

# Compactness and Convergence In the Space of Analytic Functions

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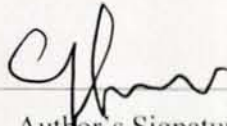
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## DECLARATION

I declare that this project has been composed by me and that no part of the project has formed the basis for the award of any Degree, Diploma, Associate ship, Fellowship or any other similar title to me.

A handwritten signature in black ink, appearing to be 'Gh...' with a stylized flourish at the end, written over a horizontal line.

Author's Signature

## PERMISSION

This is to certify that this project is compiled by **Mr. Gediyon Yemane** in the Department of Mathematics, Addis Ababa University, under my supervision. I hereby also confirm that the project can be submitted for evaluation by examiners and eventual defense.

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Advisor's Signature

## ACKNOWLEDGEMENT

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Gediyon Yemane

## ABSTRACT

This project is a reading material in all Complex Analysis. It deals mainly with the class of Analytic Functions on an open set  $G$ .

Our purpose is to describe this class of analytic functions is complete metric space with defined metric space. As a consequence we prove the set of all continuous function is complete metric space.

Let  $\mathcal{F}$  be family of analytic function in a region  $G$ . We prove that  $\mathcal{F}$  is a normal family, provided that  $\mathcal{F}$  is locally bounded.

We also briefly discuss applications to the normal families of analytic functions.

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## LIST OF MATHEMATICAL SYMBOLS

|                                   |                                                                    |
|-----------------------------------|--------------------------------------------------------------------|
| $\mathbb{C}$                      | the complex plane                                                  |
| $\mathbb{R}$                      | the real numbers                                                   |
| $\mathbb{Q}$                      | the rational numbers                                               |
| $\mathbb{Z}$                      | the integers                                                       |
| $\mathbb{N}$                      | the natural numbers                                                |
| $\mathbb{C}_\infty$               | the extended plane $\{ \mathbb{C} \cup \infty \}$                  |
| $\operatorname{Re}$               | real part                                                          |
| $\operatorname{Im}$               | imaginary part                                                     |
| $B(a; r)$                         | open ball with center $a$ and radius $r$                           |
| $\bar{B}(a; r)$                   | closed ball with center $a$ and radius $r$                         |
| $\operatorname{int} S$            | interior of a set $S$                                              |
| $\operatorname{cl} S$             | closure of a set $S$                                               |
| $S'$                              | limit point of a set $S$                                           |
| $\partial S$                      | boundary of a set $S$                                              |
| $S^c$                             | complement of a set $S$                                            |
| $\operatorname{diam} S$           | diameter of a set $S$                                              |
| $\operatorname{cis} \theta$       | $\cos \theta + i \sin \theta$                                      |
| $\gamma$                          | path                                                               |
| $\int_\gamma f(z) dz$             | path integral                                                      |
| $H(G)$                            | family of analytic functions on an open subset $G$ of $\mathbb{C}$ |
| $C(G, \mathbb{C})$                | class of continuous functions from $G$ to $\mathbb{C}$             |
| $\operatorname{ann}(a; R_1, R_2)$ | $\{z: R_1 <  z - a  < R_2\}$                                       |

## INTRODUCTION

One of the remarkable features of the space of analytic functions in one variable is that the compactness and convergence. This simple fact lies at the heart of many key results in basic complex function theory- for example the completeness of many important function spaces.

Let  $G \subset \mathbb{C}$  is a region. We wish to take the set of all analytic functions on  $G$  denoted by  $H(G)$ , and make it into a metric space. We will define a metric such that convergence in this metric space is the same as uniform convergence on compact subsets of  $G$ . we will call this the space of analytic functions on  $G$ . It is known that there exists a sequence of compact subsets  $E_n \subset G$  such that  $E_n \subset \text{int}E_{n+1}$ , such that  $\bigcup_{n=1}^{\infty} \text{int}E_n = G$  and such that if  $E$  is any compact subset of  $G$ , then  $E \subset E_n$  for some  $n$ . Now define the quantity  $\rho_n(f, g)$  for  $f, g \in H(G)$  as

$$\rho_n(f, g) := \sup_{z \in E_n} \{|f(z) - g(z)|\}.$$

We define the metric on  $H(G)$  as

$$\rho(f, g) := \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{\rho_n(f, g)}{1 + \rho_n(f, g)}.$$

This can be shown to be metric. Furthermore, it can be shown that the topology generated by this metric is independent of the choice of  $E_n$ , even though the actual values of the metric do depend on the particular  $E_n$  we have chosen. Finally, it can be shown that convergence in  $\rho$  is the same as uniform convergence on compact subsets. It is known that if you have a sequence of analytic functions on  $G$  that converge uniformly on compact subsets, then the limit is in fact analytic in  $G$ , and thus  $H(G)$  is a complete space. We can similarly define the space of continuous functions, and treat  $H(G)$  as subspaces of that. That is,  $H(G)$  would be a subspace of  $C(G, \mathbb{C})$ .

Let  $\mathcal{F}$  be a family of analytic functions on a region  $G \subset \mathbb{C}$ . We will define normal and locally bounded family of analytic functions. Finally, it can be shown that locally bounded family of analytic functions is normal, which is a famous Montel's Theorem.

This project report has two parts. In the first part we collect together various definitions of metric spaces and the topology of  $\mathbb{C}$ .

In the second part, first we prove that  $C(G, \Omega)$ , the set of all continuous functions from an open set  $G$  in  $\mathbb{C}$  to a complete metric space  $\Omega$  is complete metric space. As a consequence we prove the famous Arzela-Ascoli Theorem.

Finally, we shall see  $H(G)$ , the collection of analytic functions on an open set  $G$  in  $\mathbb{C}$ , is closed in  $C(G, \mathbb{C})$ . We complete this report proving famous Hurwitz's Theorem and Montel's Theorem.

## CHAPTER I

### PRELIMINARY CONCEPTS

#### 1.1. METRIC SPACES

##### 1.1.1. DEFINITION

A **metric space** is a pair  $(X, d)$  where  $X$  is a set and  $d$  is a mapping from  $X \times X$  into  $\mathbb{R}$  which satisfies the following conditions:

- i)  $d(x, y) \geq 0$  i.e.  $d$  is finite and non-negative real valued function.
- ii)  $d(x, y) = 0$  if and only if  $x = y$
- iii)  $d(x, y) = d(y, x)$  (symmetry)
- iv)  $d(x, z) \leq d(x, y) + d(y, z)$  (triangle inequality)  
for  $x, y, z \in X$ .

**Example:-**(a) Let  $X = \mathbb{R}$  or  $\mathbb{C}$  and defined by

$$d(z_1, z_2) = |z_1 - z_2|.$$

This metric space is called a **Euclidean metric space**.

(b) Let  $X$  be a non-empty set and defined by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y. \end{cases}$$

This metric space is called a **discrete metric space**.

**Open Ball:** Let  $(X, d)$  be a metric space. If  $a \in X$  and  $r \geq 0$ , then the set

$$\{x: x \in X, d(x, a) < r\},$$

denoted by  $B(a; r)$  is called the **open ball** with centre  $a$  and radius  $r$ .

The open ball  $B(a; r)$  on  $\mathbb{R}$  is the bounded open interval  $(a - r, a + r)$  with mid-point  $a$  and total length  $2r$ . The open ball  $B(a; r)$  on  $\mathbb{C}$  is the set



$$\{z \in \mathbb{C}: |z - a| < r\}$$

**Open Sets:** Let  $(X, d)$  be a metric space. A set  $G \subset X$  is **open** if for each  $x$  in  $G$  there is an  $r > 0$  such that  $B(x; r) \subset G$ .

**Example:-** (a) The set  $G = \{z \in \mathbb{C}: a < \operatorname{Re} z < b\}$  is open.

(b) The set  $S = \{z \in \mathbb{C}: \operatorname{Re} z < 0\} \cup \{0\}$  is not open.

Note that the empty set  $\emptyset$  and the full space  $X$  are open sets.

Observe that in any metric space  $(X, d)$ , each open ball is an open set.

**Closed Sets:** Let  $(X, d)$  be a metric space. The set  $G \subset X$  is said to be **closed** if the complement  $G^c$  is open. Equivalently,  $G \subset X$  is **closed** if  $\{z_n\} \subset G$  and  $z_n \rightarrow z$  imply  $z \in G$ .

**Interior:-** Let  $(X, d)$  be a metric space and  $S$  a subset of  $X$ . A point  $x \in S$  is called an **interior point** of  $S$  if there exist an open ball  $B(x; r)$  such that  $B(x; r) \subset S$ . The interior of  $S$ , denoted by  $\operatorname{int} S$ , is the set of all its interior points.

Observe that  $\operatorname{int} S \subset S$ . It is easy to verify that  $S$  is open if and only if  $S = \operatorname{int} S$ .

**Closure:** Let  $(X, d)$  be a metric space and  $S$  a subset of  $X$ . A point  $x \in X$  is called a **closure point** of  $S$  if every open ball centered on  $x$  contains at least one point of  $S$ . In other words, a point  $x \in X$  is a closure point of  $S$  if

$$B(x; r) \cap S \neq \emptyset \quad \text{for all } r > 0.$$

The closure of  $S$ , denoted by  $\operatorname{cl} S$ , is the set of all its closure points.

Observe that  $S \subset \operatorname{cl} S$ .

**Closed Ball:** Let  $(X, d)$  be a metric space. Let  $a \in X$  and let  $r > 0$ . Then the set

$$\{x \in X: d(x, a) \leq r\},$$

denoted by  $\bar{B}(a; r)$  is called the **closed ball** with centre  $a$  and radius  $r$ .

It can be easily proved that in a metric space  $(X, d)$ , each closed ball is a closed set.

For, let  $d(x, a) > r$ , then  $r_1 = d(x, a) - r > 0$ .

If  $d(y, x) < r_1$ , then

$$\begin{aligned} d(a, y) &\geq d(a, x) - d(y, x) \\ &> d(a, x) - r_1 \\ &= d(a, x) - [d(a, x) - r] \\ &= r. \end{aligned}$$

This shows that  $\bar{B}(a; r)^c = S^c$  is open, and thus,  $S$  is closed.

In general, it is not true that in every metric space the closure of every open ball of radius  $r$  is the closed ball of radius  $r$ .

Here is an example. Let  $(X, d)$  be a discrete metric space (i.e.,  $d(x, y) = 1$  if  $x \neq y$ ,  $d(x, y) = 0$ ).

Then

$$B(a; 1) = \{x \in X : d(x, a) < 1\} = \{a\};$$

$$\{x \in X : d(x, a) \leq 1\} = X;$$

$$\overline{B(a; 1)} = \{a\}$$

**Limit Point:** let  $(X, d)$  be a metric space and  $S$  a subset of  $X$ . A point  $x \in X$  is called a **limit point** (an accumulation point) of  $S$  if each open ball  $B(x; r)$  contains at least one point of  $S$  different from  $x$ . In other words, a point  $x \in X$  is a limit point of  $S$  if

$$B(x; r) \cap (S - \{x\}) \neq \emptyset \text{ for each } r > 0.$$

It is clear that every limit point of a set must be a closure point of that set. The set of all limit points of  $S$  is called the **derived set of  $S$** , and is denoted by  $S'$ .

Note that  $\text{cl } S = S \cup S'$ . Also, note that  $S$  is closed if and only if it contains all its limit points.

**Example:-** (a) Let  $X = \mathbb{R}$  and  $S = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$ .

0 is the limit point of  $S$ .

(b) Let  $X = \mathbb{C}$  and  $S = [0, i] \cup \{i\}$ .

The set of all points of  $[0, 1]$  is a limit point of  $S$  but  $i$  is not.

**Boundary:** Let  $(X, d)$  be metric space and  $S$  a subset of  $X$ . A point  $x \in X$  is called a **boundary point** of  $S$  if every open ball  $B(x; r)$  intersects both  $S$  and  $S^c$ . In other words, a point  $x \in X$  is a boundary point of  $S$  if

$$B(x; r) \cap S \neq \emptyset \text{ and } B(x; r) \cap S^c \neq \emptyset \text{ for all } r > 0.$$

The boundary of  $S$ , denoted by  $\partial S$ , is the set of all its boundary points.

Note that

$$\partial S = \partial S^c$$

### 1.1.2. CONVERGENCE AND COMPACTNESS

Let  $(X, d)$  be a metric space. Let  $\{x_n\}$  be a sequence in  $X$ . The sequence  $\{x_n\}$  is said to be **convergent** to  $x$  in  $X$  if for every  $\epsilon > 0$  there is an integer  $n_0$  such that

$$d(x_n, x) < \epsilon \text{ for } n \geq n_0.$$

In symbols, we write

$$\lim x_n = x$$

Note that a sequence in a metric space can have at most one limit.

**Dense Set:** Let  $(X, d)$  be a metric space and  $S$  a subset of  $X$ . The set  $S$  is called **dense** in  $X$  if  $\text{cl } S = X$ .

Example: The set  $\mathbb{Q}$  of rational numbers is dense in  $\mathbb{R}$ .

**Cauchy Sequence:** Let  $\{x_n\}$  be a sequence in  $X$ . We say that  $\{x_n\}$  is a Cauchy sequence if for every  $\epsilon > 0$  there is an integer  $n_0$  such that

$$d(x_m, x_n) < \epsilon \text{ for } m, n \geq n_0.$$

Note that a convergent sequence in  $X$  is a Cauchy sequence in  $X$  but the converse is not true. As an example, consider  $X = (0, \infty)$  with  $d(x, y) = |x - y|$ , and define the sequence by  $x_n = \frac{1}{n}$ . Then the sequence  $\{x_n\}$  is a Cauchy sequence in  $X$  that does not converge in  $X$ .

A metric space  $(X, d)$  is said to be **complete** if every Cauchy sequence in  $X$  converges in  $X$ . The simple example of complete metric space is:

$\mathbb{R}$  or  $\mathbb{C}$  with the usual metric

$$d(x, y) = |x - y|;$$

Observe that the set  $\mathbb{Q}$  of rational numbers is not complete.

**Diameter:** Let  $(X, d)$  be a metric space and  $S$  a subset of  $X$ . The **diameter** of  $S$ , denoted by  $\text{diam}S$ , is defined as

$$\text{diam}S = \sup\{d(x, y): x, y \in S\}$$

The set  $S$  is called **bounded** if  $\text{diam}S < +\infty$ .

**Uniform Convergence:** Let  $X$  be a set and  $(\Omega, \rho)$  a metric space and suppose  $f, f_1, f_2, \dots$  are functions from  $X$  into  $\Omega$ . The sequence  $\{f_n\}$  **converges uniformly** to  $f$  written  $f = \text{u-lim } f_n$  –if for every  $\epsilon > 0$  there is an integer  $N$  (depending on  $\epsilon$  alone) such that

$$\rho(f(x), f_n(x)) < \epsilon \text{ for all } x \text{ in } X, \text{ whenever } n \geq N.$$

Hence,

$$\sup \{ \rho(f(x), f_n(x)): x \in X \} < \epsilon$$

whenever  $n \geq N$ .

### 1.1.3. CONTINUOUS FUNCTIONS

Let  $(X_1, d_1)$  and  $(X_2, d_2)$  be metric spaces. The function  $f: X_1 \rightarrow X_2$  is said to be **continuous** at  $a \in X_1$  if for each  $\epsilon > 0$  there exists  $\delta > 0$  such that

$$d_2(f(x), f(a)) < \epsilon \text{ when ever } d_1(x, a) < \delta.$$

Note that  $\delta$  depends on  $\epsilon$  as well as on  $a$ .

The function  $f: X_1 \rightarrow X_2$  is said to be **continuous** if it is continuous at each point of  $X_1$ .

**Uniform Continuity:** Let  $(X_1, d_1)$  and  $(X_2, d_2)$  be metric spaces and let  $f: X_1 \rightarrow X_2$  be a function. We say that  $f$  is **uniformly continuous** if for every  $\epsilon > 0$  there exists  $\delta > 0$  (depending only on  $\epsilon$ ) such that

$$d_2(f(x_1), f(x_2)) < \epsilon \text{ when ever } d_1(x_1, x_2) < \delta.$$

Note that every uniformly continuous function is continuous but the converse is not true.

As an example, Let  $X_1 = (0,1)$  and  $X_2 = \mathbb{R}$  both with  $d(x, y) = |x - y|$ . Then  $f(x) = \frac{1}{x}$  is continuous, but not uniformly continuous.

### 1.1.4. COMPACTNESS

Let  $(X, d)$  be a metric space. A collection  $\{G_i\}$  of open subsets of  $X$  is said to be an open cover of  $X$  if  $X \subset \bigcup_{i \in I} G_i$ . Let  $S$  be a subset of  $X$ . The set  $S$  is said to be **compact** if every open cover of  $S$  has finite sub cover. In other words, there is a finite number of sets  $G_1, G_2, \dots, G_n$  in the collection  $\{G_i\}$  such that

$$S \subset G_1 \cup G_2 \cup \dots \cup G_n$$

Note that the empty set and all finite sets are compact.

The set  $S = \{z \in \mathbb{C}: |z| < 1\}$  is not compact.

A metric space  $(X, d)$  is said to be **sequentially compact** if every sequence in it has a convergent sub-sequence.

**Totally Bounded Set:** Let  $(X, d)$  be a metric space and  $E$  a subset of  $X$ . The set  $E$  is said to be **totally bounded** if for every  $r > 0$  there exists a finite number of points  $x_1, x_2, \dots, x_n$  such that

$$E \subset \bigcup_{i=1}^n B(x_i; r).$$

Note that a compact metric space is totally bounded but the converse is not necessarily true. As an example, take

$$E = (0,1), X = \mathbb{R}, \quad \text{with } d(x, y) = |x - y|.$$

Let  $(X_1, d_1)$  and  $(X_2, d_2)$  be metric spaces. Let  $f: X_1 \rightarrow X_2$  be a function, and let  $E$  be a subset of  $X_1$ . We call  $f$  an open mapping if  $f(E)$  is open whenever  $E$  is open.

Similarly, we call  $f$  a closed mapping if  $f(E)$  is closed whenever  $E$  is closed.

### 1.1.5. CONNECTEDNESS

Let  $(X, d)$  be a metric space and  $S$  a subset of  $X$ . The set  $S$  is said to be **disconnected** if there exist two disjoint non-empty open subsets  $G$  and  $H$  of  $X$  such that  $G$  intersects  $S$  and  $H$  intersects  $S$ , and  $S \subset G \cup H$ . In this case the pair  $G, H$  is said to form a disconnection of  $S$ .

A set  $S$  in a metric space  $X$  is said to be **connected** if it is not disconnected.

**Examples:-** (a) The set  $I$  of all positive integers is disconnected in  $\mathbb{R}$ .

$$\text{Let } G = \left\{x \in \mathbb{R}: x < \frac{3}{2}\right\} \text{ and } H = \left\{x \in \mathbb{R}: x > \frac{3}{2}\right\}.$$

It is immediately verified that the pair  $G, H$  form a disconnection of  $I$ .

(b) The set  $E$  of all positive rational numbers is connected in  $\mathbb{R}$ .

$$\text{Let } G = \{x \in \mathbb{R}: x < \sqrt{2}\} \text{ and } H = \{x \in \mathbb{R}: x > \sqrt{2}\}.$$

The pair  $G, H$  form a disconnection of  $E$ .

- (c) Consider the sets  $G = \{x \in \mathbb{R}: -1 < x \leq u\}$ ,  $H = \{x \in \mathbb{R}: u < x < 2\}$  where  $0 < u < 1$ . Then  $G$  and  $H$  split the interval  $[0, 1]$  into two disjoint non-empty subsets whose union is  $[0, 1]$ . But since  $G$  is not open it does not follow that  $[0, 1]$  is a connected set in  $\mathbb{R}$ .

Note that in order to show that a set is connected we need to show that no disconnection can exist.

**Component:** A subset  $D$  of a metric space  $X$  is a **component** of  $X$  if it is a maximal connected subset of  $X$ . That is,  $D$  is connected and there is no connected subset of  $X$  that properly contains  $D$ .

**Example:-** (a) A subset  $X = [0, 1) \cup \{1 + \frac{1}{n} : n \geq 1\}$  of  $\mathbb{C}$  is not connected, since there is no way to connect  $\{2\}$  and  $\{\frac{3}{2}\}$ . Therefore  $X$  is not connected. The components are  $[0, 1)$ ,  $\{2\}$ ,  $\{\frac{3}{2}\}$ ,  $\{\frac{4}{3}\}$ , ...,  $\{1 + \frac{1}{n}\}$ .

- (b) A subset  $X = \mathbb{C} - (A \cap B)$  of  $\mathbb{C}$  where  $A = [0, \infty)$  and  $B = \{z = r \operatorname{cis} \theta : r = \theta, 0 \leq \theta \leq \infty\}$  is not connected, since there is no way to connect  $(2, 1)$  and  $(1, -2)$ . The  $k$ -th component is given by  $\mathbb{C} - \{A \cap B\}$  where

$$A = [2\pi k - 2\pi, 2\pi k)$$

and

$$B = \{z = r \operatorname{cis} \theta : r = \theta, 2\pi k - 2\pi \leq \theta < 2\pi k\}, k \in \{1, 2, 3, \dots\}.$$

**Region:** Let  $S$  be a subset of the complex plane  $\mathbb{C}$ .

If  $S$  is open and connected, then  $S$  is called a **region**.

### 1.1.6. INFINITE SERIES IN $\mathbb{C}$

Consider the infinite series

$$\sum_{n=0}^{\infty} z_n = z_0 + z_1 + \dots$$

Where the  $z_n$  are arbitrary complex numbers.

Associated with this series is the sequence of its partial sums

$$S_n = z_0 + z_1 + \dots + z_{n-1} + z_n.$$

We say that the series  $\sum_{n=0}^{\infty} z_n$  converges to  $z$  if  $\lim_{n \rightarrow \infty} S_n$  exists and is equal to  $z$  if for every  $\epsilon > 0$  there is an integer  $n_0$  such that

$$\left| \sum_{k=0}^n z_k - z \right| < \epsilon \text{ whenever } n \geq n_0$$

If this is the case, we say that  $z$  is the sum of the infinite series, that is,

$$z = \sum_{k=0}^{\infty} z_k.$$

If  $f_n$  is in  $\mathbb{C}$  for every  $n \geq 0$  then the series  $\sum_{n=0}^{\infty} f_n$  converges to  $f$  if for every  $\epsilon > 0$  there is an integer  $N$  such that

$$\left| \sum_{n=0}^{\infty} f_n - f \right| < \epsilon \text{ whenever } m \geq N.$$

## 1.2. ELEMENTARY PROPERTIES OF ANALYTIC FUNCTIONS

### 1.2.1. LIMITS AND CONTINUITY

Let  $\Omega$  be an open subset of the complex plane  $\mathbb{C}$ .

Let  $f$  be a function on  $\Omega$ . Let  $A$  be a complex constant. We say that

$$\lim_{z \rightarrow z_0} f(z) = A$$

if the following condition is satisfied.

Given  $\epsilon > 0$  there exists a number  $\delta > 0$  such that if  $z \in \Omega$  and  $|z - z_0| < \delta$ , then

$$|f(z) - A| < \epsilon.$$



Note that this definition of a limit implies that  $z$  may approach  $z_0$  from any direction in the complex plane  $\mathbb{C}$ . For example, let us take

$$f(z) = \frac{(x+y)^2}{x^2+y^2}$$

It can be easily seen that

$$\lim_{x \rightarrow 0} [\lim_{y \rightarrow 0} f(z)] = 1 = \lim_{y \rightarrow 0} [\lim_{x \rightarrow 0} f(z)]$$

But along the path  $y = mx$ , we have

$$\lim_{z \rightarrow 0} f(z) = \frac{(1+m)^2}{1+m^2}$$

The limiting value here depends on  $m$  and hence

$$\lim_{z \rightarrow 0} f(z) \text{ does not exist.}$$

### 1.2.2. COMPLEX DIFFERENTIABILITY

Let  $\Omega$  be an open set in the complex plane  $\mathbb{C}$ . Suppose  $f$  is a function on  $\Omega$ . The function  $f$  is said to be **differentiable** at a point  $z_0$  of  $\Omega$  if

$$\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. The derivative is denoted by  $f'(z_0)$ . Note that the limit is independent of the path along with  $z \rightarrow z_0$  in the complex plane.

**Analytic Functions:** A function  $f$  is said to be **analytic at a point  $z_0$** , if it is differentiable throughout some  $\varepsilon$ -neighbourhood of  $z_0$ . A function  $f$  is **analytic in a region** if it is at every point of the region.

Observe that a function which is differentiable at a point, need not necessarily be analytic at that point. Example (v) below illustrates this fact.

**Example:-** (i) If  $n$  is a positive integer, then  $f(z) = z^n$  is differentiable in the entire complex plane.

(ii)  $f(z) = \operatorname{Re} z$  and  $f(z) = \operatorname{Im} z$  are not differentiable.

(iii)  $f(z) = \frac{1}{z}$  is differentiable in  $\mathbb{C} - \{0\}$ .

(iv)  $f(z) = \bar{z}$  is not differentiable.

(v)  $f(z) = |z|^2$  is differentiable only at  $z = 0$ .

A **path** in a region  $G \subset \mathbb{C}$  is a continuous function  $\gamma : [a, b] \rightarrow G$  for some interval  $[a, b]$  in  $\mathbb{R}$ .

If  $\gamma : [a, b] \rightarrow \mathbb{C}$  is a rectifiable path and  $f$  is a function defined and continuous on the trace of  $\gamma$  then the **(line) integral of  $f$  along  $\gamma$**  is

$$\int_a^b f(\gamma(t)) d\gamma(t).$$

This line integral is also denoted by  $\int_\gamma f = \int_\gamma f(z) dz$ .

An **entire function** is a function which is defined and analytic in the whole complex plane  $\mathbb{C}$ .

**Zeros of an analytic function:** If  $f: G \rightarrow \mathbb{C}$  is analytic and  $a$  in  $G$  satisfies  $f(a) = 0$  then  $a$  is a **zero of  $f$  of multiplicity  $m \geq 1$**  if there is an analytic function  $g: G \rightarrow \mathbb{C}$  such that  $f(z) = (z - a)^m g(z)$  where  $g(a) \neq 0$ .

## CHAPTER II

### COMPACTNESS AND CONVERGENCE IN THE SPACE OF ANALYTIC FUNCTIONS

#### 2.1. METRIC ON $C(G, \Omega)$

Let  $G$  be an open set in  $\mathbb{C}$  and let  $(\Omega, d)$  be a complete metric space. We denote by  $C(G, \Omega)$  the set of all continuous functions from  $G$  to  $\Omega$ .

In order to introduce a metric on  $C(G, \Omega)$  we first prove a theorem about subsets of  $\mathbb{C}$ .

**Theorem 1.** *If  $G$  is open in  $\mathbb{C}$  then there is a sequence  $\{E_n\}$  of compact subsets of  $G$  such that  $G = \bigcup_{n=1}^{\infty} E_n$ . Moreover, the sets  $E_n$  can be chosen to satisfy the following conditions:*

- (a)  $E_n \subset \text{int } E_{n+1}$ ;
- (b)  $E \subset G$  and  $E$  compact implies  $E \subset E_n$  for some  $n$ ;
- (c) Every component of  $\mathbb{C}_{\infty} - E_n$  contains a component of  $\mathbb{C}_{\infty} - G$ .

**Proof:-** For each positive integer  $n$ , define

$$E_n = \underbrace{\{z: |z| \leq n\}}_{:= \bar{B}(0; n)} \cap \underbrace{\{z: z \in G \text{ and } d(z, \mathbb{C} - G) \geq \frac{1}{n}\}}_{:= F_n};$$

since  $\bar{B}(0; n)$  is bounded then  $E_n$  is bounded and it is the intersection of two closed subsets of  $\mathbb{C}$ ,  $E_n$  is closed. Hence by Theorem 3.4 <sup>[3]</sup>  $E_n$  is compact. For

$$H_n = \{z: |z| < n + 1\} \cap \left\{z: z \in G \text{ and } d(z, \mathbb{C} - G) > \frac{1}{n+1}\right\},$$

it is clear that  $H_n$  is open and

$$E_n \subset H_n \subset E_{n+1}$$

Thus the property (a) holds. We claim that  $G = \bigcup_{n=1}^{\infty} E_n$ . To show this, suppose that  $z \in G$ , since  $G$  is open, there is  $\epsilon > 0$  such that  $B(z; \epsilon) \subset G$ . In particular, for any  $w \in \mathbb{C}_{\infty} - G$ ,  $|z - w| \geq \epsilon$  choose  $n \in \mathbb{N}$  such that  $n > \frac{1}{\epsilon}$  and  $n > |z|$ . Then  $z \in F_n$  and also

$z \in \bar{B}(0; n)$ , that is,  $z \in E_n$ . It follows that  $G = \bigcup_{n=1}^{\infty} E_n$ , as required. We also get that  $G = \bigcup_{n=1}^{\infty} \text{int} E_n$ . Further, for any compact  $E$  with  $E \subset G$ , it is clear that  $E \subset \bigcup_{n=1}^{\infty} \text{int} E_n$  and  $E \subset E_n$  for some  $n$ . This gives part (b).

To see part (c), Let  $K$  be the unbounded component of  $\mathbb{C}_{\infty} - E_n (\supset \mathbb{C}_{\infty} - G)$  and let  $F$  be the unbounded component of  $\mathbb{C}_{\infty} - G$ . Then  $K \supset \{z: |z| > n\}$ ,  $\infty \in K$  and  $K \supset F$  since  $\mathbb{C}_{\infty} - E_n \supset \mathbb{C}_{\infty} - G$ . So if  $D$  is a bounded component of  $\mathbb{C}_{\infty} - E_n$  it contains a point  $z$  with  $d(z, \mathbb{C} - G) < \frac{1}{n}$ . But, then there is  $w \in \mathbb{C} - G$  with  $|w - z| < \frac{1}{n}$  and  $z \in B(w; \frac{1}{n}) \subset \mathbb{C}_{\infty} - E_n$ . Since disks are connected and  $z$  is in the component  $D$  of  $\mathbb{C}_{\infty} - E_n$ , then  $B(w; \frac{1}{n}) \subset D$ . If  $D_1$  is the component of  $\mathbb{C}_{\infty} - G$  that contains  $w$  it follows that  $D_1 \subset D$ . ■

**Remark:-** We can find  $E_n$  for each of the following choices of  $G$ :

a)  $G$  is an open disk,

Let  $a \in \mathbb{C}$  and  $r > 0$  such that  $G = B(a, r)$ . Set

$$E_n = \overline{B(a, r - \frac{1}{n})}$$

where a ball of negative radius is to be understood as the empty set. Depending on the size of  $r$ , a finite number of  $E_n$  may be empty, not violating the condition  $E_n \subset \text{int } E_{n+1}$ , and obviously  $G = \bigcup_{n=1}^{\infty} E_n$ .

b)  $G$  is an open annulus,

Let  $a \in \mathbb{C}$  be the center of the annulus with radii  $r, R$ ,  $0 \leq r < R < \infty$  so that

$G = \text{ann}(a, r, R)$ . Define

$$E_n = \overline{\text{ann}(a, r + \frac{1}{n}, R - \frac{1}{n})}$$

again with the interpretation that an annulus with inner radius larger than the outer radius is the empty set.

c)  $G = \{z \in \mathbb{C}: |\text{Im} z| < 1\}$ ,

For this open set  $G$  define

$$E_n = \{z = x + iy \in \mathbb{C} : x, y \in \mathbb{R}, |x| \leq n\} \cap \{z = x + iy \in \mathbb{C} : x, y \in \mathbb{R}, |y| \leq 1 - \frac{1}{n}\}.$$

Although  $G$  is unbounded each  $E_n$  is bounded and closed and hence compact and

$$G = \bigcup_{n=1}^{\infty} E_n.$$

$$d) \quad G = \mathbb{C} - \mathbb{Z}.$$

Here we can define

$$E_n = \{z \in \mathbb{C} : |z| \leq n\} \cap \left( \bigcup_{j=-n}^n B(j, \frac{1}{n}) \right)^c$$

an intersection of closed set, one of which is bounded, hence  $E_n$  is compact. Also

$$E_n \subset \text{int } E_{n+1} \text{ by definition and } G = \bigcup_{n=1}^{\infty} E_n.$$

**Lemma1.** *If  $(S, d)$  is a metric space then*

$$\mu(s, t) = \frac{d(s, t)}{1 + d(s, t)}$$

*is also a metric on  $S$ . A set is open in  $(S, d)$  if and only if it is open in  $(S, \mu)$ ; a sequence is a Cauchy sequence in  $(S, d)$ ; it is a Cauchy sequence in  $(S, \mu)$ .*

**Proof:-** To show that  $\mu: S \times S \rightarrow \mathbb{R}$ ,  $\mu(s, t) = \frac{d(s, t)}{1 + d(s, t)}$  is a metric we proof that  $\mu$  satisfies the conditions of a metric function.

For  $s, t \in S$  the value  $\mu(s, t) = \frac{d(s, t)}{1 + d(s, t)}$  is well defined in the nonnegative real numbers because  $d$  is a metric. Also  $\mu(s, t) = 0$  if and only if  $d(s, t) = 0$  and since  $d$  is a metric it follows that  $s = t$ . The metric  $d$  is symmetric and therefore  $\mu$  is also. It remains to show the triangle inequality. Let therefore  $s, t, u \in S$ . The metric  $d$  satisfies the triangle inequality

$$(1) \quad d(s, u) \leq d(s, t) + d(t, u).$$

The function  $f(t) = \frac{t}{1+t}$  is a strictly increasing, concave function on the interval  $[0, \infty)$ . Thus if  $d(s, u) \leq \max\{d(s, t), d(t, u)\}$ , then also  $\mu(s, u) \leq \max\{\mu(s, t), \mu(t, u)\}$  and by non negativity of  $\mu$  also

$$\mu(s, u) \leq \mu(s, t) + \mu(t, u).$$

If instead  $\max\{d(s, t), d(t, u)\} < d(s, u)$  then

$$\frac{1}{1+d(s, u)} < \min\left\{\frac{1}{1+d(s, t)}, \frac{1}{1+d(t, u)}\right\}.$$

Together with equation (1) we have

$$\begin{aligned} \mu(s, u) &= \frac{d(s, u)}{1+d(s, u)} \leq \frac{d(s, t) + d(t, u)}{1+d(s, u)} \\ &= \frac{d(s, t)}{1+d(s, u)} + \frac{d(t, u)}{1+d(s, u)} \\ &\leq \frac{d(s, t)}{1+d(s, t)} + \frac{d(t, u)}{1+d(t, u)} \\ &= \mu(s, t) + \mu(t, u), \end{aligned}$$

which gives the triangle inequality.

To show that the metrics  $d$  and  $\mu$  are equivalent on  $S$ , let  $O$  be an open set in  $(S, d)$  and let  $x \in O$ . There is  $\varepsilon \in (0, 1)$  such that  $B_d(x, \varepsilon) \subset O$ . Choose  $\delta$  positive such that  $\delta < \frac{\varepsilon}{1+\varepsilon}$ , then  $B_\mu(x, \delta) \subset B_d(x, \varepsilon)$ . Similarly, if  $O$  is an open set in  $(S, \mu)$  and  $\delta$  is such that  $x \in B_\mu(x, \delta) \subset O$  then choose  $\varepsilon > 0$  with  $\varepsilon \leq \frac{\delta}{1-\delta}$  which implies  $B_d(x, \varepsilon) \subset B_\mu(x, \delta)$ . Since this can be done for any element  $x \in O$ , open sets in  $(S, d)$  are open in  $(S, \mu)$  and vice-versa.

The fact that exactly the open sets in  $(S, d)$  are open in  $(S, \mu)$  leads to the statement about Cauchy sequences. Assume that  $\{x_n\}$  is a Cauchy sequence in  $(S, d)$ . To see that it is Cauchy in  $(S, \mu)$  let  $\varepsilon > 0$  be arbitrary but fixed. By the above there is a  $\delta$  small enough

that  $B_d(x, \delta) \subset B_\mu(x, \varepsilon)$  and  $\delta$  depends only on  $\varepsilon$ , but not on  $x \in S$ . Since  $\{x_n\}$  is Cauchy in  $(S, d)$  there is  $N \in \mathbb{N}$  such that  $d(x_m, x_n) < \delta$  whenever  $m, n \geq N$  and therefore also  $\mu(x_m, x_n) < \varepsilon$  for  $m, n \geq N$ . The opposite statement holds with a similar argument.

**Theorem 2.** If  $G = \bigcup_{n=1}^{\infty} E_n$  where each  $E_n$  is compact and  $E_n \subset \text{int} E_{n+1}$ , define

$$(2) \quad \rho_n(f, g) = \sup \{d(f(z), g(z)) : z \in E_n\}$$

for all functions  $f$  and  $g$  in  $C(G, \Omega)$ . Also define

$$(3) \quad \rho(f, g) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{\rho_n(f, g)}{1 + \rho_n(f, g)};$$

then  $(C(G, \Omega), \rho)$  is a metric space.

**Proof:-** Since  $\frac{t}{1+t} \leq 1$  for all  $t \geq 0$ , the series in (3) is dominated by  $\sum (\frac{1}{2})^n$  and must converge. It will be shown that  $\rho$  is a metric for  $C(G, \Omega)$ . It is clear that  $\rho(f, g) = \rho(g, f)$ . Also, since each  $\rho_n$  satisfies the triangle inequality, the preceding lemma can be used to show that  $\rho$  satisfies the triangle inequality. Finally, the fact that  $G = \bigcup_{n=1}^{\infty} E_n$  gives that  $f = g$  whenever  $\rho(f, g) = 0$ . ■

**Theorem 3.** Define the metric  $\rho$  as in (3). If  $\varepsilon > 0$  is given then there is a  $\delta > 0$  and a compact set  $E \subset G$  such that for  $f$  and  $g$  in  $C(G, \Omega)$ ,

$$(4) \quad \sup \{d(f(z), g(z)) : z \in E\} < \delta \Rightarrow \rho(f, g) < \varepsilon.$$

Conversely, if  $\delta > 0$  and a compact set  $E$  are given, there is an  $\varepsilon > 0$  such that for  $f$  and  $g$  in  $C(G, \Omega)$ ,

$$(5) \quad \rho(f, g) < \varepsilon \Rightarrow \sup \{d(f(z), g(z)) : z \in E\} < \delta.$$

**Proof:-** Let  $\varepsilon > 0$  be fixed let  $m$  be a positive integer such that

$$\sum_{n=m+1}^{\infty} \left(\frac{1}{2}\right)^n < \frac{1}{2} \varepsilon$$

Put  $E = E_m$ . For suitably chosen  $\delta > 0$  we have

$$\frac{t}{1+t} < \frac{1}{2}\epsilon \quad \text{where } 0 \leq t < \delta.$$

By our hypothesis  $f$  and  $g$  are functions in  $C(G, \Omega)$  where

$$\sup \{d(f(z), g(z)) : z \in E\} < \delta.$$

Since  $E_n \subset E_m = E$  for  $1 \leq n \leq m$ ,

$$\rho_n(f, g) < \delta.$$

Therefore,

$$\frac{\rho_n(f, g)}{1 + \rho_n(f, g)} < \frac{1}{2}\epsilon \quad \text{for } 1 \leq n \leq m.$$

Hence

$$\begin{aligned} \rho(f, g) &= \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{\rho_n(f, g)}{1 + \rho_n(f, g)} \\ &= \sum_{n=1}^m \left(\frac{1}{2}\right)^n \frac{\rho_n(f, g)}{1 + \rho_n(f, g)} + \sum_{n=m+1}^{\infty} \left(\frac{1}{2}\right)^n \frac{\rho_n(f, g)}{1 + \rho_n(f, g)} \\ &< \sum_{n=1}^m \left(\frac{1}{2}\right)^n \left(\frac{1}{2}\epsilon\right) + \sum_{n=m+1}^{\infty} \left(\frac{1}{2}\right)^n \\ &< \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon. \end{aligned}$$

Conversely, Let  $E$  and  $\delta$  are given.

Since

$$G = \bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} \text{int} E_n$$

and  $E$  is compact, there is an integer  $m \geq 1$  such that  $E \subset E_m$ .

From this we have

$$\begin{aligned}\rho_m(f, g) &= \sup\{d(f(z), g(z)): z \in E_m\} \\ &\geq \sup\{d(f(z), g(z)): z \in E\}.\end{aligned}$$

Now choose  $\epsilon > 0$  such that

$$0 \leq s \leq 2^m \epsilon \text{ implies } \frac{s}{1-s} < \delta.$$

Hence

$$\frac{t}{1+t} < 2^m \epsilon \text{ implies } t < \delta.$$

Thus if  $\rho(f, g) < \epsilon$  then

$$\sum_{m=1}^{\infty} \left(\frac{1}{2}\right)^m \frac{\rho_m(f, g)}{1 + \rho_m(f, g)} < \epsilon$$

and this implies

$$\left(\frac{1}{2}\right)^m \frac{\rho_m(f, g)}{1 + \rho_m(f, g)} < \epsilon,$$

that is,

$$\frac{\rho_m(f, g)}{1 + \rho_m(f, g)} < 2^m \epsilon$$

and this gives

$$\rho_m(f, g) < \delta.$$

Hence the theorem is proved. ■

**Lemma 2.** (a) A set  $\mathcal{O} \subset (C(G, \Omega), \rho)$  is open if and only if for each  $f$  in  $\mathcal{O}$  there is a compact set  $E$  and a  $\delta > 0$  such that

$$\mathcal{O} \supset \{g: d(f(z), g(z)) < \delta, z \in E\}$$

(b) A sequence  $\{f_n\}$  in  $(C(G, \Omega), \rho)$  converges to  $f$  if and only if  $\{f_n\}$  converges to  $f$  uniformly on all compact subsets of  $G$ .

**Proof:-** (a) If  $\mathcal{O}$  is open and  $f \in \mathcal{O}$  then for some  $\epsilon > 0$ ,  $\mathcal{O} \supset \{g: \rho(f, g) < \epsilon\}$ . But now the first part of the preceding lemma says that there is a  $\delta > 0$  and a compact set  $E$  with the desired properties. Conversely, if  $\mathcal{O}$  has the stated property and  $f \in \mathcal{O}$  then the second part of the lemma gives an  $\epsilon > 0$  such that  $\mathcal{O} \supset \{g: \rho(f, g) < \epsilon\}$ ; this means that  $\mathcal{O}$  is open.

(b) Suppose first  $f_n \rightarrow f$  in the sense of the  $\rho$ -metric. For large  $n$  we have then  $\rho(f_n, f) < \epsilon$  and consequently,  $\rho_i(f_n, f) < 2^i \epsilon$ . But this implies that  $f_n \rightarrow f$  uniformly on  $E_i$  with respect to  $d$ -metric. Since every compact  $E$  is contained in an  $E_i$  it follows that the convergence is uniform on  $E$ .

Conversely, suppose this is true for all compact  $E$  subsets of  $G$ . Then it is certainly true for all  $E_i$ . Hence for all  $i$ ,

$$\frac{\rho_i(f_n, f)}{1 + \rho_i(f_n, f)} \rightarrow 0 \text{ for all } i.$$

Let  $\epsilon > 0$  and choose  $N$  so that  $\sum_{i=N+1}^{\infty} \frac{1}{2^i} < \epsilon$  choose  $n_0$  so that

$$\frac{\rho_i(f_n, f)}{1 + \rho_i(f_n, f)} < \epsilon, \quad \text{for all } n \geq n_0.$$

If  $i \leq N$ , we know  $E_i \subset E_N$  and hence

$$\rho_i(f_n, f) \leq \rho_N(f_n, f)$$

Now

$$\sum_{i=1}^N \frac{1}{2^i} \frac{\rho_i(f_n, f)}{1 + \rho_i(f_n, f)} \leq \sum_{i=1}^N \frac{\epsilon}{2^i} \leq \sum_{i=1}^{\infty} \frac{\epsilon}{2^i} = \epsilon$$

and

$$\sum_{i=N+1}^{\infty} \frac{1}{2^i} \frac{\rho_i(f_n, f)}{1 + \rho_i(f_n, f)} \leq \sum_{i=N+1}^{\infty} \frac{1}{2^i} < \epsilon$$

Thus,

$$\begin{aligned}\rho(f_n, f) &= \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\rho_i(f_n, f)}{1 + \rho_i(f_n, f)} \\ &= \sum_{i=1}^N \frac{1}{2^i} \frac{\rho_i(f_n, f)}{1 + \rho_i(f_n, f)} + \sum_{i=N+1}^{\infty} \frac{1}{2^i} \frac{\rho_i(f_n, f)}{1 + \rho_i(f_n, f)} \\ &< \epsilon + \epsilon = 2\epsilon \quad \text{if } n \geq n_0\end{aligned}$$

Since this is true for any  $E$ ,  $f_n \rightarrow f$  in  $\rho$ . ■

**Corollary 1.** *The collection of open sets is independent of the choice of the sets  $\{E_n\}$ . That is, if  $G = \bigcup_{n=1}^{\infty} E'_n$  where each  $E'_n$  is compact and  $E'_n \subset \text{int } E'_{n+1}$  and if  $\mu$  is the metric defined by the sets  $\{E'_n\}$  then a set is open in  $(C(G, \Omega), \mu)$  if and only if it is open in  $(C(G, \Omega), \rho)$ .*

**Proof:-** This is a direct consequence of part (a) of the preceding lemma since the characterization of open sets does not depend on the choice of the sets  $\{E_n\}$ . ■

**Theorem 4.**  $C(G, \Omega)$  is a complete metric space.

**Proof:-** Suppose  $\{f_n\}$  is a Cauchy sequence in  $C(G, \Omega)$ . Then the restrictions of the functions  $f_n$  to  $E$  gives a Cauchy sequence in  $C(E, \Omega)$  where  $E$  is compact and  $E \subset G$ . That is, for every  $\epsilon > 0$  there is an integer  $N$  such that

$$\rho(f_n(z), f_m(z)) < \epsilon, \text{ for all } z \text{ in } E \text{ and for } n, m \geq N.$$

By Theorem 3, for every  $\delta > 0$  there is an integer  $N$  such that

$$(6) \quad \sup \{d(f_n(z), f_m(z)) : z \in E\} < \delta \quad \text{for } n, m \geq N.$$

Hence  $\{f_n(z)\}$  is a Cauchy sequence in  $\Omega$ . Thus there exists a point  $f(z)$  in  $\Omega$  such that

$$f(z) = \lim f_n(z).$$

We now got a function  $f: G \rightarrow \Omega$  and have to prove that  $f$  is continuous and

$$\rho(f_n, f) \rightarrow 0.$$

Let  $E$  be compact and fix  $\delta > 0$ . Choose  $N$  so that

$$\sup \{d(f_n(z), f_m(z)) : z \in E\} < \delta$$

holds for  $n, m > N$ . If  $z$  is arbitrary in  $E$  but fixed then there is an integer  $m \geq N$  so that

$$d(f(z), f_m(z)) < \delta.$$

By triangle inequality, we get

$$\begin{aligned} d(f(z), f_n(z)) &\leq d(f(z), f_m(z)) + d(f_m(z), f_n(z)) \\ &< \delta + \delta = 2\delta \end{aligned}$$

for all  $n \geq N$ . Since  $N$  does not depend on  $z$  we have

$$\sup \{d(f(z), f_n(z)) : z \in E\} \rightarrow 0$$

as  $n \rightarrow \infty$ . Hence  $\{f_n\}$  converges  $f$  uniformly on every compact set in  $G$ . Thus  $f$  is continuous. Finally, it follows from Lemma 2(b) that

$$\rho(f_n, f) \rightarrow 0. \blacksquare$$

A family  $\mathcal{F}$  is said to be **normal** in  $C(G, \Omega)$  if every sequence  $\{f_n\}$  of functions  $f_n \in \mathcal{F}$  contains a subsequence which converges uniformly on every compact subset of  $G$ .

Note that this definition does not require the limit functions of the convergent subsequences to be members of  $\mathcal{F}$ .

**Lemma 3.** *A set  $\mathcal{F} \subset C(G, \Omega)$  is normal if and only if its closure is compact.*

**Theorem 5.** *A set  $\mathcal{F} \subset C(G, \Omega)$  is normal if and only if for every compact set  $E \subset G$  and  $\delta > 0$  there are functions  $f_1, \dots, f_n$  in  $\mathcal{F}$  such that for  $f$  in  $\mathcal{F}$  there is at least one  $k$ ,  $1 \leq k \leq n$ , with*

$$\sup \{d(f(z), f_k(z)) : z \in E\} < \delta.$$

**Proof:-** Suppose  $\mathcal{F}$  is normal and let  $E$  and  $\delta > 0$  be given. By Theorem 3 there is an  $\epsilon > 0$  such that (5) holds. But since  $\mathcal{F}^-$  is compact,  $\mathcal{F}$  is totally bounded. To see this, suppose  $\mathcal{F}^-$  is compact then by Theorem II 4.9<sup>[1]</sup>  $\mathcal{F}^-$  is totally bounded. Thus, since  $\mathcal{F} \subset \mathcal{F}^-$  then by Theorem 5.2.7<sup>[2]</sup>  $\mathcal{F}$  is totally bounded. So there are  $f_1, \dots, f_n$  in  $\mathcal{F}$  such that

$$\mathcal{F} \subset \bigcup_{k=1}^n \{f: \rho(f, f_k) < \epsilon\}$$

But from the choice of  $\epsilon$  this gives

$$\mathcal{F} \subset \bigcup_{k=1}^n \{f: d(f(z), f_k(z)) < \delta, z \in E\};$$

that is,  $\mathcal{F}$  satisfies the condition of the proposition.

For the converse, suppose  $\mathcal{F}$  has the stated property. Since it readily follows that  $\mathcal{F}^-$  also satisfies this condition, assume that  $\mathcal{F}$  is closed. But since  $C(G, \Omega)$  is complete  $\mathcal{F}$  must be complete. And, again using Theorem 3, it readily follows that  $\mathcal{F}$  is totally bounded. From Theorem II. 4.9<sup>[1]</sup>  $\mathcal{F}$  is compact and therefore normal. ■

**Theorem 6.** Let  $(X_n, d_n)$  be a metric space for each  $n \geq 1$  and let  $X = \prod_{n=1}^{\infty} X_n$  be their cartesian product. That is,  $X = \{\xi = \{x_n\}: x_n \in X_n \text{ for each } n \geq 1\}$ . For  $\xi = \{x_n\}$  and  $\eta = \{y_n\}$  in  $X$  define

$$(7) \quad d(\xi, \eta) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)}$$

then  $(\prod_{n=1}^{\infty} X_n, d)$ , where  $d$  is defined by (7), is a metric space. If  $\xi^k = \{x_n^k\}_{n=1}^{\infty}$  is in  $X = \prod_{n=1}^{\infty} X_n$  then  $\xi^k \rightarrow \xi = \{x_n\}$  if and only if  $x_n^k \rightarrow x_n$  for each  $n$ . Also, if each  $(X_n, d_n)$  is compact then  $X$  is compact.

**Proof:-** To show  $(\prod_{n=1}^{\infty} X_n, d)$  is a metric space we have to show that each axioms are satisfied.

i) Since  $d_n(x_n, y_n) \geq 0, n \geq 1$  therefore

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)} = d(\xi, \eta) \geq 0$$

ii) Let  $d(\xi, \eta) = 0$  then

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)} = 0,$$

this gives that  $d_n(x_n, y_n) = 0$ . So this implies that  $x_n = y_n$ ,  $n \geq 1$ , that is,  $\xi = \eta$ .

Conversely, suppose  $\xi = \eta$  then  $x_n = y_n$ ,  $n \geq 1$ . Then

$$d(\xi, \eta) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{d_n(x_n, x_n)}{1 + d_n(x_n, x_n)} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{0}{1 + 0} = 0$$

$$\text{iii) } d(\xi, \eta) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{d_n(y_n, x_n)}{1 + d_n(y_n, x_n)} = d(\eta, \xi)$$

iv) Since  $d_n$  is a metric for all  $n \geq 1$  therefore  $d_n(w_n, y_n) \leq d_n(w_n, x_n) + d_n(x_n, y_n)$ ,  $n \geq 1$

$$\begin{aligned} d(\omega, \eta) &= \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{d_n(w_n, y_n)}{1 + d_n(w_n, y_n)} \\ &\leq \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{d_n(w_n, x_n) + d_n(x_n, y_n)}{1 + d_n(w_n, x_n) + d_n(x_n, y_n)} \\ &\leq \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{d_n(w_n, x_n)}{1 + d_n(w_n, x_n)} + \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)} \\ &= d(\omega, \xi) + d(\xi, \eta) \end{aligned}$$

Hence,  $(\prod_{n=1}^{\infty} X_n, d)$  is metric space.

Suppose  $d(\xi^k, \xi) \rightarrow 0$ ; since

$$\frac{d_n(x_n^k, x_n)}{1 + d_n(x_n^k, x_n)} \leq 2^n d(\xi^k, \xi)$$

To see this

$$d(\xi^k, \xi) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{d_n(x_n^k, x_n)}{1 + d_n(x_n^k, x_n)} \text{ then } \left(\frac{1}{2}\right)^n \frac{d_n(x_n^k, x_n)}{1 + d_n(x_n^k, x_n)} \leq d(\xi^k, \xi)$$

for  $n \geq 1$

and this implies

$$\frac{d_n(x_n^k, x_n)}{1 + d_n(x_n^k, x_n)} \leq 2^n d(\xi^k, \xi)$$

we have that

$$\lim_{k \rightarrow \infty} \frac{d_n(x_n^k, x_n)}{1 + d_n(x_n^k, x_n)} = 0$$

This gives that  $x_n^k \rightarrow x_n$  for each  $n \geq 1$ .

Conversely, suppose  $x_n^k \rightarrow x_n$  for each  $n \geq 1$ . Then,

$$\frac{d_n(x_n^k, x_n)}{1 + d_n(x_n^k, x_n)} \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Thus,

$$d(\xi^k, \xi) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{d_n(x_n^k, x_n)}{1 + d_n(x_n^k, x_n)} \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Hence,  $\xi^k \rightarrow \xi$ .

Now suppose that each  $(X_n, d_n)$  is compact. To show that  $(X, d)$  is compact it suffices to show that every sequence in  $X$  has a convergent subsequence; this is accomplished by the **Cantor diagonalization process**.<sup>1</sup>

Let  $\xi^k = \{x_n^k\} \in X$  for each  $k \geq 1$  and consider the sequence of the first entries of the  $\xi^k$ ; that is, consider  $\{x_1^k\}_{k=1}^{\infty} \subset X_1$ . Since  $X_1$  is compact there is a point  $x_1$  in  $X_1$  and a

<sup>1</sup> Cantor diagonalization process is a technique of proving statements about infinite sequences, each of whose terms is an infinite sequence by operation on the  $n^{\text{th}}$  term of the  $n^{\text{th}}$  sequence for each  $n$ ; used to prove the uncountability of the real numbers

subsequence of  $\{x_1^k\}$  which converges to it. We are now faced with a problem in notation. If this subsequence of  $\{x_1^k\}_{k=1}^\infty$  is denoted by  $\{x_1^{k_j}\}_{j=1}^\infty$  there is little confusion at this stage. However, the next step in the proof is to consider the corresponding subsequence of second entries  $\{x_2^{k_j}\}_{j=1}^\infty$  and take a subsequence of this. Furthermore, it is necessary to continue this process for all the entries. It is easy to see that this is opening up a notational **Pandora's Box**.<sup>2</sup> However, there is an alternative. Denote the convergent subsequence of  $\{x_1^k\}$  by  $\{x_1^k : k \in \mathbb{N}_1\}$ , where  $\mathbb{N}_1$ , is an infinite subset of the positive integers  $\mathbb{N}$ . Consider the sequence of second entries of  $\{x_1^k : k \in \mathbb{N}_1\}$ . Then there is a point  $x_2$  in  $X_2$  and an infinite subset  $\mathbb{N}_2 \subset \mathbb{N}_1$  such that

$\lim \{x_2^k : k \in \mathbb{N}_2\} = x_2$ . (Notice that we still have  $\lim \{x_1^k : k \in \mathbb{N}_2\} = x_1$ .) Continuing this process gives a decreasing sequence of infinite subsets of  $\mathbb{N}$ ,  $\mathbb{N}_1 \supset \mathbb{N}_2 \dots$ ; and points  $x_n$  in  $X_n$  such that

$$(8) \quad \lim \{x_n^k : k \in \mathbb{N}_n\} = x_n$$

Let  $k_j$  be the  $j^{\text{th}}$  integer in  $\mathbb{N}_j$ ; and consider  $\{x_n^{k_j}\}$ ; we claim that  $x_n^{k_j} \rightarrow x_n$  as  $j \rightarrow \infty$ . To show this it suffices to show that

$$(9) \quad x_n = \lim_{k_j \rightarrow \infty} x_n^{k_j}$$

for each  $n \geq 1$ . But since  $\mathbb{N}_j \subset \mathbb{N}_n$  for  $j \geq n$ ,  $\{x_n^{k_j} : j \geq n\}$  is a subsequence of  $\{x_n^k : k \in \mathbb{N}_n\}$ . So, (9) follows from (8). ■

A set  $\mathcal{F} \subset C(G, \Omega)$  is *equicontinuous at a point*  $z_0$  in  $G$  if for every  $\epsilon > 0$  there is a  $\delta > 0$  such that for  $|z - z_0| < \delta$ ,

$$d(f(z), f(z_0)) < \epsilon$$

for every  $f$  in  $\mathcal{F}$ .  $\mathcal{F}$  is *equicontinuous over a set*  $K \subset G$  if for every  $\epsilon > 0$  there is a  $\delta > 0$  such that for  $z$  and  $z'$  in  $K$  and  $|z - z'| < \delta$ ,

$$d(f(z), f(z')) < \epsilon$$

<sup>2</sup> Pandora's Box – a complex situation with problems and pitfalls

for all  $f$  in  $\mathcal{F}$ .

For the purpose of this note, the most significant feature of equicontinuity is that it bridges the gap between point wise convergence and normal convergence.

**Lemma 4.** *Suppose  $\mathcal{F} \subset C(G, \Omega)$  is equicontinuous at each point of  $G$ ; then  $\mathcal{F}$  is equicontinuous over each compact subset of  $G$ .*

**Proof:-** Let  $E \subset G$  be compact and fix  $\epsilon > 0$ . Then for each  $w$  in  $E$  there is a  $\delta_w > 0$  such that

$$d(f(w'), f(w)) < \frac{1}{2}\epsilon$$

for all  $f$  in  $\mathcal{F}$  whenever  $|w - w'| < \delta_w$ . Now  $\{B(w; \delta_w) : w \in E\}$  forms an open cover of  $E$ ; by Lebesgue's Covering Lemma(II4.8) <sup>[1]</sup> there is a  $\delta > 0$  such that for each  $z$  in  $E$ ,  $B(z; \delta)$  is contained in one of the sets of this cover. So if  $z$  and  $z'$  are in  $E$  and  $|z - z'| < \delta$  there is a  $w$  in  $E$  with  $z' \in B(z; \delta) \subset B(w; \delta_w)$ . That is,  $|z - w| < \delta_w$  and  $|z - z'| < \delta_w$ . This gives  $d(f(z), f(w)) < \frac{1}{2}\epsilon$  and  $d(f(z'), f(w)) < \frac{1}{2}\epsilon$ ; so that

$$\begin{aligned} d(f(z), f(z')) &\leq d(f(z), f(w)) + d(f(w), f(z')) \\ &< \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon \end{aligned}$$

and  $\mathcal{F}$  is equicontinuous over  $E$ . ■

**Theorem 7. [ Arzela-Ascoli Theorem ]** *A set  $\mathcal{F} \subset C(G, \Omega)$  is normal if and only if the following two conditions are satisfied:*

(a) *for each  $z$  in  $G$ ,  $\{f(z) : f \in \mathcal{F}\}$  has compact closure in  $\Omega$ ;*

(b)  *$\mathcal{F}$  is equicontinuous at each point of  $G$ .*

**Proof:** - First assume that  $\mathcal{F}$  is normal. Notice that for each  $z$  in  $G$  the map of  $C(G, \Omega) \rightarrow \Omega$  defined by  $f \mapsto f(z)$  is continuous; since  $\mathcal{F}$  is compact its image is compact in  $\Omega$  and (a) follows. To show (b) fix a point  $z_0$  in  $G$  and let  $\epsilon > 0$ . If  $R > 0$  is chosen so

that  $E = \bar{B}(z_0; R) \subset G$  then  $E$  is compact and Theorem 5 implies there are functions  $f_1, \dots, f_n$  in  $\mathcal{F}$  such that for each  $f$  in  $\mathcal{F}$  there is at least one  $f_k$  with

$$(10) \quad \sup \{d(f(z), f_k(z)): z \in E\} < \frac{\epsilon}{3}.$$

But since each  $f_k$  is continuous there is a  $\delta$ ,  $0 < \delta < R$ , such that  $|z - z_0| < \delta$  implies that

$$d(f_k(z), f_k(z_0)) < \frac{\epsilon}{3}$$

for  $1 \leq k \leq n$ . Therefore, if  $|z - z_0| < \delta$ ,  $f \in \mathcal{F}$ , and  $k$  is chosen so that (10) holds, then

$$\begin{aligned} d(f(z), f(z_0)) &\leq d(f(z), f_k(z)) + d(f_k(z), f_k(z_0)) + d(f_k(z_0), f(z_0)) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

That is,  $\mathcal{F}$  is equicontinuous at  $z_0$ .

Now suppose  $\mathcal{F}$  satisfies conditions (a) and (b); it must be shown that  $\mathcal{F}$  is normal. Let  $\{z_n\}$  be the sequence of all points in  $G$  with rational real and imaginary parts (so for  $z$  in  $G$  and  $\delta > 0$  there is a  $z_n$  with  $|z - z_n| < \delta$ ). For each  $n \geq 1$  let

$$X_n = \{f(z_n): f \in \mathcal{F}\} \subset \Omega;$$

from part (a),  $(X_n, d)$  is a compact metric space. Thus, by Theorem 6,  $X = \prod_{n=1}^{\infty} X_n$  is a compact metric space. For  $f$  in  $\mathcal{F}$  define  $\tilde{f}$  in  $X$  by

$$\tilde{f} = \{f(z_1), f(z_2), \dots\}.$$

Let  $\{f_k\}$  be a sequence in  $\mathcal{F}$  so  $\{\tilde{f}_k\}$  is a sequence in the compact metric space  $X$ . Thus there is a  $\xi$  in  $X$  and a subsequence of  $\{\tilde{f}_k\}$  which converges to  $\xi$ . For the sake of convenient notation, assume that  $\xi = \lim_{k \rightarrow \infty} \tilde{f}_k$ . Again from Theorem 6,

$$(11) \quad \lim_{k \rightarrow \infty} f_k(z_n) = \{w_n\}$$

where  $\xi = \{w_n\}$ .

It will be shown that  $\{f_k\}$  converges to a function  $f$  in  $C(G, \Omega)$ . By (11) this function  $f$  will have to satisfy  $f(z_n) = w_n$ . The importance of (11) is that it imposes control over the behavior of  $\{f_k\}$  on a dense subset of  $G$ . We will use the fact that  $\{f_k\}$  is equicontinuous to spread this control to the rest of  $G$ .

To find the function  $f$  and show that  $\{f_k\}$  converges to  $f$  it suffices to show that  $\{f_k\}$  is a Cauchy sequence. So let  $E$  be compact set in  $G$  and let  $\epsilon > 0$ ; by Theorem 3 it suffices to find an integer  $J$  such that for  $k, j > J$ ,

$$(12) \quad \sup \left\{ d(f_k(z), f_j(z)) : z \in E \right\} < \epsilon$$

Since  $E$  is compact  $R = d(E, \partial G) > 0$ . Let  $E_I = \{z : d(z, E) \leq \frac{1}{2}R\}$ ; then  $E_I$  is compact and  $E \subset \text{int } E_I \subset E_I \subset G$ . Since  $\mathcal{F}$  is equicontinuous at each point of  $G$  it is equicontinuous on  $K_I$  by Lemma 4. So choose  $\delta$ ,  $0 < \delta < \frac{1}{2}R$ , such that

$$(13) \quad d(f(z), f(z')) < \frac{\epsilon}{3}$$

for all  $f$  in  $\mathcal{F}$  whenever  $z$  and  $z'$  are in  $E_I$  with  $|z - z'| < \delta$ . Now let  $D$  be the collection of points in  $\{z_n\}$  which are also points in  $E_I$ ; that is

$$D = \{z_n : z_n \in E_I\}$$

If  $z \in E$  then there is a  $z_n$  with  $|z - z_n| < \delta$ ; but  $\delta < \frac{1}{2}R$  gives that  $d(z_n, E) < \frac{1}{2}R$ , or that  $z_n \in E_I$ . Hence  $\{B(w; \delta) : w \in D\}$  is an open cover of  $E$ . Let  $w_1, \dots, w_n \in D$  such that

$$E \subset \bigcup_{i=1}^n B(w_i; \delta)$$

Since  $\lim_{k \rightarrow \infty} f_k(w_i)$  exists for  $1 \leq i \leq n$  (by (11) there is an integer  $J$  such that  $j, k \geq J$

$$(14) \quad d(f_k(w_i), f_j(w_i)) < \frac{\epsilon}{3}$$

for  $i = 1, \dots, n$ .

Let  $z$  be an arbitrary point in  $E$  and let  $w_i$  be such that  $|w_i - z| < \delta$ . If  $k$  and  $i$  are larger than  $J$  then (13) and (14) give

$$d(f_k(z), f_j(z)) \leq d(f_k(z), f_k(w_i)) + d(f_k(w_i), f_j(w_i)) + d(f_j(w_i), f_j(z))$$

$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

Since  $z$  was arbitrary this establishes (13). ■

## 2.2. SPACES OF ANALYTIC FUNCTIONS

The class of analytic functions in  $G$  will be denoted by  $H(G)$  where  $G$  is an open subset of  $\mathbb{C}$ . In the following theorem we will prove that  $H(G)$  is closed in  $C(G, \mathbb{C})$ .

**Theorem 8.** *Let  $\{f_n\}$  be a sequence in  $H(G)$ . Suppose that  $f$  belongs to  $C(G, \mathbb{C})$  such that  $f_n \rightarrow f$ . Then  $f$  is analytic and*

$$f_n^{(k)} \rightarrow f^{(k)} \text{ for each integer } k \geq 1.$$

**Proof:** - We will show that  $f$  is analytic by applying Morera's Theorem (IV, 5.10)<sup>[1]</sup>. Consider a disk  $D \subset G$  and take a triangle  $T$  inside  $D$ . Since  $T$  is compact,  $\{f_n\}$  converges to  $f$  uniformly over  $T$ .

Hence

$$\lim \int_T f_n = \int_T f$$

But, Since each  $f_n$  is analytic, we have

$$\lim \int_T f_n = \int_T f = 0$$

Thus  $f$  must be analytic in every disk  $D \subset G$  and this gives that  $f$  is analytic in  $G$ .

To prove that  $f_n^{(k)} \rightarrow f^{(k)}$ , consider  $D = \bar{B}(a; r) \subset G$ . Then there is a number  $R > r$  such that  $\bar{B}(a; R) \subset G$ . Let  $\gamma$  is the circle  $|z - a| = R$  then Cauchy's Integral Formula we have for  $z \in D$ ,

$$f_n^{(k)}(z) - f^{(k)}(z) = \frac{k!}{2\pi i} \int_{\gamma} \frac{f_n(w) - f(w)}{(w - z)^{k+1}} dw$$

Hence, using Cauchy's Estimate,

$$(15) \quad |f_n^{(k)}(z) - f^{(k)}(z)| \leq \frac{k! M_n R}{(R - r)^{k+1}},$$

where  $M_n = \sup \{|f_n(w) - f(w)| : |w - a| = R\}$  and  $|z - a| \leq r$ . Since  $f_n \rightarrow f$ ,  $\lim M_n = 0$ . Hence, it follows from (15) that

$$f_n^{(k)} \rightarrow f^{(k)} \text{ uniformly on } \bar{B}(a; r).$$

Now let  $E$  be an arbitrary compact subset of  $G$  and let  $0 < r < d(E, \partial G)$ . Then there are  $a_1, \dots, a_n$  in  $E$  such that  $E \subset \bigcup_{j=1}^n B(a_j; r)$ . Since  $f_n^{(k)} \rightarrow f^{(k)}$  uniformly on each  $B(a_j; r)$ , the convergence is uniform on  $E$ . Hence the theorem is proved. ■

Observe that we have considered  $H(G)$  as a subset of  $C(G, \mathbb{C})$  and the metric on  $H(G)$  is the metric which it inherits from  $C(G, \mathbb{C})$ .

Since  $C(G, \mathbb{C})$  is complete metric space we have the following results.

**Corollary 2.**  $H(G)$  is a complete metric space.

**Corollary 3.** Let  $f_n: G \rightarrow \mathbb{C}$  is analytic. If  $\sum_{n=1}^{\infty} f_n(z)$  converges uniformly on compact sets to  $f(z)$  then for a positive integer  $k$ ,

$$f^{(k)}(z) = \sum_{n=1}^{\infty} f_n^{(k)}(z).$$

Note that the above result has no analogue in the theory of functions of real variable. For example, Let  $f_n(x) = \frac{1}{n}x^n$  for  $0 \leq x \leq 1$ . Then  $0 = u - \lim f_n$ ; however the sequence of derivatives  $\{f'_n\}$  does not converge uniformly on  $[0, 1]$ .

**Theorem 9 [Hurwitz's Theorem]** Let  $G$  be a region and suppose the sequence  $\{f_n\}$  in  $H(G)$  converges to  $f$ . If  $f \not\equiv 0$ ,  $\bar{B}(a; R) \subset G$ , and  $f(z) \neq 0$  for  $|z - a| = R$  then there is an integer  $N$  such that for  $n \geq N$ ,  $f$  and  $f_n$  have the same number of zeros in  $B(a; R)$ .

**Proof:** - Since  $f(z) \neq 0$  for  $|z - a| = R$ ,

$$\delta = \inf\{|f(z)| : |z - a| = R\} > 0.$$

But  $f_n \rightarrow f$  uniformly on  $\{z : |z - a| = R\}$  so there is an integer  $N$  such that if  $n \geq N$  and  $|z - a| = R$  then

$$|f(z) - f_n(z)| < \frac{1}{2}\delta < |f(z)| \leq |f(z)| + |f_n(z)|.$$

Hence Rouché's Theorem (V.3.8)<sup>[1]</sup> implies that  $f$  and  $f_n$  have the same number of zeros in  $B(a; R)$ . ■

**Corollary 4.** If  $\{f_n\} \subset H(G)$  converges to  $f$  in  $H(G)$  and each  $f_n$  never vanishes on  $G$  then either  $f \equiv 0$  or  $f$  never vanishes.

**Proof:** - Assume that there is a  $w \in G$  with  $f(w) = 0$  and that  $f$  is not identically zero. Then there exists a circle  $\gamma$  with center  $w$  such that closure of interior of the circle  $\gamma$  ( $\text{cl int}(\gamma)$ ) is subset of  $G$  and  $f(z) \neq 0$  for all  $z \in \text{cl int}(\gamma) - \{w\}$ .

Therefore the numbers  $N$  of zeros of  $f$  in  $\text{int}(\gamma)$  satisfies

$$N = \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz \geq 1$$

But, since  $f_n(z) \neq 0$  for all  $z$  in  $G$   $\frac{f'_n(z)}{f_n(z)}$  is analytic in  $G$

$$\int_{\gamma} \frac{f'(z)}{f(z)} dz = \lim_{n \rightarrow \infty} \int_{\gamma} \frac{f'_n(z)}{f_n(z)} = 0.$$

Hence it contradicts our assumption. Therefore the required is satisfied. ■

A set  $\mathcal{F} \subset H(G)$  is **uniformly bounded** on a set  $E$  if there exists a real number  $M$  such that

$$|f(z)| \leq M \quad \text{for all } f \in \mathcal{F} \text{ and all } z \in E.$$

**Remarks:-** Certainly the uniform boundedness of a family implies that each member of the family is bounded. On the other hand, each member of the sequence  $\{f_n(z)\}$  of functions  $f_n(z) = nz$  is bounded in the disk  $|z| \leq R$ , but there is no bound that works for every member of the family.

A set  $\mathcal{F} \subset H(G)$  is **locally bounded** if for each point  $a$  in  $G$  there are constants  $M$  and  $r > 0$  such that for all  $f$  in  $\mathcal{F}$ ,

$$|f(z)| \leq M, \quad \text{for } |z - a| < r.$$

Alternately,  $\mathcal{F}$  is locally bounded if there is an  $r > 0$  such that

$$\sup\{|f(z)| : |z - a| < r, f \in \mathcal{F}\} < \infty.$$

**Remarks:-** The sequence  $f_n(z) = \frac{1}{1-z^n}$  is locally bounded, but not uniformly bounded in the disk  $|z| < 1$ .

**Lemma 5.** A set  $\mathcal{F}$  in  $H(G)$  is locally bounded if and only if  $\mathcal{F}$  is uniformly bounded on each compact subset  $G$ .

**Proof:-** Let  $\mathcal{F}$  be locally bounded and suppose  $E$  is a compact subset of  $G$ . For each point in  $E$ , choose a neighborhood in which  $\mathcal{F}$  is uniformly bounded. This provides an open cover for  $E$ . According to the Heine - Borel Theorem, there exists a finite sub cover of  $E$ . That is, there are finitely many  $z_i \in E$  and  $\epsilon_i > 0$  such that

$$E \subset \bigcup_{i=1}^n B(z_i, \epsilon_i), \quad \text{where } |f(z)| \leq M_i \text{ for all } f \in \mathcal{F} \text{ and all } z \in B(z_i; \epsilon_i).$$

Then  $\mathcal{F}$  is uniformly bounded on  $E$ , having for a bound  $M = \max\{M_1, M_2, \dots, M_n\}$ .

The converse is immediate from the fact that the closure of a neighborhood of a point is a compact set. ■

**Theorem 10[Montel's Theorem]** *A family  $\mathcal{F}$  in  $H(G)$  is normal if and only if  $\mathcal{F}$  is locally bounded.*

**Proof:-** Suppose  $\mathcal{F}$  is normal but fails to be locally bounded; then there is a compact set  $E \subset G$  such that  $\sup\{|f(z)|: z \in E, f \in \mathcal{F}\} = \infty$ . That is, there is a sequence  $\{f_n\}$  in  $\mathcal{F}$  such that for a positive integer  $n$ ,  $\sup\{|f_n(z)|: z \in E\} \geq n$ . Since  $\mathcal{F}$  is normal there is a function  $f$  in  $H(G)$  and a subsequence  $\{f_{n_k}\}$  such that  $f_{n_k} \rightarrow f$ .

But this gives that

$$\sup\{|f_{n_k}(z) - f(z)|: z \in E\} \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

If  $|f(z)| \leq M$  for  $z$  in  $E$ ,

$$n_k \leq \sup\{|f_{n_k}(z) - f(z)|: z \in E\} + M;$$

since the right hand side converges to  $M$ , this is a contradiction.

Now suppose  $\mathcal{F}$  is locally bounded; the Arzela - Ascoli Theorem will be used to show that  $\mathcal{F}$  is normal. Since condition (a) of Theorem 7 is clearly satisfied, we must show that  $\mathcal{F}$  is equicontinuous at each point of  $G$ . Fix a point  $a$  in  $G$  and  $\epsilon > 0$ ; from the hypothesis there is an  $r > 0$  and  $M > 0$  such that  $\bar{B}(a; r) \subset G$  and  $|f(z)| \leq M$  for all  $z$  in  $\bar{B}(a; r)$  and for all  $f$  in  $\mathcal{F}$ . Let  $|z - a| < \frac{1}{2}r$  and  $f \in \mathcal{F}$ ; then using Cauchy's Formula with  $\gamma(t) = a + re^{it}$ ,  $0 \leq t \leq 2\pi$ ,

$$\begin{aligned} |f(a) - f(z)| &= \frac{1}{2\pi} \left| \int_{\gamma} \frac{f(w)(a-z)}{(w-a)(w-z)} dw \right| \\ &\leq \frac{4M}{r} |a-z| \end{aligned}$$

Letting  $\delta < \min \left\{ \frac{1}{2}r, \frac{r}{4M}\epsilon \right\}$  it follows that  $|a - z| < \delta$  gives  $|f(a) - f(z)| < \epsilon$  for all  $f$  in  $\mathcal{F}$ . ■

### Some Applications of Normal Families

One can use Montel's Theorem to prove many useful results. I present two of these here.

1. A set  $\mathcal{F} \subset H(G)$  is compact if and only if it is closed and locally bounded.

**Proof:-** If  $\mathcal{F}$  is compact, then  $\mathcal{F}$  is closed (a general property that holds in any metric space). In order to show that  $\mathcal{F}$  is locally bounded, we will use the following device. Let  $E$  be any compact subset of  $G$ . Then  $f \rightarrow \sup \{|f(z)| : z \in E\}$  is a continuous map from  $H(G)$  into  $\mathbb{R}$ . Thus,  $\sup\{|f(z)| : z \in E\}$  for each  $f \in \mathcal{F}$  is a compact subset of  $\mathbb{R}$ . Then  $\mathcal{F}$  is uniformly bounded for each  $z$  in  $E$ . Hence, by lemma 5,  $\mathcal{F}$  is locally bounded. Conversely, if  $\mathcal{F}$  is closed and locally bounded, then  $\mathcal{F}$  is closed and, by Montel's theorem, normal. Therefore, by Lemma 3,  $\mathcal{F}$  is compact.

**Example:-** Let  $G$  be open and  $E = \bar{B}(a; R) \subseteq G$ . Define  $\mathcal{F}$  to be the set of all  $f \in C(G)$  such that  $|f(z)| \leq 1$  for all  $z \in G$  and  $f(z) = 0$  for  $z \in G - E$ . Show that  $\mathcal{F}$  is a closed and bounded subset of  $C(G)$ , but  $\mathcal{F}$  is not compact.

**Solution:-** Let  $f \rightarrow T(f)$  be the suggested map. Since  $|f| \leq 1$  on  $G$  and  $f = 0$  on the boundary of  $E$ , the integral over  $E$  is greater than 0 and  $T$  is well defined. If  $f_n \in \mathcal{F}$  and  $f_n \rightarrow f$ , that is,  $d(f_n, f) \rightarrow 0$ , then  $f_n \rightarrow f$  uniformly on  $E$ , hence  $T(f_n) \rightarrow T(f)$ , so that  $T$  is continuous. If  $\mathcal{F}$  were compact, then  $T(\mathcal{F})$  would be a compact, hence bounded, subset of the reals.

If  $0 < r < R$ , let  $f$  be a continuous function from  $G$  to  $[0, 1]$  such that  $f = 1$  on  $\bar{D} = \bar{B}(a, r)$  and  $f = 0$  off  $E$  (Urysohn's lemma).<sup>[4]</sup> Then

$$\int_E \int |f(x + iy)| dx dy \geq \int_{\bar{D}} \int 1 dx dy \rightarrow \int_E \int 1 dx dy$$

as  $r \rightarrow R$ . Thus  $T(\mathcal{F})$  is unbounded, a contradiction.

**2. Vitali's Theorem** If  $G$  is a region and  $\{f_n\} \subset H(G)$  is locally bounded and  $f \in H(G)$  that has the property that  $A = \{z \in G : \lim f_n(z) = f(z)\}$  has a limit point in  $G$  then  $f_n \rightarrow f$ .

**Proof:-** The sequence  $\{f_n\}$  is a locally bounded sequence in  $H(G)$  and by Montel's Theorem then  $\{f_n\}$  is normal, so there is a subsequence  $\{f_{n_k}\} \subset \{f_n\}$  that converges to  $f$  in  $H(G)$ .

Suppose that  $f_n \not\rightarrow f$  in  $H(G)$  then there must be a compact set  $E \subset G$  the convergence is not uniform on  $E$ . In other words there must be an  $\varepsilon > 0$  so that for all  $n \in \mathbb{N}$  there is  $z_n \in E$  with  $|f_n(z_n) - f(z_n)| \geq \varepsilon$ . By compactness of  $E$  extract a subsequence of  $\{z_n\}$ , say  $\{z_{n_m}\}$  that converges to a point  $z_0 \in E$ .

The sequence  $\{f_n\}$  is locally bounded, so in particular the subsequence  $\{f_{n_m}\}$  with indices corresponding to  $\{z_{n_m}\}$  is, and again by Montel's Theorem now applied to the subsequence and again the completeness of  $H(G)$  there is an analytic function  $g$  such that  $f_{n_m} \rightarrow g$  in  $H(G)$ . On the set  $A$  of points of point wise convergence  $f_n(z) \rightarrow f$  and  $f_{n_m}(z) \rightarrow g$ .

Now  $G$  is a region and  $A$  has a limit point in  $G$  so by the Identity Theorem (Corollary IV.3.8)<sup>[1]</sup> already  $f = g$  on  $G$  which gives a contradiction on the set  $E$  and the point  $z_0$  because

$$|f_{n_m}(z_{n_m}) - f(z_{n_m})| \rightarrow |g(z_0) - f(z_0)| \geq \varepsilon.$$

Hence we can conclude that  $f_n \rightarrow f$  in  $H(G)$ .

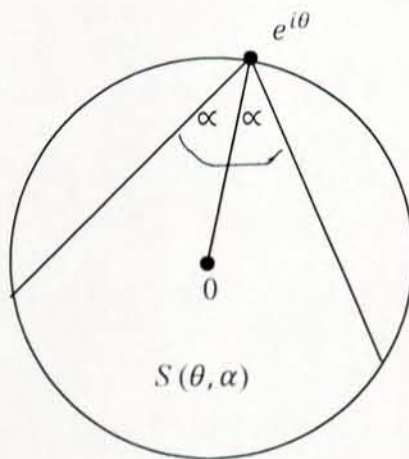
**Example:-** Let  $f$  be a locally bounded analytic function on  $D(0, 1)$  with the property that for some  $\theta$ ,  $f(re^{i\theta})$  approaches a limit  $L$  as  $r \rightarrow 1^-$ . Fix  $\alpha \in (0, \frac{\pi}{2})$  and consider the region  $S(\theta, \alpha)$  in Figure. Prove that if  $z \in S(\theta, \alpha)$  and  $z \rightarrow e^{i\theta}$ , then  $f(z) \rightarrow L$ .

**Solution:-** If  $z$  is a point on the open radial line  $S$  from 0 to  $e^{i\theta}$ , then

$$e^{i\theta} + \left(\frac{1}{n}\right)(z - e^{i\theta}) = \left(1 - \frac{1}{n}\right)e^{i\theta} + \left(\frac{1}{n}\right)z$$

also lies on  $S$ , and approaches  $e^{i\theta}$  as  $n \rightarrow \infty$ .

By hypothesis,  $f_n$  converges point wise on  $S$ . Since  $S$  certainly has a limit point in  $S(\theta, \alpha)$ , Vitali's theorem implies that  $f_n$  converges uniformly on compact subsets. Given  $\epsilon > 0$  there exists  $\delta > 0$  such that if  $z \in S(\theta, \alpha)$  and  $|z - e^{i\theta}| < \delta$ , then  $|z - w| < \epsilon$  for some  $w \in S$ . It follows that by choosing  $z$  sufficiently close to  $e^{i\theta}$ , we can make  $f(z)$  as close as we wish to  $L$ , as desired.



Figure

3. Prove that in any region  $G$  the family of analytic functions with positive real part is normal. Under what conditions is it locally bounded?

**Proof:-** Take  $e^{-f}$  and if  $\operatorname{Re} f > 0$

$$|e^{-f}| = e^{-\operatorname{Re} f} \leq 1.$$

Then the family is locally bounded.

Hence, by Montel's Theorem the family is normal.

4. If  $\mathcal{F}$  is a family of analytic functions, which is not normal in  $G$ , show that there is a point  $z_0 \in G$  such that  $\mathcal{F}$  is not normal in any neighborhood of  $z_0$ .

**Proof:-** Applying contra positive of Lemma 5 and Montel's Theorem.

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