ADDIS ABABA UNIVERSITY
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EXISTENCE AND UNIQUENESS OF SOLUTIONSOF ORDINARY DIFFERENTIAL EQUATIONS

BY

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Abstract

This thesis concentrates on questions regarding existence and uniqueness of solutions to the generic initial value problem in the theory of ordinary differential equations.That is $\frac{dy}{dx} = f(x, y);\ y(x_0) = y_0$ where f is some Lipschitz function. Compact form of existence and uniqueness theory appeared nearly 200 years after the development of the theory of differential equations. Special emphasis is given on the Lipschitz continuous functions in the discussion.

The thesis contains the following main topics:

Introduction

Chapter one

- Preliminaries, Lipschitz continuity, Metric spaces and Cauchy Sequences, Banach's fixed point theorem.

Chapter two

- Existence and Uniqueness of solutions of ODE
- Picsrd's Existence and Uniqueness Theorem
- System of Equations and Higher Order Equations
- The Existence and Uniqueness Theorem

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Introduction

Differential equations are essential for a mathematical description of nature.Many of the general laws of nature in physics,chemistry,biology,economics and engineering find their most natural expression in the language of differential equations.These equations are often quite complicated, and analytic solutions may not always be accessible. In these cases numerical aids must be used. However, this raises some delicate questions regarding the reliability and rigidity of numerical computations.For example the following questions are naturally arising;

- Given an initial value problem (IVP) is there a solution to it?

(Question of existence)

- If there is a solution is the solution unique?

(Questionof uniqueness)

- For which values of x does the solution to the IVPexist?

(The interval of existence)

In any case, a numerical algorithm, applied on some specific equation, will probably yield some kind of a plot of possible solution. We must then ask whether this plot really is the whole truth. Even if we are guaranteed the existence of a solution, we can a priori not be certain that this is the solution to the problem. There may very well be other solutions.

At early stage, Mathematicians were mostly engaged in formulating differential equations and solving them but they did not worry about the existence and uniqueness of solutions.

Differential Equations date back to the mid-seventeenth century, When Differential and Integral Calculus was discovered independently by Newton (1665) and Leibnitz (1676).

Newton had laid the foundation stone for the study of differential equations (DEs). He was followed by Leibnitz who coined the name of differential equations in 1676 to denote relationships between differentials dx and dy of two variables x and y.

Modern Mathematical physics essentially started with Newton's principle (published in 1687) in which he not only developed the calculus but also presented his three fundamental laws of motion that have made the mathematical modeling of physical phenomena possible. The fundamental law of motion in mechanics known as Newton's second law of is a differential equation to describe the state of a system. Motion of a particle of mass moving along a straight line under the influence of a specified external force $F(t,x,x')$is described by the following differential equation. (DE)

$mx'' = F(t,x,x')$; ($x' = \frac{dx}{dt}$; $x'' = \frac{d^2x}{dt^2}$) ……………………… (*)

More precisely, a differential equation (DE) is an equation that involves an unknown function and its derivatives. An ordinary differential equation (ODE) is a differential equation whose unknown is a function of single independent variable. When an ordinary differential equation (ODE) arises as a model or description of a scientific phenomenon, the independent variable is often time (t). Also, if y is a real valued function of the real variable t, then $\frac{dy}{dt}$denotes the first derivative of y. The second derivative of y is denoted by$\frac{d^2y}{dt^2}$.The third derivative of y is denoted by$\frac{d^3y}{dt^3}$. In general, for each positive integer n,$\frac{d^ny}{dt^n}$isused to denote the n^{th}derivative of the function y . A general form of an ordinary differential equation containing one independent and onedependent variableis $F(t,y,y',y'',\ldots y^{(n)}) = 0$,where F is an arbitrary function of $t,y,y',\ldots,y^n$, here t is the independent variable while y beingthedependent variable and $y^n = \frac{d^ny}{dt^n}$.The order of anordinary differential equation is the order of the highest derivative appearing in it. Equation (*) is an example of second order ordinary differential equation. On the other hand, partial differential equations (PDE) are those which have two or more independent variables.

In this thesis, we shall confine our discussion to a single scalar first order ordinary differential equation (ODE) only. The fundamental important question of existence and uniqueness of solution for an initial value problem was first answered by Rdolf Lipschitz in 1876.(Nearly 200 years later than the development of ordinary differential equations). In 1886 Giuseppe Peano discovered that the initial value problem

$$y' = f(x,y)$$

$y(x_0) = y_0$…………………………….. (**)

has a solution (it may not be unique) if f is a continuous function of (x, y). In 1890 Peano extended this theorem for system of first order ordinary differential equation using method of successive approximation. In 1890 Charles Emile Picard and Ernst Leonard Lindelöf presented existence and uniqueness theorem for the solution of the initial value problem.(**)

If we suppose for a moment that the independent variable is time, then the requirement of uniqueness has a physical interpretation. Suppose that we specify an initial condition "now", i.e. at time $t = 0$ then the existence of a unique solution means that we can use the equation to predict the future, since the solution is uniquely determined for $t > 0$. In this context, uniqueness of solutions is equivalent to the requirement that our model be deterministic.

The issues of existence and uniqueness are real, and it is possible to come up with very simple equations in which they fail. The good news is that there is a very general theorem guaranteeing existence and uniqueness, with a hypothesis which is very simple to check.

CHAPTER ONE

Preliminaries

Differential equations are broadly classified into linear and non-linear type.

Generally, linear problems are relatively 'easy' (which means that we can find an explicit solution) and non-linear problems are ' hard '. (This means that we cannot solve them explicitly except in very particular cases.)

Definition: An nth order ordinary differential equation (ODE) for $y(t)$is said to be linear if it can be written in the form

$$f(t) = a_n(t)\frac{d^n y}{dt^n} + a_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_0(t)y$$

i.e. only multiplesof y and its derivatives occur, such a linear form is called homogenous if $f(t) = 0$ and inhomogeneous if $f(t) \neq 0$.

Definition: If we assume that f is a continuous function of two variables and the differential equation

$$y' = f(t, y) \qquad \text{..............} \quad (1.1)$$

is considered, a differential function y on an interval I is said to be a solution to (1.1) on I if

$y'(t) = f\big(t, y(t)\big)$for each t in I

Example: A solution to the DE $y' = 3y$ is the function $y(t) = e^{3t}$ for all t. For if$y(t) = e^{3t}$then$y'(t) = 3e^{3t} = 3y(t)$and$y = e^{3t}$isa solution to $y' = 3y$ by definition. In fact we can verify that$y(t) = ce^{3t}$is a solution to $y' = 3y$ for any constant c.One of the fundamental problems associated with equation (1.1) is developing methods for special types of functions f that lead to the determination of the solution to (1.1) on a given interval I. In general, there are an infinite number of solutions to (1.1) on any given interval. For example, if $f(t, y) = 0,$ the $y(t) = c$ on I is a solution to (1.1) for any constant c and any interval I.

Example: consider the equation$y' = y - t$ and let I be any interval. For each real constant c the function $y = ce^t + t + 1$ for t in I is a solution to this equation, since$y'(t) = ce^t + 1$

$$= (ce^t + t + 1) - t$$

$$= y(t) - t$$

$= y - t$for all t in I

In,actuallytrying to explicitly determine a solution to equation (1.1), the simplest case is when the function f does not depend on y. Therefore, assume that g is a continuous real-valued function on an interval I and consider the equation

$y' = g(t)$ …………… (1.2)

The solutions to (1.2) on I are precisely the antderivatives of g. Thus

$y(t) = c + \int_{t_0}^{t} g(s)ds$,for all t in$I$, where c is any constant. ...(1.3)

Therefore, if y is a solution to (1.2) on I, then y has the form indicated in (1.3)for someconstant c. (and, in fact $c = y(t_0)$).

The most important case is determining a solution to (1.1) that assumes a specified value y_0 at a given time t_0 in$I$. Such a problem is called an initial value problem (IVP) and is denoted by the following manner:

$y' = f(t, y)$, $y(t_0) = y_0$………… (1.4)

Where (t_0, y_0) is some given pair in the domain of f.

1.1 Lipschitz continuity (local and global).

Understanding Lipschitz continuity is necessary to realize existence and uniqueness theory of ordinary differential equations. A function $f(x, y)$ is said to be locally Lipschitz or locally Lipschitz continuous at a point $(x_0, y_0) \in D$ (opened and connected set) if (x_0, y_0) has a neighborhood D_0 such that $f(x, y)$ satisfies:

$$|f(x, y_1) - f(x, y_2)| < L|y_1 - y_2| , \qquad \forall (x, y_1), (x, y_2) \in D_0$$

Where L being fixed over the neighborhoodD_0,afunction is said to be locally Lipschitz in a domain D if it is locally Lipschitz at every point of the domain D. We denote the set of all locally Lipschitz functions by L_l and symbolically we say $f \in L_l$ if f is locally Lipschitz.Moreover, a function is said to be globally Lipschitz or $(f \in L_g)$ if

$$|f(x, y_1) - f(x, y_2)| < L|y_1 - y_2| , \quad \forall (x, y_1), (x, y_2) \in D$$

With the same Lipschitz constant L forentire D .

Geometrically,the slope of any chord joining two points, in the neighborhoodof(x_0, y_0), is bounded then we say that f is locally Lipschitz continuous at (x_0, y_0), if this slope is bounded in the neighborhood of every point of domain D we say that f is locally Lipschitz in D. Furthermore, if this bound remains fixed over the entire domain then the function is globally Lipschitz and $f \in L_g$.

Lipschitzconditionguaranteesuniformcontinuity,butitdoesnotensure differentiability ofthefunction. Continuity is sufficientfor existence of solution and locally Lipschitz is a sufficient condition for uniqueness of the solution of an IVP of first order ordinary differential equation. A unique solution exists for IVP of a linear ODE in any interval where coefficients are continuous and bounded. Moreover, we can solve it explicitly. On the other hand, issues are not so straight forward in case of non-linear ODEs.

1.2Metric Spaces and Cauchy Sequences

Definition: Let X be a set. A function $d: X \times X \longrightarrow \mathbb{R}$ is said to be a metric if for all $x, y, z \epsilon X$.

$$i)\ d(x, y) = 0 \Leftrightarrow x = y$$

$$ii)\ d(x, y) = d(y, x)$$

$$iii)\ d(x, y) \leq d(x, z) + d(z, y)$$

If X has a defined metric d, then X sometimes written,(X, d), is said to be a metric space. Note from the definition, that a metric is always positive or zero, since for all x and y in X

$$0 = d(x, x) \leq d(x, y) + d(y, x) = 2d(x, y)$$

The notations of a distance make it possible to talk about a neighborhood of a point, i.e. the set of points on a certain distance from a given point.

We can also generalize the concept of convergence to a metric space(X, d). We say that a sequence$\{x_n\}_{n \in N}$ofpointsin X convergesto

$x \in X,\ if\ \ \forall \varepsilon > 0, \exists N\ \forall n > N,\ d(x, x_n) < \varepsilon.$

There are many examples of metrics. A more well-known example is the norm-induced metric on$\mathbb{R}^n$definedas,

$d(x,y) = |x-y| = \sqrt{(x_1-y_1)^2+\cdots+(x_n-y_n)^2}$, where $x_i, y_i,\ for,\ i = 1,\dots,n$denote the coordinates of $x, y \in \mathbb{R}^n$

Definition:(Cauchy Sequences)

Let $x_n\ _{n\in N}$ be a sequence of points in a metric space X, d . A sequence is called Cauchy if

$\forall\varepsilon > 0,\ \exists\, N\ \forall\, m,n > N,\ such\ that\ d(x_m,x_n) < \varepsilon$.

There are some apparent similarities between the definition of a convergent sequence and a Cauchy sequence. However, the definition of a Cauchy sequence is strictly weaker, which is important to realize. Of course, if $\{x_n\}_{n\in N}$ is convergent to$x \in X$, then$\forall\varepsilon > 0, \exists N$such that for all $n > N$, $d(x,x_n) < \frac{\varepsilon}{2}$ $,choose\ m > N$

$$d(x_m,x_n) \le d(x_m,x) + d(x,x_n) < \varepsilon$$

Hence every convergent sequence is Cauchy, but the converse may not always be true. However, if every Cauchy sequence in X does converge, X is said to be complete.

1.3Banach's fixed point theorem

Before we start to prove this theorem, we must first introduce some basic concepts.

Let (X,d) be a metricspace and let T be a map on X. A point $p \in X$ such that $T(p) = p$ is called a fixed point of T, and if there exist $C < 1$, such that for all $x, y \in X$, $d(T(x),T(y)) \le Cd(x,y)$,T is said to be a strict contraction on X. It is clear that a strict contraction can have at most one fixed point. To see this, assume that $p, q \in X$ are two distinct fixed points of T. Then

$$d\big(T(p),T(q)\big) = d(p,q) \le Cd(p,q)$$

Which is impossible if $d(p,q) > 0$. Hence $p = q$. We are now ready to state the famous theorem of Banach about fixed points of strict contraction on a complete metric space.

Theorem: (Banach's fixed point theorem)

Let X be a complete metric space and T a strict contraction on X. Then T has a unique fixed point in X.

Proof: the part of uniqueness is clear from above.

To prove existence, let us define the following recursion formula for

$$n \geq 0, and, x_0 \in X$$

$$x_{n+1} = T(x_n),$$

$$x(0) = x_0$$

We want to show that $\{x_n\}_{n \in N}$ converges for some initial point$x_0 \in X$. Since X is complete it is sufficient to show that this sequence is Cauchy.

It is clear that

$$d(x_2, x_1) = d(T(x_1), T(x_0)) \leq Cd(x_1, x_0)$$

Fromwhich follows inductively $\quad d(x_n, x_{n-1}) \leq C^{n-1} d(x_1, x_0)$

Using the triangle inequality, it is to show that

$d(x_{n+m}, x_n) \leq d(x_{n+m}, x_{n+m-1}) + \cdots + d(x_{n+1}, x_n)$, and from above, it follows

$$d(x_{n+m}, x_n) \leq (c^{n+m-1} + c^{n+m-2} + \cdots + c^n) d(x_1, x_0)$$

Whichisa geometrical series in C, and hence, we have

$$d(x_{n+m}, x_n) \leq \frac{c^n}{1-c} d(x_1, x_0)$$

Now, since $c < 1$, we have that for every $\varepsilon > 0$ there exists n, such that

$\frac{c^n}{1-c} d(x_1, x_0)$, and hence $\{x_n\}_{n \in N}$is Cauchy and converges to some point $p \in X$. we have

$p = T(p)$, and p is a fixed point of T.

Note that the convergence of the recursion is independent on the initial pointx_0, and that is in fact a very constructive proof.

Definition: we define the open ball of radius $r > 0$ around $p \in X$ as

$$B_p(r) = \{x \in X : d(p, x) < r\}$$

Definition: Let (X, d) be a metric space. A function $f: X \to X$ is said to be continuous, if $\forall \varepsilon > 0, \exists \delta > 0$ such that

$$d(x, y) < \delta \Rightarrow d(f(x), f(y)) < \varepsilon$$

Of course, any strict contraction will be continuous. To see this, let $\varepsilon > 0$ and choose δ such that $C\delta < \varepsilon$.We thenhas

$$d(f(x), f(y)) \leq cd(x, y) < \varepsilon$$

Notice that we never used the fact that $c < 1$. For most of our purposes, only to assume continuity is often a too weak. A stronger requirement would be differentiability.

Definition: Let (x, d_x) and (z, d_z)be metric spaces. A function $f: x \to z$ is said to be Lipschitz continuous (or Lipschitz), if

$$\exists A > 0, \forall x, y \in X \Rightarrow d_z(f(x), f(y)) < Ad_x(x, y)$$

Any strict contraction is obviously Lipschitz and copying the proof above, we see that every Lipschitz function is continuous. However there are continuous functions which are not Lipschitz. To see this, let us consider the following example

Metrics on functions spaces:

We will soon have to measure distance between certain functions. For this reason, we define a metric on the space of continuous functions on a closed bounded interval $I \subseteq \mathbb{R}$, taking values in$\mathbb{R}^n$.we call this space $C(I)$. There are several alternative metrics on$C(I)$, but we will choose one metric which also yields completeness to the space, namely

$$d(f, g) = \|f - g\| = max_{t \in I}|f(t) - g(t)|;\ f, g \in C(I)$$

Where $|.|$denotes the norm on$\mathbb{R}^n$. The reader is urged to show that this is in fact a metric. First note that the norm really is defined for all $f, g \in C(I)$, since the maximum of continuous functions always exists on a closed and bounded subset of $\mathbb{R}$.

CHAPTER TWO

Existence and Uniqueness of solutions of ODE

Let I = [-1,1] and define the open ball of radius $r > 0$ around with the norm (absolute value) $|.|$ as the defined metric, as

$$B_r(p) = \{x \in \mathbb{R}^n : |x - p| < r\}$$

We are now ready to state the fundamental theorem of about the existence and uniqueness of solutions to the generic IVP on the standard form.

Theorem 2.0: let $\varepsilon > 0$ and $x_0 \in \mathbb{R}^n$, and let $f: B_\varepsilon \times I \to \mathbb{R}^n$be Lipschitz

Then there exists a$\in$(0,1] and continuously differentiable function

$y: [-a, a] \to \mathbb{R}^n$such that

$y' = f(x, t), y(0) = x_0$

Let us introduce the notation$I_a$for the closed interval [-a,a]. Before we prove this theorem, it may be educational to examine the necessity of the restrictions above. In most elementary books on differential equations, f is assumed to be continuously differentiable as well. However, and this is a very straight forward check, every continuously differentiable map

$f: \mathbb{R}^n \longrightarrow \mathbb{R}$ is Lipschitz, so our results are strictly stronger.

Example: Let $f(t, y) = 3y^{2/3}$, and study the IVP

$$y' = f(t, y), \;\; y(0) = 0$$

As we have seen in the previous example, f is not Lipschitzat$(t, y) = (t, 0)$. Indeed, we can easily verify that both $y(t) = 0$ and $y(t) = t^3$ satisfy the equation above. These solutions are different in any neighborhood of the point 0. Hence a solution exists, but it is not unique.

A proof of the main theorem

A naive approach to solving the standard problem described above would be to integrate both sides over some interval [0, t]respecting the initial we get

$$\int_0^t y'(s)ds = \int_0^t f(t,s)ds$$

Or equivalently,

$$y(t) = y_0 + \int_0^t f(t,s)ds$$

Of course, this is just a reformulation of the original problem, and we have not gained any new information about the actual solution, if such exists. However, this new formulation is not completely worthless. We can define the operator T on$C(I)$ as

$$(Ty)(t) = y_0 + \int_0^t f(t,s)ds$$

Interms of this operator, solving the standard problem is just equivalent to finding a fixed point to the operator T. Indeed, if y is a fixed point to T, then y must be continuously differentiable, and

$$y(t) = y_0 + \int_0^t f(t,s)ds \;\; \Rightarrow y'(t) = f(t,y)$$

We will show that this operator is a strict contraction on $C(\,I_a\,)$for some small $a > 0$.

Proof: Let f be Lipschitz and $x, y \in C(I)$

$$\|Tx - Ty\| = \left\| \int_0^t f(x,s) - \int_0^t f(y,s)ds \right\|$$

$$\leq \int_0^t \|f(x,s) - f(y,s)\|ds$$

Now using the Lipschitz condition, we know that there exist A such that for all $x, y \in I$

$\|f(x,s) - f(y,s)\| \leq A(s)\|x - y\|$ Where A is some bounded function of S on an interval I_a= [-a, a], for somesmall $a > 0$. Let us set $A = sup_{s \in I_a} A(s)$

This A will of course be finite. Hence if $t = a$, we have

$$\|Tx - Ty\| \leq Aa\|x - y\|$$

We can choose ‘a’ such that $aA < 1$.This value of ‘a’ makes T into a strict contraction on. $C(I_a)$

According to the well-known fixedpoint theorem of Banach, T will have a unique fixed point in $C(I_a)$ which from our reasoning above, will be the unique solution to the standard problem.

2.1Picard's Existence and Uniqueness Theorem

In this section we consider thefirst order *Initial Value Problem (IVP)* ofordinary differential equation (not necessarily linear) of the form

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$$

The equation $y' = f(x,y)$ dictates a value of y' at each point(x,y), so one would expect there to be a unique solution curve through a given point (x_0, y_0). This fact leads us to prove the following theorem, called *Picard's Theorem*

$$\boldsymbol{Theorem\,.2.1}\ (Picard's\ Theorem)$$

Let I be a real interval (possibly $\mathbb{R}$ itself) and le *f*be a real function on $Ix\mathbb{R}$ satisfying the***Lipschitz condition***

$$|\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y_1}) - \boldsymbol{f}(\boldsymbol{x},\boldsymbol{y_2})| \le \boldsymbol{k}|\boldsymbol{y_1} - \boldsymbol{y_2}|, on\ this\ set\ I \times \mathbb{R}\ for\ k > 0$$

Let $x_0 \in$ I.Then there is a unique function y on I such that

$$y' = f\big(x, y(x)\big)\ for\ x\ \in\ I\ and\ y(x_0) = y_0$$

Note: - By Mean Value Theorem, the Lipschitz condition will be satisfied if we have

$$\left|\frac{\partial}{\partial y} f(x,y)\right| \le k \quad on\ I \times \mathbb{R}$$

Lemma.1.Let $y' = f(x,y); y(0) = 0$

It is enough to prove the result for the case$x_0 = y_0 = 0$.

Proof

Suppose the result is known for this case and $x_0, y_0$are given.

Let $g(x,y) = f(x + x_0, y + y_0)$.

By assumption, there is a unique function z such that $z' = g\big(x, z(x)\big)$when

$$x + x_0 \ \in \mathrm{I}\ and\ z(0) = 0$$

Let $y(x) = z(x - x_0) + y_0$. Then $y(x_0) = y_0$ and forevery $x \in$ I, we have

$$y'(x) = z'(x - x_0)$$

The next step is to show that the required properties of y can be equally expressed by an integral.

***Lemma*. 2.** *Let I be an interval containing 0.For a function on I, the following statements are equivalent:*

$$(\boldsymbol{i}).\ \ \boldsymbol{y'(x) = f(x, y(x))\ for\ \ x \in I\ and\ y(0) = 0};$$

$$(\boldsymbol{ii}).\ \ \boldsymbol{y(x) = \int_0^x f(t, y(t)dt\ \ for\ \ x \in I}$$

Proof: $(i) \Rightarrow (ii)$

Let $C(I)$ be the space of all continuous functions on I. We define a mapping A from $C(I)$ to itself as follows. For a given$y \in C(I)$,

$$\boldsymbol{(Ay)(x) = \int_0^x f(t, y(t))dt}$$

By Lemma.2.part (i), y is the required solution if an only if $A(y) = y$.Now we take

$$\boldsymbol{y_0 = 0\ , y_1 = A(y_0), \ldots, y_n = A(y_{n-1}) = A^n(y_0)}$$

We will show that the sequence of functions $\{y_n\}$ converges (uniformly on I) to a limit function y. One would then expect that

$$A(y) = \lim_{n\to\infty} A(y_n) = \lim_{n\to\infty} y_{n+1} = y\ , as\ required.$$

$(ii) \Rightarrow (i)$ .Conversely, suppose that y satisfies the integral equation (ii) in Lemma.3 for x ϵ I .

That is,

$$y(x) = \int_0^x f(t, y(t))dt\ \ for\ \ x \in I$$

Differentiation both sides yields

$$y'(x) = f(x, y(x))\quad for\ \ x \in I$$

***Lemma*. 3.** *Suppose that*$|u(x) - v(x)| \leq C\ for\ \ x \epsilon\ I.$

Then foreach $n \geq 1, |A^n u(x) - A^n v(x)| \leq C\,\frac{(k|x|)^n}{n!}\ for\ \ x \in I$

Proof: By the Lipschitz condition we have

$$|f(t,u(t)) - f(t,v(t))| \le ck \ \ for \ \ t \in I$$

So that

$$|(Au)(x) - (Av)(x)| = \left|\int_0^x (f(t,u(t)) - f(t,v(t)))dt\right| \le ck|x|$$

i.e. the statement holds for $n = 1$.

Now suppose that it holds for certain n. Then by the Lipschitz condition we have

$$|f(t,A^n u(t)) - f(t,A^n v(t))| \le Ck\frac{(k|t|)^n}{n!} \ for \ x > 0$$

$$|(A^{n+1})u(x) - (A^{n+1})v(x)| = \left|\int_0^x [f(t,u(t)) - f(t,v(t))]dt\right|$$

(Similarly for $x < 0$ with $|x|$ instead of x)

Proof of the theorem:For the moment assume that I is a bounded closed interval. Then $|x| \le \mathbb{R}$ (say) for $x \in I$, and continuous functions are bounded on I, so there exists M such that $|f(t,0)| \le M \quad for \ t \in I$

Uniqueness: Suppose that $AU = U \ \ and \ \ AV = V$. Then

$A^n U = U \ and \ A^n V = V \ for \ all \, n$. Also, there exists c such that $|u(x) - v(x)| \le c \ \ for \ \ x \in I$

By Lemma.3, we have

$$|u(x) - v(x)| \le C\frac{(k|x|)^n}{n!} \ for \ n \ge 1$$

But for each $x, \frac{(k|x|)^n}{n!} \to 0 \ as \ n \to \infty$

Hence $U(x) = V(x)$

Existence:Let $y_0 = 0\,, y_1 = A(y_0)\,, y_2 = A(y_1), \ldots, y_n = A^n(y_0)$.

Then$|y_1(x) - y_0(x)| = |y_1(x)| = \left|\int_0^x f(t,0)dt\right| \le M|x| \le C \ \ for \ x \in I, where \ C = MR$

By Lemma.3, $|y_{n+1}(x) - y_n(x)| \le MR\,, \quad for \ x \in I.$

Now$\sum_{n=1}^{\infty}\frac{(kR)^n}{n!}$ $is\ convergent\ (to\ e^{kR})$

So by the "M-test"$\sum_{n+1}^{\infty}(y_{n+1}-y_n)$ $is\ uniformly\ convergent\ on\ I\ (say\ to\ y)$

But , $(y_1-y_0)+(y_2-y_1)+\cdots+(y_n-y_{n-1})=y_n-y_0=y_n$

So,this implies that $y_n \to y$ uniformly on I. In other words,

$$|y_n(x)-y(x)| \le \delta_n\ for\ \ x\epsilon\ I, where\ \delta_n \to 0\ \ as\ n \to \infty.$$

By the case n=1 in Lemma.3.

$$|y_{n+1}(x)-(Ay)(x)| = |(Ay_n)(x)-(Ay)(x)| \le k|x|\delta_n\ for\ \ x\epsilon\ I$$

So,$(Ay)(x) = \lim_{n\to\infty} y_{n+1}(x) = y(x)$ $\ for\ \ x\epsilon\ I$ as required.

Finally, suppose that I is an open or unbounded interval. Then we can express it as $\bigcup_{n=1}^{\infty} I_n$, where $I_1 \subset I_2 \subset I_3 \subset \cdots$ are bounded, closed intervals with $O\ \epsilon\ I_1$.From above, for each n there is a unique solution $y_{(n)}$ on I_n with $y_{(n)}(0)=0$. By this uniqueness,$y_{(n+1)}$ agrees with $y_{(n)}$ on I_n, so it is consistent to define y on I by:

$$y(x) = y_{(n)}(x)\ \ \ for\ \ x\ \in\ \ I_n$$

Clearly, y satisfies the equation on I. □

We now give two quite simple examples to show that both parts of the theorem can fail if the Lipschitz condition is not satisfied.

Example.1. Consider the equation

$$y' = -y^2, with\ \ y(1)=1\ \ on\ the\ interval\ [-1,1].$$

Solving by elementary methods, we have

$$-\frac{1}{y^2}y' = 1$$

$$\frac{1}{y} = x + c$$

The condition $y(1)=1$ selects the solution $y=\frac{1}{x}$. This is clearly a unique solution on $(0.1]$. It cannot be extended to a differentiable function at 0, so that there is no solution on $[-1,1]$. To see that the Lipschitz condition fails, note that

$f(x,y) = -y^2$. We have

$$f(x, y+1) - f(x, y) = -(y+1)^2 + y^2 = -2y - 1\,,$$
$$which\ is\ unbounded\ on\ \mathbb{R}.$$

Example.2 Consider the equation

$$y' = 3y^{\frac{2}{3}}, with\ \ y(0) = 0\ \ on\ I = \mathbb{R}.$$

Clearly, 0 is one solution. Elementary methods give

$$\frac{1}{3y^{\frac{2}{3}}} y' = 1$$

$$y^{\frac{1}{3}} = x + c$$

$$y = (x+3)^3.$$

Hence x^3 is another solution with$y(0) = 0$. There are infinitely many solutions: for each $c > 0$, a solution is

$$y_c(x) = \begin{cases} (x-c)^3 & for\ x \geq c \\ 0 & for\ x < c \end{cases}$$

(Then y_c is differentiable at c, with derivative 0)

The Lipschitz condition fails because

$$\frac{y^{\frac{2}{3}} - 0}{y - 0} = \frac{1}{y^{\frac{1}{3}}} \to \infty \quad as\ y \to 0^+$$

$\boldsymbol{Picard's\ \ Iteration}$:

The proof of Picard’s Theorem provides a way of constructing successive approximations to the solution. With the initial condition $y_0(x) = y_0$, this means we define

$y_0(x) = y_0$, and

$$y_n(x) = y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))dt$$

Example:Consider the equation $y' = y$ with $y(0) = 1$

Of course, the solution is e^x. Picard's iteration gives the following: $f(x,y)=y$, hence

$$y_0(x)=1$$

$$y_1(x)=1+\int_0^x f(t,1)dt=1+\int_0^x 1dt=1+x$$

$$y_2(x)=1+\int_0^x f(t,1+t)dt=1+\int_0^x (1+t)dt=1+x+\frac{x^2}{2}$$

$$y_3(x)=\cdots=1+\int_0^x (1+t+\frac{t^2}{2})dt=1+t+\frac{t^2}{2}+\frac{t^3}{3!}$$

…

$$y_n(x)=1+t+\frac{t^2}{2}+\cdots+\frac{t^n}{n!}$$

$$y_n(x)=1+\int_0^x f\big(t,y_{n-1}(t)\big)dt=\int_0^x y_{n-1}(t)dt$$

Example: The equation $y'=2x+y^2, with\ y(0)=0$.

Start with $y_0(x)=0$,then

$$y_0(x)=0$$

$$y_1(x)=0+\int_0^x f\big(t,y_0(t)\big)dt=\int_0^x f(t,0)dt=\int_0^x 2tdt$$

$$y_2(x)=0+\int_0^x f\big(t,y_1(t)\big)dt=\int_0^x f(t,t^2)dt$$

…

$$y_n(x)=\int_0^x f\big(t,y_{n-1}(t)\big)dt=\int_0^x f(t,(y_{n-1}(t))^2)dt$$

Apply the Picard's iteration method to find the solution of the IVP

$$\begin{cases} y' = 3x^2(y+1) \\ y(0) = 1 \end{cases}$$

Solution:

Here $x_0 = 0,\ y_0(x) = 1$ and $f(x, y) = 3x^2(y+1)$

Then by Picard's iteration

$$y_1(x) = 1 + \int_0^x f(t,1)dt = 1 + \int_0^x 3t^2(1+1)dt = 1 + 2x^3$$

$$y_2(x) = 1 + \int_0^x f(t, 1+2t^3)dt = 1 + \int_0^x 3t^2(2t^3+2)dt = 1 + 2x^3 + x^6$$

$$y_3(x) = 1 + \int_0^x f(t, 1+2t^3+t^6)dt$$

$$= 1 + \int_0^x 3t^2(2+2t^3+t^6)dt$$

$$= 1 + 2x^2 + x^6 + \frac{2x^9}{3!}$$

$$y_4(x) = 1 + 2x^2 + x^6 + \frac{2x^9}{3!} + \frac{2x^{12}}{4!}$$

By induction, it can be shown that

$$y_n(x) = 1 + 2x^2 + x^6 + \frac{2x^9}{3!} + \frac{2x^{12}}{4!} + \cdots + \frac{x^{2n}}{n!}$$

Since f and $\frac{\partial f}{\partial y} = 3x^2$ are continuous and bounded on a region about $(0,1)$, by corollary of Picard's theorem, there exists a unique solution that can be determined by

$$\lim_{n\to\infty} y_n(x) = 2e^{x^3} - 1 \quad \text{(Applying Taylor's series expansion)}$$

To confirm this result, we integrate both sides of $\frac{y'}{y+1} = 3x^2$

$$\int \frac{dy}{y+1} = \int 3x^2 dx$$

$$\Rightarrow \ln|y+1| = x^3 + c\ , for\ \ c \in \mathcal{R}$$

$$\Rightarrow y = ce^{x^3} - 1$$

Since $x_0 = 0,\ y_0(x) = 1$, we have $1 = c - 1 \quad \Rightarrow \quad c = 2$

Thus, the solution of the given IVP becomes

$$y = 2e^{x^3} - 1 = \lim_{n\to\infty} y_n(x)$$

2.2 System of Equations and Higher Order Equations

Now consider a system of first order equations

$$y'_j(x) = f_j[x, y_1(x), \dots, y_n(x)] \qquad (1 \le j \le n)$$

In vector notation,

$$y'(x) = (y_1(x), \dots, y_n(x))$$

And the system becomes,

$$y'(x) = F(x, y(x))$$

By a vector adaptation of the proof, we shall obtain a generalization of Theorem 2.1 to the system of equations. As we have seen before, higher order equations can be seen as a special case of a system. For $y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$, write

$$\|y\| = max|y_j|$$

This is a "norm" on $\mathbb{R}^n$, giving $\|x - y\|$ as a notation of generalized "distance" between $x \; and \; y$. Note that $\|x - y\| \le M$. Simply it means that $|x_j - y_j| \le M \quad for \; each \;\; j$

Remark: Another kind of "norm" on $\mathbb{R}^n$ is given by

$$\|y\|_2 = (\sum_{j=1}^{n} {y_j}^2)^{\frac{1}{2}}$$

The two norms are "equivalent" in the sense that

$$\|y\| \le \|y\|_2 \le \sqrt{n}\|y\| \; for \; all \quad y$$

This means that either can be used to determine convergence. Though $\|.\|_2$ is arguably the "true" measure of distance, it is more complicated to use in proofs like those that follow.

Our $\|y\|$ is often denoted by $\|y\|_\infty$. Our theorem is as follows.

$\boldsymbol{Theorem\ .2.2}$

Suppose that each f_j is continuous on $I \times \mathbb{R}^n$ and

$$\left|f_j(x,y) - f_j(x,w)\right| \leq k\|y-w\| \ \ on\ I \times \mathbb{R}^n$$

Suppose that $x_0 \in I$ and real numbers $a_1, a_2, \dots, a_n$ are given. Then there is a unique vector function

$$y'(x) = F(x, y(x))\ for\ \ x\ \in I\ \ and\ \ y_j(x_0) = \alpha_j\ for\ each\ \ j.$$

As before, it is sufficient to consider the case where $x_0 = \alpha_j = 0\ \ for\ all\ \ j.$

Exactly as in Lemma.2, the required conditions equivalent to

$$y_j(x) = \int_0^x f_j(t, y(t))dt \qquad (j = 1, \dots, n)$$

Or in vector notation

$$y(x) = \int_0^x F(t, y(t))dt$$

Given $y = (y_1, y_2, \dots, y_n)$, where each y_j is in C(I), define Ay to be w, where

$$w(x) = \int_0^x F(t, y(t))dt$$

We require y such that $Ay = y$.

$\boldsymbol{Lemma.3'}$**.** Let $\mathbb{U}$ and $\mathbb{V}$ be vector functions such that

$$\|\mathbb{U}(x) - \mathbb{V}(x)\| \leq C \text{ for } \ x \in I$$

Then for eachk$\geq$ 1, $\|(A^k\mathbb{U})(x) - (A^k\mathbb{V})(x)\| \ \leq \frac{(k|x|)^k}{k!}$ for $x \in I.$

$\boldsymbol{Proof}$: Just like Lemma.3, with the notation adjusted, we describe the induction step.

Assume that the statement holds for a certain k. Then by the Lipschitz condition, for each j;

$$\left|f_j[t, (A^k\mathbb{U})(t)] - f_j[t, (A^k\mathbb{V})(t)]\right| \leq kC\frac{(k|x|)^k}{k!}$$

Where $A^{k+1}\mathbb{U} = (g_1, g_2, \dots, g_n)$ and $A^{k+1}\mathbb{V} = (h_1, h_2, \dots, yh_n)$.

This means that $g_j(x) = \int_0^x f_j[t, (A^k\mathbb{U})(t)]dt$ $(j = 1, \dots, n)$ and similarly for $h_j(x)$ with $\mathbb{V}$ instead of $\mathbb{U}$.

Exactly as in Lemma.3, we deduce that for $x \in I$

$$\left|g_j(x) - h_j(x)\right| \leq C\frac{(k|x|)^{k+1}}{(k+1)!} \qquad (j = 1, \dots, n)$$

Which is equivalentto

$$\|(A^{k+1}\mathbb{U})(x) - (A^{k+1}\mathbb{V})(x)\| \leq C\frac{(k|x|)^{k+1}}{(k+1)!} \text{ for } x \in I$$

Theorem 2.2 now follows exactly as in the case n=1.

Let $y^{(0)} = 0$ *and* $y^{(k)} = \left(y_1^{(k)}, y_2^{(k)}, \dots, y_n^{(k)}\right) = A^k(0)$. (Hence the notation $y^{(k)}$ is being used temporarily for the k-th functional sequence, not k-th derivative). As before, we see that for each j, the functions $y_j^{(k)}$converge uniformly on I,

as $k \to \infty$ (in other works $y^{(k)} \to y$ uniformly on I) and $Ay = y$.

Now let us consider a higher order differential equation of the form

$$\boldsymbol{y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y^1 + a_0(x)y = b(x)} \text{……………………..}\textbf{(2.1)}$$

Where y^i stands for the $i-th$ derivative of y with respect to x .

It is a standard practice to convert such an n^{th} order equation (2.1) into a first order system by defining

$$y = z_1$$
$$y' = z_2$$
$$y'' = z_3$$
$$\dots$$
$$\dots$$
$$y^{(n-1)} = z_n$$

Then it follows that $y' = z'_1 = z_2$

$$y'' = z'_2 = z_3$$
$$\dots$$
$$\dots$$
$$y^{(n-1)} = z'_{n-1} = z_n$$

$$y^n = z'_n = b(x) - [a_{n-1}(x)z_n + a_{n-2}(x)z_{n-1} + \cdots + a_0(x)z_1]$$

Hence the higher order differential equation (2.1) is equivalent to

$$\frac{d}{dx}\begin{pmatrix} z_1 \\ \cdots \\ z_{n-1} \\ z_n \end{pmatrix} = \begin{pmatrix} z_2 \\ \cdots . \\ z_n \\ b(x) - [a_{n-1}(x)z_n + a_{n-2}(x)z_{n-1} + \cdots + a_0(x)z_1] \end{pmatrix} \ldots\ldots\ldots(2.2)$$

If we denote vectors in $\mathcal{R}^n$ by $z = (z_1, \ldots, z_n)$, then our equation can be represented in vector form as

$$\frac{dz}{dx} = z'(x) = \begin{pmatrix} z_2 \\ \cdots . \\ z_n \\ b(x) - [a_{n-1}(x)z_n + a_{n-2}(x)z_{n-1} + \cdots + a_0(x)z_1] \end{pmatrix}$$

$$= f(x, z(x))$$

This is a first order ODE! Thus Picard's theorem implies that there exists solution for this ODE which satisfies the initial value

$\left(y(x_0), \ldots., y^{(n-1)}(x_0)\right) = (z_1(x_0), \ldots, z_n(x_0))$. Therefore the general solution of this type non-homogeneous ODE is of the form

$$c_1y_1 + c_2y_2 + \ldots . + c_ny_n + p \ where\ p\ is\ a\ prticular\ solution$$

$\boldsymbol{Theorem\ 2.3}$ Suppose that *f* is continuous on $I \times \mathbb{R}^n$ and

$$|f(x, y) - f(x, w)| \leq k\|y - w\| \quad \text{on } I \times \mathbb{R}^n$$

Suppose that $x_0 \in I$ and real numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ are given. Then there is a unique function y on I such that

$$y^{(n)}(x) = f\left[x, y(x), y'(x), \ldots, y^{(n-1)}(x)\right] for\ x\ \in I\ and\ \ y(x_0) = \alpha_{0\,,}\, y^{(j)}(x_0) = \alpha_{j,}$$

for 1$\leq j \leq n - 1$

$\boldsymbol{Proof}$: The equation is equivalent to the system

$$y'_j = y_{j+1} \quad for\ \ 0 \leq j \leq n - 1$$

$$y'_{j+1} = f(x, y_0, y_{1,\ldots,}y_{n-1})$$

In the notation of Theorem2.2 (with j ranging from 0 to n-1), we have

$f_{n-1} = f$, while for

$$0 \leq j \leq n - 1,$$

$$f_j(x, y_0, y_1, \ldots, y_{n-1}) = y_{j+1}$$

We only need to verify that the function f_j satisfy the Lipschitz condition. This is trivial for $0 \le j \le n-2$, and ensured by our hypothesis for $j = n-1$.

Finally we specialize this to linear equations, showing that the Lipschitz condition is then automatically satisfied.

***Theorem* 2 . 4** Suppose that the functions $P_0, P_1, \ldots, P_{n-1}$ *and* g are continuous on an interval I. Let $x_0 \in I$ and real numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ be given. Then there is a unique function y such that

$$y^{(n)} = P_0(x)y(x) + P_1(x)y'(x) +, \ldots + P_{n-1}(x)y^{(n-1)}(x) + g(x)$$

$for\ x \in I\ and\ y(x_0) = \alpha_0,\ y^{(j)}(x_0) = \alpha_j,$ for $1 \le j \le n-1$

Proof :Assume a bounded closed interval, I (then extend to open and unbounded intervals as before). Then each P_j is bounded on I, say $|P_j(x)| \le k \quad for\ \ x \in I$. Apply theorem2.3 with

$$f(x, y_0, y_1, \ldots, y_{n-1}) = \sum_{j=0}^{n-1} P_j(x)\, y_j + g(x)$$

Since it is a sum of products of continuous functions of single variables, f is continuous on $I \times \mathbb{R}^n$. The Lipschitz condition is satisfied, because:

$$|f(x,y) - f(x,w)| = \left|\sum_{j=0}^{n-1} P_j(x)\,(y_j - w_j)\right| \le k \sum_{j=0}^{n-1} |y_j - w_j| \le\ nk\|y - w\|$$

Given an open interval I that contains t_0, a solution of the initial value problem

$\frac{dy}{dt} = f(t,y),\ y(t_0) = y_0$ …………….. (2.3)

is a continuous function $y(t)$ defined on I.

with $y(t_0) = y_0$ *and* $y'(t) = f(t,y)$

2.3 The existence and uniqueness Theorem

If $f(t,y)$and $\frac{\partial f}{\partial y}(t,y)$ are continuous for $a < y < b$ and $c < t < d$ then for any $y_0 \in (a,b)$ and $t_0 \in (c,d)$ then the initial value problem (2.3) has a unique solution on some open interval I containing t_0.

Proof: (an outline version of the proof)It is the necessary that the function f is a Lipchitz continuous function on x, which means that

$|f(t,x) - f(t,y)| \leq L|x - y|$ ………………….. (2.4)

for some constant L. Any function with continuous first derivative is Lipchitz continuous has continuous first derivative, $eg. f(x) = |x|$

Proof: The proof of existence of solutions is much more involved than the proof of their uniqueness. We will consider here the slightly simpler case

$y' = f(y)$with, $y(0) = y_0$ ………….(2.5)

Assuming that $|f(x) - f(y)| \leq |x - y|$………. (2.6)

The first step is to convert the DE in to an integral equation that is easier to deal with:

We integrate both sides of (2.5) between times o and t to give

$y(t) = y_0 + \int_0^t f(y(\tilde{t}))d\tilde{t}$ ………………... (2.7)

This integral equation is equivalent to the original DE: any solution of (2.7) will solve (2.5) and vice versa.

The idea behind the method is to use the right-hand side of (2.7) as a means of refining any 'guesses' of the solution $y_n(t)$ by replacing it with

$y_{n+1}(t) = y_0 + \int_0^t f(y_n(\tilde{t}))d\tilde{t}$ ………… ….. (2.8)

We start with $y_0(t) = y_0$,for all t, set

$y_1(t) = y_0 + \int_0^t f(y_0)d\tilde{t}$ and continue in the way using (2.8).

The hope is that $y_n(t)$ will converge to the solution of the differential equation as $n \to \infty$.

(i) Use (2.6) to show that

$$|y_{n+1}(t) - y_n(t)| \leq L \int_0^t \left|y_n(\tilde{t}) - y_{n-1}(\tilde{t})\right| d\tilde{t}$$

And deduce that

$$max_{t\in[0,\frac{1}{2L}]}|y_{n+1}(t) - y_n(t)| \leq \frac{1}{2} max_{t\in[0,\frac{1}{2L}]}|y_n(t) - y_{n-1}(t)| \ldots. \quad (2.9)$$

(ii) Using (2.9) show that

$$max_{t\in[0,\frac{1}{2L}]}|y_{n+1}(t) - y_n(t)| \leq \frac{1}{2^{n-1}} max_{t\in[0,\frac{1}{2L}]}|y_1(t) - y_0(t)|$$

(iii) By writing

$$y_n(t) = [y_n(t) - y_{n-1}(t)] + [y_{n-1}(t) - y_{n-2}(t)] + \cdots + [y_1(t) - y_0(t)] + y_0(t)$$

Deduce that

$$max_{t\in[0,\frac{1}{2L}]}|y_n(t) - y_m(t)| \leq \frac{1}{2^{N-2}} max_{t\in\left[0,\frac{1}{2L}\right]}|y_1(t) - y_0(t)|, \quad \forall\, n, m \geq N \ldots(2.10)$$

It follows that $y_n(t)$ converges to some function $y_\infty(t)$as $n \to \infty$, and therefore taking limits in both sides of (2.8) implies that

$$y_\infty(t) = y_0 + \int_0^t f(y_\infty(\tilde{t}))d\tilde{t}$$

That $y_\infty(t)$satisfies (2.7), and so is a solution of the differential equation.

Summary

The proof of Picard's Existence and Uniqueness Theorem provides a way of constructing successive approximations (Picard's Iteration) to a solution. Picard's Existence and Uniqueness Theorem is a general theorem guaranteeing existence and uniqueness of ordinary differential equations, with a hypothesis which is very simple to check.

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