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ANALYSIS OF FOURIER TRANSFORM IN L^1 SPACE AND ITS INVERSION

By

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ABSTRACT

This project discusses the concept of Fourier transform of a function f in $L^1(\mathbb{R})$ Space with its properties theorem, inversion theorem, Fourier sine and cosine transforms theorem, Plancherel's and Parseval's identities theorem and the applications of Fourier transform in partial differential equations, Shannon's sampling theorem and Heisenberg's inequality.

Therefore the purpose of this project is to solving certain problems in partial differential equations like for example Heat equation, Wave equation , and Laplace equation, to solve some complicated integrals shortly and simply, and it works in Shannon's sampling theorem and Heisenberg's inequality.

This project uses some definitions and theorems as a preliminary from some real analysis and Fourier analysis books.

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Chapter 1

Introduction

The Fourier transform is a mathematical operation that decomposes a function in to its constituent frequencies, known as a frequency spectrum. For instance, the transform of a musical chord made up of pure notes is a mathematical representation of the amplitudes (and phase) of the individual notes that make it up. The composite wave form depends on time, and therefore is called the time domain representation. The frequency spectrum is a function of frequency and is called the frequency domain representation. Each value of the function is a complex number (called complex amplitude) that encodes both a magnitude and phase component. The term Fourier transforms refers to both the transform operation and to the complex-valued function it produces. Frequency is the number of occurrences of a repeating event per unit time. It is also referred to as temporal frequency. The period is the duration of one cycle in a repeating event, so the period is the reciprocal of the frequency.

Fourier theory is a branch of mathematics first invented to solve certain problems in partial differential equations. The most well-known of these equations are:

Laplace's equation,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0, \qquad \text{for } u(x,y) \text{ a function of two variables}$$

The wave equation,

$$\frac{\partial^2 u}{\partial t^2}-c^2\frac{\partial^2 u}{\partial x^2}=0, \qquad \text{for } u(x,t) \text{ a function of two variables,}$$

The heat equation,

$$\frac{\partial u}{\partial t}-k\frac{\partial^2 u}{\partial x^2}=0\ , \qquad \text{for } u(x,t) \text{ a function of two variables,}$$

In the heat equation, x represents the position along the bar measured from some origin, t represents time, $u(x,t)$ the temperature at position x, time t. Fourier was initially concerned with the heat equation. Incidentally, the same equation describes the concentration of a dye diffusing in a liquid such as water. For this reason the equation is sometimes called the diffusion equation.

In the wave equation, x represents the position along an elastic string under tension, measured from some origin, t represents time, $u(x,t)$ the displacement of the string from equilibrium at position x, time t.

In Laplace's equation, $u(x,y)$ represents the steady temperature of a flat conducting plate at the position (x,y) in the plane.

Since both the heat equation and the wave equation involve a single space variable x, we sometime refer to them as the one dimensional heat equation and the one dimensional wave equation respectively.

Laplace's equation involves two spatial variables and is therefore sometimes called the two-dimensional Laplace equation. Laplace's equation is connected to the theory of analytic functions of a complex variable. If $f(z) = u(x,y) + iv(x,y)$, the real and imaginary parts $u(x,y), v(x,y)$ satisfy the Cauchy-Riemann equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

Then

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} = -\frac{\partial^2 u}{\partial y^2}$$

Or

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Similarly,

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

Heat conduction and wave propagation usually occur in three space dimensions and are described by the following versions of Laplace's equation, the heat equation and the wave equation;

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0,$$

$$\frac{\partial u}{\partial t}-k\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)=0,$$

$$\frac{\partial^2 u}{\partial t^2}-c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)=0.$$

The Fourier transform is a generalization of the complex Fourier series in the limit as period p approaches to infinity.

I use some definitions and some theorems like for example Fubini's theorem and Tonelli's theorem as a preliminary to understand the topic Fourier transform.

Fourier transform has an application in partial differential equations like for example to solve Laplace equation, Heat equation and Wave equation and also used to solve many complicated differential equations that are not easily solved using the other methods, so in this project I used Fourier transform to solve partial differential equations of Laplace equation, Heat equation, and Wave equation.

In this project I included the definition and properties of Fourier transform and theorems like inversion theorem, Fourier sine and cosine transforms theorem, Parseval's and Plancherel's theorem, and Shannon's sampling theorem and Heisenberg's Inequality.

Chapter 2

Preliminaries

2.1 Piecewise continuous and piecewise smooth functions

Definition: For a given function $f(x)$ the right-hand and the left-hand limits at the point x_0 are defined as follows:

$$f({x_0}^+) = \lim_{\substack{x\to x_0 \\ x>x_0}} f(x) = \lim_{x\to {x_0}^+} f(x),$$

and

$$f({x_0}^-) = \lim_{\substack{x\to x_0 \\ x<x_0}} f(x) = \lim_{x\to {x_0}^-} f(x),$$

We say $f(x)$ has the limit at $x = x_0$ provided $f({x_0}^+) = f({x_0}^-)$. The function $f(x)$ is said to be continuous at a point x_0 provided $\lim_{x\to x_0} f(x) = f(x_0)$ and both $\lim_{x\to x_0} f(x)$, $f(x_0)$ exists, and it is continuous in an interval $\alpha < x < \beta$ if it is continuous at each point x for $\alpha < x < \beta$.

Definition: A function $f(x)$ is called piecewise continuous (sectionally continuous) on an interval $\alpha < x < \beta$ if there are finitely many points $\alpha = x_0 < x_1 < \cdots < x_n = \beta$ such that:

1. $f(x)$ is continuous on each sub- interval $x_0 < x < x_1, x_1 < x < x_2, \dots, x_{n-1} < x < x_n$, and
2. On each sub interval (x_{k-1}, x_k) both $f({x_{k-1}}^+)$ and$f({x_k}^-)$ exist, that is, are finite.

Note that the function $f(x)$ need not be defined at the points x_k.

We shall denote the class of piecewise continuous functions on $\alpha < x < \beta$ by $C_P(\alpha, \beta)$.

Definition: A function $f(x), \alpha < x < \beta$, is said to be piece wise smooth (sectionally smooth) if both $f(x)$ and $f'(x)$ are piece wise continuous on $\alpha < x < \beta$. The class of piecewise smooth functions on $\alpha < x < \beta$ is denoted by ${C^1}_P(\alpha, \beta)$.

Note that $f'(x)$ is piece wise continuous means that $f'(x)$ is continuous except at $s_0, \dots, s_m$ (these points include $x_0, x_1, \dots, x_n$ where $f(x)$ is not continuous) and on each sub interval (x_{k-1}, x_k) both $f'(x_{k-1}{}^{+})$ and$f'(x_k{}^{-})$ exist. Here $f'(s_j{}^{+}) = \lim_{x\to s_j{}^{+}} f'(x)$ and $f'(s_j{}^{-}) = \lim_{x\to s_j{}^{-}} f'(x)$.

2.2 Measure theory

1. The measure of open intervals:
   We define
   $$\mu(\mathrm{I}) = b - a$$
   Where I denotes the open interval (a, b).
2. The measure of open sets:
   Define
   $$\mu(G) = \sum \mu(\mathrm{I_k})$$
   Where G is an open set and $\{\mathrm{I_k}\}$ is the sequence of component intervals of G. If one of the components unbounded, we let $\mu(G) = \infty$. [If $G \neq \emptyset$, then G can be expressed as a finite or countably infinite disjoint union of open intervals: $G = \bigcup \mathrm{I_k}$. If $G = \emptyset$, the empty set, define $\mu(G) = 0$] this definition is a natural one; it conforms to our intuitive requirement that "the whole is equal to the sum of the parts."
3. The measure of bounded closed sets:
   Define
   $$\mu(\mathrm{E}) = b - a - \mu\big((a, b) \setminus_{\mathrm{E}}\big)$$
   Where E is abounded closed set and $[a, b]$ is the smallest closed interval containing E. Since $[a, b] = \mathrm{E} \cup ([a, b] \setminus_{\mathrm{E}})$, our intuition would demand that:
   $\mu(\mathrm{E}) + \mu\big((a, b) \setminus_{\mathrm{E}}\big) = b - a$ and this becomes our definition.

Definition: Let X be any set, and let $\mathcal{A}$ be a non empty family of subsets of X. We say $\mathcal{A}$ is an algebra of sets if it satisfies the following conditions:

1. $\emptyset \in \mathcal{A}$
2. If $A \in \mathcal{A}$ and $B \in \mathcal{A}$, then $A \cup B \in \mathcal{A}$.
3. If $A \in \mathcal{A}$, then $\mathrm{X} \setminus_{\mathrm{A}} \in \mathcal{A}$

Definition: let X be a set, and let $\mathcal{M}$ be a family of subsets of a set X. We say that $\mathcal{M}$ is a σ-algebra of sets if $\mathcal{M}$ is an algebra of sets and $\mathcal{M}$ is closed under countable unions; that is, if $\{A_k\} \subset \mathcal{M}$, then

$$\bigcup_{k=1}^{\infty} A_k \in \mathcal{M}$$

Definition: the collection $\mathfrak{B}$ of borel sets is the smallest σ-algebra which contains all of the open sets.

It is also the smallest σ-algebra which contains all closed sets and the smallest σ-algebra which contains the open intervals.

A set which is a countable union of closed sets is called an $\mathcal{F}_\sigma$ ($\mathcal{F}$ for closed, σ for sum). Thus every countable set is an $\mathcal{F}_\sigma$, as is, of course, every closed set. A countable union of sets in $\mathcal{F}_\sigma$ is again in $\mathcal{F}_\sigma$.

Since

$$(a, b) = \bigcup_{n=1}^{\infty} \left[a + \frac{1}{n}, b - \frac{1}{n}\right]$$

Each open interval is an $\mathcal{F}_\sigma$, and hence each open set is an $\mathcal{F}_\sigma$.

We say that a set is a $\mathcal{G}_\delta$ if it is the intersection of a countable collection of open sets ($\mathcal{G}$ for open, δ for durchschnitt). Thus the complement of an $\mathcal{F}_\sigma$ is a $\mathcal{G}_\delta$, and conversely.

The $\mathcal{F}_\sigma$ and $\mathcal{G}_\delta$ are relatively simple types of borel sets. We could also consider sets of type $\mathcal{F}_{\sigma\delta}$, which are the intersections of countable collections of sets each of which is an $\mathcal{F}_\sigma$. Similarly, we can construct the classes $\mathcal{G}_{\delta\sigma}, \mathcal{F}_{\sigma\delta\sigma}$, etc. Thus the classes in the two sequences:

$$\mathcal{F}_\sigma, \mathcal{F}_{\sigma\delta}, \mathcal{F}_{\sigma\delta\sigma}, \dots, \mathcal{G}_\delta, \mathcal{G}_{\delta\sigma}, \mathcal{G}_{\delta\sigma\delta}, \dots$$

are all classes of borel sets. However, not every borel set belongs to one of these classes.

Definition: Let $\mathcal{M}$ be a σ-algebra of subsets of a set X, and let μ be an extended real-valued set function on $\mathcal{M}$.we say μ is a signed measure if $\mu(\emptyset) = 0$, and whenever $\{A_k\}$ is a sequence of pairwise disjoint elements of $\mathcal{M}$, then $\sum_{n=1}^{\infty}\mu(A_n)$ is defined as an extended real number with $\mu(\bigcup_{k=1}^{\infty}A_k)=\sum_{n=1}^{\infty}\mu(A_n)$.

If $\mu(\mathrm{A})\geq 0$ for all $A\in\mathcal{M}$, we say that μ is a measure. In this case we call the triple $(\mathrm{X},\mathcal{M},\mu)$ a measure space. The members of $\mathcal{M}$ are called measurable sets.

Note: μ is translation invariant; that is, if E is a set for which μ is defined and if $\mathrm{E}+\mathrm{y}$ is the set

$$\{x+y:x\in\mathrm{E}\}$$

Obtained by replacing each point x in E by the point $x+y$, then

$$\mu(\mathrm{E}+y)=\mu\mathrm{E}.$$

Definition: Let $(\mathrm{X},\mathcal{M},\mu)$ be a measure space. The measure μ is called complete if the conditions $\mathrm{Z}\subset A$ and $\mu(A)=0$ imply that $\mathrm{Z}\in\mathcal{M}$. In that case, $(\mathrm{X},\mathcal{M},\mu)$ is called a complete measure space.

Definition: Let $(\mathrm{X},\mathcal{M},\mu)$ be a measure space. If $\mu(\mathrm{X})<\infty$, then we say that the measure space is finite. If $\mathrm{X}=\bigcup_{\mathrm{n}=1}^{\infty}\mathrm{X_n}$ with $\mu(\mathrm{X_n})<\infty$ for all $n\in\mathbb{N}$, then we say that the space is σ-finite.
Definition: let X be a set and let $\rho:\mathrm{X}\longrightarrow\mathbb{R}$. If ρ satisfies the following conditions, then we say that ρ is a metric on X and call the pair (X,ρ) a metric space.

1. $\rho(x,y)\geq 0$ for all $x,y\in\mathrm{X}$
2. $\rho(x,y)=0$ if and only if $x=y$
3. $\rho(x,y)=\rho(y,x)$ for all $x,y\in\mathrm{X}$
4. $\rho(x,z)\ \leq\rho(x,y)+\rho(y,z)$ for all $x,y,z\in\mathrm{X}$

Definition: Let $(\mathrm{X},\mathcal{M},\mu)$ be a measure space, and let

$$f:\mathrm{X}\longrightarrow[-\infty,\infty]$$

The function f is measurable if for every $\alpha\in\mathbb{R}$ the set:

$$\mathrm{E}_{\alpha}(f)=\{x:f(x)>\ \alpha\}$$

is a measurable set.

Definition: Let X be a metric space, and let

$$f\colon \mathrm{X} \longrightarrow [-\infty, \infty]$$

The function f is a borel function or is borel measurable if the set

$$\mathrm{E}_\alpha(f) = \{x \colon f(x) > \alpha\}$$

is a borel set for every $\alpha \in \mathbb{R}$.

Theorem: Let $(\mathrm{X}, \mathcal{M}, \mu)$ be a measure space and f areal-valued function on X. Then f is measurable if and only if $f^{-1}(\mathfrak{B}) \in \mathcal{M}$ for every borel set $\mathfrak{B} \subset \mathbb{R}$.

Theorem: Let $(\mathrm{X}, \mathcal{M}, \mu)$ be a measure space. The following conditions on a function f are equivalent,

1. f is measurable
2. For all $\alpha \in \mathbb{R}$,the set $\{x \colon f(x) \geq \alpha\} \in \mathcal{M}$
3. For all $\alpha \in \mathbb{R}$, the set $\{x \colon f(x) < \alpha\} \in \mathcal{M}$
4. For all $\alpha \in \mathbb{R}$, the set $\{x \colon f(x) \leq \alpha\} \in \mathcal{M}$

2.3 Integration

2.3.1 The integral of a non negative function

Theorem (monotone convergence theorem): Let $\langle f_n \rangle$ be an increasing sequence of non negative measurable functions, and let $f(x) = \underline{\lim} f_n(x)$, then:

$$\int f = \lim \int f_n$$

Definition: a non negative measurable function f is called integrable over the measurable set E if

$$\int_{\mathrm{E}} f < \infty.$$

Theorem: Let f be a non negative function which is integrable over a set E. Then given $\varepsilon > 0$ there is a $\delta > 0$ such that for every set $A \subset \mathrm{E}$ with $\mu\mathrm{A} < \delta$ we have:

$$\int_{\mathrm{A}} f < \varepsilon.$$

Proof: The proposition would be trivial if f were bounded. Set:

$$f_n(x) = \begin{cases} f(x) & ; \text{ if } f(x) \le n \\ n & ; \text{ otherwise} \end{cases}$$

Then each f_n is bounded and f_n converges to f at each point. By the monotone convergence theorem there is an N such that:

$$\int_E f_N > \int_E f - \frac{\varepsilon}{2} \text{ , and}$$

$$\int_E f - f_N < \frac{\varepsilon}{2}. \text{ choose } \delta < \frac{\varepsilon}{2N}.$$

If $\mu A < \delta$, we have

$$\int_A f = \int_A (f - f_N) + \int_A f_N$$

$$< \int_E (f - f_N) + N\mu A < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

2.3.2 The general lebesgue integral

By the positive part f^+ of a function f we mean the function $f^+ = f \vee 0$; that is:

$$f^+(x) = \max\{f(x), 0\}.$$

Similarly, we define the negative part f^- by $f^- = (-f) \vee 0$. If f is measurable, so are f^+ and f^-. We have:

$$f = f^+ - f^- \text{ and } |f| = f^+ + f^-$$

With these notions in mind we make the following definition:

Definition: A measurable function f is said to be integrable over E if f^+ and f^- are both integrable over E. In this case we define:

$$\int_E f = \int_E f^+ - \int_E f^-$$

Theorem (Fubini): Let $(X, \mathcal{M}, \mu)$ and (Y, β, ν) be two complete measure spaces and f an integrable function on $X \times \mathbf{Y}$. Then**:**

1. For almost all x the function f_x defined by $f_x(y) = f(x, y)$ is an integrable function on Y;
2. For almost all y the function f_y defined by $f_y(x) = f(x, y)$ is an integrable function on X;
3. $\int_Y f(x, y)\, d\nu(y)$ is an integrable function on X;
4. $\int_X f(x, y)\, d\mu(x)$ is an integrable function on Y;
5. $\int_X \left[\int_Y f\, d\nu\right] d\mu = \int_{X\times Y} f\, d(\mu \times \nu) = \int_Y \left[\int_X f\, d\mu\right] d\nu.$

Theorem (Tonelli): Let $(X, \mathcal{M}, \mu)$ and (Y, β, ν) be two σ-finite measure spaces, and let f be a non negative measurable function on $X \times \mathbf{Y}$. Then**:**

1. For almost all x the function f_x defined by $f_x(y) = f(x, y)$ is a measurable function on Y;
2. For almost all y the function f_y defined by $f_y(x) = f(x, y)$ is a measurable function on X;
3. $\int_Y f(x, y)\, d\nu(y)$ is a measurable function on X;
4. $\int_X f(x, y)\, d\mu(x)$ is a measurable function on Y;
5. $\int_X \left[\int_Y f\, d\nu\right] d\mu = \int_{X\times Y} f\, d(\mu \times \nu) = \int_Y \left[\int_X f\, d\mu\right] d\nu.$

Lemma: Let h and g be integrable functions on X and Y respectively, and define $f(x, y) = h(x)g(y)$. Then f is integrable on $X \times Y$ and

$$\int_{X\times Y} f\, d(\mu \times \nu) = \int_X h d\mu \int_Y g d\nu$$

(Note: we do not need to assume that μ and ν are σ-finite.)

Proof: Let h and g be integrable functions on X and Y respectively, and define $f(x,y) = h(x)g(y)$

If $h = \chi_A$ and $g = \chi_B$ where $A \subset$ X and $B \subset$ Y are measurable sets, then $f = \chi_{A\times B}$ where $A \times B$ is a measurable rectangle. Thus f is integrable on X × Y and

$$\int_{X\times Y} f\, d(\mu \times \nu) = (\mu \times \nu)(A \times B) = \mu(A)\nu(B) = \int_X h d\mu \int_Y g d\nu$$

It follows that the result holds for simple functions and thus non negative integrable functions. For general integrable functions h and g, note that $f^+ = h^+g^+ + h^-g^-$ and $f^- = h^+g^- + h^-g^+$. Thus f is integrable on X × Y and

$$\int_{X\times Y} f\, d(\mu \times \nu) = \int_{X\times Y} f^+\, d(\mu \times \nu) - \int_{X\times Y} f^-\, d(\mu \times \nu)$$

$$= \int_X h^+ d\mu \int_Y g^+ d\nu + \int_X h^- d\mu \int_Y g^- d\nu - \int_X h^+ d\mu \int_Y g^- d\nu - \int_X h^- d\mu \int_Y g^+ d\nu$$

$$= \int_X h^+ d\mu \int_Y g d\nu - \int_X h^- d\mu \int_Y g d\nu = \int_X h d\mu \int_Y g d\nu$$

Lemma: Let X = Y = $\mathbb{R}$ and let $\mu = \nu =$ Lebesgue measure. Then $\mu \times \nu$ is two-dimensional Lebesgue measure on X × Y = $\mathbb{R}^2$. We often write $dx\,dy$ for $d(\mu \times \nu)$.

a. For each measurable subset E of $\mathbb{R}$, let

$$\sigma(E) = \{\langle x, y\rangle : x - y \in E\}$$

Show that $\sigma(E)$ is a measurable subset of $\mathbb{R}^2$.

b. If f is a measurable function on $\mathbb{R}$, the function F defined by $F(x,y) = f(x-y)$ is a measurable function on $\mathbb{R}^2$.
c. If f and g are integrable functions on $\mathbb{R}$, then for almost all x the function φ given by $\varphi(y) = f(x-y)g(y)$ is integrable. If we denote its integral by $h(x)$, then h is integrable and

$$\int |h| \le \int |f| \int |g|.$$

Proof (a): If E is an open set, then $\sigma(\mathrm{E})$ is open and thus measurable.

If E is a $\mathcal{G}_\delta$ with $\mathrm{E} = \cap\, \mathrm{E_i}$, where each $\mathrm{E_i}$ is open, then $\sigma(\mathrm{E}) = \cap\, \sigma(\mathrm{E_i})$, which is measurable.

If E is a set of measure zero, then $\sigma(\mathrm{E})$ is a set of measure zero and is thus measurable.

A general measurable set E is the difference of a $\mathcal{G}_\delta$ set A and a set B of measure zero and $\sigma(\mathrm{E}) = {}^{\sigma(\mathrm{B})}\!/_{\sigma(\mathrm{A})}$ so $\sigma(\mathrm{E})$ is measurable.

(b) Let f be a measurable function on $\mathbb{R}$, and define the function F by $F(x, y) = f(x - y)$. For any α, we have

$$\{\langle x, y\rangle : F(x, y) > \alpha\} = \{\langle x, y\rangle : f(x - y) > \alpha\}$$

$$= \{\langle x, y\rangle : x - y \in f^{-1}[(\alpha, \infty)]\}$$

$$= \sigma(f^{-1}[(\alpha, \infty)]).$$

The interval (α, ∞) is a borel set so $f^{-1}[(\alpha, \infty)]$ is measurable. It follows from part (a) that $\{\langle x, y\rangle : F(x, y) > \alpha\}$ is measurable. Hence F is a measurable function on $\mathbb{R}^2$.

(c) Let f and g be integrable functions on $\mathbb{R}$ and define the function φ by $\varphi(y) = f(x - ygy.$ By Tonelli's Theorem,

$$\int_{\mathrm{X}\times\mathrm{Y}} |f(x - y)g(y)|\, dx\, dy = \int_{\mathrm{Y}} \left[\int_{\mathrm{X}} |f(x - y)g(y)|\, dx\right] dy$$

$$= \int_{\mathrm{Y}} \left[\int_{\mathrm{X}} |f(x - y)|\, dx\right] |g(y)|\, dy$$

$$= \int_{\mathrm{Y}} \left[\int_{\mathrm{X}} |f(x)|\, dx\right] |g(y)|\, dy$$

$$= \int |f| \int |g|$$

Thus the function $|f(x-y)g(y)|$ is integrable. By Fubini's Theorem, for almost all x, the function φ is integrable. Let $h = \int_{Y} \varphi$. Then

$$\int |h| = \int_{X} \left| \int_{Y} \varphi \right| \leq \int_{X} \int_{Y} |\varphi| = \int_{X\times Y} |\varphi| \leq \int |f| \int |g|.$$

2.4 The L^{P} space

Definition: Let $(X, \mathcal{M}, \mu)$ be a measure space. If f is a measurable function on X and $0 < P < \infty$, we define:

$$\|f\|_{P} = \left[\int |f|^{P}\, d\mu\right]^{\frac{1}{P}}$$

(Allowing the possibility that $\|f\|_{P} = \infty$), and we define:

$$L^{P}(X, \mathcal{M}, \mu) = \{f : X \longrightarrow \mathbb{C}\,;\, f \text{ is measurable and } \|f\|_{P} < \infty\}$$

We abbreviate $L^{P}(X, \mathcal{M}, \mu)$ by $L^{P}(\mu)$, $L^{P}(X)$, or simply L^{P} when this will cause no confusion.

Theorem (Holder inequality): If P and q are non negative extended real numbers suchthat

$$\frac{1}{P} + \frac{1}{q} = 1$$

and if $f \in L^{P}$ and $g \in L^{q}$, then $f.g \in L^{1}$ and,

$$\int |fg| \leq \|f\|_{P}.\|g\|_{q}$$

Equality holds if and only if, for some non zero constants α and β, we have $\alpha|f|^{P} = \beta|g|^{q}$ a.e.

Theorem (Minkowski Inequality): If f and g are in L^{P}, then so is $f+g$ and

$$\|f+g\|_{P} \leq \|f\|_{P} + \|g\|_{P}$$

2.5 Convolution

Definition: Let f and g be real valued functions, then the convolutions of f and g, is defined as:

$$(f*g)(x) = f*g(x) = \int_{\mathbb{R}} f(x-y)g(y)\,dy\ , \qquad x,y \in \mathbb{R}.$$

Lemma: Convolution obeys the same algebraic laws as ordinary multiplication:

1. $f*(ag+bh) = a(f*g) + b(f*h)$
2. $f*g = g*f$
3. $f*(g*h) = (f*g)*h$

Lemma: If $f, g \in L^1(\mathbb{R})$, then $f*g \in L^1(\mathbb{R})$, and $\|f*g\|_1 \le \|f\|_1\|g\|_1$.

Proof: since, the function:

$$F(x,y) = f(x-y)g(y)$$

is a measurable function with respect to two-dimensional Lebesgue measure in $\mathbb{R}^2$.

Thus we can apply Tonelli's theorem to obtain:

$$\int_{\mathbb{R}} |f*g(x)|\,dx = \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(x-y)g(y)\,dy \right| dx$$

$$\le \int_{\mathbb{R}} \left[\int_{\mathbb{R}} |f(x-y)g(y)|\,dy \right] dx$$

$$= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} |f(x-y)||g(y)|\,dy \right] dx$$

$$= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} |f(x-y)|\,dx \right] |g(y)|\,dy$$

$$= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} |f(x)|\,dx \right] |g(y)|\,dy$$

$$= \int_{\mathbb{R}} |f(x)|\, dx \int_{\mathbb{R}} |g(y)|\, dy$$

Thus $f * g \in L^1(\mathbb{R})$ and $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$.

2.6 Fourier series

Definition: A function $f(x),\ f: \mathbb{R} \longrightarrow \mathbb{R}$ is called periodic if $f(x + \mathrm{P}) = f(x)$ for all $x \in \mathbb{R}$. $\mathrm{P} > 0$ is called the period of f.

Suppose $f(x)$ is periodic with period2π, then an important question is whether $f(x)$ has a Fourier series expansion of the form:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)\,, 0 < x < 2\pi$$

The constant term is taken as $\frac{a_0}{2}$ as a matter of convenience.

The formulas:

$$e^{ix} = \cos x + i \sin x,$$

$$e^{-ix} = \cos x - i \sin x.$$

Can be used to write the Fourier series expansion as:

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{inx}$$

Where the coefficients are:

$$c_n = \frac{a_n - ib_n}{2}\,, c_{-n} = \frac{a_n + ib_n}{2}\,, n = 1,2, \dots, c_0 = \frac{a_0}{2}.$$

Assuming for the moment that the 2π-periodic function f has a Fourier series expansion, the Fourier coefficients c_n will be determined using the following orthogonality property of the complex exponentials $e^{inx}, n = 0, \pm 1, \pm 2, \dots$,.

Lemma:

$$\int_{-\pi}^{\pi} e^{inx}e^{imx} = \begin{cases} 0 & ; n \neq m \\ 2\pi & ; n = m \end{cases}$$

Chapter 3

Fourier transform

3.1 Definitions of Fourier transform

The Fourier integral (Fourier transform) is a natural extension of Fourier trigonometric series in the sense that it represents a piece wise smooth function whose domain is semi-infinite or infinite.

Let $f_P(x)$ be aperiodic function of 2P that can be represented by a Fourier trigonometric series:

$$f_P(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(\omega_n x) + b_n \sin(\omega_n x))$$

$$\text{Where, } \omega_n = \frac{n\pi}{2} \;;$$

$$a_n = \frac{1}{P}\int_{-P}^{P} f_P(t)\cos(\omega_n t)\,dt\,, \qquad n \geq 0$$

$$b_n = \frac{1}{P}\int_{-P}^{P} f_P(t)\sin(\omega_n t)\,dt\,, \qquad n \geq 1$$

Now insert a_n and b_n in $f_P(x)$, then

$$f_P(x) = \frac{1}{2P}\int_{-P}^{P} f_P(t)\,dt + \frac{1}{P}\sum_{n=1}^{\infty}\left(\cos(\omega_n x)\int_{-P}^{P} f_P(t)\cos(\omega_n t)\,dt + \sin(\omega_n x)\int_{-P}^{P} f_P(t)\sin(\omega_n t)\,dt\right)$$

We now set:

$$\Delta\omega = \omega_{n+1} - \omega_n = \frac{(n+1)\pi}{P} - \frac{n\pi}{P} = \frac{\pi}{P}\,, then\ \frac{1}{P} = \frac{\Delta\omega}{\pi}, and\ write:$$

$$f_P(x)$$
$$= \frac{1}{2\mathrm{P}}\int_{-\mathrm{P}}^{\mathrm{P}} f_P(t)\,dt$$
$$+ \frac{1}{\pi}\sum_{n=1}^{\infty}\Bigg(\cos(\omega_n\, x)\Delta\omega\int_{-\mathrm{P}}^{\mathrm{P}} f_P(t)\cos(\omega_n\, t)\,dt$$
$$+ \sin(\omega_n x)\Delta\omega\int_{-\mathrm{P}}^{\mathrm{P}} f_P(t)\sin(\omega_n t)\,\mathrm{dt}\Bigg) \qquad (1)$$

We now let $P \longrightarrow \infty$ and assume that the resulting non periodic function $f(x) = \lim_{P\to\infty} f_P\,(x)$

is absolutely integrable on the $x - axis$,that is,$\int_{-\infty}^{\infty}|f(x)|\,dx\ < \infty$ .Then $\frac{1}{P} \to 0$, and the value of the first term on the right side of equation (1) approaches zero, also, $\Delta\boldsymbol{\omega} = \frac{\boldsymbol{\pi}}{\boldsymbol{P}} \longrightarrow \mathbf{0}$ and the infinite series in equation (1) becomes an integral from 0 to ∞, which represents $f(x)$, that is ,

$$f(x) = \frac{1}{\pi}\int_0^{\infty}[\cos(\,\omega\, x)\int_{-\infty}^{\infty} f(t)\cos(\omega\, t)\,dt + \sin(\omega x)\int_{-\infty}^{\infty} f(t)\sin(\omega t)\,\mathrm{dt}]\,\mathrm{d}\omega$$

$$= \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\,[\cos(\,\omega\, x)\cos(\omega\, t) + \sin(\omega x)\sin(\omega t)]\,\mathrm{dt}\,\mathrm{d}\omega$$

$$= \frac{1}{\pi}\int_0^{\infty}\left[\int_{-\infty}^{\infty} f(t)\cos(\omega x - \omega t)\,dt\right]\mathrm{d}\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t)\cos(\omega x - \omega t)\,dt\right]\mathrm{d}\omega \qquad (2)$$

Since $\int_{-\infty}^{\infty} f(t)\cos(\omega x - \omega t)\,dt$ is an even function of ω, because $\cos(\omega x - \omega t)$ is an even function of **ω**, the function f does not depend on **ω**, and we integrate with respect to t (not **ω**).

From the above argument it is clear that:

$$\frac{\mathrm{i}}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t)\sin(\omega x - \omega t)\,dt\right]\mathrm{d}\omega = 0 \qquad (3)$$

Adding (2) and (3) gives:

$f(x) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t)e^{i\omega(x-t)}\,dt\right]\mathrm{d}\omega$

This is called the complex Fourier integral.

Then $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt\right]\mathrm{e}^{i\omega x}\mathrm{d}\omega$

The expressions in brackets is a function of ω, denoted by $\mathcal{F}(\omega)$ *or* $\mathcal{F}(f(x))$ *or* $\hat{f}(\omega)$, and is called the Fourier transform of f .Now writing x for t, we get :

$\mathcal{F}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx$, and

$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{F}(\omega)e^{i\omega x}\,d\omega$ is called inverse Fourier transform of $\mathcal{F}(\omega)$.

Definition: The Fourier transform of a function $f \in L^1(\mathbb{R})$ is defined to be:

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-i\xi x}\,dx \ ,\xi \in \mathbb{R}$$

and

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\xi)e^{ix\xi}\,d\xi \ \ ,\xi \in \mathbb{R}$$

is called the inverse Fourier transform of $\hat{f}(\xi)$.

Lemma: for any function f on $\mathbb{R}$ and every $y \in \mathbb{R}$, let f_y be the translate of f defined by:

$$f_y(x) = f(x-y) \ , \qquad x \in \mathbb{R}$$

If $1 \le \mathrm{P} < \infty$, and if $f \in L^{\mathrm{P}}$, the mapping $y \longrightarrow f_y$ is a uniformly continuous mapping of $\mathbb{R}$ into $L^{\mathrm{P}}(\mathbb{R})$.

Proof: Fix $\varepsilon > 0$. Since $f \in L^{\mathrm{P}}$ there exists a continuous function g whose support lies in a bounded interval $[-A, A]$, such that

$$\|f-g\|_{\mathrm{P}} < \varepsilon$$

The uniform continuity of g shows that there exists a $\delta \in (0, A)$, such that $|s-t| < \delta$ implies

$$|g(s)-g(t)|<(3A)^{-\frac{1}{\mathrm{P}}}\varepsilon.$$

If $|s-t|<\delta$, it follows that

$$\int_{-\infty}^{\infty}|g(x-s)-g(x-t)|^{\mathrm{P}}\,dx<(3A)^{-1}\varepsilon^{\mathrm{P}}(2A+\delta)<\varepsilon^{\mathrm{P}},$$

So that $\|g_s-g_t\|_{\mathrm{P}}<\varepsilon$.

Note that L^{P} −norms (relative to Lebesgue measure) are translation invariant:$\|f\|_{\mathrm{P}}=\|f_s\|_{\mathrm{P}}$. Thus

$$\|f_s-f_t\|_{\mathrm{P}}\le\|f_s-g_s\|_{\mathrm{P}}+\|g_s-g_t\|_{\mathrm{P}}+\|g_t-f_t\|_{\mathrm{P}}$$

$$=\|(f-g)_s\|_{\mathrm{P}}+\|g_s-g_t\|_{\mathrm{P}}+\|(g-f)_t\|_{\mathrm{P}}<3\varepsilon$$

Whenever$|s-t|<\delta$. This completes the proof.

3.2 Lemma (Riemann-Lebesgue):

The Fourier transform of a function $f\in L^1(\mathbb{R})$ is continuous function on $\mathbb{R}$ and $\hat{f}(\xi)\longrightarrow 0$ as $|\xi|\longrightarrow\infty$

Proof: Assume that $f\in L^1(\mathbb{R})$.

To prove $\hat{f}(\xi)\longrightarrow 0$ as $|\xi|\longrightarrow\infty$.

Let $\xi\in\mathbb{R}$,$\xi\neq 0$.Then

$$\hat{f}(\xi)=\int_{-\infty}^{\infty}f(x)e^{-i\xi x}\,dx$$

$$=-e^{-\pi i}\int_{-\infty}^{\infty}f(x)e^{-i\xi x}\,dx$$

$$=-\int_{-\infty}^{\infty}e^{-\pi i}f(x)e^{-i\xi x}\,dx\;=-\int_{-\infty}^{\infty}f(x)e^{-i\xi\left(x+\frac{\pi}{\xi}\right)}\,dx$$

$$= -\int_{-\infty}^{\infty} f\left(x - \frac{\pi}{\xi}\right) e^{-i\xi x}\, dx\,.$$

Therefore

$$2\hat{f}(\xi) = \hat{f}(\xi) + \hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-i\xi x}\, dx - \int_{-\infty}^{\infty} f\left(x - \frac{\pi}{\xi}\right) e^{-i\xi x}\, dx$$

It implies that

$$2\left|\hat{f}(\xi)\right| = \left|\int_{-\infty}^{\infty} f(x) e^{-i\xi x}\, dx - \int_{-\infty}^{\infty} f\left(x - \frac{\pi}{\xi}\right) e^{-i\xi x}\, dx\right|$$

$$= \left|\int_{-\infty}^{\infty} \left(f(x) - f\left(x - \frac{\pi}{\xi}\right)\right) e^{-i\xi x}\, dx\right|$$

$$\leq \int_{-\infty}^{\infty} \left|\left(f(x) - f\left(x - \frac{\pi}{\xi}\right)\right) e^{-i\xi x}\right|\, dx$$

$$= \int_{-\infty}^{\infty} \left|f(x) - f\left(x - \frac{\pi}{\xi}\right)\right| \left|e^{-i\xi x}\right|\, dx$$

$$= \int_{-\infty}^{\infty} \left|f(x) - f\left(x - \frac{\pi}{\xi}\right)\right|\, dx \longrightarrow 0 \;\; as \;\; |\xi| \longrightarrow \infty$$

(by the above Lemma.)

To prove continuity of $\hat{f}$:

Let $\varepsilon > o$ be given and $\alpha > 0$ chosen such that

$$\int_{|x|>\alpha} |f(x)|\, dx < \frac{\varepsilon}{4}$$

and $\delta > 0$ Chosen such that

$$2\alpha\delta \int_{|x|<\alpha} |f(x)|\, dx < \varepsilon$$

Then for $|\eta| < \delta$,

$$\left|\hat{f}(\xi+\eta)-\hat{f}(\xi)\right| = \left|\int_{-\infty}^{\infty}\left(f(x)e^{-i(\xi+\eta)x} - f(x)e^{-i\xi x}\right)dx\right|$$

$$\leq \int_{-\infty}^{\infty}|f(x)|\left|e^{-i(\xi+\eta)x} - e^{-i\xi x}\right|dx$$

$$= \int_{-\infty}^{\infty}|f(x)|\left|e^{-i\xi x}\left(e^{-i\eta x}-1\right)\right|dx$$

$$= 2\int_{-\infty}^{\infty}|f(x)|\left|\sin\left(\frac{\eta x}{2}\right)\right|dx$$

$$\leq 2\int_{|x|>\alpha}|f(x)|\left|\sin\left(\frac{\eta x}{2}\right)\right|dx + 2\int_{|x|<\alpha}|f(x)|\left|\sin\left(\frac{\eta x}{2}\right)\right|dx$$

$$\leq 2\int_{|x|>\alpha}|f(x)|\,dx + 2\int_{|x|<\alpha}|f(x)|\left|\frac{\eta x}{2}\right|dx$$

$$\leq 2\int_{|x|>\alpha}|f(x)|\,dx + \alpha\delta\int_{|x|<\alpha}|f(x)|\,dx$$

$$< \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$$

Therefore $\hat{f}(\xi)$ is uniformly continuous on $\mathbb{R}$.

3.3 Theorem (Properties of Fourier transform)

1. Linearity: For any constants a, b and if $f \in L^1(\mathbb{R})$, then the following inequality holds:

$$\mathcal{F}\big(af(x)+bg(x)\big) = a\mathcal{F}\big(f(x)\big) + b\mathcal{F}\big(g(x)\big)$$

Proof:

$$\mathcal{F}\big(af(x)+bg(x)\big) = \int_{-\infty}^{\infty}\big(af(x)+bg(x)\big)e^{-i\xi x}\,dx$$

$$= \int_{-\infty}^{\infty} af(x)e^{-i\xi x}\,dx + \int_{-\infty}^{\infty} bg(x)e^{-i\xi x}\,dx$$

$$= a\int_{-\infty}^{\infty} f(x)e^{-i\xi x}\,dx + b\int_{-\infty}^{\infty} g(x)e^{-i\xi x}\,dx$$

$$= a\mathcal{F}\big(f(x)\big) + b\mathcal{F}\big(g(x)\big)$$

2. Translation
   a. Let $f: \mathbb{R} \longrightarrow \mathbb{R}$ and $h \in \mathbb{R}$ .The translate of f by h is the function $\tau_h f$ defined by $(\tau_h f)(x) = f(x-h), x \in \mathbb{R}$, then $\widehat{(\tau_h f)}(\xi) = e^{-i\xi h}\hat{f}(\xi)$
   b. For $c \in \mathbb{R}$,
   $\big(\widehat{e^{icx}f(x)}\big)(\xi) = \big(\widehat{\tau_{(-c)}f}\big)(\xi)$

Proof: a. The Fourier transform of $\tau_h f$ is:

$$\big(\widehat{\tau_h f}\big)(\xi) = \int_{-\infty}^{\infty} (\tau_h f)\,(x) e^{-i\xi x}\,dx$$

$$= \int_{-\infty}^{\infty} f(x-h)\, e^{-i\xi x}\,dx$$

$$= \int_{-\infty}^{\infty} f(x)\, e^{-i\xi(x+h)}\,dx$$

$$= e^{-i\xi h}\int_{-\infty}^{\infty} f(x) e^{-i\xi x}\,dx$$

$$= e^{-i\xi h}\hat{f}(\xi)$$

b. The Fourier transform of $e^{icx}f(x)$ is:

$$\big(\widehat{e^{icx}f(x)}\big)(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x}\left(e^{icx}f(x)\right)dx$$

$$= \int_{-\infty}^{\infty} f(x)\, e^{-i\xi(x-c)}\,dx$$

$$= \hat{f}(\xi + c) = \big(\widehat{\tau_{(-c)}f}\big)(\xi)$$

3. Dilation: Let $\lambda \in \mathbb{R}, \lambda > 0$,the dilation of f by λ is defined as $\delta_\lambda f$ where $(\delta_\lambda f)(x) = \lambda^{\frac{-1}{2}} f(\lambda^{-1}x), x \in \mathbb{R}$, then $\widehat{(\delta_\lambda f)}(\xi) = (\widehat{\delta_{\lambda^{-1}} f})(\xi)$
   Proof: The Fourier transform of $(\delta_\lambda f)(x)$is:

$$\widehat{(\delta_\lambda f)}(\xi) = \int_{-\infty}^{\infty} (\delta_\lambda f)(x)\, e^{-i\xi x}\, dx$$

$$= \lambda^{\frac{-1}{2}} \int_{-\infty}^{\infty} f(\lambda^{-1}x)\, e^{-i\xi x}\, dx$$

$$= \lambda^{\frac{-1}{2}} \int_{-\infty}^{\infty} f(y)\, e^{-i(\xi\lambda)y}\, \lambda dy$$

$$= \lambda^{\frac{1}{2}} \hat{f}(\lambda\xi) = (\widehat{\delta_{\lambda^{-1}} f})(\xi)$$

4. Differentiation: Let $f(x)$ be continuous on ℝ and $f(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$. Furthermore let $f'(x) \in L^1(\mathbb{R})$.Then $\widehat{f'}(\xi) = i\xi \hat{f}(\xi)$.

Proof: Integrating by parts and using $f(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$.we obtain

$$\widehat{f'}(\xi) = \int_{-\infty}^{\infty} f'(x)\, e^{-i\xi x}\, dx$$

$$= \left(\left[f(x) e^{-i\xi x} \right]_{-\infty}^{\infty} - (-i\xi) \int_{-\infty}^{\infty} f(x) e^{-i\xi x}\, dx \right)$$

$$= i\xi \hat{f}(\xi)$$

Similarly,$\widehat{f''}(\xi) = i\xi \widehat{f'}(\xi) = (i\xi)^2 \hat{f}(\xi) = -\xi^2 \hat{f}(\xi)$

Examples:

a) Show that

$$\left(\widehat{x e^{-x^2}}\right) = \frac{-i\xi}{2\sqrt{2}} e^{\frac{-\xi^2}{4}}$$

Proof:

$$\left(\widehat{x e^{-x^2}}\right) = \left(\widehat{\frac{-1}{2} (e^{-x^2})'} \right)$$

$$= \frac{-1}{2} \left(\widehat{(e^{-x^2})'} \right)$$

$$= \frac{-1}{2} i\xi \left(\widehat{e^{-x^2}} \right)$$

$$= \frac{-1}{2} i\xi \frac{1}{\sqrt{2}} e^{\frac{-\xi^2}{4}}$$

$$= \frac{-i\xi}{2\sqrt{2}} e^{\frac{-\xi^2}{4}}$$

b) The property of Fourier transform of derivatives can be used for solution of differential equations:

$$y' - 4y = H(t)e^{-4t}$$

$$H(t) = \begin{cases} 0 & ;\ t < 0 \\ 1 & ;\ t \geq 0 \end{cases}$$

$$\mathcal{F}(y') - 4\mathcal{F}(y) = \mathcal{F}(H(t)e^{-4t}) = \frac{1}{4 + iw}$$

Setting $\mathcal{F}\big(y(t)\big) = Y(w)$, we have $iwY(w) - 4Y(w) = \frac{1}{4+iw}$

Then $Y(w) = \frac{1}{(4+iw)(-4+iw)} = -\frac{1}{16+w^2}$

Therefore

$$y(w) = \mathcal{F}^{-1}\big(Y(w)\big) = -\frac{1}{8}e^{-4|t|}$$

5. Multiplication: we denote by ∂ the differential operator $\partial = \frac{\partial}{\partial\xi}$ .If f and $xf \in L^1(\mathbb{R})$ then

$\partial\hat{f}(\xi) = \big(\widehat{(-\imath\xi)f}\big)(\xi).$

Proof: Let $g(x) = (-ix)f(x), \partial = \frac{\partial}{\partial\xi}$

Now

$$\partial\hat{f}(\xi) = \frac{\partial}{\partial\xi}\hat{f}(\xi) = \frac{\partial}{\partial\xi}\int_{-\infty}^{\infty} f(x)e^{-i\xi x}\,dx$$

$$= \int_{-\infty}^{\infty} \frac{\partial}{\partial\xi} f(x)e^{-i\xi x}\,dx$$

$$= \int_{-\infty}^{\infty} (-ix)f(x)e^{-i\xi x}\,dx$$

$$= \int_{-\infty}^{\infty} g(x)e^{-i\xi x}\,dx$$

$$= \hat{g}(\xi)$$

$$= \left((\widehat{-\iota\xi)f}\right)(\xi)$$

Similarly it is possible to show using the principle of induction:

$\partial^k \hat{f}(\xi) = ((\widehat{-\iota\xi)^k f})(\xi), k = 1,2, \ldots$

6. Symmetry: let $f \in L^1(\mathbb{R})$, then $\mathcal{F}\big(\mathcal{F}(t)\big) = 2\pi f(-w)$

   Proof: The inverse Fourier transform is:

$$f(t) = \mathcal{F}^{-1}\big(f(w)\big) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{F}(w)e^{iwt}\,dw$$

   Therefore

$$2\pi f(-w) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{F}(t)e^{-iwt}\,dt = \mathcal{F}\big(\mathcal{F}(t)\big)$$

7. Modulation: let $f \in L^1(\mathbb{R})$, then

$$\mathcal{F}(f(t)\cos(w_0 t)) = \frac{1}{2}[\mathcal{F}(w + w_0) + \mathcal{F}(w - w_0)]$$

$$\mathcal{F}(f(t)\sin(w_0 t)) = \frac{1}{2}[\mathcal{F}(w + w_0) - \mathcal{F}(w - w_0)]$$

   Proof: Using Euler formula, properties 1(linearity) and properties 2b (shifting):

$$\mathcal{F}(f(t)\cos(w_0 t)) = \tfrac{1}{2}\left[\mathcal{F}\left(e^{iw_0 t}f(t)\right)\right] + \tfrac{1}{2}\left[\mathcal{F}\left(e^{-iw_0 t}f(t)\right)\right]$$

$$= \tfrac{1}{2}[\mathcal{F}(w + w_0) + \mathcal{F}(w - w_0)]$$

8. Convolution: Let $f, g \in L^1(\mathbb{R})$, then so is $f * g$ and $(\widehat{f * g})(\xi) = \hat{f}(\xi)\hat{g}(\xi)$,where $f * g$ is the convolution of real valued functions f and g such that

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x - y)g(y)\,dy$$

   Proof: Using Tonell's Theorem, we have:

$$(\widehat{f * g})(\xi) = \int_{-\infty}^{\infty} (f * g)(x)e^{-i\xi x}\,dx$$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x - y)g(y)\,dy\right]e^{-i\xi x}\,dx$$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x - y)\,e^{-i\xi x}\,dx\right]g(y)\,dy$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x-y)\, e^{-i\xi(x-y)}\, dx \right] e^{-i\xi y}\, g(y)\, dy$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x)\, e^{-i\xi(x)}\, dx \right] e^{-i\xi y}\, g(y)\, dy$$

$$= \int_{-\infty}^{\infty} f(x)\, e^{-i\xi x}\, dx \int_{-\infty}^{\infty} e^{-i\xi y}\, g(y)\, dy \; = \hat{f}(\xi)\hat{g}(\xi)$$

Note: By taking inverse Fourier transforms of both sides equation $(\widehat{f * g})(\xi) = \hat{f}(\xi)\hat{g}(\xi)$, we obtain:

$$(f * g)(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)\hat{g}(\xi) e^{i\xi x}\, d\xi$$

Examples:

a) what function f has the fourier transform

$$\hat{f}(w) = \frac{1}{(1+w^2)^2}$$

Solution: we start from the formula

$\hat{g}(w) = \frac{2}{1+w^2}$ if $g(t) = e^{-|t|}$.

By the convolution theorem, we get:

$$\mathcal{F}(g * g)(w) = \left(\hat{g}(w)\right)^2 = \frac{4}{(1+w^2)^2} = 4\hat{f}(w)$$

Thus $f = \frac{1}{4}(g * g)$

Now for $t > 0$ we get:

$$4f(t) = g * g(t) = \int_{-\infty}^{\infty} e^{-|t-y|}\, e^{-|y|}\, dy$$

$$= \int_{-\infty}^{0} e^{-(t-y)} e^{y}\, dy + \int_{0}^{t} e^{-(t-y)} e^{-y} dy + \int_{t}^{\infty} e^{t-y} e^{-y} dy$$

$$= e^{-t} \int_{-\infty}^{0} e^{2y}\, dy + e^{-t} \int_{0}^{t} dy + e^{t} \int_{t}^{\infty} e^{-2y} dy$$

$$= (1+t)e^{-t}$$

Since $\hat{f}$ is an even function, f is also an even function, and so we must have

$$f(t) = \frac{1}{4}(1 + |t|)e^{-|t|}$$

b) If $f(t) = \frac{1}{(1+t^2)}$,then find $f * f$.

Solution: put $g = f * f$.

Computing the convolution directly is toil some. Instead, we make use of convolution theorem.

Let us start from the fact that $\hat{f}(w) = \pi e^{-|w|}$ or $\int_{-\infty}^{\infty} \frac{e^{-iwt}}{(1+t^2)} dt = \pi e^{-|w|}$, then convolution Theorem gives:

$$\hat{g}(w) = \left(\hat{f}(w)\right)^2 = \pi^2 e^{-|2w|}$$

We now exchange w for $2w$, multiply by π and make the change of variable $2t = y$:

$$\hat{g}(w) = \pi^2 e^{-|2w|}$$

$$= \int_{-\infty}^{\infty} \frac{\pi e^{-it.2w}}{1 + t^2} dt$$

$$= \int_{-\infty}^{\infty} \frac{\pi e^{-iyw}}{1 + \left(\frac{y}{2}\right)^2} \frac{dy}{2} = \int_{-\infty}^{\infty} \frac{2\pi e^{-iwt}}{4 + t^2} dt$$

We find that $g(t) = \frac{2\pi}{4+t^2}$. Thus, we have proved the formula:

$$\int_{-\infty}^{\infty} \frac{1}{(1 + y^2)(1 + (t - y)^2)} dy = \frac{2\pi}{4 + t^2}, t \in \mathbb{R}$$

Lemma: $\int_0^{\infty} \frac{\sin Au}{u} du = \frac{\pi}{2}$ for $A > 0$.

Proof: It is easy to check that, by the change of variable $Au = t$, that the integral is independent of the A(if $A > 0$), so one can just as well assume that $A = 1$.

Since $\frac{1}{u} = \int_0^{\infty} e^{-ux} dx$, which might be substituted in to the integral:

$$\int_0^\infty \frac{\sin u}{u} du = \int_0^\infty \left[\int_0^\infty e^{-ux}\, dx\right] \sin u\, du = \int_0^\infty \left[\int_0^\infty \sin u\, e^{-ux}\, du\right] dx$$

$$= \int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}$$

3.4 Theorem (Inversion Theorem):

Let $f, \hat{f} \in L^1(\mathbb{R})$ and let f be piecewise smooth on $\mathbb{R}$ .Then for every $x \in \mathbb{R}$,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}\,(\xi)e^{ix\xi}\, d\xi = \frac{1}{2}[f(x^-) + f(x^+)].$$

Proof: Assume $f \in L^1(\mathbb{R})$, f is piecewise smooth on $\mathbb{R}$ and $\hat{f} \in L^1(\mathbb{R})$, define

$$f_a(x) = \frac{1}{2\pi}\int_{-a}^{a} \hat{f}\,(\xi)e^{ix\xi}\, d\xi$$

$$= \frac{1}{2\pi}\int_{-a}^{a}\left[\int_{-\infty}^{\infty} f(y)\, e^{-i\xi y}\, dy\right] e^{ix\xi}\, d\xi$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-a}^{a} e^{ix\xi}\, d\xi\right] f(y)e^{-i\xi y}\, dy$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-a}^{a} e^{ix\xi}\, e^{-i\xi y} d\xi\right] f(y)\, dy$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-a}^{a} e^{i(x-y)\xi}\, d\xi\right] f(y)\, dy$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{e^{ia(x-y)} - e^{-ia(x-y)}}{i(x-y)}\right] f(y)\, dy$$

$$= \int_{-\infty}^{\infty} D_a(x-y)f(y)dy$$

$$= \int_{-\infty}^{\infty} D_a(y)f(x-y)dy$$

Where $D_a(x) = \frac{\sin ax}{\pi x}$

Now $f_a(x) - \frac{1}{2}[f(x^-) + f(x^+)]$

$$= \int_{-\infty}^{\infty} D_a(y) f(x-y)dy - \frac{1}{2}[f(x^-) + f(x^+)]$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin ay}{\pi y} f(x-y)dy - \frac{1}{2}[f(x^-) + f(x^+)]$$

$$= \frac{1}{\pi}\int_{0}^{\infty} \frac{\sin ay}{\pi y} f(x-y)dy + \frac{1}{\pi}\int_{-\infty}^{0} \frac{\sin ay}{\pi y} f(x-y)dy - \frac{1}{2}[f(x^-) + f(x^+)]$$

$$= \frac{1}{\pi}\int_{0}^{\infty} \frac{\sin ay}{\pi y} f(x-y)dy - \frac{1}{2}\, f(x^-) + \frac{1}{\pi}\int_{-\infty}^{0} \frac{\sin ay}{\pi y} f(x-y)dy - \frac{1}{2}\, f(x^+)$$

$$= \frac{1}{\pi}\int_{0}^{\infty} \frac{\sin ay}{\pi y} f(x-y)dy - \frac{1}{\pi}\left(\frac{\pi}{2}\right) f(x^-) + \frac{1}{\pi}\int_{-\infty}^{0} \frac{\sin ay}{\pi y} f(x-y)dy - \frac{1}{\pi}\left(\frac{\pi}{2}\right)\, f(x^+)$$

$$= \frac{1}{\pi}\int_{0}^{\infty} \frac{\sin ay}{\pi y} f(x-y)dy - \frac{1}{\pi}\left(\int_{0}^{\infty} \frac{\sin ay}{y} dy\right) f(x^-) + \frac{1}{\pi}\int_{-\infty}^{0} \frac{\sin ay}{\pi y} f(x-y)dy$$

$$-\frac{1}{\pi}\left(\int_{0}^{\infty} \frac{\sin ay}{y} dy\right)\, f(x^+)$$

$$= \frac{1}{\pi}\int_{0}^{\infty} [f(x-y) - f(x^-)]\frac{\sin ay}{y} dy + \frac{1}{\pi}\int_{0}^{\infty} \frac{\sin ay}{\pi y} f(x+y)dy - \frac{1}{\pi}\left(\int_{0}^{\infty} \frac{\sin ay}{y} f(x^+)dy\right)$$

$$= \frac{1}{\pi}\int_{0}^{\infty} [f(x-y) - f(x^-)]\frac{\sin ay}{y} dy + \frac{1}{\pi}\int_{0}^{\infty} [f(x+y) - f(x^+)]\frac{\sin ay}{y} dy$$

In the first integral,

$$\frac{1}{\pi}\int_{0}^{\infty} [f(x-y) - f(x^-)]\frac{\sin ay}{y} dy$$

$$= \frac{1}{\pi}\int_{0}^{k} [f(x-y) - f(x^-)]\frac{\sin ay}{y} dy + \frac{1}{\pi}\int_{k}^{\infty} [f(x-y) - f(x^-)]\frac{\sin ay}{y} dy$$

$$= \frac{1}{\pi}\int_{0}^{k} [f(x-y) - f(x^-)]\frac{\sin ay}{y} dy + \frac{1}{\pi}\int_{k}^{\infty} f(x-y)\frac{\sin ay}{y} dy - \frac{1}{\pi}\int_{k}^{\infty} f(x^-)\frac{\sin ay}{y} dy$$

If $k \geq 1$,

$$\left|\frac{1}{\pi}\int_{k}^{\infty} f(x-y)\frac{\sin ay}{y}dy\right| \leq \int_{k}^{\infty}|f(x-y)|\,dy \leq \left|\int_{k}^{\infty}|f(x-y)|\,dy\right|, and$$

$$\left|\frac{1}{\pi}\int_{k}^{\infty} f(x^{-})\frac{\sin ay}{y}dy\right| = \left|\frac{1}{\pi}f(x^{-})\right|\left|\int_{k}^{\infty}\frac{\sin ay}{y}dy\right|$$

Since $\int_{o}^{\infty}|f(x)|dx, \int_{o}^{\infty}\frac{\sin ay}{y}dx$ are both convergent integrals, then $\int_{k}^{\infty}\frac{\sin ay}{y}dy \longrightarrow 0,$ and

$\int_{k}^{\infty}|f(x-y)|\,dy \longrightarrow 0 \;\; as \;\; k \longrightarrow \infty$.

For the integrals over $[o, k]$,

$$\frac{1}{\pi}\int_{0}^{k}[f(x-y) - f(x^{-})]\frac{\sin ay}{y}dy$$

$$= \int_{0}^{k}\left(\frac{e^{iay} - e^{-iay}}{2i}\right)[f(x-y) - f(x^{-})]dy$$

$$= \frac{1}{2i}\int_{0}^{k}[f(x-y) - f(x^{-})]e^{iay}\,dy - \frac{1}{2i}\int_{0}^{k}[f(x-y) - f(x^{-})]e^{-iay}\,dy$$

$$= \frac{1}{2i}\int_{-\infty}^{\infty} g(y)e^{iay}\,dy - \frac{1}{2i}\int_{-\infty}^{\infty} g(y)e^{-iay}\,dy$$

$$= \frac{1}{2i}\left(\hat{g}(-a)\right) - \frac{1}{2i}\hat{g}(a)$$

Where $g(y) = \begin{cases} \frac{f(x-y)-f(x^{-})}{y} & ; o < y < k \\ 0 & ; otherwise \end{cases}$

Since f is piecewise smooth, $f'(x^{-})$ exists for all $x \in \mathbb{R}$ and $\lim_{y\to 0^{-}} g(y) = f'(x^{-})$ Therefore g is bounded on $[o, k]$ and hence $g \in L^{1}(\mathbb{R})$. By the Riemann –Lebesgue Lemma, $\hat{g}$ exists, is continuous and $\hat{g}(\pm a) \longrightarrow 0$, as $a \longrightarrow \infty$ and therefore

$\int_{0}^{k}[f(x-y) - f(x^{-})]\frac{\sin ay}{y}dy \longrightarrow 0$ as $a \longrightarrow \infty$ for $k \geq 1$

In the second integral,

$$\frac{1}{\pi}\int_0^{\infty}[f(x+y)-f(x^+)]\frac{\sin ay}{y}dy$$

$$=\frac{1}{\pi}\int_0^{k}[f(x+y)-f(x^+)]\frac{\sin ay}{y}dy+\frac{1}{\pi}\int_k^{\infty}[f(x+y)-f(x^+)]\frac{\sin ay}{y}dy$$

If $k \geq 1$,

$$\left|\frac{1}{\pi}\int_k^{\infty}[f(x+y)-f(x^+)]\frac{\sin ay}{y}dy\right| \leq \int_k^{\infty}|f(x+y)|\,dy, and$$

$$\int_k^{\infty} f(x^+)\frac{\sin ay}{y}dy = f(x^+)\int_k^{\infty}\frac{\sin ay}{y}dy$$

Since $\int_o^{\infty}|f(x)|dx, \int_o^{\infty}\frac{\sin ay}{y}dx$ are both convergent integrals, then $\int_k^{\infty}\frac{\sin ay}{y}dy \longrightarrow 0,$ and

$$\int_k^{\infty}|f(x+y)|\,dy \longrightarrow 0 \;\; as \;\; k \longrightarrow \infty.$$

For the integrals over $[o,k]$

$$\int_0^{k}[f(x+y)-f(x^+)]\frac{\sin ay}{y}dy$$

$$=\int_{-\infty}^{\infty}\left(\frac{e^{iay}-e^{-iay}}{2i}\right)g(y)dy$$

$$=\frac{1}{2i}[\hat{g}(-a)-\hat{g}(a)]$$

Where $g(y) = \begin{cases} \frac{f(x+y)-f(x^+)}{y} & ; o<y<k \\ 0 & ; otherwise \end{cases}$

Since f is piecewise smooth, $f'(x^+)$ exists for all $x \in \mathbb{R}$ and $\lim_{y\to 0^+} g(y) = f'(x^+)$
Therefore g is bounded on $[o,k]$ and hence $g \in L^1(\mathbb{R})$. By the Riemann –Lebesgue Lemma, $\hat{g}$ exists, is continuous $\hat{g}(\pm a) \longrightarrow 0$, as $a \longrightarrow \infty$ and therefore

$\int_0^{k}[f(x+y)-f(x^+)]\frac{\sin ay}{y}dy \longrightarrow 0$ as $a \longrightarrow \infty$ for $k \geq 1$.

Hence

$$\left(f_a(x) - \frac{1}{2}[f(x^-) + f(x^+)]\right) \longrightarrow 0 \text{ as } a \longrightarrow \infty$$

Or

$$\frac{1}{2}[f(x^-) + f(x^+)] = \lim_{a\to\infty} f_a(x)$$

$$= \lim_{a\to\infty} \frac{1}{2\pi}\int_{-a}^{a} \hat{f}(\xi)e^{ix\xi}\, d\xi$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)e^{ix\xi}\, d\xi$$

Examples:

1. For $f(t) = e^{-|t|}$, we have $\hat{f}(w) = \frac{2}{1+w^2}$. Since f is piecewise smooth, it follows that :

$$e^{-|t|} = \frac{1}{\pi}\lim_{A\to\infty}\int_{-A}^{A} \frac{e^{iwt}}{1+w^2}dw$$

In this case $\hat{f}$ happens to be absolutely integrable and we can write simply:

$$e^{-|t|} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{iwt}}{1+w^2}dw$$

By changing the variable $t = w$, we get:

$$e^{-|w|} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{iwt}}{1+t^2}dt$$

It implies that:

$$\pi e^{-|w|} = \pi e^{-|-w|} = \int_{-\infty}^{\infty} \frac{e^{-iwt}}{1+t^2}dt$$

In this way we have found the Fourier transform of $\frac{1}{1+t^2}$, which is rather difficult to reach by other methods:

Therefore $\mathcal{F}\left(\frac{1}{1+t^2}\right) = \pi e^{-|w|}$.

2. For the function

$$f(t) = \begin{cases} 1 & ; \ |t| < 1 \\ 0 & ; \ |t\,| > 1 \end{cases}$$

Then clearly $f \in L^1(\mathbb{R})$, and $\hat{f}(w) = \frac{2\sin w}{w}$. In this case the inversion integral is not absolutely convergent. The theorem here says that:

$$\lim_{A\to\infty} \int_{-A}^{A} \frac{\sin w}{w} e^{iwt} dw = \begin{cases} 1 & ; \quad |t\,| < 1 \\ \frac{1}{2} & ; \quad t = \pm 1 \\ 0 & ; \quad |t| > 1 \end{cases}$$

3.5 Theorem (Fourier sine and cosine transforms):

If $f(x) \in L^1(\mathbb{R})$ is an even function ,then

$$\hat{f}(\xi) = 2\int_0^\infty f(x)\cos(\xi x)\, dx$$

$$f(x) = \frac{1}{\pi}\int_0^\infty \hat{f}(\xi)\cos(\xi x)\, d\xi$$

are called Fourier cosine transforms.

If $f(x) \in L^1(\mathbb{R})$ is an odd function ,then

$$\hat{f}(\xi) = 2\int_0^\infty f(x)\sin(\xi x)\, dx$$

$$f(x) = \frac{1}{\pi}\int_0^\infty \hat{f}(\xi)\sin(\xi x)\, d\xi$$

are called Fourier sine transforms.

Proof: Let $f(x) \in L^1(\mathbb{R})$ is an even function, then the Fourier transforms of $f(x)$ is given by:

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-i\xi x}\, dx$$

$$= \int_{-\infty}^{\infty} f(x)[\cos(\xi x) - i\sin(\xi x)]\,dx$$

$$= \int_{-\infty}^{\infty} f(x)\cos(\xi x)\,dx - i\int_{\infty}^{\infty} f(x)\sin(\xi x)\,dx$$

Since $f(x)$ is even, then $f(x)\cos(\xi x)$ is even and $f(x)\sin(\xi x)$ is odd. Thus the second integral on the right hand side of the last equation is zero and we have:

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\cos(\xi x)\,dx$$

$$= 2\int_{0}^{\infty} f(x)\cos(\xi x)\,dx$$

$\hat{f}(\xi)$ is an even function ,since $\hat{f}(-\xi) = \hat{f}(\xi)$,then the inverse Fourier transforms of $\hat{f}(\xi)$ is given by:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)\,e^{ix\xi}\,d\xi$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)[\cos(\xi x) + i\sin(\xi x)]\,d\xi$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)\cos(\xi x)\,d\xi + i\frac{1}{2\pi}\int_{\infty}^{\infty} \hat{f}(\xi)\sin(\xi x)\,d\xi$$

Since $\hat{f}(\xi)$ is even, so $\hat{f}(\xi)\cos(\xi x)$ is even and $\hat{f}(\xi)\sin(\xi x)$ is odd .Thus the second integral on the right hand side of the last equation is zero, and we have:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)\cos(\xi x)\,d\xi$$

$$= \frac{1}{\pi}\int_{0}^{\infty} \hat{f}(\xi)\cos(\xi x)\,d\xi$$

Similarly we can prove Fourier sine transforms by replacing the cosine by the sine.

Example: The Fourier transform of

$$f(t) = \begin{cases} 1 & ; \quad |t| \le 1 \\ 0 & ; \quad |t| > 1 \end{cases}$$ is given by:

$$\hat{f}(w) = \int_{-1}^{1} e^{-iwt}\, dt = \begin{cases} 2\dfrac{\sin w}{w} & ; \; w \neq 0 \\ 2 & ; \; w = 0 \end{cases}$$

The inverse Fourier transform is:

$$\frac{2}{2\pi}\int_{-\infty}^{\infty} e^{iwx}\frac{\sin w}{w}\, dw = \frac{2}{\pi}\int_{0}^{\infty} \cos wx \frac{\sin w}{w}\, dw$$

$$= \frac{1}{\pi}\int_{0}^{\infty} \frac{\sin\big(w(x+1)\big)}{w}\, dw - \frac{1}{\pi}\int_{0}^{\infty} \frac{\sin(\,w(x-1))}{w}\, dw$$

$$= \begin{cases} 1 & ; \quad |x| < 1 \\ \dfrac{1}{2} & ; \quad |x| = 1 \\ 0 & ; \quad |\,x| > 1 \end{cases}$$

3.6 Theorem (Plancherels' and Parsevals' Identities):

If $f, g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then

$$\int_{-\infty}^{\infty} f(y)\overline{g(y)}\, dy = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)\overline{\hat{g}(\xi)}\, d\xi$$

called Plancherels' Identity. When $f = g$ we obtain Parsevals' Identity,

$$\int_{-\infty}^{\infty} |f(y)|^2\, dy = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\hat{f}(\xi)\right|^2 d\xi$$

Proof: Method 1

By the convolution theorem, for $f, g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, $(\widehat{f * g}) = \hat{f}\hat{g}$

Therefore

$$f * g = \widetilde{(\hat{f}\hat{g})}$$

Or

$$\int_{-\infty}^{\infty} f(x-y)g(y)\,dy = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}\,(\xi)\hat{g}(\xi)e^{ix\xi}d\xi$$

Set $x = 0$,

$$\int_{-\infty}^{\infty} f(-y)g(y)\,dy = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}\,(\xi)\hat{g}(\xi)d\xi$$

Replacing $g(x)$ by $\overline{g(-x)}$, then the Fourier transform $\hat{g}(\xi)$ is replaced by $\overline{\hat{g}(\xi)}$, hence

$$\int_{-\infty}^{\infty} f(-y)\overline{g(-y)}\,dy = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}\,(\xi)\,\overline{\hat{g}(\xi)}d\xi$$

Or

$$\int_{-\infty}^{\infty} f(y)\overline{g(y)}\,dy = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}\,(\xi)\,\overline{\hat{g}(\xi)}d\xi$$

This is Plancherels' Identity .When $f = g$ we obtain Parsevals' Identity:

$$\int_{-\infty}^{\infty} |f(y)|^2\,dy = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\hat{f}(\xi)\right|^2 d\xi$$

Method 2: Since $g(x) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then so is $\overline{g(x)}$ and

$$\overline{\hat{g}(\xi)} = \int_{-\infty}^{\infty} \overline{g(x)}e^{i\xi x}\,dx$$

Now using Fubini's theorem, we obtain:

$$\int_{-\infty}^{\infty} f(y)\overline{g(y)}\,dy = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{\infty}^{\infty}\hat{f}(\xi)e^{i\xi y}\,d\xi\right)\overline{g(y)}\,dy$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{\infty}^{\infty}\overline{g(y)}e^{i\xi y}\,dy\right)\hat{f}(\xi)\,d\xi$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{\hat{g}(\xi)}\hat{f}(\xi)\,d\xi$$

Examples:

1. Let $f(x) = \chi_{a(x)} = \begin{cases} 1 & ;|x| < a \\ 0 & ;\ otherwise \end{cases}$

   Then $\hat{f}(\xi) = \frac{2\sin a\xi}{\xi}$, and by Parsevals' Identity ,

$$\int_{-\infty}^{\infty} |f(y)|^2\, dy = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\hat{f}(\xi)\right|^2 d\xi$$

Or

$$\int_{-\infty}^{\infty} \left|\chi_{a(x)}\right|^2 dy = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\frac{2\sin a\xi}{\xi}\right|^2 d\xi$$

Or

$$2\pi a = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\frac{2\sin a\xi}{\xi}\right|^2 d\xi$$

Or

$$4\pi a = \int_{-\infty}^{\infty} \left(\frac{2\sin a\xi}{\xi}\right)^2 d\xi$$

Therefore

$$\int_{-\infty}^{\infty} \left(\frac{2\sin a\xi}{\xi}\right)^2 d\xi = \pi a$$

2. Let $f(x) = e^{-a|x|}$, then $\hat{f}(\xi) = \frac{2a}{a^2+\xi^2}$ and by Parsevals' Identity

$$\int_{-\infty}^{\infty} |f(y)|^2\, dy = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\hat{f}(\xi)\right|^2 d\xi$$

Or

$$\int_{-\infty}^{\infty} \left|e^{-a|x|}\right|^2 dy = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\frac{2a}{a^2+\xi^2}\right|^2 d\xi$$

Or

$$\int_{-\infty}^{\infty} \left(e^{-a|x|}\right)^2 dy = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{2a}{a^2+\xi^2}\right)^2 d\xi$$

Or

$$\frac{1}{a} = \frac{2a^2}{\pi} \int_{-\infty}^{\infty} \frac{1}{(a^2+\xi^2)^2} d\xi$$

Therefore

$$\int_{-\infty}^{\infty} \frac{1}{(a^2+\xi^2)^2} d\xi = \frac{\pi}{2a^3}$$

Fourier transform

Chapter 4

Applications of Fourier transform

4.1 Applications in partial differential equations

1. The wave equation:

Consider the initial value- problem for the wave equation:

$u_{tt} = c^2 u_{xx}$ $, -\infty < x < \infty , t > 0 , c > 0$

u and u_x finite as $|x| \longrightarrow \infty , t > 0$ (1.1)

$u(x,0) = f_1(x), -\infty < x < \infty$

$u_t(x,0) = f_2(x), -\infty < x < \infty$

Where $\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0, -\infty < x < \infty , t > 0 , c > 0$ describes the vertical vibrations of an infinite stretched elastic string, where $u(x,t)$ is the vertical displacement of the string from its rest position x, time t, and $u(x,0) = f_1(x), -\infty < x < \infty$ and $u_t(x,0) = f_2(x), -\infty < x < \infty$ describes the initial displacement and velocity respectively.

Where the functions $f_1(x)$ and $f_2(x)$ are piecewise smooth and absolutely integrable in $(-\infty, \infty)$.

To find the solution this problem, we introduce the Fourier transforms:

$$\widehat{f_j}(\xi) = \int_{-\infty}^{\infty} f_j(x) e^{-i\xi x}\, dx \ , j = 1,2.$$

And its inversion formulas

$$f_j(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f_j}(\xi) e^{i\xi x}\, d\xi \ , j = 1,2.$$

We also need the Fourier representation of the solution $u(x,t)$:

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}(\xi, t) e^{i\xi x}\, d\xi \ ,$$

Where $\hat{u}(\xi, t)$ is an unknown function, which we will now determine. For this, we substitute this in to the differential equation (1.1) to obtain:

$$0 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{\partial^2\hat{u}(\xi,t)}{\partial t^2} + c^2\xi^2\hat{u}(\xi,t)\right]e^{i\xi x}\,d\xi$$

Thus $\hat{u}$ must be a solution of the differential equation:

$$\frac{\partial^2\hat{u}(\xi,t)}{\partial t^2} + c^2\xi^2\hat{u}(\xi,t) = 0$$

Whose solution can be written as:

$$\hat{u}(\xi,t) = c_1(\xi)\cos(\xi ct) + c_2(\xi)\sin(\xi ct).$$

To find $c_1(\xi)$ and $c_2(\xi)$, we note that:

$$f_1(x) = u(x,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} c_1(\xi)e^{i\xi x}\,d\xi$$

$$f_2(x) = u_t(x,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \xi c c_2(\xi)e^{i\xi x}\,d\xi$$

and hence $\widehat{f_1}(\xi) = c_1(\xi)$ and $\widehat{f_2}(\xi) = \xi c c_2(\xi)$.

Therefore, it follows that:

$$\hat{u}(\xi,t) = \widehat{f_1}(\xi)\cos(\xi ct) + \frac{\widehat{f_2}(\xi)}{\xi c}\sin(\xi ct)$$

and hence the Fourier representation of the solution is:

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\widehat{f_1}(\xi)\cos(\xi ct) + \frac{\widehat{f_2}(\xi)}{\xi c}\sin(\xi ct)\right]e^{i\xi x}\,d\xi \qquad (1.2)$$

Now since $\cos\theta = \left(e^{i\theta} + e^{-i\theta}\right)/2$, $\sin\theta = \left(e^{i\theta} - e^{-i\theta}\right)/2i$, then we have:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{f_1}(\xi)(\cos(\xi ct))e^{i\xi x}\,d\xi$$

$$= \frac{1}{2} \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f_1}(\xi) \left(e^{i\xi ct} + e^{-i\xi ct} \right) e^{i\xi x} \, d\xi$$

$$= \frac{1}{2} \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f_1}(\xi) \left(e^{i\xi(x+ct)} + e^{i\xi(x-ct)} \right) d\xi$$

$$= \frac{1}{2} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f_1}(\xi) \, e^{i\xi(x+ct)} \, d\xi + \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f_1}(\xi) \, e^{i\xi(x-ct)} d\xi \right]$$

$$= \frac{1}{2} [f_1(x+ct) + f_1(x-ct)]$$

Similarly,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f_2}(\xi) \left(\frac{\sin(\xi ct)}{\xi c} \right) e^{i\xi x} \, d\xi$$

$$= \frac{1}{2} \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f_2}(\xi) \left(\frac{e^{i\xi ct} - e^{-i\xi ct}}{i\xi c} \right) e^{i\xi x} \, d\xi$$

$$= \frac{1}{2} \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f_2}(\xi) \left(\frac{e^{i\xi(x+ct)} - e^{i\xi(x-ct)}}{i\xi c} \right) d\xi$$

$$= \frac{1}{2c} \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f_2}(\xi) \left(\int_{x-ct}^{x+ct} e^{i\xi w} \, dw \right) d\xi$$

$$= \frac{1}{2c} \int_{x-ct}^{x+ct} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f_2}(\xi) \, e^{i\xi w} d\xi \right] dw$$

$$= \frac{1}{2c} \int_{x-ct}^{x+ct} f_2(w) \, dw$$

Putting these together yields d'Alembert's formula:

$$u(x,t) = \frac{1}{2} [f_1(x+ct) + f_1(x-ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} f_2(w) \, dw \qquad (1.3)$$

2. Laplace's equation:

2.1 Consider Laplace's equation in two variables on the upper half-plane $y > 0$. Then

$$u_{xx} + u_{yy} = 0 \quad , -\infty < x < \infty , y > 0$$
$$u(x, 0) = f(x) \quad , -\infty < x < \infty \qquad (2.1.1)$$
$$|u(x, y)| \le M \quad , -\infty < x < \infty , y > 0$$

Where the function f is piecewise smooth and absolutely integrable in $(-\infty, \infty)$.

If $f(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$,then we also have the implied boundary conditions

$\lim_{|x| \longrightarrow \infty} u(x, y) = 0$, $\lim_{y \longrightarrow +\infty} u(x, y) = 0$.

To find the solution of this problem, we introduce the Fourier transforms :

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-i\xi x} \, dx$$

and its inversion formulas:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{i\xi x} \, d\xi$$

We also need the Fourier representation of the solution $u(x, t)$:

$$u(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}(\xi, y) e^{i\xi x} \, d\xi \,,$$

Where $\hat{u}(\xi, t)$ is an unknown function, which we will now determine. For this, we substitute this in to the differential equation (2.1.1), to obtain:

$$0 = u_{xx} + u_{yy} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[(i\xi)^2 \hat{u}(\xi, y) + \frac{\partial^2 \hat{u}(\xi, y)}{\partial y^2} \right] e^{i\xi x} \, d\xi$$

Thus $\hat{u}$ must satisfy:

$$(i\xi)^2 \hat{u}(\xi, y) + \frac{\partial^2 \hat{u}(\xi, y)}{\partial y^2} = 0, -\infty < x < \infty, y > 0$$

Or $\frac{\partial^2 \hat{u}(\xi, y)}{\partial y^2} - \xi^2 \hat{u}(\xi, y) = 0$

This has solutions:

$$\hat{u}(\xi, y) = A(\xi)e^{\xi y} + B(\xi)e^{-\xi y}$$

We need two conditions to determine the functions $A(\xi)$ and $B(\xi)$:

In addition to $u(x,0) = f(x)$, $-\infty < x < \infty$, let $\frac{\partial u(x,0)}{\partial y} = g(x)$, $-\infty < x < \infty$ for some function $g \in L^1(\mathbb{R})$. we will find g such that the solution $u(x,y)$ is bounded for $y > 0$, in fact such that $u(x,y) \longrightarrow 0$ as $y \longrightarrow \infty$.

Taking transforms we get :

$$\hat{u}(\xi, 0) = \hat{f}(\xi) \ , \ \frac{\partial \hat{u}(\xi,0)}{\partial y} = \hat{g}(\xi) \ ,$$

Or $A(\xi) + B(\xi) = \hat{f}(\xi)$, $-\xi A(\xi) - \xi B(\xi) = \hat{g}(\xi)$

Solving for $A(\xi)$, and $B(\xi)$ we get:

$$A(\xi) = \frac{1}{2}\left(\hat{f}(\xi) + \frac{\hat{g}(\xi)}{\xi}\right)$$

$$B(\xi) = \frac{1}{2}\left(\hat{f}(\xi) - \frac{\hat{g}(\xi)}{\xi}\right)$$

and

$$\hat{u}(\xi, y) = \frac{1}{2}\left(\hat{f}(\xi) + \frac{\hat{g}(\xi)}{\xi}\right)e^{\xi y} + \frac{1}{2}\left(\hat{f}(\xi) - \frac{\hat{g}(\xi)}{\xi}\right)e^{-\xi y}$$

For $\xi > 0, \hat{u}(\xi, y) \longrightarrow 0$ as $y \longrightarrow 0$ if and only if $\hat{f}(\xi) + \frac{\hat{g}(\xi)}{\xi} = 0$, and

For $\xi < 0, \hat{u}(\xi, y) \longrightarrow 0$ as $y \longrightarrow 0$ if and only if $\hat{f}(\xi) - \frac{\hat{g}(\xi)}{\xi} = 0$.

Therefore

$$\hat{g}(\xi) = \begin{cases} -\xi\hat{f}(\xi) & ; \xi > 0 \\ \xi\hat{f}(\xi) & ; \xi < 0 \end{cases}$$

$$= -|\xi|\hat{f}(\xi)$$

Hence

$$\hat{u}(\xi, y) = \begin{cases} \hat{f}(\xi)e^{-\xi y} & ;\ \xi > 0 \\ \hat{f}(\xi)e^{\xi y} & ;\xi < 0 \end{cases}$$

$$= \hat{f}(\xi)e^{-|\xi|y}$$

Since, $e^{-|\xi|y} = \left(\widehat{\frac{y}{\pi(x^2+y^2)}}\right)$, by convolution theorem,

$$u(x, y) = \int_{-\infty}^{\infty} \hat{u}(\xi, y)e^{i\xi x}\, d\xi$$

$$= \int_{-\infty}^{\infty} \hat{f}(\xi)e^{-|\xi|y}e^{i\xi x}\, d\xi$$

$$= \int_{-\infty}^{\infty} \hat{f}(\xi)\left(\widehat{\frac{y}{\pi(x^2 + y^2)}}\right)e^{i\xi x}\, d\xi$$

$$= \left(\frac{y}{\pi(x^2 + y^2)} * f\right)$$

$$= \int_{-\infty}^{\infty} \frac{y}{\pi((x - s)^2 + y^2)} f(s)\, ds \qquad (2.1.2)$$

This representation is known as Poisson's formula.

In particular, for

$$u(x, 0) = f(x) = \begin{cases} 1 & ;\ a < x < b \\ 0 & ;\ otherwise \end{cases}$$

Then (2.1.2) becomes:

$$u(x, y) = \int_{a}^{b} \frac{y}{\pi((x - s)^2 + y^2)} ds$$

$$= \int_a^b \frac{\frac{ds}{y}}{\pi\left(\frac{(x-s)^2}{y^2}+1\right)}$$

Using the substitution $v = (s-x)/y$, we have $ds = ydv$, so that

$$u(x,y) = \int_{(a-x)/y}^{(b-x)/y} \frac{1}{\pi(v^2+1)} dv$$

$$= \frac{1}{\pi}\left(\tan^{-1}\frac{b-x}{y} - \tan^{-1}\frac{a-x}{y}\right)$$

2.2 Consider the Laplace equation in a semi-infinite strip

$$u_{xx} + u_{yy} = 0 \ , \ 0 < x < \infty$$
$$u(x,0) = f(x) \ , \quad 0 < x < \infty$$
$$u(0,y) = 0 \quad , \ 0 < y < b$$
$$u(x,b) = 0 \ , \ 0 < x < \infty$$

Where the function f is piecewise smooth and absolutely integrable in $[0,\infty)$ .we shall also need the boundary conditions $\lim_{x\to\infty} u(x,y) = 0$ and $\lim_{x\to\infty} u_x(x,y) = 0$.

To solve this problem, we let

$$f(x) = \frac{1}{\pi}\int_0^\infty \widehat{f_s}(\xi) \sin \xi x \, d\xi$$

$$\widehat{f_s}(\xi) = 2\int_0^\infty f(x) \sin \xi x \, dx$$

$$u(x,y) = \frac{1}{\pi}\int_0^\infty \hat{u}_s(\xi,y) \sin \xi x \, d\xi$$

This as in problem (2.1), leads to the same differential equation

$(\hat{u}_s)'' = \xi^2 \hat{u}_s$, and hence

$\hat{u}_s(\xi,y) = c_1(\xi) \cos h\xi y + c_2(\xi) \sin h\xi y$

Now the boundary conditions $\hat{u}_s(\xi, b) = 0$ yields:

$$c_1(\xi) = -c_2(\xi)\frac{\sin h\xi b}{\cos h\xi b}$$

Thus we have

$$\hat{u}_s(\xi, y) = -c_2(\xi)\frac{\sin h\xi b}{\cos h\xi b}\cos h\xi y + c_2(\xi)\sin h\xi y$$

$$= c_2(\xi)\frac{\sin h\xi(y-b)}{\cos h\xi b}$$

Now since, $\hat{u}_s(\xi, 0) = \widehat{f_s}(\xi)$,we find:

$c_2(\xi) = -\widehat{f_s}(\xi)\frac{\cos h\xi b}{\sin h\xi b}$, and therefore

$$\hat{u}_s(\xi, y) = \widehat{f_s}(\xi)\frac{\sin h\xi(b-y)}{\sin h\xi b}$$

This gives the solution:

$$u(x, y) = \frac{1}{\pi}\int_0^\infty \widehat{f_s}(\xi)\frac{\sin h\xi(b-y)}{\sin h\xi b}\sin \xi x\, d\xi$$

$$= \frac{2}{\pi}\int_0^\infty \int_0^\infty f(t)\sin \xi t \frac{\sin h\xi(b-y)}{\sin h\xi b}\sin \xi x\, dtd\xi$$

3. The Heat equation:
   3.1. consider the heat flow problem of an infinitely long thin bar insulted on its lateral surface ,which is modeled by the following initial value problem:

$u_t = c^2 u_{xx}$,$-\infty < x < \infty$, $t > 0$, $c > 0$

u and u_x finite as $|x| \longrightarrow \infty$, $t > 0$ (3.1.1)

$u(x, 0) = f(x), -\infty < x < \infty$

Where the function f is piecewise smooth and absolutely integrable in $(-\infty , \infty)$.

Let $\hat{u}(\xi, t)$ be the fourier transform of $u(x, t)$.Thus from the fourier pair ,we have :

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(\xi,t)e^{i\xi x}\,d\xi$$

$$\hat{u}(\xi,t) = \int_{-\infty}^{\infty} u(x,t)e^{-i\xi x}\,dx$$

Assuming that the derivatives can be taken under the integral ,we get:

$$\frac{\partial u}{\partial t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\partial \hat{u}(\xi,t)}{\partial t} e^{i\xi x}\,d\xi$$

$$\frac{\partial u}{\partial x} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(\xi,t)(i\xi)e^{i\xi x}\,d\xi$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(\xi,t)(i\xi)^2 e^{i\xi x}\,d\xi$$

In order for $u(x,t)$ to satisfy the heat equation, we must have:

$$0 = \frac{\partial u}{\partial t} - c^2\frac{\partial^2 u}{\partial x^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{\partial \hat{u}(\xi,t)}{\partial t} + c^2\xi^2\hat{u}(\xi,t)\right]e^{i\xi x}\,d\xi$$

Thus $\hat{u}$ must be a solution of the differential equation:

$$\frac{\partial \hat{u}(\xi,t)}{\partial t} + c^2\xi^2\hat{u}(\xi,t) = 0$$

and this has solutions:

$$\hat{u}(\xi,t) = A(\xi)e^{-\xi^2 c^2 t}$$

The initial condition $A(\xi)$ is determined by:

$$\hat{u}(\xi,0) = \int_{-\infty}^{\infty} u(x,0)e^{-i\xi x}\,dx$$

$$= \int_{-\infty}^{\infty} f(x)e^{-i\xi x}\,dx = \hat{f}(\xi).$$

Therefore, we have:

$\hat{u}(\xi,t) = \hat{f}(\xi)e^{-\xi^2c^2t}$, and hence:

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)e^{-\xi^2c^2t}e^{i\xi x}\,d\xi \tag{3.1.2}$$

Now since:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\xi^2c^2t}e^{i\xi x}\,d\xi = 2\pi\frac{e^{-x^2/(4c^2t)}}{\sqrt{4\pi c^2 t}},$$

Then by convolution theorem, and if we denote $\hat{g}(\xi) = e^{-\xi^2c^2t}$, then from equation(3.1.2) it follows that:

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)\hat{g}(\xi)e^{i\xi x}\,d\xi$$

$$= (f * g)(x)$$

$$= \int_{-\infty}^{\infty} f(\mu)\frac{e^{-(x-\mu)^2/(4c^2t)}}{\sqrt{4\pi c^2 t}}\,d\mu \tag{3.1.3}$$

This formulas due to Gauss and Weierstrass.

For each μ the function $(x,t) \longrightarrow \frac{e^{-(x-\mu)^2/(4c^2t)}}{\sqrt{4\pi c^2 t}}$ is a solution of the heat equation and is called the fundamental solution. Thus,(3.1.3) gives a representation of the solution as a continuous super position of the fundamental solution.

3.2. Consider the heat equation:

$$\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = 0 \;, -\infty < x < \infty, t > 0$$

For $u(x,t)$ a function of two variables, with initial condition:

$$u(x,t) = f(x)\,, -\infty < x < \infty$$

For $f \in L^1(\mathbb{R}), f$ continuous and bounded, can be solved using fourier transforms.

Taking transforms in the variable x,

$$\frac{\partial \hat{u}(\xi,t)}{\partial t} - k(i\xi)^2\hat{u}(\xi,t) = 0$$

Or

$$\frac{\partial \hat{u}(\xi,t)}{\partial t} + k\xi^2\hat{u}(\xi,t) = 0$$

Solving this first order ordinary differential equation:

$$\hat{u}(\xi,t) = \hat{u}(\xi,0)e^{-k\xi^2 t} = \hat{f}(\xi)e^{-k\xi^2 t}$$

Now $\left(\widehat{e^{\frac{-x^2}{2}}}\right) = \sqrt{2\pi}e^{\frac{-\xi^2}{2}}$ and using the dilation property of Fourier transforms:

$$\left(\widehat{\lambda^{\frac{-1}{2}}e^{\frac{-(\lambda^{-1}x)^2}{2}}}\right) = \sqrt{2\pi}\,\lambda^{\frac{1}{2}}e^{\frac{-(\lambda\xi)^2}{2}}$$

$$e^{\frac{-(\lambda\xi)^2}{2}} = \frac{1}{\lambda\sqrt{2\pi}}\left(\widehat{e^{\frac{-(\lambda^{-1}x)^2}{2}}}\right)$$

Let $\frac{\lambda^2}{2} = kt$ or$\lambda = \sqrt{2kt}$, then

$$e^{\frac{-k\xi^2 t}{2}} = \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2kt}}\left(\widehat{e^{\frac{-1}{2}\left(\frac{x}{\sqrt{2kt}}\right)^2}}\right)$$

$$= \frac{1}{\sqrt{4\pi kt}}\left(\widehat{e^{\frac{-1}{2}\left(\frac{x}{\sqrt{2kt}}\right)^2}}\right)$$

Therefore from the convolution theorem,

Since $\hat{u}(\xi,t) = \hat{f}(\xi)\frac{1}{\sqrt{4\pi kt}}\left(\widehat{e^{\frac{-1}{2}\left(\frac{x}{\sqrt{2kt}}\right)^2}}\right)$

$$u(x,t) = \left(f * \frac{1}{\sqrt{4\pi kt}}e^{\frac{-1}{2}\left(\frac{x}{\sqrt{2kt}}\right)^2}\right)(x)$$

$$= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4kt}} f(y)\, dy$$

The function $h(x,t) = \frac{e^{\frac{-x^2}{4kt}}}{\sqrt{4\pi kt}}$ is called the heat kernel. We can then write:

$u(x,t) = \int_{-\infty}^{\infty} h(x-y,t)\, f(y)\, dy.$

Definition: The finite Fourier sine transform of $f(x)$, $0 < x < L$ is defined as:

$$\widehat{f_s}(n) = \int_0^L f(x) \sin\frac{n\pi x}{L} dx \qquad (1)$$

Where $n \geq 1$ is an integer. The function $f(x)$ is then called the inverse finite fourier sine transform of $\widehat{f_s}(n)$ and is given by:

$$f(x) = \frac{2}{L} \sum_{n=1}^{\infty} \widehat{f_s}(n) \sin\frac{n\pi x}{L} \qquad (2)$$

Definition: the finite Fourier cosine transform of $f(x)$, $0 < x < L$ is defined as:

$$\widehat{f_c}(n) = \int_0^L f(x) \cos\frac{n\pi x}{L} dx \qquad (3)$$

Where $n \geq 0$ is an integer. The function $f(x)$ is then called the inverse finite Fourier cosine transform of $\widehat{f_c}(n)$ and is given by:

$$f(x) = \frac{1}{L}\widehat{f_c}(0) + \frac{2}{L} \sum_{n=1}^{\infty} \widehat{f_c}(n) \cos\frac{n\pi x}{L} \qquad (4)$$

Finite Fourier transforms are useful in solving partial differential equations. For this we note that:

$$\int_0^L \frac{\partial u}{\partial x} \sin\frac{n\pi x}{L} dx$$

$$= u(x,t)\sin\frac{n\pi x}{L}\Big|_0^L - \frac{n\pi}{L}\int_0^L u(x,t)\cos\frac{n\pi x}{L}\,dx$$

and hence

$$\mathcal{F}_s\left(\frac{\partial u}{\partial x}\right) = -\frac{n\pi}{L}\widehat{f_c}(n) \qquad (5)$$

and similarly,

$$\mathcal{F}_c\left(\frac{\partial u}{\partial x}\right) = \frac{n\pi}{L}\widehat{f_s}(n) - [u(0,t) - u(L,t)\cos n\pi] \qquad (6)$$

$$\mathcal{F}_s\left(\frac{\partial^2 u}{\partial x^2}\right) = -\frac{n\pi}{L}\widehat{f_c}(n)$$

$$= -\frac{(n\pi)^2}{L^2}\widehat{f_s}(n) + \frac{n\pi}{L}[u(0,t) - u(L,t)\cos n\pi] \qquad (7)$$

$$\mathcal{F}_c\left(\frac{\partial^2 u}{\partial x^2}\right) = -\frac{(n\pi)^2}{L^2}\widehat{f_c}(n) - [u_x(0,t) - u_x(L,t)\cos n\pi] \qquad (8)$$

Example: we will use finite Fourier sine transforms to find the solution of the problem:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}\ , \qquad 0 < x < 4, \qquad t > 0$$

$$u(x,0) = 2x\ , \qquad 0 < x < 4$$

$$u(0,t) = u(4,t) = 0\ , \qquad t > 0$$

Taking the finite Fourier sine transform with L=4 of both sides of the partial differential equation gives:

$$\int_0^4 \frac{\partial u}{\partial t}\sin\frac{n\pi x}{4}\,dx = \int_0^4 \frac{\partial^2 u}{\partial x^2}\sin\frac{n\pi x}{4}\,dx$$

Writing $\hat{u}$ for $\widehat{f_s}(n)$ and using (7) with $u(0,t) = u(4,t) = 0$ leads to:

$$\frac{d\hat{u}(n,t)}{dt} = -\frac{n^2\pi^2}{16}\hat{u},$$

Which can be solved to obtain:

$$\hat{u}(n,t) = ce^{-\frac{n^2\pi^2 t}{16}}$$

Now taking the finite Fourier sine transform of the condition $u(x,0) = 2x$, we have:

$$\hat{u}(n,0) = \int_0^4 2x \sin\frac{n\pi x}{4} dx$$

$$= \left[2x\left(-\frac{\cos\frac{n\pi x}{4}}{\frac{n\pi}{4}}\right) - 2\left(-\frac{\sin\frac{n\pi x}{4}}{\frac{n^2\pi^2}{16}}\right)\right]\Bigg|_0^4$$

$$= -\frac{32}{n\pi}\cos n\pi$$

Since $c = \hat{u}(n,0)$ it follows that:

$$\hat{u}(n,t) = -\frac{32}{n\pi}\cos n\pi e^{-\frac{n^2\pi^2 t}{16}}$$

Thus from (2) we get:

$$u(x,t) = \frac{-16}{\pi}\sum_{n=1}^{\infty}\frac{\cos(\mathrm{n}\pi)}{\mathrm{n}} e^{-\frac{n^2\pi^2 t}{16}} \sin\frac{n\pi x}{4}$$

4.2 Band limited functions and Shannon's sampling theorem:

The Fourier transform variable has the role frequency and $\hat{f}(\xi)$ is referred to as the frequency representation of $f(x)$.

If $\hat{f}(\xi) = 0$ for $|\xi| > \xi_c > 0$, then $f(x)$ is called a band-limited function and ξ_c is called the cut-off frequency.

Many functions from science and technology are band-limited. For example, human hearing is assumed to be limited to frequencies below about 20KHZ. Therefore acoustic signals recorded on compact discs are limited to a band width of 22KHZ.

A first step in the processing of signals, is sampling. A signal represented by a continuous function$f(x)$, is replaced by its samples at regular intervals, $\{f(nL), n = \pm 0, \pm 1, \pm 2, \ldots\}$.

Shannon's theorem shows that it is possible to exactly recover the band-limited continuous function $f(x)$ from knowledge of its samples, provided the sampling interval L, is sufficiently short.

Theorem (Shannon's theorem)

Let $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ be continuous and band-limited to$|\xi| \leq \xi_c$. Then

$$f(x) = \sum_{n=-\infty}^{\infty} f(nL) \frac{\sin \xi_c (x - nL)}{\xi_c (x - nL)}, \qquad L = \frac{\pi}{\xi_c}.$$

Proof: Consider the functions $\widehat{\Phi}_n(\xi)$ given by:

$$\widehat{\Phi}_n(\xi) = \begin{cases} 0 & ; \quad |\xi| > \xi_c \\ \frac{1}{\sqrt{2\xi_c}} e^{-\frac{in\pi\xi}{\xi_c}} ; & |\xi| \leq \xi_c \end{cases}$$

The inverse Fourier transforms of $\widehat{\Phi}_n(\xi)$ are given by:

$$\Phi_n(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{\Phi}_n(\xi) e^{ix\xi}\, d\xi$$

$$2\pi\Phi_n(x) = \int_{-\xi_c}^{\xi_c} \frac{1}{\sqrt{2\xi_c}} e^{-\frac{in\pi\xi}{\xi_c}} e^{ix\xi}\, d\xi$$

$$= \frac{1}{\sqrt{2\xi_c}} \int_{-\xi_c}^{\xi_c} e^{i\xi\left(x-\frac{n\pi}{\xi_c}\right)}\, d\xi$$

$$= \frac{1}{\sqrt{2\xi_c}} \left[\frac{e^{i\xi\left(x-\frac{n\pi}{\xi_c}\right)}}{i\left(x-\frac{n\pi}{\xi_c}\right)}\right]_{-\xi_c}^{\xi_c}$$

$$= \sqrt{2\xi_c}\left(\frac{e^{i(\xi_c x - n\pi)} - e^{-i(\xi_c x - n\pi)}}{2i(\xi_c x - n\pi)}\right)$$

$$= \sqrt{2\xi_c}\,\frac{\sin(\xi_c x - n\pi)}{\xi_c x - n\pi}$$

$$= \sqrt{\frac{2\pi}{L}}\,\frac{\sin \xi_c (x - nL)}{\xi_c (x - nL)}$$

Where $L = \frac{\pi}{\xi_c}$.

Consider the inner products:

$$\langle \widehat{\Phi}_n ; \widehat{\Phi}_m \rangle = \int_{-\infty}^{\infty} \widehat{\Phi}_n(\xi)\, \overline{\widehat{\Phi}_m(\xi)} d\xi$$

$$= \frac{1}{2\xi_c} \int_{-\xi_c}^{\xi_c} e^{-\frac{in\pi\xi}{\xi_c}} \overline{\left(e^{-\frac{im\pi\xi}{\xi_c}}\right)} d\xi$$

$$= \frac{1}{2\xi_c} \int_{-\xi_c}^{\xi_c} e^{\frac{i(m-n)\pi\xi}{\xi_c}}\, d\xi$$

$$= \begin{cases} 0 \ ; \ m \neq n \\ 1 \ ; \ m = n \end{cases}$$

So the functions $\{\widehat{\Phi}_n(\xi); n = \pm 0, \pm 1, \pm 2, \dots\}$, form an orthogonal set in $L^2(-\xi_c, \xi_c)$.

Since $\hat{f}(\xi)$ is band-limited to $|\xi| \leq \xi_c$, it has a Fourier series:

$$\hat{f}(\xi) = \sum_{n=-\infty}^{\infty} c_n \widehat{\Phi}_n(\xi)$$

$$= \sum_{n=-\infty}^{\infty} c_n \frac{1}{\sqrt{2\xi_c}} e^{-\frac{in\pi\xi}{\xi_c}}$$

Where the Fourier coefficients c_n are given by:

$$c_n = \langle \hat{f};\ \widehat{\Phi}_n \rangle = \int_{-\xi_c}^{\xi_c} \hat{f}(\xi) \frac{1}{\sqrt{2\xi_c}} e^{\frac{in\pi\xi}{\xi_c}} d\xi$$

By plancherel's theorem

$$c_n = \langle \hat{f};\ \widehat{\Phi}_n \rangle = 2\pi \langle f;\ \Phi_n \rangle$$

So

$$\hat{f}(\xi) = \sum_{n=-\infty}^{\infty} c_n \widehat{\Phi}_n(\xi) = \sum_{n=-\infty}^{\infty} 2\pi \langle f;\ \Phi_n \rangle \widehat{\Phi}_n(\xi)$$

Taking inverse Fourier transforms

$$f(x) = \sum_{n=-\infty}^{\infty} \langle f;\ \Phi_n \rangle 2\pi \Phi_n(x) = \sum_{n=-\infty}^{\infty} \langle f;\ \Phi_n \rangle \sqrt{\frac{2\pi}{L}} \frac{\sin \xi_c(x - nL)}{\xi_c(x - nL)}$$

The samples at intervals of length $L = \frac{\pi}{\xi_c}$ are:

$$f(kL) = \sum_{n=-\infty}^{\infty} \langle f;\ \Phi_n \rangle \sqrt{\frac{2\pi}{L}} \frac{\sin \xi_c(kL - nL)}{\xi_c(kL - nL)}$$

$$= \sum_{n=-\infty}^{\infty} \langle f;\ \Phi_n \rangle \sqrt{\frac{2\pi}{L}} \frac{\sin(k - n)}{(k - n)\pi}$$

$$= \langle f;\ \Phi_k \rangle \sqrt{\frac{2\pi}{L}}$$

So $\langle f;\ \Phi_k \rangle = f(kL)\sqrt{\frac{L}{2\pi}}$, $k = 0, \pm 1, \pm 2, \ldots$

Finally therefore,

$$f(x) = \sum_{n=-\infty}^{\infty} \langle f;\ \Phi_n \rangle \sqrt{\frac{2\pi}{L}} \frac{\sin \xi_c(x - nL)}{\xi_c(x - nL)} = \sum_{n=-\infty}^{\infty} f(nL) \frac{\sin \xi_c(x - nL)}{\xi_c(x - nL)}$$

The relationship $\omega L = 2\pi$, where ω is the frequency of sampling in cycles per unit length, shows that from Shannon's theorem, to reconstruct a band-limited function, it suffices to sample at a frequency, $\omega = \frac{2\pi}{L} = 2\xi_c$, twice the cut-off frequency.

4.3 Heisenberg's Inequality

Let $f \in L^2(\mathbb{R})$, $xf \in L^2(\mathbb{R})$. Then the quantity:

$$\Delta_a f = \left(\frac{\int_{-\infty}^{\infty}(x-a)^2\,|f(x)|^2\,dx}{\int_{-\infty}^{\infty}|f(x)|^2 dx}\right)$$

is called the dispersion about the point $x = a$ of f. The reasoning behind the definition is that if $f(x)$ is concentrated near$x = a$, then $\Delta_a f$ is smaller than when f is not close zero far from $x = a$.

Example: Consider the characteristic function

$$\chi_b(x) = \begin{cases} 1\ ; & |x| \le b \\ 0\ ; & otherwise \end{cases}$$

Which has Fourier transform:

$$\widehat{(\chi_b)}(\xi) = \frac{2\sin\xi}{\xi}$$

Notice that χ_b is concentrated near $x = 0$ for small b. The dispersion about the origin is:

$$\Delta_0 \chi_b = \left(\frac{\int_{-b}^{b} x^2\,dx}{\int_{-b}^{b} dx}\right) = \frac{b^2}{3}$$

and it is clear that the dispersion increases as b increases.

Notice that the Fourier transform $\widehat{\chi_b}$ does not have a finite dispersion about the origin, since

$$\int_{-\infty}^{\infty} \xi^2\,|\widehat{\chi_b}(\xi)|^2\,d\xi = \int_{-\infty}^{\infty} \xi^2 \left(\frac{2\sin(b\xi)}{\xi}\right)^2 d\xi$$

$$= 4\int_{-\infty}^{\infty} \sin^2(b\xi)\,d\xi = \infty$$

This indicates that $\widehat{\chi_b}$ is spread out away from $x = 0$.

The following result shows that there is a type of inverse relationship between the dispersion of a function and that of its Fourier transform.

Theorem: Let $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then for all $a, \alpha \in \mathbb{R}$,

$$(\Delta_a f)(\Delta_\alpha \hat{f}) = \left(\frac{\int_{-\infty}^{\infty} x^2 \, |f(x)|^2 \, dx}{\int_{-\infty}^{\infty} |f(x)|^2 dx}\right)\left(\frac{\int_{-\infty}^{\infty} \xi^2 \, \left|\hat{f}(\xi)\right|^2 d\xi}{\int_{-\infty}^{\infty} \left|\hat{f}(\xi)\right|^2 d\xi}\right) \geq \frac{1}{4}$$

and equality holds if and only if $f(x) = ce^{-kx^2}$ for constants $c \in \mathbb{R}$ and $k > 0$.

Proof: we firstly prove the result for$a = \alpha = 0$.

$\int_{-\infty}^{\infty} x^2 \, |f(x)|^2 \, dx$ and $\int_{-\infty}^{\infty} \xi^2 \, \left|\hat{f}(\xi)\right|^2 d\xi$ are both assumed finite since otherwise the result is trivial.

Let $f^*(x) = \left(\widetilde{\imath\xi\hat{f}(\xi)}\right)$ or$\widehat{f^*}(\xi) = i\xi\hat{f}(\xi)$. Then $f^* \in L^2(\mathbb{R})$ and

$$\left(\int_{-\infty}^{\infty} x^2 \, |f(x)|^2 \, dx\right)\left(\int_{-\infty}^{\infty} \xi^2 \, \left|\hat{f}(\xi)\right|^2 d\xi\right)$$

$$= \left(\int_{-\infty}^{\infty} x^2 \, |f(x)|^2 \, dx\right)\left(\int_{-\infty}^{\infty} \left|(i\xi)\hat{f}(\xi)\right|^2 d\xi\right)$$

$$= \left(\int_{-\infty}^{\infty} x^2 \, |f(x)|^2 \, dx\right)\left(\int_{-\infty}^{\infty} \left|\widehat{f^*}(\xi)\right|^2 d\xi\right)$$

$$= 2\pi\left(\int_{-\infty}^{\infty} x^2 \, |f(x)|^2 \, dx\right)\left(\int_{-\infty}^{\infty} |f^*(x)|^2 \, dx\right)$$

(By Parseval's identity)

Since

$$\left[\int_{-\infty}^{\infty} \frac{1}{2} x\left(f^*(x)\overline{f(x)} + \overline{f^*(x)}f(x)\right) dx\right]^2$$

$$= \left[\int_{-\infty}^{\infty} x R_e\left(\overline{f^*(x)}f(x)\right) dx\right]^2$$

$$= \left[R_e \int_{-\infty}^{\infty} x\, f(x) \overline{f^*(x)} dx \right]^2$$

$$\leq \left| \int_{-\infty}^{\infty} x\, f(x) \overline{f^*(x)} dx \right|^2$$

$$\leq \left(\int_{-\infty}^{\infty} x^2 \, |f(x)|^2 \, dx \right) \left(\int_{-\infty}^{\infty} |f^*(x)|^2 \, dx \right)$$

(by Cauchy-Schwartz inequality)

We will show that:

$$-\int_{-\infty}^{\infty} x \left(f^*(x) \overline{f(x)} + \overline{f^*(x)} f(x) \right) dx = \int_{-\infty}^{\infty} |f(x)|^2 \, dx$$

From which the result follows, for then:

$$\left(\int_{-\infty}^{\infty} x^2 \, |f(x)|^2 \, dx \right) \left(\int_{-\infty}^{\infty} \xi^2 \left| \hat{f}(\xi) \right|^2 d\xi \right)$$

$$= 2\pi \left(\int_{-\infty}^{\infty} x^2 \, |f(x)|^2 \, dx \right) \left(\int_{-\infty}^{\infty} |f^*(x)|^2 \, dx \right)$$

$$\geq 2\pi \left[\int_{-\infty}^{\infty} \frac{1}{2} x \left(f^*(x) \overline{f(x)} + \overline{f^*(x)} f(x) \right) dx \right]^2$$

$$= \frac{\pi}{2} \left[\int_{-\infty}^{\infty} |f(x)|^2 \, dx \right]^2$$

$$= \left(\int_{-\infty}^{\infty} |f(x)|^2 \, dx \right) \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \hat{f}(\xi) \right|^2 d\xi \right)$$

$$= \frac{1}{4} \left(\int_{-\infty}^{\infty} |f(x)|^2 \, dx \right) \left(\int_{-\infty}^{\infty} \left| \hat{f}(\xi) \right|^2 d\xi \right).$$

To complete the proof, we assume that f is continuous and piece wise smooth, this assumption can be removed since functions in $L^1(\mathbb{R})$ are the uniform limit of such functions.

Then from the property of Fourier transform,

$$f^*(x) = f'(x)$$

Whenever the derivative exists. Then for any interval $[a, b]$,

$$b|f(b)|^2 - a|f(a)|^2 = \int_a^b \frac{d}{dx}(x|f(x)|^2)\,dx$$

$$= \int_a^b \left(xf'(x)\overline{f(x)} + xf(x)\overline{f'(x)} + |f(x)|^2\right) dx$$

$$= \int_a^b x\left(f^*(x)\overline{f(x)} + f(x)\overline{f^*(x)}\right) dx + \int_a^b |f(x)|^2\,dx$$

The assumption $f \in L^2(\mathbb{R})$ implies that $b|f(b)|^2 \longrightarrow 0$ as $b \longrightarrow \infty$ and $a|f(a)|^2 \longrightarrow 0$ as $a \longrightarrow -\infty$.

Since otherwise $|f(x)| > c|x|^{-\frac{1}{2}}$ as $|x| \longrightarrow \infty$, which is not integrable. Taking the limit as $b \longrightarrow \infty$ and $a \longrightarrow -\infty$,

$0 = \int_a^b x\left(f^*(x)\overline{f(x)} + f(x)\overline{f^*(x)}\right) dx + \int_a^b |f(x)|^2\,dx$ as required.

As for the case of equality in Heisenberg's inequality, this holds if and only if $f(x)\overline{f^*(x)}$ is real and $f^*(x) = Kxf(x)$ for some complex constants$K$. That is,

$f(x)\overline{f^*(x)} = f(x)\overline{Kxf(x)} = x|f(x)|^2\overline{K}$ is real. Therefore K is real.

The differential equation:

$f'(x) = Kxf(x)$ has solutions of the form $f(x) = ce^{-\frac{Kx^2}{2}}$, c is any real constant, and

$f(x) = ce^{-\frac{Kx^2}{2}} \in L^2(\mathbb{R})$ if and only if $K > 0$. Therefore equality holds in Heisenberg's inequality only if $f(x) = ce^{-\frac{Kx^2}{2}}$, for constants $c \in \mathbb{R}$ and $K > 0$.

Conversely, let $f(x) = e^{-\frac{Kx^2}{2}}$ for constant$K > 0$. Then

$$\left(\int_{-\infty}^{\infty} x^2\,|f(x)|^2\,dx\right)\left(\int_{-\infty}^{\infty} \xi^2\,\left|\hat{f}(\xi)\right|^2\,d\xi\right)$$

$$= 2\pi\left(\int_{-\infty}^{\infty} x^2\,|f(x)|^2\,dx\right)\left(\int_{-\infty}^{\infty} |f^*(x)|^2\,dx\right)$$

$$= 2\pi\left(\int_{-\infty}^{\infty} x^2\,|f(x)|^2\,dx\right)\left(\int_{-\infty}^{\infty} |Kxf(x)|^2\,dx\right)$$

$$= 2\pi K^2\left(\int_{-\infty}^{\infty} x^2\,|f(x)|^2\,dx\right)^2$$

$$= 2\pi K^2\left(\int_{-\infty}^{\infty} x^2\,e^{-Kx^2}\,dx\right)^2$$

$$= 2\pi K^2\left(\int_{-\infty}^{\infty} \left(\frac{x}{-2K}\right)\left(-2Kxe^{-Kx^2}\right)dx\right)^2$$

$$= 2\pi K^2\left(\int_{-\infty}^{\infty} \left(\frac{x}{-2K}\right)\left(\frac{d}{dx}\left(e^{-Kx^2}\right)\right)dx\right)^2$$

$$= 2\pi K^2\left(\int_{-\infty}^{\infty} \left(\frac{1}{2K}\right)e^{-Kx^2}\,dx\right)^2 = \frac{\pi}{2}\left(\int_{-\infty}^{\infty} e^{-Kx^2}\,dx\right)^2$$

(integration by parts)

$$= \frac{\pi}{2K}\left(\int_{-\infty}^{\infty} e^{-t^2}\,dt\right)^2 = \frac{\pi^2}{2K}$$

Whereas,

$$\frac{1}{4}\left(\int_{-\infty}^{\infty} |f(x)|^2\,dx\right)\left(\int_{-\infty}^{\infty} \left|\hat{f}(\xi)\right|^2\,d\xi\right)$$

$$= \frac{1}{4}\left(\int_{-\infty}^{\infty} |f(x)|^2\,dx\right)2\pi\left(\int_{-\infty}^{\infty} |f(x)|^2\,d\xi\right)$$

$$=\frac{\pi}{2}\left(\int_{-\infty}^{\infty}|f(x)|^2\,dx\right)^2=\frac{\pi}{2}\left(\int_{-\infty}^{\infty}e^{-Kx^2}\,dx\right)^2$$

and equality holds .

Let $f\in L^1(\mathbb{R})\cap L^2(\mathbb{R})$. Then for all $a,\alpha\in\mathbb{R}$,

$$(\Delta_a f)(\Delta_\alpha \hat{f})=\left(\frac{\int_{-\infty}^{\infty}x^2\,|f(x)|^2\,dx}{\int_{-\infty}^{\infty}|f(x)|^2dx}\right)\left(\frac{\int_{-\infty}^{\infty}\xi^2\,\left|\hat{f}(\xi)\right|^2\,d\xi}{\int_{-\infty}^{\infty}\left|\hat{f}(\xi)\right|^2d\xi}\right)\geq\frac{1}{4}$$

and equality holds if and only if $f(x)=ce^{-kx^2}$ for constants $c\in\mathbb{R}$ and $k>0$.

4.4 Conclusion

The Fourier transform of a function $f \in L^1(\mathbb{R})$ is defined to be:

$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-i\xi x}\, dx$, where $\xi \in \mathbb{R}$ and

$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)e^{ix\xi}\, d\xi$ is called the inverse Fourier transform of $\hat{f}(\xi)$.

The properties of Fourier transform are linearity, translation, dilation, differentiation, multiplication, symmetry, modulation, and convolution.

Let $f, \hat{f} \in L^1(\mathbb{R})$ and let f be piecewise smooth on $\mathbb{R}$ .Then for every, $x \in \mathbb{R}$,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}\,(\xi)e^{ix\xi}\, d\xi = \frac{1}{2}[f(x^-) + f(x^+)].$$

is called inversion theorem.

If $f, g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then $\int_{-\infty}^{\infty} f(y)\overline{g(y)}\, dy = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)\overline{\widehat{g(\xi)}}\, d\xi$ called Plancherels' Identity. When $f = g$ we obtain Parsevals' Identity, $\int_{-\infty}^{\infty}|f(y)|^2\, dy = \frac{1}{2\pi}\int_{-\infty}^{\infty}|f(\xi)|^2 d\xi$.

Fourier transforms have applications in partial differential equation like to solve wave equation, laplace equation and heat equation and so on.

Fourier transform is used to study band-limited functions.

Let $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ be continuous and band-limited to $|\xi| \leq \xi_c$. Then

$$f(x) = \sum_{n=-\infty}^{\infty} f(nL)\frac{\sin \xi_c(x - nL)}{\xi_c(x - nL)}\ , \qquad L = \frac{\pi}{\xi_c}.$$

is called Shannon's Theorem.

Let $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then for all $a, \alpha \in \mathbb{R}$,

$$(\Delta_a f)(\Delta_\alpha \hat{f}) = \left(\frac{\int_{-\infty}^{\infty} x^2 \, |f(x)|^2 \, dx}{\int_{-\infty}^{\infty} |f(x)|^2 dx}\right)\left(\frac{\int_{-\infty}^{\infty} \xi^2 \, \left|\hat{f}(\xi)\right|^2 \, d\xi}{\int_{-\infty}^{\infty} \left|\hat{f}(\xi)\right|^2 d\xi}\right) \geq \frac{1}{4}$$

and equality holds if and only if $f(x) = ce^{-kx^2}$ for constants $c \in \mathbb{R}$ and $k > 0$.

is called Heisenberg's inequality.

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Declaration

This project is my original work, has not been presented for a degree in any other University and that all the sources of material used for the project have been dully acknowledged.

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