SUPERPOSITION OF SQUEEZED STATES

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Abstract

We discuss the statistical and squeezing properties of squeezed coherent state, squeezed vacuum state and displaced squeezed vacuum state. We also study the quantum properties of the superposition of squeezed coherent and squeezed vacuum states as well as the superposition of displaced squeezed vacuum and squeezed vacuum states.

It so turns out that the mean photon number of the superposed state is the sum of the mean photon number of the separate squeezed states. In addition, we note that the superposition of a coherent state and a squeezed vacuum state is the displaced squeezed vacuum state.

Furthermore, the superposition leads to a two-fold increase in the degree of squeezing. Unlike the individual squeezed states, the squeezing occurs for $0 < r < \frac{1}{2}\ln 2$.

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1. Introduction

Developing new states of light has been an important topic in quantum optics and quantum information theories [7]. Squeezing is one of the interesting nonclassical features of light that has been attracting attention and studied by many authors [1-10]. Hence, there has been considerable interest in attempts to produce purely quantum states of light such as squeezed states and the analysis of the quantum properties of these states generated by various quantum optical systems.

In squeezed light the fluctuations in one quadrature is below the coherent level at the expense of enhanced fluctuations in the other quadrature, with the product of the uncertainties in the two quadratures satisfying the uncertainty relation [1, 4].

Squeezed states are produced by various optical processes such as subharmonic generation and second harmonic generation [1, 2]. These states exhibit the nonclassical feature of squeezing.

Squeezed states have several potential applications such as in low-noise communications, precise measurements and detection of weak signals [8-10].

We discuss the statistical and squeezing properties of squeezed coherent state $|\gamma, r\rangle$, squeezed vacuum state $|r\rangle$ and displaced squeezed vacuum state $|\lambda, r\rangle$.

We also study the quantum properties of the superposition of squeezed coherent and squeezed vacuum states as well as the superposition of displaced squeezed vacuum and squeezed vacuum states.

2. Squeezed States

It is possible to generate states of light in which the fluctuations in one quadrature is below the coherent state level at the expense of enhanced fluctuations in the canonically conjugate quadrature such that the uncertainity relation is not violated. Such states of light are called squeezed states.

We consider here squeezed coherent state, squeezed vacuum state and displaced squeezed vacuum state. We find the mean photon number, the normally-ordered variance of the photon number and the quadrature variance of these squeezed states .

Moreover, using the antinormally-ordered characteristic function, we calculate the Q function of various squeezed states. Finally, with the help of the resulting Q function, we determine the photon number distribution.

2.1 Squeezed Coherent State

A squeezed coherent state is defined by

$$|\gamma, r\rangle = \hat{s}(r)|\gamma\rangle, \tag{2.0.1}$$

where γ is a complex number and

$$\hat{s}(r) = e^{\frac{r}{2}(\hat{a}^2 - \hat{a}^{\dagger 2})} \tag{2.0.2}$$

is a squeeze operator, with r being the squeeze parameter taken to be real and positive for convienence.

Observable quantities associated with a single-mode light are represented by operators formed by taking Hermitian combinations of $\hat{a}$ and $\hat{a}^\dagger$. The most important of these are the plus and minus quadrature operators defind by

$$\hat{a}_+ = \hat{a}^\dagger + \hat{a}, \tag{2.0.3}$$

$$\hat{a}_- = i(\hat{a}^\dagger - \hat{a}). \tag{2.0.4}$$

These operators satisfy the commutation relation

$$[\hat{a}_+, \hat{a}_-] = 2i. \tag{2.0.5}$$

A single-mode light is said to be in a squeezed state if either $\Delta\hat{a}_+ < 1$ or $\Delta\hat{a}_- < 1$ such that $\Delta\hat{a}_+\Delta\hat{a}_- \geq 1$. A squeezed state for which the equality holds is called a minimum uncertainity state.

2.1.1 The Q Function

The Q function is an important tool in quantum optics. Knowing this function, all the nonclassical effect can be predicted and the different moments of the operators can be evaluated. We now proceed to determine the Q function for a squeezed coherent state. The Q function is expressible interms of the antinormally-ordered characteristic function as

$$Q(\alpha^*, \alpha) = \frac{1}{\pi^2}\int d^2z\phi_a(z)e^{(z^*\alpha - z\alpha^*)}, \tag{2.0.6}$$

where the antinormally-ordered characteristic function $\phi_a(z)$ is defined by

$$\phi_a(z) = Tr(\hat{\rho}e^{-z^*\hat{a}}e^{z\hat{a}^\dagger}). \tag{2.0.7}$$

This can be rewritten as

$$\phi_a(z) = Tr(|\gamma, r\rangle\langle\gamma, r|e^{-z^*\hat{a}}e^{z\hat{a}^\dagger}), \tag{2.0.8}$$

where

$$\hat{\rho} = |\gamma, r\rangle\langle\gamma, r| \tag{2.0.9}$$

is the density operator for a squeezed coherent state. Using the unitarity property of the squeeze operator and applying the trace operation, we obtain

$$\phi_a(z) = \langle\gamma|e^{-z^*\hat{a}(r)}e^{z\hat{a}^\dagger(r)}|\gamma\rangle, \tag{2.0.10}$$

where $\hat{a}(r)$ and $\hat{a}^\dagger(r)$ are defined by

$$\hat{a}(r) = \hat{s}^\dagger(r)\hat{a}\hat{s}(r) \tag{2.0.11}$$

and

$$\hat{a}^\dagger(r) = \hat{s}^\dagger(r)\hat{a}^\dagger\hat{s}(r). \tag{2.0.12}$$

Applying the Baker-Hausdorff identity

$$e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}+\frac{1}{2}[\hat{A},\hat{B}]}, \tag{2.0.13}$$

one finds

$$\phi_a(z) = e^{-\frac{1}{2}z^*z}\langle\gamma|exp[z\hat{a}^\dagger(r) - z^*\hat{a}(r)]|\gamma\rangle. \tag{2.0.14}$$

Furthermore, taking the derivative of (2.0.11) with respect to r, we have

$$\frac{d}{dr}\hat{a}(r) = -\frac{1}{2}[\hat{a}(r), \hat{a}^{\dagger 2}(r)]. \tag{2.0.15}$$

In view of the fact that

$$[\hat{a}(r), \hat{a}^{\dagger 2}(r)] = 2\hat{a}^\dagger(r), \tag{2.0.16}$$

we see that

$$\frac{d}{dr}\hat{a}(r) = -\hat{a}^\dagger(r) \tag{2.0.17}$$

and

$$\frac{d}{dr}\hat{a}^\dagger(r) = -\hat{a}(r). \tag{2.0.18}$$

These are coupled differential equations. In order to decouple these diferential equations, we defferentiate once more (2.0.17) with respect to r and then we see

$$\frac{d^2}{dr^2}\hat{a}(r) = \hat{a}(r). \tag{2.0.19}$$

The solution of this second order differential equation can be put in the form

$$\hat{a}(r) = \hat{b}_1e^r + \hat{b}_2e^{-r}. \tag{2.0.20}$$

We then see that

$$\hat{a}(0) = \hat{b}_1 + \hat{b}_2 = \hat{a} \tag{2.0.21}$$

and

$$\frac{d}{dr}\hat{a}(r)|_{r=0} = \hat{b}_1 - \hat{b}_2 = -\hat{a}^\dagger. \tag{2.0.22}$$

This leads to

$$\hat{b}_1 = \frac{1}{2}(\hat{a} - \hat{a}^\dagger) \tag{2.0.23}$$

and

$$\hat{b}_2 = \frac{1}{2}(\hat{a} + \hat{a}^\dagger). \tag{2.0.24}$$

Substituting (2.0.23) and (2.0.24) into (2.0.20), we obtain

$$\hat{a}(r) = \hat{a}\cosh r - \hat{a}^\dagger \sinh r \tag{2.0.25}$$

and its complex conjugate is taken to be

$$\hat{a}^\dagger(r) = \hat{a}^\dagger \cosh r - \hat{a}\sinh r. \tag{2.0.26}$$

Now inserting (2.0.25) and (2.0.26) into (2.0.14), we get

$$\phi_a(z) = e^{-\frac{1}{2}z^*z}\langle\gamma|exp[(z\cosh r + z^*\sinh r)\hat{a}^\dagger - (z\sinh r + z^*\cosh r)\hat{a}]|\gamma\rangle. \tag{2.0.27}$$

Applying the Baker-Hausdorff identity once more, we have

$$\begin{aligned}\phi_a(z) = \quad & exp[-z^*z\cosh^2 r - \cosh r\sinh r(z^{*2} + z^2)/2] \\ & \times\langle\gamma|exp(z\cosh r + z^*\sinh r)\hat{a}^\dagger \\ & \times exp - (z\sinh r + z^*\cosh r)\hat{a}|\gamma\rangle.\end{aligned} \tag{2.0.28}$$

We then see that

$$\begin{aligned}\phi_a(z) = \quad & exp[-z^*z\cosh^2 r - \cosh r\sinh r(z^{*2} + z^2)/2 \\ & +(\gamma^*\cosh r - \gamma\sinh r)z - (\gamma\cosh r - \gamma^*\sinh r)z^*],\end{aligned} \tag{2.0.29}$$

so that on combinig this result with (2.0.6), there follows

$$\begin{aligned}Q(\alpha^*,\alpha) = \quad & \frac{1}{\pi}\int\frac{d^2z}{\pi}exp[-z^*z\cosh^2 r - \cosh r\sinh r(z^{*2} + z^2)/2 \\ & +(\gamma^*\cosh r - \gamma\sinh r - \alpha^*)z - (\gamma\cosh r - \gamma^*\sinh r - \alpha)z^*].\end{aligned} \tag{2.0.30}$$

Thus on performing the integration employing the relation

$$\int \frac{d^2z}{\pi} exp[-az^*z + bz + cz^* + Az^2 + Bz^{*2}]$$
$$= \left[\frac{1}{a^2 - 4AB}\right]^{1/2} exp\left[\frac{abc + Ac^2 + Bb^2}{a^2 - 4AB}\right], a > 0 \tag{2.0.31}$$

we obtain

$$\begin{aligned} Q(\alpha^*, \alpha) = \quad & \frac{sechr}{\pi} exp[-\alpha^*\alpha - \tanh r(\alpha^2 + \alpha^{*2} - \gamma^2 - \gamma^{*2})/2 \\ & + sechr(\gamma^*\alpha + \gamma\alpha^*) - \gamma^*\gamma]. \end{aligned} \tag{2.0.32}$$

2.1.2 Mean Photon number

We seek to determine the mean photon number for a squeezed coherent state. To this end, the expectaton value of $\hat{n}$ is expressible interms of a squeezed coherent state basis as

$$\langle \hat{n} \rangle = \langle \gamma, r | \hat{a}^\dagger \hat{a} | \gamma, r \rangle. \tag{2.0.33}$$

On account of the definition given by Eq. (2.0.1), this expression can be rewritten as

$$\langle \hat{n} \rangle = \langle \gamma | \hat{s}^\dagger(r) \hat{a}^\dagger \hat{a} \hat{s}(r) | \gamma \rangle. \tag{2.0.34}$$

Applying the unitarity property of the squeeze operator

$$\hat{s}(r)\hat{s}^\dagger(r) = \hat{I} \tag{2.0.35}$$

and using the definitions given by Eq. (2.0.11) and (2.0.12), we have

$$\langle \hat{n} \rangle = \langle \gamma | \hat{a}^\dagger(r) \hat{a}(r) | \gamma \rangle. \tag{2.0.36}$$

In view of (2.0.25) and (2.0.26), we get

$$\begin{aligned} \langle \hat{n} \rangle = \quad & \langle \gamma | \hat{a}^\dagger \hat{a} | \gamma \rangle \cosh^2 r - \langle \gamma | \hat{a}^{\dagger 2} | \gamma \rangle \cosh r \sinh r \\ & - \langle \gamma | \hat{a}^2 | \gamma \rangle \cosh r \sinh r + \langle \gamma | \hat{a}\hat{a}^\dagger | \gamma \rangle \sinh^2 r. \end{aligned} \tag{2.0.37}$$

Arranging the operators in normal order employing the commutation relation

$$[\hat{a}, \hat{a}^\dagger] = 1, \tag{2.0.38}$$

one readily finds

$$\langle \hat{n} \rangle = |\gamma \cosh r - \gamma^* \sinh r|^2 + \sinh^2 r. \tag{2.0.39}$$

2.1.3 Normally-ordered variance of the Photon number

The normally-ordered variance of the photon number is defined by

$$:(\Delta n)^2 := \langle : \hat{n}^2 : \rangle - \langle \hat{n} \rangle^2, \tag{2.0.40}$$

where :: denotes normal ordering of the quantum mechanical operators. For a squeezed coherent state, the expectation value of $: \hat{n}^2 :$ is expressible as

$$\langle : \hat{n}^2 : \rangle = \langle \gamma, r | \hat{a}^{\dagger 2} \hat{a}^2 | \gamma, r \rangle. \tag{2.0.41}$$

In view of (2.0.1), we write

$$\langle : \hat{n}^2 : \rangle = \langle \gamma | \hat{s}^\dagger(r) \hat{a}^{\dagger 2} \hat{a}^2 \hat{s}(r) | \gamma \rangle. \tag{2.0.42}$$

Now applying the identity given by (2.0.35), for the squeeze operator, we have

$$\langle : \hat{n}^2 : \rangle = \langle \gamma | \hat{s}^\dagger(r) \hat{a}^\dagger \hat{s}(r) \hat{s}^\dagger(r) \hat{a}^\dagger \hat{s}(r) \hat{s}^\dagger(r) \hat{a} \hat{s}(r) \hat{s}^\dagger(r) \hat{a} \hat{s}(r) | \gamma \rangle. \tag{2.0.43}$$

On account of (2.0.11) and (2.0.12), we see

$$\langle : \hat{n}^2 : \rangle = \langle \gamma | \hat{a}^{\dagger 2}(r) \hat{a}^2(r) | \gamma \rangle. \tag{2.0.44}$$

Now taking into account (2.0.25) and (2.0.26), we can write

$$\langle : \hat{n}^2 : \rangle = \langle \gamma | [\hat{a}^\dagger \cosh r - \hat{a} \sinh r]^2 \times [\hat{a} \cosh r - \hat{a}^\dagger \sinh r]^2 | \gamma \rangle. \tag{2.0.45}$$

Upon multiplication, there follows

$$\begin{aligned}
\langle : \hat{n}^2 : \rangle = \quad & 4\langle \gamma | \hat{a}^\dagger \hat{a} | \gamma \rangle \cosh^2 r \sinh^2 r - 5 \cosh r \sinh^3 r [\langle \gamma | \hat{a}^{\dagger 2} | \gamma \rangle + \langle \gamma | \hat{a}^2 | \gamma \rangle] \\
& + \langle \gamma | \hat{a}^{\dagger 4} | \gamma \rangle \cosh^2 r \sinh^2 r + 4 \langle \gamma | \hat{a}^{\dagger 2} \hat{a}^2 | \gamma \rangle \cosh^2 r \sinh^2 r + 2 \sinh^4 r \\
& + \langle \gamma | \hat{a}^{\dagger 2} \hat{a}^2 | \gamma \rangle \cosh^4 r - 2 \cosh^3 r \sinh r [\langle \gamma | \hat{a}^{\dagger 3} \hat{a} | \gamma \rangle + \langle \gamma | \hat{a}^\dagger \hat{a}^3 | \gamma \rangle] \\
& + \langle \gamma | \hat{a}^4 | \gamma \rangle \cosh^2 r \sinh^2 r + 4 \langle \gamma | \hat{a}^\dagger \hat{a} | \gamma \rangle \sinh^4 r + \cosh^2 r \sinh^2 r \\
& - \langle \gamma | \hat{a}^2 | \gamma \rangle \cosh^3 r \sinh r - \langle \gamma | \hat{a}^{\dagger 2} | \gamma \rangle \cosh^3 r \sinh r.
\end{aligned} \tag{2.0.46}$$

It then follows that

$$\begin{aligned}\langle : \hat{n}^2 : \rangle = \quad & -2\cosh^3 r \sinh r(\gamma^*\gamma^3 + \gamma^{*3}\gamma) + 4\gamma^{*2}\gamma^2 \cosh^2 r \sinh^2 r \\ & +\gamma^{*2}\gamma^2 \cosh^4 r - \cosh^3 r \sinh r(\gamma^{*2} + \gamma^2) + 4\gamma^*\gamma \sinh^4 r \\ & + \cosh^2 r \sinh^2 r(\gamma^4 + \gamma^{*4}) - 5\cosh r \sinh^3 r(\gamma^2 + \gamma^{*2}) \\ & +4\gamma^*\gamma \cosh^2 r \sinh^2 r + 2\sinh^4 r + \cosh^2 r \sinh^2 r.\end{aligned} \tag{2.0.47}$$

Employing this result along with (2.0.39), Eq. (2.0.40) can be put in the form

$$\begin{aligned}: (\Delta n)^2 := \quad & -\cosh^3 r \sinh r(\gamma^{*2} + \gamma^2) - \cosh r \sinh^3 r(4\gamma^{*2} + 3\gamma^2) \\ & +2\gamma^*\gamma \sinh^4 r + 2\gamma^*\gamma \cosh^2 r \sinh^2 r + \cosh^2 r \sinh^2 r \\ & -2\cosh r \sinh^3 r(\gamma^{*3}\gamma + \gamma^*\gamma^3) + \gamma^{*2}\gamma^2 \sinh^4 r + \sinh^4 r.\end{aligned} \tag{2.0.48}$$

Now rearranging or collecting similar terms, the normally-ordered variance of the photon number for a squeezed coherent state finally takes the form

$$\begin{aligned}: (\Delta n)^2 := \quad & \sinh^2 r(\cosh^2 r + \sinh^2 r) - \cosh^3 r \sinh r(\gamma^{*2} + \gamma^2) \\ & - \cosh r \sinh^3 r[2(\gamma^{*3}\gamma + \gamma^*\gamma^3) + (4\gamma^{*2} + 3\gamma^2)] \\ & +\gamma^*\gamma \sinh^2 r(2\sinh^2 r + 2\cosh^2 r + \gamma^*\gamma \sinh^2 r).\end{aligned} \tag{2.0.49}$$

2.1.4 Photon number distribution

The photon number distribution $P(n)$ is the probability of finding n photons in a given single-mode light. So that, the photon number distribution is expressible interms of the Q function as[1]

$$P(n) = \frac{\pi}{n!}\frac{\partial^{2n}}{\partial\alpha^{*n}\partial\alpha^{n}}[Q(\alpha^*,\alpha)e^{\alpha^*\alpha}]|_{\alpha^*=\alpha=0}. \tag{2.0.50}$$

Now in view of (2.0.32), this expression can be rewritten as

$$\begin{aligned} P(n) = &\quad \frac{sechr}{n!}exp[-\gamma\gamma^* + \tanh r(\gamma^2+\gamma^{*2})/2]\frac{\partial^{2n}}{\partial\alpha^{*n}\partial\alpha^{n}} \\ &\times exp[-a(\alpha^2+\alpha^{*2}) + b(\gamma^*\alpha+\gamma\alpha^*)]_{\alpha^*=\alpha=0}, \end{aligned} \tag{2.0.51}$$

where

$$a = \frac{1}{2}\tanh r, \tag{2.0.52}$$

$$b = sechr. \tag{2.0.53}$$

On account of the power series expansions

$$e^{-a\alpha^2} = \sum_i \frac{(-a)^i}{i!}\alpha^{2i}, \tag{2.0.54}$$

$$e^{-a\alpha^{*2}} = \sum_j \frac{(-a)^j}{j!}\alpha^{*2i}, \tag{2.0.55}$$

$$e^{b\gamma^*\alpha} = \sum_k \frac{(b\gamma^*)^k}{k!}\alpha^{k}, \tag{2.0.56}$$

$$e^{b\gamma\alpha^*} = \sum_l \frac{(b\gamma)^l}{l!}\alpha^{*l}, \tag{2.0.57}$$

expression (2.0.51) can be put in the form

$$\begin{aligned} P(n) = &\frac{sechr}{n!}exp^{[-\gamma\gamma^*+\tanh r(\gamma^2+\gamma^{*2})/2]} \\ &\times \sum_{ijkl}\frac{\gamma^l\gamma^{*k}(-a)^{i+j}b^{k+l}}{i!j!k!l!}\frac{\partial^{2n}}{\partial\alpha^{n}\partial\alpha^{*n}}\left(\alpha^{2i+k}\alpha^{*2j+l}\right)|_{\alpha=\alpha^*=0}, \end{aligned} \tag{2.0.58}$$

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[1] For the derivatin of Eq. (2.0.50) see K. Fesseha, fundamentals of quantum optics(Lulu, United States of America, 2008).

so that carrying out the differentiation using the identity

$$\frac{\partial^n}{\partial x^n}x^m = \frac{m!}{(m-n)!}x^{m-n} \tag{2.0.59}$$

and applying the condition $\alpha = \alpha^* = 0$, we find

$$\begin{aligned} P(n) = \quad & \frac{sechr}{n!}exp^{[-\gamma\gamma^*+\tanh r(\gamma^2+\gamma^{*2})/2]} \\ & \times \sum_{ijkl} \frac{\gamma^l\gamma^{*k}(-a)^{i+j}b^{k+l}}{i!j!k!l!}\frac{(2i+k)!}{(2i+k-n)!}\frac{(2j+l)!}{(2j+l-n)!}\delta_{2i+k,n}\delta_{2j+l,n}. \end{aligned} \tag{2.0.60}$$

Hence in view of the property of the Kroncker delta , we see

$$k = n - 2i, \tag{2.0.61}$$

$$l = n - 2j, \tag{2.0.62}$$

and on taking into account that a factorial is defined for non negative integers, we have

$$\begin{aligned} P(n) \quad &= \frac{\tanh^n r}{n!2^n\cosh r}exp[-\gamma\gamma^* + \tanh r(\gamma^2+\gamma^{*2})/2] \\ & \left(\sum_{i=0}^{[n]}\frac{(-1)^i n!}{i!(n-2i)!}(\frac{\gamma^* b}{\sqrt{a}})^{n-2i}\right)\left(\sum_{j=0}^{[n]}\frac{(-1)^j n!}{j!(n-2j)!}(\frac{\gamma b}{\sqrt{a}})^{n-2j}\right), \end{aligned} \tag{2.0.63}$$

where $[n] = \frac{n}{2}$ for even n and $[n] = \frac{(n-1)}{2}$ for odd n. And hence, the photon number distribution can be put in the form

$$\begin{aligned} P(n) = \quad & \frac{\tanh^n r}{n!2^n\cosh r}exp[-\gamma\gamma^* + \tanh r(\gamma^2+\gamma^{*2})/2] \\ & \left|H_n(\frac{\gamma}{\sqrt{2\cosh r\sinh r}})\right|^2, \end{aligned} \tag{2.0.64}$$

where

$$H_n(x) = \sum_{k}^{[n]}\frac{(-1)^k n!}{k!(n-2k)!}(2x)^{n-2k} \tag{2.0.65}$$

is a Hermite polynomial of order n.

2.1.5 Quadrature variance

The variance of the quadrature operators $\hat{a}_+$ and $\hat{a}_-$ are defined as

$$(\Delta a_+)^2 = \langle \hat{a}_+^2 \rangle - \langle \hat{a}_+ \rangle^2 \tag{2.0.66}$$

and

$$(\Delta a_-)^2 = \langle \hat{a}_-^2 \rangle - \langle \hat{a}_- \rangle^2 \tag{2.0.67}$$

respectively. On account of (2.0.3) and (2.0.4), these quadrature variances can be rewritten in the form

$$(\Delta a_+)^2 = 1 + 2\langle \hat{a}^\dagger \hat{a} \rangle + \langle \hat{a}^{\dagger 2} \rangle + \langle \hat{a}^2 \rangle - \langle \hat{a} \rangle^2 - \langle \hat{a}^\dagger \rangle^2 - 2\langle \hat{a}^\dagger \rangle \langle \hat{a} \rangle \tag{2.0.68}$$

and

$$(\Delta a_-)^2 = 1 + 2\langle \hat{a}^\dagger \hat{a} \rangle - \langle \hat{a}^{\dagger 2} \rangle - \langle \hat{a}^2 \rangle + \langle \hat{a} \rangle^2 + \langle \hat{a}^\dagger \rangle^2 - 2\langle \hat{a}^\dagger \rangle \langle \hat{a} \rangle. \tag{2.0.69}$$

In a squeezed coherent state, the expectation values of $\hat{a}^{\dagger 2}$ is expressible as

$$\langle \hat{a}^{\dagger 2} \rangle = \langle \gamma, r | \hat{a}^{\dagger 2} | \gamma, r \rangle. \tag{2.0.70}$$

In view of Eq. (2.0.1), we write

$$\langle \hat{a}^{\dagger 2} \rangle = \langle \gamma | \hat{s}^\dagger(r) \hat{a}^{\dagger 2} \hat{s}(r) | \gamma \rangle. \tag{2.0.71}$$

Now applying the unitarity property of the squeeze operator given by Eq. (2.0.35), we see that

$$\langle \hat{a}^{\dagger 2} \rangle = \langle \gamma | \hat{s}^\dagger(r) \hat{a}^\dagger \hat{s}(r) \hat{s}^\dagger(r) \hat{a}^\dagger \hat{s}(r) | \gamma \rangle. \tag{2.0.72}$$

On account of (2.0.12), we can write

$$\langle \hat{a}^{\dagger 2} \rangle = \langle \gamma | \hat{a}^{\dagger 2}(r) | \gamma \rangle. \tag{2.0.73}$$

In view of Eq. (2.0.26), we see that

$$\begin{aligned} \langle \hat{a}^{\dagger 2} \rangle = \quad & \langle \gamma | \hat{a}^{\dagger 2} | \gamma \rangle \cosh^2 r - 2\langle \gamma | \hat{a}^\dagger \hat{a} | \gamma \rangle \cosh r \sinh r \\ & + \langle \gamma | \hat{a}^2 | \gamma \rangle \sinh^2 r - \cosh r \sinh r. \end{aligned} \tag{2.0.74}$$

It then follows that

$$\begin{aligned}\langle\hat{a}^{\dagger 2}\rangle = \quad & \gamma^{*2}\cosh^2 r - 2\gamma^*\gamma\cosh r\sinh r \\ & +\gamma^2\sinh^2 r - \cosh r\sinh r. \end{aligned} \tag{2.0.75}$$

On the other hand, the expectation value of $\hat{a}^\dagger$ in a squeezed coherent state basis is expressible as

$$\langle\hat{a}^\dagger\rangle = \langle\gamma, r|\hat{a}^\dagger|\gamma, r\rangle. \tag{2.0.76}$$

This expression is equivalently written as

$$\langle\hat{a}^\dagger\rangle = \langle\gamma|\hat{s}^\dagger(r)\hat{a}^\dagger\hat{s}(r)|\gamma\rangle. \tag{2.0.77}$$

In view of (2.0.12), we write

$$\langle\hat{a}^\dagger\rangle = \langle\gamma|\hat{a}^\dagger(r)|\gamma\rangle. \tag{2.0.78}$$

Taking into account Eq. (2.0.26), we see that

$$\langle\hat{a}^\dagger\rangle = \langle\gamma|\hat{a}^\dagger|\gamma\rangle\cosh r - \langle\gamma|\hat{a}|\gamma\rangle\sinh r. \tag{2.0.79}$$

This expression yields

$$\langle\hat{a}^\dagger\rangle = \gamma^*\cosh r - \gamma\sinh r. \tag{2.0.80}$$

Following a similar procedure and in view of (2.0.11), (2.0.25) or taking the complex conjugate of Eq. (2.0.75) and (2.0.80), one can show that

$$\begin{aligned}\langle\hat{a}^2\rangle = \quad & \gamma^2\cosh^2 r - 2\gamma^*\gamma\cosh r\sinh r \\ & +\gamma^{*2}\sinh^2 r - \cosh r\sinh r \end{aligned} \tag{2.0.81}$$

and

$$\langle\hat{a}\rangle = \gamma\cosh r - \gamma^*\sinh r. \tag{2.0.82}$$

Upon substitution of (2.0.39), (2.0.80), (2.0.81) and (2.0.82) into (2.0.68) and (2.0.69), the variance of the plus quadrature takes the form

$$(\Delta a_+)^2 = 2\sinh^2 r - 2\cosh r\sinh r + 1, \tag{2.0.83}$$

from which, using the definition of hyperbolic sine and cosine, we have

$$\sinh^2 r = \frac{e^{2r} - 2 + e^{-2r}}{4}, \tag{2.0.84}$$

$$\cosh r \sinh r = \frac{e^{2r} - e^{-2r}}{4}. \tag{2.0.85}$$

In view of theses relation, expression (2.0.83) finally takes the form

$$(\Delta a_+)^2 = e^{-2r} \tag{2.0.86}$$

and the variance of the minus quadrature turns out to be

$$(\Delta a_-)^2 = e^{2r}. \tag{2.0.87}$$

We easily observe that, for $r > 0$, the variance of the quadrature operators show that the fluctuations in the first quadrature are below the coherent and vacuum noise level with enhanced flactuations in the second quadrature. It then immidiately follows that a squeezed coherent states are minimum uncertainity states having unequal uncertainity associated with each quadrature.

2.2 Squeezed Vacuum State

A squeezed vacuum state is generated by the action of the squeeze operator on the vacuum state:

$$|r\rangle = \hat{s}(r)|0\rangle, \tag{2.0.88}$$

where $\hat{s}(r)$ is the usual squeeze operator.

2.2.1 The Q Function

We proceed to detemine the Q function for a squeezed vacuum state. To this end, the Q function for a single-mode squeezed vacuum state is expressible as

$$Q(\alpha^*, \alpha) = \frac{1}{\pi^2}\int d^2z\phi_a(z)e^{(z^*\alpha - z\alpha^*)}, \tag{2.0.89}$$

where the antinormally-ordered characteristic function $\phi_a(z)$ is defined by

$$\phi_a(z) = Tr\big(\hat{\rho}e^{-z^*\hat{a}}e^{z\hat{a}^\dagger}\big). \tag{2.0.90}$$

This can be written as

$$\phi_a(z) = Tr\big(|r\rangle\langle r|e^{-z^*\hat{a}}e^{z\hat{a}^\dagger}\big), \tag{2.0.91}$$

where

$$\hat{\rho} = |r\rangle\langle r| \tag{2.0.92}$$

is the density operator for a squeezed vacuum state. Applying the trace operation and using the definition for a squeezed vacuum state given by Eq. (2.0.88) and in view of the identity (2.0.35), for a squeeze operator, we obtain

$$\phi_a(z) = \langle 0|e^{-z^*\hat{a}(r)}e^{z\hat{a}^\dagger(r)}|0\rangle, \tag{2.0.93}$$

where $\hat{a}(r)$ and $\hat{a}^\dagger(r)$ are defined by (2.0.11) and (2.0.12). Applying the Baker-Hausdorff identity given by Eq. (2.0.13), one finds

$$\phi_a(z) = e^{-\frac{1}{2}zz^*}\langle 0|exp[z\hat{a}^\dagger(r) - z^*\hat{a}(r)]|0\rangle, \tag{2.0.94}$$

So that introduction of (2.0.25) and (2.0.26) into (2.0.94), yields

$$\phi_a(z) = e^{-\frac{1}{2}zz^*}\langle 0|exp[(z\cosh r + z^*\sinh r)\hat{a}^\dagger - (z^*\cosh r + z\sinh r)\hat{a}]|0\rangle. \qquad (2.0.95)$$

Now applying the Baker-Hausdorff identity once more, we have

$$\begin{aligned}\phi_a(z) = \quad & exp\big[-zz^*\cosh^2 r - \cosh r\sinh r(z^{*2}+z^2)\big]\\ & \times\langle 0|exp(z\cosh r + z^*\sinh r)\hat{a}^\dagger\\ & \times exp - (z\sinh r + z^*\cosh r)\hat{a}|0\rangle. \end{aligned} \qquad (2.0.96)$$

It then follows that

$$\phi_a(z) = exp\big[-zz^*\cosh^2 r - \cosh r\sinh r(z^{*2}+z^2)\big]. \qquad (2.0.97)$$

On account of this result along with Eq. (2.0.89), there follows

$$Q(\alpha^*,\alpha) = \frac{1}{\pi^2}\int d^2z exp\big[-zz^*\cosh^2 r + z^*\alpha - z\alpha^* - \cosh r\sinh r(z^{*2}+z^2)/2\big]. \quad (2.0.98)$$

Thus on performing the integration employing the relation (2.0.31), we find

$$Q(\alpha^*,\alpha) = \frac{sechr}{\pi}exp\big[-\alpha^*\alpha - \tanh r(\alpha^{*2}+\alpha^2)/2\big]. \qquad (2.0.99)$$

One can also arrive at the same result by setting $\gamma = 0$ in expression (2.0.32).

2.2.2 Mean Photon number

We next proceed to derive the mean photon number for a squeezed vacuum state. In view of (2.0.88), the mean photon number of the squeezed vacuum state is expressible as

$$\langle\hat{n}\rangle = \langle r|\hat{n}|r\rangle, \qquad (2.0.100)$$

which can be rewritten as

$$\langle\hat{n}\rangle = \langle 0|\hat{s}^\dagger(r)\hat{a}^\dagger\hat{a}\hat{s}(r)|0\rangle, \qquad (2.0.101)$$

where

$$\hat{n} = \hat{a}^\dagger\hat{a} \qquad (2.0.102)$$

is the number operator. Applying the unitarity property of the squeeze operator and on account of (2.0.11) and (2.0.12), we have

$$\langle \hat{n} \rangle = \langle 0|\hat{a}^{\dagger}(r)\hat{a}(r)|0\rangle. \tag{2.0.103}$$

In view of (2.0.25) and (2.0.26), we write

$$\begin{aligned} \langle \hat{n} \rangle = \quad & \langle 0|\hat{a}^{\dagger}\hat{a}|0\rangle \cosh^2 r - \langle 0|\hat{a}^{\dagger 2}|0\rangle \cosh r \sinh r + \sinh^2 r \\ & -\langle 0|\hat{a}^{2}|0\rangle \cosh r \sinh r + \langle 0|\hat{a}^{\dagger}\hat{a}|0\rangle \sinh^2 r. \end{aligned} \tag{2.0.104}$$

It then follows that

$$\langle \hat{n} \rangle = \sinh^2 r. \tag{2.0.105}$$

2.2.3 Normally-ordered variance of the photon number

We seek to determine the normally-ordered variance of the photon number for a squeezed vacuum state. To this end, the expectation value of $\langle : \hat{n}^2 : \rangle$ in a squeezed vacuum state basis can be expressed as

$$\langle : \hat{n}^2 : \rangle = \langle r|\hat{a}^{\dagger 2}\hat{a}^2|r\rangle, \tag{2.0.106}$$

so that using Eq. (2.0.88), we equivalently rewrite in the form

$$\langle : \hat{n}^2 : \rangle = \langle 0|\hat{s}^{\dagger}(r)\hat{a}^{\dagger 2}\hat{a}^2\hat{s}(r)|0\rangle. \tag{2.0.107}$$

Applying the unitarity property of the squeeze operator and taking into account (2.0.11) and (2.0.12), we have

$$\langle : \hat{n}^2 : \rangle = \langle 0|\hat{a}^{\dagger 2}(r)\hat{a}^2(r)|0\rangle. \tag{2.0.108}$$

In view of (2.0.25) and (2.0.26), we write

$$\langle : \hat{n}^2 : \rangle = \langle 0|(\hat{a}^{\dagger} \cosh r - \hat{a} \sinh r)^2 \times (\hat{a} \cosh r - \hat{a}^{\dagger} \sinh r)^2|0\rangle. \tag{2.0.109}$$

Upon multiplying we are left with

$$\begin{aligned}\langle : \hat{n}^2 : \rangle = \quad & +8\langle 0|\hat{a}^{\dagger}\hat{a}|0\rangle \cosh^2 r \sinh^2 r + 4\langle 0|\hat{a}^{\dagger 2}\hat{a}^2|0\rangle \cosh^2 r \sinh^2 r \\ & +\langle 0|\hat{a}^{\dagger 4}|0\rangle\rangle \cosh^2 r \sinh^2 r + 2\sinh^4 r + \cosh^2 r \sinh^2 r \\ & -4\langle 0|\hat{a}^{\dagger 2}|0\rangle \cosh r \sinh^3 r - \langle 0|\hat{a}\hat{a}^{\dagger 3}|0\rangle \cosh r \sinh^3 r \\ & -2\langle 0|\hat{a}^{\dagger 2}|0\rangle \cosh r \sinh^3 r - \langle 0|\hat{a}^{\dagger 3}\hat{a}|0\rangle \cosh r \sinh^3 r \\ & -2\langle 0|\hat{a}^{\dagger}\hat{a}^3|0\rangle \cosh^3 r \sinh r - \langle 0|\hat{a}^3|0\rangle \cosh^3 r \sinh r \\ & +\langle 0|\hat{a}^4|0\rangle \cosh^2 r \sinh^2 r - 6\langle 0|\hat{a}^2|0\rangle \cosh r \sinh^3 r \\ & -\langle 0|\hat{a}^{\dagger 3}|0\rangle \cosh^3 r \sinh r - \langle 0|\hat{a}^{\dagger 2}|0\rangle \cosh^3 r \sinh r \\ & -2\langle 0|\hat{a}^{\dagger}\hat{a}^3|0\rangle \cosh r \sinh^3 r + 4\langle 0|\hat{a}^{\dagger}\hat{a}|0\rangle \sinh^4 r \\ & +\langle 0|\hat{a}^{\dagger 2}\hat{a}^2|0\rangle \cosh^4 r - \langle 0|\hat{a}^{\dagger 3}\hat{a}|0\rangle \cosh^3 r \sinh r. \end{aligned} \tag{2.0.110}$$

Putting the operators in normal order using commutation relation given by (2.0.38), we obtain

$$\langle : \hat{n}^2 : \rangle = \cosh^2 r \sinh^2 r + 2\sinh^4 r. \tag{2.0.111}$$

Employing (2.0.105) and (2.0.111) into (2.0.40), the normally-ordered variance of the photon number for a squeezed vacuum state takes the form

$$: (\Delta n)^2 := \cosh^2 r \sinh^2 r + \sinh^4 r. \tag{2.0.112}$$

In view of (2.0.105), this expression can be put in the form

$$: (\Delta n)^2 := \bar{n}(2\bar{n} + 1). \tag{2.0.113}$$

We can also arrive at the same result by setting $\gamma = 0$ in the normally-ordered variance of a squeezed coherent state (2.0.49).

2.2.4 Photon number distribution

Now applying (2.0.99) along with (2.0.50), the photon number distribution is expressible as

$$P(n) = \frac{sechr}{n!}\frac{\partial^{2n}}{\partial\alpha^n\partial\alpha^{*n}}\left[e^{-a(\alpha^2+\alpha^{*2})}\right]|_{\alpha=\alpha^*=0}, \tag{2.0.114}$$

In which

$$a = \frac{1}{2}\tanh r. \tag{2.0.115}$$

On account of the power series expansions

$$e^{-a\alpha^2} = \sum_i \frac{(-a)^i}{i!}\alpha^{2i}, \tag{2.0.116}$$

$$e^{-a\alpha^{*2}} = \sum_j \frac{(-a)^j}{j!}\alpha^{*2j}, \tag{2.0.117}$$

expression (2.0.114) can be put in the form

$$P(n) = \frac{sechr}{n!}\sum_{ij}\frac{(-a)^{i+j}}{i!j!}\frac{\partial^{2n}}{\partial\alpha^n\partial\alpha^{*n}}\left(\alpha^{2i}\alpha^{*2j}\right)|_{\alpha=\alpha^*=0}. \tag{2.0.118}$$

So that carrying out the differentiation using the identity given by (2.0.59) and applying the condition $\alpha = \alpha^* = 0$, one finds

$$P(n) = \frac{\sec hr}{n!}\sum_{ij}\frac{(-a)^{i+j}}{i!j!}\frac{(2i)!}{(2i-n)!}\frac{(2j)!}{(2j-n)!}\delta_{2i,n}\delta_{2j,n}. \tag{2.0.119}$$

Hence in view of the property of the Kronecker delta, we see that

$$i = \frac{n}{2}, \tag{2.0.120}$$

$$j = \frac{n}{2}, \tag{2.0.121}$$

and on taking into account that a factorial is defined for non-negative integers, we have

$$P(n) = \frac{(-1)^n n! \tanh^n r}{2^n(\frac{n}{2}!)^2\cosh r} \tag{2.0.122}$$

for $n = 0, 2, 4...$ and

$$P(n) = 0 \tag{2.0.123}$$

for $n = 1, 3, 5...$

We then observe that the probability of finding odd number of photons in a squeezed vacuum state is zero due to the fact that the photons are always generated in pairs.

2.2.5 Quadrature variance

We next proceed to determine the quadrature variance for a squeezed vacuum state. To this end, we find the expectation value $\hat{a}^{\dagger 2}$. In a squeezed vacuum state, the expectation values of $\hat{a}^{\dagger 2}$ can be expressed as

$$\langle \hat{a}^{\dagger 2} \rangle = \langle r | \hat{a}^{\dagger 2} | r \rangle. \tag{2.0.124}$$

Using (2.0.88), we write

$$\langle \hat{a}^{\dagger 2} \rangle = \langle 0 | \hat{s}^{\dagger}(r) \hat{a}^{\dagger 2} \hat{s}(r) | 0 \rangle. \tag{2.0.125}$$

Applying the unitarity property of the squeeze operator given by (2.0.35), we have

$$\begin{aligned} \langle \hat{a}^{\dagger 2} \rangle = \quad & \langle 0 | \hat{a}^{\dagger 2} | 0 \rangle \cosh^2 r - 2 \langle 0 | \hat{a}^{\dagger} \hat{a} | 0 \rangle \cosh r \sinh r \\ & + \langle 0 | \hat{a}^2 | 0 \rangle \sinh^2 r - \cosh r \sinh r. \end{aligned} \tag{2.0.126}$$

It then follows that

$$\langle \hat{a}^{\dagger 2} \rangle = - \cosh r \sinh r. \tag{2.0.127}$$

In a similar way , one can show that

$$\langle \hat{a}^{\dagger} \rangle = 0. \tag{2.0.128}$$

Applying (2.0.127) and (2.0.128) along with their complex conjugate together with Eq. (2.0.105) into Eq. (2.0.68) and (2.0.69), the quadrature variance for the plus quadrature takes the form

$$(\Delta a_+)^2 = 2 \sinh^2 r - 2 \cosh r \sinh r + 1. \tag{2.0.129}$$

In view of (2.0.84) and (2.0.85), the quadrature variance for the plus quadrature finally takes the form

$$(\Delta a_+)^2 = e^{-2r} \tag{2.0.130}$$

and the quadrature variance for the minus quadrature turns out to be

$$(\Delta a_-)^2 = e^{2r} \tag{2.0.131}$$

showing that, for $r > 0$, the squeezing occurs in the plus quadrature.

2.3 Displaced squeezed vacuum state

A displaced squeezed vacuum state is defined as

$$|\lambda, r\rangle = \hat{D}(\lambda)\hat{S}(r)|0\rangle, \tag{2.0.132}$$

where

$$\hat{D}(\lambda) = e^{\lambda\hat{a}^{\dagger} - \lambda^{*}\hat{a}} \tag{2.0.133}$$

and

$$\hat{S}(r) = e^{\frac{r}{2}(\hat{a}^{2} - \hat{a}^{\dagger 2})} \tag{2.0.134}$$

are the usual displacement and squeeze operators respectively.

2.3.1 The Q function

We now proceed to derive the Q function for a displaced squeezed vacuum state. To this end, the Q function for a displaced squeezed vacuum state is expressible as

$$Q(\alpha^{*}, \alpha) = \frac{1}{\pi^{2}} \int d^{2}z\phi_{a}(z)exp^{(z^{*}\alpha - z\alpha^{*})}, \tag{2.0.135}$$

where the antinormally-ordered characteristic function $\phi_a(z)$ is defined by

$$\phi_{a}(z) = Tr(\hat{\rho}e^{-z^{*}\hat{a}}e^{z\hat{a}^{\dagger}}). \tag{2.0.136}$$

This can be rewritten as

$$\phi_{a}(z) = Tr(|\lambda, r\rangle\langle r, \lambda|e^{-z^{*}\hat{a}}e^{z\hat{a}^{\dagger}}), \tag{2.0.137}$$

where

$$\hat{\rho} = |\lambda, r\rangle\langle r, \lambda| \tag{2.0.138}$$

is the density operator for a displaced squeezed vacuum state.

Applying the trace operation and using the definition for a displaced squeezed vacuum state given by (2.0.132), we write

$$\phi_{a}(z) = \langle 0|\hat{s}^{\dagger}(r)\hat{D}^{\dagger}(\lambda)e^{-z^{*}\hat{a}}e^{z\hat{a}^{\dagger}}\hat{D}(\lambda)\hat{s}(r)|0\rangle. \tag{2.0.139}$$

Using the unitarity property of the squeeze and displacement operators, this expression can be inturn written as

$$\phi_a(z) = \langle 0|\hat{s}^\dagger(r)\hat{D}^\dagger(\lambda)e^{-z^*\hat{a}}\hat{D}(\lambda)\hat{s}(r)s^\dagger(r)\hat{D}^\dagger(\lambda)e^{z\hat{a}^\dagger}\hat{D}(\lambda)\hat{s}(r)|0\rangle. \qquad (2.0.140)$$

In view of the transformations

$$\hat{s}^\dagger(r)\hat{D}^\dagger(\lambda)\hat{a}\hat{D}(\lambda)\hat{s}(r) = \hat{s}^\dagger(r)(\hat{a}+\lambda)\hat{s}(r) = \hat{a}(r)+\lambda, \qquad (2.0.141)$$

$$\hat{s}^\dagger(r)\hat{D}^\dagger(\lambda)\hat{a}^\dagger\hat{D}(\lambda)\hat{s}(r) = \hat{s}^\dagger(r)(\hat{a}^\dagger+\lambda^*)\hat{s}(r) = \hat{a}^\dagger(r)+\lambda^*, \qquad (2.0.142)$$

Eq. (1.0.140) can be expressed in the form

$$\phi_a(z) = \langle 0|exp(-z^*\lambda - z^*\hat{a}(r))exp(z\lambda^* + z\hat{a}^\dagger(r))|0\rangle. \qquad (2.0.143)$$

On account of the identity given by (2.0.13), we write

$$\begin{aligned}\phi_a(z) = \quad & exp(-\frac{1}{2}z^*z - z^*\lambda + z\lambda^*)\\ & \times\langle 0|exp[-z^*\hat{a}(r) + z\hat{a}^\dagger(r)]|0\rangle. \end{aligned} \qquad (2.0.144)$$

Taking into account Eq. (2.0.25) and (2.0.26), we have

$$\begin{aligned}\phi_a(z) = \quad & exp(-\frac{1}{2}z^*z - z^*\lambda + z\lambda^*)\\ & \times\langle 0|exp\big[(z\cosh r + z^*\sinh r)\hat{a}^\dagger\\ & -(z^*\cosh r + z\sinh r)\hat{a}\big]|0\rangle. \end{aligned} \qquad (2.0.145)$$

Applying Baker-Hausdorff identity once more, we obtain

$$\begin{aligned}\phi_a(z) = \quad & exp[z\lambda^* - z^*\lambda - z^*z\cosh^2 r\\ & -\cosh r\sinh r(z^2 + z^{*2})/2]\\ & \times\langle 0|exp(z\cosh r + z^*\sinh r)\hat{a}^\dagger\\ & \times exp - (z^*\cosh r + z\sinh r)\hat{a}|0\rangle. \end{aligned} \qquad (2.0.146)$$

It then follows that

$$\phi_a(z) = exp[z\lambda^* - z^*\lambda - z^*z\cosh^2 r - \cosh r\sinh r(z^2 + z^{*2})/2], \qquad (2.0.147)$$

so that on combinig this result along with (2.0.135), we have

$$Q(\alpha^*, \alpha) = \frac{1}{\pi}\int \frac{d^2z}{\pi} exp[-z^* z\cosh^2 r + (\lambda^* - \alpha^*)z \\ +(\alpha - \lambda)z^* - \cosh r \sinh r(z^2 + z^{*2})/2]. \tag{2.0.148}$$

Thus on performing the integration employing the relation given by (2.0.31), the Q function for a displaced squeezed vacuum state finally takes the form

$$Q(\alpha^*, \alpha) = \frac{sechr}{\pi} exp[-\alpha^*\alpha - \lambda^*\lambda + \tanh r(\lambda^*\alpha^* + \lambda\alpha) \\ +\lambda\alpha^* + \lambda^*\alpha - \tanh r(\alpha^2 + \alpha^{*2} + \lambda^2 + \lambda^{*2})/2]. \tag{2.0.149}$$

On the other hand, the displaced squeezed vacuum state can be casted into the form of a squeezed coherent state. From the definition of a displaced squeezed vacuum state, we have

$$|\lambda, r\rangle = \hat{D}(\lambda)\hat{s}(r)|0\rangle. \tag{2.0.150}$$

Using the unitarity property of the squeeze operator

$$\hat{S}(r)\hat{S}^\dagger(r) = \hat{I} \tag{2.0.151}$$

and expanding the displacement operator in normal order

$$\hat{D}(\lambda) = e^{-\frac{1}{2}\lambda^*\lambda} e^{\lambda\hat{a}^\dagger} e^{-\lambda^*\hat{a}}, \tag{2.0.152}$$

the displaced squeezed vacuum state can be rewritten as

$$|\lambda, r\rangle = e^{-\frac{1}{2}\lambda^*\lambda}\hat{S}(r)\hat{S}^\dagger(r)e^{\lambda\hat{a}^\dagger}\hat{S}(r)\hat{S}^\dagger(r)e^{-\lambda^*\hat{a}}\hat{S}(r)|0\rangle, \tag{2.0.153}$$

so that on account of (2.0.11) and (2.0.12), we have

$$|\lambda, r\rangle = e^{-\frac{1}{2}\lambda^*\lambda}\hat{S}(r)e^{\lambda\hat{a}^\dagger(r)}e^{-\lambda^*\hat{a}(r)}|0\rangle. \tag{2.0.154}$$

Applying the Baker-Hausdorff identity and noting that

$$[\lambda\hat{a}^\dagger(r), -\lambda^*\hat{a}(r)] = \lambda^*\lambda, \tag{2.0.155}$$

one obtains

$$|\lambda, r\rangle = \hat{S}(r)e^{[\lambda\hat{a}^{\dagger}(r)-\lambda^{*}\hat{a}(r)]}|0\rangle. \tag{2.0.156}$$

With the aid of (2.0.25) and (2.0.26), the displaced squeezed vacuum state finally takes the form

$$|\lambda, r\rangle = \hat{S}(r)|\gamma\rangle, \tag{2.0.157}$$

where

$$\gamma = \lambda\cosh r + \lambda^{*}\sinh r. \tag{2.0.158}$$

This shows that the squeezing and the statistical properties of the displaced squeezed vacuum state can be obtained from the squeezed coherent state by making use of Eq. (1.0.158). So that one can obtain Eq. (2.0.149) by making use of (2.0.158) into (2.0.32).

We next seek to show that the displaced squeezed vacuum state is the superposition of coherent state and a squeezed vacuum state. To this end, the Q function for the superposition of coherent state and a squeezed vacuum state is expressible as

$$\begin{aligned} Q(\alpha^{*}, \alpha) = \quad & \frac{1}{\pi}\int d^{2}\eta d^{2}\gamma Q(\eta^{*}, \eta + \frac{\partial}{\partial\eta^{*}})Q(\gamma^{*}, \gamma + \frac{\partial}{\partial\gamma^{*}}) \\ & \times exp(-\alpha^{*}\alpha - \eta * \eta - \gamma^{*}\gamma + \alpha^{*}\eta + \alpha\eta^{*} \\ & + \alpha^{*}\gamma + \alpha\gamma^{*} - \eta^{*}\gamma - \eta\gamma^{*}). \end{aligned} \tag{2.0.159}$$

Using (1.0.99), one can write the Q function for a squeezed vacuum state to be

$$\begin{aligned} Q(\gamma^{*}, \gamma + \frac{\partial}{\partial\gamma^{*}}) = \quad & \frac{sechr}{\pi}exp[-\gamma^{*}\gamma - \tanh r(\gamma^{*2} + \gamma^{2})/2] \\ & \times exp[-\gamma^{*}\frac{\partial}{\partial\gamma^{*}} - \tanh r(2\gamma\frac{\partial}{\partial\gamma^{*}} + \frac{\partial^{2}}{\partial\gamma^{*2}})/2]. \end{aligned} \tag{2.0.160}$$

On the other hand, we note that, the Q function for coherent states is given by[2]

$$Q(\alpha^{*}, \alpha) = \frac{1}{\pi}exp[-\alpha^{*}\alpha + \lambda^{*}\alpha + \lambda\alpha^{*} - \lambda^{*}\lambda]. \tag{2.0.161}$$

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[2] for the derivation of Eq.(2.0.161) and (2.0.159), see K.Fesseha, fundamentals of quantum optics(Lulu, United states of America, 2008).

This expression is equivalently rewritten as

$$Q(\eta^*, \eta + \frac{\partial}{\partial\eta^*}) = \frac{1}{\pi} exp(\eta^*\eta + \lambda^*\eta + \lambda\eta^* - \eta^*\frac{\partial}{\partial\eta^*} + \lambda^*\frac{\partial}{\partial\eta^*} - \lambda^*\lambda). \tag{2.0.162}$$

Then in view of Eq. (2.0.160) and (2.0.162) expression (2.0.159) can be written in the form

$$\begin{aligned} Q(\alpha^*, \alpha) = \quad & \frac{\sec hr}{\pi} exp^{(-\alpha^*\alpha - \lambda^*\lambda)} \\ & \times \int \frac{\partial^2\eta}{\pi}\frac{\partial^2\gamma}{\pi} exp[-\eta^*\eta - \gamma^*\gamma + \lambda^*\eta + \lambda\eta^* + \alpha^*\eta \\ & + \alpha^*\gamma + a(\gamma^2 + \gamma^{*2})] \\ & \times exp(-\eta^* + \lambda^*)\frac{\partial}{\partial\eta^*} exp(-\eta + \alpha - \gamma)\eta^* \\ & \times exp[(-\gamma^* + 2a\gamma)\frac{\partial}{\partial\gamma^*} + a\frac{\partial^2}{\partial\gamma^*}] exp(-\gamma + \alpha - \eta)\gamma^*, \end{aligned} \tag{2.0.163}$$

where

$$a = -\frac{1}{2}\tanh r. \tag{2.0.164}$$

On account of the power series expansion

$$\begin{aligned} & exp[(-\eta^* + \lambda^*)\frac{\partial}{\partial\eta^*}] exp^{(-\eta+\alpha-\gamma)\eta^*} \\ & = \sum_i \frac{(-\eta^* + \lambda^*)^i}{i!}\frac{\partial^i}{\partial\eta^{*i}} exp^{(-\eta+\alpha-\gamma)\eta^*} \end{aligned} \tag{2.0.165}$$

and carrying out the differentiation, we write

$$exp(-\eta^* + \lambda^*)\frac{\partial}{\partial\eta^*} exp(-\eta + \alpha - \gamma)\eta^* = exp(-\lambda^*\eta + \lambda^*\alpha - \lambda^*\gamma). \tag{2.0.166}$$

In a similar fashion, we have

$$\begin{aligned} & exp[(-\gamma^* + 2a\gamma)\frac{\partial}{\partial\gamma^*} + a\frac{\partial^2}{\partial\gamma^{*2}}] exp(-\gamma + \alpha - \eta)\gamma^* \\ & = \sum_j \frac{[(-\gamma^* + 2a\gamma)\frac{\partial}{\partial\gamma^*} + a\frac{\partial^2}{\partial\gamma^{*2}}]^j}{j!} exp^{(-\gamma+\alpha-\eta)\gamma^*}. \end{aligned} \tag{2.0.167}$$

Using Bionomial expansion, this expression can be written as

$$\begin{aligned}&exp[(-\gamma^* + 2a\gamma)\frac{\partial}{\partial\gamma^*} + a\frac{\partial^2}{\partial\gamma^{*2}}]exp(-\gamma + \alpha - \eta)\gamma^* \\ &= \sum_j \frac{1}{j!}\sum_k \frac{j!}{k!(j-k)!}(-\gamma^* + 2a\gamma)^{j-k}\frac{\partial^{j-k}}{\partial\gamma^{*j-k}}a^k\frac{\partial^{2k}}{\partial\gamma^{*2k}} \\ &\times exp(-\gamma + \alpha - \eta)\gamma^*. \qquad (2.0.168)\end{aligned}$$

Performing the differentiation, we obtain

$$\begin{aligned}&exp[(-\gamma^* + 2a\gamma)\frac{\partial}{\partial\gamma^*} + a\frac{\partial^2}{\partial\gamma^{*2}}]exp(-\gamma + \alpha - \eta)\gamma^* \\ &= exp(-a\gamma^2 + a\eta^2 + a\alpha^2 - 2a\alpha\eta). \qquad (2.0.169)\end{aligned}$$

Appling Eq. (2.0.166) and (2.0.169) into (2.0.163), we get

$$\begin{aligned}Q(\alpha^*, \alpha) = \quad &\frac{\sec hr}{\pi}exp(-\alpha^*\alpha - \lambda^*\lambda + \alpha\lambda^* + a\alpha^2) \\ &\times \int \frac{d^2\eta}{\pi}exp(-\eta^*\eta + \lambda\eta^* + \alpha^*\eta - 2a\alpha\eta + a\eta^2) \qquad (2.0.170) \\ &\times \int \frac{d^2\gamma}{\pi}exp(-\gamma^*\gamma + \alpha^*\gamma - \lambda^*\gamma + a\gamma^{*2}). \qquad (2.0.171)\end{aligned}$$

Now performing the integration employing the relation given by Eq. (2.0.31) with respect to γ we have

$$\begin{aligned}Q(\alpha^*, \alpha) = \quad &\frac{sechr}{\pi}exp[-\alpha^*\alpha - \lambda^*\lambda + \alpha\lambda^* + a(\alpha^2 + \alpha^{*2}) + a\lambda^{*2} - 2a\alpha^*\lambda^*] \\ &\times \int \frac{d^2\eta}{\pi}exp[-\eta^*\eta + (\alpha^* - 2a\alpha)\eta + \lambda\eta^* + a\eta^2]. \qquad (2.0.172)\end{aligned}$$

carrying out the integration once more with respect to η and replacing the value of a, we left with

$$\begin{aligned}Q(\alpha^*, \alpha) = \quad &\frac{\sec hr}{\pi}exp[-\alpha^*\alpha - \lambda^*\lambda + \alpha^*\lambda + \alpha\lambda^* + \tanh r(\alpha^*\lambda^* + \alpha\lambda) \\ &- \tanh r(\alpha^2 + \alpha^{*2} + \lambda^2 + \lambda^{*2})/2] \qquad (2.0.173)\end{aligned}$$

which is in agreement with the Q function obtained in Eq. (2.0.149) verifying that, a displaced squeezed vacuum state is the superposition of coherent state and a squeezed vacuum state.

2.3.2 Mean photon number

We are now in a position to determine the mean photon number for a displaced squeezed vacuum state. To this end, the expectation value of the number operator $\hat{n}$ in a displaced squeezed vacuum state basis is expressible as

$$\langle\hat{n}\rangle = \langle r,\lambda|\hat{a}^{\dagger}\hat{a}|\lambda,r\rangle. \tag{2.0.174}$$

In view of (2.0.132), we write

$$\langle\hat{n}\rangle = \langle 0|\hat{s}^{\dagger}(r)\hat{D}^{\dagger}(\lambda)\hat{a}^{\dagger}\hat{a}\hat{D}(\lambda)\hat{s}(r)|0\rangle. \tag{2.0.175}$$

Applying the unitarity property of the displacement operator, we have

$$\langle\hat{n}\rangle = \langle 0|\hat{s}^{\dagger}(r)[\hat{a}^{\dagger}\hat{a} + \lambda\hat{a}^{\dagger} + \lambda^{*}\hat{a} + \lambda^{*}\lambda]\hat{s}(r)|0\rangle. \tag{2.0.176}$$

Moreover, using the unitarity property of the squeeze operator once again and taking into account (2.0.11) and (2.0.12), we get

$$\langle\hat{n}\rangle = \langle 0|\hat{a}^{\dagger}(r)\hat{a}(r)|0\rangle + \lambda\langle 0|\hat{a}^{\dagger}(r)|0\rangle + \lambda^{*}\langle 0|\hat{a}(r)|0\rangle + \lambda^{*}\lambda. \tag{2.0.177}$$

Applying (2.0.25) and (2.0.26), we see that

$$\begin{aligned}\langle\hat{n}\rangle = \quad & \langle 0|\hat{a}^{\dagger}\hat{a}|0\rangle\cosh^{2} + \langle 0|\hat{a}^{\dagger}\hat{a}|0\rangle\sinh^{2} r - \langle 0|\hat{a}^{\dagger 2}|0\rangle\cosh r\sinh r \\ & -\langle 0|\hat{a}^{2}|0\rangle\cosh r\sinh r + \lambda\langle 0|\hat{a}^{\dagger}|0\rangle\cosh r - \lambda\langle 0|\hat{a}|0\rangle\sinh r \\ & +\lambda^{*}\langle 0|\hat{a}|0\rangle\cosh r - \lambda^{*}\langle 0|\hat{a}^{\dagger}|0\rangle\sinh r + \lambda^{*}\lambda + \sinh^{2} r.\end{aligned} \tag{2.0.178}$$

It then follows that

$$\langle\hat{n}\rangle = \lambda^{*}\lambda + \sinh^{2} r, \tag{2.0.179}$$

which is the sum of the mean photon number for a coherent state $|\lambda\rangle$ and the mean photon number of a squeezed vacuum state $|r\rangle$, found in (2.0.105).

2.3.3 Normally-ordered variance of the photon number

The expectation value of $:\hat{n}^2:$ for a displaced squeezed vacuum state is found to be

$$\langle :\hat{n}^2: \rangle = \langle r, \lambda|\hat{a}^{\dagger 2}\hat{a}^2|\lambda, r\rangle. \tag{2.0.180}$$

In view of (2.0.132), we write

$$\langle :\hat{n}^2: \rangle = \langle 0|\hat{s}^{\dagger}(r)\hat{D}^{\dagger}(\lambda)\hat{a}^{\dagger 2}\hat{a}^2\hat{D}(\lambda)\hat{s}(r)|0\rangle. \tag{2.0.181}$$

Applying the unitarity property of the displacement operator and in view of the transformations given by Eq. (2.0.141) and (2.0.142), we have

$$\begin{aligned}\langle :\hat{n}^2: \rangle = \quad & \langle 0|\hat{s}^{\dagger}(r)[\hat{a}^{\dagger 2}\hat{a}^2 + 2\lambda\hat{a}^{\dagger 2}\hat{a} + \lambda^2\hat{a}^{\dagger 2} + 2\lambda^*\hat{a}^{\dagger}\hat{a}^2 + 4\lambda^*\lambda\hat{a}^{\dagger}\hat{a} \\ & +2\lambda^*\lambda^2\hat{a}^{\dagger} + \lambda^{*2}\hat{a}^2 + 2\lambda^{*2}\hat{a} + \lambda^{*2}\lambda^2]\hat{s}(r)|0\rangle. \end{aligned} \tag{2.0.182}$$

Moreover, using the unitarity property of the squeeze operator and taking into account (2.0.11) and (2.0.12), we get

$$\begin{aligned}\langle :\hat{n}^2: \rangle = \quad & 4\lambda^*\lambda\langle 0|\hat{a}^{\dagger}(r)\hat{a}(r)|0\rangle + 2\lambda^*\lambda^2\langle 0|\hat{a}^{\dagger}(r)|0\rangle \\ & +\lambda^{*2}\langle 0|\hat{a}^2|0\rangle + 2\lambda^{*2}\lambda\langle 0|\hat{a}(r)|0\rangle + \lambda^{*2}\lambda^2 \\ & +2\lambda^*\langle 0|\hat{a}^{\dagger}(r)\hat{a}^2(r)|0\rangle + \langle 0|\hat{a}^{\dagger 2}(r)\hat{a}^2(r)|0\rangle \\ & +2\lambda\langle 0|\hat{a}^{\dagger 2}(r)\hat{a}(r)|0\rangle + \lambda^2\langle 0|\hat{a}^{\dagger 2}(r)|0\rangle. \end{aligned} \tag{2.0.183}$$

On account of (2.0.25) and (2.0.26), one can see that

$$\begin{aligned}\langle : \hat{n}^2 : \rangle = \quad & \langle 0|[\hat{a}^{\dagger 4}\cosh^2 r\sinh^2 r - \hat{a}^{\dagger}\hat{a}^3\cosh^3 r\sinh r + \hat{a}^{\dagger}\hat{a}^2\hat{a}^{\dagger}\cosh^2 r\sinh^2 r \\ & +\hat{a}\hat{a}^{\dagger 2}\hat{a}\cosh^2 r\sinh^2 r - \hat{a}\hat{a}^{\dagger 3}\cosh r\sinh^3 r + \hat{a}^4\cosh^2 r\sinh^2 r - 2\lambda^*\hat{a}^3\cosh^2 r\sinh r \\ & +\hat{a}^{\dagger 2}\hat{a}^2\cosh^4 r - \hat{a}^{\dagger 2}\hat{a}\hat{a}^{\dagger}\cosh^3 r\sinh r - \hat{a}^{\dagger 3}\hat{a}\cosh^3 r\sinh r + 2\lambda^*\lambda^2\hat{a}^{\dagger}\cosh r \\ & +\hat{a}^{\dagger}\hat{a}\hat{a}^{\dagger}\hat{a}\cosh^2 r\sinh^2 r - \hat{a}^{\dagger}\hat{a}\hat{a}^{\dagger 2}\cosh r\sinh^3 r - 2\lambda^*\lambda^2\hat{a}\sinh r + \lambda^{*2}\hat{a}^2\cosh^2 r \\ & -\hat{a}\hat{a}^{\dagger}\hat{a}^2\cosh^3 r\sinh r + \hat{a}\hat{a}^{\dagger}\hat{a}\hat{a}^{\dagger}\cosh^2 r\sinh^2 r - \lambda^{*2}\hat{a}\hat{a}^{\dagger}\cosh r\sinh r \\ & -\hat{a}^3\hat{a}^{\dagger}\cosh r\sinh^3 r - \hat{a}^2\hat{a}^{\dagger}\hat{a}\cosh r\sinh^3 r - 2\lambda^{*2}\lambda\hat{a}^{\dagger}\sinh r + \lambda^{*2}\lambda^2 \\ & +\hat{a}^2\hat{a}^{\dagger 2}\sinh^4 r + 2\lambda\hat{a}^{\dagger 2}\hat{a}\cosh^3 r - 2\lambda\hat{a}^{\dagger 3}\cosh^2 r\sinh r - \lambda^{*2}\hat{a}^{\dagger}\hat{a}\cosh r\sinh r \\ & -2\lambda\hat{a}^{\dagger}\hat{a}^2\cosh^2 r\sinh r + 2\lambda\hat{a}^{\dagger}\hat{a}\hat{a}^{\dagger}\cosh r\sinh^2 r + \lambda^{*2}\hat{a}^{\dagger 2}\sinh^2 r + 2\lambda^{*2}\lambda\hat{a}\cosh r \\ & -2\lambda\hat{a}\hat{a}^{\dagger}\hat{a}\cosh^2 r\sinh r + 2\lambda\hat{a}\hat{a}^{\dagger 2}\cosh r\sinh^2 r - 4\lambda^*\lambda\hat{a}^2\cosh r\sinh r + 4\lambda^*\lambda\hat{a}\hat{a}^{\dagger}\sinh^2 r \\ & +2\lambda\hat{a}^3\cosh r\sinh^2 r - 2\lambda\hat{a}^2\hat{a}^{\dagger}\sinh^3 r + \lambda^2\hat{a}^{\dagger 2}\cosh^2 r - 4\lambda^*\lambda\hat{a}^{\dagger 2}\cosh r\sinh r \\ & -\lambda^2\hat{a}^{\dagger}\hat{a}\cosh r\sinh r - \lambda^2\hat{a}\hat{a}^{\dagger}\cosh r\sinh r + \lambda^2\hat{a}^2\sinh^2 r + 4\lambda^*\lambda\hat{a}^{\dagger}\hat{a}\cosh^2 r \\ & +2\lambda^*\hat{a}^{\dagger}\hat{a}^2\cosh^3 r - 2\lambda^*\hat{a}^{\dagger}\hat{a}\hat{a}^{\dagger}\cosh^2 r\sinh r + 2\lambda^*\hat{a}\hat{a}^{\dagger}\hat{a}\cosh r\sinh^2 r - 2\lambda^*\hat{a}\hat{a}^{\dagger 2}\sinh^2 r \\ & -2\lambda^*\hat{a}^{\dagger 2}\hat{a}\cosh^2 r\sinh r + 2\lambda^*\hat{a}^{\dagger 3}\cosh r\sinh^2 + 2\lambda^*\hat{a}^2\hat{a}^{\dagger}\cosh r\sinh^2 r]|0\rangle. \qquad (2.0.184)\end{aligned}$$

Arranging the operators in normal order using the commutation relation given by Eq. (2.0.38), one readily obtains

$$\begin{aligned}\langle : \hat{n}^2 : \rangle = \quad & \cosh^2 r\sinh^2 r + 2\sinh^4 r + 4\lambda^*\lambda\sinh^2 r \\ & +\lambda^{*2}\lambda^2 - \cosh r\sinh r(\lambda^{*2} + \lambda^2). \qquad (2.0.185)\end{aligned}$$

Employing this result along with (2.0.179) into Eq. (2.0.40), the normally-ordered variance of the photon number for a displaced squeezed vacuum state takes the form

$$\begin{aligned}: (\Delta n)^2 := \quad & \cosh^2 r\sinh^2 r + \sinh^4 r + 2\lambda^*\lambda\sinh^2 r \\ & -\cosh r\sinh r(\lambda^{*2} + \lambda^2), \qquad (2.0.186)\end{aligned}$$

which can be inturn rewritten in the form

$$: (\Delta n)^2 := |\lambda\cosh r - \lambda^*\sinh r|^2 - \lambda^*\lambda + \cosh^2 r\sinh^2 r + \sinh^4 r. \qquad (2.0.187)$$

From this, we see that, when we set $\lambda = 0$, we easily obtain the normally-ordered variance of the photon number for a squeezed vacuum state and on the other hand, when we put $r = 0$, we arrive at the same result of the normally-ordered variance of the photon number for a coherent state.

2.3.4 Photon number distribution

Since the squeezing and the statistical properties of the displaced squeezed vacuum state can be obtained from those of the squeezed coherent state by making use of (2.0.158), one can obtain the photon number distribution for a displaced squeezed vacuum state to be

$$\begin{aligned} P(n) = & \frac{\tanh^n r}{n! 2^n \cosh r} exp[-\lambda^* \lambda - \tanh r (\lambda^2 + \lambda^{*2})/2] \\ & \times \left| H_n \left(\frac{\lambda \cosh r + \lambda^* \sinh r}{\sqrt{2 \cosh r \sinh r}} \right) \right|^2 , \end{aligned} \qquad (2.0.188)$$

where $H_n(x)$ is a Hermite polynomial given by Eq. (2.0.65).

2.3.5 Quadrature variance

We next seek to determine the quadrature variance for a displaced squeezed vacuum state. To do so, we first determine the expectation value of $\hat{a}^{\dagger 2}$. The expectation value of $\hat{a}^{\dagger 2}$ in a displaced squeezed vacuum state basis is expressible as

$$\langle \hat{a}^{\dagger 2} \rangle = \langle r, \lambda | \hat{a}^{\dagger 2} | \lambda, r \rangle. \tag{2.0.189}$$

Using the definition given by (2.0.132), this can be rewritten as

$$\langle \hat{a}^{\dagger 2} \rangle = \langle 0 | \hat{s}^{\dagger}(r) \hat{D}^{\dagger}(\lambda) \hat{a}^{\dagger 2} \hat{D}(\lambda) \hat{s}(r) | 0 \rangle. \tag{2.0.190}$$

Applying the unitarity property of the displacement and squeeze operator, we write

$$\langle \hat{a}^{\dagger 2} \rangle = \langle 0 | \hat{s}^{\dagger}(r) \hat{D}^{\dagger}(\lambda) \hat{a}^{\dagger} \hat{D}(\lambda) \hat{s}(r) \hat{s}^{\dagger}(r) \hat{D}^{\dagger}(\lambda) \hat{a}^{\dagger} \hat{D}(\lambda) \hat{s}(r) | 0 \rangle. \tag{2.0.191}$$

On account of the transformation given by Eq. (2.0.142), expresion (2.0.191) can be put in the form

$$\langle \hat{a}^{\dagger 2} \rangle = \langle 0 | \hat{a}^{\dagger 2}(r) | 0 \rangle + 2\lambda^{*} \langle 0 | \hat{a}^{\dagger}(r) | 0 \rangle + \lambda^{*2}. \tag{2.0.192}$$

Moreover, on account of (2.0.25), we have

$$\begin{aligned} \langle \hat{a}^{\dagger 2} \rangle = \quad & \langle 0 | \hat{a}^{\dagger 2} | 0 \rangle \cosh^{2} r - 2 \langle 0 | \hat{a}^{\dagger} \hat{a} | 0 \rangle \cosh r \sinh r \\ & + \langle 0 | \hat{a}^{2} | 0 \rangle \sinh^{2} r - \cosh r \sinh r + \lambda^{*2} \\ & + 2\lambda^{*} \langle 0 | \hat{a}^{\dagger} | 0 \rangle \cosh r - 2\lambda^{*} \langle 0 | \hat{a} | 0 \rangle \sinh r. \end{aligned} \tag{2.0.193}$$

It then follows that

$$\langle \hat{a}^{\dagger 2} \rangle = \lambda^{*2} - \cosh r \sinh r. \tag{2.0.194}$$

On the other hand, the expectation value of $\hat{a}^{\dagger}$ is expressible as

$$\langle \hat{a}^{\dagger} \rangle = \langle r, \lambda | \hat{a}^{\dagger} | \lambda, r \rangle, \tag{2.0.195}$$

which can be inturn written as

$$\langle \hat{a}^{\dagger} \rangle = \langle 0 | \hat{s}^{\dagger}(r) \hat{D}^{\dagger}(\lambda) \hat{a}^{\dagger} \hat{D}(\lambda) \hat{s}(r) | 0 \rangle. \tag{2.0.196}$$

In view of (2.0.142), we have

$$\langle \hat{a}^\dagger \rangle = \langle 0|\hat{s}^\dagger(r)[\hat{a}^\dagger + \lambda^*]\hat{s}(r)|0\rangle. \tag{2.0.197}$$

We note the that

$$\hat{s}^\dagger(r)\hat{a}^\dagger\hat{s}(r) = \hat{a}^\dagger(r), \tag{2.0.198}$$

so that the above expression can be put in the form

$$\langle \hat{a}^\dagger \rangle = \langle 0|\hat{a}^\dagger(r)|0\rangle + \lambda^*. \tag{2.0.199}$$

On account of (2.0.25), we write

$$\langle \hat{a}^\dagger \rangle = \langle 0|\hat{a}^\dagger|0\rangle \cosh r - \langle 0|\hat{a}|0\rangle \sinh r + \lambda^*. \tag{2.0.200}$$

It then follows that

$$\langle \hat{a}^\dagger \rangle = \lambda^*. \tag{2.0.201}$$

Applying (2.0.194) and (2.0.201) along with their complex conjugate together with Eq. (2.0.179) into (2.0.68) and (2.0.69), we get

$$(\Delta a_+)^2 = 2\sinh^2 r - 2\cosh r \sinh r + 1. \tag{2.0.202}$$

On account of (2.0.84) and (2.0.85), the quadrature variance of the plus quadrature finally takes the form

$$(\Delta a_+)^2 = e^{-2r} \tag{2.0.203}$$

and the quadrature variance for the minus quadrature turns out to be

$$(\Delta a_-)^2 = e^{2r}, \tag{2.0.204}$$

which is the same result found in (2.0.86) and (2.0.87) for a squeezed coherent state $|\gamma, r\rangle$ and a squeezed vacuum state, $|r\rangle$ found in (2.0.130) and (2.0.131), showing that the squeezing is occured in the plus quadrature.

3. Superposition of Squeezed States

Great attention has been given to produce the superposition of squeezed states. These states have more interesting and distinicte features such as squeezing. We consider here the superposition of a squeezed coherent state and a squeezed vacuum state as well as the superposition of a displaced squeezed vacuum state and a squeezed vacuum state. To this end, we find the Q function using the method of finding the Q function for the superposition of two different states but having the same frequency.

Using the resulting Q function, We consider the possibility of describing the mean photon number, the photon number distribution and the quadrature variance for the superposition of the above mentioned states.

3.1 Superposition of Squeezed vacuum States and Squeezed Coherent State

In section (2.1), (2.2) and 2.3, we introduced, the statistical properties of a squeezed coherent state, a squeezed vacuum state and a displaced squeezed vacuum such as the Q function, the mean photon number, the normally-ordered variance of the photon number, the photon number distribution and the quadrature variance.

In this section we seek to determine the Q function for the superposition of a squeezed vacuum state and a squeezed coherent state. With the aid of the resulting Q function, we calculate the mean photon number, the photon number distribution and the quadrature variance.

3.1.1 The Q Function

We now proceed to determine the Q function for the superposition of a squeezed coherent state and a squeezed vacuum state. To this end, the Q function for the superposition of two light beams in different states but having the same frequency is expressible as[3]

$$\begin{aligned} Q(\alpha^*,\alpha) = \quad & \frac{1}{\pi}\int d^2\beta d^2\eta Q(\beta^*,\beta+\frac{\partial}{\partial\beta^*})Q(\eta^*,\eta+\frac{\partial}{\partial\eta^*}) \\ & \times exp\big[-\alpha^*\alpha-\beta^*\beta-\eta^*\eta+\alpha^*\beta+\alpha\beta^*+\alpha^*\eta+\alpha\eta^* \\ & -\beta^*\eta-\beta\eta^*\big]. \end{aligned} \tag{3.0.205}$$

Using Eq. (2.0.32), one can write

$$\begin{aligned} Q(\beta^*,\beta+\frac{\partial}{\partial\beta^*}) = \quad & \frac{sechr}{\pi}exp[-\beta^*\beta-\gamma^*\gamma-\tanh r(\beta^2+\beta^{*2}-\gamma^2-\gamma^{*2})/2 \\ & +sechr(\gamma^*\beta+\gamma\beta^*)] \\ & \times exp[-\beta^*\frac{\partial}{\partial\beta^*}-\tanh r(2\beta\frac{\partial}{\partial\beta^*}+\frac{\partial^2}{\partial\beta^{*2}})/2 \\ & +\gamma^* sechr\frac{\partial}{\partial\beta^*}]. \end{aligned} \tag{3.0.206}$$

On the other hand, using Eq. (2.0.99), we have

$$\begin{aligned} Q(\eta^*,\eta+\frac{\partial}{\partial\eta^*}) = \quad & \frac{sechr}{\pi}exp[-\eta^*\eta-\tanh r(\eta^{*2}+\eta^2)/2] \\ & \times exp[-\eta^*\frac{\partial}{\partial\eta^*}-\tanh r(2\eta\frac{\partial}{\partial\eta^*}+\frac{\partial^2}{\partial\eta^{*2}})/2]. \end{aligned} \tag{3.0.207}$$

Introducing (3.0.206) and (3.0.207) into Eq. (3.0.205), we get

$$\begin{aligned} Q(\alpha^*,\alpha) = \quad & \frac{sech^2r}{\pi}exp[-\alpha^*\alpha-\gamma^*\gamma+\tanh r(\gamma^2+\gamma^{*2})/2] \\ & \times\int\frac{d^2\beta}{\pi}\frac{d^2\eta}{\pi}exp[-\beta^*\beta-\eta^*\eta-\tanh r(\beta^2+\beta^{*2}+\eta^2+\eta^{*2})/2 \\ & +sechr(\gamma^*\beta+\gamma\beta^*)+\alpha^*\beta+\alpha^*\eta] \\ & \times exp[(-\beta^*+2a\beta+b\gamma^*)\frac{\partial}{\partial\beta^*}+a\frac{\partial^2}{\partial\beta^{*2}}]exp(\alpha-\beta-\eta)\beta^* \\ & \times exp[(-\eta^*+2a\eta)\frac{\partial}{\partial\eta^*}+a\frac{\partial^2}{\partial\eta^{*2}}]exp(\alpha-\eta-\beta)\eta^*, \end{aligned} \tag{3.0.208}$$

where

$$a=-\frac{1}{2}\tanh r, \tag{3.0.209}$$

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[3] for the derivation of Eq. (3.0.205), see K. Fesseha, Fundamentals of quantum optics(Lulu, United states of America, 2008).

$$b = sechr. \tag{3.0.210}$$

Expanding in power series, we get

$$\begin{aligned} & exp[(-\beta^* + 2a\beta + b\gamma^*)\frac{\partial}{\partial\beta^*} + a\frac{\partial^2}{\partial\beta^{*2}}]e^{(\alpha-\beta-\eta)\beta^*} \\ & = \sum_l \frac{1}{l!}[(-\beta^* + 2a\beta + b\gamma^*)\frac{\partial}{\partial\beta^*} + a\frac{\partial^2}{\partial\beta^{*2}}]^l e^{(\alpha-\beta-\eta)\beta^*}. \end{aligned} \tag{3.0.211}$$

This can be inturn rewritten using Binomial expansion as

$$\begin{aligned} & exp[(-\beta^* + 2a\beta + b\gamma^*)\frac{\partial}{\partial\beta^*} + a\frac{\partial^2}{\partial\beta^{*2}}]e^{(\alpha-\beta-\eta)\beta^*} \\ & = \sum_l \frac{1}{l!}\sum_k^l \frac{l!(-\beta^* + 2a\beta + b\gamma^*)^{l-k}}{k!(l-k)!}\frac{\partial^{l-k}}{\partial\beta^{*l-k}}a^k\frac{\partial^{2k}}{\partial\beta^{*2k}}e^{(\alpha-\beta-\eta)\beta^*}. \end{aligned} \tag{3.0.212}$$

Carrying out the differentiation one finds

$$\begin{aligned} & exp[(-\beta^* + 2a\beta + b\gamma^*)\frac{\partial}{\partial\beta^*} + a\frac{\partial^2}{\partial\beta^{*2}}]e^{(\alpha-\beta-\eta)\beta^*} \\ & = exp(b\gamma^*\alpha - b\gamma^*\beta - b\gamma^*\eta - a\beta^2 + a\alpha^2 + a\eta^2 - 2a\alpha\eta). \end{aligned} \tag{3.0.213}$$

In a similar manner, we write

$$\begin{aligned} & exp[(-\eta^* + 2a\eta)\frac{\partial}{\partial\eta^*} + a\frac{\partial^2}{\partial\eta^{*2}}]e^{(\alpha-\eta-\beta)\eta^*} \\ & = exp(a\alpha^2 - a\eta^2 + a\beta^2 - 2a\alpha\beta). \end{aligned} \tag{3.0.214}$$

Applying Eq. (3.0.213) and (3.0.214) into (3.0.208), we obtain

$$\begin{aligned} Q(\alpha^*, \alpha) = \quad & \frac{sech^2 r}{\pi}exp[-\alpha^*\alpha - \gamma^*\gamma + \tanh r(\gamma^2 + \gamma^{*2})/2 + \alpha\gamma^* sechr - \alpha^2\tanh r] \\ & \times\int\frac{d^2\beta}{\pi}exp[-\beta^*\beta + (\alpha\tanh r + \alpha^*)\beta + \beta^*\gamma sechr - \tanh r(\beta^2 + \beta^{*2})/2] \\ & \times\int\frac{d^2\eta}{\pi}exp[-\eta^*\eta + (\alpha\tanh r + \alpha^* - \gamma^*\sec hr)\eta - \tanh r(\eta^2 + \eta^{*2})] \end{aligned} \tag{3.0.215}$$

Thus on performing the integration employing the relation given by (2.0.31), the Q function for the superposition of squeezed coherent state and a squeezed vacuum state finally takes the form

$$\begin{aligned} Q(\alpha^*,\alpha) = \quad & \frac{1}{\pi} exp[-(\cosh^2 r + \sinh^2 r)\alpha^*\alpha - \gamma^*\gamma + \cosh r(\gamma^*\alpha + \gamma\alpha^*) \\ & + sechr(\gamma^*\alpha^* + \gamma\alpha) - \cosh r \sinh r(\alpha^2 + \alpha^{*2})]. \end{aligned} \tag{3.0.216}$$

3.1.2 Mean photon number

We next seek to calculate the mean photon number of the superposition of a squeezed coherent state and a squeezed vacuum state. To this end, the expectation value of $\hat{n}$ interms of the Q function is expressible as

$$\langle \hat{n} \rangle = \int d^2\alpha Q(\alpha^*,\alpha) n_a(\alpha^*,\alpha), \tag{3.0.217}$$

where

$$n_a(\alpha^*,\alpha) = \alpha\alpha^* - 1 \tag{3.0.218}$$

is the c-number function corresponding to the operator function $\hat{n}(\hat{a}^\dagger, \hat{a})$ in the antinormal order. On account of Eq. (3.0.216) and (3.0.218), expression (3.0.217) can be written as

$$\begin{aligned} \langle \hat{n} \rangle = \quad & e^{-\gamma^*\gamma} \\ & \times \int \frac{d^2\alpha}{\pi} exp[-(\cosh^2 r + \sinh^2 r)\alpha^*\alpha + \cosh r(\gamma^*\alpha + \gamma\alpha^*) \\ & + sechr(\gamma^*\alpha^* + \gamma\alpha) - \cosh r \sinh r(\alpha^2 + \alpha^{*2})]\alpha\alpha^* - 1. \end{aligned} \tag{3.0.219}$$

This is equivalently rewritten in the form

$$\begin{aligned} \langle \hat{n} \rangle = \quad & e^{-\gamma^*\gamma} \\ & \times \frac{d}{da}\frac{d}{db} \int \frac{d^2\alpha}{\pi} exp[-(\cosh^2 r + \sinh^2 r)\alpha^*\alpha + (\gamma^* \cosh r + \gamma \sinh r + a)\alpha \\ & + (\gamma \cosh r + \gamma^* \sinh r + b)\alpha^* - \cosh r \sinh r(\alpha^2 + \alpha^{*2})] - 1_{a=b=0}, \end{aligned} \tag{3.0.220}$$

so that performing the integration employing the relation given by (2.0.31), we see

$$\begin{aligned}\langle\hat{n}\rangle = \quad & \frac{d}{da}\frac{d}{db}\Bigg[exp\big[-\gamma^*\gamma + (\cosh^2 r + \sinh^2 r)[\gamma^*\gamma\cosh^2 r + \gamma^2\cosh r\sinh r \\ & +\gamma^{*2}\cosh r\sinh r + b\gamma^*\cosh r + a\gamma\cosh r + a\gamma^*\sinh r + ab \\ & +\gamma^*\gamma\sinh^2 r + b\gamma\sinh r] - \cosh r\sinh r[\gamma^2\cosh^2 r + 4\gamma^*\gamma\cosh r\sinh r \\ & +2b\gamma\cosh r + \gamma^{*2}\sinh^2 r + 2b\gamma^*\sinh r + b^2 + \gamma^{*2}\cosh^2 r \\ & +2a\gamma^*\cosh r + \gamma^2\sinh^2 r + 2a\gamma\sinh r + a^2]\big]\Bigg] - 1_{a=b=0}. \qquad (3.0.221)\end{aligned}$$

Hence, carrying out the differentiation and applying the condition $a = b = 0$, one easily finds

$$\langle\hat{n}\rangle = \big|\gamma\cosh r - \gamma^*\sinh r\big|^2 + 2\sinh^2 r, \qquad (3.0.222)$$

which is the sum of the mean photon numbers for a squeezed coherent state $|\gamma, r\rangle$, found in (2.0.39), and for the squeezed vacuum state $|r\rangle$, found in (2.0.105).

3.1.3 Photon number distribution

We next proceed to determine the photon number distribution for the superposition of a squeezed coherent state and a squeezed vacuum state. To this end, applying Eq. (2.0.50) together with (3.0.216), the photon number distribution can be written as

$$\begin{aligned}P(n) = \quad & \frac{1}{n!}exp(-\frac{\gamma^*\gamma}{\cosh^2 r + \sinh^2 r})\frac{\partial^{2n}}{\partial\alpha^{*n}\partial\alpha^n} \\ & \times exp[a\alpha + a^*\alpha^* - b(\alpha^2 + \alpha^{*2})]_{\alpha=\alpha^*=0}, \qquad (3.0.223)\end{aligned}$$

in which

$$a = \frac{\gamma\cosh r + \gamma^*\sinh r}{\cosh^2 r + \sinh^2 r} \qquad (3.0.224)$$

and

$$b = \frac{\cosh r\sinh r}{\cosh^2 r + \sinh^2 r}. \qquad (3.0.225)$$

On account of the power series expansions

$$e^{-b\alpha^2} = \sum_i \frac{(-b)^i}{i!}\alpha^{2i}, \qquad (3.0.226)$$

$$e^{-b\alpha^{*2}} = \sum_{j} \frac{(-b)^j}{j!}\alpha^{*2j}, \tag{3.0.227}$$

$$e^{a\alpha} = \sum_{k} \frac{(a)^k}{k!}\alpha^{k}, \tag{3.0.228}$$

$$e^{a^*\alpha^*} = \sum_{l} \frac{(a)^{*l}}{k!}\alpha^{*l}, \tag{3.0.229}$$

expression (3.0.223) can be put in the form

$$\begin{aligned} P(n) = \quad & \frac{1}{n!} exp(-\frac{\gamma^*\gamma}{\cosh^2 r + \sinh^2 r}) \\ & \times \sum_{ijkl} \frac{(-b)^{i+j} a^k a^{*l}}{i!j!k!l!} \frac{\partial^{2n}}{\partial\alpha^{*n}\partial\alpha^n}\left[\alpha^{2i+k}\alpha^{*2j+l}\right]_{\alpha=\alpha^*=0}, \end{aligned} \tag{3.0.230}$$

so that carrying out the differentiation using the identity given by (2.0.59) and applying the condition $\alpha = \alpha^* = 0$, one easily finds

$$\begin{aligned} P(n) = \quad & \frac{1}{n!} exp(-\frac{\gamma^*\gamma}{\cosh^2 r + \sinh^2 r}) \\ & \times \sum_{ijkl} \frac{(-b)^{i+j} a^k a^{*l}}{i!j!k!l!} \frac{(2i+k)!}{(2i+k-n)!} \frac{(2j+l)!}{(2j+l-n)!} \delta_{2i+k,n}\delta_{2j+l,n}. \end{aligned} \tag{3.0.231}$$

Hence in view of the property of the Kronecker delta, we see that

$$k = n - 2i, \tag{3.0.232}$$

$$l = n - 2j, \tag{3.0.233}$$

and on taking into account that a factorial is defined for non-negative integers, we have

$$\begin{aligned} P(n) = \quad & \frac{\cosh^n r \sinh^n r}{n!(\cosh^2 r + \sinh^2 r)^n} exp(-\frac{\gamma^*\gamma}{\cosh^2 r + \sinh^2 r}) \\ & \times \Bigg(\sum_{i=0}^{[n]} \frac{(-1)^i n!}{i!(n-2i)!}\Big(\frac{a}{\sqrt{b}}\Big)^{n-2i}\Bigg)\Bigg(\sum_{j=0}^{[n]} \frac{(-1)^j n!}{j!(n-2j)}\Big(\frac{a^*}{\sqrt{b}}\Big)^{n-2j}\Bigg), \end{aligned} \tag{3.0.234}$$

where $[n]$ is defined by $[n] = \frac{n}{2}$ for even n and $[n] = \frac{n-1}{2}$ for odd n.

Finally the photon number distribution can be put in the form

$$\begin{aligned} P(n) = \quad & \frac{\cosh^n r \sinh^n r}{n!(\cosh^2 r + \sinh^2 r)^n} exp(-\frac{\gamma^*\gamma}{\cosh^2 r + \sinh^2 r}) \\ & \times \left| H_n\Bigg(\frac{\gamma \cosh r + \gamma^* \sinh r}{\sqrt{4(\cosh^3 r \sinh r + \cosh r \sinh^3 r)}}\Bigg)\right|^2, \end{aligned} \tag{3.0.235}$$

where

$$H_n(x) = \sum_{k=0}^{[n]} \frac{(-1)^k n!}{k!(n-2k)!}(2x)^{n-2k} \tag{3.0.236}$$

is a Hermite polynomial of order n.

3.1.4 Quadrature variance

We now proceed to determine the quadrature variance for the superposition of a squeezed coherent state and a squeezed vacuum state. The variance of the plus quadrature is defined by

$$(\Delta a_+)^2 = \langle \hat{a}_+^2 \rangle - \langle \hat{a}_+ \rangle^2. \tag{3.0.237}$$

In view of (2.0.3), this expression can be put in the form

$$(\Delta a_+)^2 = 1 + 2\langle \hat{a}^\dagger \hat{a} \rangle + \langle \hat{a}^{\dagger 2} \rangle + \langle \hat{a}^2 \rangle - \langle \hat{a} \rangle^2 - \langle \hat{a}^\dagger \rangle^2 - 2\langle \hat{a}^\dagger \rangle \langle \hat{a} \rangle. \tag{3.0.238}$$

On the other hand, on account of (2.0.4), the variance of the minus quadrature is turns out to be

$$(\Delta a_-)^2 = 1 + 2\langle \hat{a}^\dagger \hat{a} \rangle - \langle \hat{a}^{\dagger 2} \rangle - \langle \hat{a}^2 \rangle + \langle \hat{a} \rangle^2 + \langle \hat{a}^\dagger \rangle^2 - 2\langle \hat{a}^\dagger \rangle \langle \hat{a} \rangle. \tag{3.0.239}$$

The expectation value of $\hat{a}^{\dagger 2}$ interms of the Q function is expressible as

$$\langle \hat{a}^{\dagger 2} \rangle = \int d^2\alpha Q(\alpha^*, \alpha)\alpha^{*2}, \tag{3.0.240}$$

where α^{*2} is the c-number variable corresponding to the operator $\hat{a}^{\dagger 2}$. On account of (3.0.216), expression (3.0.240) can be written in the form

$$\begin{aligned} \langle \hat{a}^{\dagger 2} \rangle = \quad & e^{-\gamma^*\gamma} \\ & \times \int \frac{d^2\alpha}{\pi} exp[-(\cosh^2 r + \sinh^2 r)\alpha^*\alpha + \cosh r(\alpha\gamma^* + \alpha^*\gamma) \\ & + \sinh r(\alpha^*\gamma^* + \alpha\gamma) - \cosh r \sinh r(\alpha^2 + \alpha^{*2})]\alpha^{*2}. \end{aligned} \tag{3.0.241}$$

This can be inturn written as

$$\begin{aligned} \langle \hat{a}^{\dagger 2} \rangle = \quad & e^{-\gamma^*\gamma} \\ & \times \frac{d^2}{da^2} \int \frac{d^2\alpha}{\pi} exp[-(\cosh^2 r + \sinh^2 r)\alpha^*\alpha + (\gamma^* \cosh r + \gamma \sinh r)\alpha \\ & + (\gamma \cosh r + \gamma^* \sinh r + a)\alpha^* - \cosh r \sinh r(\alpha^2 + \alpha^{*2})]_{a=0}. \end{aligned} \tag{3.0.242}$$

Now carrying out the integration using the relation given by (2.0.31), we obtain

$$\begin{aligned}\langle\hat{a}^{\dagger 2}\rangle = \quad & e^{-\gamma^*\gamma} \\ & \times\frac{d^2}{da^2}\bigg[exp[(\cosh^2 r+\sinh^2 r)[\gamma^*\gamma\cosh^2 r+\gamma^{*2}\cosh r\sinh r \\ & +a\gamma^*\sinh r+\gamma^2\cosh r\sinh r+\gamma^*\gamma\sinh^2 r+a\gamma\sinh r] \\ & -\cosh r\sinh r[\gamma^2\cosh^2 r+4\gamma^*\gamma\cosh r\sinh r+2a\gamma\cosh r \\ & +\gamma^{*2}\sinh^2 r+2a\gamma^*\sinh r+a^2+\gamma^{*2}\cosh^2 r+\gamma^2\sinh^2 r]]\bigg]_{a=0} \quad (3.0.243)\end{aligned}$$

Upon differentiation and applying the condition $a = 0$, one readily finds

$$\begin{aligned}\langle\hat{a}^{\dagger 2}\rangle = \quad & \gamma^{*2}\cosh^6 r+\gamma^2\sinh^6 r+\gamma^{*2}\cosh^2 r\sinh^4 r \\ & -2\gamma^{*2}\cosh^4 r\sinh^2 r-2\gamma^*\gamma\cosh^5 r\sinh r \\ & -2\gamma^*\gamma\cosh r\sinh^5 r+\gamma^2\cosh^4 r\sinh^2 r \\ & -\gamma^2\cosh^2 r\sinh^4 r+4\gamma^*\gamma\cosh^3 r\sinh^3 r \\ & -2\cosh r\sinh r. \qquad (3.0.244)\end{aligned}$$

Following the same procedure, we obtain the expectation value

$$\begin{aligned}\langle\hat{a}^{\dagger}\rangle = \quad & \gamma^*\cosh^3 r+\gamma^*\cosh r\sinh r+\gamma\cosh^2 r\sinh r \\ & +\gamma\sinh^3 r-2\gamma\cosh^2 r\sinh r-2\gamma^*\cosh r\sinh^2 r. \qquad (3.0.245)\end{aligned}$$

Thus upon substitution of (3.0.245) and (3.0.244) along with their complex conjugate together with Eq. (3.0.222) into (3.0.239) and (3.0.238), the variance of the plus quadrature finally takes the form

$$(\Delta a_+)^2 = 2e^{-2r} - 1 \qquad (3.0.246)$$

and the variance of the minus quadrature turns out to be

$$(\Delta a_-)^2 = 2e^{2r} - 1. \qquad (3.0.247)$$

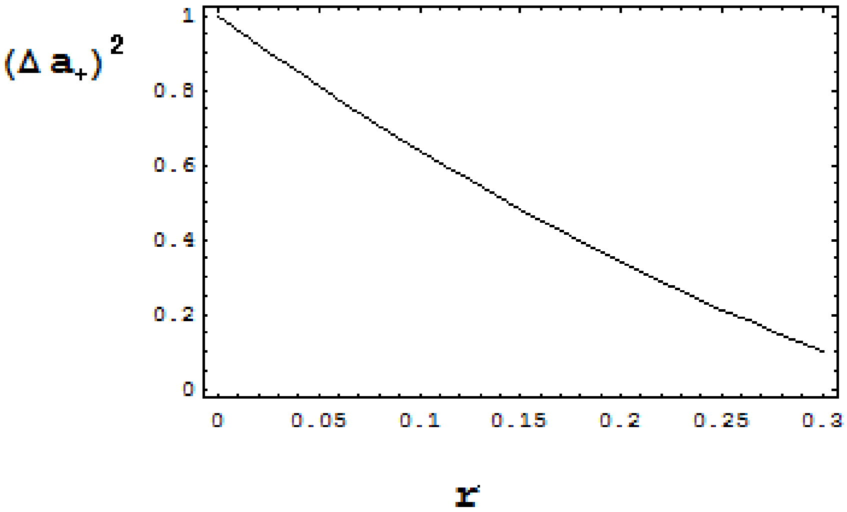

Figure 3.1: The graph for the quadrature variance $(\Delta a_+)^2$ vs the squeeze parameter r for the superposition of squeezed coherent state and a squeezed vacuum state.

It is worth mentioning that for $0 \leq r < \frac{1}{2}\ln 2$, the squeezing occurs in the plus quadrature showing that for the superposition of a squeezed coherent state and a squeezed vacuum state, r does not take all values between 0 and ∞ as the quadrature variance of the individual squeezed states.

3.2 Superposition of Squeezed vacuum and displaced Squeezed vacuum states

In this section, we determine the Q function, using Eq. (3.0.205) for the superposition of the displaced sqeezed vacuum state and a squeezed vacuum state. With the aid of the resulting Q function, we find the mean photon number, the photon number distribution and the quadrature variance for the superposition of these two squeezed states.

3.2.1 The Q function

We seek here to determine the Q function for the superposition of a displaced squeezed vacuum state and a squeezed vacuum state. To this end, in view of (2.0.149), the Q function for a displaced squeezed vacuum state can be written as

$$\begin{aligned} Q(\beta^*, \beta + \frac{\partial}{\partial \beta^*}) = \quad & \frac{\sec hr}{\pi} exp[-\beta^*\beta - \lambda^*\lambda + \beta^*\lambda + \beta\lambda^* - 2a(\beta^*\lambda^* + \beta\lambda) \\ & + a(\beta^2 + \beta^{*2} + \lambda^2 + \lambda^{*2})] \\ & \times exp[(-2a\lambda - \beta^* + \lambda^* + 2a\beta)\frac{\partial}{\partial \beta^*} + a\frac{\partial^2}{\partial \beta^{*2}}], \end{aligned} \tag{3.0.248}$$

where

$$a = -\frac{1}{2}\tanh r. \tag{3.0.249}$$

Using this expression along with Eq. (3.0.207) into Eq. (3.0.205), we have

$$\begin{aligned} Q(\alpha^*, \alpha) = \quad & \frac{sechr}{\pi} exp^{[-\alpha^*\alpha - \lambda^*\lambda + a(\lambda^2 + \lambda^{*2})]} \\ & \times \int \frac{\partial^2\beta}{\pi}\frac{\partial^2\eta}{\pi} exp[-\beta^*\beta + a(\beta^2 + \beta^{*2} + \eta^2 + \eta^{*2}) \\ & -\eta^*\eta + \beta^*\lambda + \beta\lambda^* - 2a\beta^*\lambda^* - 2a\beta\lambda + \alpha^*\beta + \alpha^*\eta] \\ & \times exp[(-\beta^* - 2a\lambda + \lambda^* + 2a\beta)\frac{\partial}{\partial \beta^*} + a\frac{\partial^2}{\partial \beta^{*2}}] exp(\alpha - \beta - \eta)\beta^* \\ & \times exp[(-\eta^* + 2a\eta)\frac{\partial}{\partial \eta^*} + a\frac{\partial^2}{\partial \eta^{*2}}] exp(\alpha - \eta - \beta)\eta^*. \end{aligned} \tag{3.0.250}$$

Expanding in power series, we have

$$\begin{aligned}&exp[(-\beta^* - 2a\lambda + \lambda^* + 2a\beta)\frac{\partial}{\partial\beta^*} + a\frac{\partial^2}{\partial\beta^{*2}}]\\&= \sum_i \frac{1}{i!}[(-\beta^* - 2a\lambda + \lambda^* + 2a\beta)\frac{\partial}{\partial\beta^*} + a\frac{\partial^2}{\partial\beta^{*2}}]^i\\&\times exp(\alpha - \beta - \eta)\beta^*.\end{aligned} \tag{3.0.251}$$

This can be inturn rewritten using Binomial expansion as

$$\begin{aligned}&exp[(-\beta^* - 2a\lambda + \lambda^* + 2a\beta)\frac{\partial}{\partial\beta^*} + a\frac{\partial^2}{\partial\beta^{*2}}]\\&= \sum_i \frac{1}{i!}\sum_j^i \frac{i!(-\beta^* - 2a\lambda + \lambda^* + 2a\beta)^{i-j}}{j!(i-j)!}\frac{\partial^{i-j}}{\partial\beta^{*i-j}}a^j\frac{\partial^{2j}}{\partial\beta^{*2j}}\\&\times exp(\alpha - \beta - \eta)\beta^*.\end{aligned} \tag{3.0.252}$$

Carrying out the differentiation, one readily finds

$$\begin{aligned}&exp[(-\beta^* - 2a\lambda + \lambda^* + 2a\beta)\frac{\partial}{\partial\beta^*} + a\frac{\partial^2}{\partial\beta^{*2}}]\\&= \sum_i \frac{1}{i!}\sum_j^i \frac{i![(-\beta^* - 2a\lambda + \lambda^* + 2a\beta)(\alpha - \beta - \eta)]^{i-j}}{j!(i-j)!}\\&\times a^j(\alpha - \beta - \eta)^{2j}e^{(\alpha-\beta-\eta)\beta^*}.\end{aligned} \tag{3.0.253}$$

This can be rewritten as

$$\begin{aligned}&exp[(-\beta^* - 2a\lambda + \lambda^* + 2a\beta)\frac{\partial}{\partial\beta^*} + a\frac{\partial^2}{\partial\beta^{*2}}]\\&= \sum_i \frac{1}{i!}[(-\beta^* - 2a\lambda + \lambda^* + 2a\beta)(\alpha - \beta - \eta) + a(\alpha - \beta - \eta)^2]^i\\&\times exp^{(\alpha-\beta-\eta)\beta^*}.\end{aligned} \tag{3.0.254}$$

It then follows that

$$\begin{aligned}&exp[(-\beta^* - 2a\lambda + \lambda^* + 2a\beta)\frac{\partial}{\partial\beta^*} + a\frac{\partial^2}{\partial\beta^{*2}}]\\&= exp(-2a\alpha\lambda + 2a\beta\lambda + 2a\eta\lambda + \lambda^*\alpha - \beta\lambda^*\\&-\lambda^*\eta - a\beta^2 + a\alpha^2 + a\eta^2 - 2a\eta\alpha).\end{aligned} \tag{3.0.255}$$

Using this expression along with Eq. (3.0.214) into Eq. (3.0.250), we obtain

$$\begin{aligned} Q(\alpha^*,\alpha) = \quad & \frac{sech^2 r}{\pi} exp[-\alpha^*\alpha - \lambda^*\lambda + \lambda^*\alpha - 2a\lambda\alpha + 2a\alpha^2 + a(\lambda^2 + \lambda^{*2})] \\ & \times \int \frac{d^2\beta}{\pi} exp[-\beta^*\beta + (\alpha^* - 2a\alpha)\beta + (\lambda - 2a\lambda^*)\beta^* + a(\beta^2 + \beta^{*2})] \\ & \times \int \frac{d^2\eta}{\pi} exp[-\eta^*\eta + (2a\lambda + \alpha^* - \lambda^* - 2a\alpha)\eta + a(\eta^2 + \eta^{*2})]. \quad (3.0.256) \end{aligned}$$

Thus performing the integration employing the relation given by (2.0.31), the Q function for the superposition of a displaced squeezed vacuum state and a squeezed vacuum state finally takes the form

$$\begin{aligned} Q(\alpha^*,\alpha) = \quad & \frac{1}{\pi} exp[-(\cosh^2 r + \sinh^2 r)(\alpha^*\alpha + \lambda^*\lambda - \lambda^*\alpha - \lambda\alpha^*) \\ & + \cosh r \sinh r(2\lambda\alpha + 2\lambda^*\alpha^* - \alpha^2 - \alpha^{*2} - \lambda^2 - \lambda^{*2})]. \quad (3.0.257) \end{aligned}$$

One can also obtain the same result by making use of Eq. (2.0.158) into (2.0.216).

3.2.2 Mean photon number

We next proceed to determine the mean photon number for the superposition of a displaced squeezed vacuum state and a squeezed vacuum state. To do so, the expectation value of $\hat{n}$ interms of the Q function is expressible as

$$\langle \hat{n} \rangle = \int d^2\alpha Q(\alpha^*,\alpha) n_a(\alpha^*,\alpha), \quad (3.0.258)$$

where

$$n_a(\alpha^*,\alpha) = \alpha\alpha^* - 1 \quad (3.0.259)$$

is the c-number function corresponding to the operator function $\hat{n}(\hat{a}^\dagger, \hat{a})$ in the antinormal order. In view of (3.0.257) and (3.0.259), expression (3.0.258) can be written in the form

$$\begin{aligned} \langle \hat{n} \rangle = \quad & exp[(\cosh^2 r + \sinh^2 r)\lambda^*\lambda - \cosh r \sinh r(\lambda^2 + \lambda^*)] \\ & \times \int \frac{d^2\alpha}{\pi} exp\Big[- (\cosh^2 r + \sinh^2 r)\alpha^*\alpha + \big((\cosh^2 r + \sinh^2 r)\lambda^* \\ & + 2\lambda \cosh r \sinh r\big)\alpha + \big((\cosh^2 r + \sinh^2 r)\lambda + 2\lambda^* \cosh r \sinh r\big)\alpha^* \\ & - \cosh r \sinh r(\alpha^2 + \alpha^{*2})\Big] \alpha\alpha^* - 1. \quad (3.0.260) \end{aligned}$$

This can be inturn written in the form

$$\begin{aligned}\langle\hat{n}\rangle = \quad & exp[(\cosh^2 r+\sinh^2 r)\lambda^*\lambda-\cosh r\sinh r(\lambda^2+\lambda^{*2})] \\ & \times\frac{d}{da}\frac{d}{db}\int\frac{d^2\alpha}{\pi}exp\bigg[-(\cosh^2 r+\sinh^2 r)\alpha^*\alpha+\big((\cosh^2 r+\sinh^2 r)\lambda^* \\ & +a+2\cosh r\sinh r\big)\alpha+\big((\cosh^2 r+\sinh^2 r)\lambda+b+2\lambda^*\cosh r\sinh r\big)\alpha^* \\ & -\cosh r\sinh r(\alpha^2+\alpha^{*2})\bigg]-1_{a=b=0}, \qquad (3.0.261)\end{aligned}$$

so that performing the integration employing the relation given by (2.0.31), we get

$$\begin{aligned}\langle\hat{n}\rangle = \quad & exp\bigg[-(\cosh^2 r+\sinh^2 r)^3\lambda^*\lambda+4(\cosh^2 r+\sinh^2 r)\cosh^2 r\sinh^2 r \\ & -(\cosh^2 r+\sinh^2 r)^2\cosh r\sinh r(\lambda^2+\lambda^{*2})+4\cosh^3 r\sinh^3 r(\lambda^2+\lambda^{*2})\bigg] \\ & \times\frac{d}{da}\frac{d}{db}exp\bigg[(\cosh^2 r+\sinh^2 r)^3\lambda^*\lambda+(\cosh^2 r+\sinh^2 r)^2\cosh r\sinh r\lambda^{*2} \\ & -4\lambda^{*2}\cosh^3 r\sinh^3 r-b^2\cosh r\sinh r-4\lambda^*\lambda(\cosh^2 r+\sinh^2 r)\cosh^2 r\sinh^2 r \\ & +a\lambda(\cosh^2 r+\sinh^2 r)^2+ab(\cosh^2 r+\sinh^2 r)-4b\lambda^*\cosh^2 r\sinh^2 r \\ & +b\lambda^*(\cosh^2 r+\sinh^2 r)^2+\lambda^2(\cosh^2 r+\sinh^2 r)^2\cosh r\sinh r \\ & -4a\lambda\cosh^2 r\sinh^2 r-4\lambda^2\cosh^3 r\sinh^3 r-a^2\cosh r\sinh r\bigg]-1_{a=b=0}. \quad (3.0.262)\end{aligned}$$

Carrying out the differentiation and applying the condition $a=b=0$, the mean photon number for the superposition of the displaced squeezed vacuum state and a squeezed vacuum state turns out to be

$$\langle\hat{n}\rangle=\lambda^*\lambda+2\sinh^2 r, \qquad (3.0.263)$$

which is the sum of the mean photon number of the displaced squeezed vacuum $|\lambda,r\rangle$ found in Eq. (2.0.179) and a squeezed vacuum state $|r\rangle$ found in Eq. (2.0.105) Showing that, the mean photon number is increased than the mean photon number of the individual superposed states.

3.2.3 Photon number distribution

We nex seek to determine the photon number distribution for the superposition of squeezed vacuum state and a displaced squeezed vacuum state. To this end, applying Eq.

(2.0.50) together with (3.0.257), the photon number distribution can be written as

$$\begin{aligned} P(n) = \quad & \frac{1}{n!} exp[-\lambda^*\lambda - a(\lambda^2 + \lambda^{*2})] \\ & \times \frac{\partial^{2n}}{\partial\alpha^{*n}\partial\alpha^n} exp\big[(\lambda^* + 2a\lambda)\alpha + (\lambda + 2a\lambda^*)\alpha^* - a(\alpha^2 + \alpha^{*2})\big], \end{aligned} \quad (3.0.264)$$

where

$$a = \frac{\cosh r \sinh r}{\cosh^2 r + \sinh^2 r}. \quad (3.0.265)$$

On acount of the power series expansions

$$exp(-a\alpha^2) = \sum_i \frac{(-a)^i}{i!}\alpha^{2i}, \quad (3.0.266)$$

$$exp(-a\alpha^{*2}) = \sum_j \frac{(-a)^j}{j!}\alpha^{*2j}, \quad (3.0.267)$$

$$exp(\lambda^* + 2a\lambda)\alpha = \sum_k \frac{(\lambda^* + 2a\lambda)^k}{k!}\alpha^k, \quad (3.0.268)$$

$$exp(\lambda + 2a\lambda^*)\alpha^* = \sum_l \frac{(\lambda + 2a\lambda^*)^l}{l!}\alpha^{*2}, \quad (3.0.269)$$

expression (3.0.264) can be put in the form

$$\begin{aligned} P(n) = \quad & \frac{1}{n!} exp[-\lambda^*\lambda - a(\lambda^2 + \lambda^{*2})] \\ & \times \sum_{ijkl} \frac{(-a)^{i+j}(\lambda^* + 2a\lambda)^k(\lambda + 2a\lambda^*)^l}{i!j!k!l!} \frac{\partial^{2n}}{\partial\alpha^{*n}\partial\alpha^n}\left(\alpha^{2i+k}\alpha^{*2j+l}\right)_{\alpha=\alpha^*=0} \end{aligned} \quad (3.0.270)$$

so that carrying out the differentiaiton using the identity given by Eq. (2.0.59) and applying the condition $\alpha = \alpha^* = 0$, we easily obtain

$$\begin{aligned} P(n) = \quad & \frac{1}{n!} exp[-\lambda^*\lambda - a(\lambda^2 + \lambda^{*2})] \\ & \times \sum_{ijkl} \frac{(-a)^{i+j}(\lambda^* + 2a\lambda)^k(\lambda + 2a\lambda^*)^l}{i!j!k!l!} \\ & \times \frac{(2i+k)!}{(2i+k-n)!}\frac{(2j+l)!}{(2j+l-n)!}\delta_{2i+k,n}\delta_{2j+l,n}. \end{aligned} \quad (3.0.271)$$

Hence in view of the property of Kronecker delta, we see that

$$k = n - 2i, \quad (3.0.272)$$

$$l = n - 2j, \tag{3.0.273}$$

and on taking into account that a factorial is defined for non-negative integers, we obtain

$$\begin{aligned} P(n) = \quad & \frac{a^n}{n!} exp[-\lambda^*\lambda - a(\lambda^2 + \lambda^{*2})] \\ & \times \left(\sum_i \frac{(-1)^i n!}{i!(n-2i)!} (\frac{\lambda^* + 2a\lambda}{\sqrt{a}})^{n-2i} \right) \left(\sum_j \frac{(-1)^j n!}{j!(n-2j)!} (\frac{\lambda + 2a\lambda^*}{\sqrt{a}})^{n-2j} \right), \end{aligned} \tag{3.0.274}$$

where $[n] = \frac{n}{2}$ for even n and $[n] = \frac{(n-1)}{2}$ for odd n. Finally the photon number distribution can be put in the form

$$\begin{aligned} P(n) = \quad & \frac{\cosh^n r \sinh^n r}{n!(\cosh^2 r + \sinh^2 r)^n} exp[-\lambda^*\lambda - a(\lambda^2 + \lambda^{*2})] \\ & \times \left| H_n \left(\frac{\lambda(\cosh^2 r + \sinh^2 r) + 2\lambda^* \cosh r \sinh r}{\sqrt{4(\cosh^3 r \sinh r + \cosh r \sinh^3 r)}} \right) \right|^2, \end{aligned} \tag{3.0.275}$$

where $H_n(x)$ is a Hermite polynomial of order n given by Eq. (3.0.236). One can also arrive at the same result by making use of Eq. (2.0.158) into (3.0.235).

3.2.4 Quadrature variance

We seek to determine the quadrature variance for the superposition of a displaced squeezed vacuum state and a squeezed vacuum state. To this end, the expectation value of $\hat{a}^{\dagger 2}$ interms of the Q function is expressible as

$$\langle \hat{a}^{\dagger 2} \rangle = \int d^2\alpha Q(\alpha^*, \alpha)\alpha^{*2}, \tag{3.0.276}$$

where α^{*2} is the c- number variable corresponding to the operator $\hat{a}^{\dagger 2}$. In view of (3.0.257), one can write

$$\begin{aligned} \langle \hat{a}^{\dagger 2} \rangle = \quad & exp[-(\cosh^2 r + \sinh^2 r)\lambda^*\lambda - \cosh r \sinh r(\lambda^2 + \lambda^{*2})] \\ & \times \int \frac{d^2\alpha}{\pi} exp[-(\cosh^2 r + \sinh^2 r)(\alpha^*\alpha - \lambda^*\alpha - \lambda\alpha^*) \\ & + \cosh r \sinh r(2\lambda\alpha + 2\lambda^*\alpha^* - \alpha^2 - \alpha^{*2})]\alpha^{*2}. \end{aligned} \tag{3.0.277}$$

This can be rewritten as

$$\begin{aligned} \langle \hat{a}^{\dagger 2} \rangle = \quad & exp[-(\cosh^2 r + \sinh^2 r)\lambda^*\lambda - \cosh r \sinh r(\lambda^2 + \lambda^{*2})] \\ & \times \frac{d^2}{da^2} \int \frac{d^2\alpha}{\pi} exp[-(\cosh^2 r + \sinh^2 r)\alpha^*\alpha + [(\cosh^2 r + \sinh^2 r)\lambda^* \\ & + 2\cosh r \sinh r\lambda]\alpha + [(\cosh^2 r + \sinh^2 r)\lambda + 2\cosh r \sinh r\lambda^* + a]\alpha^* \\ & - \cosh r \sinh r(\alpha^2 + \alpha^{*2})]_{a=0}. \end{aligned} \tag{3.0.278}$$

Thus performing the integration employing the relation (2.0.31), we get

$$\begin{aligned}\langle\hat{a}^{\dagger 2}\rangle = \quad & exp[-(\cosh^2 r+\sinh^2 r)^3\lambda^*\lambda+4(\cosh^2 r+\sinh^2 r)\cosh^2 r\sinh^2 r\lambda^*\lambda \\ & -(\cosh^2 r+\sinh^2 r)^2\cosh r\sinh r(\lambda^2+\lambda^{*2})+4\cosh^3 r\sinh^3 r(\lambda^2+\lambda^{*2})] \\ & \times\frac{d^2}{da^2}exp[(\cosh^2 r+\sinh^2 r)^3\lambda^*\lambda+(\cosh^2 r+\sinh^2 r)^2\cosh r\sinh r\lambda^{*2} \\ & +a\lambda^*(\cosh^2 r+\sinh^2 r)^2+(\cosh^2 r+\sinh^2 r)^2\cosh r\sinh r\lambda^2-4a\lambda^*\cosh^2 r\sinh^2 r \\ & -4\lambda^{*2}\cosh^3 r\sinh^3 r-a^2\cosh r\sinh r-4\lambda^*\lambda(\cosh^2 r+\sinh^2 r)\cosh^2 r\sinh^2 r \\ & -4\lambda^2\cosh^3 r\sinh^3 r]_{a=0}. \qquad (3.0.279)\end{aligned}$$

Now carrying out the differentiation and applying the condition $a = 0$, one easily finds

$$\begin{aligned}\langle\hat{a}^{\dagger 2}\rangle = \quad & (\cosh^2 r+\sinh^2 r)^4\lambda^{*2}-8(\cosh^2 r+\sinh^2 r)^2\cosh^2 r\sinh^2 r\lambda^{*2} \\ & +16\cosh^4 r\sinh^4 r\lambda^{*2}-2\cosh r\sinh r. \qquad (3.0.280)\end{aligned}$$

Following the same procedure, one can show that

$$\langle\hat{a}^{\dagger}\rangle = (\cosh^2 r+\sinh^2 r)^2\lambda^*-4\cosh^2 r\sinh^2 r\lambda^*. \qquad (3.0.281)$$

Employing Eq. (3.0.280) and (3.0.281) along with their complex conjugate together with Eq. (3.0.263) into Eqs. (3.0.238) and (3.0.239), one readily finds

$$(\Delta a_+)^2 = 2e^{-2r}-1 \qquad (3.0.282)$$

and

$$(\Delta a_-)^2 = 2e^{2r}-1. \qquad (3.0.283)$$

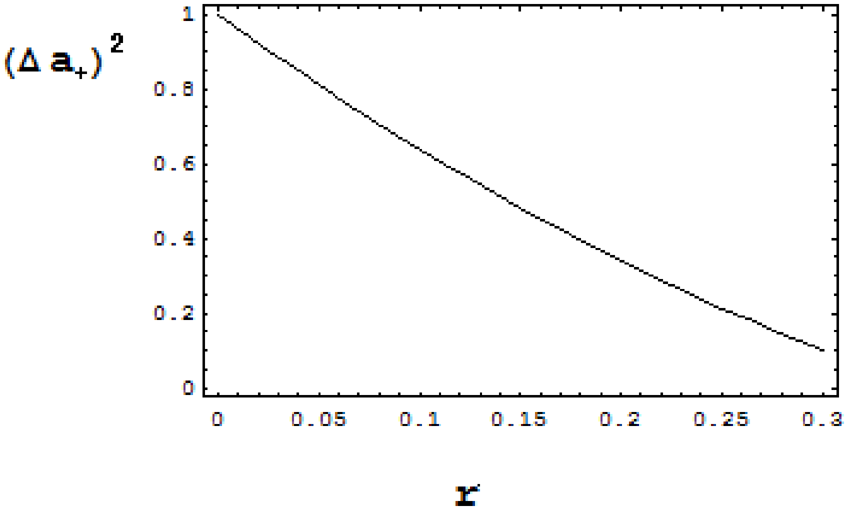

Figure 3.2: The graph for the quadrature variance $(\Delta a_+)^2$ vs the squeeze parameter r for the superposition of displaced squeezed vacuum and a squeezed vacuum state.

We see that the squeezing occurs in the plus quadrature for $0 \leq r < \frac{1}{2}\ln 2$.

On the other hand, the degree of squeezing of the superposed states is expressible as

$$1 - (\Delta a_+)^2, \tag{3.0.284}$$

where $(\Delta a_+)^2$ is the quadrature variance of the superposed states. In view of Eq. (3.0.282), the degree of squeezing can be rewritten in the form

$$2(1 - e^{-2r}), \tag{3.0.285}$$

where

$$1 - e^{-2r} \tag{3.0.286}$$

is the degree of squeezing for the individual states. This shows that the superposition leads to a two-fold increase in the degree of squeezing.

4. Conclusion

We have discussed the statistical properties of the squeezed coherent state, the squeezed vacuum state, the displaced squeezed vacuum state, the superposition of squeezed coherent state and the squeezed vacuum state as well as the superposition of a displaced squeezed vacuum state with that of the squeezed vacuum state.

We verified that a displaced squeezed vacuum state is the superposition of coherent state and a squeezed vacuum state where, as $r \to 0$, we obtain the coherent state. On the other hand, when $\lambda \to 0$, we left with the squeezed vacuum state verifying that the displaced squeezed vacuum state is the superposition of the coherent state and a squeezed vacuum state.

We noted that the mean photon number of the superposed state is the sum of the mean photon number of the separate squeezed states.

We also observed that, for the superposition of a squeezed coherent state and a squeezed vacuum state as well as for the superposition of the displaced squeezed vacuum and a squeezed vacuum state, the squeezing occurs in the plus quadrature for $0 \leq r < \frac{1}{2} \ln 2$ showing that, like the separate squeezed states, r does not take all values between 0 and ∞.

In addition, we determined that the superposition leads to a two-fold increase in the degree of squeezing.

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Declaration

This thesis is my original work, has not been presented for a degree in any other University and that all the sources of material used for the thesis have been dully acknowledged.

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