *for all ball* B, *when* $p=1$.

Definition 3.8. *Let* $\Omega \subset \mathbb{R}^2$ *be open set,* $1 \leq p<\infty$ *and* k *a nonnegative integer. Suppose that the weight* $\omega \in \mathcal{B}_p(\Omega)$. *We define the weighted Sobolev space* $W^{k,p}(\Omega, \omega)$ *as the set of functions* $u \in L^p(\Omega, \omega)$ *with weak derivatives* $D^\alpha u \in L^p(\Omega, \omega)$ *for* $|\alpha| \leq k$. *The norm of* $u \in W^{k,p}(\Omega, \omega)$ *is given by*

$$\|u\|_{W^{k,p}(\Omega, \omega)}=\left(\sum_{|\alpha| \leq k} \int_{\Omega}\left|D^\alpha u\right| \omega dx\right)^{1 / p}$$

Chapter 4

Laplace Equation And The Fundamental Solution

4.1 Laplaces Equation

Definition 4.1. *A second order linear PDE that is written in the following form is called the Laplace's equation:*

$$\triangle u = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) u(x) = 0 \tag{4.1}$$

A function $u(x)$ which has continuous second partial derivatives and solves Laplace's equation (4.1) is called a **harmonic function**.

Note That 4.1. *The inhomogeneous version of Laplaces Equation is the Poisson's equation, and it can be written as:*

$$-\triangle u = f(x)$$

The Fundamental Solution

Consider Laplace's equation in $\mathbb{R}^2$,

$$\Delta u = 0 \quad x \in \mathbb{R}^2.$$

It is evident that a large number of functions u satisfy this equation. Any constant function is, in particular, harmonic. Furthermore, a solution can be any function of the form $u(x) = a_1 x_1 + a_2 x_2$ for constants a_1, a_2. But in this case, our goal is to identify a specific Laplace equation solution that will enable us to resolve Poisson's problem. We search for a radial solution because Laplace's equation is symmetrical. That is, we look for a harmonic function u on $\mathbb{R}^2$ such that $u(x) = w(|x|)$. Radial solutions are a logical choice since they convert a

PDE into an ODE, which is typically simpler to solve, in addition to being a logical choice because of the symmetry of Laplace's equation. As a result, we search for a radial solution.
If $u(x) = w(|x|)$, then

$$u_{x_i} = \frac{x_i}{|x|}w'(|x|) \quad |x| \neq 0 \; where \; i = 1, 2$$

which implies

$$u_{x_i x_i} = \frac{1}{|x|}w'(|x|) - \frac{x_i^2}{|x|^3}w'(|x|) + \frac{x_i^2}{|x|^2}w''(|x|) \quad |x| \neq 0$$

By definition we have, $\triangle u = u_{x_1x_1} + u_{x_2x_2}$
Substitute the expressions for $u_{x_1x_1}$ and $u_{x_2x_2}$ into this equation:

$$\triangle u = \left(\frac{1}{|x|}w'(|x|) - \frac{x_1^2}{|x|^3}w'(|x|) + \frac{x_1^2}{|x|^2}w''(|x|)\right) + \left(\frac{1}{|x|}w'(|x|) - \frac{x_2^2}{|x|^3}w'(|x|) + \frac{x_2^2}{|x|^2}w''(|x|)\right)$$

Now,by putting all like terms together we obtain:

$$\Delta u = \frac{1}{|x|}w'(|x|) + w''(|x|)$$

Letting $r = |x|$, we see that $u(x) = w(|x|)$ is a radial solution of Laplace's equation implies w satisfies

$$\frac{1}{r}w'(r) + w''(r) = 0$$

Therefore,

$$\begin{aligned} w'' &= \frac{-1}{r}w' \\ \implies \frac{w''}{w'} &= \frac{-1}{r} \\ \implies \ln w' &= -\ln r + C \\ \implies w'(r) &= \frac{C}{r} \end{aligned}$$

which implies $\quad w(x) = e_1 \ln r + e_2$
From these calculations, we see that for any constants e_1, e_2, the function

$$u(x) \equiv e_1 \ln|x| + e_2 \tag{4.2}$$

for $x \in \mathbb{R}^2, |x| \neq 0$ is a solution of Laplace's equation in $\mathbb{R}^2 - \{0\}$. We notice that the function u defined in (4.2) satisfies $\Delta u(x) = 0$ for $x \neq 0$, but at $x = 0, \Delta u(0)$ is undefined. We claim that we can choose constants e_1 and e_2 appropriately so that

$$-\Delta_x u = \delta$$

in the sense of distributions. Recall that δ is the distribution which is defined as follows. For all $\phi \in \mathcal{D}$,

$$(\delta, \phi) = \phi(0)$$

Below, we will prove this claim. For now, though, assume we can prove this. That is, assume we can find constants e_1, e_2 such that u defined in (4.2) satisfies

$$-\Delta_x u = \delta \tag{4.3}$$

Let Φ denote the solution of (4.3). Then, define

$$w(x) = \int_{\mathbb{R}^2} \Phi(x-y) f(y)\,dy$$

Formally, we compute the Laplacian of w as follows,

$$\begin{aligned} -\Delta_x w &= -\int_{\mathbb{R}^2} \Delta_x \Phi(x-y) f(y)\,dy \\ &= -\int_{\mathbb{R}^2} \Delta_y \Phi(x-y) f(y)\,dy \\ &= \int_{\mathbb{R}^2} \delta_x f(y)\,dy = f(x) \end{aligned}$$

That is, w is a solution of Poisson's equation! Of course, this set of equalities above is entirely formal. We have not proven anything yet. However, we have motivated a solution formula for Poisson's equation from a solution to (4.3). We now return to using the radial solution (4.2) to find a solution of (4.3).
Define the function Φ as follows. For $|x| \neq 0$, let

$$\Phi(x) = \frac{1}{2\pi} \ln |x| \tag{4.4}$$

We see that Φ satisfies Laplace's equation on $\mathbb{R}^2 - \{0\}$. As we will show in the following claim, Φ satisfies $-\Delta_x \Phi = \delta$. For this reason, we call Φ the **fundamental solution** of Laplace's equation.

Theorem 4.1. *For Φ defined in (4.4), Φ satisfies*

$$-\Delta_x \Phi = \delta$$

in the sense of distributions. That is, for all $f \in \mathcal{D}$,

$$\int_{\mathbb{R}^2} \Phi(x) \Delta_x f(x)\,dx = f(0)$$

Proof. Let T_Φ be the distribution associated with the fundamental solution Φ. That is, let $T_\Phi : \mathcal{D} \to \mathbb{R}$ be defined such that

$$(T_\Phi, f) = \int_{\mathbb{R}^2} \Phi(x) f(x)\,dx$$

for all $f \in \mathcal{D}$. Recall that the derivative of a distribution T is defined as

$$(T, f) = -(T, f')$$

for all $f \in \mathcal{D}$. Therefore, the distributional Laplacian of Φ is defined as the distribution $T_{\Delta\Phi}$ such that

$$(T_{\Delta\Phi}, f) = (T_\Phi, \Delta f)$$

for all $f \in \mathcal{D}$. We will show that

$$(T_\Phi, \Delta f) = (\delta, f) = f(0)$$

and, therefore,

$$(T_{\Delta\Phi}, f) = f(0)$$

which means $-\Delta_x \Phi = \delta$ in the sense of distributions.

By definition,

$$(T_\Phi, \Delta f) = \int_{\mathbb{R}^2} \Phi(x)\Delta f(x)dx$$

Now, we would like to apply the divergence theorem that can be found after the end of the proof on (Theorem 4.2), but Φ has a singularity at $x = 0$. We get around this, by breaking up the integral into two pieces: one piece consisting of the ball of radius δ about the origin, $B(0, \delta)$ and the other piece consisting of the complement of this ball in $\mathbb{R}^2$. Therefore, we have

$$\begin{aligned}(T_\Phi, \Delta f) &= \int_{\mathbb{R}^2} \Phi(x)\Delta f(x)dx \\ &= \int_{B(0,\delta)} \Phi(x)\Delta f(x)dx + \int_{\mathbb{R}^2 - B(0,\delta)} \Phi(x)\Delta f(x)dx \\ &= K + L\end{aligned}$$

We look first at term K. the term K is bounded as follows,

$$\begin{aligned}\left|\int_{B(0,\delta)} \frac{1}{2\pi}\ln|x|\Delta f(x)dx\right| &\leq C^{'}|\Delta f|_{L^\infty}\left|\int_{B(0,\delta)} \ln|x|dx\right| \\ &\leq C\left|\int_0^{2\pi}\int_0^\delta \ln|r|rdrd\theta\right| \\ &\leq C\left|\int_0^\delta \ln|r|rdr\right| \\ &\leq C\ln|\delta|\delta^2\end{aligned}$$

Therefore, as $\delta \to 0, |K| \to 0$.

Next, we look at term L. Applying the divergence theorem, we have

$$\begin{aligned}\int_{\mathbb{R}^2-B(0,\delta)} \Phi(x)\Delta_x f(x)dx =& \int_{\mathbb{R}^2-B(0,\delta)} \Delta_x\Phi(x)f(x)dx - \int_{\partial(\mathbb{R}^2-B(0,\delta))} \frac{\partial\Phi}{\partial n}f(x)dS(x)\\ &+ \int_{\partial(\mathbb{R}^2-B(0,\delta))} \Phi(x)\frac{\partial f}{\partial n}dS(x)\\ =& -\int_{\partial(\mathbb{R}^2-B(0,\delta))} \frac{\partial\Phi}{\partial n}f(x)dS(x) + \int_{\partial(\mathbb{R}^2-B(0,\delta))} \Phi(x)\frac{\partial f}{\partial n}dS(x)\\ \equiv& L1 + L2\end{aligned}$$

using the fact that $\Delta_x\Phi(x) = 0$ for $x \in \mathbb{R}^2 - B(0,\delta)$. We first look at term $L1$. Now, by assumption, $f \in \mathcal{D}$, and, therefore, f vanishes at ∞. Consequently, we only need to calculate the integral over $\partial B(0,\epsilon)$ where the normal derivative n is the outer normal to $\mathbb{R}^2 - B(0,\delta)$. With a simple computation, we can observe that

$$\nabla_x\Phi(x) = \frac{x}{2\pi|x|^2}$$

The outer unit normal to $\mathbb{R}^2 - B(0,\delta)$ on $B(0,\delta)$ is given by

$$n = \frac{x}{|x|}$$

Therefore, the normal derivative of Φ on $B(0,\delta)$ is given by

$$\frac{\partial\Phi}{\partial n} = \left(\frac{x}{2\pi|x|^2}\right)\cdot\left(\frac{x}{|x|}\right) = \frac{1}{2\pi|x|}$$

Therefore, $L1$ can be written as

$$\int_{\partial B(0,\delta)} \frac{1}{2\pi|x|}f(x)dS(x) = \frac{1}{2\pi\delta}\int_{\partial B(0,\delta)} f(x)dS(x)$$

Now since f is a continuous function, then

$$\frac{1}{2\pi\delta}\int_{\partial B(0,\delta)} f(x)dS(x) \to f(0) \quad \text{as } \delta \to 0$$

Lastly, we look at term $L2$. Now using the fact that f vanishes as $|x| \to +\infty$, we only need to integrate over $\partial B(0,\delta)$. Using the fact that $f \in \mathcal{D}$, and, therefore, infinitely differentiable, we have

$$\begin{aligned}\left|\int_{\partial B(0,\delta)} \Phi(x)\frac{\partial f}{\partial n}dS(x)\right| &\le \left|\frac{\partial f}{\partial n}\right|_{L^\infty(\partial B(0,\delta))}\int_{\partial B(0,\delta)} |\Phi(x)|dS(x)\\ &\le C\int_{\partial B(0,\delta)} |\Phi(x)|dS(x)\end{aligned}$$

$$
\begin{aligned}
&= \frac{C}{2\pi} \int_{\partial B(0,\delta)} |\ln|x||dS(x) \\
&= \frac{C}{2\pi} |\ln|\delta|| \int_{\partial B(0,\delta)} dS(x) \\
&= \frac{C}{2\pi} |\ln|\delta||(2\pi\delta) \\
&= C\delta|\ln|\delta||
\end{aligned}
$$

Therefore, we conclude that term $L2$ is bounded in absolute value by

$$C\delta|\ln|\delta||$$

Therefore, $|L2| \to 0$ as $\delta \to 0$. Combining these estimates, we see that

$$\int_{\mathbb{R}^2} \Phi(x)\Delta_x f(x)dx = \lim_{\delta\to 0} K + L1 + L2 = f(0)$$

Therefore, our theorem is proved. □

Remark 4.1. *From our discussion before the above theorem, we expect the function*

$$w(x) = \int_{\mathbb{R}^2} \Phi(x-y)f(y)dy = \Phi * f$$

to give us a solution of Poisson's equation.

4.2 Greens Identities

Let us first state the Divergence theorem as the following:

Theorem 4.2 (Divergence Theorem)**.** *Let Ω be a bounded solid region with a C^1 boundary curve $\partial\Omega$. Let n be the unit outward normal vector on $\partial\Omega$. Let F be any C^1 vector field on $\bar{\Omega} = \partial\Omega \bigcup \Omega$ Then*

$$\int_\Omega \nabla.F dV = \int_{\partial\Omega} F.n dS$$

where dV is the volume element in Ω and dS is the surface element on $\partial\Omega$.

Greens Identities

Green's Identities form an important tool in the analysis of Laplace equation. Let $u, w \in C^2(\bar{\Omega})$. Then we have

Theorem 4.3 (First Green's Identity)**.**

$$\int_\Omega u\Delta w dx = -\int_\Omega \nabla u \cdot \nabla w dx + \int_{\partial\Omega} u\frac{\partial w}{\partial n} dS$$

Proof.

$$\begin{aligned}\nabla(u\nabla w) &= \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(u\frac{\partial w}{\partial x_i}\right)\\ &= \sum_{i=1}^{2} \frac{\partial u}{\partial x_i}\cdot\frac{\partial w}{\partial x_i} + \sum_{i=1}^{2} u\frac{\partial^2 w}{\partial x_i^2}\\ &= \nabla u\nabla w + u\Delta w\end{aligned}$$

Integrating with respect to dx on Ω,

$$\begin{aligned}\int_\Omega \nabla u\cdot\nabla w dx + \int_\Omega u\Delta w dx &= \int_\Omega \nabla(u\nabla w)dx\\ &= \int_{\partial\Omega} u\nabla w\cdot n dS\\ &= \int_{\partial\Omega} u\frac{\partial w}{\partial n}dS\end{aligned}$$

Therefore,

$$\int_\Omega u\Delta w dx = -\int_\Omega \nabla u\cdot\nabla w dx + \int_{\partial\Omega} u\frac{\partial w}{\partial n}dS \tag{4.5}$$

□

Theorem 4.4 (Second Green's Identity:)**.**

$$\int_\Omega (u\Delta w - w\Delta u)dx = \int_{\partial\Omega}\left(u\frac{\partial w}{\partial n} - w\frac{\partial u}{\partial n}\right)dS$$

Proof. First consider the following equation,

$$\int_\Omega w\Delta u dx = -\int_\Omega \nabla u\cdot\nabla w dx + \int_{\partial\Omega} w\frac{\partial u}{\partial n}dS \tag{4.6}$$

Subtract equation (4.5) from equation (4.6), we get second Green's identity

$$\int_\Omega (u\Delta w - w\Delta u)dx = \int_{\partial\Omega}\left(u\frac{\partial w}{\partial n} - w\frac{\partial u}{\partial n}\right)dS$$

□

Theorem 4.5 (Third Green's Identity:)**.**

$$u - \int_{\partial\Omega} u\frac{\partial\Phi}{\partial n}dS + \int_{\partial\Omega}\Phi\frac{\partial u}{\partial n}dS = \int_\Omega \Phi f dx$$

Proof. Assume that u is the solution of a PDE $\Delta u = f$ and using $\Phi(x,y)\ as\ w(x)$ in the second Greens Identity we obtain the following third Greens Identity as:

$$\begin{aligned}
\int_{\Omega}(u\Delta\Phi-\Phi\Delta u)dx &= \int_{\partial\Omega}\left(u\frac{\partial\Phi}{\partial n}-\Phi\frac{\partial u}{\partial n}\right)dS\\
\Leftrightarrow \int_{\Omega}u\Delta\Phi dx-\int_{\Omega}\Phi\Delta u dx &= \int_{\partial\Omega}u\frac{\partial\Phi}{\partial n}dS-\int_{\partial\Omega}\Phi\frac{\partial u}{\partial n}dS\\
\Leftrightarrow \int_{\Omega}u(x)\delta(x-y)dx-\int_{\Omega}\Phi\Delta u dx &= \int_{\partial\Omega}u\frac{\partial\Phi}{\partial n}dS-\int_{\partial\Omega}\Phi\frac{\partial u}{\partial n}dS\\
\Leftrightarrow u(y)-\int_{\partial\Omega}u\frac{\partial\Phi}{\partial n}dS+\int_{\partial\Omega}\Phi\frac{\partial u}{\partial n}dS &= \int_{\Omega}\Phi f dx
\end{aligned}$$

□

4.3 Potential Theory

Double and Single Layer Potentials

The Dirichlet, Neumann, and Mixed BVPs are solved in potential theory using the single and double layer potentials. Thus, in a moment, we shall define the potentials for both the single and double layers. It is known that

$$\Phi(x-y)=\begin{cases}\frac{1}{2\pi}\ln|x-y| & n=2\\ \frac{-1}{n(n-2)\alpha(n)}\cdot\frac{1}{|x-y|^{n-2}} & n\geq 3\end{cases}$$

is the fundamental solution of Laplace equation $\Delta u=0$ in $\mathbb{R}^n$, where $|x-y|$ is the distance between two points $x=(x_1,x_2,\dots,x_n)$ and $y=(y_1,y_2,\dots,y_n)$ in $\mathbb{R}^n, \alpha(n)$ is the volume of unit sphere in $\mathbb{R}^n$; which can be obtained similarly as (4.2) obtained. under this particular section we consider the general $\mathbb{R}^n$ case, which ofcourse works for $\mathbb{R}^2$.
Let h be a continuous function on $\partial\Omega$. The single layer potential with h is defined as

$$V_{\Delta}(x)=-\int_{\partial\Omega}h(y)\Phi(x-y)dS(y) \qquad x\in\mathbb{R}^2\backslash\partial\Omega$$

The double layer potential with h is defined as

$$W_{\Delta}(x)=-\int_{\partial\Omega}h(y)\frac{\partial\Phi}{\partial n_y}(x-y)dS(y) \qquad x\in\mathbb{R}^2\backslash\partial\Omega. \tag{4.7}$$

Theorem 4.6. *For a continuous function h on $\partial\Omega$, $\Delta V_{\Delta}(x)=\Delta W_{\Delta}(x)=0$ for all $x\notin\partial\Omega$.*

Proof. We will prove $W_{\Delta}(x)$ is harmonic for all $x\notin\partial\Omega$. Similar proof work for $V_{\Delta}(x)$. Fix $x\in\mathbb{R}^2\backslash\partial\Omega$, $\frac{\partial\Phi}{\partial n_y}(x-y)$ is smooth function for all $y\in\partial\Omega$ and $\Phi(x-y)$ is harmonic for all $x\neq y$ implies that $\Delta_x\frac{\partial\Phi}{\partial n_y}(x-y)=0$ for all $y\in\partial\Omega$.

Therefore, using the fact that our integral is finite and $\frac{\partial \Phi}{\partial n_y}(x-y)$ is smooth, we conclude that

$$\begin{aligned}\Delta_x W_\Delta(x) &= -\Delta_x \int_{\partial\Omega} h(y)\frac{\partial \Phi}{\partial n_y}(x-y)dS(y)\\ &= -\int_{\partial\Omega} h(y)\Delta_x \frac{\partial \Phi}{\partial n_y}(x-y)dS(y)\\ &= 0\end{aligned}$$

□

Lemma 4.1 (Gauss's Lemma). *Consider the double layer potential,*

$$W_\Delta(x) = -\int_{\partial\Omega} \frac{\partial \Phi}{\partial n_y}(x-y)dS(y)$$

Then,

$$W_\Delta(x) = \begin{cases} 0 & x \in \Omega^+ \\ 1 & x \in \Omega \\ 1/2 & x \in \partial\Omega \end{cases}$$

Proof. First, for $x \in \Omega^+$,

$$\begin{aligned}W_\Delta(x) &= -\int_{\partial\Omega} \frac{\partial \Phi}{\partial n_y}(x-y)dS(y)\\ &= -\int_{\Omega} \Delta_y \Phi(x-y)dS(y)\\ &= 0\end{aligned}$$

using the Divergence Theorem and the fact that $\Phi(x-y)$ is smooth for $y \in \Omega, x \in \Omega^+$.

Now, for $x \in \Omega, \Phi(x-y)$ is not smooth for all $y \in \Omega$. In order to overcome this problem, we fix $\epsilon > 0$ sufficiently small such that $B(x,\epsilon)$ is contained within Ω. Then on the region $\Omega - B(x,\epsilon), \Phi(x-y)$ is smooth, and, consequently, we can say

$$\begin{aligned}0 &= \int_{\Omega - B(x,\epsilon)} \Delta_y \Phi(x-y)dS(y)\\ &= \int_{\partial(\Omega - B(x,\epsilon))} \frac{\partial \Phi}{\partial n_y}(x-y)dS(y)\\ &= \int_{\partial\Omega} \frac{\partial \Phi}{\partial n_y}(x-y)dS(y) + \int_{\partial B(x,\epsilon)} \frac{\partial \Phi}{\partial n_y}(x-y)dS(y)\end{aligned}$$

where n is outer unit normal to $\Omega - B(x,\epsilon)$.

As mentioned above,

$$\Phi_{y_i}(x-y) = \frac{y_i - x_i}{n\alpha(n)|y-x|^n}$$

For $y \in \partial B(x,\epsilon)$, the outer unit normal to $\Omega - B(x,\epsilon)$ is given by

$$n(y) = \frac{y-x}{|y-x|}$$

Therefore, for $y \in \partial B(x,\epsilon)$,

$$\begin{aligned}
\frac{\partial \Phi}{\partial n_y}(x-y) &= \nabla_y \Phi(y-x) \cdot n(y) \\
&= \frac{y-x}{n\alpha(n)|y-x|^n} \cdot \frac{y-x}{|y-x|} \\
&= \frac{|y-x|^2}{n\alpha(n)|y-x|^{n+1}} \\
&= \frac{1}{n\alpha(n)|y-x|^{n-1}}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_{\partial B(x,\epsilon)} \frac{\partial \Phi}{\partial n_y}(x-y)dS(y) &= \int_{\partial B(x,\epsilon)} \frac{1}{n\alpha(n)|y-x|^{n-1}} dS(y) \\
&= \frac{1}{n\alpha(n)\epsilon^{n-1}} \int_{\partial B(x,\epsilon)} dS(y) \\
&= 1
\end{aligned}$$

Therefore, we conclude that

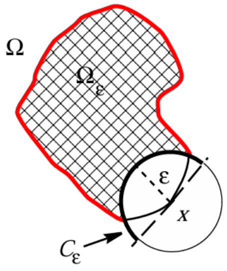
$$\begin{aligned}
0 &= \int_{\partial\Omega} \frac{\partial \Phi}{\partial n_y}(x-y)dS(y) + \int_{\partial B(x,\epsilon)} \frac{\partial \Phi}{\partial n_y}(x-y)dS(y) \\
&= \int_{\partial\Omega} \frac{\partial \Phi}{\partial n_y}(x-y)dS(y) + 1
\end{aligned}$$

which implies

$$-\int_{\partial\Omega} \frac{\partial \Phi}{\partial n_y}(x-y)dS(y) = 1$$

Last, we consider the case $x \in \partial\Omega$. In this case, $\frac{\partial\Phi}{\partial n_y}(x-y)$ is not defined at $y = x$. Fix $x \in \partial\Omega$. Let $B(x,\epsilon)$ be the ball of radius ϵ about x.
Let

$$\Omega_\epsilon \equiv \Omega - \Omega \cap B(x,\epsilon), \quad C_\epsilon \equiv \{y \in \partial B(x,\epsilon) : n(x).y < 0\}, \quad \tilde{C}_\epsilon \equiv \partial\Omega_\epsilon \cap C_\epsilon, \quad \tilde{C}_\Omega = \partial\left(\Omega \cap B(x,\epsilon)\right) - \tilde{C}_\epsilon$$



First, we note that

$$
\begin{aligned}
0 &= \int_{\Omega_\epsilon} \Delta_y \Phi(x-y)\,dy && (4.8)\\
&= \int_{\partial\Omega_\epsilon} \frac{\partial \phi}{\partial n_y}(x-y)\,dS(y)\\
&= \int_{\partial\Omega - \tilde{C}_\Omega} \frac{\partial \Phi}{\partial n_y}(x-y)\,dS(y) + \int_{\tilde{C}_\epsilon} \frac{\partial \Phi}{\partial n_y}(x-y)\,dS(y)
\end{aligned}
$$

where n_y is the outer unit normal to Ω_ϵ.

Now, first we recall that

$$\nabla_y \Phi(x-y) = \frac{x-y}{n\alpha(n)|x-y|^n}$$

For all $y \in \tilde{C}_\epsilon$, the outer unit normal is given by

$$n(y) = \frac{y-x}{|y-x|}$$

Therefore,

$$
\begin{aligned}
\int_{\tilde{C}_\epsilon} \frac{\partial \Phi}{\partial n_y}(x-y)\,dS(y) &= \int_{\tilde{C}_\epsilon} \frac{1}{n\alpha(n)|y-x|^{n-1}}\,dS(y)\\
&= \frac{1}{n\alpha(n)\epsilon^{n-1}} \int_{\tilde{C}_\epsilon} dS(y)
\end{aligned}
$$

Next, we use the fact that

$$\int_{\tilde{C}_\epsilon} dS(y) \approx \int_{C_\epsilon} dS(y)$$

In fact, as we will show below,

$$\int_{\tilde{C}_\epsilon} dS(y) = \int_{C_\epsilon} dS(y) + O\left(\epsilon^n\right) \qquad (4.9)$$

We omit the proof of (4.9) for now and will return to it below. Assuming this fact for now, we have

$$\int_{\tilde{C}_\epsilon} dS(y) = \frac{1}{2}n\alpha(n)\epsilon^{n-1} + O\left(\epsilon^n\right)$$

which implies

$$\begin{aligned}\int_{\tilde{C}_\epsilon} \frac{\partial \Phi}{\partial n_y}(x-y)dS(y) &= \frac{1}{n\alpha(n)\epsilon^{n-1}}\left[\frac{1}{2}n\alpha(n)\epsilon^{n-1} + O\left(\epsilon^n\right)\right] \qquad (4.10)\\ &= \frac{1}{2} + \frac{1}{n\alpha(n)}O(\epsilon^n)\end{aligned}$$

Combing (4.8) and (4.10), we have

$$0 = \int_{\partial\Omega - \tilde{C}_\Omega} \frac{\partial \Phi}{\partial n_y}(x-y)dS(y) + \frac{1}{2} + \frac{1}{n\alpha(n)}O(\epsilon^n)$$

which implies

$$\int_{\partial\Omega - \tilde{C}_\Omega} \frac{\partial \Phi}{\partial n_y}(x-y)dS(y) = -\frac{1}{2} - \frac{1}{n\alpha(n)}O(\epsilon^n)$$

Taking the limit as $\epsilon \to 0^+$, we have

$$-\int_{\partial\Omega} \frac{\partial \Phi}{\partial n_y}(x-y)dS(y) = \frac{1}{2}$$

From the fact that $\tilde{C}_\Omega \to 0$.
Now we will prove (4.9).

Claim: For $\tilde{C}_\epsilon$ and C_ϵ as defined above, we have

$$\int_{\tilde{C}_\epsilon} dS(y) = \int_{C_\epsilon} dS(y) + O\left(\epsilon^n\right)$$

Proof. We just need to show that the surface area of $C_\epsilon - \tilde{C}_\epsilon$ is $O\left(\epsilon^n\right)$. The surface area is approximately the surfaces area of the base times height. Now the surface area of the base is $O\left(\epsilon^{n-2}\right)$. Therefore, we just need to show that the height is $O\left(\epsilon^2\right)$.

Without loss of generality, we let $x = 0$. Now, by assumption, $\partial\Omega$ is C^2. Therefore, $\partial\Omega$ can be written as the graph of a C^2 function $f : \mathbb{R}^{n-1} \to \mathbb{R}$ such that $f(0) = 0$ and $\nabla f(0) = 0$. Therefore, if $y \in C_\epsilon - \tilde{C}_\epsilon$, then

$$|y_n| \leq |f\left(y_1, y_2, \ldots, y_{n-1}\right) \leq C|\left(y_1, y_2, \ldots, y_{n-1}\right)|^2 \leq C|y|^2 \leq C\epsilon^2$$

using Taylor's theorem. Therefore, the height is $O\left(\epsilon^2\right)$ and the claim follows. □

Theorem 4.7. *Let h be a continuous function on $\partial\Omega$ and $x_0 \in \partial\Omega$. Then*

(*i*) $\lim_{x\in\Omega\to x_0} V_\Delta(x) = V_\Delta(x_0)$

(*ii*) $\lim_{x\in\Omega\to x_0} \frac{\partial V_\Delta(x)}{\partial n_x} = \frac{1}{2}h(x_0) + \frac{\partial V_\Delta(x_0)}{\partial n_x}, \lim_{x\in\Omega^+\to x_0} \frac{\partial V_\Delta(x)}{\partial n_x} = -\frac{1}{2}h(x_0) + \frac{\partial V_\Delta(x_0)}{\partial n_x}$

(*iii*) $\lim_{x\in\Omega^+\to x_0} W_\Delta(x) = -\frac{1}{2}h(x_0) + W_\Delta(x_0), \lim_{x\in\Omega\to x_0} W_\Delta(x) = \frac{1}{2}h(x_0) + W_\Delta(x_0)$

Proof. (i) Let $x \in \Omega, x_0 \in \partial\Omega$. We have

$$\begin{aligned} V_\Delta(x) &= -\int_{\partial\Omega} \Phi(x,y)h(y)dS(y) \text{ and} \\ V_\Delta(x_0) &= -\int_{\partial\Omega} \Phi(x_0,y)h(y)dS(y) \end{aligned}$$

We need to show that

$$\lim_{x\in\Omega\to x_0} V_\Delta(x) = V_\Delta(x_0)$$

That is for all $\epsilon > 0$ there exists a $\delta > 0$ such that $|V_\Delta(x) - V_\Delta(x_0)| < \epsilon$ for $|x - x_0| < \delta$. Now,

$$V_\Delta(x) - V_\Delta(x_0) = -\int_{\partial\Omega} h(y)\left[\Phi(x,y) - \Phi(x_0,y)\right]dS(y)$$

By assumption, h is continuous, and as we know $\Phi(x,y)$ is smooth for $y \neq x$. Therefore, to get a bound on $|V_\Delta(x) - V_\Delta(x_0)|$, we divide $\partial\Omega$ into two pieces:

(1) $B(x_0,\gamma) \cap \partial\Omega$

(2) $\partial\Omega - B(x_0,\gamma) \cap \partial\Omega$. We look at these two pieces below. First for (1),

$$|V_\Delta(x) - V_\Delta(x_0)| \leq |h(y)|_{L^\infty}\left(B(x_0,\gamma)\cap\partial\Omega\right)\left|\int_{B(x_0,\gamma)\cap\partial\Omega} \Phi(x,y) - \Phi(x_0,y)\,dS(y)\right|$$

By assumption, h is continuous. Therefore, for all $\tilde{\epsilon} > 0$ there exists a $\gamma > 0$ such that $|h(y)| < \tilde{\epsilon}$ for $|y| < \gamma$ and $-\gamma < x_0 < \gamma$. In addition that

$$\left|\int_{B(x_0,\gamma)\cap\partial\Omega} \Phi(x,y) - \Phi(x_0,y)\,dS(y)\right| \leq C$$

using the fact that V_Δ is defined for all $x \in \mathfrak{R}$. Therefore, we conclude that for any $\tilde{\epsilon} > 0, |(1)| \leq C_1\tilde{\epsilon}$ for γ chosen appropriately small. Next, for (2), we use the fact that $\Phi(x,y)$ is continuous in x for x away from y. Consequently,by making $(\partial\Omega - B(x_0,\gamma)\cap\partial\Omega) = \partial\Omega^*$ we have

$$\begin{aligned} &|V_\Delta(x) - V_\Delta(x_0)| \leq \\ &|h(y)|_{L^\infty}(\partial\Omega^*)\,|\Phi(x,y) - \Phi(x_0,y)|\,L^\infty(\partial\Omega^*)\,|\int_{\partial\Omega^*} dS(y) \end{aligned}$$

Now, first h is bounded on $\partial\Omega$. Therefore, $|h(y)| \leq C$. Next, $\left|\int dS(y)\right| \leq C'$. Lastly, using the fact that $\Phi(x-y)$ is continuous in x uniformly for y, we conclude that there exists a δ such that

$$|\Phi(x,y) - \Phi(x_0,y)| \, L^{\infty}(\partial\Omega^*) \leq \tilde{\epsilon}$$

for $|x-x_0| < \delta$. Therefore, $|(2)| \leq C_2\tilde{\epsilon}$ if $|x-x_0| < \gamma$ where δ is chosen appropriately small. Consequently, for $\epsilon > 0$ choose $\tilde{\epsilon} > 0$ such that $C_1\tilde{\epsilon}+C_2\tilde{\epsilon} < \epsilon$. Then choosing $\gamma > 0$ sufficiently small such that $|(1)| \leq C_1\tilde{\epsilon}$ and $\delta > 0$ sufficiently small such that $|(2)| \leq C_2\tilde{\epsilon}$ when $|x-x_0| < \delta$, we conclude that

$$|V_\Delta(x) - V_\Delta(x_0)| \leq C_1\tilde{\epsilon} + C_2\tilde{\epsilon} < \epsilon \text{ for } |x-x_0| < \delta$$

Proof of equation (ii) is similar to the following Proof of equation (iii).

We will prove only the first case, when $x \in \Omega^+$. The second case works similarly. Let $x \in \Omega^+, x_0 \in \partial\Omega$. We have

$$\begin{aligned} W_\Delta(x) &= -\int_{\partial\Omega} h(y)\frac{\partial\Phi}{\partial n_y}(x-y)dS(y) \\ &= -\int_{\partial\Omega} h(y)\frac{\partial\Phi}{\partial n_y}(x-y)dS(y) + h(x_0)\int_{\partial\Omega}\frac{\partial\Phi}{\partial n_y}(x-y)dS(y) - h(x_0)\int_{\partial\Omega}\frac{\partial\Phi}{\partial n_y}(x-y)dS(y) \\ &= -\int_{\partial\Omega} [h(y) - h(x_0)]\frac{\partial\Phi}{\partial n_y}(x-y)dS(y) \\ &\equiv I(x) \end{aligned}$$

using the fact that

$$-\int_{\partial\Omega}\frac{\partial\Phi}{\partial n_y}(x-y)dS(y) = 0$$

for $x \in \Omega^+$, proven in Gauss' lemma. Similarly,

$$\begin{aligned} W_\Delta(x_0) &= -\int_{\partial\Omega} h(y)\frac{\partial\Phi}{\partial n_y}(x_0-y)\,dS(y) \\ &= -\int_{\partial\Omega} [h(y) - h(x_0)]\frac{\partial\Phi}{\partial n_y}(x_0-y)\,dS(y) - h(x_0)\int_{\partial\Omega}\frac{\partial\Phi}{\partial n_y}(x_0-y)\,dS(y) \\ &= -\int_{\partial\Omega} [h(y) - h(x_0)]\frac{\partial\Phi}{\partial n_y}(x_0-y)\,dS(y) + \frac{1}{2}h(x_0) \\ &\equiv I(x_0) + \frac{1}{2}h(x_0) \end{aligned}$$

again using Gauss' lemma. Therefore,

$$W_\Delta(x) - W_\Delta(x_0) = I(x) - I(x_0) - \frac{1}{2}h(x_0)$$

which implies

$$W_{\Delta}(x)=I(x)-I\left(x_{0}\right)-\frac{1}{2} h\left(x_{0}\right)+W_{\Delta}\left(x_{0}\right)$$

Therefore, to prove our theorem, we need only show that

$$\lim _{x \in \Omega^{+} \rightarrow x_{0}}\left[I(x)-I\left(x_{0}\right)\right]=0$$

where

$$I(x) \equiv-\int_{\partial \Omega}\left[h(y)-h\left(x_{0}\right)\right] \frac{\partial \Phi}{\partial n_{y}}(x-y) d S(y)$$

Now,

$$I(x)-I\left(x_{0}\right)=-\int_{\partial \Omega}\left[h(y)-h\left(x_{0}\right)\right]\left[\frac{\partial \Phi}{\partial n_{y}}(x-y)-\frac{\partial \Phi}{\partial n_{y}}\left(x_{0}-y\right)\right] d S(y)$$

We need to show that for all $\epsilon>0$ there exists a $\delta>0$ such that $|I(x)-I(x0)|<\epsilon$ for $\left|x-x_{0}\right|<\delta$.

By assumption, h is continuous, and as we know $\Phi(x-y)$ is smooth for $y \neq x$. Therefore, to get a bound on $\left|I(x)-I\left(x_{0}\right)\right|$, we divide $\partial \Omega$ into two pieces:

(1) $B\left(x_{0}, \gamma\right) \cap \partial \Omega$

(2) $\partial \Omega-B\left(x_{0}, \gamma\right) \cap \partial \Omega$. We look at these two pieces below. First for (1),

$$\begin{aligned}
&\left|-\int_{\partial \Omega}\left[h(y)-h\left(x_{0}\right)\right]\left[\frac{\partial \Phi}{\partial n_{y}}(x-y)-\frac{\partial \Phi}{\partial n_{y}}\left(x_{0}-y\right)\right] d S(y)\right| \\
&\leq\left|h(y)-h\left(x_{0}\right)\right|_{L^{\infty}\left(B\left(x_{0}, \gamma\right) \cap \partial \Omega\right)}\left|\int_{B\left(x_{0}, \gamma\right) \cap \partial \Omega} \frac{\partial \Phi}{\partial n_{y}}(x-y)-\frac{\partial \Phi}{\partial n_{y}}\left(x_{0}-y\right) d S(y)\right|
\end{aligned}$$

By assumption, h is continuous. Therefore, for all $\tilde{\epsilon}>0$ there exists a $\gamma>0$ such
that if $\left|h(y)-h\left(x_{0}\right)\right|<\tilde{\epsilon}$ if $\left|y-x_{0}\right|<\gamma$. In addition,

$$\left|\int_{B\left(x_{0}, \gamma\right) \cap \partial \Omega} \frac{\partial \Phi}{\partial n_{y}}(x-y)-\frac{\partial \Phi}{\partial n_{y}}\left(x_{0}-y\right) d S(y)\right| \leq C$$

using the fact that $W_{\Delta}(x)$ is defined for all $x \in \Re$. Therefore, we conclude that for any $\tilde{\epsilon}>0$,

$$|(1)| \leq C 1 \tilde{\epsilon}$$

for γ chosen appropriately small. Next, for (2), we use the fact that $\frac{\partial \Phi}{\partial n_{y}}(x-y)$ is continuous in x for x away from y. Consequently, we have

$$\begin{aligned}&\left|-\int_{\partial\Omega-B(x_0,\gamma)\cap\partial\Omega}[h(y)-h(x_0)]\left[\frac{\partial\Phi}{\partial n_y}(x-y)-\frac{\partial\Phi}{\partial n_y}(x_0-y)\right]dS(y)\right|\\&\leq|h(y)-h(x_0)|_{L^\infty}\left|\frac{\partial\Phi}{\partial n_y}(x-y)-\frac{\partial\Phi}{\partial n_y}(x_0-y)\right|_{L^\infty(\partial\Omega-B(x_0,\gamma)\cap\partial\Omega)}\left|\int dS(y)\right|\end{aligned}$$

Now, first h is bounded on $\partial\Omega$. Therefore, $|h(y)-h(x_0)|\leq C$. Next, $\left|\int dS(y)\right|\leq C'$. Lastly, using the fact that $\frac{\partial\Phi}{\partial n_y}(x-y)$ is continuous in x uniformly for y, we conclude that there exists a $\delta>0$ such that

$$\left|\frac{\partial\Phi}{\partial n_y}(x-y)-\frac{\partial\Phi}{\partial n_y}(x_0-y)\right|_{L^\infty(\partial\Omega-B(x_0,\gamma)\cap\partial\Omega)}\leq\tilde{\epsilon}$$

for $|x-x_0|<\delta$. Therefore,

$$|(2)|\leq C_2\tilde{\epsilon}$$

if $|x-x_0|<\delta$ where δ is chosen appropriately small. Consequently, for $\epsilon>0$ choose $\tilde{\epsilon}>0$ such that

$$C_1\tilde{\epsilon}+C_2\tilde{\epsilon}<\epsilon$$

Then choosing $\gamma>0$ sufficiently small such that

$$|(1)|\leq C_1\tilde{\epsilon}$$

and $\delta>0$ sufficiently small such that

$$|(2)|\leq C_2\tilde{\epsilon}$$

when $|x-x_0|<\delta$, we conclude that

$$|I(x)-I(x_0)|\leq C_1\tilde{\epsilon}+C_2\tilde{\epsilon}\leq\epsilon$$

for $|x-x_0|<\delta$, implies that

$$|I(x)-I(x_0)|\leq\epsilon$$

Therefore, we have shown that

$$\lim_{x\in\Omega^+\to x_0}[I(x)-I(x_0)]=0$$

Consequently,

$$\begin{aligned}&\lim_{x\in\Omega^+\to x_0}W_\Delta(x)\\&=\lim_{x\in\Omega^+\to x_0}\left[I(x)-I(x_0)-\frac{1}{2}h(x_0)+W_\Delta(x_0)\right]\\&=-\frac{1}{2}h(x_0)+W_\Delta(x_0)\end{aligned}$$

□

Therefore, we have the following jump relation. Where the proof can be found on [24, *Theorems* 3.17 *and* 3.18.]

Theorem 4.8. *let* $g_1 \in H^{-\frac{1}{2}}(\Omega),\ g_2 \in H^{\frac{1}{2}}(\Omega)$. *Then*

$$\begin{aligned}
\gamma^{\pm} V_{\Delta} g_1(y) &:= \mathcal{V}_{\Delta} g_1(y) \\
\gamma^{\pm} W_{\Delta} g_2(y) &:= \mp\frac{1}{2} g_2(y) + \mathcal{W}_{\Delta} g_2(y) \\
T^{\pm} V_{\Delta} g_1(y) &:= \pm\frac{1}{2} g_1(y) + \mathcal{W}'_{\Delta} g_1(y)
\end{aligned}$$

where $y \in \partial\Omega$.

Chapter 5

BDIEs for Variable-Coefficient Mixed BVP in 2D Unbounded Domain

5.1 Basic Notations and Function Spaces

Let $\Omega = \Omega^{+}$be an unbounded open domain in $\mathbb{R}^2$ such that the complement $\Omega^{-} := \mathbb{R}^2 \backslash \bar{\Omega}$ is bounded open domain. Let the boundary $\partial\Omega = \partial\Omega^{-}$be closed and infinitely smooth curve. The space of infinitely differentiable functions having compact support in Ω is denotes by $\mathcal{D}(\Omega)$ and its dual space, the space of distributions, by $\mathcal{D}'(\Omega)$, while $\mathcal{D}(\bar{\Omega})$ is the set of restrictions on $\bar{\Omega}$ of functions from $\mathcal{D}\left(\mathbb{R}^2\right)$. The spaces $H^s(\Omega), H^s(\partial\Omega)$ denote the Sobolev (Bessel potential) spaces. We also denote $\widetilde{H}^s(\Gamma_1) = \{g : g \in H^s(\Gamma), supp\ g \subset \overline{\Gamma_1}\}$,$H^s(\Gamma_1) = \{r_{\Gamma_1} g : g \in H^s(\Gamma)\}$,where Γ_1 is a proper submanifold of a closed surface Γ and r_{Γ_1} is the restriction operator on Γ_1.Moreover for $s = -\frac{1}{2}$ we define the subspace $H_{**}^{-\frac{1}{2}}(\Gamma_1)$ of $H^{-\frac{1}{2}}(\Gamma_1)$ as :$H_{**}^{-\frac{1}{2}}(\Gamma_1) := \{r_{\Gamma_1} g : g \in H^{-\frac{1}{2}}(\Gamma) : \langle g, 1\rangle_\Gamma = 0\}$ We shall consider the following second order partial differential equation, with variable coefficient

$$Au(x) := \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(a(x)\frac{\partial u(x)}{\partial x_i}\right) = f(x) \quad x \in \Omega \tag{5.1}$$

where u is unknown function; $f(x)$ and $a(x) > a_0 > 0$ are given functions in Ω.

We will further use the weighted Sobolev spaces. Let

$$\rho(x) := \left(1 + |x|^2\right)^{1/2} \ln\left(2 + |x|^2\right)$$

For any real α, we denote by $L^2\left(\rho^\alpha ; \Omega\right)$ the weighted Lebesgue space (see, e.g., [12]) consisting of all measurable functions $g(x)$ on Ω such that $g \rho^\alpha \in L^2(\Omega)$, i.e.,

$$\|g\|_{L^2\left(\rho^\alpha ; \Omega\right)}=\left[\int_{\Omega}\left|g(x) \rho^\alpha(x)\right|^2 d x\right]^{\frac{1}{2}}<\infty$$

The space $L^2\left(\rho^\alpha ; \Omega\right)$, equipped with the norm $\|\cdot\|_{L^2\left(\rho^\alpha ; \Omega\right)}$ and appropriate inner product, is a Hilbert space. The weighted Sobolev space $\mathcal{H}^1(\Omega)$ is defined by

$$\mathcal{H}^1(\Omega):=\left\{g \in L^2\left(\rho^{-1} ; \Omega\right): \nabla g \in L^2(\Omega)\right\} \tag{5.2}$$

and for its norm we have $\|g\|_{\mathcal{H}^1(\Omega)}^2:=\|g\|_{L^2\left(\rho^{-1} ; \Omega\right)}^2+\|\nabla g\|_{L^2(\Omega)}^2$, while $|g|_{\mathcal{H}^1(\Omega)}^2:=\sum_{i=1}^2 \int_{\Omega}\left|\frac{\partial g}{\partial x_i}\right|^2 d x=\|\nabla g\|_{L^2(\Omega)}^2$ is the square of the semi-norm. The space $\mathcal{D}\left(\mathbb{R}^2\right)$ is dense in $\mathcal{H}^1\left(\mathbb{R}^2\right)$, see e.g., [1, Theorem 7.2]. This implies that the dual space of $\mathcal{H}^1\left(\mathbb{R}^2\right)$, denoted by $\mathcal{H}^{-1}\left(\mathbb{R}^2\right)$, is a space of distributions. Using the corresponding property for the space $H^1(\Omega)$, one can prove that $\mathcal{D}(\bar{\Omega})$ is dense in $\mathcal{H}^1(\Omega)$. The trace operator γ^{+}on $\partial \Omega$ defined on functions from $\mathcal{H}^1(\Omega)$, satisfies the usual trace theorems. This allows to define in particular the subspace

$$\mathcal{H}_0^1(\Omega)=\left\{g \in \mathcal{H}^1(\Omega): \gamma^{+} g=0\right\}$$

It can be showed that $\mathcal{D}(\Omega)$ is dense in $\mathcal{H}_0^1(\Omega)$ and as a result, its dual space is a space of distributions. Let us denote by $\widetilde{\mathcal{H}}^1(\Omega)$ a completion of $\mathcal{D}(\Omega)$ in $\mathcal{H}^1\left(\mathbb{R}^2\right)$, and $\widetilde{\mathcal{H}}^{-1}(\Omega):=\left[\mathcal{H}^1(\Omega)\right]^{\prime}, \quad \mathcal{H}^{-1}(\Omega):=\left[\widetilde{\mathcal{H}}^1(\Omega)\right]^{\prime}$ are the corresponding dual spaces. The inclusion $L^2(\rho ; \Omega) \subset \mathcal{H}^{-1}(\Omega)$ holds and a distribution f in the dual space $\widetilde{\mathcal{H}}^{-1}(\Omega)$ has the form $f=\sum_{i=1}^2 \frac{\partial g_i}{\partial x_i}+f_0$, where $g_i \in L^2\left(\mathbb{R}^2\right)$ and is zero outside $\Omega, f_0 \in L^2(\rho ; \Omega)$, cf. e.g., [15, Eq. (2.5.129)]. This implies that $\mathcal{D}(\Omega)$ is dense in $\widetilde{\mathcal{H}}^{-1}(\Omega)$ and $\mathcal{D}\left(\mathbb{R}^2\right)$ is dense in $\mathcal{H}^{-1}\left(\mathbb{R}^2\right)$.

Lemma 5.1. *The space $\mathcal{H}^1(\Omega)$ contains constant functions.*

Proof. Let C $\in \mathbb{R}$ then from (5.2) the result follows. $\square$

Note That 5.1. *Lemma 5.1 implies that, the space of real constants, $\mathbb{R}$, is a closed subspace of $\mathcal{H}^1(\Omega)$. Thus we can define the quotient space $\mathcal{H}^1(\Omega) / \mathbb{R}$, which is a Banach space, and its norm is given by $\|u+\mathbb{R}\|_{\mathcal{H}^1(\Omega) / \mathbb{R}}=\inf_{c \in \mathbb{R}}\| u+c\|_{\mathcal{H}^1(\Omega)}$. The dual space $\left(\mathcal{H}^1(\Omega) / \mathbb{R}\right)^{\prime}$ is identified with $\widetilde{\mathcal{H}}^{-1}(\Omega) \perp \mathbb{R}$, i.e., $\left(\mathcal{H}^1(\Omega) / \mathbb{R}\right)^{\prime}=\widetilde{\mathcal{H}}^{-1}(\Omega) \perp \mathbb{R}$ since they are isometrically isomorphic (see e.g., [13,Lemma 2.12(ii)]). Similarly, $\left(\widetilde{\mathcal{H}}^1(\Omega) / \mathbb{R}\right)^{\prime}=\mathcal{H}^{-1}(\Omega) \perp \mathbb{R}$.*

The following Poincaré-type inequalities hold (cf. [2, Theorems 1.1 and 1.2]).

Theorem 5.1. *(i) The semi-norm $|\cdot|_{\mathcal{H}^1(\Omega)}$ defined on $\mathcal{H}^1(\Omega) / \mathbb{R}$ is a norm equivalent to the quotient norm, i.e., there exist positive constants k_1, K_1 such that*

$$k_1|v|_{\mathcal{H}^1(\Omega)} \leq \|v\|_{\mathcal{H}^1(\Omega)/\mathbb{R}} \leq K_1|v|_{\mathcal{H}^1(\Omega)}$$

(ii) Moreover, the semi-norm $|\cdot|_{\mathcal{H}^1(\Omega)}$ *is a norm on* $\mathcal{H}_0^1(\Omega)$ *equivalent to the norm* $\|\cdot\|_{\mathcal{H}^1(\Omega)}$*, i.e., there exist positive constants* k_2, K_2 *such that*

$$k_2|v|_{\mathcal{H}^1(\Omega)} \leq \|v\|_{\mathcal{H}_0^1(\Omega)} \leq K_2|v|_{\mathcal{H}^1(\Omega)}$$

For $u \in \mathcal{H}^1(\Omega)$ and the coefficient $a(x) \in L_\infty(\Omega)$, PDE (5.1) is well defined in the distributional sense as $\langle Au, v\rangle_\Omega := -\langle a\nabla u, \nabla v\rangle_\Omega = -\mathcal{E}(u,v)$, for any $v \in \mathcal{D}(\Omega)$, where $\mathcal{E}(u,v) := \int_\Omega E(u,v)(x)dx, \quad E(u,v)(x) := \nabla v(x) \cdot a(x)\nabla u(x)$. Unless stated otherwise we henceforth assume that there are some constants a_0, a_1 such that

$$a \in L_\infty\left(\mathbb{R}^2\right) \text{ and } 0 < a_0 < a(x) < a_1 < \infty \quad \text{for a.e } x \in \mathbb{R}^2 \tag{5.3}$$

To obtain boundary-domain integral equations, we will also always consider the coefficient a such that

$$a \in C^1\left(\mathbb{R}^2\right) \text{ and } \rho\nabla a \in L^\infty\left(\mathbb{R}^2\right) \tag{5.4}$$

(see, e.g., [10]) for $u \in H^1(\Omega)$, if $u \in \mathcal{H}^1(\Omega^+)$, then from the trace theorem it follows that, $\gamma^+ u \in H^{\frac{1}{2}}(\partial\Omega)$, where $\gamma^+ = \gamma^+_{\partial\Omega}$is the trace operator on $\partial\Omega$ from the exterior domain Ω^+.

For the operator A, similar to [5] for the three dimensional case, we introduce the space, $\mathcal{H}^{1,0}(\Omega; A) := \left\{g \in \mathcal{H}^1(\Omega) : Ag \in L^2\left(\rho; \Omega\right)\right\}$, where the norm is given by its square, $\|g\|^2_{\mathcal{H}^{1,0}(\Omega;A)} := \|g\|^2_{\mathcal{H}^1(\Omega)} + \|Ag\|^2_{L^2(\rho;\Omega)}$. For $u \in \mathcal{H}^{1,0}(\Omega; A)$, as in the 3D case, [5], we define the canonical co-normal derivative $T^+u \in H^{-\frac{1}{2}}(\partial\Omega)$ similar to, for example in [7, Lemma 3.2] and [13, Lemma 4.3] as

$$\left\langle T^+u, \omega\right\rangle_{\partial\Omega} := \int_\Omega \left[\left(\gamma^+_{-1}\omega\right) Au + E\left(u, \gamma^+_{-1}\omega\right)\right] dx \quad \forall\omega \in H^{\frac{1}{2}}(\partial\Omega)$$

where $\gamma^+_{-1} : H^{\frac{1}{2}}(\partial\Omega) \to \mathcal{H}^1(\Omega)$ is a bounded right inverse to the trace operator $\gamma^+ : \mathcal{H}^1(\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$, and $\langle\cdot,\cdot\rangle_{\partial\Omega}$ represents the duality brackets among the spaces $H^{-\frac{1}{2}}(\partial\Omega)$ and $H^{\frac{1}{2}}(\partial\Omega)$ which extends the $L_2(\partial\Omega)$ scalar product. The operator $T^+ : \mathcal{H}^{1,0}(\Omega; A) \to H^{-\frac{1}{2}}(\partial\Omega)$ is continuous and gives the continuous extension to $\mathcal{H}^{1,0}(\Omega; A)$ of the classical co-normal derivative operator $a\frac{\partial}{\partial n}$, where $\frac{\partial}{\partial n} = \gamma^+\nabla \cdot n$ and $n = n^+$is normal vector on $\partial\Omega$ directed outward the exterior domain Ω. When $a \equiv 1$,we employ for T^+ the notation T^+_Δ, which is the continuous extension on $\mathcal{H}^{1,0}(\Omega; \Delta)$ of the classical normal derivative operator ∂_n.Similar to the proofs available in [7, Lemma 3.4] (see also [23] for the spaces $H^{s,t}(\Omega; A)$), one can prove that for $u \in \mathcal{H}^{1,0}(\Omega; A)$ and $v \in \mathcal{H}^1(\Omega)$ the first Green identity

$$\left\langle T^+u, \gamma^+v\right\rangle_{\partial\Omega} = \int_\Omega [vAu + E(u,v)]dx \quad \forall v \in \mathcal{H}^1(\Omega) \tag{5.5}$$

holds true. Then, for any functions $u, v \in \mathcal{H}^{1,0}(\Omega; A)$ we have the second Green identity,

$$\int_{\Omega} [vAu - uAv]dx = \langle T^{+}u, \gamma^{+}v \rangle_{\partial\Omega} - \langle T^{+}v, \gamma^{+}u \rangle_{\partial\Omega} \tag{5.6}$$

Remark 5.1. *If a satisfies condition (5.3) and the second condition in (5.4), then* $\|ga\|_{\mathcal{H}^1(\Omega)} \leq C_1 \|g\|_{\mathcal{H}^1(\Omega)}$, $\left\|g\frac{1}{a}\right\|_{\mathcal{H}^1(\Omega)} \leq C_2 \|g\|_{\mathcal{H}^1(\Omega)}$, *where the constant* C_1 *and* C_2 *are independent of* $g \in \mathcal{H}^1(\Omega)$, *this means, a and* $1/a$ *are multipliers in the space* $\mathcal{H}^1(\Omega)$.

5.2 Mixed BVP in Exterior Domain

Let $\partial\Omega = \overline{\partial\Omega_D} \cup \overline{\partial\Omega_N}$, where $\partial\Omega_D$ and $\partial\Omega_N$ are relatively open, non-empty and non-intersecting parts of $\partial\Omega$ with an infinitely smooth boundary curve $\ell = \overline{\partial\Omega_N} \bigcap \overline{\partial\Omega_D}$ We will derive and analyze the system of BDIEs for the following mixed BVP:

Given $f \in L^2(\rho; \Omega) \perp \mathbb{R}$, $\psi_0 \in H_{**}^{-\frac{1}{2}}(\partial\Omega_N)$ and $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega_D)$, find a function $u \in \mathcal{H}^{1,0^{\perp}}(\Omega; A)$ such that:

$$Au = f \quad \text{in } \Omega \tag{5.7}$$

$$\gamma^{+}u = \varphi_0 \quad \text{on } \partial\Omega_D \tag{5.8}$$

$$T^{+}u = \psi_0 \quad \text{on } \partial\Omega_N \tag{5.9}$$

Let us denote by $\mathcal{A}_M = [A, T^{+}, \gamma^{+}]^T : \mathcal{H}^{1,0}(\Omega; A) \to L^2(\rho; \Omega) \perp \mathbb{R} \times H_{**}^{-\frac{1}{2}}(\partial\Omega_N) \times H^{\frac{1}{2}}(\partial\Omega_D)$, the left hand side operator, which is evidently continuous. Similar to the proof in [5] for the three-dimensional case, one can prove the following assertion in the 2D case.

Theorem 5.2. *Under conditions (5.3), the Mixed problem (5.7)-(5.9) is uniquely solvable and its solution can be written as* $u = \mathcal{A}_M^{-1}(f, \psi_0, \varphi_0)^T$, *where the operator* $\mathcal{A}_M^{-1} : L^2(\rho; \Omega) \times H^{-\frac{1}{2}}(\partial\Omega_N) \times H^{\frac{1}{2}}(\partial\Omega_D) \to \mathcal{H}^{1,0}(\Omega; A)$ *is continuous.*

5.3 Parametrix-Based Potentials in Exterior Domain

A function $P(x, y)$ is a parametrix (Levi function) for the operator A if $A_x P(x, y) = \delta(x - y) + R(x, y)$, where δ is the Dirac-delta distribution, while $R(x, y)$ is a remainder possessing at most a weak (integrable) singularity at $x = y$. In particular, see e.g.,[20] the function

$$P(x, y) = \frac{\ln|x - y|}{2\pi a(y)}, \quad x, y \in \mathbb{R}^2 \tag{1}$$

is a parametrix for the operator $A(x, \partial_x)$ given by:

$$A(x, \partial_x) P(x, y) = R(x, y) + \delta(x - y), \tag{2}$$

where

$$R(x, y) = \sum_{i=1}^{2} \frac{x_i - y_i}{2\pi a(y)|x - y|^2} \frac{\partial a(x)}{\partial x_i}, \quad x, y \in \mathbb{R}^2 \tag{3}$$

Let $u \in \mathcal{D}(\bar{\Omega})$. For any fixed $y \in \Omega$, let $B_\varepsilon(y)$ be an open ball centered at y with a sufficiently small radius $\varepsilon > 0$, and let $B_r(0)$ be an open ball centered at the origin with a radius r large enough to contain $\partial\Omega$ and the support of u, put $\Omega_\varepsilon := (\Omega \cap B_r(0)) \backslash B_\varepsilon(y)$, we have $R(\cdot, y) \in L^2(\rho; \Omega_\varepsilon)$ and thus $P(\cdot, y) \in \mathcal{H}^{1,0}(\Omega_\varepsilon)$ by (2). Applying the second Green identity (5.6) in Ω_ε with $v = P(y, \cdot)$ and taking usual limits as $\varepsilon \to 0$, cf. [14], we get the third Green identity in $\Omega_r := \Omega \cap B_r(0)$,

$$u + \mathcal{R}u - V\left(T^+ u\right) + W\left(\gamma^+ u\right) = \mathcal{P}Au \tag{5.10}$$

for $u \in \mathcal{D}(\bar{\Omega})$. Here,

$$\mathcal{P}g(y) := \int_\Omega P(x, y) g(x) dx, \quad \mathcal{R}g(y) := \int_\Omega R(x, y) g(x) dx, \quad y \in \mathbb{R}^2 \tag{5.11}$$

are, respectively, the parametrix-based Newtonian and remainder potentials, while

$$Vg(y) := -\int_{\partial\Omega} P(x, y) g(x) dS_x, Wg(y) := -\int_{\partial\Omega} \left[T_x P(x, y)\right] g(x) dS_x, y \in \mathbb{R}^2 \backslash \partial\Omega, \tag{5.12}$$

are the parametrix-based single layer and double layer potentials. Deducing (5.10) we took into account that $u \equiv 0$ in $\Omega \backslash B_r(0) \subset \Omega \backslash$ supp u. Since no term in (5.10) depends on r if r is sufficiently large, we obtain that (5.10) is valid in the whole domain Ω for any $u \in \mathcal{D}(\bar{\Omega})$.

From definitions (1)-(3) and (5.11)-(5.12) one can obtain representations of the parametrix-based potential operators in terms of their counterparts for $a = 1$ (i.e., associated with the Laplace operator Δ), cf. [4, 5],

$$\mathcal{P}g = \frac{1}{a} \mathcal{P}_\Delta g, Vg = \frac{1}{a} V_\Delta g, \ Wg = \frac{1}{a} W_\Delta(ag), \ \mathcal{R}g = -\frac{1}{a} \sum_{j=1}^{2} \partial_j \left[\mathcal{P}_\Delta \left(g \partial_j a\right)\right] \tag{5.13}$$

The Newtonian and the remainder potential operators given by (5.11) for $\Omega = \mathbb{R}^2$ will be denoted as $\mathbf{P}$ and $\mathbf{R}$, respectively, and the relations similar to (5.13) hold for them as well.

Invertibility of the Single Layer Potential Operator

The boundary integral operator $\mathcal{V}_\Delta : H^{-1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$ is Fredholm operator of index zero [13 Theorem 7.6]. Thus the relation (5.13), leads to the same result for single layer potential $\mathcal{V}$. For the three dimensional case, the following holds. For $\psi^* \in H^{-1/2}(\partial\Omega)$, if $V\psi^*(y) = 0, y \in \Omega$, then $\psi^* = 0$, which implies the invertibilty of single layer potential operator mapping from $H^{-1/2}(\partial\Omega)$ to $H^{1/2}(\partial\Omega)$. But it is not true the two dimensional case. It is well know *see*, [3 *Remark* 1.42(*ii*)], [16, *proof of thm* 6.22] for some $2D$ domains the kernel of the operator $\mathcal{V}_\Delta$ is non-zero, which by (5.13) also implies that $\ker \mathcal{V} \neq \{0\}$ for the same domains. The following example illustrates this fact.

Example 5.1. *Take the density function $\phi \equiv 1$ and $\Omega = B(0,R)$ to be a disc of radius R centered at the origin and $\partial\Omega = \partial B(0,R)$ be the circular boundary of the disc. We can show that* $a(y)V\phi(y) = V_\Delta\phi(y) = \begin{cases} R\ln|y|, & \textit{for } |y| > R, \\ R\ln R, & \textit{for } |y| \leq R. \end{cases}$

Proof. Let $\phi \equiv 1$. Then

$$V_\Delta\phi(y) = \frac{1}{2\pi}\int_{|x|=R} \ln|y-x|dS_x$$

If $|y| > R$, then the function $g(x) = \ln|y-x|$ is harmonic in the disk $B(0,R)$. Then $g(x)$ has the mean value property,

$$\ln|y| = g(0) = \frac{1}{2\pi R}\int_{|x|=R} g(x)dS_x$$

Therefore,

$$\frac{1}{2\pi}\int_{|x|=R} \ln|y-x|dS_x = R\ln|y|, \qquad \text{for } |y| > R \tag{5.14}$$

For $|y| \leq R$, in particular take $y = 0$,

$$\left(V_\Delta\phi\right)(0) = \frac{1}{2\pi}\int_{|x|=R} \ln|x|dS_x = R\ln R$$

The relation (5.14) implies that, the limit of the value of the potential when $|y|$ approach the boundary from exterior is given by

$$\lim_{|y|\to R^+} \left(V_\Delta\phi\right)(y) = R\ln R \text{ for } |y| = R$$

Furthermore, since the single layer potential is continuous on $\Re^2$ we have

$$\left(V_\Delta\phi\right)(y) = R\ln R \quad \text{for } |y| = R$$

To determine the value of the potential inside the disc for $y \neq 0$, we use the maximum/minimum principle. Since the single layer potential is harmonic on Ω it has neither maximum nor minimum in the disc. Let

$$C_0 = (V_\Delta \phi)(y_0) \quad \text{for } 0 < |y_0| < R$$

If we assume $C_0 \neq R \ln R$, i.e., C_0 is different from the value of potential on the boundary, we will arrive contradiction of the maximum principle. Thus $(V_\Delta \phi)(y)$ is constant on $\bar{\Omega}$. Therefore, $(V_\Delta \phi)(y) = R \ln R, \quad$ for $|y| \leq R$. $\square$

Remark 5.2. *In the above example, if we take the value of* $R = 1$*, and since* $a(y) \neq 0$*, then* $(V\phi)(y) = 0$ *in* $\bar{\Omega}$.

Example 5.1 shows that, the kernel of the operator $\mathcal{V} : H^{-1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$ contains non zero element for a unit ball, i.e., ker $\mathcal{V} \neq \{0\}$ for $\Omega = B(0,1)$, which means, the operators $\mathcal{V}$ is not one to one for this particular domain. Furthermore, the given example applies to the bounded case, and one can also find similar examples for the unbounded case.

Theorem 5.3. *The following spaces are subspaces of* $L^2(\rho;\Omega), \mathcal{H}^{1,0}(\Omega;A)$ *and* $H^s(\partial\Omega), \widetilde{H}^s(\Gamma_1)$, *respectively, Where* $\Gamma_1 \subset \partial\Omega$

(i) $L^2(\rho;\Omega) \perp \mathbb{R} := \left\{ f \in L^2(\rho;\Omega) : \langle f, 1 \rangle_\Omega = 0 \right\}$

(ii) $\mathcal{H}^{1,0^\perp}(\Omega;A) := \left\{ g \in \mathcal{H}^1(\Omega) : Ag \in L^2(\rho;\Omega) \perp \mathbb{R} \right\}$

(iii) $H^s_{**}(\partial\Omega) := \{ \psi \in H^s(\partial\Omega) : \langle \psi, 1 \rangle_{\partial\Omega} = 0 \}$, $\widetilde{H}^s_{**}(\Gamma_1) := \{ \psi \in \widetilde{H}^s(\Gamma_1) : \langle \psi, 1 \rangle_{\Gamma_1} = 0 \}$

Proof. (i) let f and g be in $L^2(\rho;\Omega) \perp \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}$ then $\alpha f + \beta g \in L^2(\rho;\Omega)$ then

$$\begin{aligned} \langle \alpha f + \beta g, 1 \rangle_\Omega &= \langle \alpha f, 1 \rangle_\Omega + \langle \beta g, 1 \rangle_\Omega \\ &= \alpha \langle f, 1 \rangle_\Omega + \beta \langle g, 1 \rangle_\Omega \\ &= 0 \end{aligned}$$

(ii) let f and g be in $\mathcal{H}^{1,0\perp}(\Omega;A)$ and $\alpha, \beta \in \mathbb{R}$ then $\alpha f + \beta g \in \mathcal{H}^{1,0}(\Omega;A)$ then by linearity of an operator A and $L^2(\rho;\Omega) \perp \mathbb{R}$ above,

$$\begin{aligned} A(\alpha f + \beta g) &= A(\alpha f) + A(\beta g) \\ &= \alpha A f + \beta A g \\ &\in L^2(\rho;\Omega) \perp \mathbb{R} \end{aligned}$$

(iii) let ψ and φ be in $H^s_{**}(\partial\Omega)$ and $\alpha, \beta \in \mathbb{R}$ then $\alpha\psi + \beta\varphi \in H^s(\partial\Omega)$ then

$$\begin{aligned} \langle \alpha\psi + \beta\varphi, 1 \rangle_{\partial\Omega} &= \langle \alpha\psi, 1 \rangle_{\partial\Omega} + \langle \beta\varphi, 1 \rangle_{\partial\Omega} \\ &= \alpha \langle \psi, 1 \rangle_{\partial\Omega} + \beta \langle \varphi, 1 \rangle_{\partial\Omega} \\ &= 0 \end{aligned}$$

Similarly the right handside of (iii) follows from the proof of item (iii). $\square$

In order to have invertibility for the single layer potential operator in $2D$, we consider the following theorem.

Theorem 5.4. *If* $\psi \in H_{**}^{-\frac{1}{2}}(\partial\Omega)$ *satisfies* $\mathcal{V}\psi = 0$ *on* $\partial\Omega$*, then* $\psi = 0$.

Proof. The theorem holds for the operator $\mathcal{V}_\Delta$ see, [13 corollary 8.11(ii)],

$$\begin{aligned}
&\mathcal{V}\psi = 0 \\
&\Rightarrow \frac{1}{a(y)}\mathcal{V}_\Delta \psi = 0 \\
&\Rightarrow \psi = 0, \quad (\text{ since } a(y) \neq 0, \Rightarrow \mathcal{V}_\Delta \neq 0)
\end{aligned}$$

□

Lemma 5.2. *If* $u \in \mathcal{H}^{1,0^\perp}(\Omega; A)$ *then* $T^+u \in H_{**}^{-\frac{1}{2}}(\partial\Omega)$.

Proof. Employing the first Green identity (5.5) with $v = 1$, we have:

$$\begin{aligned}
\left\langle T^+u, 1\right\rangle_{\partial\Omega} &= \int_\Omega 1Au \; dx \\
&= \langle Au, 1\rangle_\Omega \\
&= 0 \; ; \quad \mathit{since} \quad Au \in L^2(\rho;\Omega) \perp \mathbb{R}
\end{aligned}$$

□

In addition to conditions (5.3) and (5.4) on the coefficient a, we will sometimes also need the condition

$$\rho^2 \Delta a \in L^\infty\left(\mathbb{R}^2\right) \tag{5.15}$$

Employing that the corresponding mapping properties hold true for the potentials associated with the Laplace operator Δ, cf. eg. Section 8 in [19] and references therein, relations (5.13) lead to the following assertion. *see* [27, *Theorem* 3] and *also* [5, *Theorem* 4.1]

Theorem 5.5. *The following operators are continuous under conditions (5.4).*

$$\begin{aligned}
&\mathbf{P} : \mathcal{H}^{-1}(\mathfrak{R}^2) \perp \mathbb{R} \to \mathcal{H}^1(\mathbb{R}^2) \\
&\mathcal{P} : \widetilde{\mathcal{H}}^{-1}(\Omega) \perp \mathbb{R} \to \mathcal{H}^1(\mathbb{R}^2) \\
&\mathbf{R} : L^2\left(\rho^{-1}; \mathbb{R}^2\right) \to \mathcal{H}^1(\mathbb{R}^2) \\
&V : H_{**}^{-\frac{1}{2}}(\partial\Omega) \to \mathcal{H}^1(\Omega) \\
&W : H^{\frac{1}{2}}(\partial\Omega) \to \mathcal{H}^1(\Omega)
\end{aligned}$$

while The following operators are continuous under conditions (5.4) and (5.15).

$$\begin{aligned}
&\mathcal{P} : L^2(\rho;\Omega) \perp \mathbb{R} \to \mathcal{H}^{1,0}\left(\mathbb{R}^2; A\right) \\
&\mathcal{R} : \mathcal{H}^1(\Omega) \to \mathcal{H}^{1,0}(\Omega; A) \\
&V : H_{**}^{-\frac{1}{2}}(\partial\Omega) \to \mathcal{H}^{1,0}(\Omega; A) \\
&W : H^{\frac{1}{2}}(\partial\Omega) \to \mathcal{H}^{1,0}(\Omega; A)
\end{aligned}$$

Remark 5.3. *Similar to [23,Theorem 3.12] one can prove that $\mathcal{D}(\bar{\Omega})$ is dense in $\mathcal{H}^{1,0}(\Omega; A)$ also in $\mathcal{H}^{1,0^{\perp}}(\Omega; A)$ which then implies by theorem 5.5 and lemma 5.2, (5.10) holds for any $u \in \mathcal{H}^{1,0^{\perp}}(\Omega; A)$.*

The boundary integral (pseudo-differential) operators of the direct values and of the co-normal derivatives of the single and double layer potentials are defined by

$$\mathcal{V}g(y) := -\int_{\Gamma} P(x,y)g(x)ds_x \mathcal{W}g(y) := -\int_{\Gamma} T_x P(x,y)g(x)ds_x \quad y \in \Gamma \quad (5.16)$$

$$\mathcal{W}'g(y) := -\int_{\Gamma} T_y P(x,y)g(x)ds_x \quad \mathcal{L}^{\pm}g(y) := T_y^{\pm} W g(y) \quad y \in \Gamma \quad (5.17)$$

The mapping and jump properties of the operators (5.16)-(5.17) follow from relations (5.13) and are described in details in [27].

Applying the trace and co-normal derivative operators to the third Green identity (5.10), and using the jump relations for the potential operators we obtain for $u \in \mathcal{H}^{1,0^{\perp}}(\Omega; A)$,

$$\frac{1}{2}\gamma^{+}u + \gamma^{+}\mathcal{R}u - \mathcal{V}T^{+}u + \mathcal{W}\gamma^{+}u = \gamma^{+}\mathcal{P}Au \quad \text{on } \partial\Omega \quad (5.18)$$

$$\frac{1}{2}T^{+}u + T^{+}\mathcal{R}u - \mathcal{W}'T^{+}u + \mathcal{L}^{+}\gamma^{+}u = T^{+}\mathcal{P}Au \quad \text{on } \partial\Omega \quad (5.19)$$

Conditions (5.4) are assumed to hold for (5.18) and conditions (5.4) and (5.15) for (5.19). For some functions f, Ψ and Φ let us consider a more general indirect integral relation associated with equation (5.10).

$$u + \mathcal{R}u - V\Psi + W\Phi = \mathcal{P}_f \quad \text{in } \Omega \quad (5.20)$$

Lemma 5.3. *Let $u \in \mathcal{H}^{1,0^{\perp}}(\Omega; A), f \in L^2(\rho; \Omega) \perp \mathbb{R}, \Psi \in H_{**}^{-\frac{1}{2}}(\partial\Omega)$, and $\Phi \in H^{\frac{1}{2}}(\partial\Omega)$ satisfy equation (5.20) and let conditions (5.4) and (5.15) hold. Then, u is a solution of the equation*

$$Au = f \quad \text{in } \Omega \quad (5.21)$$

while

$$V\left(\Psi - T^{+}u\right) - W\left(\Phi - \gamma^{+}u\right) = 0, \quad \text{in } \Omega \quad (5.22)$$

Proof. Since $u \in \mathcal{H}^{1,0^{\perp}}(\Omega; A)$, by Remark 5.3 we can write the third Green identity (5.10) for the function u. Then subtracting (5.20) from it, we obtain

$$-V\Psi^{*} + W\Phi^{*} = \mathcal{P}[Au - f] \quad \text{in } \Omega \quad (5.23)$$

where $\Psi^{*} := T^{+}u - \Psi$ and $\Phi^{*} := \gamma^{+}u - \Phi$. Multiplying equality (5.23) by $a(y)$ we get

$$-V_{\Delta}\Psi^{*} + W_{\Delta}\left(a\Phi^{*}\right) = \mathcal{P}_{\Delta}[Au - f] \quad \text{in } \Omega$$

Equation (5.21) is obtained by applying the Laplace operator Δ to the previous equation and considering that the functions on the left are harmonic potentials, while the function on the right is the classical Newtonian potential. (5.22) is obtained by Substituting (5.21) into (5.23). $\square$

Lemma 5.4. *Let conditions (5.4) and (5.15) hold.*
(i) If $\Psi^* \in H_{**}^{-\frac{1}{2}}(\partial\Omega)$ *and* $V\Psi^* = 0$ *in* Ω*, then* $\Psi^* = 0$.
(ii) If $\Phi^* \in H^{\frac{1}{2}}(\partial\Omega)$ *and* $W\Phi^*(y) = 0$ *in* Ω*, then* $\Phi^*(x) = C/a(x)$*, where* C *is a constant.*
(iii) let $\partial\Omega = \overline{\Gamma_1} \bigcup \overline{\Gamma_2}$*, where* Γ_1 *and* Γ_2 *are nonempty non intersecting simply connected sub-curves of* $\partial\Omega$ *with infinitely smooth boundaries.*
if $\Psi^* \in \widetilde{H}_{**}^{-\frac{1}{2}}(\Gamma_1)$*,* $\Phi^* \in \widetilde{H}^{\frac{1}{2}}(\Gamma_2)$ *and* $V\Psi^*(y)$*-*$W\Psi^*(y)$*=0 in* Ω*, then* Ψ^**=0 and* Φ^**=0 on* $\partial\Omega$*.*

Proof. The proof of item (i) follows from theorem 5.4, while the proof of item (iii) is similar to the proof of [25, Lemma 2.12].

To prove item (ii), from the first Green identity (5.5) for the interior domain Ω^- employing for $v(x) = C, A = \Delta, u = \frac{\ln|x-y|}{2\pi}$ and for any $y \in \Omega$, the function $\Phi_\Delta = C$ satisfies the equation $W_\Delta \Phi_\Delta = 0$ in the exterior domain Ω for any $C = \text{const}$. Now let us check there is no other solution of the equation in Ω in $H^{\frac{1}{2}}(\partial\Omega)$. By the Lyapunov-Tauber theorem $T_\Delta^+ W_\Delta \Phi_\Delta = T_\Delta^- W_\Delta \Phi_\Delta = 0$ on $\partial\Omega$, which implies $W_\Delta \Phi_\Delta = const$ inthe interior domain Ω^- due to the uniqueness up to a constant of the solution of the Neumann problem in $H^{\frac{1}{2}}(\Omega^-)$. Then by the jump property of the double layer $\Phi_\Delta = const$. Applying the relation $Wg = \frac{1}{a} W_\Delta(ag)$ completes the proof of item (ii). $\square$

5.4 BDIEs for Exterior Mixed BVP

To reduce the variable-coefficient Mixed BVP (5.7)-(5.9) to a segregated boundary domain integral equation systems, Let us fix an extension $\Phi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ of the given function φ_0 in the condition (5.8) from $\partial\Omega_D$ to the whole of $\partial\Omega$ and an extension $\Psi_0 \in H_{**}^{-\frac{1}{2}}(\partial\Omega)$ of the given function ψ_0 in the condition (5.9) from $\partial\Omega_N$ to the whole of $\partial\Omega$. moreover Φ_0 and Ψ_0 are considered as known.

For a given function f in $L^2(\rho;\Omega) \perp \mathbb{R}$, assume that the function u satisfies the PDE $Au = f$ in Ω. Then, we can reduce the BVP (5.7)-(5.9) to a system of Boundary-Domain Integral Equations (BDIEs) and in all of them we represent in (5.10), (5.18) and (5.19) the trace of the function u and in its co-normal derivative as

$$\gamma^+ u = \Phi_0 + \varphi, \quad \varphi \in \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N); \quad T^+ u = \Psi_0 + \psi, \quad \psi \in \widetilde{H}_{**}^{-\frac{1}{2}}(\partial\Omega_D)$$

and will regard the new unknown functions φ and ψ as formally segregated of $u \in \mathcal{H}^{1,0^\perp}(\Omega; A)$. Thus we will look for the triplet

$$\mathcal{U} = (u, \psi, \varphi)^\top := \mathcal{H}^{1,0^\perp}(\Omega; A) \times \widetilde{H}_{**}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$$

BDIE system (M11) . Obtained under conditions (5.4) and (5.15), using equation (5.10) in Ω, the restriction of equation (5.18) on $\partial\Omega_D$, and the restriction of equation (5.19) on $\partial\Omega_N$, we arrive at the BDIE system (M11) of three equations for the triplet of unknowns, (u, ψ, φ),

$$u + \mathcal{R}u - V\psi + W\varphi = F_0 \quad \text{in } \Omega \tag{5.24}$$

$$\gamma^+\mathcal{R}u - \mathcal{V}\psi + \mathcal{W}\varphi = \gamma^+F_0 - \Phi_0 \quad \text{on} \quad \partial\Omega_D \tag{5.25}$$

$$T^+\mathcal{R}u - \mathcal{W}'\psi + \mathcal{L}^+\varphi = T^+F_0 - \Psi_0 \quad \text{on} \quad \partial\Omega_N \tag{5.26}$$

where

$$F_0 := \mathcal{P}f + V\Psi_0 - W\Phi_0 \quad \text{in} \quad \Omega \tag{5.27}$$

We denote the matrix operator of the left hand side of the systems (M11) as

$$\mathcal{M}^{11} := \begin{bmatrix} I+\mathcal{R} & -V & W \\ r_{\partial\Omega_D}\gamma^+\mathcal{R} & -r_{\partial\Omega_D}\mathcal{V} & r_{\partial\Omega_D}\mathcal{W} \\ r_{\partial\Omega_N}T^+\mathcal{R} & -r_{\partial\Omega_N}\mathcal{W}' & r_{\partial\Omega_N}\mathcal{L}^+ \end{bmatrix}, \mathcal{F}^{11} := \begin{bmatrix} F_0 \\ r_{\partial\Omega_D}\gamma^+F_0 - \varphi_0 \\ r_{\partial\Omega_N}T^+F_0 - \psi_0 \end{bmatrix}$$

Note That 5.2. *Due to the mapping properties of operators involved in* $\mathcal{M}^{11}$ *,The operator* $\mathcal{M}^{11} : \mathcal{H}^{1,0^\perp}(\Omega; A) \times \widetilde{H}_{**}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N) \to \mathcal{H}^{1,0}(\Omega; A) \times H^{\frac{1}{2}}(\partial\Omega_D) \times H^{-\frac{1}{2}}(\partial\Omega_N)$ *is bounded. And also* $\mathcal{F}^{11} = 0$ *if and only if* $(f, \Phi_0, \Psi_0) = 0$

Proof. $(\Leftarrow)$ evidently true.
$(\Rightarrow)$ from equation (5.27) we have that $F_0 \in \mathcal{H}^{1,0}(\Omega; A)$ and by our assumption $0 = F_0$ implies $F_0 \in \mathcal{H}^{1,0^\perp}(\Omega; A)$, Lemma 5.3 with $F_0 = 0$ for u implies $f = 0$ and $V(\Psi_0) - W(\Phi_0) = 0$, in Ω and The equalities $\gamma^+F_0 = \Phi_0$ on $\partial\Omega_D$ and $T^+F_0 = \Psi_0$ on $\partial\Omega_N$, implies $\Phi_0 = \varphi_0 = 0$ on $\partial\Omega_D$ and $\Psi_0 = \psi_0 = 0$ on $\partial\Omega_N$ that is,$\Psi_0 \in \widetilde{H}_{**}^{-\frac{1}{2}}(\partial\Omega_D)$ and $\Phi_0 \in \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$. Lemma 5.4 (iii) implies $\Phi_0 = \Psi_0 = 0$ ☐

BDIE system (M12). Obtained under conditions (5.4) and using equation (5.10) in Ω and equation (5.18) on the whole of $\partial\Omega$, we arrive at the BDIE system (M12) of two equations for the triplet (u, ψ, φ),

$$u + \mathcal{R}u - V\psi + W\varphi = F_0 \quad \text{in } \Omega$$
$$\frac{1}{2}\varphi + \gamma^+\mathcal{R}u - \mathcal{V}\psi + \mathcal{W}\varphi = \gamma^+F_0 - \Phi_0 \quad \text{on} \quad \partial\Omega$$

The left hand side matrix operator of the system is

$$\mathcal{M}^{12} := \begin{bmatrix} I+\mathcal{R} & -V & W \\ \gamma^+\mathcal{R} & -\mathcal{V} & \frac{1}{2}I + \mathcal{W} \end{bmatrix}, \mathcal{F}^{12} := \begin{bmatrix} F_0 \\ \gamma^+F_0 - \Phi_0 \end{bmatrix}$$

Note That 5.3. *Due to the mapping properties of operators involved in* $\mathcal{M}^{12}$ *, The operator* $\mathcal{M}^{12}:\mathcal{H}^{1,0^{\perp}}(\Omega;A)\times\widetilde{H}_{**}^{-\frac{1}{2}}(\partial\Omega_D)\times\widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)\to\mathcal{H}^{1,0}(\Omega;A)\times H^{\frac{1}{2}}(\partial\Omega)$ *is bounded.And also* $\mathcal{F}^{12}=0$ *if and only if* $(f,\Phi_0,\Psi_0)=0$

Proof. ($\Leftarrow$) evidently true.
($\Rightarrow$) from equation (5.27) we have that $F_0\in\mathcal{H}^{1,0}(\Omega;A)$ and by our assumption $0=F_0$ implies $F_0\in\mathcal{H}^{1,0^{\perp}}(\Omega;A)$, Lemma 5.3 with $F_0=0$ for u implies $f=0$ and $V(\Psi_0)-W(\Phi_0)=0$, in Ω and The equalities $\gamma^{+}F_0=\Phi_0$ on $\partial\Omega$, implies $\Phi_0=0$.Lemma 5.4 (i) implies $\Psi_0=0$ $\square$

BDIE system (M21). Obtained under conditions (5.4) and (5.15) and Using equation (5.10) in Ω and equation (5.19) on the whole of $\partial\Omega$, we arrive at the BDIE system (M21) of two equations for the triplet (u,ψ,φ),

$$u+\mathcal{R}u-V\psi+W\varphi=F_0 \quad \text{in } \Omega \tag{5.28}$$

$$\frac{1}{2}\psi+T^{+}\mathcal{R}u-\mathcal{W}'\psi+\mathcal{L}^{+}\varphi=T^{+}F_0-\Psi_0 \quad \text{on} \quad \partial\Omega \tag{5.29}$$

The left hand side matrix operator of the system is

$$\mathcal{M}^{21}:=\begin{bmatrix} I+\mathcal{R} & -V & W \\ T^{+}\mathcal{R} & \frac{1}{2}I-\mathcal{W}' & \mathcal{L}^{+}\end{bmatrix},\mathcal{F}^{21}:=\begin{bmatrix} F_0 \\ T^{+}F_0-\Psi_0\end{bmatrix}$$

Note That 5.4. *Due to the mapping properties of operators involved in* $\mathcal{M}^{21}$ *,The operator* $\mathcal{M}^{21}:\mathcal{H}^{1,0^{\perp}}(\Omega;A)\times\widetilde{H}_{**}^{-\frac{1}{2}}(\partial\Omega_D)\times\widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)\to\mathcal{H}^{1,0}(\Omega;A)\times H^{-\frac{1}{2}}(\partial\Omega)$ *is bounded.*

BDIE system (M22). Obtained under conditions (5.4) and (5.15) and using equation (5.10) in Ω, the restriction of equation (5.19) on $\partial\Omega_D$, and the restriction of equation (5.18) on $\partial\Omega_N$, we arrive for the triplet (u,ψ,φ) at the BDIE system (M22) of three equations of "almost" the second kind (up to the spaces),

$$\begin{aligned} u+\mathcal{R}u-V\psi+W\varphi&=F_0 && \text{in } \Omega\\ \frac{1}{2}\psi+T^{+}\mathcal{R}u-\mathcal{W}'\psi+\mathcal{L}^{+}\varphi&=T^{+}F_0-\Psi_0 && \text{on} \quad \partial\Omega_D\\ \frac{1}{2}\varphi+\gamma^{+}\mathcal{R}u-\mathcal{V}\psi+\mathcal{W}\varphi&=\gamma^{+}F_0-\Phi_0 && \text{on} \quad \partial\Omega_N \end{aligned}$$

The matrix operator of the left hand side of the system (M22) takes form

$$\mathcal{M}^{22}:=\begin{bmatrix} I+\mathcal{R} & -V & W \\ r_{\partial\Omega_D}T^{+}\mathcal{R} & r_{\partial\Omega_D}\left(\frac{1}{2}I-\mathcal{W}'\right) & r_{\partial\Omega_D}\mathcal{L}^{+} \\ r_{\partial\Omega_N}\gamma^{+}\mathcal{R} & -r_{\partial\Omega_N}\mathcal{V} & r_{\partial\Omega_N}\left(\frac{1}{2}I+\mathcal{W}\right)\end{bmatrix},\mathcal{F}^{22}:=\begin{bmatrix} F_0 \\ r_{\partial\Omega_D}\{T^{+}F_0-\Psi_0\} \\ r_{\partial\Omega_N}\{\gamma^{+}F_0-\Phi_0\}\end{bmatrix}$$

Note That 5.5. *Due to the mapping properties of operators involved in* $\mathcal{M}^{22}$ *,The operator* $\mathcal{M}^{22}: \mathcal{H}^{1,0^{\perp}}(\Omega;A)\times H_{**}^{-\frac{1}{2}}(\partial\Omega_D)\times H^{\frac{1}{2}}(\partial\Omega_N)\rightarrow \mathcal{H}^{1,0}(\Omega;A)\times H^{-\frac{1}{2}}(\partial\Omega_D)\times H^{\frac{1}{2}}(\partial\Omega_N)$ *is bounded.And also* $\mathcal{F}^{22}=0$ *if and only if* $(f,\Phi_0,\Psi_0)=0$

Proof. The proof follows in the similar way as in the Note that 5.2 proof □

5.5 Equivalence and Uniqueness Theorems

Theorem 5.6. *Let* $\varphi_0\in H^{\frac{1}{2}}(\partial\Omega_D),\psi_0\in H_{**}^{-\frac{1}{2}}(\partial\Omega_N), L^2(\rho;\Omega)\perp\mathfrak{R}$ *and let* $\Phi_0\in H^{\frac{1}{2}}(\partial\Omega)$ *and* $\Psi_0\in H_{**}^{-\frac{1}{2}}(\partial\Omega)$ *be some extensions of* φ_0 *and* ψ_0*, respectively, and conditions (5.4) and (5.15) hold.*
(i) If a function $u\in\mathcal{H}^{1,0^{\perp}}(\Omega;A)$ *solves the BVP (5.7)-(5.9), then the triplet* (u,ψ,φ)*, where*

$$T^{+}u-\Psi_0=\psi\in\widetilde{H}_{**}^{-\frac{1}{2}}(\partial\Omega_D),\gamma^{+}u-\Phi_0=\varphi\in\widetilde{H}^{\frac{1}{2}}(\partial\Omega_N) \tag{5.30}$$

solves the BDIE systems (M11), (M12), (M21) and (M22).

(ii) If a triplet $(u,\psi,\varphi)\in\mathcal{H}^{1,0^{\perp}}(\Omega;A)\times\widetilde{H}_{**}^{-\frac{1}{2}}(\partial\Omega_D)\times\widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$ *solves one of the BDIE systems (M11), (M12) or (M22), then this solution is unique and solves all the systems, including (M21), while* u *solves BVP (5.7)-(5.9) and relations (5.30) hold.*

Proof. (i) immediately follows from the deduction of the BDIE systems (M11), (M12), (M21) and (M22).

(ii) Let a triplet $(u,\psi,\varphi)^T$ solve BDIE system (M11) , (M12) or (M22). The hypotheses of Lemma 5.3 are satisfied for the first equation in BDIE system, implying that u solves PDE (5.7) in Ω, while the following equation holds:

$$V\Psi^{*}-W\Phi^{*}=0 \quad \text{in} \quad \Omega \tag{5.31}$$

where $\Psi^{*}=\Psi_0+\psi-T^{+}u$ and $\Phi^{*}=\Phi_0+\varphi-\gamma^{+}u$.

Suppose first that the triplet $(u,\psi,\varphi)^T$ solves BDIE system (M11). Taking trace of (5.24) on $\partial\Omega_D$ using the jump relations of Theorem 5.6, and subtracting (5.25) from it, we obtain

$$\gamma^{+}u=\varphi_0 \quad \text{on} \quad \partial\Omega_D \tag{5.32}$$

i.e., u satisfies the Dirichlet condition (5.8). Taking the co-normal derivative of Eq. (5.24) on $\partial\Omega_N$, using the jump relations on Theorem 5.6 and subtracting Eq. (5.26) from it, we obtain

$$T^{+}u=\psi_0 \quad \text{on} \quad \partial\Omega_N, \tag{5.33}$$

i.e., u satisfies the Neumann condition (5.9). Hence u solves the mixed BVP (5.7)-(5.9).

Taking into account $\varphi = 0, \Phi_0 = \varphi_0$ on $\partial\Omega_D$ and $\psi = 0, \Psi_0 = \psi_0$ on $\partial\Omega_N$, (5.32) and (5.33) imply that the first equation in (5.30) is satisfied on $\partial\Omega_N$ and the second equation in (5.30) is satisfied on $\partial\Omega_D$. Thus we have $\Psi^* \in \widetilde{H}_{**}^{-\frac{1}{2}}(\partial\Omega_D)$ and $\Phi^* \in \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$ in (5.31). Let $\Gamma_1 = \partial\Omega_D, \Gamma_2 = \partial\Omega_N$. Then Lemma 5.4 (iii) implies $\Psi^* = \Phi^* = 0$, which completes the proof of conditions in (5.30). Uniqueness of the solution to BDIE systems (M11) follows from (5.30) along with Note That 5.2 and Theorem 5.2.

Finally, item (i) implies that triplet $(u, \psi, \varphi)^T$ solves also BDIE systems (M12), (M21) and (M22).

Similar arguments work if we suppose that instead of the BDIE systems (M11), the triplet $(u, \psi, \varphi)^T$ solves BDIE systems (M12) or (M22).

$\square$

The situation with uniqueness and equivalence for system (M21) differs from the one for other systems and from its counterpart BDIE system (M21) in [25], particularly because item (ii) of Lemma 5.4 is different from its analog, Lemma 2.11 (ii) in [25]. This leads to the following assertion.

Theorem 5.7. *Let $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega_D), \psi_0 \in H_{**}^{-\frac{1}{2}}(\partial\Omega_N), f \in L^2(\rho;\Omega) \perp \mathbb{R}$ and let $\Phi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ and $\Psi_0 \in H_{**}^{-\frac{1}{2}}(\partial\Omega)$ be some extensions of φ_0 and ψ_0, respectively, and conditions (5.4) and (5.15) hold.*

(i) Homogeneous BDIE system (M21) admits only one linearly independent solution $\left(u^0, \psi^0, \varphi^0\right) \in \mathcal{H}^{1,0^\perp}(\Omega; A) \times \widetilde{H}_{**}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$, where u^0 is the solution of the mixed BVP

$$Au^0 = 0 \qquad in \quad \Omega \tag{5.34}$$

$$r_{\partial\Omega_D}\gamma^+ u^0 = \frac{1}{a(x)} \qquad on \quad \partial\Omega_D \tag{5.35}$$

$$r_{\partial\Omega_N} T^+ u^0 = 0 \qquad on \quad \partial\Omega_N \tag{5.36}$$

while

$$\psi^0 = T^+ u^0, \quad \varphi^0 = \gamma^+ u^0 - 1/a(x) \qquad on \quad \partial\Omega \tag{5.37}$$

(ii) The non-homogeneous BDIE systems (M21) is solvable, and any its solution $(u, \psi, \varphi) \in \mathcal{H}^{1,0^\perp}(\Omega; A) \times \widetilde{H}_{**}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$ can be represented as

$$u = \tilde{u} + Cu^0 \qquad in \quad \Omega \tag{5.38}$$

where $\tilde{u}$ solves the BVP (5.7)-(5.9) and C is a constant, while

$$\psi = T^+\tilde{u} - \Psi_0 + C\psi^0, \quad \varphi = \gamma^+\tilde{u} - \Phi_0 + C\varphi^0 \qquad on \quad \partial\Omega \tag{5.39}$$

Proof. Problem (5.34)-(5.36) is uniquely solvable in $\mathcal{H}^{1,0^{\perp}}(\Omega; A)$ by Theorem 5.2. Consequently, the third Green identity (5.10) is applicable to u^0, leading to

$$u^0 + \mathcal{R}u^0 - V\psi^0 + W\varphi^0 = 0 \quad \text{in} \quad \Omega \tag{5.40}$$

with notations (5.37), if we take into account that $W(1/a(x)) = 0$ in Ω due to the second relation in (5.13) and the equality $W_{\Delta}1 = 0$ in Ω (cf.the proof of Lemma 5.4(ii)). Taking the co-normal derivative of (5.40) and substituting the first equation of (5.37) again, we arrive at

$$\frac{1}{2}\psi^0 + T^+\mathcal{R}u^0 - \mathcal{W}'\psi^0 + \mathcal{L}^+\varphi^0 = 0 \quad \text{on} \quad \partial\Omega \tag{5.41}$$

Equations (5.40)-(5.41) mean that the triplet $\left(u^0, \psi^0, \varphi^0\right)$ solves the homogeneous BDIE system (M21). We use the following steps to show item (ii) and confirm that the homogeneous BDIE system (M21) has just one linearly independent solution. First, we note that the deduction of the system (M21) and the solvability of the BVP (5.7)-(5.9) in $\mathcal{H}^{1,0\perp}(\Omega; A)$ entail the solvability of the non-homogeneous system (M21).

Let now a triplet $(u, \psi, \varphi)^\top \in \mathcal{H}^{1,0^{\perp}}(\Omega; A) \times \widetilde{H}_{**}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$ solve (generally non-homogeneous) BDIE system (M21). Take the co-normal derivative of equation (5.28) on $\partial\Omega$ and subtract it from equation (5.29) to obtain

$$\psi + \Psi_0 - T^+u = 0 \quad \text{on} \quad \partial\Omega \tag{5.42}$$

Taking into account that $\psi = 0$ and $\Psi_0 = \psi_0$ on $\partial\Omega_N$, this implies that u satisfies condition (5.9).

Equations (5.28) and (5.27) and Lemma 5.3 with $\Psi = \psi + \Psi_0, \Phi = \varphi + \Phi_0$ imply that u is a solution of equation (5.7) and

$$V\left(\Psi_0 + \psi - T^+u\right) - W\left(\Phi_0 + \varphi - \gamma^+u\right) = 0 \quad \text{in} \quad \Omega \tag{5.43}$$

Due to (5.42) the first term vanishes in (5.43), and by Lemma 5.4(ii) we obtain

$$\Phi_0 + \varphi - \gamma^+u = -C/a(x) \quad \text{on} \quad \partial\Omega \tag{5.44}$$

where C is a constant. Taking into account that $\varphi = 0$ on $\partial_D\Omega$ and $\Phi_0 = \varphi_0$ on $\partial\Omega_D$, we conclude that u satisfies the Dirichlet condition

$$\gamma^+u = \varphi_0 + C/a(x) \quad \text{on} \quad \partial\Omega_D \tag{5.45}$$

instead of (5.8). Introducing notation $\tilde{u}$ by (5.38) in (5.42), (5.44) and (5.45) and taking into account (5.34)-(5.36) prove the claim of item (ii). The case $\varphi_0 = 0, \Phi_0 = 0, \psi_0 = 0, \Psi_0 = 0, f = 0$ leading to the homogeneous BDIE system (M21) also implies that $\tilde{u}$ for this case satisfies homogeneous BVP (5.7)-(5.9) and thus $\tilde{u} = 0$ in (5.38) and (5.39) meaning that the triplet $\left(u^0, \psi^0, \varphi^0\right)$ is the only linearly independent solution of the homogeneous BDIE system (M21). This completes the proof of item (i) and of the whole theorem. □

Conclusion

In this thesis, we have considered a second-order elliptic partial differential equation with a variable coefficient in a 2D Unbounded domain, in appropriate weighted Sobolev space. The right-hand side functions were from $L^2(\rho;\Omega) \perp \mathbb{R}$ and the Mixed data from the space $H_{**}^{-\frac{1}{2}}(\partial\Omega_N)$ and $H^{\frac{1}{2}}(\partial\Omega_D)$. The BVP was reduced to four systems of Boundary Domain Integral Equations and their equivalence to the original BVP and Uniqueness property was shown.

The properties of a parametrix-based potential operator that contain logarithmic singularity were investigated. Unlike properties in 3D case, The single layer potential needs special consideration to be invertible, which is critical on this study.

future work

Future work on the BDIEs for Variable Coefficient Mixed BVP in 2D Unbounded Domain, will consider the Fredholm properties and invertibility of the corresponding BDIOs in weighted Sobolev spaces. and we will also consider the Direct segregated systems of BDIEs for the Neumann BVPs for a scalar second order divergent elliptic PDEs with a variable coefficient in an exterior two-dimensional domain. and we will again consider the equivalence of BDIE system to the original boundary value problems and the Fredholm properties and invertibility of the corresponding BDIOs are to be analyzed in weighted Sobolev spaces for in the future.

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