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ON THE GENERAL ECCENTRIC CONNECTIVITY INDEX OF GRAPHS

MSC THESIS BY:
Hana Adugna

RESEARCH SUPERVISOR: **Mesfin Masre(PhD)**

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Certificate

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As a member of the Board of Examiners of the MSc Thesis-Open Defense Examination, this is to certify that the MSc Graduate thesis prepared by Hana Adugna, entitled: **On the general eccentric connectivity index of graphs** and submitted in fulfillment of the requirements for the Degree of Master of Science in Mathematics (Combinatory).

Date:____________________

Signed by the Examining Committee:

Supervisor:

Mesfin Masre (PhD), Signature:________________

(Department of Mathematics, Addis Ababa University).

Internal Examiner:

1. Name: ____________________________,Signature:________________
   (Department of Mathematics, Addis Ababa University).
2. Name: ____________________________,Signature:________________
   (Department of mathematics, Addis Ababa University).

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Declaration

I declare and attest that my research paper is original with no submissions to other academic institutions, certification reviews, or postgraduate degree completions.

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Hana Adugna

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Abstract

The general eccentric connectivity index of graphs is the main topic of this study. For a connected graph G, the general eccentric connectivity index of graph G is defined by

$$ECI_a(G) = \sum_{v \in V(G)} ecc_G(v) d_G^a(v)$$

for $a \in \mathbb{R}$, where the degree of v in G is $d_G(v)$, the eccentricity of vertex v is $ecc_G(v)$, and the vertex set of G is $V(G)$.

In this thesis, we study the general degree-eccentricity index of graphs. Among all the unicyclic graphs of a particular order and matching number, we identify the unicyclic graphs with the largest and smallest general eccentric connectivity index.

Chapter 1

Introduction and Preliminaries

This chapter presents a mathematical model called topological indices and outlines the standard ideas from graph theory that will be used in the thesis. Next, we provide some key background results and explain the motivation for our investigation. The following chapters will define the terms not described in this chapter as needed.

1.1 Graph theory terminologies

This section provides an introduction to certain graph properties and definitions [1] [33] [41] [32] [43] [45] [37].

Definition of Graph A graph G is an ordered pair $(V(G), E(G))$ consisting of a set $V(G)$ of vertices and a set $E(G)$, disjoint from $V(G)$, of edges, together with an incidence function ψ_G that associates with each edge of G an unordered pair of (not necessarily distinct) vertices of G. If e is an edge and u and v are vertices such that $\psi_G(e) = u, v$, then e is said to join u and v, and the vertices u and v are called the ends of e. We denote the numbers of vertices and edges in G by $v(G)$ and $e(G)$; these two basic parameters are called the order and size of G, respectively.

Finite and Simple graph A graph is finite if both its vertex set and edge set are finite. A graph is simple if it has no loops and no two of its links join the same pair of vertices.

In this thesis, we mainly study simple and finite graphs, and the term ‘graph’ always means ‘simple graph’ and ‘finite graph’ which is denoted by G.

Isomorphic graphs Two graphs G and H are said to be isomorphic (written $G \cong H$) if there are bijections $\theta : V(G) \to V(H)$ and $\phi : E(G) \to E(H)$ such that $\psi_G(e) = uv$ if and only if $\psi_H(\phi(e)) = \theta(u)\theta(v)$; such a pair (θ, ϕ) of mappings is called an isomorphism between G and H.

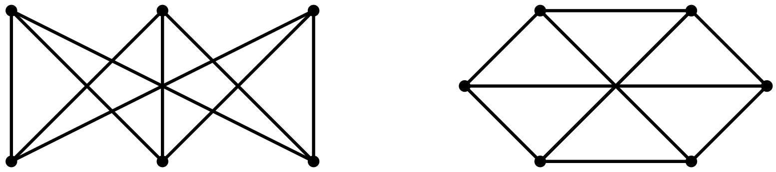

Figure 1.1: Isomorphic graphs

Neighborhood of a vertex Each edge in E(G) connects two vertices from V (G), called the ends or endpoints of this edge. For an edge e with endpoints u and v, we say that u and v are adjacent and that u and v are incident to e. The set of all vertices adjacent to a specific vertex v is called the (open) neighborhood of v and is denoted by $N_G(v)$ (or just N(v) for short). The set $N[v] = N(v) \cup v$ that also includes v itself is called the closed neighborhood.

Degree of a vertex The degree $d_G(v)$ of a vertex v in G is the number of edges of G incident with v, each loop counting as two edges. We denote by $\delta(G)$ and $\Delta(G)$ the minimum and maximum degrees, respectively, of vertices of G.

A vertex with degree one is referred to as a pendant or leaf, a vertex with degree two or more is referred to as an internal vertex, and a vertex with degree three or more is referred to as a branching vertex.

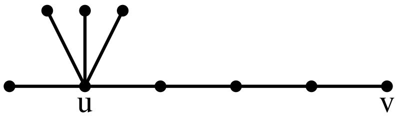

Figure 1.2: Graph Example

Regular graph A graph G is regular if all vertices of G have the same degree. In particular, if the degree of each vertex is r, then G is said to be a r-regular graph.

Walk, trial, path, and cycle A walk of graph G is an alternating sequence of vertices and edges beginning and ending with vertices, in which each line is incident with two vertices immediately proceeding and following it. This walk joins v_0 and v_n. It is sometimes called

$v_0 - v_n$ walk. It is a closed walk if $v_0 = v_n$ and is open otherwise. A walk is called a trial if all the edges in the walk are distinct and a path if all the vertices are distinct.

Cycle graph A walk is closed if it has a positive length and its origin and terminus are the same. A closed trail with a distinct origin and internal vertices is a cycle (denoted as C_n for n vertices). Number of vertices and edges in C_n are same(n = m) and the degree of each vertex is 2.

Complete graph A simple graph G is considered a complete graph if every pair of distinct vertices of G are adjacent in G and denoted by K_n, where n is the order of G.

Bipartite graph A bipartite graph is a graph whose vertex set V can be partitioned into two non-empty and disjoint sets subsets V_1 and V_2 in such a way that each edge of the graph joins a vertex in the first set V_1 to a vertex in the second set V_2.

Complete bipartite graph A complete bipartite graph is a bipartite graph in which each vertex in the first set is joined to each vertex in the second set by exactly one edge. The complete bipartite graph with $n = |V_1|$ and $r = |V_2|$ vertices is denoted by $K_{n,r}$.

Connected graph A graph G is connected if each pair of vertices in G belongs to a path, otherwise, G is disconnected.
Star graph A complete bipartite graph of the form $K_{1,r}$ is called a star graph.

Trees Graph A connected acyclic graph is called a tree. A forest is another name for a graph that is only acyclic and not necessarily connected, it can be thought of as a union of trees.

Two trees of order n will occur particularly frequently: the path P_n is the only tree with only two leaves, and the star S_n is the only tree with n - 1 leaves (Figure 1.3). Intuitively, the path is the most “stretched out” among all trees of the same order, and the star is the most “compact” among all trees of the same order. The path and the star turn out to be the extremal structures in the studies of many topics in chemical graph theory.

When trees with specific constraints are considered, many problems become much more complicated and various special trees need to be defined. An example of this kind is the class of caterpillars: a caterpillar is a tree with the property that a path remains when all leaves are removed; see Figure 1.4.

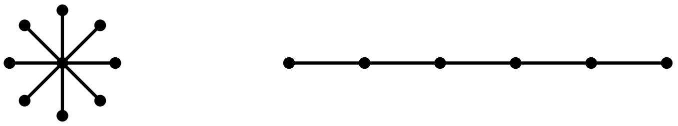

Figure 1.3: Star and path graph

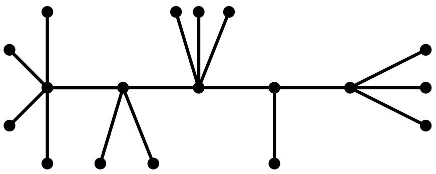

Figure 1.4: Caterpillar Graph

Unicyclic graph A unicyclic graph is a connected graph containing exactly one cycle. The number of edges and vertices of a unicyclic graph are equal.

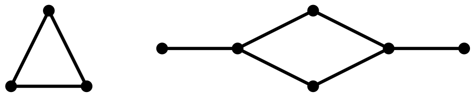

Figure 1.5: Unicyclic Graph

Distance in graph The distance $d_G(u,v)$ between two vertices $u,v \in V(G)$ is the minimum number of edges on a path in G between u and v. If u and v are in different components of G, then $d_G(u,v) = \infty$.

Eccentricity of a vertex The eccentricity of a vertex v in a graph G is the maximum distance from v to any vertex in G. Define as $ecc_G(v) = max_{v_i \in V(G)} d_G(v_i, v_j)$.

If u is a vertex such that $d_G(u,v) = ecc_G(v)$, then u is called an eccentric vertex of v in G. If there is only one such vertex u, then u is called the unique eccentric point v. A vertex in the graph with the minimum eccentricity is called the central vertex and denoted by $C(G)$.

Lemma 1.1. *Let T be a tree with three vertices or more.*

1. $ecc(w) = ecc(v) - 1$, *if v is a leaf of T and u is its neighbor.*
2. $d(v) \geq 2$, *if v is a central vertex of T.*

Corollary 1.1. *Let T be a tree with n-vertices. Then, either a single vertex or a single edge makes up the center $C(G)$.*

Diameter of a graph The diameter of G is the maximum distance between any two vertices in G. Defined as $d(G) = max_{\{u,v\} \subseteq V(G)} d_G(u,v)$.

Girth of a graph where there is a cycle in G containing an edge then the edge is said to be cyclic. The girth $k(e)$ of a cyclic edge e is defined as the length of the shortest cycle containing e; we set $k(e) = \infty$ if e is a bridge. The girth $k(G)$ of G is the size of the shortest cycle in G; we set $k(G) = \infty$ if G is a forest.

Independent set An independent set in a simple graph G is a subset of V(G) in which no two vertices are adjacent to each other. The independence number of G, usually denoted by $\alpha(G)$, is the size of a maximum independent set of G.

Matching in a graph A matching in a simple graph G is a set of edges without common vertices. A maximum matching is a matching that contains the largest possible number of edges. The matching number of G, denoted by $M(G)$, is the size of a maximum matching of G. For any tree of order n and matching number M, we have $1 \leq M \leq \frac{n}{2}$.

Lemma 1.2. *For any given graph G,* $\alpha(G) + M(G) = |V(G)|$.

Lemma 1.3. *[6] Let* $G \in U_{2M,M} (M \geq 3)$, *and T be a tree in G that is rooted at r. If* $v \in V(T)$ *is a vertex furthest from the root r with* $d_G(v,r) \geq 2$, *then v is a pendant vertex adjacent to a vertex u of degree* 2.

Lemma 1.4. *[13]Let U be a unicyclic graph having n vertices, and matching number M other than a cycle, where* $n > 2M$. *Then U contains a maximum matching M and a pendant vertex u, so M does not match u.*

The subsequent Lemma about matching numbers in a tree was proved by Hou and Li[48].

Lemma 1.5. *Let T be any tree of order* $n \geq 3$, *and matching number* M, *where* $n > 2M$. *Then, T has a maximum matching M and a pendant vertex v such that M does not match v.*

1.2 Topological Indices

Topological indices are invariants computed using the topological information within a molecule's graph structure. Topological indices, which frequently graph invariants, are numerical attributes of a graph that describe its composition.

Topological data about a molecule consists of the arrangement and, occasionally, the types of atoms connected by bonds. Such topological descriptors correlate with certain compound properties and activities. Numerous topological indices have been employed in theoretical chemistry, particularly in the quantitative structure-property relationship (QSPR), and quantitative structure-activity relationship (QSAR) investigations, numbering in the hundreds.

The first topological index based on distance, known as the Wiener index, was created by the chemist Harold Wiener (1947) [14] for examining the alkanes' boiling point. The total of all the distances measured between the graph's vertices defines this index. Referring to the distances between a graph's vertices, the Wiener index was initially described as

$$W(G) = \sum_{u,v \in V(G)} d_G(u,v).$$

Subsequently, numerous distance-based topological indices have been examined; these include:

- Hyper-Wiener index: Milan Randi first presented the acyclic graph's hyper-Wiener index in 1993 [35]. Then Klein. [17], extended Randic's idea for all connected graphs, as oversight of the Wiener index. As with the Wiener index, the hyper-Wiener index is conventionally represented by WW. For a graph G, the hyper-Wiener index defined as

$$WW = \frac{1}{2} \sum_{u,v \in V(G)} (d(u,v) + d^2(u,v)).$$

- The Harary index of a graph G, represented by H(G), has been presented by Plavšić et al.[7] and by Ivanciuc et al. [34] in 1993. In honor of Professor Frank Harary, 70^{th} birthday, it has been named by his name. The Hararyindex is defined as follows:

$$H = H(G) = \sum_{u,v \in V(G)} \frac{1}{d_G(u,v)}.$$

- The Gutman index in a graph G, denoted by Gut(G), introduced in 1994 by Gutman [9], it is described as:

$$Gut(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u) d_G(v) d_G(u,v).$$

The first degree-based topological indices with two molecular characterizations are the first and second Zagreb index $M_1(G)$,$M_2(G)$, they initially showed up in the topological formula for conjugated molecules' total $\pi-$electron energy, they were obtained by Gutman in 1972, and Trinajsti[10]. The first and second Zagreb indices are described as follows:

$$M_1(G) = \sum_{v\in V(G)} d_G^2(v) = \sum_{uv\in E(G)} [d_G(u) + d_G(u)], \qquad M_2(G) = \sum_{uv\in E(G)} d_G(u)d_G(v).$$

Additional topological indexes based on the degree that exists are:

- One of the degree-based indices, the Randić index is also referred to as the branching index or the connectivity index. Randic proposed this index [36] in 1975 to measure the degree of branching in the carbon-atom skeleton of saturated hydrocarbons. Additionally, there is a strong association between the Randic index and many alkane physical-chemical characteristics, such as boiling temperatures, energy levels, surface areas, etc. Bollobás and Erdös [5] expanded this index in 1998, naming it the universal Randic index, by substituting the power of any real number for the square root.For a graph G, the Randić index R(G) is defined as

$$R(G) = \sum_{uv\in E(G)} \frac{1}{\sqrt{d_G(u)d_G(v)}}.$$

- The popularity of the Randic index led to several generalizations and modifications, including the sum-connectivity index and the universal sum-connectivity index. Another variant of the Randić index is called the harmonic index H(G), which is defined as

$$H(G) = \sum_{uv\in E(G)} \frac{2}{d_G(u) + d_G(v)}.$$

One of the most investigated topological indices is the eccentric-based topological indices. The distance and adjacency topological descriptor known as the eccentric connectivity index was initially proposed in 1997 by Sharma, Goswami, and Madan [2]. Their correlation coefficients about the eccentric connectivity index for different databases containing graph $G's$ physical attributes varied from 0.95 to 0.99. They claimed that correlations were far higher than the first and most well-known topological index which is the Wiener index. The eccentric connectivity index is defined as

$$\xi(G) = \sum_{u\in V(G)} d_G(u) ecc_G(u)$$

It's been applied to the progression of several mathematical models for predicting biological processes with different characteristics.

The overall eccentricity of graph G, represented by $\xi(G)$, equals the eccentricity of each vertex added together. i.e.,

$$\xi(G) = \sum_{v\in V(G)} ecc_G(v).$$

The connective eccentricity index is a new adjacency-cum-path length-based topological descriptor that was first presented by Gupta, Singh, and Madan in 2000. The predictability of this index about antihypertensive action was examined by the authors using nonpeptide N-benzylimidazole derivatives to assess the index's potential for biological activity prediction. They demonstrated that the index produced better results than Balaban's mean square distance index did for the same values and that the prediction accuracy in the active range was approximately 80%.

For each graph G, the definition of the Connective Eccentricity Index (CEI) is

$$\varepsilon^{ce}(G) = \sum_{v\in V(G)} \frac{d_G(v)}{ecc_G(v)}.$$

Since then, extensive research has been done on the mathematical characteristics of topological indices based on graph eccentricity. A general eccentricity-based index known as the general eccentric connectivity index was first proposed in 2020 by Masre and Vetrík in [42]. The index was investigated for trees and unicyclic graphs. In the same year, Masre and Vetrík in [25] introduced a more general eccentricity-based topological indices, called the general degree-eccentricity index of graphs. They specified the general degree-eccentricity index for a G as

$$DEI_{a,b}(G) = \sum_{v\in V(G)} d_G^a(v) ecc_G^b(v).$$

This index generalized several eccentricity-based topological indices. Some of them are:

- $DEI_{a,1}(G)$ is the general eccentric connectivity index,
- $DEI_{1,1}(G)$ is the classical eccentric connectivity index,
- $DEI_{1,-1}(G)$ is the connective eccentricity index,

- $DEI_{0,1}(G)$ is the total eccentricity index, and
- $DEI_{0,2}(G)$ is the first Zagreb eccentricity index of G.

1.3 Literature Review

Over the past few years, several graph theory methods have been developed to analyze and predict molecules' physicochemical, environmental, and biomedical properties. In chemistry, a molecular graph considers the connections between atoms to depict the structure of a molecule. A graph can model this, where the point represents the atoms, and the edges represent the covalent connections[4]. Numerous studies have been published on such graphs invariant 'topological indices' since the first distance-based topological index was presented by the chemist Harold Wiener (1947) for investigating the boiling point of alkanes. The total of all the distances separating the vertices of the graph under consideration defines this index [44].

Since the first and second Zagreb indices which are the first degree-based topological indices, are the first and second Zagreb indices, a substantial amount of study done on degree-based topological indices. Many findings on Zagreb indices have been published over the past few decades. Li et al. [20, 21] presented the expanded version of the first Zagreb index in 2004 and 2005. One of the most widely used effective molecular descriptors in studies of structure-property and structure-activity connections is the Randić index. Numerous studies have been conducted on the mathematical characteristics of this graph invariant (see book [36] and investigation [19]). Owing to the Randić index's widespread use, several overviews and adaptations, including the general sum-connectivity index [39] and the sum-connectivity index [46, 38]. The harmonic index is one variation of the Randić index. The relationship between the eigenvalues of graphs was examined by Favaron et al. [23] and, Zhong [22] determined the smallest and largest values of the harmonic index on simple connected graphs and trees also described the corresponding extremal graphs.

Sharma in 1997 proposed the eccentric-connectivity index, a substantial eccentricity-based topological indicator [3]. These indicators have demonstrated exceptional predictive capacity for medicinal properties, such as chemical component anti-HIV activity prediction. Because eccentricity-based indexes are widely used, their mathematical characteristics have been researched. Morgan provided strict limits on the eccentric connectivity index for a certain graph order.[30], and the same authors derived a lower bound in [29]for graphs with the specified diameter and order. Zhang got the greatest eccentric connectivity index of graphs in a given order and size. Among graphs of the specified order and pendant vertex count, the graphs with the lowest eccentric connectivity index were identified by Devillez [52][8], Wu and Chen [51] provided a lower bound on the eccentric connectivity index for graphs of a given order with minimum degree; Mukungunugwa and Mukwembi [31] obtained asymptotically sharp upper limits on graphs of a given order and vertex connectivity; Zhang presented sharp lower limits for bipartite graphs with a specified diameter, size, and order. Zhang examined graphs with a diameter of two and [54] [53]. Liu provided the precise values of polyacenic nanotubes' eccentric connectivity index. [16]; Venkatakrishnan, Balachandran, and Kannan [40] provided actual values for generalized thorn graphs; Malik and Farooq [24] reported accurate results for certain 3-fence graphs and associated line graphs. Yu and Feng [11] provided upper constraints on the connective eccentricity index for graphs with a specific order and corresponding matching number, however, Xu provided bounds within the index for graphs with specified parameters explored in this study. Wang established a lower limit [18] for graphs with a particular order and corresponding clique number. According to Li [47], there are restrictive upper bounds on bipartite graphs of a given order with matching numbers, dimensions, and vertex connectivity. For several families of path-thorn graphs, Javaid [12] supplied exact values. [15]. Ili ´c obtained bounds on the vertex connectivity and the eccentric distance sum of graphs that contain a certain order combined with a particular independence number. [49].

Numerous mathematicians have researched and expanded on the topic of eccentricity-based Topological indices. Masre and Vetrik [25] have provided it in a general-degree-eccentricity index form. They claimed that this general index is a particular case of several significant eccentricity-based indices. Among these are the Zagreb eccentricity index of G, the index of connective eccentricity, the total eccentricity index, the general eccentric connectivity index, and as well as the classical eccentric connectivity index. Additional limits for eccentric-based special indices have been established and released. In the same study, they examined the number of pendant vertices and both side limit constraints regarding the general eccentric connection index for trees with a specified order, diameter, and order. Masre and Vetrik discovered both side limits on the general eccentric connectivity index for unicyclic graphs with a particular order and girth in [4]. For $a > 0$ and $b = 1$, the upper limit holds for trees with a specified order, diameter, and number of pendent vertices. The authors of [50] examined the parameters of order, radius, independence number, eccentricity, pendent vertices, and cut edges about the boundaries of the general eccentric connectivity index. They also described the extremal graphs that reached the boundaries.

The paper is set up as follows: Within Chapter 2, we will see pertinent findings about the general degree-eccentricity index of tree graphs. In Chapter 3, we explore the general eccentric connectivity index of unicyclic graphs. This chapter concerns the general eccentric connectivity index's boundaries for unicyclic graphs with a particular order and matching number. In Chapter 4, We investigate the general degree-eccentricity index for graphs with various supplied graph parameters. In Chapter 5, we give conclusions and future works.

Chapter 2

General Degree-Eccentricity Index of Tree

This chapter explores extremal graphs and values of the general degree-eccentricity index of tree graphs, regarding graph-theoretic parameters, which different authors study. We use Lemma 2.1 to prove our main results.

Lemma 2.1. *[42]Let $1 \leq x < y$ and $c > 0$. For $a > 1$ or $a < 0$, we have*

$$(x+c)^a - x^a < (y+c)^a - y^a.$$

If $0 < a < 1$, then

$$(x+c)^a - x^a > (y+c)^a - y^a.$$

2.1 Trees of given order

This section covers the topics on the boundaries of the general degree-eccentricity index of a tree graph having n vertices.

Theorem 2.1. *[28] Assume that T is any tree with n vertices, where $n \geq 4$. For $0 < a < 1$ and $b > 0$,*

$$DEI_{a,b}(S_n) \leq DEI_{a,b}(T) \leq DEI_{a,b}(P_n).$$

Theorem 2.2. *[28] Assume that T is any tree with n vertices, where $n \geq 4$. For $a > 1$ and $b < 0$,*

$$DEI_{a,b}(P_n) \leq DEI_{a,b}(T) \leq DEI_{a,b}(S_n).$$

2.2 Trees of given order and diameter

This section covers the topics on boundaries of the general degree-eccentricity index of trees having n vertices and diameter d.

Declare that all n-vertex trees with diameter d belong to the set $T_{n,d}$. Clearly, $T_{n,2} = \{S_n\}$ and $T_{n,n-1} = P_n$. Therefore, we take into consideration $T_{n,d}$ for $3 \leq d \leq n-2$.

Let $V_{n,d}$ be a tree for a positive integer d. Its central vertex is adjacent to all other $n-d-1$ vertices and consists a path of length d. Demonstrates it [26] that among trees with n vertices and diameter d, $V_{n,d}$ is the only tree with the minimum general eccentric connectivity index. See Figure 2.1

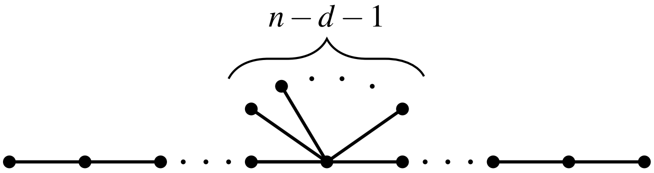

Figure 2.1: $V_{n,d}$

Let $B_{n,d}$ be a tree with all the other $n-d-1$ vertices connected to $v_1$and the path $v_0v_1v_2\dots v_d$. In [26], the general eccentric connectivity index for trees with n vertices and diameter d is shown with a sharp upper bound. See Figure 2.2.

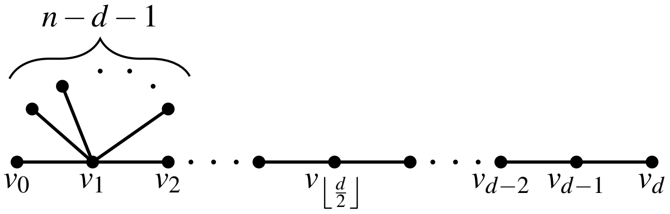

Figure 2.2: $B_{n,d}$

Theorem 2.3. *[26] Assume $T \in T_{n,d}$. Consequently, for $a \geq 1$ and $b < 0$, we have*

$$DEI_{a,b}(T) \leq 2^{a+1} \sum_{r=\frac{d+2}{2}}^{d-1} r^b + (n-d+1)^a \left(\frac{d}{2}\right)^b + (n-d-1)\left(\frac{d+2}{2}\right)^b + 2d^b,$$

if d is even,

$$DEI_{a,b}(T) \leq 2^{a+1} \sum_{r=\frac{d+2}{2}}^{d-1} r^b + ((n-d+1)^a + 2^a)\left(\frac{d+1}{2}\right)^b + (n-d-1)\left(\frac{d+3}{2}\right)^b + 2d^b,$$

if d is odd, and for $0 < a \leq 1$ *and* $b > 0$

$$DEI_{a,b}(T) \geq 2^{a+1} \sum_{r=\frac{d+2}{2}}^{d-1} r^b + (n-d+1)^a \left(\frac{d}{2}\right)^b + (n-d-1)\left(\frac{d+2}{2}\right)^b + 2d^b,$$

if d is even,

$$DEI_{a,b}(T) \geq 2^{a+1} \sum_{r=\frac{d+2}{2}}^{d-1} r^b + ((n-d+1)^a + 2^a)\left(\frac{d+1}{2}\right)^b + (n-d-1)\left(\frac{d+3}{2}\right)^b + 2d^b,$$

if d is odd, then T must be $V_{n,d}$ *if and only if equalities exist.*

Theorem 2.4. *[26] Let* $T \in T_{n,d}$*. Then for* $a > 1$ *and* $b > 0$*, we have*

$$DEI_{a,b}(T) \leq 2^{a+1} \sum_{r=\frac{d+1}{2}} r^a + ((n-d+1)^a + 2^a)(d-1)^b + (n-d+1)d^b + 2^a \left(\frac{d}{2}\right)^b,$$

if d is even,

$$DEI_{a,b}(T) \leq 2^{d-2} \sum_{r=\frac{d+1}{2}} r^b + ((n-d+1)^a + 2^a)(d-1)^b + (n-d+1)d^b$$

if d is odd, for $0 < a < 1$ *and* $b < 0$*, we have*

$$DEI_{a,b}(T) \geq 2^{a+1} \sum_{r=\frac{d+1}{2}} r^a + ((n-d+1)^a + 2^a)(d-1)^b + (n-d+1)d^b + 2^a \left(\frac{d}{2}\right)^b,$$

if d is even,

$$DEI_{a,b}(T) \geq 2^{d-2} \sum_{r=\frac{d+1}{2}} r^b + ((n-d+1)^a + 2^a)(d-1)^b + (n-d+1)d^b$$

If d is odd, then T must be $B_{n,d}$ *if equalities exist.*

2.3 Trees of Given Order and Matching Number

This section deals with the boundaries of the general degree-eccentricity index of trees having n vertices including a matching number of m.

Lemma 2.2. *[28] Suppose T is a tree of order n, matching number m, and maximum degree* $\triangle$*. Then,* $\triangle \leq n - m$*.*

For a positive integer n and m, where $3 \leq m \leq \frac{n}{2}$, let $T_{n,m}$ be the tree that includes one vertex of degree $n-m$, where $n-2m+1$ pendant vertices are adjacent to v and $m-1$ vertices of degree 2; thus, each vertex of degree 2 is adjacent to one pendant vertex.$T_{n,m}$ then has matching number m and order n; see Figure 2.3.

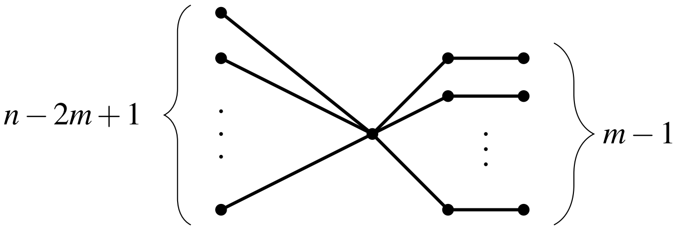

Figure 2.3: $T_{n,m}$

Theorem 2.5. *[28] Suppose T is any tree of order $2m$ with matching number $m \geq 3$. Then, for $b > 0$ and $0 < a \leq 1$, we have*

$$DEI_{a,b}(T) \geq m^a 2^b + (m-1)(2^a 3^b + 4^b) + 3^b,$$

for $a \geq 1$ and $b < 0$, we have

$$DEI_{a,b}(T) \leq m^a 2^b + (m-1)(2^a 3^b + 4^b) + 3^b$$

with equality if and only if T is $T_{2m,m}$.

Theorem 2.6. *[28] Suppose T is any tree of order n with matching number m, where $3 \leq m \leq \frac{n}{2}$. Then, for $b > 0$, and $0 < a \leq 1$, we have*

$$DEI_{a,b}(T) \leq (n-m)^a 2^b + (m-1)(2^a 3^b + 4^b) + (n-2m+1)3^b,$$

for $b < 0$ and $a \geq 1$, we have

$$DEI_{a,b}(T) \geq (n-m)^a 2^b + (m-1)(2^a 3^b + 4^b) + (n-2m+1)3^b$$

equality holds if and only if T is $T_{n,m}$.

For a positive integer r, t, and l let $P_l(r,t)$ be the tree that includes the path of length l, with r pedant vertices adjacent to one of the end vertex and t vertices adjacent to the other end vertex of the path. See Figure 2.4.

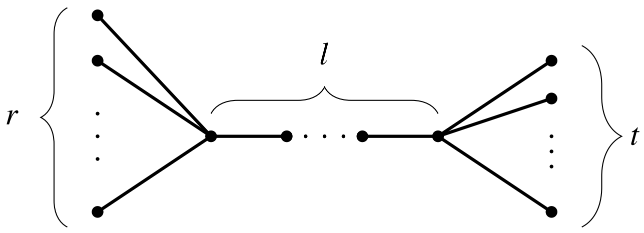

Figure 2.4: $P_l(r,t)$

Theorem 2.7. *[28] Suppose T is any tree of order $n \geq 4$ with matching number 2. Then for $0 < a < 1$ and $b > 0$, we have*

$$DEI_{a,b}(T) \geq [(n-2)^a + 2^b]2^b + (n-2)3^b,$$

for $a > 1$ and $b < 0$, we have

$$DEI_{a,b}(T) \leq [(n-2)^a + 2^b]2^b + (n-2)3^b$$

with equality if and only if T is $P_2(n-3,1)$.

Theorem 2.8. *[28] Suppose T is any tree of order n with independence number α, where $\frac{n}{2} \leq \alpha \leq n-3$. Then, for $b > 0$ and $0 < a \leq 1$, we have*

$$DEI_{a,b}(T) \geq \alpha^a 2^b + (n-\alpha-1)(2^a 3^b + 4^b) + (2\alpha n + 1)3^b,$$

for $b < 0$ and $a \geq 1$, we have

$$DEI_{a,b}(T) \leq \alpha^a 2^b + (n-\alpha-1)(2^a 3^b + 4^b) + (2\alpha n + 1)3^b$$

equality holds if and only if T is $T_{n,n-\alpha}$.

Theorem 2.9. *[28] Suppose T is any tree of order $n \geq 4$ with independence number $n-2$. Then, for $b > 0$ and $0 < a < 1$, we have*

$$DEI_{a,b}(T) \geq [(n-2)^a + 2^a]2^b + (n-2)3^b,$$

For $b < 0$ and $a > 1$, we have

$$DEI_{a,b}(T) \leq [(n-2)^a + 2^a]2^b + (n-2)3^b,$$

equalities for T is $P_2(n-3,1)$.

Chapter 3

General Eccentric Connectivity Index of Unicyclic Graphs

In this chapter, we deal with the general eccentric connectivity index of unicyclic graphs within different parameters of a graph and visualize the extremal graphs.

3.1 Unicyclic graphs of given order

This part deals with the limits of the general eccentric connectivity index for a unicyclic graph having n vertices.

For $k \geq 3$ suppose C_k be a cycle graph with $V(C_k) = \{c_0, c_1, \ldots, c_{k-1}\}$ and $E(C_k) = \{c_0c_1, c_1c_2, \ldots, c_{k-2}c_{k-1}, c_{k-1}c_0\}$. Suppose $C_k(n-k)$ be a unicyclic graph made up of the cycle $C_{k,}$ to which $n-k$ pendant vertices are connected at one of its vertices. See Figure 3.1.

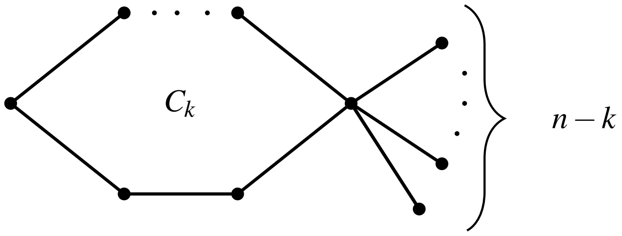

Figure 3.1: $C_k(n-k)$

Theorem 3.1. *[42] Suppose U is any unicyclic graph of order $n \geq 4$. Then for $0 < a < 1$, we have*

$$ECI_{a,b}(U) \geq 2^{a+2} + (n-1)^a + 2n - 6$$

equality holds if and only if U is $C_3(n-3)$.

Theorem 3.2. *[42] Suppose U is any unicyclic graph of order* $n \geq 4$. *Then for* $0 < a \leq 1$, *we have*

$$ECI_{a,b}(U) \leq \left\lfloor \frac{3n^2-12n+17}{2} \right\rfloor 2^{a-1} + (n-3)3^a + n - 2$$

with equality if and only if U is $C_3 \star P_{n-2}$.

Corollary 3.1. *[50] Let G be a unicyclic graph of order n, then for* $a < 0$, *we have*

$$ECI_{a,b}(G) \leq n(2^a+1) + 2^{a+1} - 2^a - 1$$

and the equality holds if and only $G \cong C_{n,1}$.

3.2 Unicyclic graphs of given order and girth

This part deals along the boundaries of the general degree-eccentricity index for unicyclic graph having n vertices and girth k.

Theorem 3.3. *[42] Suppose U is any unicyclic graph with n vertices and girth k, where* $3 \leq k \leq n-1$. *Then for* $0 < a < 1$, *we have*

$$ECI_a(U) \geq \begin{cases} (k^2-k+2)2^{a-1} + \dfrac{k(n-k+2)^a}{2} + \dfrac{(k+2)(n-k)}{2} & \text{if } k \text{ is even,} \\ (k^2-2k+5)2^{a-1} + \dfrac{(k-1)(n-k+1)^a}{2} + \dfrac{(k+2)(n-k)}{2} & \text{if } k \text{ is odd,} \end{cases}$$

with equalities if and only if U is $C_k(n-k)$.

Assume that the unicyclic graph $C_k \star P_{n-k+1}$ is derived from the cycle C_k and the path P_{n-k+1} by associating one vertex of the cycle with an end-vertex of the path. See Figure 3.2.

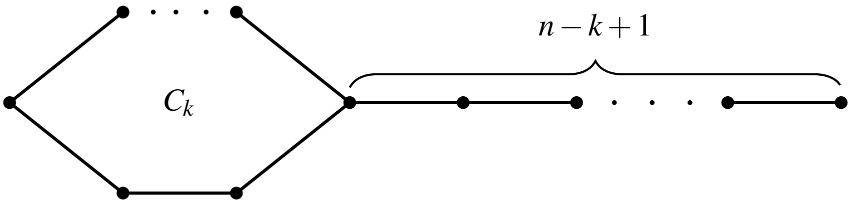

Figure 3.2: $C_k \star P_{n-k+1}$

Theorem 3.4. *[42] Suppose U is any unicyclic graph with n vertices and girth k, where $3 \leq k \leq n-1$. Then for $0 < a \leq 1$, we have*

$$ECI_a(U) \leq ECI_a(C_K \star P_{n-k+1})$$

equality holds if and only if U is $C_k \star P_{n-k+1}$.

3.3 Unicyclic graphs of given matching number

This section will examine the thesis's main results, which are based on the study of the unicyclic graphs' general eccentric connectivity index for a specific order and matching number.

Theorem 3.5. *Suppose U is a unicyclic graph with $2m$ vertices, girth k, and perfect matching m, where $m \geq 2$ and $3 \leq k \leq 2m-1$. Then for $0 < a \leq 1$, we have*

$$ECI_a(U) \leq ECI_a(C_k \star P_{2m-k+1}),$$

equality holds if and only if U is $C_k \star P_{2m-k+1}$.

Proof. Assume U' is a unicyclic graph with $2m$ vertices, girth k, and a perfect matching m having the largest ECI_a among unicyclic graphs of order $2m$, perfect matching and girth k. Let $D = u_0u_1u_2\ldots u_d$ be a diametral path in U' and $C_k = c_0c_1\ldots c_{k-1}c_0$ be the cycle in U'. Since U' is not C_{2m}, we have either u_0 or u_d is pendant vertex.

Claim 3.5.1. *For every vertex outside of D, the degree is at least 2.*

Contrarily, assume that there is a vertex $v \notin D$ and $d_{U'}(v) = 1$. Then by Lemma 1.3, v is adjacent to a vertex of degree 2, say u, let w be the unique neighbor of u other than v, where $d_{U'}(w) = s \geq 2$. Let $V(U_1) = V(U^{'})$ and $E(U_1) = E(U^{'}) \backslash \{uw\} \cup \{uu_0\}$. We can easily observe that U_1 is a unicyclic graph of order $2m$, girth k, and matching number m. We have $d_{U'}(u_0) = 1, d_{U_1}(u_0) = 2$, $d_{U'}(w) = s, d_{U'}(w) = s-1$ and $d_{U_1}(x) = d_{U_1}(x)$ for all $x \in V(U^{'})$. We know that $ecc_{U_1}(y) \geq ecc_{U'}(y)$ for all $y \in V(U^{'})$. More over, $ecc_{U'}(u_0) = ecc_{U_1}(u_0) = d$,

$ecc_{U'}(v) \leq d, ecc_{U_1}(v) = d+2, ecc_{U'}(w) \leq d-2$ and $ecc_{U_1}(w) \leq d-1$. Thus

$$\begin{aligned}
ECI_a(U^{'}) - ECI_a(U_1) &\leq ecc_{U'}(u_0)d^a_{U'}(u_0) - ecc_{U_1}(u_0)d^a_{U_1}(u_0) + ecc_{U'}(w)d^a_{U'}(w) \\
&\quad - ecc_{U_1}(w)d^a_{U_1}(w) + ecc_{U'}(v)d^a_{U'}(v) - ecc_{U_1}(v)d^a_{U_1}(v) \\
&\leq d(1^a - 2^a) + ecc_{U_1}(w)(s^a - (s-1)^a) + d \cdot 1^a - (d+2) \cdot 1^a \\
&< d(1 - 2^a + (s^a - (s-1)^a)) - 2 \leq 0,
\end{aligned}$$

since by lemma 2.1, $2^a - 1 > s^a - (s-1)^a$ for $s \geq 3$. Hence $ECI_a(U^{'}) < ECI_a(U_1)$, which is contradiction. Therefore, U' has at most two pendant vertices.

From claim 3.5.1, we have shown that at most two vertices of the cycle, C_k have a degree of at least three.

Claim 3.5.2. *If T is a tree connected to c_0, then T is a path.*

On the contrary, say that T is not a path. Then T has a vertex of degree at least 3, say v. Then T contains at least two pendant paths say $vv_1v_2\dots v_{p-1}v_p$ and $vv^{'}_1v^{'}_2\dots v_{q-1}v^{'}_q$, where $p, q \geq 1$. Since U' has a perfect matching either p or q must be even. We consider the following two cases:

Case 1: v is a vertex of C_k. That is $v = c_0$.
If k is even, then $p+q$ must be even. That is, both are even or both are odd. If p and q are odd, the graph has no perfect matching. So, both p and q must be even. If k is odd, then $p+q$ must be odd. That is, one is even and the other is odd. In any of the cases, we have the matching number of U' is $m = \lfloor \frac{k}{2} \rfloor + \lceil \frac{p+q}{2} \rceil$. Let $V(U_2) = V(U^{'})$ and $E(U_2) = E(U^{'}) \backslash \{c_0c_1, c_0c_{k-1}\} \cup \{v_pc_1, v_pc_{k-1}\}$. Then U_2 is a unicyclic graph of order $2m$ and girth k. We can observe that, the matching number of U_2 is also $m = \lfloor \frac{k}{2} \rfloor + \lceil \frac{p+q}{2} \rceil$.
Sub case 1: If p is odd and q is even c_0v_1 must be in the perfect matching in $U^{'}$, then since $\{c_1v_p, v_pc_{k-1}, c_0v^{'}_1\} \notin M^{'}$ and $\{v_pv_{p-1}, c_0v_1\} \in M^{'}$, we have matching number $m^{'} = m = \lfloor \frac{k}{2} \rfloor + \lfloor \frac{p}{2} \rfloor + \frac{q}{2}$ in U_2.
Sub case 2: If q is odd and p is even $c_0v^{'}_1$ must be in the perfect matching in $U^{'}$, then since $\{c_1v_p, v_pc_{k-1}, c_0v_1\} \notin M^{'}$ and $\{v_pv_{p-1}, c_0v^{'}_1\} \in M^{'}$ we have matching number $m^{'} = m = \lfloor \frac{k}{2} \rfloor + \lfloor \frac{p}{2} \rfloor + \frac{q}{2}$ in U_2.
Sub case 3: If p and q are both even c_1c_0 or $c_{k-1}c_0$ is in the matching set in $U^{'}$, then on the path $\{v_pv_{p-1}, \cdots, v_1c_0, c_0v^{'}_1, \cdots, v^{'}_{q-1}v^{'}_q\}$ if we start selecting edges from $v^{'}_{q-1}v^{'}_q$ by escaping an edge we get $\lfloor \frac{p}{2} \rfloor + \lfloor \frac{q}{2} \rfloor$ number of edges in $M^{'}$ which implies in U_2 the matching number is $m^{'} = m = \lfloor \frac{k}{2} \rfloor + \lfloor \frac{p}{2} \rfloor + \lfloor \frac{q}{2} \rfloor$ in all cases. Hence U_2 is a unicyclic graph having a perfect

matching m, girth k, and order 2m.

We have $d_{U'}(v_p) = 1, d_{U_2}(v_p) = 3, d_{U'}(c_0) = 4, d_{U_2}(c_0) = 2$ and $d_{U'}(x) = d_{U_2}(x)$ for all $x \in V(U^{'})\backslash\{c_0, v_p\}$. Note that by Claim 3.5.1, the diametral path of $U^{'}$ is $v_p v_{p-1} \ldots v_1 v v_1^{'} \ldots v_{q-1}^{'} v_q^{'}$ accordingly $ecc_{U_2}(y) = d_{U'}(y, v_p)$ or $d_{U'}(y, v_q)$ for any vertex $y \in V(U^{'})$. Therefore $ecc_{U_2}(y) \geq ecc_{U'}(y)$ for all $y \in V(U^{'})$. More over, we know that $ecc_{U'}(v_p) = ecc_{U_2}(v_p) = p + q, ecc_{U'}(v) = p$ and $ecc_{U_2}(v) = p + \lfloor \frac{k}{2} \rfloor \leq p + q$. Thus

$$\begin{aligned}
ECI_a(U^{'}) - ECI_a(U_2) &\leq ecc_{U'}(v_P) d^a_{U'}(v_p) - ecc_{U_2}(v_P) d^a_{U_2}(v_p) + ecc_{U'}(c_0) d^a_{U'}(c_0) \\
&\quad - ecc_{U_2}(c_0) d^a_{U_2}(c_0) \\
&\leq (p+q)(1-3^a) + p(4^a) - \left(p + \left\lfloor \frac{k}{2} \right\rfloor\right) 2^a \\
&\leq (p+q)(1-3^a) + \left(p + \left\lfloor \frac{k}{2} \right\rfloor\right) (4^a - 2^a) \\
&\leq (p+q)(1 - 3^a + 4^a - 2^a) \leq 0,
\end{aligned}$$

since by lemma 2.1, $3^a - 1 > 4^a - 2^a$ for $0 < a < 1$ and $3^a - 1 + 4^a - 2^a = 0$ for $a = 1$. Hence $ECI_a(U^{'}) - ECI_a(U_2) < 0$ and $ECI_a(U^{'}) < ECI_a(U_2)$, which is contradiction.

Case 2: v is not a vertex of C_k. That is $d(c_0, v) \geq 1$.

Let we identify c_0 with v_0 and v_i with v and $v_0^{'}$, where $1 \leq i \leq p-1$ then without loss of generality, suppose that $p - i \geq q$. Let $V(U_3) = V(U^{'})$ and $E(U_3) = E(U^{'})\backslash\{v_{i-1}v_i\} \cup \{v_p v_{i-1}\}$, then the matching number in $U^{'}$ is $m = \lfloor \frac{i+k}{2} \rfloor + \left\lfloor \frac{p-i}{2} \right\rfloor + \lfloor \frac{q}{2} \rfloor$, and in U_3 for $M^{'}$ is the matching set of U_3 and $|M^{'}| = m^{'}$ we have three cases:

Sub case 1: If $p - i$ is odd and q is even $v_i v_1$ must be in the perfect matching in $U^{'}$. Since $\{v_{i-1}v_p\} \notin M$ also in $M^{'}$, we have matching number $m = m^{'} = \lfloor \frac{k+i}{2} \rfloor + \left\lfloor \frac{p-i}{2} \right\rfloor + \frac{q}{2}$ in U_3.

Sub case 2: If q is odd and $p - i$ is even $v_i v_1^{'}$ must be in the perfect matching in $U^{'}$. Same to case 1 since $\{v_{i-1}v_p\} \notin M$ also in $M^{'}$, we will have matching number $m = m^{'} = \lfloor \frac{k+i}{2} \rfloor + \left\lfloor \frac{p-i}{2} \right\rfloor + \frac{q}{2}$ in U_3.

Sub case 3: If $p - i$ and q are both even $v_i v_{i-1}$ is in the matching number in $U^{'}$. And in U_3 since $v_i v_{1+1}$ is in $M^{'}$ instead of $v_p v_{p-1}$ we have $m = m^{'} = \lfloor \frac{k+i}{2} \rfloor + \left\lfloor \frac{p-i}{2} \right\rfloor + \lfloor \frac{q}{2} \rfloor$ in all cases.

We have $d_{U'}(v_p) = 1, d_{U_3}(v_p) = 2, d_{U'}(v_i) = 3, d_{U_3}(v_i) = 2$ and $d_{U'}(x) = d_{U_3}(x)$ for all $x \in V(U^{'})\backslash\{v_p, v\}$. Note that by Claim 3.5.1, the diametral path is $v_p v_{p-1} \ldots v_1 v_i v_1^{'} \ldots v_{q-1}^{'} v_q^{'}$ accordingly $ecc_{U'}(y) = d_{U'}(y, v_P)$ or $d_{U'}(y, v_q^{'})$ for all $y \in V(U^{'})$. Hence $ecc_{U_3}(y) \geq ecc_{U'}(y)$ for all $y \in V(U^{'})$. More over $ecc_{U'}(v_p) = q + i = ecc_{U_3}(v_p), ecc_{U'}(v_i) = p - i$ and $ecc_{U_3}(v_i) =$

$p+\lfloor \frac{k}{2} \rfloor \leq q+i$. Thus

$$\begin{aligned}
ECI_a(U^{'})-ECI_a(U_3) &\leq ecc_{U^{'}}(v_P)d^a_{U^{'}}(v_p)-ecc_{U_3}(v_P)d^a_{U_3}(v_p)+ecc_{U^{'}}(v_i)d^a_{U^{'}}(v_i)\\
&\quad -ecc_{U_3}(v_i)d^a_{U_3}(v_i)\\
&=(q+i)(1^a-2^a)+(p-i)\cdot 3^a-\left(p+\left\lfloor \frac{k}{2}\right\rfloor\right)(2^a)\\
&<(q+i)(1-2^a)+p(3^a-2^a)\\
&<(q+i)(1-2^a+3^a-2^a)\leq 0,
\end{aligned}$$

since by lemma 2.1, $2^a-1>3^a-2^a$ for $0<a<1$. Hence $ECI_a(U^{'})-ECI_a(U_3)<0$ and $ECI_a(U^{'})<ECI_a(U_3)$, which is contradiction.

Claim 3.5.3. *C_k contains exactly one vertex of degree 3.*

Conversely, say that C_k has at least two vertices of degree 3. Let c_0 and c_d be any vertices of C_k with degree 3 not far from to each other. Then C_k contain pendant paths $c_0v_1v_2\ldots v_{p-1}v_p$ and $c_dv^{'}_1v^{'}_2\ldots v^{'}_{q-1}v^{'}_q$, where $q,p\geq 1$. Keeping generalities intact, we presume $p\geq q$. The matching number of $U^{'}$ is $m=\lfloor \frac{k}{2} \rfloor+\lceil \frac{p+q}{2} \rceil$. We distinguish three cases:

For $k\geq 4$ and $q=1$. Let $V(U^{'})=V(U_4)$ and $E(U_4)=E(U^{'})\backslash\{c_dc_{d+1},c_{k-1}c_0\}\cup\{v^{'}_1c_{d+1}, c_1c_{k-1}\}$. The matching number of U_4 is $m=\lfloor \frac{k}{2} \rfloor+\lceil \frac{p+q}{2} \rceil$. Hence, U_4 is a unicyclic graph having a perfect matching m, girth k, and $U^{'}$ and U_4 have the same diametral path (connecting v_p and v_q). We have $ecc_{U^{'}}(x)=ecc_{U_4}(x)$ for all $x\in V(U^{'})\backslash\{c_{d+1},c_{d+2},\ldots,c_{k-1}\}$ and $\sum_{i=d+1}^{k-1}(ecc_{U^{'}}(c_i)-ecc_{U_4}(c_i)<0$ from [42].

If $d\neq 1$, then $d_{U^{'}}(c_d)=3$, $d_{U_4}(c_d)=2, d_{U^{'}}(v^{'}_1)=2$, $d_{U_4}(v^{'}_1)=3$, $d_{U^{'}}(c_1)=2$, $d_{U_4}(c_1)=3$, $d_{U^{'}}(c_0)=3$, $d_{U_4}(c_0)=2$ and $d_{U^{'}}(x)=d_{U_4}(x)$ for all $x\in V(U^{'})\backslash\{c_d,v^{'}_1,c_0,c_1\}$. Thus

$$\begin{aligned}
DEI_{a.b}(U^{'})-ECI_a(U_4) &< ecc_{U^{'}}(v^{'}_1)d^a_{U^{'}}(v^{'}_1)-ecc_{U_4}(v^{'}_1)d^a_{U_4}(v^{'}_1)+ecc_{U^{'}}(c_d)d^a_{U^{'}}(c_d)\\
&\quad -ecc_{U_4}(c_d)d^a_{U_4}(c_d)+ecc_{U^{'}}(c_1)d^a_{U^{'}}(c_1)-ecc_{U_4}(c_1)d^a_{U_4}(c_1)\\
&\quad +ecc_{U^{'}}(c_0)d^a_{U^{'}}(c_0)-ecc_{U_4}(c_0)d^a_{U_4}(c_0)\\
&=ecc_{U_4}(v^{'}_1)(2^a-3^a)+ecc_{U_4}(c_d)(3^a-2^a)+ecc_{U^{'}}(c_1)(3^a-2^a)\\
&\quad +ecc_{U^{'}}(c_0)(3^a-2^a)\\
&=[ecc_{U^{'}}(c_0)-ecc_{U^{'}}(c_1)+ecc_{U^{'}}(c_d)-ecc_{U^{'}}(v^{'}_1)](3^a-2^a)
\end{aligned}$$

Since $ecc_{U^{'}}(v^{'}_1)=ecc_{U_{,}}(c_d)+1$,$ecc_{U^{'}}(c_0)\leq ecc_{U_{,}}(c_1)+1$, thus $ECI_a(U^{'})<ECI_a(U_4)$.

If $d=1$, then $d_{U^{'}}(c_0)=3,d_{U_4}(c_0)=2,d_{U^{'}}(v^{'}_1)=2,d_{U_4}(v^{'}_1)=3$ and $d_{U^{'}}(x)=d_{U_4}(x)$ for all $x \in V(U^{'})\backslash\{c_0, v_1^{'}\}$. Thus

$$\begin{aligned}ECI_a(U^{'}) - ECI_a(U_4) < {} & ecc_{U'}(c_0)d_{U'}^a(c_0) - ecc_{U_4}(c_0)d_{U_4}^a(c_0) + ecc_{U'}(v_1^{'})d_{U'}^a(v_1^{'}) - \\ & ecc_{U_4}(v_1^{'})d_{U_4}^a(v_1^{'}) \\ & = ecc_{U'}(c_0)(3^a - 2^a) + ecc_{U_4}(v_1^{'})(2^a - 3^a) \\ & = ecc_{U'}(c_0) - ecc_{U_4}(v_1^{'})(3^a - 2^a) < 0,\end{aligned}$$

Since $ecc_{U'}(v_1^{'}) = p+2$ and $ecc_{U'}(c_0) = max\{p, q+1\} < p+2$. Thus $DEI_{a.b}(U^{'}) < ECI_a(U_4)$, which is a contradiction.

For $k \geq 4$ and $q \geq 2$. Let $V(U^{'}) = V(U_4)$ and $E(U_4) = E(U^{'})\backslash\{c_dc_{d+1}, c_{k-1}c_0\} \cup \{v_2^{'}c_{d+1}, c_2c_{k-1}\}$. Then the matching number of U_4 is $m = \lfloor \frac{k}{2} \rfloor + \lceil \frac{p+q}{2} \rceil$. Hence, U_4 is a unicyclic graph having a perfect matching m, girth k, and $U^{'}$ and U_4 have the same diametral path(connecting v_p and v_q). We've got $ecc_{U'}(x) = ecc_{U_4}(x)$ for all $x \in V(U')\backslash\{c_{d+1}, c_{d+2}, \ldots, c_{k-1}\}$ and $\sum_{i=d+1}^{k-1}(ecc_{U'}(c_i) - ecc_{U_4}(c_i) < 0$ from [42].

If $d \neq 1$, then $d_{U'}(c_d) = 3$, $d_{U_4}(c_d) = 2, d_{U'}(v_2^{'}) = 2$, $d_{U_4}(v_2^{'}) = 3$, $d_{U'}(c_2) = 2$, $d_{U_4}(c_2) = 3$, $d_{U'}(c_0) = 3$, $d_{U_4}(c_0) = 2$ and $d_{U'}(x) = d_{U_4}(x)$ for all $x \in V(U^{'})\backslash\{c_d, v_2^{'}, c_0, c_2\}$. Thus

$$\begin{aligned}ECI_a(U^{'}) - ECI_a(U_4) < {} & ecc_{U'}(v_2^{'})d_{U'}^a(v_2^{'}) - ecc_{U_4}(v_2^{'})d_{U_4}^a(v_2^{'}) + ecc_{U'}(c_d)d_{U'}^a(c_d) \\ & - ecc_{U_4}(c_d)d_{U_4}^a(c_d) + ecc_{U'}(c_2)d_{U'}^a(c_2) - ecc_{U_4}(c_2)d_{U_4}^a(c_2) \\ & + ecc_{U'}(c_0)d_{U'}^a(c_0) - ecc_{U_4}(c_0)d_{U_4}^a(c_0) \\ & = ecc_{U_4}(v_2^{'})(2^a - 3^a) + ecc_{U_4}(c_d)(3^a - 2^a) + ecc_{U'}(c_2)(3^a - 2^a) \\ & + ecc_{U'}(c_0)(3^a - 2^a) \\ & = [ecc_{U'}(c_0) - ecc_{U'}(c_2) + ecc_{U'}(c_d) - ecc_{U'}(v_2^{'})]\,(3^a - 2^a)\end{aligned}$$

Since $ecc_{U'}(v_2^{'}) = ecc_{U_,}(c_d) + 2$,$ecc_{U'}(c_0) \leq ecc_{U_,}(c_2) + 2$, thus $ECI_a(U^{'}) < ECI_a(U_4)$.

If d = 1, then $d_{U'}(c_0) = 3, d_{U_4}(c_0) = 2, d_{U'}(v_2^{'}) = 2, d_{U_4}(v_2^{'}) = 3$ and $d_{U'}(x) = d_{U_4}(x)$ for all $x \in V(U^{'})\backslash\{c_0, v_2^{'}\}$. Thus

$$\begin{aligned}ECI_a(U^{'}) - ECI_a(U_4) < {} & ecc_{U'}(c_0)d_{U'}^a(c_0) - ecc_{U_4}(c_0)d_{U_4}^a(c_0) + ecc_{U'}(v_2^{'})d_{U'}^a(v_2^{'}) - \\ & ecc_{U_4}(v_2^{'})d_{U_4}^a(v_2^{'}) \\ & = ecc_{U'}(c_0)(3^a - 2^a) + ecc_{U_4}(v_2^{'})(2^a - 3^a) \\ & = ecc_{U'}(c_0) - ecc_{U_4}(v_2^{'})(3^a - 2^a) < 0,\end{aligned}$$

Since $ecc_{U'}(v_2^{'}) = p+3$ and $ecc_{U'}(c_0) = max\{p, q+1\} < p+2$. Thus $DEI_{a.b}(U^{'}) <$

$ECI_a(U_4)$, which is a contradiction.

For k = 3. We have $c_1 = c_d$ and $c_2 = c_{d+1} = c_{k-1,}$ thus $E(U_4) = \{c_2v_1^{'}\} \cup E(U^{'})\backslash\{c_0c_2\}$. Then the matching number of U_4 is $m = \lfloor \frac{k}{2} \rfloor + \lceil \frac{p+q}{2} \rceil$. Hence, U_4 is a unicyclic graph having a perfect matching m, girth k = 3, and $U^{'}$ and U_4 have the same diametral path(connecting v_p and v_q)

Then $d_{U'}(c_0) = 3, d_{U_4} = 2, d_{U'}(v_1^{'}) = 2, d_{U_4}(v_1^{'}) = 3$ and $d_{U'}(x) = d_{U_4}(x)$ for all $x \in V(U^{'})\backslash\{c_0, v_1^{'}\}$. We have $eccc_{U'}(c_2) = p+1, ecc_{U_4}(c_2) = p+2$ and $ecc_{U'}(y) = ecc_{U_4}(y)$ for all $y \in V(U^{'})\backslash\{c_2\}$. Note that $ecc_{U'}(v_1^{'}) = p+2$ and $ecc_{U'}(c_0) = max\{p, q+1\} < p+2$.

$$\begin{aligned} ECI_a(U^{'}) - ECI_a(U_4) &< ecc_{U'}(c_0)d_{U'}^a(c_0) - ecc_{U_4}(c_0)d_{U_4}^a(c_0) + ecc_{U'}(v_1^{'})d_{U'}^a(v_1^{'}) \\ &\quad - ecc_{U_4}(v_1^{'})d_{U_4}^a(v_1^{'}) + ecc_{U'}(c_2)d_{U'}^a(c_2) - ecc_{U_4}(c_2)d_{U_4}^a(c_2) \\ &= ecc_{U'}(c_0)(3^a - 2^a) + (p+2)(2^a - 3^a) + (p+1)(2^a) - (p+2)(2^a) \\ &< (p+2)[(3^a - 2^a) + 2^a - 3^a] - 2^a < 0. \end{aligned}$$

Therefore, $ECI_a(U^{'}) < ECI_a(U_4)$. Hence, we have a contradiction. So $U^{'}$ is $C_k \star P_{2m-k+1}$.

□

Theorem 3.6. *Suppose U is any unicyclic graph with $2m$ vertices and a perfect matching m, where $m \geq 2$. Then for $0 < a < 1$, we have*

$$ECI_a(U) \leq ECI_a(C_3 \star P_{2m-2}),$$

equality holds if and only if U is $C_3 \star P_{2m-2}$.

Proof. Suppose $U^{'}$ is the extremal graph. Then from Theorem 3.4, we know that $U^{'}$ is $C_k \star P_{2m-k+1}$ for some $3 \leq k \leq 2m-1$. The matching number of U' is $m = \lfloor \frac{k}{2} \rfloor + \lceil \frac{2m-k}{2} \rceil$. We prove that $U^{'}$ is $C_3 \star P_{2m-2.}$

To the contrary say that $U^{'}$ is not $C_3 \star P_{2m-2}$. That is, $U^{'}$ is $C_k \star P_{2m-k+1}$ for some $4 \leq k \leq 2m-1$. Consider two cases: $4 \leq k \leq 2m-1$ and $k = 2m$.

Case 1: $4 \leq k \leq 2m-1$. Let $V(U_1) = V(U^{'})$ and $E(U_1) = E(U^{'})\backslash\{c_0c_1\} \cup \{c_1c_3\}$. The matching number of U_1 is $m = 1 + \lceil \frac{2m-2}{2} \rceil$. Hence, U_1 is a unicyclic graph of order $2m$ having matching number m. Then we have $d_{U_1}(c_3) = 3, d_{U'}(c_3) = 2, d_{U_1}(c_0) = 3, d_{U'}(c_0) = 2$ and $d_{U_1}(x) = d_{U'}(x)$ for all $x \in V(U^{'})\backslash\{c_0, c_3\}$, $ecc_{U_1}(y) \geq ecc_{U'}(y)$, for all $y \in V(U^{'})$. More over

$ecc_{U_1}(c_3) \geq ecc_{U_1}(c_0)$ and $ecc_{U'}(c_3) \geq ecc_{U'}(c_0)$. Thus

$$\begin{aligned}
ECI_a(U^{'}) - ECI_a(U_1) &\leq ecc_{U'}(c_0)d^a_{U'}(c_0) - ecc_{U_1}(c_0)d^a_{U_1}(c_0) + ecc_{U'}(c_3)d^a_{U'}(c_3) \\
&\quad - ecc_{U_1}(c_3)d^a_{U_1}(c_3) \\
&\leq ecc_{U'}(c_0)(3^a) - ecc_{U_1}(c_0)(2^a) + ecc_{U'}(c_3)(2^a) - ecc_{U_1}(c_3)(3^a) \\
&\leq ecc_{U'}(c_3)(3^a + 2^a) - ecc_{U_1}(c_3)(3^a + 2^a) \\
&= (ecc_{U'}(c_0) - ecc_{U_1}(c_3))(3^a + 2^a) \leq 0,
\end{aligned}$$

as $ecc_{U'}(c_0) \leq ecc_{U_1}(c_3)$. Hence, $ECI_a(U_1) \geq ECI_a(U^{'})$, which contradicts itself.

Case 2: k = 2m. Let $V(U_2) = V(C_{2m})$ and $E(U_2) = E(C_{2m})\backslash\{c_0c_1\} \cup \{c_1c_3\}$. Then we have $d_{U_2}(c_0) = 1, d_{C_{2m}}(c_0) = 2, d_{U_2}(c_3) = 3, d_{C_{2m}}(c_3) = 2$ and $d_{U_2}(x) = d_{C_{2m}}(x)$ for all $x \in V(C_{2m})\backslash\{c_0, c_3\}, ecc_{U_2}(y) \geq ecc_{C_{2m}}(y)$ for all $y \in V(C_{2m})$ except one central vertex in the path $c_1c_3c_4 \dots c_{k-1}c_0$ and it minimize only by one. Hence,

$$\sum_{w \in V(C_{2m})\backslash\{c_0c_3\}} ecc_{C_{2m}}(w) - ecc_{U_2}(w) \leq 0.$$

Thus

$$\begin{aligned}
ECI_a(C_{2m}) - ECI_a(U_2) &\leq ecc_{C_{2m}}(c_0)d^a_{C_{2m}}(c_0) - ecc_{U_2}(c_0)d^a_{U_2}(c_0) + ecc_{C_{2m}}(c_3)d^a_{C_{2m}}(c_3) \\
&\quad - ecc_{U_2}(c_3)d^a_{U_2}(c_3) \\
&= ecc_{C_{2m}}(c_0)2^a - ecc_{U_2}(c_0)1^a + ecc_{C_{2m}}(c_3)2^a - ecc_{U_2}(c_3)3^a \\
&\leq ecc_{U_2}(c_0)(2^a - 1) + ecc_{U_2}(c_3)(2^a - 3^a) \\
&\leq ecc_{U_2}(c_0)[2^a - 1 + 2^a - 3^a] \leq 0,
\end{aligned}$$

for $0 < a < 1$ and since lemma 2.1, $(2^a - 1) - (3^a - 2^a)$. Thus $ECI_a(U_2) > ECI_a(C_{2m})$. □

Let $U_{2m,m}$ be a unicyclic graph with cycle three with one pendant vertex and m-2 paths length two connected to a single vertex. See Figure 3.3.

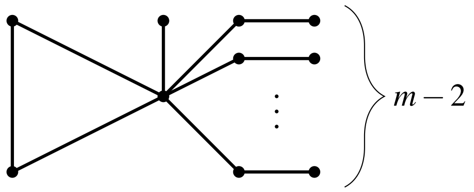

Figure 3.3: $U_{2m,m}$

Theorem 3.7. *Suppose U is a unicyclic graph order 2m and a perfect matching m, where* $m \geq 3$. *Then for* $0 < a < 1$, *we have*

$$ECI_a(U) \geq 2(m+1)^a + 3m \cdot 2^a + 4m - 5,$$

equality holds if and only if U is $U_{2m,m}$.

Proof: One vertex of degree $m+1$, *m vertices of degree* 2, *and* $m-1$ *pendant vertices make up the graph* $U_{2m,m}$. *Thus*

$$ECI_a(U_{2m,m}) = 2(m+1)^a + 3m \cdot 2^a + 4m - 5.$$

Observe that if m = 2, then either $U \cong C_4$ *or* $U \cong U_{4,2}$. *Then the result follows as* $ECI_a(U_{4,2})$ $= 2^a \cdot 4 + 3^a + 2 < 8 \cdot 2^a = ECI_a(U_4)$, *where* $0 < a < 1$. *So, we suppose that* $m > 3$. *Let* $U^{'}$ *be a unicyclic graph possessing the highest* ECI_a *among unicyclic graphs with order 2m and matching number m. We now prove by induction on m.*

Claim 3.7.1. $U^{'}$ *is a cycle graph. We know* $ECI_a(C_{2m}) = (2m^2)2^a$. *Then we compare as:*

$$\begin{aligned} ECI_a(C_{2m}) - ECI_a(U_{2m,m}) &= (2m^2)2^a - [2(m+1)^a + 3m \cdot 2^a + 4m - 5] \\ &= (2m^2 - 3m)2^a - 2(m+1)^a - 4m + 5, \end{aligned}$$

for $0 < a < 1$, *let* $F(m) = (2m^2 - 3m)2^a - 2(m+1)^a - 4m + 5$. *Then*

$$\begin{aligned} F^{'}(m) &= (4m-3)2^a - 2a(m+1)^{a-1} - 4 \\ &\geq 4m - 7 - 2a(m+1)^{a-1} \\ &\geq 4m - 9 \geq 0, \ \ m \geq 3 \end{aligned}$$

Hence, F(m) is an increasing function on $0 < a < 1$. *And since* $m > 3$,

$$\begin{aligned} F(m) > F(3) &= 9 \cdot 2^a - 2 \cdot 4^a - 7 \\ &= 9 \cdot 2^a - 2^{2a+1} - 7 \\ &= 2^a(9 - 2^{a+1}) - 7 > 0. \end{aligned}$$

Then we have

$$2m^2 - 7m + 3 \leq ECI_a(C_{2m}) - ECI_a(U_{2m,m}) \leq 4m^2 - 12m + 3.$$

Thus, $ECI_a(C_{2m,m}) - ECI_a(U_{2m,m}) \geq 0,$ *for* $m > 3$ *then,* $ECI_a(C_{2m,m}) \geq ECI_a(U_{2m,m})$.

Claim 3.7.2. *$U^{'}$ is a cycle graph with exactly one pendant vertex attached to each of the vertices of the cycle. Let $C_k = c_0c_1\dots c_{k-1}c_0$ be the cycle, then $k = \frac{2m}{2}$, since the graph's order is 2m and each vertex of the cycle are attached to a single pendant vertex. Thus for*

$$ECI_a(U^{'}) = \left\lfloor \frac{m}{2} \right\rfloor \cdot m \cdot (3^a + 1) + m \cdot 3^a + 2m,$$

we obtain

$$\begin{aligned} ECI_a(U^{'}) - ECI_a(U_{2m,m}) = &\left\lfloor \frac{m}{2} \right\rfloor \cdot m \cdot (3^a + 1) + m \cdot 3^a + 2m - [2(m+1)^a + 3m \cdot 2^a \\ &+ 4m - 5] \end{aligned}$$

Now for $0 < a < 1$, *let*

$$\begin{aligned} F(m) &= \left\lfloor \frac{m}{2} \right\rfloor \cdot m \cdot (3^a + 1) + m \cdot 3^a + 2m - [2(m+1)^a + 3m \cdot 2^a + 4m - 5], \\ &= \left\lfloor \frac{m}{2} \right\rfloor \cdot m \cdot (3^a + 1) + m \cdot 3^a - 2(m+1)^a - 3m \cdot 2^a - 2m + 5. \end{aligned}$$

Since

$$m \cdot \left\lfloor \frac{m}{2} \right\rfloor = \begin{cases} \frac{m^2}{2} & \text{if } m = even \\ \frac{m^2 - m}{2} & \text{if } m = odd \end{cases}$$

For m is even we have:

$$\begin{aligned} F'(m) &= m \cdot (3^a + 1) + 3^a - 2a(m+1)^{a-1} - 3 \cdot 2^a - 2 \\ &= (m+1) \cdot 3^a - 3 \cdot 2^a + m - 2 - 2a(m+1)^{a-1} \\ &\geq m + 1 - 3 + m - 2 - 2a(m+1)^{a-1} \\ &\geq 2m - 4 \geq 0, m \geq 3 \end{aligned}$$

For m is odd we have:

$$\begin{aligned} F'(m) &= \left(\frac{2m-1}{2} \right) (3^a + 1) + 3^a - 2a(m+1)^{a-1} - 3 \cdot 2^a - 2 \\ &= \left(m + \frac{1}{2} \right) \cdot 3^a - 3 \cdot 2^a + m - \left(\frac{5}{2} \right) - 2a(m+1)^{a-1} \\ &\geq m + \frac{1}{2} - 3 + m - \frac{5}{2} - 2a(m+1)^{a-1} \\ &\geq 2m - 6 \geq 0, m \geq 3 \end{aligned}$$

Hence, F(m) is an increasing function on $0 < a < 1$. *And since* $m \geq 3$,

$$\begin{aligned} F(m) \geq F(3) &= 9 \cdot 3^a - 9 \cdot 2^a - 2 \cdot 4^a + 5 \\ &= 9 \cdot 3^a - 9 \cdot 2^a - 2^{2a+1} + 5 \\ &= 9 \cdot 3^a - 2^a(9 - 2^{a+1}) + 5 > 0. \end{aligned}$$

Then we have

$$2m \cdot \left\lfloor \frac{m}{2} \right\rfloor - 4m + 3 \leq ECI_a(U^{'}) - ECI_a(U_{2m,m}) \leq 4m \cdot \left\lfloor \frac{m}{2} \right\rfloor - 7m + 3$$

Thus $ECI_a(U^{'}) - ECI_a(U_{2m,m}) \geq 0$ *for* $m > 3$. *Hence,* $ECI_a(U^{'}) \geq ECI_a(U_{2m,m})$.

Claim 3.7.3. $U^{'}$ *is not* C_{2m} *or the graph in claim 3.7.2. Suppose* $U^{'}$ *contain a pendant vertex which is the furthest one from the root vertex of the tree attached to it, say v. Let N(v) = w, then by Lemma1.3 d(w) = 2, such that there exists a unique vertex* $z \neq v$, *where* $zw \in E(U^{'})$. *Suppose* $V(U^{''}) = V(U^{'}) \backslash \{v, w\}$ *and* $E(U^{''}) = E(U^{'}) \backslash \{vw, wz\}$. *Which is* $U^{''}$ *has order 2m-2 and a perfect matching m-1. Then we have* $d(z) = p \geq 2$ *and since at most one edge incident to z is in the perfect matching M,* $U^{'}$ *has at least m-1 edges(which are in M). Such that* $2m \geq p + m - 1$, *implies that* $p \leq m + 1$.

Let

$$E = \sum_{i \in V(U^{''}) \backslash \{z\}} ecc_{U^{''}}(i) d^a_{U^{''}}(i) \leq \sum_{i \in V(U^{'}) \backslash \{z,v,w\}} ecc_{U^{'}}(i) d^a_{U^{'}}(i) = F$$

Note that $d_{U^{''}}(z) = p - 1$. *Then by induction hypothesis*

$$\begin{aligned} ECI_a(U^{''}) = E + ecc_{U^{''}}(z) d^a_{U^{''}}(z) &\geq 2(m - 1 + 1)^a + (3m - 3)2^a + 4(m - 1) - 5. \\ &= 2(m^a) + (3m - 3)2^a + 4m - 9. \end{aligned} \tag{3.1}$$

Then we obtain

$$F \geq E \geq 2(m^a) + (3m - 3)2^a + 4m - 9 - ecc_{U^{''}}(z) d_{U^{''}}(z).$$

Thus

$$\begin{aligned}ECI_a(U^{'}) =& ecc_{U'}(z)(p^a)+F+ecc_{U'}(v)+ecc_{U'}(w)2^a\\ &\geq ecc_{U'}(z)(p^a)+ecc_{U'}(v)+ecc_{U'}(w)2^a+2(m^a)+(3m-3)2^a+4m-9\\ &-ecc_{U''}(z)(p-1)^a\\ &\geq ecc_{U''}(z)(p^a-(p-1)^a)+ecc_{U'}(v)+2(m^a)+(3m-3+ecc_{U'}(w))2^a\\ &+4m-9. \qquad (3.2)\end{aligned}$$

And since, $2 \leq p \leq m+1$

$$\begin{aligned}ECI_a(U^{'}) \geq & ecc_{U''}(z)((m+1)^a-m^a)+ecc_{U'}(v)+2(m^a)+(3m-3+ecc_{U'}(w))2^a+\\ & 4m-9\\ &= ecc_{U''}(z)(m+1)^a+ecc_{U'}(v)+(2-ecc_{U''}(z))m^a+(3m-3+ecc_{U'}(w))2^a\\ &+4m-9. \qquad (3.3)\end{aligned}$$

The equality in (3.1) and (3.2) holds if and only if $U^{''}$ *is* $U_{2m,m}$*, Thus the equality in (3.3) holds if and only if* $U^{''}$ *is* $U_{2m,m}$*,* $d_{U'}(z) = m+1$*, and* $ecc_{U'}(v) = 3 = ecc_{U'}(w), ecc_{U'}(z) = 2$*, which implies that* $U^{'}$ *is* $U_{2m,m}$*.*

Suppose $U_{n,m}$ be a unicyclic graph having cycle three with $n-2m+1$ pendant vertices and $m-2$ paths length two attached to one of the vertices of the cycle. See Figure 3.4

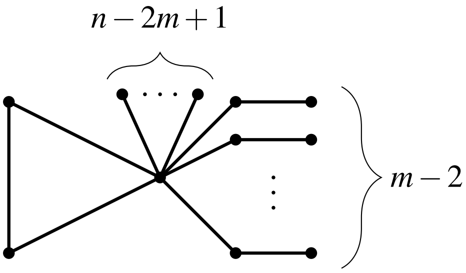

Figure 3.4: $U_{n,m}$

Theorem 3.8. *Suppose U be any unicyclic graph order n and the matching number* $m \geq 2$*, where* $n \geq 2m$*. Then for* $0 < a < 1$*, we have*

$$ECI_a(U) \geq 2(n-m+1)^a + (3m)2^a + 3n - 2m - 5,$$

equality holds if and only if U is $U_{n,m}$.

Proof: Theorem 3.7 shows that $U_{2m,m}$ is a minimal graph among the unicyclic graph having order 2m and matching number m. Hence in our proof, we only consider for $n > 2m$. $U_{n,m}$ has one vertex of degree n - m + 1, m vertices of degree 2, and n - m + 1 pendant vertices. Thus

$$ECI_a(U_{n,m}) = 2(n-m+1)^a + (3m)2^a + 3n - 2m - 5.$$

We will prove by induction on order n. Let the Theorem hold for all unicyclic graphs with an order less than n. Let U be a unicyclic graph having order n and matching number m,

If U does not contain a pendant vertex then $U \cong C_n$ and n = 2m + 1. We get

$$ECI_a(U) = \left(n \cdot \left\lfloor \frac{n}{2} \right\rfloor\right) \cdot 2^a = \left((2m+1) \cdot \left\lfloor \frac{2m+1}{2} \right\rfloor\right) \cdot 2^a > 2 \cdot (m+2)^a + (3m) \cdot 2^a + 4m - 2$$

$$= ECI_a(U_{2m+1,m}),$$

since $0 < a < 1$.

So we can assume that U has a pendant vertex, say v. By lemma 1.4, U contains a maximum M which doesn't match v. Let w be the neighbor of v in U. Let $U^{'}$ be a unicyclic graph such that $V(U^{'}) = V(U) \backslash \{v\}$ and $E(U^{'}) = E(U) \backslash \{vw\}$. Then $U^{'}$ has order n - 1 and matching number m. We have $d_u(w) = p \geq 2$ and since at most one edge incident to w is in M, U contains at least m - 1 other edges (which are in M). Hence, $n \geq p + m - 1$, implies $p \leq n - m + 1$.

Let

$$E = \sum_{i \in V(U') \backslash \{w\}} ecc_{U'}(i) d^a_{U'}(i) \leq \sum_{i \in V(U) \backslash \{v,w\}} ecc_U(i) d^a_U(i) = F.$$

Note that $d_{U'}(w) = p - 1$. Then by induction hypothesis

$$ECI_a(U^{'}) = ecc_{U'}(w)(p-1)^a + E \geq 2(n-1-m+1)^a + 3m \cdot 2^a + 3(n-1) - 2m - 5 \tag{3.4}$$

Then we obtain

$$F \geq E \geq 2(n-m)^a + (3m)2^a + 3n - 2m - 8 - ecc_{U'}(w)(p-1)^a.$$

Thus

$$\begin{aligned} ECI_a(U) = 5ecc_U(w)\cdot p^a + ecc_U(v) + F &\geq ecc_U(w)\cdot p^a + ecc_U(v) + 2(n-m)^a + (3m)2^a \\ &+ 3n - 2m - 8 - ecc_{U'}(w)(p-1)^a \\ &\geq ecc_{U'}(w)(p^a - (p-1)^a) + 2(n-m)^a + (3m)2^a + 3n - 2m - 8 + ecc_U(v). \end{aligned} \tag{3.5}$$

And since $2 \leq p \leq n-m+1$,

$$\begin{aligned} ECI_a(U) &\geq ecc_{U'}(w)[(n-m+1)^a - (n-m+1-1)^a] + 2(n-m)^a + (3m)2^a + 3n - 2m \\ &- 8 + ecc_U(v) \\ &= ecc_{U'}(w)[(n-m+1)^a - (n-m)^a] + 2(n-m)^a + (3m)2^a + 3n - 2m - 8 \\ &+ ecc_U(v). \end{aligned} \tag{3.6}$$

The equality in (3.4) and (3.5) holds if and only if $U^{'}$ *is* $U_{n-1,m}$. *Thus equality in (3.6) holds if and only if* $U^{'}$ *is* $U_{n-1,m}$ *and* $d_{U(w)} = n-m+1, ecc_U(w) = 2, ecc_U(v) = 3$. *which implies that* U *is* $U_{n,m}$.

Chapter 4

General Degree-Eccentricity Index Of Graphs

Using various graph characteristics(parameters), we address the general-degree-eccentricity index of graphs in this chapter.

4.1 Graphs of given order

This part deals with the boundaries for a graph's general degree-eccentricity index having n vertices.

Lemma 4.1. *[50] Suppose u and v are non-adjacent vertices in G, then for $a < 0$, and $b = 1$ we have*

$$ECI_a(G) < ECI_a(G+uv).$$

Theorem 4.1. *[50] Suppose G is a connected simple graph on n vertices and m edges, then for $a < 0$, and $b = 1$ we have*

$$ECI_a(G) \leq 2m \leq n(n-1)$$

equality holds if and only if $G \cong K_n$.

Let n_1 indicate the number of vertices in G that have an eccentricity of one. Suppose $K_p - qe$ be the graph attained from K_p by elimination of q independent edges for $0 \leq q \leq \lfloor \frac{n}{2} \rfloor$.

Theorem 4.2. *[50] Suppose G be a connected graph of order n with $n_1 \geq 1$ number of* *vertices with eccentricity one and for* $a < 0$, *and* $b = 1$ *we have*

$$ECI_a(G) \leq n2^a(n-2) + n_1(n-1-(n-2)2^a)$$

and the equality holds if and only if $G \cong K_{n_1} + (K_{n-pn_1} - \frac{n-n_1}{2}e)$, *where* $n - n_1$ *is even.*

4.2 Graph of a given matching number

Theorem 4.3. *[27] Suppose* G *be a connected graph of order* n *with matching number* m, *where* $2 \leq m \leq \frac{n}{2}$. *Then for* $a \geq 1$ *and* $b < 0$,

$$ECI_a(G) \leq ECI_a(K_m \vee \overline{K_{n-m}})$$

equality holds if and only if G *is* $K_m \vee \overline{K_{n-m}}$.

Let G_1 and G_2 be two graphs retrieved from $K_{n-\alpha} + \overline{K_\alpha}$ by elimination of an edge joining two vertices in $K_{n-\alpha}$ and K_α and eliminating an edge incident two vertices in $K_{n-\alpha}$, respectively.

Theorem 4.4. *[50] Let* G *be a graph with order* $n > 5$ *having independence number* α. *Then, for* $a < 0$, *and* $b = 1$ *we have*

1. $ECI_a(G) \leq (n-\alpha)(n-1) + \alpha(n-\alpha)2^a$ *and the equality holds if and only if* $G \cong K_{n-\alpha} + \overline{K_\alpha}$,
2. $ECI_a(G) \leq (n+\alpha-1)(n-1) + (n-2)2^a + (n-\alpha-1)2^a + (\alpha-1)(n-\alpha)2^a$ *and the equality holds if and only if* $G \cong G_1$.

Chapter 5

Conclusion and Future Work

5.1 Conclusion

In this thesis, the first chapter introduced graph theoretical terms and topological indices that are helpful in the thesis. In the second chapter, we explored the general degree-eccentricity index of a tree graph in a given order whose bounds were also mentioned in the first section. In sections 2 and 3 we stated bounds for ECI_a of a tree in a given order with a given diameter, and in a given order and a corresponding matching number. The third Chapter deals with ECI_a of a unicyclic graph. Bounds on a unicyclic graph in a given order and a given order with girth were stated in the first and the second sections of Chapter 3. In section 3 We provided both lower and upper limits on ECI_a for unicyclic graphs of a given matching number, where $0 < a < 1, b = 1$. In the fourth chapter we explored ECI_a of a graph in a given order, and a given order with a given matching number and we also stated some extremal values.

5.2 Future Research

We list a few difficult open issues on the general degree-eccentricity index for unicyclic graphs.

Problem 5.1 Determine the upper and lower boundaries on $ECI_a(U)$ for unicyclic graph U of a given order and matching number, where $a > 1, b = 1$

Problem 5.2 Determine the upper and lower boundaries on $ECI_a(U)$ for unicyclic graph U of a given order and matching number, where $0 < a < 1, b > 0$

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